On Triangulated Orbit Categories

Dedicated to Claus Michael Ringel

on the occasion of his sixtieth birthday

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Abstract. We show that the category of orbits of the bounded derived category of a hereditary category under a well-behaved autoequivalence is canonically triangulated. This answers a question by Aslak Buan, Robert Marsh and Idun Reiten which appeared in their study [8] with M. Reineke and G. Todorov of the link between tilting theory and cluster algebras (*cf.* also [16]) and a question by Hideto Asashiba about orbit categories. We observe that the resulting triangulated orbit categories provide many examples of triangulated categories with the Calabi-Yau property. These include the category of projective modules over a preprojective algebra of generalized Dynkin type in the sense of Happel-Preiser-Ringel [29], whose triangulated structure goes back to Auslander-Reiten's work [6], [44], [7].

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1 Introduction

Let $\mathcal{T}$ be an additive category and $F : \mathcal{T} \to \mathcal{T}$ an automorphism (a standard construction allows one to replace a category with autoequivalence by a category with automorphism). Let $F^{\mathbf{Z}}$ denote the group of automorphisms generated by F. By definition, the *orbit category* $\mathcal{T}/F = \mathcal{T}/F^{\mathbf{Z}}$ has the same objects as $\mathcal{T}$ and its morphisms from X to Y are in bijection with

$$\bigoplus_{n\in\mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(X, F^n Y).$$

The composition is defined in the natural way (*cf.* [17], where this category is called the skew category). The canonical projection functor $\pi : \mathcal{T} \to \mathcal{T}/F$ is endowed with a natural isomorphism $\pi \circ F \xrightarrow{\sim} \pi$ and 2-universal among such functors. Clearly $\mathcal{T}/F$ is still an additive category and the projection is an additive functor. Now suppose that $\mathcal{T}$ is a triangulated category and that F is a triangle functor. Is there a triangulated structure on the orbit category such that the projection functor becomes a triangle functor? One can show that in general, the answer is negative. A closer look at the situation even gives one the impression that quite strong assumptions are needed for the answer to be positive. In this article, we give a sufficient set of conditions. Although they are very strong, they are satisfied in certain cases of interest. In particular, one obtains that the cluster categories of [8], [16] are triangulated. One also obtains that the category of projective modules over the preprojective algebra of a generalized Dynkin diagram in the sense of Happel-Preiser-Ringel [29] is triangulated, which is also immediate from Auslander-Reiten's work [6], [44], [7]. More generally, our method yields many easily constructed examples of triangulated categories with the Calabi-Yau property.

Our proof consists in constructing, under quite general hypotheses, a 'triangulated hull' into which the orbit category $\mathcal{T}/F$ embeds. Then we show that under certain restrictive assumptions, its image in the triangulated hull is stable under extensions and hence equivalent to the hull.

The contents of the article are as follows: In section 3, we show by examples that triangulated structures do not descend to orbit categories in general. In section 4, we state the main theorem for triangulated orbit categories of derived categories of hereditary algebras. We give a first, abstract, construction of the triangulated hull of an orbit category in section 5. This construction is based on the formalism of dg categories as developped in [32] [21] [57]. Using the natural t-structure on the derived category of a hereditary category we prove the main theorem in section 6.

We give a more concrete construction of the triangulated hull of the orbit category in section 7. In some sense, the second construction is 'Koszul-dual' to the first: whereas the first construction is based on the tensor algebra

$$T_A(X) = \oplus_{n=0}^{\infty} X^{\otimes_A n}$$

of a (cofibrant) differential graded bimodule X over a differential graded algebra A, the second one uses the 'exterior algebra'

$$A \oplus X^{\wedge}[-1]$$

on its dual $X^{\wedge} = \mathsf{RHom}_A(X_A, A)$ shifted by one degree. In the cases considered by Buan et al. [8] and Caldero-Chapoton-Schiffler [16], this also yields an

interesting new description of the orbit category itself in terms of the stable category [40] of a differential graded algebra.
In section 8, we observe that triangulated orbit categories provide easily constructed examples of triangulated categories with the Calabi-Yau property. Finally, in section 9, we characterize our constructions by universal properties in the 2-category of enhanced triangulated categories. This also allows us to examine their functoriality properties and to formulate a more general version of the main theorem which applies to derived categories of hereditary categories which are not necessarily module categories.

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3 Examples

Let $\mathcal{T}$ be a triangulated category, $F : \mathcal{T} \to \mathcal{T}$ an autoequivalence and $\pi : \mathcal{T} \to \mathcal{T}/F$ the projection functor. In general, a morphism

$$u : X \to Y$$

of $\mathcal{T}/F$ is given by a morphism

$$X \to \bigoplus_{i=1}^{N} F^{n_i} Y$$

with N non vanishing components $u_1, \ldots, u_N$ in $\mathcal{T}$. Therefore, in general, u does not lift to a morphism in $\mathcal{T}$ and it is not obvious how to construct a 'triangle'

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow SZ$$

in $\mathcal{T}/F$. Thus, the orbit category $\mathcal{T}/F$ is certainly not trivially triangulated. Worse, in 'most' cases, it is impossible to endow $\mathcal{T}/F$ with a triangulated structure such that the projection functor becomes a triangle functor. Let us consider three examples where $\mathcal{T}$ is the bounded derived category $\mathcal{D}^b(A) = \mathcal{D}^b(\mathsf{mod}A)$ of the category of finitely generated (right) modules $\mathsf{mod}A$ over an algebra A of finite dimension over a field k. Thus the objects of $\mathcal{D}^b(A)$ are the complexes

$$M = (\ldots \to M^p \to M^{p+1} \to \ldots)$$

of finite-dimensional A-modules such that $M^p = 0$ for all large $|p|$ and morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms. The suspension functor S is defined by $SM = M[1]$, where $M[1]^p = M^{p+1}$ and $d_{M[1]} = -d_M$, and the triangles are constructed from short exact sequences of complexes.

Suppose that A is hereditary. Then the orbit category $\mathcal{D}^b(A)/S^2$, first introduced by D. Happel in [27], is triangulated. This result is due to Peng and Xiao [43], who show that the orbit category is equivalent to the homotopy category of the category of 2-periodic complexes of projective A-modules.

On the other hand, suppose that A is the algebra of dual numbers $k[X]/(X^2)$. Then the orbit category $\mathcal{D}^b(A)/S^2$ is not triangulated. This is an observation due to A. Neeman (unpublished). Indeed, the endomorphism ring of the trivial module k in the orbit category is isomorphic to a polynomial ring $k[u]$. One checks that the endomorphism $1 + u$ is monomorphic. However, it does not admit a left inverse (or else it would be invertible in $k[u]$). But in a triangulated category, each monomorphism admits a left inverse.

One might think that this phenomenon is linked to the fact that the algebra of dual numbers is of infinite global dimension. However, it may also occur for algebras of finite global dimension: Let A be such an algebra. Then, as shown by D. Happel in [27], the derived category $\mathcal{D}^b(A)$ has Auslander-Reiten triangles. Thus, it admits an autoequivalence, the Auslander-Reiten translation τ, defined by

$$\mathsf{Hom}(?, S\tau M) \xrightarrow{\sim} D\mathsf{Hom}(M, ?)\,,$$

where D denotes the functor $\mathsf{Hom}_k(?, k)$. Now let Q be the Kronecker quiver

$$1 \rightrightarrows 2\,.$$

The path algebra $A = kQ$ is finite-dimensional and hereditary so Happel's theorem applies. The endomorphism ring of the image of the free module A_A in the orbit category $\mathcal{D}^b(A)/\tau$ is the preprojective algebra $\Lambda(Q)$ (*cf.* section 7.3). Since Q is not a Dynkin quiver, it is infinite-dimensional and in fact contains a polynomial algebra (generated by any non zero morphism from the simple projective P_1 to $\tau^{-1}P_1$). As above, it follows that the orbit category does not admit a triangulated structure.

4 The main theorem

Assume that k is a field, and $\mathcal{T}$ is the bounded derived category $\mathcal{D}^b(\mathsf{mod}A)$ of the category of finite-dimensional (right) modules $\mathsf{mod}A$ over a finite-dimensional k-algebra A. Assume that $F : \mathcal{T} \to \mathcal{T}$ is a standard equivalence

[48], *i.e.* F is isomorphic to the derived tensor product

$$? \overset{L}{\otimes}_A X : \mathcal{D}^b(\mathsf{mod}A) \to \mathcal{D}^b(\mathsf{mod}A)$$

for some complex X of A-A-bimodules. All autoequivalences with an 'algebraic construction' are of this form, *cf.* section 9.

Theorem 1 *Assume that the following hypotheses hold:*

1) *There is a hereditary abelian k-category $\mathcal{H}$ and a triangle equivalence*

$$\mathcal{D}^b(\mathsf{mod}A) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{H})$$

In the conditions 2) and 3) below, we identify $\mathcal{T}$ with $\mathcal{D}^b(\mathcal{H})$.

2) *For each indecomposable U of $\mathcal{H}$, only finitely many objects F^iU, $i \in \mathbf{Z}$, lie in $\mathcal{H}$.*

3) *There is an integer $N \geq 0$ such that the F-orbit of each indecomposable of $\mathcal{T}$ contains an object S^nU, for some $0 \leq n \leq N$ and some indecomposable object U of $\mathcal{H}$.*

Then the orbit category $\mathcal{T}/F$ admits a natural triangulated structure such that the projection functor $\mathcal{T} \to \mathcal{T}/F$ is triangulated.

The triangulated structure on the orbit category is (most probably) not unique. However, as we will see in section 9.6, the orbit category is the associated triangulated category of a dg category, the *exact (or pretriangulated) dg orbit category*, and the exact dg orbit category is unique and functorial (in the homotopy category of dg categories) since it is the solution of a universal problem. Thus, although perhaps not unique, the triangulated structure on the orbit category is at least canonical, insofar as it comes from a dg structure which is unique up to quasi-equivalence.

The construction of the triangulated orbit category $\mathcal{T}/F$ via the exact dg orbit category also shows that there is a triangle equivalence between $\mathcal{T}/F$ and the stable category $\underline{\mathcal{E}}$ of some Frobenius category $\mathcal{E}$.

In sections 7.2, 7.3 and 7.4, we will illustrate the theorem by examples. In sections 5 and 6 below, we prove the theorem. The strategy is as follows: First, under very weak assumptions, we embed $\mathcal{T}/F$ in a naturally triangulated ambient category $\mathcal{M}$ (whose intrinsic interpretation will be given in section 9.6). Then we show that $\mathcal{T}/F$ is closed under extensions in the ambient category $\mathcal{M}$. Here we will need the full strength of the assumptions 1), 2) and 3).

If $\mathcal{T}$ is the derived category of an abelian category which is not necessarily a module category, one can still define a suitable notion of a standard equivalence $\mathcal{T} \to \mathcal{T}$, *cf.* section 9. Then the analogue of the above theorem is true, *cf.* section 9.9.

5 Construction of the triangulated hull $\mathcal{M}$

The construction is based on the formalism of dg categories, which is briefly recalled in section 9.1. We refer to [32], [21] and [57] for more background.

5.1 The dg orbit category

Let $\mathcal{A}$ be a dg category and $F : \mathcal{A} \to \mathcal{A}$ a dg functor inducing an equivalence in $H^0\mathcal{A}$. We define the *dg orbit category* $\mathcal{B}$ to be the dg category with the same objects as $\mathcal{A}$ and such that for $X, Y \in \mathcal{B}$, we have

$$\mathcal{B}(X,Y) \xrightarrow{\sim} \mathsf{colim}_p \bigoplus_{n\geq 0} \mathcal{A}(F^n X, F^p Y)\,,$$

where the transition maps of the colim are given by F. This definition ensures that $H^0\mathcal{B}$ is isomorphic to the orbit category $(H^0\mathcal{A})/F$.

5.2 The projection functor and its right adjoint

¿From now on, we assume that for all objects X, Y of $\mathcal{A}$, the group

$$(H^0\mathcal{A})(X, F^n Y)$$

vanishes except for finitely many $n \in \mathbf{Z}$. We have a canonical dg functor $\pi : \mathcal{A} \to \mathcal{B}$. It yields an $\mathcal{A}$-$\mathcal{B}$-bimodule

$$(X, Y) \mapsto \mathcal{B}(\pi X, Y).$$

The standard functors associated with this bimodule are

- the derived tensor functor (=induction functor)

$$\pi_* : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$$

- the derived Hom-functor (right adjoint to π_*), which equals the restriction along π:

$$\pi_\rho : \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{A}$$

For $X \in \mathcal{A}$, we have

$$\pi_*(X^\wedge) = (\pi X)^\wedge\,,$$

where $X^\wedge$ is the functor represented by X. Moreover, we have an isomorphism in $\mathcal{D}\mathcal{A}$

$$\pi_\rho \pi_*(X^\wedge) = \bigoplus_{n\in\mathbf{Z}} F^n(X^\wedge),$$

by the definition of the morphisms of $\mathcal{B}$ and the vanishing assumption made above.

5.3 Identifying objects of the orbit category

The functor $\pi_\rho : \mathcal{DB} \to \mathcal{DA}$ is the restriction along a morphism of dg categories. Therefore, it detects isomorphisms. In particular, we obtain the following: Let $E \in \mathcal{DB}$, $Z \in \mathcal{DA}$ and let $f : Z \to \pi_\rho E$ be a morphism. Let $g : \pi_* Z \to E$ be the morphism corresponding to f by the adjunction. In order to show that g is an isomorphism, it is enough to show that $\pi_\rho g : \pi_\rho \pi_* Z \to \pi_\rho E$ is an isomorphism.

5.4 The ambient triangulated category

We use the notations of the main theorem. Let X be a complex of A-A-bimodules such that F is isomorphic to the total derived tensor product by X. We may assume that X is bounded and that its components are projective on both sides. Let $\mathcal{A}$ be the dg category of bounded complexes of finitely generated projective A-modules. The tensor product by X defines a dg functor from $\mathcal{A}$ to $\mathcal{A}$. By abuse of notation, we denote this dg functor by F as well. The assumption 2) implies that the vanishing assumption of subsection 5.2 is satisfied. Thus we obtain a dg category $\mathcal{B}$ and an equivalence of categories

$$\mathcal{D}^b(\mathsf{mod}A)/F \xrightarrow{\sim} H^0\mathcal{B}.$$

We let the ambient triangulated category $\mathcal{M}$ be the triangulated subcategory of $\mathcal{DB}$ generated by the representable functors. The Yoneda embedding $H^0\mathcal{B} \to \mathcal{DB}$ yields the canonical embedding $\mathcal{D}^b(\mathsf{mod}A)/F \to \mathcal{M}$. We have a canonical equivalence $\mathcal{D}(\mathsf{Mod}A) \xrightarrow{\sim} \mathcal{DA}$ and therefore we obtain a pair of adjoint functors (π_*, π_ρ) between $\mathcal{D}(\mathsf{Mod}A)$ and $\mathcal{DB}$. The functor π_* restricts to the canonical projection $\mathcal{D}^b(\mathsf{mod}A) \to \mathcal{D}^b(\mathsf{mod}A)/F$.

6 The orbit category is closed under extensions

Consider the right adjoint π_ρ of π_*. It is defined on $\mathcal{DB}$ and takes values in the unbounded derived category of all A-modules $\mathcal{D}(\mathsf{Mod}A)$. For $X \in \mathcal{D}^b(\mathsf{mod}A)$, the object $\pi_*\pi_\rho X$ is isomorphic to the sum of the translates F^iX, $i \in \mathbf{Z}$, of X. It follows from assumption 2) that for each fixed $n \in \mathbf{Z}$, the module $H^n_{\mathsf{mod}A} F^i X$ vanishes for almost all $i \in \mathbf{Z}$. Therefore the sum of the F^iX lies in $\mathcal{D}(\mathsf{mod}A)$. Consider a morphism $f : \pi_*X \to \pi_*Y$ of the orbit category $\mathcal{T}/F = \mathcal{D}^b(\mathsf{mod}A)/F$. We form a triangle

$$\pi_*X \to \pi_*Y \to E \to S\pi_*X$$

in $\mathcal{M}$. We apply the right adjoint π_ρ of π_* to this triangle. We get a triangle

$$\pi_\rho\pi_*X \to \pi_\rho\pi_*Y \to \pi_\rho E \to S\pi_\rho\pi_*X$$

in $\mathcal{D}(\mathsf{Mod}A)$. As we have just seen, the terms $\pi_\rho\pi_*X$ and $\pi_\rho\pi_*Y$ of the triangle belong to $\mathcal{D}(\mathsf{mod}A)$. Hence so does $\pi_\rho E$. We will construct an object $Z \in \mathcal{D}^b(\mathsf{mod}A)$ and an isomorphism $g : \pi_*Z \to E$ in the orbit category. Using proposition 6.1 below, we extend the canonical t-structure on

$\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{D}^b(\mathsf{mod}A)$ to a t-structure on all of $\mathcal{D}(\mathsf{mod}A)$. Since $\mathcal{H}$ is hereditary, part b) of the proposition shows that each object of $\mathcal{D}(\mathsf{mod}A)$ is the sum of its $\mathcal{H}$-homology objects placed in their respective degrees. In particular, this holds for $\pi_\rho E$. Each of the homology objects is a finite sum of indecomposables. Thus $\pi_\rho E$ is a sum of shifted copies of indecomposable objects of $\mathcal{H}$. Moreover, $F\pi_\rho E$ is isomorphic to $\pi_\rho E$, so that the sum is stable under F. Hence it is a sum of F-orbits of shifted indecomposable objects. By assumption 3), each of these orbits contains an indecomposable direct factor of

$$\bigoplus_{0 \leq n \leq N} (H^n \pi_\rho E)[-n].$$

Thus there are only finitely many orbits involved. Let $Z_1, \ldots, Z_M$ be shifted indecomposables of $\mathcal{H}$ such that $\pi_\rho E$ is the sum of the F-orbits of the Z_i. Let f be the inclusion morphism

$$Z = \bigoplus_{i=1}^{M} Z_i \to \pi_\rho E$$

and $g : \pi_* Z \to E$ the morphism corresponding to f under the adjunction. Clearly $\pi_\rho g$ is an isomorphism. By subsection 5.3, the morphism g is invertible and we are done.

6.1 Extension of t-structures to unbounded categories

Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ an aisle [35] in $\mathcal{T}$. Denote the associated t-structure [10] by $(\mathcal{U}_{\leq 0}, \mathcal{U}_{\geq 0})$, its heart by $\mathcal{U}_0$, its homology functors by $H^n_{\mathcal{U}} : \mathcal{T} \to \mathcal{U}_0$ and its truncation functors by $\tau_{\leq n}$ and $\tau_{>n}$. Suppose that $\mathcal{U}$ is *dominant*, *i.e.* the following two conditions hold:

1) a morphism s of $\mathcal{T}$ is an isomorphism iff $H^n_{\mathcal{U}}(s)$ is an isomorphism for all $n \in \mathbf{Z}$ and

2) for each object $X \in \mathcal{T}$, the canonical morphisms

$$\mathsf{Hom}(?, X) \to \mathsf{lim}\mathsf{Hom}(?, \tau_{\leq n} X) \text{ and } \mathsf{Hom}(X, ?) \to \mathsf{lim}\mathsf{Hom}(\tau_{>n} X, ?)$$

are surjective.

Let $\mathcal{T}^b$ be the full triangulated subcategory of $\mathcal{T}$ whose objects are the $X \in \mathcal{T}$ such that $H^n_{\mathcal{U}}(X)$ vanishes for all $|n| \gg 0$. Let $\mathcal{V}^b$ be an aisle on $\mathcal{T}^b$. Denote the associated t-structure on $\mathcal{T}^b$ by $(\mathcal{V}_{\leq n}, \mathcal{V}_{>n})$, its heart by $\mathcal{V}_0$, the homology functor by $H^n_{\mathcal{V}^b} : \mathcal{B} \to \mathcal{V}_0$ and its truncation functors by $(\sigma_{\leq 0}, \sigma_{>0})$.
Assume that there is an $N \gg 0$ such that we have

$$H^0_{\mathcal{V}^b} \xrightarrow{\sim} H^0_{\mathcal{V}^b} \tau_{>-n} \text{ and } H^0_{\mathcal{V}^b} \tau_{\leq n} \xrightarrow{\sim} H^0_{\mathcal{V}^b}$$

for all $n \geq N$. We define $H^0_{\mathcal{V}} : \mathcal{T} \to \mathcal{V}_0$ by

$$H^0_{\mathcal{V}}(X) = \mathsf{colim} H^0_{\mathcal{V}^b} \tau_{>-n} \tau_{\leq m} X$$

and $H^n_{\mathcal{V}}(X) = H^0_{\mathcal{V}} S^n X$, $n \in \mathbf{Z}$. We define $\mathcal{V} \subset \mathcal{T}$ to be the full subcategory of $\mathcal{T}$ whose objects are the $X \in \mathcal{T}$ such that $H^n_{\mathcal{V}}(X) = 0$ for all $n > 0$.

Proposition 1 *a) $\mathcal{V}$ is an aisle in $\mathcal{T}$ and the associated t-structure is dominant.*

b) If $\mathcal{V}^b$ is hereditary, i.e. *each triangle*

$$\sigma_{\leq 0} X \to X \to \sigma_{>0} X \to S\sigma_{\leq 0} X \ , \ X \in \mathcal{T}^b$$

splits, then $\mathcal{V}$ is hereditary and each object $X \in \mathcal{T}$ is (non canonically) isomorphic to the sum of the $S^{-n} H^n_{\mathcal{V}}(X)$, $n \in \mathbf{Z}$.

The proof is an exercise on t-structures which we leave to the reader.

7 Another construction of the triangulated hull of the orbit category

7.1 The construction

Let A be a finite-dimensional algebra of finite global dimension over a field k. Let X be an A-A-bimodule complex whose homology has finite total dimension. Let F be the functor

$$? \overset{L}{\otimes}_A X : \mathcal{D}^b(\mathsf{mod} A) \to \mathcal{D}^b(\mathsf{mod} A).$$

We suppose that F is an equivalence and that for all L, M in $\mathcal{D}^b(\mathsf{mod} A)$, the group

$$\mathsf{Hom}(L, F^n M)$$

vanishes for all but finitely many $n \in \mathbf{Z}$. We will construct a triangulated category equivalent to the triangulated hull of section 5.

Consider A as a dg algebra concentrated in degree 0. Let B be the dg algebra with underlying complex $A \oplus X[-1]$, where the multiplication is that of the trivial extension:

$$(a, x)(a', x') = (aa', ax' + xa').$$

Let $\mathcal{D}B$ be the derived category of B and $\mathcal{D}^b(B)$ the *bounded derived category*, *i.e.* the full subcategory of $\mathcal{D}B$ formed by the dg modules whose homology has finite total dimension over k. Let $\mathsf{per}(B)$ be the *perfect* derived category of B, *i.e.* the smallest subcategory of $\mathcal{D}B$ containing B and stable under shifts, extensions and passage to direct factors. By our assumption on A and X, the perfect derived category is contained in $\mathcal{D}^b(B)$. The obvious morphism $B \to A$ induces a restriction functor $\mathcal{D}^b A \to \mathcal{D}^b B$ and by composition, we obtain a functor

$$\mathcal{D}^b A \to \mathcal{D}^b B \to \mathcal{D}^b(B)/\mathsf{per}(B)$$

Theorem 2 *The category $\mathcal{D}^b(B)/\mathsf{per}(B)$ is equivalent to the triangulated hull (*cf. *section 5) of the orbit category of $\mathcal{D}^b(A)$ under F and the above functor identifies with the projection functor.*

Proof. If we replace X by a quasi-isomorphic bimodule, the algebra B is replaced by a quasi-isomorphic dg algebra and its derived category by an equivalent one. Therefore, it is no restriction of generality to assume, as we will do, that X is cofibrant as a dg A-A-bimodule. We will first compute morphisms in $\mathcal{D}B$ between dg B-modules whose restrictions to A are cofibrant. For this, let C be the dg submodule of the bar resolution of B as a bimodule over itself whose underlying graded module is

$$C = \coprod_{n\geq 0} B \otimes_A X^{\otimes_A n} \otimes_A B.$$

The bar resolution of B is a coalgebra in the category of dg B-B-bimodules (*cf. e.g.* [32]) and C becomes a dg subcoalgebra. Its counit is

$$\varepsilon : C \to B \otimes_A B \to B$$

and its comultiplication is given by

$$\Delta(b_0, x_1, \ldots, x_n, b_{n+1}) = \sum_{i=0}^{n} (b_0, x_1, \ldots, x_i) \otimes 1 \otimes 1 \otimes (x_{i+1}, \ldots, b_{n+1}).$$

It is not hard to see that the inclusion of C in the bar resolution is an homotopy equivalence of left (and of right) dg B-modules. Therefore, the same holds for the counit $\varepsilon : C \to B$. For an arbitrary right dg B-module L, the counit ε thus induces a quasi-isomorphism $L \otimes_B C \to L$. Now suppose that the restriction of L to A is cofibrant. Then $L \otimes_A B \otimes_A B$ is cofibrant over B and thus $L \otimes_B C \to L$ is a cofibrant resolution of L. Let $\mathcal{C}_1$ be the dg category whose objects are the dg B-modules whose restriction to A is cofibrant and whose morphism spaces are the

$$\mathsf{Hom}_B(L \otimes_B C, M \otimes_B C).$$

Let $\mathcal{C}_2$ be the dg category with the same objects as $\mathcal{C}_1$ and whose morphism spaces are

$$\mathsf{Hom}_B(L \otimes_B C, M).$$

By definition, the composition of two morphisms f and g of $\mathcal{C}_2$ is given by

$$f \circ (g \otimes \mathbf{1}_C) \circ (\mathbf{1}_L \otimes \Delta).$$

We have a dg functor $\Phi : \mathcal{C}_2 \to \mathcal{C}_1$ which is the identity on objects and sends $g : L \to M$ to

$$(g \otimes \mathbf{1}_C) \circ (\mathbf{1}_L \otimes \Delta) : L \otimes_B C \to M \otimes_B C.$$

The morphism

$$\mathsf{Hom}_B(L \otimes_B C, M) \to \mathsf{Hom}_B(L \otimes_B C, M \otimes_B C)$$

given by Φ is left inverse to the quasi-isomorphism induced by $M \otimes_B C \to M$. Therefore, the dg functor Φ yields a quasi-isomorphism between $\mathcal{C}_2$ and $\mathcal{C}_1$ so that we can compute morphisms and compositions in $\mathcal{D}B$ using $\mathcal{C}_2$. Now suppose that L and M are cofibrant dg A-modules. Consider them as dg B-modules via restriction along the projection $B \to A$. Then we have natural isomorphisms of complexes

$$\mathsf{Hom}_{\mathcal{C}_2}(L, M) = \mathsf{Hom}_B(L \otimes_B C, M) \xrightarrow{\sim} \mathsf{Hom}_A(\coprod_{n \geq 0} L \otimes_A X^{\otimes_A n}, M).$$

Moreover, the composition of morphisms in $\mathcal{C}_2$ translates into the natural composition law for the right hand side. Now we will compute morphisms in the quotient category $\mathcal{D}B/\mathsf{per}(B)$. Let M be as above. For $p \geq 0$, let $C_{\leq p}$ be the dg subbimodule with underlying graded module

$$\coprod_{n=0}^{p} B \otimes_A X^{\otimes_A n} \otimes_A B.$$

Then each morphism

$$P \to M \otimes_B C$$

of $\mathcal{D}B$ from a perfect object P factors through

$$M \otimes_B C_{\leq p} \to M \otimes_B C$$

for some $p \geq 0$. Therefore, the following complex computes morphisms in $\mathcal{D}B/\mathsf{per}(B)$:

$$\mathsf{colim}_{p \geq 1} \mathsf{Hom}_B(L \otimes_B C, M \otimes_B C/C_{\leq p-1}).$$

Now it is not hard to check that the inclusion

$$M \otimes X^{\otimes_A p} \otimes_A A \to M \otimes_B (C/C_{\leq p-1})$$

is a quasi-isomorphism of dg B-modules. Thus we obtain quasi-isomorphisms

$$\mathsf{Hom}_B(L \otimes_B C, M \otimes X^{\otimes_A p} \otimes_A A) \to \mathsf{Hom}_B(L \otimes_B C, M \otimes_B C/C_{\leq p-1})$$

and

$$\prod_{n \geq 0} \mathsf{Hom}_A(L \otimes_A X^{\otimes_A n}, M \otimes_A X^{\otimes_A p}) \to \mathsf{Hom}_B(L \otimes_B C, M \otimes_B C/C_{\leq p-1}).$$

Moreover, it is not hard to check that if we define transition maps

$$\prod_{n \geq 0} \mathsf{Hom}_A(L \otimes_A X^{\otimes_A n}, M \otimes_A X^{\otimes_A p}) \to \prod_{n \geq 0} \mathsf{Hom}_A(L \otimes_A X^{\otimes_A n}, M \otimes_A X^{\otimes_A (p+1)})$$

by sending f to $f \otimes_A \mathbf{1}_X$, then we obtain a quasi-isomorphism of direct systems of complexes. Therefore, the following complex computes morphisms in $\mathcal{D}B/\mathsf{per}(B)$:

$$\mathsf{colim}_{p\geq 1} \prod_{n\geq 0} \mathsf{Hom}_A(L\otimes_A X^{\otimes_A n}, M\otimes_A X^{\otimes_A p}).$$

Let $\mathcal{C}_3$ be the dg category whose objects are the cofibrant dg A-modules and whose morphisms are given by the above complexes. If L and M are cofibrant dg A-modules and belong to $\mathcal{D}^b(\mathsf{mod}A)$, then, by our assumptions on F, this complex is quasi-isomorphic to its subcomplex

$$\mathsf{colim}_{p\geq 1} \coprod_{n\geq 0} \mathsf{Hom}_A(L\otimes_A X^{\otimes_A n}, M\otimes_A X^{\otimes_A p}).$$

Thus we obtain a dg functor

$$\mathcal{B} \to \mathcal{C}_3$$

(where $\mathcal{B}$ is the dg category defined in 5.1) which induces a fully faithful functor $H^0(\mathcal{B}) \to \mathcal{D}B/\mathsf{per}(B)$ and thus a fully faithful functor $\mathcal{M} \to \mathcal{D}^b(B)/\mathsf{per}(B)$. This functor is also essentially surjective. Indeed, every object in $\mathcal{D}^b(B)$ is an extension of two objects which lie in the image of $\mathcal{D}^b(\mathsf{mod}A)$. The assertion about the projection functor is clear from the above proof.

7.2 The motivating example

Let us suppose that the functor F is given by

$$M \mapsto \tau S^{-1} M\ ,$$

where τ is the Auslander-Reiten translation of $\mathcal{D}^b(A)$ and S the shift functor. This is the case considered in [8] for the construction of the cluster category. The functor F^{-1} is isomorphic to

$$M \mapsto S^{-2}\nu M$$

where ν is the Nakayama functor

$$\nu = ?\overset{L}{\otimes}_A DA\ ,\ DA = \mathsf{Hom}_k(A,k).$$

Thus F^{-1} is given by the bimodule $X = (DA)[-2]$ and $B = A \oplus (DA)[-3]$ is the trivial extension of A with a non standard grading: A is in degree 0 and DA in degree 3. For example, if A is the quiver algebra of an alternating quiver whose underlying graph is A_n, then the underlying ungraded algebra of B is the quadratic dual of the preprojective algebra associated with A_n, *cf.* [15]. The algebra B viewed as a differential graded algebra was investigated by Khovanov-Seidel in [36]. Here the authors show that $\mathcal{D}^b(B)$ admits a canonical action by the braid group on $n+1$ strings, a result which was obtained independently in a similar context by Zimmermann-Rouquier [52]. The canonical generators of the braid group act by triangle functors T_i endowed with morphisms $\phi_i : T_i \to \mathbf{1}$. The cone on each ϕ_i belongs to $\mathsf{per}(B)$ and $\mathsf{per}(B)$ is in fact equal to its smallest triangulated subcategory stable under direct factors and containing these cones. Thus, the action becomes trivial in $\mathcal{D}^b(B)/\mathsf{per}(B)$ and in a certain sense, this is the largest quotient where the ϕ_i become invertible.

7.3 Projectives over the preprojective algebra

Let A be the path algebra of a Dynkin quiver, *i.e.* a quiver whose underlying graph is a Dynkin diagram of type A, D or E. Let C be the associated preprojective algebra [26], [20], [50]. In Proposition 3.3 of [7], Auslander-Reiten show that the category of projective modules over C is equivalent to the stable category of maximal Cohen-Macaulay modules over a representation-finite isolated hypersurface singularity. In particular, it is triangulated. This can also be deduced from our main theorem: Indeed, it follows from D. Happel's description [27] of the derived category $\mathcal{D}^b(A)$ that the category $\mathsf{proj}C$ of finite dimensional projective C-modules is equivalent to the orbit category $\mathcal{D}^b(A)/\tau$, *cf.* also [25]. Moreover, by the theorem of the previous section, we have an equivalence

$$\mathsf{proj}C \xrightarrow{\sim} \mathcal{D}^b(B)/\mathsf{per}(B)$$

where $B = A \oplus (DA)[-2]$. This equivalence yields in fact more than just a triangulated structure: it shows that $\mathsf{proj}C$ is endowed with a canonical Hochschild 3-cocycle m_3, *cf.* for example [11]. It would be interesting to identify this cocycle in the description given in [23].

7.4 Projectives over $\Lambda(L_n)$

The category of projective modules over the algebra $k[\varepsilon]/(\varepsilon^2)$ of dual numbers is triangulated. Indeed, it is equivalent to the orbit category of the derived category of the path algebra of a quiver of type A_2 under the Nakayama autoequivalence ν. Thus, we obtain examples of triangulated categories whose Auslander-Reiten quiver contains a loop. It has been known since Riedtmann's work [49] that this cannot occur in the stable category (*cf.* below) of a self-injective finite-dimensional algebra. It may therefore seem surprising, *cf.* [58], that loops do occur in this more general context. However, loops already do occur in stable categories of finitely generated reflexive modules over certain non commutative generalizations of local rings of rational double points, as shown by Auslander-Reiten in [6]. These were completely classified by Reiten-Van den Bergh in [46]. In particular, the example of the dual numbers and its generalization below are among the cases covered by [46].

The example of the dual numbers generalizes as follows: Let $n \geq 1$ be an integer. Following [30], the *generalized Dynkin graph* L_n is defined as the graph

$$\circlearrowleft 1 —— 2 —— \quad \cdots \quad —— n-1 —— n$$

Its edges are in natural bijection with the orbits of the involution which exchanges each arrow α with $\overline{\alpha}$ in the following quiver:

$$\begin{tikzcd} 1 \arrow[loop left, "\varepsilon=\overline{\varepsilon}"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a_1}"] \arrow[r, shift left, "a_2"] & \cdots \arrow[l, shift left, "\overline{a_2}"] \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a_{n-2}}"] \arrow[r, shift left, "a_{n-1}"] & n \arrow[l, shift left, "\overline{a_{n-1}}"] \end{tikzcd} .$$

The associated preprojective algebra $\Lambda(L_n)$ of generalized Dynkin type L_n is defined as the quotient of the path algebra of this quiver by the ideal generated by the relators

$$r_v = \sum \alpha\, \overline{\alpha}\,,$$

where, for each $1 \leq v \leq n$, the sum ranges over the arrows α with starting point v. Let A be the path algebra of a Dynkin quiver with underlying Dynkin graph A_{2n}. Using D. Happel's description [27] of the derived category of a Dynkin quiver, we see that the orbit category $\mathcal{D}^b(A)/(\tau^n S)$ is equivalent to the category of finitely generated projective modules over the algebra $\Lambda(L_n)$. By the main theorem, this category is thus triangulated. Its Auslander-Reiten quiver is given by the ordinary quiver of $\Lambda(L_n)$, *cf.* above, endowed with $\tau = \mathbf{1}$: Indeed, in $\mathcal{D}^b(A)$, we have $S^2 = \tau^{-(2n+1)}$ so that in the orbit category, we obtain

$$\mathbf{1} = (\tau^n S)^2 = \tau^{2n} S^2 = \tau^{-1}.$$

8 On the Calabi-Yau property

8.1 Serre functors and localizations

Let k be a field and $\mathcal{T}$ a k-linear triangulated category with finite-dimensional Hom-spaces. We denote the suspension functor of $\mathcal{T}$ by S. Recall from [47] that a *right Serre functor* for $\mathcal{T}$ is the datum of a triangle functor $\nu : \mathcal{T} \to \mathcal{T}$ together with bifunctor isomorphisms

$$D\mathsf{Hom}_{\mathcal{T}}(X, ?) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{T}}(?, \nu X)\ ,\ X \in \mathcal{T}\ ,$$

where $D = \mathsf{Hom}_k(?, k)$. If ν exists, it is unique up to isomorphism of triangle functors. Dually, a *left Serre functor* is the datum of a triangle functor $\nu' : \mathcal{T} \to \mathcal{T}$ and isomorphisms

$$D\mathsf{Hom}_{\mathcal{T}}(?, X) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{T}}(\nu', ?)\ ,\ X \in \mathcal{T}.$$

The category $\mathcal{T}$ *has Serre duality* if it has both a left and a right Serre functor, or equivalently, if it has a onesided Serre functor which is an equivalence, *cf.* [47] [13]. The following lemma is used in [9].

Lemma 1 *Suppose that $\mathcal{T}$ has a left Serre functor ν'. Let $\mathcal{U} \subset \mathcal{T}$ be a thick triangulated subcategory and $L : \mathcal{T} \to \mathcal{T}/\mathcal{U}$ the localization functor.*

a) If L admits a right adjoint R, then $L\nu' R$ is a left Serre functor for $\mathcal{T}/\mathcal{U}$.

b) More generally, if the functor $\nu' : \mathcal{T} \to \mathcal{T}$ admits a total right derived functor $\mathbf{R}\nu' : \mathcal{T}/\mathcal{U} \to \mathcal{T}/\mathcal{U}$ in the sense of Deligne [19] with respect to the localization $\mathcal{T} \to \mathcal{T}/\mathcal{U}$, then $\mathbf{R}\nu'$ is a left Serre functor for $\mathcal{T}/\mathcal{U}$.

Proof. a) For X, Y in $\mathcal{T}$, we have

$$\begin{aligned}\mathsf{Hom}_{\mathcal{T}/\mathcal{U}}(L\nu' RX, Y) &= \mathsf{Hom}_{\mathcal{T}}(\nu' RX, RY)\\ &= D\mathsf{Hom}_{\mathcal{T}}(RY, RX) = D\mathsf{Hom}_{\mathcal{T}/\mathcal{U}}(Y, X).\end{aligned}$$

Here, for the last isomorphism, we have used that R is fully faithful (L is a localization functor).
b) We assume that $\mathcal{T}$ is small. Let $\mathsf{Mod}\mathcal{T}$ denote the (large) category of functors from $\mathcal{T}^{op}$ to the category of abelian groups and let $h : \mathcal{T} \to \mathsf{Mod}\mathcal{T}$ denote the Yoneda embedding. Let L^* be the unique right exact functor $\mathsf{Mod}\mathcal{T} \to \mathsf{Mod}(\mathcal{T}/\mathcal{U})$ which sends hX to hLX, $X \in \mathcal{T}$. By the calculus of (right) fractions, L^* has a right adjoint R which takes an object Y to

$$\mathsf{colim}_{\Sigma_Y} hY' ,$$

where the colim ranges over the category Σ_Y of morphisms $s : Y \to Y'$ which become invertible in $\mathcal{T}/\mathcal{U}$. Clearly L^*R is isomorphic to the identity so that R is fully faithful. By definition of the total right derived functor, for each object $X \in \mathcal{T}/\mathcal{U}$, the functor

$$\mathsf{colim}_{\Sigma_X} h(L\nu' X') = L^* \nu'^* Rh(X)$$

is represented by $\mathbf{R}\nu'(X)$. Therefore, we have

$$\begin{aligned}\mathsf{Hom}_{\mathcal{T}/\mathcal{U}}(\mathbf{R}\nu'(X), Y) &= \mathsf{Hom}_{\mathsf{Mod}\mathcal{T}/\mathcal{U}}(L^*\nu'^* Rh(X), h(Y))\\ &= \mathsf{Hom}_{\mathsf{Mod}\mathcal{T}}(\nu'^* Rh(X), Rh(Y)).\end{aligned}$$

Now by definition, the last term is isomorphic to

$$\mathsf{Hom}_{\mathsf{Mod}\mathcal{T}}(\mathsf{colim}_{\Sigma_X} h(\nu' X'), \mathsf{colim}_{\Sigma_Y} hY') = \mathsf{lim}_{\Sigma_X} \mathsf{colim}_{\Sigma_Y} \mathsf{Hom}_{\mathcal{T}}(\nu' X', Y')$$

and this identifies with

$$\begin{aligned}\mathsf{lim}_{\Sigma_X} \mathsf{colim}_{\Sigma_Y} D\mathsf{Hom}_{\mathcal{T}}(Y', X') &= D(\mathsf{colim}_{\Sigma_X} \mathsf{lim}_{\Sigma_Y} \mathsf{Hom}_{\mathcal{T}}(Y', X'))\\ &= D\mathsf{Hom}_{\mathcal{T}/\mathcal{U}}(Y, X).\end{aligned}$$

8.2 Definition of the Calabi-Yau property

Keep the hypotheses of the preceding section. By definition [39], the triangulated category $\mathcal{T}$ is *Calabi-Yau of CY-dimension* d if it has Serre duality and there is an isomorphism of triangle functors

$$\nu \xrightarrow{\sim} S^d.$$

By extension, if we have $\nu^e \xrightarrow{\sim} S^d$ for some integer $e > 0$, one sometimes says that $\mathcal{T}$ is Calabi-Yau of fractional dimension d/e. Note that $d \in \mathbf{Z}$ is only determined up to a multiple of the order of S. It would be interesting to link the CY-dimension to Rouquier's [51] notion of dimension of a triangulated category.

The terminology has its origin in the following example: Let X be a smooth projective variety of dimension d and let $\omega_X = \Lambda^d T_X^*$ be the canonical bundle. Let $\mathcal{T}$ be the bounded derived category of coherent sheaves on X. Then the classical Serre duality

$$D\mathsf{Ext}^i_X(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathsf{Ext}^{d-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X)\,,$$

where $\mathcal{F}$, $\mathcal{G}$ are coherent sheaves, lifts to the isomorphism

$$D\mathsf{Hom}_{\mathcal{T}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{T}}(\mathcal{G}, \mathcal{F} \otimes \omega_X[d])\,,$$

where $\mathcal{F}$, $\mathcal{G}$ are bounded complexes of coherent sheaves. Thus $\mathcal{T}$ has Serre duality and $\nu = ? \otimes \omega_X[d]$. So the category $\mathcal{T}$ is Calabi-Yau of CY-dimension d iff ω_X is isomorphic to $\mathcal{O}_X$, which means precisely that the variety X is Calabi-Yau of dimension d.

If $\mathcal{T}$ is a Calabi-Yau triangulated category 'of algebraic origin' (for example, the derived category of a category of modules or sheaves), then it often comes from a Calabi-Yau A_∞-category. These are of considerable interest in mathematical physics, since, as Kontsevich shows [38], [37], *cf.* also [18], a topological quantum field theory is associated with each Calabi-Yau A_∞-category satisfying some additional assumptions[1].

8.3 Examples

(1) If A is a finite-dimensional k-algebra, then the homotopy category $\mathcal{T}$ of bounded complexes of finitely generated projective A-modules has a Nakayama functor iff DA is of finite projective dimension. In this case, the category $\mathcal{T}$ has Serre duality iff moreover A_A is of finite injective dimension, *i.e.* iff A is Gorenstein, *cf.* [28]. Then the category $\mathcal{T}$ is Calabi-Yau (necessarily of CY-dimension 0) iff A is symmetric.

(2) If Δ is a Dynkin graph of type A_n, D_n, E_6, E_7 or E_8 and h is its Coxeter number (*i.e.* $n+1$, $2(n-1)$, 12, 18 or 30, respectively), then for the bounded derived category of finitely generated modules over a quiver with underlying graph Δ, we have isomorphisms

$$\nu^h = (S\tau)^h = S^h\tau^h = S^{(h-2)}.$$

[1] Namely, the associated triangulated category should admit a generator whose endomorphism A_∞-algebra B is compact (*i.e.* finite-dimensional), smooth (*i.e.* B is perfect as a bimodule over itself), and whose associated Hodge-de Rham spectral sequence collapses (this property is conjectured to hold for all smooth compact A_∞-algebras over a field of characteristic 0).

Hence this category is Calabi-Yau of fractional dimension $(h-2)/h$.
(3) Suppose that A is a finite-dimensional algebra which is selfinjective (*i.e.* A_A is also an injective A-module). Then the category $\mathsf{mod}A$ of finite-dimensional A-modules is Frobenius, *i.e.* it is an abelian (or, more generally, an exact) category with enough projectives, enough injectives and where an object is projective iff it is injective. The stable category $\underline{\mathsf{mod}}\,A$ obtained by quotienting $\mathsf{mod}A$ by the ideal of morphisms factoring through injectives is triangulated, *cf.* [27]. The inverse of its suspension functor sends a module M to the kernel ΩM of an epimorphism $P \to M$ with projective P. Let $\mathcal{N}M = M \otimes_A DA$. Then $\underline{\mathsf{mod}}\,A$ has Serre duality with Nakayama functor $\nu = \Omega \circ \mathcal{N}$. Thus, the stable category is Calabi-Yau of CY-dimension d iff we have an isomorphism of triangle functors

$$\Omega^{(d+1)} \circ \mathcal{N} = \mathbf{1}.$$

For this, it is clearly sufficient that we have an isomorphism

$$\Omega^{d+1}_{A^e}(A) \otimes_A DA \xrightarrow{\sim} A\,,$$

in the stable category of A-A-bimodules, *i.e.* modules over the selfinjective algebra $A \otimes A^{op}$. For example, we deduce that if A is the path algebra of a cyclic quiver with n vertices divided by the ideal generated by all paths of length $n-1$, then $\underline{\mathsf{mod}}\,A$ is Calabi-Yau of CY-dimension 3.
(4) Let A be a dg algebra. Let $\mathsf{per}(A) \subset \mathcal{D}(A)$ be the subcategory of perfect dg A-modules, *i.e.* the smallest full triangulated subcategory of $\mathcal{D}(A)$ containing A and stable under forming direct factors. For each P in $\mathsf{per}(A)$ and each $M \in \mathcal{D}(A)$, we have canonical isomorphisms

$$D\mathsf{RHom}_A(P, M) \xrightarrow{\sim} \mathsf{RHom}_A(M, D\mathsf{RHom}_A(P, A))$$

and

$$P \overset{L}{\otimes}_A DA \xrightarrow{\sim} D\mathsf{RHom}_A(P, A).$$

So we obtain a canonical isomorphism

$$D\mathsf{RHom}_A(P, M) \xrightarrow{\sim} \mathsf{RHom}_A(M, P \overset{L}{\otimes}_A DA).$$

Thus, if we are given a quasi-isomorphism of dg A-A-bimodules

$$\phi : A[n] \to DA\,,$$

we obtain

$$D\mathsf{RHom}_A(P, M) \xrightarrow{\sim} \mathsf{RHom}_A(M, P[n])$$

and in particular $\mathsf{per}(A)$ is Calabi-Yau of CY-dimension n.
(5) To consider a natural application of the preceding example, let B be the symmetric algebra on a finite-dimensional vector space V of dimension n and $\mathcal{T} \subset \mathcal{D}(B)$ the localizing subcategory generated by the trivial B-module k (*i.e.* the smallest full triangulated subcategory stable under infinite sums and

containing the trivial module). Let $\mathcal{T}^c$ denote its subcategory of compact objects. This is exactly the triangulated subcategory of $\mathcal{D}(B)$ generated by k, and also exactly the subcategory of the complexes whose total homology is finite-dimensional and supported in 0. Then $\mathcal{T}^c$ is Calabi-Yau of CY-dimension n. Indeed, if

$$A = \mathsf{RHom}_B(k, k)$$

is the Koszul dual of B (thus, A is the exterior algebra on the dual of V concentrated in degree 1; it is endowed with $d = 0$), then the functor

$$\mathsf{RHom}_B(k, ?) : \mathcal{D}(B) \to \mathcal{D}(A)$$

induces equivalences from $\mathcal{T}$ to $\mathcal{D}(A)$ and $\mathcal{T}^c$ to $\mathsf{per}(A)$, *cf.* for example [32]. Now we have a canonical isomorphism of A-A-bimodules $A[n] \xrightarrow{\sim} DA$ so that $\mathsf{per}(A)$ and $\mathcal{T}$ are Calabi-Yau of CY-dimension n. As pointed out by I. Reiten, in this case, the Calabi-Yau property even holds more generally: Let $M \in \mathcal{D}(B)$ and denote by $M_{\mathcal{T}} \to M$ the universal morphism from an object of $\mathcal{T}$ to M. Then, for $X \in \mathcal{T}^c$, we have natural morphisms

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}(B)}(M, X[n]) \to \mathsf{Hom}_{\mathcal{T}}(M_{\mathcal{T}}, X[n]) &\xrightarrow{\sim} D\mathsf{Hom}_{\mathcal{T}}(X, M_{\mathcal{T}}) \\ &\xrightarrow{\sim} D\mathsf{Hom}_{\mathcal{D}(B)}(X, M). \end{aligned}$$

The composition

$$(*) \qquad \mathsf{Hom}_{\mathcal{D}(B)}(M, X[n]) \to D\mathsf{Hom}_{\mathcal{D}(B)}(X, M)$$

is a morphism of (co-)homological functors in $X \in \mathcal{T}^c$ (resp. $M \in \mathcal{D}(B)$). We claim that it is an isomorphism for $M \in \mathsf{per}(B)$ and $X \in \mathcal{T}^c$. It suffices to prove this for $M = B$ and $X = k$. Then one checks it using the fact that

$$\mathsf{RHom}_B(k, B) \xrightarrow{\sim} k[-n].$$

These arguments still work for certain non-commutative algebras B: If B is an Artin-Schelter regular algebra [2] [1] of global dimension 3 and type A and $\mathcal{T}$ the localizing subcategory of the derived category $\mathcal{D}(B)$ of non graded B-modules generated by the trivial module, then $\mathcal{T}^c$ is Calabi-Yau and one even has the isomorphism $(*)$ for each perfect complex of B-modules M and each $X \in \mathcal{T}^c$, *cf.* for example section 12 of [41].

8.4 Orbit categories with the Calabi-Yau property

The main theorem yields the following

Corollary 1 *If $d \in \mathbf{Z}$ and Q is a quiver whose underlying graph is Dynkin of type A, D or E, then*

$$\mathcal{T} = \mathcal{D}^b(kQ)/\tau^{-1}S^{d-1}$$

is Calabi-Yau of CY-dimension d. In particular, the cluster category $\mathcal{C}_{kQ}$ is Calabi-Yau of dimension 2 and the category of projective modules over the preprojective algebra $\Lambda(Q)$ is Calabi-Yau of CY-dimension 1.

The category of projective modules over the preprojective algebra $\Lambda(L_n)$ of example 7.4 does not fit into this framework. Nevertheless, it is also Calabi-Yau of CY-dimension 1, since we have $\tau = \mathbf{1}$ in this category and therefore $\nu = S\tau = S$.

8.5 Module categories over Calabi-Yau categories

Calabi-Yau triangulated categories turn out to be 'self-reproducing': Let $\mathcal{T}$ be a triangulated category. Then the category $\mathsf{mod}\mathcal{T}$ of finitely generated functors from $\mathcal{T}^{op}$ to $\mathsf{Mod}k$ is abelian and Frobenius, *cf.* [24], [42]. If we denote by Σ the exact functor $\mathsf{mod}\mathcal{T} \to \mathsf{mod}\mathcal{T}$ which takes $\mathsf{Hom}(?, X)$ to $\mathsf{Hom}(?, SX)$, then it is not hard to show [24] [25] that we have

$$\Sigma \xrightarrow{\sim} S^3$$

as triangle functors $\underline{\mathsf{mod}}\mathcal{T} \to \underline{\mathsf{mod}}\mathcal{T}$. One deduces the following lemma, which is a variant of a result which Auslander-Reiten [7] obtained using dualizing R-varieties [4] and their functor categories [3], *cf.* also [5] [45]. A similar result is due to Geiss [25].

Lemma 2 *If $\mathcal{T}$ is Calabi-Yau of CY-dimension d, then the stable category $\underline{\mathsf{mod}}\,\mathcal{T}$ is Calabi-Yau of CY-dimension $3d-1$. Moreover, if the suspension of $\mathcal{T}$ is of order n, the order of the suspension functor of $\underline{\mathsf{mod}}\,\mathcal{T}$ divides $3n$.*

For example, if A is the preprojective algebra of a Dynkin quiver or equals $\Lambda(L_n)$, then we find that the stable category $\underline{\mathsf{mod}}\,A$ is Calabi-Yau of CY-dimension $3 \times 1 - 1 = 2$. This result, with essentially the same proof, is due to Auslander-Reiten [7]. For the preprojective algebras of Dynkin quivers, it also follows from a much finer result due to Ringel and Schofield (unpublished). Indeed, they have proved that there is an isomorphism

$$\Omega^3_{A^e}(A) \xrightarrow{\sim} DA$$

in the stable category of bimodules, *cf.* Theorems 4.8 and 4.9 in [15]. This implies the Calabi-Yau property since we also have an isomorphism

$$DA \otimes_A DA \xrightarrow{\sim} A$$

in the stable category of bimodules, by the remark following definition 4.6 in [15]. For the algebra $\Lambda(L_n)$, the analogous result follows from Proposition 2.3 of [12].

9 Universal properties

9.1 The homotopy category of small dg categories

Let k be a field. A *differential graded (=dg) k-module* is a $\mathbf{Z}$-graded vector space

$$V = \bigoplus_{p \in \mathbf{Z}} V^p$$

endowed with a differential d of degree 1. The *tensor product* $V \otimes W$ of two dg k-modules is the graded space with components

$$\bigoplus_{p+q=n} V^p \otimes W^q \,, \; n \in \mathbf{Z} \,,$$

and the differential $d \otimes \mathbf{1} + \mathbf{1} \otimes d$, where the tensor product of maps is defined using the Koszul sign rule. A *dg category* [32] [21] is a k-category $\mathcal{A}$ whose morphism spaces are dg k-modules and whose compositions

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$$

are morphisms of dg k-modules. For a dg category $\mathcal{A}$, the *category* $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and has morphism spaces $H^0\mathcal{A}\,(X, Y)$, $X, Y \in \mathcal{A}$. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ between dg categories is a functor compatible with the grading and the differential on the morphism spaces. It is a *quasi-equivalence* if it induces quasi-isomorphisms in the morphism spaces and an equivalence of categories from $H^0(\mathcal{A})$ to $H^0(\mathcal{B})$. We denote by dgcat the category of small dg categories. The *homotopy category* of small dg categories is the localization $\mathsf{Ho}(\mathsf{dgcat})$ of dgcat with respect to the class of quasi-equivalences. According to [56], the category dgcat admits a structure of Quillen model category (*cf.* [22], [31]) whose weak equivalences are the quasi-equivalences. This implies in particular that for $\mathcal{A}, \mathcal{B} \in \mathsf{dgcat}$, the morphisms from $\mathcal{A}$ to $\mathcal{B}$ in the localization $\mathsf{Ho}(\mathsf{dgcat})$ form a set.

9.2 The bimodule bicategory

For two dg categories $\mathcal{A}$, $\mathcal{B}$, we denote by $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ the full subcategory of the derived category $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$, *cf.* [32], whose objects are the dg $\mathcal{A}$-$\mathcal{B}$-bimodules X such that $X(?, A)$ is isomorphic to a representable functor in $\mathcal{D}(\mathcal{B})$ for each object A of $\mathcal{A}$. We think of the objects of $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ as 'representations up to homotopy' of $\mathcal{A}$ in $\mathcal{B}$. The *bimodule bicategory* rep, *cf.* [32] [21], has as objects all small dg categories; the morphism category between two objects $\mathcal{A}$, $\mathcal{B}$ is $\mathsf{rep}(\mathcal{A}, \mathcal{B})$; the composition bifunctor

$$\mathsf{rep}(\mathcal{B}, \mathcal{C}) \times \mathsf{rep}(\mathcal{A}, \mathcal{B}) \to \mathsf{rep}(\mathcal{A}, \mathcal{B})$$

is given by the derived tensor product $(X, Y) \mapsto X \overset{L}{\otimes}_{\mathcal{B}} Y$. For each dg functor $F : \mathcal{A} \to \mathcal{B}$, we have the dg bimodule

$$X_F : A \mapsto \mathcal{B}\,(?, FA) \,,$$

which clearly belongs to $\mathsf{rep}(\mathcal{A}, \mathcal{B})$. One can show that the map $F \mapsto X_F$ induces a bijection, compatible with compositions, from the set of morphisms from $\mathcal{A}$ to $\mathcal{B}$ in $\mathsf{Ho}(\mathsf{dgcat})$ to the set of isomorphism classes of bimodules X in $\mathsf{rep}(\mathcal{A}, \mathcal{B})$. In fact, a much stronger result by B. Toën [57] relates rep to the Dwyer-Kan localization of dgcat.

9.3 Dg orbit categories

Let $\mathcal{A}$ be a small dg category and $F \in \mathsf{rep}(\mathcal{A}, \mathcal{A})$. We assume, as we may, that F is given by a cofibrant bimodule. For a dg category $\mathcal{B}$, define $\widetilde{\mathsf{eff}}_0(\mathcal{A}, F, \mathcal{B})$ to be the category whose objects are the pairs formed by an $\mathcal{A}$-$\mathcal{B}$-bimodule P in $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ and a morphism of dg bimodules

$$\phi : P \to PF.$$

Morphisms are the morphisms of dg bimodules $f : P \to P'$ such that we have $\phi' \circ f = (fF) \circ \phi$ in the category of dg bimodules. Define $\mathsf{eff}_0(\mathcal{A}, F, \mathcal{B})$ to be the localization of $\widetilde{\mathsf{eff}}_0(\mathcal{A}, F, \mathcal{B})$ with respect to the morphisms f which are quasi-isomorphisms of dg bimodules. Denote by $\mathsf{eff}(\mathcal{A}, F, \mathcal{B})$ the full subcategory of $\mathsf{eff}_0(\mathcal{A}, F, \mathcal{B})$ whose objects are the (P, ϕ) where ϕ is a quasi-isomorphism. It is not hard to see that the assignments

$$\mathcal{B} \mapsto \mathsf{eff}_0(\mathcal{A}, F, \mathcal{B}) \quad \text{and} \quad \mathcal{B} \mapsto \mathsf{eff}(\mathcal{A}, F, \mathcal{B})$$

are 2-functors from rep to the category of small categories.

Theorem 3 *a) The* 2*-functor* $\mathsf{eff}_0(\mathcal{A}, F, ?)$ *is* 2*-representable,* i.e. *there is small dg category* $\mathcal{B}_0$ *and a pair* (P_0, ϕ_0) *in* $\mathsf{eff}(\mathcal{A}, F, \mathcal{B}_0)$ *such that for each small dg category* $\mathcal{B}$*, the functor*

$$\mathsf{rep}(\mathcal{B}_0, \mathcal{B}) \to \mathsf{eff}_0(\mathcal{A}, F, \mathcal{B}_0)\ ,\ G \mapsto G \circ P_0$$

is an equivalence.

b) The 2*-functor* $\mathsf{eff}(\mathcal{A}, F, ?)$ *is* 2*-representable.*

c) For a dg category $\mathcal{B}$*, a pair* (P, ϕ) *is a* 2*-representative for* $\mathsf{eff}_0(\mathcal{A}, F, ?)$ *iff* $H^0(P) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ *is essentially surjective and, for all objects* A, B *of* $\mathcal{A}$*, the canonical morphism*

$$\bigoplus_{n \in \mathbf{N}} \mathcal{A}\,(F^n A, B) \to \mathcal{B}\,(PA, PB)$$

is invertible in $\mathcal{D}(k)$*.*

d) For a dg category $\mathcal{B}$*, a pair* (P, ϕ) *is a* 2*-representative for* $\mathsf{eff}(\mathcal{A}, F, ?)$ *iff* $H^0(P) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ *is essentially surjective and, for all objects* A, B *of* $\mathcal{A}$*, the canonical morphism*

$$\bigoplus_{c \in \mathbf{Z}} \mathsf{colim}_{r \gg 0} \mathcal{A}\,(F^{p+r} A, F^{p+c+r} B) \to \mathcal{B}\,(PA, PB)$$

is invertible in $\mathcal{D}(k)$*.*

We *define* $\mathcal{A}/F$ to be the 2-representative of $\mathsf{eff}(\mathcal{A}, F, ?)$. For example, in the notations of 5.1, $\mathcal{A}/F$ is the dg orbit category $\mathcal{B}$. It follows from part d) of the theorem that we have an equivalence

$$H^0(\mathcal{A})/H^0(F) \to H^0(\mathcal{A}/F).$$

Proof. We only sketch a proof and refer to [54] for a detailed treatment. Define $\mathcal{B}_0$ to be the dg category with the same objects as $\mathcal{A}$ and with the morphism spaces

$$\mathcal{B}_0\,(A,B) = \bigoplus_{n\in\mathbf{N}} \mathcal{A}\,(F^n A, B).$$

We have an obvious dg functor $P_0 : \mathcal{A} \to \mathcal{B}_0$ and an obvious morphism ϕ : $P_0 \to P_0 F$. The pair (P_0, ϕ_0) is then 2-universal in rep. This yields a) and c). For b), one adjoins a formal homotopy inverse of ϕ to $\mathcal{B}_0$. One obtains d) by computing the homology of the morphism spaces in the resulting dg category.

9.4 Functoriality in $(\mathcal{A}, F)$

Let a square of rep

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{A}' \arrow[d, "F'"] \\ \mathcal{A} \arrow[r, "G"] & \mathcal{A}' \end{tikzcd}$$

be given and an isomorphism

$$\gamma : F'G \to GF$$

of $\mathsf{rep}(\mathcal{A}, \mathcal{A}')$. We assume, as we may, that $\mathcal{A}$ and $\mathcal{A}'$ are cofibrant in dgcat and that F, F' and G are given by cofibrant bimodules. Then $F'G$ is a cofibrant bimodule and so $\gamma : F'G \to GF$ lifts to a morphism of bimodules

$$\widetilde{\gamma} : F'G \to GF.$$

If $\mathcal{B}$ is another dg category and (P, ϕ) an object of $\mathsf{eff}_0(\mathcal{A}', F', \mathcal{B})$, then the composition

$$PG \xrightarrow{\phi G} PF'G \xrightarrow{P\widetilde{\gamma}} PGF$$

yields an object $PG \to PGF$ of $\mathsf{eff}_0(\mathcal{A}, F, \mathcal{B})$. Clearly, this assignment extends to a functor, which induces a functor

$$\mathsf{eff}(\mathcal{A}', F', \mathcal{B}) \to \mathsf{eff}(\mathcal{A}, F, \mathcal{B}).$$

By the 2-universal property of section 9.3, we obtain an induced morphism

$$\overline{G} : \mathcal{A}/F \to \mathcal{A}'/F'.$$

One checks that the composition of two pairs (G, γ) and (G', γ') induces a functor isomorphic to the composition of $\overline{G}$ with $\overline{G'}$.

9.5 The bicategory of enhanced triangulated categories

We refer to [33, 2.1] for the notion of an *exact dg category.* We also call these categories *pretriangulated* since if $\mathcal{A}$ is an exact dg category, then $H^0(\mathcal{A})$ is triangulated. More precisely, $\mathcal{E} = Z^0(\mathcal{A})$ is a Frobenius category and $H^0(\mathcal{A})$ is its associated stable category $\underline{\mathcal{E}}$ (*cf.* example (3) of section 8.3 for these notions). The inclusion of the full subcategory of (small) exact dg categories into $\mathsf{Ho}(\mathsf{dgcat})$ admits a left adjoint, namely the functor $\mathcal{A} \mapsto \mathsf{pretr}(\mathcal{A})$ which maps a dg category to its 'pretriangulated hull' defined in [14], *cf.* also [33, 2.2]. More precisely, the adjunction morphism $\mathcal{A} \to \mathsf{pretr}(\mathcal{A})$ induces an equivalence of categories

$$\mathsf{rep}\,(\mathsf{pretr}(\mathcal{A}), \mathcal{B}) \to \mathsf{rep}(\mathcal{A}, \mathcal{B})$$

for each exact dg category $\mathcal{B}$, *cf.* [55].
The *bicategory* enh *of enhanced [14] triangulated categories*, *cf.* [32] [21], has as objects all small exact dg categories; the morphism category between two objects $\mathcal{A}$, $\mathcal{B}$ is $\mathsf{rep}(\mathcal{A}, \mathcal{B})$; the composition bifunctor

$$\mathsf{rep}(\mathcal{B}, \mathcal{C}) \times \mathsf{rep}(\mathcal{A}, \mathcal{B}) \to \mathsf{rep}(\mathcal{A}, \mathcal{C})$$

is given by the derived tensor product $(X, Y) \mapsto X \overset{L}{\otimes}_{\mathcal{B}} Y$.

9.6 Exact dg orbit categories

Now let $\mathcal{A}$ be an exact dg category and $F \in \mathsf{rep}(\mathcal{A}, \mathcal{A})$. Then $\mathcal{A}/F$ is the dg orbit category of subsection 5.1 and $\mathsf{pretr}(\mathcal{A}/F)$ is an exact dg category such that $H^0\mathsf{pretr}(\mathcal{A}/F)$ is the triangulated hull of section 5. In particular, we obtain that the triangulated hull is the stable category of a Frobenius category. From the construction, we obtain the universal property:

Theorem 4 *For each exact dg category $\mathcal{B}$, we have an equivalence of categories*

$$\mathsf{rep}(\mathsf{pretr}(\mathcal{A}/F), \mathcal{B}) \to \mathsf{eff}(F, \mathcal{B}).$$

9.7 An example

Let A be a finite-dimensional algebra of finite global dimension and TA the trivial extension algebra, *i.e.* the vector space $A \oplus DA$ endowed with the multiplication defined by

$$(a, f)(b, g) = (ab, ag + fb)\ ,\ (a, f), (b, g) \in TA\ ,$$

and the grading such that A is in degree 0 and DA in degree 1. Let $F : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$ equal $\tau S^2 = \nu S$ and let $\widetilde{F}$ be the dg lift of F given by $? \otimes_A R[1]$, where R is a projective bimodule resolution of DA. Let $\mathcal{D}^b(A)_{dg}$ denote a dg category quasi-equivalent to the dg category of bounded complexes of finitely generated projective A-modules.

Theorem 5 *The following are equivalent*

(i) The k-category $\mathcal{D}^b(A)/F$ is naturally equivalent to its 'triangulated hull'

$$H^0(\mathsf{pretr}(\mathcal{D}^b(A)_{dg}/\widetilde{F})).$$

(ii) Each finite-dimensional TA-module admits a grading.

Proof. We have a natural functor

$$\mathsf{mod}A \to \mathsf{grmod}TA$$

given by viewing an A-module as a graded TA-module concentrated in degree 0. As shown by D. Happel [27], *cf.* also [34], this functor extends to a triangle equivalence Φ from $\mathcal{D}^b(A)$ to the stable category $\underline{\mathsf{grmod}}\,TA$, obtained from $\mathsf{grmod}TA$ by killing all morphisms factoring through projective-injectives. We would like to show that we have an isomorphism of triangle functors

$$\Phi \circ \tau S^2 \xrightarrow{\sim} \Sigma \circ \Phi$$

where Σ is the grading shift functor for graded TA-modules: $(\Sigma M)^p = M^{p+1}$ for all $p \in \mathbf{Z}$. ¿From [27], we know that $\tau S \xrightarrow{\sim} \nu$, where $\nu = ? \overset{L}{\otimes}_A DA$. Thus it remains to show that

$$\Phi \circ \nu S \xrightarrow{\sim} \Sigma \circ \Phi.$$

As shown in [34], the equivalence Φ is given as the composition

$$\begin{aligned} \mathcal{D}^b(\mathsf{mod}A) \to \quad & \mathcal{D}^b(\mathsf{grmod}TA) \to \\ & \mathcal{D}^b(\mathsf{grmod}TA)/\mathsf{per}(\mathsf{grmod}TA) \to \underline{\mathsf{grmod}}\,TA\,, \end{aligned}$$

where the first functor is induced by the above inclusion, the notation $\mathsf{per}(\mathsf{grmod}TA)$ denotes the triangulated subcategory generated by the projective-injective TA-modules and the last functor is the 'stabilization functor' *cf.* [34]. We have a short exact sequence of graded TA-modules

$$0 \to \Sigma^{-1}(DA) \to TA \to A \to 0.$$

We can also view it as a sequence of left A and right graded TA-modules. Let P be a bounded complex of projective A-modules. Then we obtain a short exact sequence of complexes of graded TA-modules

$$0 \to \Sigma^{-1}(P \otimes_A DA) \to P \otimes_A TA \to P \to 0$$

functorial in P. It yields a functorial triangle in $\mathcal{D}^b(\mathsf{grmod}A)$. The second term belongs to $\mathsf{per}(\mathsf{grmod}TA)$. Thus in the quotient category

$$\mathcal{D}^b(\mathsf{grmod}TA)/\mathsf{per}(\mathsf{grmod}TA)\,,$$

the triangle reduces to a functorial isomorphism

$$P \xrightarrow{\sim} S\Sigma^{-1}\nu P.$$

Thus we have a functorial isomorphism

$$\Phi(P) \xrightarrow{\sim} S\Sigma^{-1}\Phi(\nu P).$$

Since A is of finite global dimension, $\mathcal{D}^b(\mathsf{mod}A)$ is equivalent to the homotopy category of bounded complexes of finitely generated projective A-modules. Thus we get the required isomorphism

$$\Sigma\Phi \xrightarrow{\sim} \Phi S\nu.$$

More precisely, one can show that $\underline{\mathsf{grmod}}\, TA$ has a canonical dg structure and that there is an isomorphism

$$(\mathcal{D}^b(A))_{dg} \xrightarrow{\sim} (\underline{\mathsf{grmod}}\, TA)_{dg}$$

in the homotopy category of small dg categories which induces Happel's equivalence and under which Σ corresponds to the lift $\widetilde{F}$ of $F = S\nu$. Hence the orbit categories $\mathcal{D}^b(\mathsf{mod}A)/\tau S^2$ and $\underline{\mathsf{grmod}}\, TA/\Sigma$ are equivalent and we are reduced to determining when $\underline{\mathsf{grmod}}\, TA/\Sigma$ is naturally equivalent to its triangulated hull. Clearly, we have a full embedding

$$\underline{\mathsf{grmod}}\, TA/\Sigma \to \underline{\mathsf{mod}}\, TA$$

and its image is formed by the TA-modules which admit a grading. Now $\underline{\mathsf{mod}}\, TA$ is naturally equivalent to the triangulated hull. Therefore, condition (i) holds iff the embedding is an equivalence iff each finite-dimensional TA-module admits a grading.
In [53], A. Skowroński has produced a class of examples where condition (ii) does not hold. The simplest of these is the algebra A given by the quiver

$$\begin{tikzcd}
& \bullet \arrow[dl, "\beta"'] \arrow[dd] \\
\bullet \arrow[dr, "\alpha"'] & \\
& \bullet
\end{tikzcd}$$

with the relation $\alpha\beta = 0$. Note that this algebra is of global dimension 2.

9.8 Exact categories and standard functors

Let $\mathcal{E}$ be a small exact k-category. Denote by $\mathcal{C}^b(\mathcal{E})$ the category of bounded complexes over $\mathcal{E}$ and by $\mathcal{A}c^b(\mathcal{E})$ its full subcategory formed by the acyclic bounded complexes. The categories with the same objects but whose morphisms are given by the morphism *complexes* are denoted respectively by $\mathcal{C}^b(\mathcal{E})_{dg}$ and $\mathcal{A}c^b(\mathcal{E})_{dg}$. They are exact dg categories and so is the dg quotient [33] [21]

$$\mathcal{D}^b(\mathcal{E})_{dg} = \mathcal{C}^b(\mathcal{E})_{dg}/\mathcal{A}c^b(\mathcal{E})_{dg}.$$

Let $\mathcal{E}'$ be another small exact k-category. We call a triangle functor $F : \mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}')$ a *standard functor* if it is isomorphic to the triangle functor induced by a morphism

$$\widetilde{F} : \mathcal{D}^b(\mathcal{E})_{dg} \to \mathcal{D}^b(\mathcal{E}')_{dg}$$

of $\mathsf{Ho}(\mathsf{dgcat})$. Slightly abusively, we then call $\widetilde{F}$ a *dg lift* of F. Each exact functor $\mathcal{E} \to \mathcal{E}'$ yields a standard functor; a triangle functor is standard iff it admits a lift to an object of $\mathsf{rep}(\mathcal{D}^b(\mathcal{E})_{dg}, \mathcal{D}^b(\mathcal{E}')_{dg})$; compositions of standard functors are standard; an adjoint (and in particular, the inverse) of a standard functor is standard.
If $F : \mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E})$ is a standard functor with dg lift $\widetilde{F}$, we have the dg orbit category $\mathcal{D}^b(\mathcal{E})_{dg}/\widetilde{F}$ and its pretriangulated hull

$$\mathcal{D}^b(\mathcal{E})_{dg}/\widetilde{F} \to \mathsf{pretr}(\mathcal{D}^b(\mathcal{E})_{dg}/\widetilde{F}).$$

The examples in section 3 show that this functor is not an equivalence in general.

9.9 Hereditary categories

Now suppose that $\mathcal{H}$ is a small hereditary abelian k-category with the Krull-Schmidt property (indecomposables have local endomorphism rings and each object is a finite direct sum of indecomposables) where all morphism and extension spaces are finite-dimensional. Let

$$F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})$$

be a standard functor with dg lift $\widetilde{F}$.

Theorem 6 *Suppose that F satisfies assumptions 2) and 3) of the main theorem in section 4. Then the canonical functor*

$$\mathcal{D}^b(\mathcal{H})/F \to H^0(\mathsf{pretr}(\mathcal{D}^b(\mathcal{H})_{dg}/\widetilde{F}))$$

is an equivalence of k-categories. In particular, the orbit category $\mathcal{D}^b(\mathcal{H})/F$ admits a triangulated structure such that the projection functor becomes a triangle functor.

The proof is an adaptation, left to the reader, of the proof of the main theorem.

Suppose for example that $\mathcal{D}^b(\mathcal{H})$ has a Serre functor ν. Then ν is a standard functor since it is induced by the tensor product with bimodule

$$(A, B) \mapsto D\mathsf{Hom}_{\mathcal{D}^b(\mathcal{H})_{dg}}(B, A) \, ,$$

where $D = \mathsf{Hom}_k(?, k)$. The functor $\tau^{-1} = S\nu^{-1}$ induces equivalences $\mathcal{I} \xrightarrow{\sim} S\mathcal{P}$ and $\mathcal{H}_{ni} \to \mathcal{H}_{np}$, where $\mathcal{P}$ is the subcategory of projectives, $\mathcal{I}$ the subcategory of injectives, $\mathcal{H}_{np}$ the subcategory of objects without a projective direct summand and $\mathcal{H}_{ni}$ the subcategory of objects without an injective direct summand. Now let $n \geq 2$ and consider the autoequivalence $F = S^n\nu^{-1} = S^{n-1}\tau^{-1}$ of $\mathcal{D}^b(\mathcal{H})$. Clearly F is standard. It is not hard to see that F satisfies the hypotheses 2) and 3) of the main theorem in section 4. Thus the orbit category

$$\mathcal{D}^b(\mathcal{H})/F = \mathcal{D}^b(\mathcal{H})/S^n\nu^{-1}$$

is triangulated. Note that we have excluded the case $n = 1$ since the hypotheses 2) and 3) are not satisfied in this case, in general, as we see from the last example in section 3.

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