# Rational Curves on Homogeneous Cones

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Abstract. A homogeneous cone $X$ is the cone over a homogeneous variety $G/P$ embedded thanks to an ample line bundle $L$. In this article, we describe the irreducible components of the scheme of morphisms of class $\alpha \in A_1(X)$ from a rational curve to X.

The situation depends on the line bundle L : if the projectivised tangent space to the vertex contains lines then the irreducible components are described by the difference between Cartier and Weil divisors. On the contrary if there is no line in the projectivised tangent space to the vertex then there are new irreducible components corresponding to the multiplicity of the curve through the vertex.

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In this text we study the scheme of morphisms from $\mathbb{P}^1$ to any homogeneous cone that is to say a cone $X$ over a homogeneous variety $G/P$.

This study is motivated by the more general problem of describing the irreducible components of the scheme of morphisms from $\mathbb{P}^1$ to a variety $X$ endowed with the action of a solvable group with finitely many orbits (for example homogeneous varieties, Schubert varieties or spherical varieties). For homogeneous varieties, this is already known (see [Th], [KP] or [P1]). For Schubert varieties it is not known. We describe in [P2] the irreducible components of this scheme of morphisms when $X$ is a minuscule Schubert variety. The singularities of minuscule Schubert varities are locally isomorphic to cones over homogeneous varieties (see [BP]) so it is natural to address this problem on a general cone over a homogeneous variety. Moreover some non minuscule Schubert varieties are cones over homogeneous varieties. We prove (theorem 0.1) that the situation is more complicated than in the minuscule case but still has a nice description in terms of the combinatorial data of $G$.

More precisely, let $G/P$ be a homogeneous variety and let $L$ be a very ample divisor on $G/P$. We may embed $G/P$ in $\mathbb{P}(H^0L)$. If $V$ is a $n$-dimensional vector space, it defines a linear subspace $\mathbb{P}(V)$ in $\mathbb{P}(H^0L \oplus V)$. Let us denote by $X = \mathfrak{C}_{L,n}(G/P)$ the cone above $G/P$ whose vertex is $\mathbb{P}(V)$. Now let $U$ be the open subset of $X$ complementary to $Y = \mathbb{P}(V)$. We have a surjective morphism (see paragraph 1):

$$s : \mathrm{Pic}(U)^\vee \to A_1(X).$$

For any class $\alpha \in A_1(X)$, we can consider the following morphism:

$$i : \coprod_{s(\beta)=\alpha} \mathbf{Hom}_\beta(\mathbb{P}^1, U) \to \mathbf{Hom}_\alpha(\mathbb{P}^1, X)$$

where $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ is the scheme of morhisms $f : \mathbb{P}^1 \to X$ with $f_*[\mathbb{P}^1] = \alpha$ and $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ is the scheme of morhisms $g : \mathbb{P}^1 \to U$ such that $[g] = \beta$ where $[g]$ is the linear function $L \mapsto \deg(g^*L)$ on $\mathrm{Pic}(U)$. As $Y = X \setminus U$ lies in codimension 2, we expect the image of this morphism to be dense. For example we prove in [P2] that it is true for $X$ a minuscule Schubert variety and $U$ the smooth locus.

In our case the situation will be more complicated. Let us first describe the "expected" components in the case where $i$ is dominant. In this case we may apply the results of [P1] to prove that $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ is irreducible as soon as it is non empty and the images of the irreducible varieties $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ will give the irreducible components of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$. The expected components are thus indexed by the subset $\mathfrak{ne}(\alpha)$ of $\mathrm{Pic}(U)^\vee$ given by elements $\beta$ such that $s(\beta) = \alpha$ and $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ is non empty.

This set can be discribed in terms of roots: the ample divisor $L$ is a dominant weight in the facet of the parabolic $P$. An element $\alpha \in A_1(X)$ is completely determined by $\alpha \cdot L = d \in \mathbb{Z}$. Denote by $\mathfrak{ne}_B(\alpha)$ the set of all elements $\beta$ in the cone generated by the positive roots such that $\langle \beta^\vee, L \rangle = d$. This is a subset of $A_1(G/B)$. Then $\mathfrak{ne}(\alpha)$ is its image in $A_1(G/P)$ (see paragraph 1 for a more details). We prove the

Theorem 0.1. — *Let $R$ be the root lattice.*

(i) *If $L(R) = \mathbb{Z}$, then the irreducible components of the scheme of morphisms $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ are indexed by $\mathfrak{ne}(\alpha)$.*

(ii) *If $L(R) \neq \mathbb{Z}$ (i.e. if we have $L > c_1(T_{G/P})$), then the irreducible components of the scheme $\mathbf{Hom}_\alpha(\mathbb{P}^1, \mathfrak{C}(G/P))$ are indexed by $\coprod_{\alpha' \leq \alpha} \mathfrak{ne}(\alpha')$.*

We will see (paragraph 1) that $A_1(X) \simeq \mathbb{Z}$ so that $\alpha' \leq \alpha$ in $A_1(X)$ means that the same inequality holds in $\mathbb{Z}$. In the second case, a general curve can meet the vertex of the cone. The integer $\alpha - \alpha'$ is then the multiplicity of the curve at the vertex.

Remark 0.2. — (i) *The condition $L(R) = \mathbb{Z}$ is exactly equivalent to the fact that there exists lines on $G/P$ embedded with $L$. In other words there exists lines in the projectivized tangent cone to the singularity.*
*We studied in [P2] the same problem for minuscule Schubert varieties where the multiplicity in the singularity did not appear. If one consider more generaly quasi-minuscule Schubert varieties of non minuscule type (see [LMS] for a definition, the case of quasi-minuscule Schubert varieties of minuscule type should be very similar to the case of minuscule Scubert varieties) we recover this condition of the existence of lines in the projectivised tangent cone to the singularity.*
(ii) *If $P = B$ is a Borel subgroup and if we choose for $L$ the Plücker embedding (or equivalently $L = \rho$ as a weight where $\rho$ is half the sum of the positive roots) then $L(R) = \mathbb{Z}$ and the set $\mathfrak{ne}(\alpha)$ is in bijection with the set of irreducible integrable representations of level exactly $\alpha \cdot L$ of the affine Lie algebra $\hat{\mathfrak{g}}$ (see paragraph 1 for the general case).*

Here is an outline of the paper. In the first paragraph we define the surjective map $s$ of the introduction and the set $\mathfrak{ne}(\alpha)$ for a homogeneous cone $X$. In the second paragraph we study the scheme of morphisms from $\mathbb{P}^1$ to the blowing-up $\widetilde{X}$ of the cone $X$ and prove a smoothing result. In the last paragraph we prove our main result.

The key point as indicated above is to study the surjectivity of the map $i$ that is to say to study the following problem: can any morphism $f : \mathbb{P}^1 \to X$ be factorised in $U$ (modulo deformation). We do this by lifting $f$ in $\widetilde{f}$ on $\widetilde{X}$ and the problem becomes: does the lifted curve $\widetilde{f}$ of a general curve $f$ meet the exceptional divisor $E$. If it is the case then we add a "line" $\Gamma \subset E$ (this is possible only when $L(R) = \mathbb{Z}$) with $\Gamma \cdot E = -1$ and smooth the union $\widetilde{f}(\mathbb{P}^1) \cup \Gamma$. The intersection with $E$ is lowered by one in the operation. We conclude by induction on the number of intersection of $\widetilde{f}$ with $E$.

We end with a discussion on the dimensions of the components, in particular the variety $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ is equidimensional if and only if $L = \frac{1}{2}c_1(G/P)$ or $L = c_1(G/P)$.

## 1 Preliminary

In this paragraph we explain the results on cycles used in the introduction. We describe the surjective morphism $s : \mathrm{Pic}(U)^\vee \to A_1(X)$ and define the set of classes $\mathfrak{ne}(\alpha)$ for $\alpha \in A_1(X)$.

If $X$ is a scheme of dimension $n$, we denote by $Z_*(X)$ the group of 1-cycles on $X$ and by $Z_*^{\equiv}(X)$ and $Z_*^r(X)$ the subgroups of cycles trivial for the numerical and rational equivalence. Let us denote by $N_*(X)$ and $A_*(X)$ the corresponding quotients. The Picard group is the image in $A_{n-1}(X)$ of the subgroup of Cartier divisors in $Z_{n-1}(X)$.

Lemma 1.1. — *Let $X = \mathfrak{C}_{L,n}(G/P)$ be a cone over a homogeneous variety $G/P$ then*
(i) $\mathrm{Pic}(X) \simeq N^1(X)$,
(ii) $A_1(X) \simeq N_1(X)$.
*In particular we have $A_1(X) \simeq \mathrm{Pic}(X)^\vee$ and they are isomorphic to $\mathbb{Z}$.*

Proof. Consider the decomposition $V \oplus H^0L$. The following group

$$G' = \begin{pmatrix} GL(V) & \mathrm{Hom}(H^0L, V) \\ 0 & G \end{pmatrix}$$

acts on $X$ and the unipotent part $U(G')$ acts on $X$ with finitely many orbits. Remark that it is not the case if we only take the unipotent part $U$ of $GL(V)\times G$. Indeed, take for example $V = \mathbb{C}$ and $G = GL(V) = GL_1(\mathbb{C})$. Then $G/P$ is a point and $X$ is a projective line but $U$ is the trivial group and has infinitely many orbits in $X$. The same problem appears in the general case.
(i) Thanks to the results of [FMcPSS] the groups $A_*(X)$ are free generated by invariant subvarieties. The Picard group is contained in $A_{n-1}(X)$ and is in particular free. Thanks to [Fu] Example 19.3.3. this implies that $\mathrm{Pic}(X) \simeq N^1(X)$.
(ii) The results of [FMcPSS] also imply that $A_1(X)$ is generated by the one-dimensional invariant subvarieties. The only such subvariety is

- the fibre of the cone over the 0-dimensional orbit in $G/P$ if $\dim V = 1$;
- the 1 dimensional orbit in $\mathbb{P}(V)$ if $\dim V \geq 2$.

We get the isomorphism $A_1(X) \simeq \mathbb{Z}$ with this variety as generator. This generator is clearly numerically free (for example its degree is 1) so we get the result.
The duality comes from general duality between $N_1(X)$ and $N^1(X)$. □

Let $U$ be the smooth locus of $X$, it is also the dense orbit under $G'$ in $X$. Let $Y$ be the complementary of $U$ in $X$, it is of codimension at least 2 (at least when $\dim(G/P) > 0$). This in particular implies that $\mathrm{Pic}(U) = A_{n-1}(U) \simeq A_{n-1}(X)$. We now have the following inclusion:

$$\mathrm{Pic}(X) \subset A_{n-1}(X) \simeq \mathrm{Pic}(U)$$

giving the surjection

$$s : \mathrm{Pic}(U)^\vee \to A_1(X).$$

With these notations we make the following:

Definition 1.2. — *Let $\alpha \in A_1(X)$. We define the set $\mathfrak{ne}(\alpha) \subset A_{n-1}(X)^\vee$. Let us make the identification $A_{n-1}(X) \simeq \mathrm{Pic}(U)$. The elements of $\mathfrak{ne}(\alpha)$ are the elements $\beta \in \mathrm{Pic}(U)^\vee$ such that $s(\beta) = \alpha$ and there exists a complete curve $C \subset U$ with $[C] = \beta$ as a linear form on $\mathrm{Pic}(U)$ ($\beta$ is effective).*

Let us describe $\mathfrak{ne}(\alpha)$ explicitly: the smooth part $U$ is a vector bundle over $G/P$. In particular we have $\mathrm{Pic}(U) \simeq \mathrm{Pic}(G/P)$.
Let us fix $T$ a maximal Torus in $G$, fix $B$ a Borel subgroup containing $T$ and suppose that $B \subset P$. Let us denote by $\Delta$ the set of all roots, by $\Delta^+$ (resp. $\Delta^-$) the set of positive (resp. negative) roots and by $S$ the set of simple roots associated to the data $(G, T, B)$.
Denote by $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{p}$ the Lie algebras of $G$, $T$ and $P$ and define

$$\alpha(\mathfrak{p}) = \Big\{\alpha \in S \ / \ \mathfrak{g}_\alpha \subset \mathfrak{p} \text{ and } \mathfrak{g}_{-\alpha} \not\subset \mathfrak{p}\Big\}.$$

Now set $\mathfrak{t}(\mathfrak{p})^*$ as the subvector space of $\mathfrak{t}^*$ generated by the roots in $\alpha(\mathfrak{p})$, we have

$$\mathrm{Pic}(G/P) \simeq \mathfrak{t}(\mathfrak{p}) \cap Q$$

where $\mathfrak{t}(\mathfrak{p})$ is the dual of $\mathfrak{t}(\mathfrak{p})^*$ in $\mathfrak{t}$ and $Q$ is the weight lattice. The Picard group of $X$ in $\mathrm{Pic}(U) \simeq \mathrm{Pic}(G/P) \simeq \mathfrak{t}(\mathfrak{p}) \cap Q$ is given by the the line generated by $\lambda$ (the weight associated to $L$). We have

$$\mathrm{Pic}(U)^\vee \simeq \mathfrak{t}^*/\mathfrak{t}(\mathfrak{p})^* \cap R$$

where $R$ is the root lattice. Furthermore, an element $\beta \in \mathrm{Pic}(U)^\vee$ gives an effective element if and only if it is in the image of the cone generated by positive roots i.e. in the cone $\mathfrak{t}^*/\mathfrak{t}(\mathfrak{p})^* \cap R^+$ (see [P1]). Then we have

$$\mathfrak{ne}(\alpha) = \big\{\beta \in \mathfrak{t}^*/\mathfrak{t}(\mathfrak{p})^* \cap R^+ \ / \ \langle \beta^\vee, \lambda\rangle = \alpha \cdot L\big\}$$

where the integer $\langle \beta^\vee, \lambda\rangle$ is well defined because $\lambda \in \mathfrak{t}(\mathfrak{p}) \cap Q$.

Example 1.3. — *Choose for $L$ (or for $\lambda$) the smallest ample sheave on $X$. This is possible: the picard group $\mathrm{Pic}(U) = \mathfrak{t}(\mathfrak{p}) \cap Q$ is a direct sum of weight lattices of semi-simple Lie algebras $(\mathfrak{g}_i)_{i\in[1,r]}$. We just have to take*

$$\lambda = \sum_{i\in[1,r]} \rho_i$$

*where $\rho_i$ is half the sum of positive roots in $\mathfrak{g}_i$.
Let us denote by $\mathfrak{ir}_{\hat{\mathfrak{g}}_i}(\ell)$ the set of isomorphism classes of irreductible integrable representations of level exactely $\ell$ of the affine Lie algebra $\hat{\mathfrak{g}}_i$. Then we have*

$$\mathfrak{ne}(\alpha) = \prod_{\ell_1+\cdots+\ell_r = \alpha\cdot L} \mathfrak{ir}_{\hat{\mathfrak{g}}_i}(\ell_i).$$

*In particular if $P = B$ is a Borel subgroup of $G$ then $r = 1$ and $G_1 = G$ and we recover the example of the introduction:*

$$\mathfrak{ne}(\alpha) = \mathfrak{ir}_{\hat{\mathfrak{g}}}(\alpha \cdot L).$$

Remark 1.4. — (i) *The scheme $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ is the scheme of morphisms from $\mathbb{P}^1$ to $X$ of class $\alpha$ (for more details see [Gr] and [Mo]).*
*In general, this will just mean that $\alpha \in A_1(X)$ and that $f_*[\mathbb{P}^1] = \alpha$ but sometimes (in particular in the introduction for the open part $U$) we consider $\alpha \in \mathrm{Pic}(X)^\vee$ and the class of a morphism $f : \mathbb{P}^1 \to X$ will be the linear form $\mathrm{Pic}(X) \to \mathbb{Z}$ given by $L \mapsto \deg(f^*L)$. In the case of a homogeneous cone $X$ the two notion coincide because of the previous lemma.*
(ii) *If $X$ is a variety, $\alpha \in A_1(X)$ and $F$ a vector bundle on $X$ we will denote $\alpha \cdot F = \int_\alpha c_1(F)$ by abuse of notation.*

## 2 Resolution

Recall that we denote by $X$ the cone $\mathfrak{C}_{L,n}(G/P)$. Let $\widetilde{X}$ be the blowing-up of $X$ in $\mathbb{P}(V)$. It is smooth and isomorphic to

$$\mathbb{P}_{G/P}((V \otimes \mathcal{O}_{G/P}) \oplus L).$$

Let us denote by $p$ the projection from $\widetilde{X}$ to $G/P$ and by $\pi : \widetilde{X} \to X$ the blowing-up. The morphism $p$ has natural sections given by points of $\mathbb{P}(V)$ or equivalently by surjective morphisms $L \oplus (V \otimes \mathcal{O}_{G/P}) \to V \otimes \mathcal{O}_{G/P} \to \mathcal{O}_{G/P}$.

### 2.1 Cycles on $\widetilde{X}$

Lemma 2.1. — (i) *Rational and numerical equivalences coincide on $\widetilde{X}$. In particular we have $A_1(\widetilde{X}) \simeq \mathrm{Pic}(\widetilde{X})^\vee \simeq A_{n-1}(\widetilde{X})^\vee$.*
(ii) *We have $\mathrm{Pic}(\widetilde{X}) \simeq \mathrm{Pic}(G/P) \oplus \mathbb{Z}$ with the factor $\mathbb{Z}$ generated by a $p$-relative ample class.*

Proof. (i) Rational and numerical equivalence coincide on $G/P$. Moreover the fibration in projetive spaces $\widetilde{X} \to G/P$ has sections so that rational and numerical equivalences coincide on $\widetilde{X}$. This in particular implies that $\mathrm{Pic}(\widetilde{X}) = A_{n-1}(\widetilde{X}) = N^1(\widetilde{X})$ and $A_1(\widetilde{X}) = N_1(\widetilde{X})$ and the duality follows.
(ii) The variety $\widetilde{X}$ is a $\mathbb{P}^n$-bundle over $G/P$ with sections so we get that

$$\mathrm{Pic}(\widetilde{X}) \simeq \mathrm{Pic}(G/P) \oplus \mathbb{Z}$$

with the factor $\mathbb{Z}$ generated by a $p$-relative ample class, the relative tangent sheaf $T_p$ of $p$ is $n+1$ times this class on the factor $\mathbb{Z}$. $\square$

Any element $\widetilde{\alpha} \in A_1(\widetilde{X}) \simeq \mathrm{Pic}(\widetilde{X})^\vee$ is given by the class $\beta = p_*\widetilde{\alpha} \in A_1(G/P)$ and the relative degree $d = \widetilde{\alpha} \cdot T_p$. We will use the notation $\ell = \beta \cdot L = \widetilde{\alpha} \cdot p^*L$. Let us denote by $E$ the exceptional divisor on $\widetilde{X}$, it is a trivial $\mathbb{P}^{n-1}$ bundle over $G/P$ given by the surjection $L \oplus (V \otimes \mathcal{O}_{G/P}) \to V \otimes \mathcal{O}_{G/P}$. Then we have:

$$\widetilde{\alpha} \cdot E = \frac{d - n\ell}{n+1},$$

it has to be an integer so that $d \equiv n\ell \mod n+1$.

Let us consider the following morphism still denoted by $p$:

$$p : \mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X}) \to \mathbf{Hom}_\beta(\mathbb{P}^1, G/P).$$

Proposition 2.2. — *Thanks to the morphism $p$, the scheme $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ is an open subset of a projective bundle over $\mathbf{Hom}_\beta(\mathbb{P}^1, G/P)$.*

Proof. This generalises proposition 4 of [P1] in the case where the relative degree $\widetilde{\alpha} \cdot T_p$ is negative. This is possible because the vector bundle associated to the $\mathbb{P}^n$ fibration has a decomposition $L \oplus (V \otimes \mathcal{O}_{G/P})$. We only describe the fibers, for the structure of projective bundle see [P1] proposition 4.
Let $f : \mathbb{P}^1 \to G/P$, we have to calculate the fiber of $p$ above $f$. The fiber is given by sections of the $\mathbb{P}^n$-bundle $f^*(p) : \mathbb{P}_{\mathbb{P}^1}((V \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ whose relative degree is $d$. In other words the fiber is given by surjections $(V \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell) \to \mathcal{O}_{\mathbb{P}^1}(x)$ modulo scalar multiplication where $d = (n+1)x - \ell$. The fiber is therefore isomorphic to an open subset of $\mathbb{P}(\mathrm{Hom}((V \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell), \mathcal{O}_{\mathbb{P}^1}(x))$.
Let us remark that if $\mathbf{Hom}_\beta(\mathbb{P}^1, G/P)$ is not empty then we have $\ell \geq 0$ and in this case $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ is not empty if and only if $x \geq 0$ when $n \geq 2$ and if and only if $x = 0$ or $x \geq \ell$ when $n = 1$. In terms of $d$ this means that $d = -\ell$ or $d \geq n\ell$ if $n = 1$ and $d \geq -\ell$ if $n \geq 2$. In any cases, if $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ is not empty then $x \geq 0$.
There are two cases:

- If $x < \ell$ then any section is included in the exceptional divisor and the dimension of the fiber is:

$$\frac{n}{n+1}(\ell + d) + n - 1.$$

- If $x \geq \ell$ then the fiber is of dimension $d + n$. $\square$

Let $\widetilde{\alpha} \in A_1(\widetilde{X})$ such that $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ is not empty. This is equivalent to the fact that $\beta \in A_1(G/P)$ is positive (see [P1], it is equivalent to the fact that $\mathbf{Hom}_\beta(\mathbb{P}^1, G/P)$ is non empty) and such that $d = -\ell$ or $d \geq n\ell$ if $n = 1$, $d \geq -\ell$ if $n \geq 2$ (recall that $\ell = \beta \cdot L$).

Corollary 2.3. — *The scheme* $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ *is irreducible of dimension*

- $\displaystyle\int_{\widetilde{\alpha}} c_1(T_{\widetilde{X}}) + \dim(\widetilde{X})$ *if* $d \geq n\ell$
- $\displaystyle\int_{\widetilde{\alpha}} c_1(T_{\widetilde{X}}) + \dim(\widetilde{X}) - \widetilde{\alpha} \cdot E - 1$ *if* $d < n\ell$.

Proof. We just use the preceding proposition and the fact proved in [P1] that the scheme $\mathbf{Hom}_\beta(\mathbb{P}^1, G/P)$ is irreductible of dimension $\int_\beta c_1(T_{G/P}) + \dim(G/P)$. Remark that in the last case we have $n\ell > d$ so that the dimension of $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ is still greater than the expected dimension $\displaystyle\int_{\widetilde{\alpha}} c_1(T_{\widetilde{X}}) + \dim(\widetilde{X})$. □

## 2.2 Smoothing curves on $\widetilde{X}$

In this paragraph we will prove some results on curves on $\widetilde{X}$.

Proposition 2.4. — *Assume that* $L(R) = \mathbb{Z}$.
*Let* $\widetilde{\alpha} \in A_1(\widetilde{X})$, $\widetilde{f} \in \mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ *such that* $\widetilde{f}(\mathbb{P}^1) \not\subset E$ *and* $\widetilde{\alpha} \cdot E > 0$. *Assume that the image of* $p \circ \widetilde{f} : \mathbb{P}^1 \to G/P$ *is not a line in the embedding given by* $L$. *Then there exists a deformation* $\widetilde{f}'$ *of* $\widetilde{f}$ *and a curve* $\Gamma \subset \widetilde{X}$ *contracted by* $\pi$ *with* $\Gamma \cdot E = -1$ *such that the curve* $\widetilde{f}'(\mathbb{P}^1) \cup \Gamma$ *can be smoothed. The smoothed curve is the image of a morphism* $\widehat{f} : \mathbb{P}^1 \to \widetilde{X}$.

Proof. Let $(x, v) \in E \simeq G/P \times \mathbb{P}(V)$ be a point in the intersection $\widetilde{f}(\mathbb{P}^1) \cap E$. Let us first remark that we may assume (after deformation) that $\widetilde{f}(\mathbb{P}^1)$ is a nodal curve. Indeed, because $p \circ \widetilde{f} : \mathbb{P}^1 \to G/P$ is not a line, this implies in particular that $G/P$ is not $\mathbb{P}^1$ so it is of dimension at least 2. The results of [P1] prove that a general curve in $G/P$ is nodal and so is $\widetilde{f}(\mathbb{P}^1)$ if $\widetilde{f}$ is general.

Lemma 2.5. — *There exists a deformation* $\widetilde{f}'$ *of* $\widetilde{f}$ *and a rational curve* $\Gamma$ *in* $\widetilde{X}$ *such that* $[\Gamma] \cdot E = -1$, $[\Gamma] \cdot L = 1$ *and meeting* $\widetilde{f}'(\mathbb{P}^1)$ *in exactly one point.*

Proof. Let us consider the lines in $G/P$ that is to say the rational curves $\Gamma'$ in $G/P$ such that $[\Gamma'] \cdot L = 1$. Such curves exist because we have $L(R) = \mathbb{Z}$. Let $\Gamma'$ be such a line passing through $p(x, v) = x \in G/P$ and let $\Gamma$ be the section of $\Gamma'$ in $E$ given by the point $v \in \mathbb{P}(V)$. This curve is contracted by $\pi$ to the point $v \in \mathbb{P}(V)$, its intersection with $E$ is given by $-[\Gamma'] \cdot L = -1$.
As we assumed that $p \circ \widetilde{f}(\mathbb{P}^1)$ is not a line then $\Gamma'$ meets $p \circ \widetilde{f}(\mathbb{P}^1)$ in a finite number of points: $x$ and other points $(x_i)$. The morphism $\widetilde{f}$ is given by a section of the projective bundle over $p \circ \widetilde{f}$ that is to say by a surjection

$$s : (V \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{O}_{\mathbb{P}^1}(\ell) \to \mathcal{O}_{\mathbb{P}^1}\left(\frac{d+\ell}{n+1}\right).$$

To deform $\widetilde{f}$ we can deform this surjection $s$ in $s'$ such that at $x$, we have $s'_x = s_x$ and at $x_i$ we have $s'_{x_i} \neq s_{x_i}$ for all $i$. This gives the required deformation. $\square$

Lemma 2.6. — *The curve $\widetilde{f}'(\mathbb{P}^1) \cup \Gamma$ can be smoothed. The smoothed curve is the image of a morphism $\widehat{f} : \mathbb{P}^1 \to \widetilde{X}$ of class $\widehat{\alpha}$ with*

$$\widehat{\alpha} \cdot (p^*L + E) = \widetilde{\alpha} \cdot (p^*L + E) \quad \text{and} \quad \widehat{\alpha} \cdot E < \widetilde{\alpha} \cdot E.$$

Proof. If the smoothing exists then we have $\widehat{\alpha} = \widetilde{\alpha} + [\Gamma]$ so $\widehat{\alpha} \cdot E = \widetilde{\alpha} \cdot E - 1$. Furthermore we have $p^*L + E = \pi^*L$ so that

$$\widehat{\alpha} \cdot (p^*L + E) = \widehat{\alpha} \cdot \pi^*L = \pi_*\widehat{\alpha} \cdot L = \pi_*\widetilde{\alpha} \cdot L = \widetilde{\alpha} \cdot \pi^*L = \widetilde{\alpha} \cdot (p^*L + E).$$

This simply comes from the fact that $\pi_*[\Gamma] = 0$. Let us note that the curves $f' = \pi \circ \widetilde{f}$ and $f'' = \pi \circ \widehat{f}$ have the same degrees but the curve $f''$ meets the vertex in one point less than $f'$.

To smooth $\widetilde{f}'(\mathbb{P}^1) \cup \Gamma$ we use the following result proved in [HH] for $\mathbb{P}^3$ but valid for any smooth projective variety:

Theorem 2.7. — *Let $Z$ be a smooth projective variety and let $C$ be a nodal curve in $Z$. Assume that the cohomology group $H^1 T_Z|_C$ is trivial then $C$ can be smoothed.*

As $\widetilde{f}(\mathbb{P}^1)$ is nodal and thanks to the previous lemma we know that $\widetilde{f}'(\mathbb{P}^1) \cup \Gamma$ is nodal. We just have to prove that the cohomology group $H^1(T_{\widetilde{X}}|_{\widetilde{f}'(\mathbb{P}^1) \cup \Gamma})$ is trivial. We have the exact sequences

$$0 \to T_p \to T_{\widetilde{X}} \to p^*T_{G/P} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{\widetilde{f}'(\mathbb{P}^1)}(-Q) \to \mathcal{O}_{\widetilde{f}'(\mathbb{P}^1) \cup \Gamma} \to \mathcal{O}_\Gamma \to 0$$

where $Q$ is the intersection point of $\widetilde{f}'(\mathbb{P}^1)$ and $\Gamma$. We just have to prove the vanishing of the following cohomology groups:

$$H^1(p^*T_{G/P}|_\Gamma) \ ; \ H^1(p^*T_{G/P}|_{\widetilde{f}'(\mathbb{P}^1)}(-Q)) \ ; \ H^1(T_p|_\Gamma) \text{ and } H^1(T_p|_{\widetilde{f}'(\mathbb{P}^1)}(-Q)).$$

The first two groups are respectively equal to $H^1(T_{G/P}|_{\Gamma'})$ and $H^1(T_{G/P}|_{p(\widetilde{f}'(\mathbb{P}^1))}(-Q))$ where we denoted $\Gamma' = p(\Gamma)$. They are trivial because $T_{G/P}$ is globally generated and $\Gamma'$ and $p(\widetilde{f}'(\mathbb{P}^1))$ are rational curves. Let us denote by $\mathcal{O}_p(1)$ the tautological quotient of the projective bundle associated to $(V \otimes \mathcal{O}_X) \oplus L$, the relative tangent sheaf is given by $T_p = \mathrm{Coker}(\mathcal{O}_{\widetilde{X}} \to ((V^\vee \otimes \mathcal{O}_{\widetilde{X}}) \oplus L^\vee) \otimes \mathcal{O}_p(1))$. In particular we have:

$$T_p|_\Gamma = \mathrm{Coker}(\mathcal{O}_{\mathbb{P}^1} \to (V^\vee \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \qquad \text{and}$$

$$T_p|_{\widetilde{f}'(\mathbb{P}^1)} = \mathrm{Coker}\left(\mathcal{O}_{\mathbb{P}^1} \to \left(V^\vee \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{d+\ell}{n+1}\right)\right) \oplus \mathcal{O}_{\mathbb{P}^1}\left(\frac{d-n\ell}{n+1}\right)\right).$$

This proves that the group $H^1(T_\pi|_\Gamma)$ vanishes. Furthermore, since $\widetilde{f}$ exists we must have $\frac{d+\ell}{n+1} \geq 0$ (see proposition 2.2) and $\frac{d-n\ell}{n+1} = \widetilde{\alpha} \cdot E > 0$ so that $H^1(T_p|_{\widetilde{f}'(\mathbb{P}^1)}(-Q))$ also vanishes. $\square$

## 3 Homogeneous cones

Recall that we denote by $X$ the cone $\mathfrak{C}_{L,n}(G/P)$. In this paragraph we study the irreducible components of the scheme $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ where $\alpha \in A_1(X)$. Recall that $A_1(X) \simeq \mathbb{Z}$ and under this identification $\alpha$ is just the degree of the corresponding curve.

### 3.1 The case $L(R) = \mathbb{Z}$

Theorem 3.1. — *Assume that $L(R) = \mathbb{Z}$, let $\alpha \in A_1(X)$ and $f \in \mathbf{Hom}_\beta(\mathbb{P}^1, X)$. Then there exists a deformation $f'$ of $f$ such that $f'$ does not meet the vertex $\mathbb{P}(V)$ of the cone $X$.*

Proof. Let us begin with the following:

Lemma 3.2. — *Let $f \in \mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ such that $f$ factors through the vertex $\mathbb{P}(V)$ of the cone. Then there exists a deformation $f'$ of $f$ in $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ such that $f'(\mathbb{P}^1)$ does not factor through the vertex.*

Proof. Let $x \in G/P$ and consider the linear subspace generated by $x$ and $\mathbb{P}(V)$. It is a projective space contained in $X$ containing $\mathbb{P}(V)$ as a hyperplane and containing $f(\mathbb{P}^1)$. In this projective space we can deform the morphism $f$ so that it does not factor through $\mathbb{P}(V)$ any more. $\square$

A general morphism $f \in \mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ does not factor through the vertex $\mathbb{P}(V)$ of the cone so it can be lifted in a morphism $\widetilde{f} : \mathbb{P}^1 \to \widetilde{X}$. Let $\widetilde{\alpha} \in A_1(\widetilde{X})$ the class of $\widetilde{f}$, we have $\pi_*\widetilde{\alpha} = \alpha$. Because $f$ does not factor through the vertex, the morphism $\widetilde{f}$ does not factor through the exceptional divisor $E$ so we have: $\widetilde{\alpha} \cdot E \geq 0$. If $\widetilde{\alpha} \cdot E = 0$, then $\widetilde{f}(\mathbb{P}^1)$ does not meet $E$ thus $f$ does not meet the vertex and we are done. Let us assume that $\widetilde{\alpha} \cdot E > 0$. We proceed by induction on $\widetilde{\alpha} \cdot E$. Consider the morphism $p \circ \widetilde{f} : \mathbb{P}^1 \to G/P$.

Lemma 3.3. — *If the image of $p \circ \widetilde{f}$ is a line in the projective embedding given by $L$ then there exists a deformation $f' \in \mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ of $f$ not meeting the vertex.*

Proof. Indeed, if the image of $p \circ \widetilde{f}$ is a line then $f$ factors through the linear subspace generated by the vertex and this line. It is a $\mathbb{P}^{n+1}$ and the vertex is a linear subspace of codimension 2. There exists a deformation $f'$ of $f$ in this projective space not meeting the vertex. $\square$

Let us now assume that the image of $p \circ \widetilde{f}$ is not a line, we may apply proposition 2.4 so that there exists a deformation $\widetilde{f}'$ of $\widetilde{f}$ and a curve $\Gamma \subset \widetilde{X}$ contracted by $\pi$ with $\Gamma \cdot E = -1$ such that the curve $\widetilde{f}'(\mathbb{P}^1) \cup \Gamma$ can be smoothed. The

smoothed curve is the image of a morphism $\widehat{f}:\mathbb{P}^1\to\widetilde{X}$ of class $\widehat{\alpha}$. Let us consider $f'=\pi\circ\widetilde{f}'$ and $f''=\pi\circ\widehat{f}$. Then $f'$ is a deformation of $f$ and because $\Gamma$ is contracted by $\pi$ the map $f''$ is a deformation of $f'$ and a fortiori of $f$.
We have to prove the result on $f''$ whose lifting is $\widehat{f}$ of class $\widehat{\alpha}$. But we have $\widehat{\alpha}\cdot E=\widetilde{\alpha}\cdot E-1$ so the result is true by induction. □

Theorem 3.4. — *Assume $L(R)=\mathbb{Z}$ and let $\alpha\in A_1(X)$ then the irreducible components of the scheme $\mathbf{Hom}_\alpha(\mathbb{P}^1,X)$ are indexed by $\mathfrak{ne}(\alpha)$. For $\widetilde{\alpha}\in\mathfrak{ne}(\alpha)$ the dimension of the corresponding component is*

$$\int_{\widetilde{\alpha}}c_1(T_{\widetilde{X}})+\dim(X).$$

Proof. Theorem 3.1 proves that the set of morphisms $f:\mathbb{P}^1\to X$ whose image does not meet the vertex $\mathbb{P}(V)$ is a dense open subset of $\mathbf{Hom}_\alpha(\mathbb{P}^1,X)$. It is enough to study this open set. Any curve is this open set comes from a unique lifting $\widetilde{f}:\mathbb{P}^1\to\widetilde{X}$ whose image does not meet $E$. Let $\widetilde{\alpha}\in A_1(\widetilde{X})$ the class of $\widetilde{f}$, since $\widetilde{\alpha}\cdot E=0$ we have $\widetilde{\alpha}\in\mathrm{Pic}(U)^\vee$ and in fact $\widetilde{\alpha}\in\mathfrak{ne}(\alpha)$. The morphism

$$\pi_*:\coprod_{\widetilde{\alpha}\in\mathfrak{ne}(\alpha)}\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})\to\mathbf{Hom}_\alpha(\mathbb{P}^1,X)$$

is thus dominant and birational (the inverse is given by lifting morphisms). What is left to prove is that for each $\widetilde{\alpha}\in\mathfrak{ne}(\alpha)$ the image of $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ (which is an irreducible scheme) forms an irreducible component of $\mathbf{Hom}_\alpha(\mathbb{P}^1,X)$. To prove this it is enough to prove that for any $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ in $\mathfrak{ne}(\alpha)$ the image of $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ is not contained in the closure of $\mathbf{Hom}_{\widetilde{\alpha}'}(\mathbb{P}^1,\widetilde{X})$ in $\mathbf{Hom}_\alpha(\mathbb{P}^1,X)$. This would be trivial if the scheme $\coprod_{\widetilde{\alpha}\in\mathfrak{ne}(\alpha)}\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ was equidimensional (it is the case if $L=\frac{1}{2}c_1(G/P)$). In general, suppose there exist $f\in\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ such that $f$ does not meet the vertex and such that $f$ is the limit of a familly $f'_t$ of morphisms in $\mathbf{Hom}_{\widetilde{\alpha}'}(\mathbb{P}^1,\widetilde{X})$. Because the condition of meeting the vertex is closed me may assume that the elements $f'_t$ do not meet the vertex. In particular projecting on $G/P$ gives a deformation from $p(f'_t)$ to $p(f)$. This implies that $p_*\widetilde{\alpha}=p_*\widetilde{\alpha}'$ but as $\widetilde{\alpha}\cdot E=0=\widetilde{\alpha}'\cdot E$ we have $\widetilde{\alpha}=\widetilde{\alpha}'$. The dimension comes from corollary 2.3. □

### 3.2 The case $L(R)\neq\mathbb{Z}$

We begin with the following lemma on root systems:

Lemma 3.5. — *Let $G$ be a semi-simple Lie group, $P\subset G$ a parabolic subgroup, $L$ a dominant weight in the facet defined by $P$ and $R$ the root lattice, then we have the equivalence*

$$L(R)\neq\mathbb{Z}\Longleftrightarrow L\geq c_1(G/P)$$

*where* $c_1(G/P) \in \mathrm{Pic}(G/P)$ *is considered as a weight and the order is given by the positivity on simple roots.*

Proof. Let us first describe $c_1(G/P)$ as a weight. Consider the set $\alpha(\mathfrak{p})$ of simple root and the lattice $\mathfrak{t}(\mathfrak{p}) \cap Q$ (which is isomorphic to $\mathrm{Pic}(G/P)$) defined in paragraph 1. The lattice $\mathfrak{t}(\mathfrak{p}) \cap Q$ decomposes into a direct sum of root lattices $R_i$. Let $\rho_i$ be half the sum of positive roots of the root system corresponding to $R_i$. Then we have

$$c_1(G/P) = 2\sum_i \rho_i.$$

If $L \geq c_1(G/P)$ then for any simple root $\alpha$ we have

$$\langle \alpha^\vee, L\rangle \geq \langle \alpha^\vee, c_1(G/P)\rangle = \sum_i \langle \alpha^\vee, \rho_i\rangle = \left\{ \begin{array}{ll} 0 & \text{if } \alpha \notin \alpha(\mathfrak{p}) \\ 2 & \text{if } \alpha \in \alpha(\mathfrak{p}) \end{array}\right.$$

and in particular $1 \notin L(R)$.
Conversely, suppose that $L(R) \neq \mathbb{Z}$. Because $L$ is in the facet of $P$ we have $\langle \alpha^\vee, L\rangle = 0$ for any simple root $\alpha \notin \alpha(\mathfrak{p})$. If $\alpha$ is a simple root in $\alpha(\mathfrak{p})$ then $\langle \alpha^\vee, L\rangle \geq 2$ (otherwise $L(R) = \mathbb{Z}$). We see that for any simple root $\langle \alpha^\vee, L\rangle \geq \langle \alpha^\vee, c_1(G/P)\rangle$ thus $L \geq c_1(G/P)$. □

Remark 3.6. — *Let* $\widetilde{\alpha} \in A_1(\widetilde{X})$ *such that* $\widetilde{\alpha} \cdot E \geq 0$. *Recall the notations* $\beta = p_*\widetilde{\alpha}$, $d = \widetilde{\alpha} \cdot T_p$ *is the relative degree and* $\ell = \widetilde{\alpha} \cdot p^*L = \beta \cdot L$. *Let* $\alpha = \pi_*\widetilde{\alpha}$ *considered as an integer. Then the dimension of* $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ *is given by*

$$\begin{array}{rcl} \displaystyle\int_{\widetilde{\alpha}} c_1(T_{\widetilde{X}}) + \dim(\widetilde{X}) & = & \displaystyle\int_\beta c_1(T_{G/P}) + d + \dim(\widetilde{X}) \\ & = & \displaystyle\int_\beta c_1(T_{G/P}) + (n+1)\widetilde{\alpha} \cdot E + n\ell + \dim(\widetilde{X}) \\ & = & \beta \cdot (c_1(T_{G/P}) - L) + (n+1)\widetilde{\alpha} \cdot (E + p^*L) + \dim(\widetilde{X}) \\ & = & \beta \cdot (c_1(T_{G/P}) - L) + (n+1)\alpha + \dim(\widetilde{X}). \end{array}$$

*So we have the formula*

$$\dim(\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})) = \int_{\widetilde{\alpha}} p^*(c_1(T_{G/P}) - L) + (n+1)\alpha + \dim(\widetilde{X}).$$

Theorem 3.7. — *Assume* $L(R) \neq \mathbb{Z}$ *and let* $\alpha \in A_1(X)$. *Then the irreducible components of* $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ *are indexed by* $\displaystyle\coprod_{\alpha' \leq \alpha} \mathfrak{ne}(\alpha')$.

Proof. Thanks to lemma 3.2 (this lemma works without the hypothesis $L(R) = \mathbb{Z}$) there exists a dense open subset of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ given by morphisms $f$ that do not factor through the vertex of the cone. It is enough to study this open set. In particular we know that the morphism

$$\pi_* : \coprod_{\widetilde{\alpha}\in A_1(\widetilde{X}),\ \pi_*\widetilde{\alpha}=\alpha} \mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X}) \to \mathbf{Hom}_{\alpha}(\mathbb{P}^1,X)$$

is dominant. The classes $\widetilde{\alpha}$ can even be choosen such that $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ is not empty. However the intersection $\widetilde{\alpha}\cdot E$ need not to be 0. In particular the classes $\widetilde{\alpha}$ can be choosen in

$$A(\alpha) = \coprod_{\alpha'\leq\alpha} \mathfrak{ne}(\alpha')$$

where $\alpha' = p_*\widetilde{\alpha}\cdot L$ and $\alpha-\alpha' = \widetilde{\alpha}\cdot E$. Indeed let $\widetilde{\alpha}\in A_1(\widetilde{X})$ and set as usual $\beta = p_*\widetilde{\alpha}$. Then there exists a unique element $\widetilde{\alpha}'\in A_1(\widetilde{X})$ such that $p_*\widetilde{\alpha}' = \beta$ and $\widetilde{\alpha}'\cdot E = 0$ (take $n\beta\cdot L$ for the relative degree). If $\widetilde{\alpha}$ is such that $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ is not empty then $\beta$ is effective and because of the value of the relative degree we have that $\mathbf{Hom}_{\widetilde{\alpha}'}(\mathbb{P}^1,\widetilde{X})$ is not empty. In particular $\widetilde{\alpha}'\in\mathfrak{ne}(\alpha')$ for $\alpha' = \pi_*\widetilde{\alpha}' = p_*\widetilde{\alpha}\cdot L$ and we have $\widetilde{\alpha}\cdot E = \pi_*\widetilde{\alpha}-\pi_*\widetilde{\alpha}'$. The element $\widetilde{\alpha}$ is uniquely determined by $\widetilde{\alpha}'$ and $\widetilde{\alpha}\cdot E$.

It is enough to prove that the images by $\pi_*$ of the irreducible schemes $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ for $\widetilde{\alpha}\in A(\alpha)$ are the irreducible components. In other words we have to prove that for any $\widetilde{\alpha}$ and $\widehat{\alpha}$ in $A(\alpha)$ the image of $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ is not contained in the closure of the image of $\mathbf{Hom}_{\widehat{\alpha}}(\mathbb{P}^1,\widetilde{X})$ in $\mathbf{Hom}_{\alpha}(\mathbb{P}^1,X)$.

Let $\widetilde{f}\in\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ a generic point and $\widehat{f}_t$ a familly of morphisms in $\mathbf{Hom}_{\widehat{\alpha}}(\mathbb{P}^1,\widetilde{X})$ such that $\pi\circ\widehat{f}_t$ converges to $\pi\circ\widetilde{f}$. In the compactification of $\mathbf{Hom}_{\widehat{\alpha}}(\mathbb{P}^1,\widetilde{X})$ by stable maps (see for example [FP]), the familly $\widehat{f}_t$ has a limit say $\widehat{f}$ which is a morphism from a tree $\cup_i D_i$ of rational curves to $\widetilde{X}$. Then we must have $\pi\circ\widehat{f} = \pi\circ\widetilde{f}$ as stable maps. In particular all but one of the images by $\widehat{f}$ of the irreducible components of the tree are contracted by $\pi$. To fix notation say that $D_i$ is contracted by $\pi$ for $i\geq 2$ and $\pi\circ\widehat{f}|_{D_1} = \pi\circ\widetilde{f}$. Because $\widetilde{f}$ is generic, it is not contained in the exceptional divisor so that the equality $\pi\circ\widehat{f}|_{D_1} = \pi\circ\widetilde{f}$ implies that $\widehat{f}|_{D_1} = \widetilde{f}$. We see that $\widehat{f}_*[D_1] = \widetilde{\alpha}$ so that

$$\widehat{\alpha} = \widehat{f}_*[D_1] + \sum_{i\geq 2}\widehat{f}_*[D_i] = \widetilde{\alpha} + \sum_{i\geq 2}\widehat{f}_*[D_i].$$

In particular we have $\widehat{\beta} = p_*\widehat{\alpha} \geq p_*\widetilde{\alpha} = \widetilde{\beta}$ and because $L(R)\neq\mathbb{Z}$ we know thanks to lemma 3.5 that $L\geq c_1(G/P)$ and we get

$$\widehat{\beta}\cdot(c_1(T_{G/P})-L) \leq \widetilde{\beta}\cdot(c_1(T_{G/P})-L).$$

As $\alpha = \pi_*\widehat{\alpha} = \pi_*\widetilde{\alpha}$ we see that

$$\dim(\mathbf{Hom}_{\widehat{\alpha}}(\mathbb{P}^1,\widetilde{X})) \leq \dim(\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})).$$

But the morphism $\pi_*$ is generically injective on $\mathbf{Hom}_{\widehat{\alpha}}(\mathbb{P}^1,\widetilde{X})$ and $\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X})$ so that the scheme $\pi_*(\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1,\widetilde{X}))$ cannot be in the closure of $\pi_*(\mathbf{Hom}_{\widehat{\alpha}}(\mathbb{P}^1,\widetilde{X}))$. $\square$

REMARK 3.8. — *Let us end with a discussion on the dimensions of the irreducible components of* $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ *for* $\alpha \in A_1(X)$.
(I) *In the first case:* $L(R) = \mathbb{Z}$, *these irreducible components are indexed by elements* $\widetilde{\alpha}$ *in* $\mathfrak{ne}(\alpha)$. *For such an element we have* $\widetilde{\alpha} \cdot E = 0$ *and the dimension of the component is given by*

$$\dim(\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})) = \int_{\widetilde{\alpha}} p^*(c_1(T_{G/P}) - L) + (n+1)\alpha + \dim(\widetilde{X}).$$

*The "variable" part in this dimension is the first one and it is given by*

$$\beta \cdot (c_1(T_{G/P}) - L)$$

*with* $\beta = p_*\widetilde{\alpha}$ *and we have* $\alpha = \beta \cdot L$ *so that the "variable" part is* $\beta \cdot c_1(T_{G/P})$. *The element* $\beta$ *ranges in the subset of the positive cone in the root lattice* $R$ *(in the projection of* $R$ *in* $\mathrm{Pic}(G/P)$*) given by the condition* $\beta \cdot L = \alpha$. *In particular if* $L$ *is not collinear to* $c_1(G/P)$ *the dimensions of the irreducible components are not equal. In this case the variety* $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ *is equidimensional if and only if* $L = \frac{1}{2}c_1(G/P)$.
(II) *In the second case:* $L(R) \neq \mathbb{Z}$, *these irreducible components are indexed by elements* $\widetilde{\alpha} \in \coprod_{\alpha' \leq \alpha} \mathfrak{ne}(\alpha')$. *For such an element we have* $\widetilde{\alpha} \cdot E \geq 0$ *and the dimension of the component is given by*

$$\dim(\mathbf{Hom}_{\widetilde{\alpha}}(\mathbb{P}^1, \widetilde{X})) = \int_{\widetilde{\alpha}} p^*(c_1(T_{G/P}) - L) + (n+1)\alpha + \dim(\widetilde{X}).$$

*The "variable" part in this dimension is the first one and it is given by*

$$\beta \cdot (c_1(T_{G/P}) - L)$$

*with* $\beta = p_*\widetilde{\alpha}$. *In this case we have* $\beta \cdot L = \alpha' \leq \alpha$. *The element* $\beta$ *ranges in the subset of the positive cone in the root lattice* $R$ *(in the projection of* $R$ *in* $\mathrm{Pic}(G/P)$*) given by the condition* $\beta \cdot L \leq \alpha$. *In particular if* $L$ *is not collinear to* $c_1(G/P)$ *the dimensions of the irreducible components are not equal (look at the* $\beta$ *such that* $\beta \cdot L = \alpha$*). Furthermore even if* $L$ *is collinear to* $c_1(G/P)$ *the dimensions of the irreducible components are not equal unless* $L = c_1(G/P)$. *In this case the variety* $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ *is equidimensional if and only if* $L = c_1(G/P)$.

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