On the Values of Equivariant Zeta Functions of Curves over Finite Fields

David Burns

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Abstract. Let K/k be a finite Galois extension of global function fields of characteristic p. Let C_K denote the smooth projective curve that has function field K and set $G := \mathrm{Gal}(K/k)$. We conjecture a formula for the leading term in the Taylor expansion at zero of the G-equivariant truncated Artin L-functions of K/k in terms of the Weil-étale cohomology of $\mathbb{G}_m$ on the corresponding open subschemes of C_K. We then prove the ℓ-primary component of this conjecture for all primes ℓ for which either $\ell \neq p$ or the relative algebraic K-group $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell)$ is torsion-free. In the remainder of the manuscript we show that this result has the following consequences for K/k: if $p \nmid |G|$, then refined versions of all of Chinburg's 'Ω-Conjectures' in Galois module theory are valid; if the torsion subgroup of $K^\times$ is a cohomologically-trivial G-module, then Gross's conjectural 'refined class number formula' is valid; if K/k satisfies a certain natural class-field theoretical condition, then Tate's recent refinement of Gross's conjecture is valid.

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1. Introduction

Let K/k be a finite Galois extension of global function fields of characteristic p. Let C_K be the unique geometrically irreducible smooth projective curve which has function field equal to K and set $G := \mathrm{Gal}(K/k)$. For each finite non-empty set S of places of k that contains all places which ramify in K/k, we write $\mathcal{O}_{K,S}$ for the subring of K consisting of those elements that are integral at all places of K which do not lie above an element of S and we set $U_{K,S} := \mathrm{Spec}(\mathcal{O}_{K,S})$. With R denoting either $\mathbb{Z}$ or $\mathbb{Z}_\ell$ for some prime ℓ and E an extension field of the field of fractions of R, we write $K_0(R[G], E)$ for the relative algebraic K-group defined by Swan in [46].

In §2 we formulate a conjectural equality $\mathrm{C}(K/k)$ between an element of $K_0(\mathbb{Z}[G],\mathbb{R})$ constructed from the leading term in the Taylor expansion at $s=0$ of the G-equivariant Artin L-function of $U_{K,S}$ and the refined Euler characteristic of a pair comprising the Weil-étale cohomology of $\mathbb{G}_m$ on $U_{K,S}$ (considered as an object of an appropriate derived category) and a natural logarithmic regulator mapping. This conjecture is motivated both by the general approach described by Lichtenbaum in [40, §8] and also by analogy to a special case of the equivariant refinement of the Tamagawa Number Conjecture of Bloch and Kato (which was formulated by Flach and the present author in [13]). The equality $\mathrm{C}(K/k)$ can be naturally reinterpreted as a conjectural equality in $K_0(\mathbb{Z}[G],\mathbb{Q})$ involving the leading term at $t=1$ of the G-equivariant Zeta-function of $U_{K,S}$ and in §3 we shall prove the validity, resp. the validity modulo torsion, of the projection of the latter conjectural equality to $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ for all primes $\ell\neq p$, resp. for $\ell=p$ (this is Theorem 3.1). If $\ell\neq p$, then our proof combines Grothendieck's formula for the Zeta-function in terms of the action of frobenius on ℓ-adic cohomology together with a non-commutative generalisation of a purely algebraic observation of Kato in [35] (this result may itself be of some independent interest) and an explicit computation of certain Bockstein homomorphisms in ℓ-adic cohomology. In the case that $\ell=p$ we are able to deduce our result from Bae's verification of the 'Strong-Stark Conjecture' [3] which in turn relies upon results of Milne [43] concerning relations between Zeta-functions and p-adic cohomology.

In the remainder of the manuscript we show that $\mathrm{C}(K/k)$ provides a universal approach to the study of several well known conjectures. A key ingredient in all of our results in this direction is a previous observation of Flach and the present author which allows an interpretation in terms of Weil-étale cohomology of the canonical extension classes defined using class field theory by Tate in [49].

In §4 we consider connections between $\mathrm{C}(K/k)$ and the central conjectures of classical Galois module theory. To be specific, we prove that $\mathrm{C}(K/k)$ implies the validity for K/k of a strong refinement of the '$\Omega(3)$-Conjecture' formulated by Chinburg in [18, §4.2]. Taken in conjunction with Theorem 3.1 this result allows us to deduce that if $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p)$ is torsion-free, resp. $p\nmid|G|$, then the $\Omega(3)$-Conjecture, resp. the $\Omega(1)$-, $\Omega(2)$- and $\Omega(3)$-Conjectures, formulated by Chinburg in loc. cit., are valid for K/k. This is a strong refinement of previous results in this area.

We assume henceforth that G is abelian. In this case Gross has conjectured a 'refined class number formula' which expresses an explicit congruence relation between the values at $s=0$ of the Dirichlet L-functions associated to K/k [31]. This conjecture has attracted much attention and indeed Tate has recently formulated a strong refinement in the case that G is cyclic [51]. However, whilst the conjecture of Gross has already been verified in several interesting cases [31, 1, 47, 37, 39], much of this evidence is obtained either by careful analysis of special cases or by induction on $|G|$ and, as yet, no coherent overview of or systematic approach to these conjectures of Gross and Tate has emerged. In contrast, in §5 we shall use the general approach of algebraic height pairings developed by Nekovář in [44] to interpret the integral regulator mapping of Gross as a Bockstein homomorphism in Weil-étale cohomology, and we shall also apply this interpretation to prove that if the torsion subgroup μ_K of $K^\times$ is a cohomologically-trivial G-module (a condition that is automatically satisfied if, for example, $|\mu_K|$ is coprime to $|G|$), then $\mathrm{C}(K/k)$ implies the validity of Gross's conjecture for K/k. Under a certain natural class-field theoretical assumption on K/k we shall also show (in §6) that $\mathrm{C}(K/k)$ implies the validity of Tate's refinement of Gross's conjecture. When combined with Theorem 3.1 (and earlier results of Tan concerning p-extensions) these observations allow us to deduce the validity of Gross's conjecture for all extensions K/k for which μ_K is a cohomologically-trivial G-module and also to prove the validity of Tate's refinement of Gross's conjecture for a large family of extensions.

A further development of the approach used here should allow the removal of any hypothesis on μ_K (indeed, in special cases this is already achieved in the present manuscript). However, even at this stage, our results constitute a strong improvement of previous results in this area and also provide a philosophical underpinning to the conjectures of Gross and Tate that was not hitherto apparent. Indeed, the approach presented here leads to the formulation of natural analogues of these conjectures concerning the values of (higher order) derivatives of L-functions that vanish at $s = 0$. These developments have in turn led to a proof of Tate's conjecture under the hypothesis only that $|\mu_K|$ is coprime to $|G|$ and have also provided new insight into Gross's 'refined p-adic abelian Stark conjecture' as well as several other conjectures due, for example, to Rubin, to Darmon, to Popescu and to Tan. For more details of this aspect of the theory the reader is referred to [10, 34].

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2. The leading term conjecture

2.1. Relative Algebraic K-Theory. In this subsection we quickly recall certain useful constructions in algebraic K-theory.

If Λ is any ring, then all modules are to be understood as left modules. We write $\zeta(\Lambda)$ for the centre of Λ, $K_1(\Lambda)$ for the Whitehead group of Λ and $K_0(\Lambda)$ for the Grothendieck group of the category of finitely generated projective Λ-modules. We also write $\mathcal{D}(\Lambda)$ for the derived category of complexes of Λ-modules with only finitely many non-zero cohomology groups, and we let $\mathcal{D}^{\mathrm{fpd}}(\Lambda)$, resp. $\mathcal{D}^{\mathrm{perf}}(\Lambda)$,denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$

consisting of those complexes that are quasi-isomorphic to a bounded complex of projective Λ-modules, resp. to a bounded complex of finitely generated projective Λ-modules.

We let R denote either $\mathbb{Z}$ or $\mathbb{Z}_\ell$ for some prime ℓ, E and F denote extension fields of the field of fractions of R and we fix a finite group G. For finitely generated $E[G]$-modules V and W we write $\mathrm{Is}_{E[G]}(V,W)$ for the set of $E[G]$-module isomorphisms from V to W. The relative algebraic K-group $K_0(R[G],E)$ is an abelian group with generators (X,ϕ,Y), where X,Y are finitely generated projective $R[G]$-modules and ϕ is an element of $\mathrm{Is}_{E[G]}(X\otimes_R E, Y\otimes_R E)$. For the defining relations we refer to [46, p. 215]. We systematically use the following facts: there is a long exact sequence of relative K-theory (cf. [46, Th. 15.5])

$$K_1(R[G]) \to K_1(E[G]) \xrightarrow{\partial^1_{R[G],E}} K_0(R[G],E) \xrightarrow{\partial^0_{R[G],E}} K_0(R[G]) \to K_0(E[G]);$$

if $E \subseteq F$, then there is a natural injective 'inclusion' homomorphism $K_0(R[G],E) \subseteq K_0(R[G],F)$; for each rational prime ℓ the assignment $(X,\phi,Y) \mapsto (X\otimes_{\mathbb{Z}}\mathbb{Z}_\ell, \phi\otimes_{\mathbb{Q}}\mathbb{Q}_\ell, Y\otimes_{\mathbb{Z}}\mathbb{Z}_\ell)$ induces a homomorphism

$$\rho_\ell : K_0(\mathbb{Z}[G],\mathbb{Q}) \to K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$$

and the product of these homomorphisms over all primes ℓ induces an isomorphism (cf. the discussion following [26, (49.12)])

$$\prod_\ell \rho_\ell : K_0(\mathbb{Z}[G],\mathbb{Q}) \cong \bigoplus_\ell K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell). \tag{1}$$

Let A be a finite dimensional central simple F-algebra, F' an extension of F which splits A and e an indecomposable idempotent of $A\otimes_F F'$. If V is any finitely generated A-module and $\phi\in\mathrm{End}_A(V)$, then the 'reduced determinant' of ϕ is defined by setting $\mathrm{detred}_A(\phi) := \det_{F'}(\phi\otimes_F \mathrm{id}_{F'}|e(V\otimes_F F'))$. This is an element of F which is independent of the choices of F' and e. This construction extends to finite-dimensional semi-simple F-algebras in the obvious way. In particular, the group $K_1(E[G])$ is generated by symbols of the form $[\phi]$ with $\phi\in\mathrm{Aut}_{E[G]}(V)$ and the assignment $[\phi]\mapsto\mathrm{detred}_{E[G]}(\phi)$ induces a well-defined injective 'reduced norm' homomorphism $\mathrm{nr}_{E[G]} : K_1(E[G]) \to \zeta(E[G])^\times$ [26, §45A]. For each ℓ the map $\mathrm{nr}_{\mathbb{Q}_\ell[G]}$ is bijective and so there exists a unique homomorphism $\delta_\ell : \zeta(\mathbb{Q}_\ell[G])^\times \to K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ with $\partial^1_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell} = \delta_\ell\circ\mathrm{nr}_{\mathbb{Q}_\ell[G]}$. (When we need to be more precise we write $\delta_{G,\ell}$ rather than δ_ℓ.) The map $\mathrm{nr}_{\mathbb{R}[G]}$ is not in general surjective, but nevertheless there exists a canonical 'extended boundary' homomorphism $\delta : \zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G],\mathbb{R})$ which satisfies $\partial^1_{\mathbb{Z}[G],\mathbb{R}} = \delta\circ\mathrm{nr}_{\mathbb{R}[G]}$ and is such that $\zeta(\mathbb{Q}[G])^\times$ is the full pre-image of $K_0(\mathbb{Z}[G],\mathbb{Q})$ under δ (cf. [13, Lem. 9]).

The map $\mathrm{nr}_{E[G]}$ induces an equivalence relation '$\sim$' on each set $\mathrm{Is}_{E[G]}(V,W)$ in the following way: $\phi\sim\phi'$ if $\mathrm{nr}_{E[G]}([\phi'\circ\phi^{-1}]) = 1$. In the sequel we shall often not distinguish between an element of $\mathrm{Is}_{E[G]}(V,W)$ and its associated equivalence class in $\mathrm{Is}_{E[G]}(V,W)/\sim$.

For each $\mathbb{Z}$-graded module $C^\cdot$ we write C^{all}, C^- and C^+ for the direct sum of C^i as i ranges over all, all odd and all even integers respectively.

An 'E-trivialisation' of an object $C^\cdot$ of $\mathcal{D}^{\mathrm{perf}}(R[G])$ is an element τ of $\mathrm{Is}_{E[G]}(H^+(C^\cdot)\otimes_R E, H^-(C^\cdot)\otimes_R E)/\sim$. In [9] it is shown that a variant of the classical construction of Whitehead torsion allows one to associate to each such pair $(C^\cdot,\tau)$ a canonical 'refined Euler characteristic' element $\chi_{R[G],E}(C^\cdot,\tau)$ which belongs to $K_0(R[G],E)$ and has image under $\partial^0_{R[G],E}$ equal to the Euler characteristic of $C^\cdot$ in $K_0(R[G])$. Further details of this construction are recalled in the Appendix.

In the sequel we shall use the following notation and conventions. We abbreviate 'cohomologically-trivial' to 'c-t', '$\chi_{\mathbb{Z}[G],\mathbb{R}}$' to '$\chi$', '$\chi_{\mathbb{Z}[G],\mathbb{Q}}$' to '$\chi_{\mathbb{Q}}$' and '$\chi_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}$' to '$\chi_\ell$' (or to '$\chi_{G,\ell}$' when we need to be more precise); if X is any scheme of finite type over the finite field $\mathbb{F}_p$ of cardinality p and $\mathcal{F}$ is any étale (pro-) sheaf, resp. Weil-étale sheaf, on X, then we abbreviate $R\Gamma(X_{\text{ét}},\mathcal{F})$, resp. $R\Gamma(X_{\text{Weil-ét}},\mathcal{F})$ to $R\Gamma(X,\mathcal{F})$, resp. $R\Gamma_{\mathcal{W}}(X,\mathcal{F})$, and we also use similar abbreviations on cohomology; for any commutative ring Λ we write $x\mapsto x^\#$ for the Λ-linear involution of $\zeta(\Lambda[G])$ that is induced by setting $g^\# := g^{-1}$ for each $g\in G$; for any group H and any H-module M we write M^H, resp. M_H, for the maximal submodule, resp. quotient, of M upon which H acts trivially; for any abelian group A we let A_{tors} denote its torsion subgroup; unless explicitly indicated otherwise, all tensor products and exterior powers are to be considered as taken in the category of abelian groups.

2.2. Formulation of the conjecture. We assume henceforth that S is a finite non-empty set of places of k containing all places which ramify in K/k. We let $\mathrm{Irr}_{\mathbb{C}}(G)$ denote the set of irreducible finite dimensional complex characters of G. For each $\chi\in\mathrm{Irr}_{\mathbb{C}}(G)$ we write $L_S(\chi,s)$ for the associated S-truncated Artin L-function and $L_S^*(\chi,0)$ for the leading term in the Taylor expansion of $L_S(\chi,s)$ at $s=0$. Recalling that $\zeta(\mathbb{C}[G])$ identifies with $\prod_{\mathrm{Irr}_{\mathbb{C}}(G)}\mathbb{C}$, we define a $\zeta(\mathbb{C}[G])$-valued meromorphic function of a complex variable s by setting

$$\theta_{K/k,S}(s) := (L_S(\chi,s))_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)}.$$

The leading term $\theta^*_{K/k,S}(0)$ in the Taylor expansion of $\theta_{K/k,S}(s)$ at $s=0$ is equal to $(L_S^*(\chi,0))_{\chi\in\mathrm{Irr}(G)}$ and hence belongs to $\zeta(\mathbb{R}[G])^\times$. In this subsection we follow the philosophy introduced by Lichtenbaum in [40] to formulate a conjectural description of $\delta(\theta^*_{K/k,S}(0)^\#)$ in terms of Weil-étale cohomology.

For any intermediate field F of K/k we write $Y_{F,S}$ for the free abelian group on the set of places $S(F)$ of F which lie above those in S and $X_{F,S}$ for the kernel of the homomorphism $Y_{F,S}\to\mathbb{Z}$ that sends each element of $S(F)$ to 1. We write $\mathcal{O}_{F,S}$ for the ring of $S(F)$-integers in F and $\mathcal{O}^\times_{F,S}$ for its unit group. We also set $U_{F,S} := \mathrm{Spec}(\mathcal{O}_{F,S})$ and $A_{F,S} := \mathrm{Pic}(\mathcal{O}_{F,S})$.

Lemma 1.

i) *Let $j: U_{K,S}\to C_K$ denote the natural open immersion. Then there exists a canonical isomorphism in $\mathcal{D}(\mathbb{Z}[G])$ of the form*

$$R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)\cong R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{\mathcal{W}}(C_K,j_!\mathbb{Z}),\mathbb{Z}[-2]).$$

ii) *There exists a natural distinguished triangle in* $\mathcal{D}(\mathbb{Z}[G])$ *of the form*

$$X_{K,S}\otimes\mathbb{Q}[-2]\to R\Gamma(U_{K,S},\mathbb{G}_m)\to R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)\to X_{K,S}\otimes\mathbb{Q}[-1]$$

where the map induced on cohomology (in degree 2*) by the first morphism is the composite of the projection* $X_{K,S}\otimes\mathbb{Q}\to X_{K,S}\otimes\mathbb{Q}/\mathbb{Z}$ *and the canonical identification* $X_{K,S}\otimes\mathbb{Q}/\mathbb{Z}\cong H^2(U_{K,S},\mathbb{G}_m)$.

iii) $R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$ *is an object of* $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ *that is acyclic outside degrees* 0 *and* 1*. One has a canonical identification* $H^0_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)=\mathcal{O}^\times_{K,S}$ *and a natural exact sequence of* G*-modules*

$$0\to A_{K,S}\to H^1_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)\to X_{K,S}\to 0.$$

iv) *If* J *is any normal subgroup of* G*, then there exists a natural isomorphism in* $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G/J])$ *of the form*

$$R\Gamma_{\mathcal{W}}(U_{K^J,S},\mathbb{G}_m)\cong R\operatorname{Hom}_{\mathbb{Z}[J]}(\mathbb{Z},R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)).$$

With respect to the descriptions of cohomology given in iii) (for both K *and* K^J*) the displayed isomorphism induces the natural identification* $\mathcal{O}^\times_{K^J,S}=(\mathcal{O}^\times_{K,S})^J$ *and also identifies* $X_{K^J,S}$ *with a submodule of* $X_{K,S}$ *by means of the map that sends each place* v *of* $S(K^J)$ *to* $\sum_{j\in J}j(w)$ *where* w *is any place of* K *lying above* v.

Proof. Claim i) is proved by the argument of [40, proof of Th. 6.5].
The existence of the distinguished triangle in claim ii) can be proved by comparing the spectral sequences of [40, Prop. 2.3(f)] or by using the approach of Geisser in [30, Th. 6.1].
The descriptions of the groups $H^i_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$ given in claim iii) are proved by Lichtenbaum in [40, Th. 7.1c)]. They follow from the long exact sequence of cohomology associated to the triangle in claim ii), the canonical identifications $H^0(U_{K,S},\mathbb{G}_m)\cong\mathcal{O}^\times_{K,S}, H^1(U_{K,S},\mathbb{G}_m)\cong\operatorname{Pic}(\mathcal{O}_{K,S})$ and $H^2(U_{K,S},\mathbb{G}_m)\cong X_{K,S}\otimes\mathbb{Q}/\mathbb{Z}$ and the fact that $H^i(U_{K,S},\mathbb{G}_m)=0$ if $i>2$. Since each cohomology group of $R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$ is finitely generated, a standard argument of homological algebra shows that this complex belongs to $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ if and only if it belongs to $\mathcal{D}^{\mathrm{fpd}}(\mathbb{Z}[G])$ (cf. [11, proof of Prop. 1.20, Steps 3 and 4]). On the other hand, any G-module that is c-t has finite projective dimension as a $\mathbb{Z}[G]$-module and so it suffices to show that $R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$ is isomorphic to a bounded complex of G-modules which are each c-t. Now the G-module $X_{K,S}\otimes\mathbb{Q}$ is c-t and so the distinguished triangle of claim ii) implies that we need only prove that $R\Gamma(U_{K,S},\mathbb{G}_m)$ is isomorphic to a bounded complex of G-modules which are each c-t. But this is true because the natural morphism $\pi:U_{K,S}\to U_{k,S}$ is étale and $\mathbb{G}_m=\pi^*\mathbb{G}_m$ on $(U_{K,S})_{\text{ét}}$ (cf. [11, proof of Prop. 1.20, Steps 1 and 2]).
Claim iv) follows from the triangle of claim ii) and the description of cohomology given in iii) (for both K and K^J) together with an explicit computation of the maps induced on cohomology by the natural isomorphism $R\Gamma(U_{K^J,S},\mathbb{G}_m)\cong R\operatorname{Hom}_{\mathbb{Z}[J]}(\mathbb{Z},R\Gamma(U_{K,S},\mathbb{G}_m))$ in $\mathcal{D}(\mathbb{Z}[G/J])$ (for more details as to the latter see, for example, the proof of [12, Lem. 12]). $\square$

For each place w of K we let $|\cdot|_w$ denote the absolute value of w normalised as in [50, Chap. 0, 0.2]. We write $\mathrm{R}_{K,S}$ for the $\mathbb{R}[G]$-equivariant isomorphism $\mathcal{O}_{K,S}^\times \otimes \mathbb{R} \longrightarrow X_{K,S} \otimes \mathbb{R}$ which at each $u \in \mathcal{O}_{K,S}^\times$ satisfies

$$\mathrm{R}_{K,S}(u) = -\sum_{w \in S(K)} \log |\, u\,|_w \cdot w. \tag{2}$$

We also denote by $\mathrm{R}_{K,S}$ the $\mathbb{R}$-trivialisation of $R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$ that is induced by $\mathrm{R}_{K,S}$ and the descriptions of Lemma 1iii).
We can now state the central conjecture of this paper.

Conjecture C(K/k): *In $K_0(\mathbb{Z}[G], \mathbb{R})$ one has an equality*

$$\delta(\theta^*_{K/k,S}(0)^\#) = \chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S}).$$

Remark 1. Lemma 1i) shows that C(K/k) can be naturally rephrased in terms of $R\Gamma_{\mathcal{W}}(C_K, j_!\mathbb{Z})$. We have chosen to work in terms of $\mathbb{G}_m$ rather than $j_!\mathbb{Z}$ for the purposes of explicit computations that we make in subsequent sections (see also Remark 3 in this regard).

Remark 2. If G is abelian, then the equality of C(K/k) is equivalent to a formula for the $\mathbb{Z}[G]$-submodule of $\mathbb{R}[G]$ which is generated by $\theta^*_{K/k,S}(0)^\#$ in terms of the $\mathbb{Z}[G]$-equivariant graded determinant of $R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$ (see Remark A1 in the Appendix). By using this observation in conjunction with Remark 1 it can be shown that C(k/k) is equivalent to a special case of the conjecture formulated by Lichtenbaum in [40, Conj. 8.1e)].

Remark 3. Let $j : U_{K,S} \to C_K$ denote the natural open immersion. Then the Poincaré Duality Theorem of [42, Chap. II, Th. 3.1] gives rise to a commutative diagram in $\mathcal{D}(\mathbb{Z}[G])$ of the form

$$\begin{tikzcd}
X_{K,S} \otimes \mathbb{Q}[-2] \arrow[r] \arrow[d, equal] & R\Gamma(U_{K,S}, \mathbb{G}_m) \arrow[r] \arrow[d] & R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \\
X_{K,S} \otimes \mathbb{Q}[-2] \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(R\Gamma(C_K, j_!\mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-3]) \arrow[d] & \\
& (\hat{\mathcal{O}}^\times_{K,S}/\mathcal{O}^\times_{K,S})[0] &
\end{tikzcd}$$

where the top row is as in Lemma 1ii), $\hat{\mathcal{O}}^\times_{K,S}$ denotes the profinite completion of $\mathcal{O}^\times_{K,S}$ and the second column is a distinguished triangle. This diagram implies that $R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$ is a precise analogue of the complex Ψ_S that occurs in [12, Rem. following Prop. 3.1] and [7, Prop. 2.1.1]. For this reason, C(K/k) is an analogue of the conjectural vanishing of the element $T\Omega(K/k, 0)$ defined in [7, Th. 2.1.2], where K/k is a Galois extension of number fields of group G, and also coincides in the abelian case with the function field case of [8, Conj. 2.1]. We recall that the vanishing of the element $T\Omega(K/k, 0)$ is equivalent to the validity of the 'Lifted Root Number Conjecture' of Gruenberg, Ritter and

Weiss [33] (see [7, Th. 2.3.3] for a proof of this fact) and also to the validity of the 'Equivariant Tamagawa Number Conjecture' of [13, Conj. 4(iv)] as applied to the pair $(h^0(\mathrm{Spec}\ K), \mathbb{Z}[G])$ where $h^0(\mathrm{Spec}\, K)$ is considered as a motive that is defined over k and has coefficients $\mathbb{Q}[G]$ (see [7, Th. 2.4.1] or [14, §3] for different proofs of this fact). We further recall that [13, Conj. 4(iv)] is itself a natural equivariant version of the seminal conjecture of Bloch and Kato from [6], and that if G is abelian, then it refines the 'Generalized Iwasawa Main Conjecture' formulated by Kato in [35, §3.2] (cf. [14, §2] in this regard). Finally we recall that strong evidence in favour of [13, Conj. 4(iv)] has recently been obtained in [15, 16].

By a change of variable we now remove all of the transcendental terms which occur in $\mathrm{C}(K/k)$ and then decompose the conjecture according to (1).
To do this we set $t := p^{-s}$ and then define a $\zeta(\mathbb{C}[G])$-valued function of the complex variable t by means of the equality $Z_{K/k,S}(t) := \theta_{K/k,S}(s)$. For each place $w \in S(K)$ we write val_w and $k(w)$ for its valuation and residue field and let $\deg(w)$ denote the degree of the field extension $k(w)/\mathbb{F}_p$. We write $\mathrm{D}_{K,S} : \mathcal{O}_{K,S}^\times \to X_{K,S}$ for the homomorphism which at each $u \in \mathcal{O}_{K,S}^\times$ satisfies

$$\mathrm{D}_{K,S}(u) = \sum_{w \in S(K)} \mathrm{val}_w(u) \deg(w) \cdot w.$$

Lemma 2. *Let $e : \mathrm{Spec}(\zeta(\mathbb{R}[G])) \to \mathbb{Z}$ denote the algebraic order of $Z_{K/k,S}(t)$ at $t = 1$ (which we regard as an element of $\mathbb{Z}^{\pi_0(\mathrm{Spec}(\zeta(\mathbb{R}[G])))}$ in the natural way). Then the element*

$$Z^*_{K/k,S}(1) := \lim_{t \to 1} (1-t)^{-e} Z_{K/k,S}(t)$$

belongs to $\zeta(\mathbb{Q}[G])^\times$ and $\mathrm{C}(K/k)$ is valid if and only if in $K_0(\mathbb{Z}[G], \mathbb{Q})$ one has

$$\delta(Z^*_{K/k,S}(1)^\#) = \chi_{\mathbb{Q}}(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{D}_{K,S} \otimes \mathbb{Q}). \tag{3}$$

Proof. The algebraic order of $\theta_{K/k,S}(s)^\#$ at $s = 0$ is equal to e. In addition, by an explicit computation one verifies that

$$\begin{aligned}\theta^*_{K/k,S}(0)^\# &= \lim_{s \to 0} s^{-e} \theta_{K/k,S}(s)^\# \\ &= (\log(p))^e \cdot Z^*_{K/k,S}(1)^\#.\end{aligned}$$

When combined with the known validity of Stark's Conjecture for K/k [50, p. 111], this equality proves that $Z^*_{K/k,S}(1)$ belongs to $\zeta(\mathbb{Q}[G])^\times$. Also, since $\chi_{\mathbb{Q}}(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{D}_{K,S} \otimes \mathbb{Q})$ is equal to $\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{D}_{K,S} \otimes \mathbb{R})$ in $K_0(\mathbb{Z}[G], \mathbb{R})$, the above equality shows that $\mathrm{C}(K/k)$ is equivalent to (3) provided that in $K_0(\mathbb{Z}[G], \mathbb{R})$ one has

$$\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S}) = \chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{D}_{K,S} \otimes \mathbb{R}) + \delta((\log(p))^e).$$

The validity of this equality follows directly from [7, Prop. 1.2.1(ii)] in conjunction with the equality $\mathrm{R}_{K,S}(u) = \log(p) \cdot \mathrm{D}_{K,S}(u)$ for each $u \in \mathcal{O}_{K,S}^\times$ and the fact that the reduced rank (as defined in [13, §2.6]) of the $\mathbb{R}[G]$-module $X_{K,S} \otimes \mathbb{R}$ is equal to e [50, Chap. I, Prop. 3.4]. □

From Lemma 2 and the bijectivity of the map (1) it is clear that $\mathrm{C}(K/k)$ is valid if and only if, for each prime ℓ, the following conjecture is valid.

Conjecture $\mathrm{C}_\ell(K/k)$: *The image of (3) under ρ_ℓ is valid.*

Remark 4. There are several useful functorial properties of $\mathrm{C}(K/k)$ that can be proved directly or by combining Remark 3 with the relevant arguments from either [7] or [13, §4.4-5]. For example, in this way it can be shown that the validity of $\mathrm{C}(K/k)$ is independent of the choice of S (cf. [7, Th. 2.1.2]). In addition, if ℓ is any prime and H is any subgroup of G, then it can be shown that the validity of the image of the equality of $\mathrm{C}_\ell(K/k)$ under the natural restriction map $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell) \to K_0(\mathbb{Z}_\ell[H], \mathbb{Q}_\ell)$ is equivalent to the validity of $\mathrm{C}_\ell(K/K^H)$ (cf. [7, Prop. 2.1.4(i)]). In a similar way, if J is any normal subgroup of G, then Lemma 1iv) implies that the validity of the image of the equality of $\mathrm{C}_\ell(K/k)$ under the natural coinflation map $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell) \to K_0(\mathbb{Z}_\ell[G/J], \mathbb{Q}_\ell)$ is equivalent to the validity of $\mathrm{C}_\ell(K^J/k)$ (cf. [7, Prop. 2.1.4(ii)]).

3. Evidence

In this section we shall provide the following evidence in support of $\mathrm{C}(K/k)$.

Theorem 3.1. *Let K/k be a finite Galois extension of global function fields of characteristic p and set $G := \mathrm{Gal}(K/k)$.*

i) *If $\ell \neq p$, then $\mathrm{C}_\ell(K/k)$ is valid.*
ii) *$\mathrm{C}_p(K/k)$ is valid modulo the torsion subgroup of $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$.*

Corollary 1. *$\mathrm{C}(K/k)$ is valid modulo the torsion subgroup of $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$.*

Proof. Clear. □

Remark 5. The group $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ is torsion-free if $p \nmid |G|$ (cf. [13, proof of Lem. 11c)]) and also if $p = 2$ and G is either of order 2 or is dihedral of order congruent to 2 modulo 4 [5, Lem. 8.2].

3.1. The descent formalism. In this subsection we prepare for the proof of Theorem 3.1i) by proving a purely algebraic result. This provides a natural generalisation of several results that have already been used elsewhere (cf. Remark 6) and so the material of this subsection may well itself be of some independent interest.

We fix an arbitrary rational prime ℓ and for each $\mathbb{Z}_\ell$-module M we set $M_{\mathbb{Q}_\ell} := M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We say that an endomorphism ψ of a finitely generated $\mathbb{Z}_\ell[G]$-module M is 'semi-simple at 0' if the natural composite homomorphism

$$\ker(\psi) \xrightarrow{\subseteq} M \twoheadrightarrow \mathrm{cok}(\psi) \tag{4}$$

has both finite kernel and finite cokernel. We note that this condition is satisfied if and only if there exists a $\mathbb{Q}_\ell[G][\psi]$-equivariant direct complement to the submodule $\ker(\psi)_{\mathbb{Q}_\ell}$ of $M_{\mathbb{Q}_\ell}$.

Let t be an indeterminate. Then for any element f of $\zeta(\mathbb{Q}_\ell[G])[[t]]$ we write $e_f : \mathrm{Spec}(\zeta(\mathbb{Q}_\ell[G])) \to \mathbb{Z}$ for the algebraic order of $f(t)$ at $t = 1$. We identify e_f with an element of $\mathbb{Z}^{\pi_0(\mathrm{Spec}(\zeta(\mathbb{Q}_\ell[G])))}$ in the natural way and then set

$$f^*(1) := \lim_{t\to 1}(1-t)^{-e_f} f(t) \in \zeta(\mathbb{Q}_\ell[G])^\times.$$

In particular, if θ is any endomorphism of a finitely generated $\mathbb{Z}_\ell[G]$-module M for which $1-\theta$ is semi-simple at 0 and

$$f(t) = \mathrm{detred}_{\mathbb{Q}_\ell[G]}(1-\theta\cdot t : M_{\mathbb{Q}_\ell}),$$

then we set

$$\begin{aligned}\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta : M_{\mathbb{Q}_\ell}) :=& f^*(1)\\ =& \mathrm{detred}_{\mathbb{Q}_\ell[G]}(1-\theta : D)\end{aligned}$$

where D is any choice of a $\mathbb{Q}_\ell[G][\theta]$-equivariant direct complement to the submodule $\ker(1-\theta)_{\mathbb{Q}_\ell}$ of $M_{\mathbb{Q}_\ell}$.

We now suppose given a bounded complex of finitely generated projective $\mathbb{Z}_\ell[G]$-modules $P^\cdot$ and a $\mathbb{Z}_\ell[G]$-equivariant endomorphism θ of $P^\cdot$ which is such that the induced endomorphism $H^i(1-\theta)$ of $H^i(P^\cdot)$ is semi-simple at 0 in each degree i.

We let $\mathrm{C}(\theta)^\cdot$ denote the -1-shift of the mapping cone of the endomorphism of $P^\cdot$ induced by $1-\theta$. Then from the long exact sequence of cohomology that is associated to the distinguished triangle

$$P^\cdot \xrightarrow{1-\theta} P^\cdot \to C(\theta)^\cdot[1] \to P^\cdot[1]$$

one obtains in each degree i a short exact sequence

$$0 \to \mathrm{cok}(H^{i-1}(1-\theta)) \to H^i(\mathrm{C}(\theta)^\cdot) \to \ker(H^i(1-\theta)) \to 0.$$

Upon combining these sequences with the isomorphisms

$$\ker(H^i(1-\theta))_{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathrm{cok}(H^i(1-\theta))_{\mathbb{Q}_\ell}$$

induced by (4) (with $\psi = H^i(1-\theta)$ and $M = H^i(P^\cdot)$) one obtains a well-defined $\mathbb{Q}_\ell$-trivialisation τ_θ of $\mathrm{C}(\theta)^\cdot$.

Proposition 3.1. *Let $P^\cdot$ be a bounded complex of finitely generated projective $\mathbb{Z}_\ell[G]$-modules and θ a $\mathbb{Z}_\ell[G]$-equivariant endomorphism of $P^\cdot$ for which $H^i(1-\theta)$ is semi-simple at 0 in each degree i. Then in $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ one has*

$$\chi_\ell(\mathrm{C}(\theta)^\cdot, \tau_\theta) = \sum_{i\in\mathbb{Z}}(-1)^i\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta : H^i(P^\cdot)_{\mathbb{Q}_\ell})).$$

Proof. We shall argue by induction on the quantity

$$|P^\cdot| := \max\{i : P^i \neq 0\} - \min\{j : P^j \neq 0\}.$$

We first assume that $|P^\cdot| = 0$ so that $P^\cdot$ has only one non-zero term. To be specific, we assume that $P^\cdot = P^n[-n]$ (so that $H^n(P^\cdot) = P^n$). In this case $C(\theta)^\cdot$

is equal to the complex $P^n \xrightarrow{1-\theta^n} P^n$, where the first term is placed in degree n. In addition, upon choosing a $\mathbb{Q}_\ell[G][\theta^n]$-equivariant direct complement D to $\ker(1-\theta^n)_{\mathbb{Q}_\ell}$ in $P^n_{\mathbb{Q}_\ell}$, and using (4) to identify $H^n(C(\theta)^\cdot)_{\mathbb{Q}_\ell} = \ker(1-\theta^n)_{\mathbb{Q}_\ell}$ with $H^{n+1}(C(\theta)^\cdot)_{\mathbb{Q}_\ell} = \mathrm{cok}(1-\theta^n)_{\mathbb{Q}_\ell}$, the trivialisation τ_θ is induced by the identity map on cohomology. Hence, from Lemma A1, one has

$$\begin{aligned}\chi_\ell(C(\theta)^\cdot,\tau_\theta) =&(-1)^n\partial^1_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}([\mathrm{id}_{\ker(1-\theta^n)_{\mathbb{Q}_\ell}}\oplus(1-\theta^n)\mid_D])\\ =&(-1)^n\partial^1_{\mathbb{Z}_\ell[G],\mathbb{Q}_\ell}([1-\theta^n\mid_D])\\ =&(-1)^n\delta_\ell(\mathrm{detred}_{\mathbb{Q}_\ell[G]}(1-\theta^n:D))\\ =&(-1)^n\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta:H^n(P^n[-n])_{\mathbb{Q}_\ell})),\end{aligned}$$

as required.

We now assume that $|P^\cdot| = n$ and, to fix notation, that $\min\{j : P^j \neq 0\} = 0$. We also assume that the claimed formula is true for any pair of the form $(Q^\cdot,\phi)$ where $Q^\cdot$ is a bounded complex of finitely generated projective $\mathbb{Z}_\ell[G]$-modules for which $|Q^\cdot| \leq n-1$ and ϕ is a $\mathbb{Z}_\ell[G]$-equivariant endomorphism of $Q^\cdot$ for which $H^i(1-\phi)$ is semi-simple at 0 in each degree i. For any complex $C^\cdot$ and any integer i we write $B^i(C^\cdot), Z^i(C^\cdot)$ and $d^i(C^\cdot)$ for the boundaries, cycles and differential in degree i. If necessary, we use the argument of [25, Lem. 7.10] to change θ by a homotopy in order to ensure that, in each degree i, the restriction of $1-\theta^i$ to $B^i(P^\cdot)$ induces an automorphism of $B^i(P^\cdot)_{\mathbb{Q}_\ell}$. We shall make frequent use of this assumption (without explicit comment) in the remainder of this argument.

We henceforth let $Q^\cdot$ denote the naive truncation in degree $n-1$ of $P^\cdot$ (so $Q^i = P^i$ if $i \leq n-1$ and $Q^n = 0$). Then one has a tautological short exact sequence of complexes $0 \to P^n[-n] \to P^\cdot \to Q^\cdot \to 0$. From the associated long exact cohomology sequence we deduce that $H^i(Q^\cdot) = H^i(P^\cdot)$ if $i < n-1$ and that there are commutative diagrams of short exact sequences of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}(P^\cdot) & \longrightarrow & H^{n-1}(Q^\cdot) & \longrightarrow & B^n & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle H^{n-1}(1-\theta)} & & \Big\downarrow{\scriptstyle H^{n-1}(1-\phi)} & & \Big\downarrow{\scriptstyle 1-\theta^n} & & \\ 0 & \longrightarrow & H^{n-1}(P^\cdot) & \longrightarrow & H^{n-1}(Q^\cdot) & \longrightarrow & B^n & \longrightarrow & 0\end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^n & \longrightarrow & H^n(P^n[-n]) & \longrightarrow & H^n(P^\cdot) & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle 1-\theta^n} & & \Big\downarrow{\scriptstyle H^n(1-\theta^n[-n])} & & \Big\downarrow{\scriptstyle H^n(1-\theta)} & & \\ 0 & \longrightarrow & B^n & \longrightarrow & H^n(P^n[-n]) & \longrightarrow & H^n(P^\cdot) & \longrightarrow & 0.\end{array}$$

We write ϕ, resp. $\theta^n[-n]$, for the endomorphism of $Q^\cdot$, resp. $P^n[-n]$, which is induced by θ. Then the above diagrams imply that $\ker(H^i(1-\phi))_{\mathbb{Q}_\ell} = \ker(H^i(1-\theta))_{\mathbb{Q}_\ell}$ and $\mathrm{cok}(H^i(1-\phi))_{\mathbb{Q}_\ell} = \mathrm{cok}(H^i(1-\theta))_{\mathbb{Q}_\ell}$ for all $i < n$ and also that $\ker(H^n(1-\theta^n[-n]))_{\mathbb{Q}_\ell} = \ker(H^n(1-\theta))_{\mathbb{Q}_\ell}$ and $\mathrm{cok}(H^n(1-\theta^n[-n]))_{\mathbb{Q}_\ell} = \mathrm{cok}(H^n(1-\theta))_{\mathbb{Q}_\ell}$. This implies that $1-\phi$ and $1-\theta^n[-n]$ induce endomorphisms of $H^i(Q^\cdot)$ and $H^i(P^n[-n])$ respectively which are each semi-simple at 0 in all degrees i.

We set $C := \mathrm{Cone}(1-\theta^n[-n])[-1]$, $D := C(\theta)^{\cdot} = \mathrm{Cone}(1-\theta)[-1]$ and $E := \mathrm{Cone}(1-\phi)[-1]$ so that there is a natural short exact sequence of complexes

$$0 \to C \xrightarrow{\alpha} D \xrightarrow{\beta} E \to 0. \tag{5}$$

Now, since $|Q^{\cdot}| < n$, our inductive hypothesis implies that

$$\begin{aligned}\chi_\ell(E,\tau_\phi) =& \sum_{i=0}^{n-1}(-1)^i\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\phi : H^i(Q^{\cdot})_{\mathbb{Q}_\ell}))\\ =&(-1)^{n-1}\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\phi : H^{n-1}(Q^{\cdot})_{\mathbb{Q}_\ell}))\\ &+\sum_{i=0}^{n-2}(-1)^i\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta : H^i(P^{\cdot})_{\mathbb{Q}_\ell})).\end{aligned}$$

In addition, since $|P^n[-n]| = 0$, our earlier argument proves that

$$\chi_\ell(C,\tau_{\theta^n[-n]}) = (-1)^n\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta^n[-n] : H^n(P^n[-n])_{\mathbb{Q}_\ell}).$$

From the commutative diagrams displayed above, one also has

$$\begin{aligned}&\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\phi : H^{n-1}(Q^{\cdot})_{\mathbb{Q}_\ell}) =\\ &\qquad \mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta^n : B^n_{\mathbb{Q}_\ell})\cdot \mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta : H^{n-1}(P^{\cdot})_{\mathbb{Q}_\ell})\end{aligned}$$

and

$$\begin{aligned}&\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta^n[-n] : H^n(P^n[-n])_{\mathbb{Q}_\ell}) =\\ &\qquad \mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta^n : B^n_{\mathbb{Q}_\ell})\cdot \mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta : H^n(P^{\cdot})_{\mathbb{Q}_\ell}).\end{aligned}$$

Upon combining the last four displayed formulas we obtain an equality

$$\chi_\ell(C,\tau_{\theta^n[-n]}) + \chi_\ell(E,\tau_\phi) = \sum_{i\in\mathbb{Z}}(-1)^i\delta_\ell(\mathrm{detred}^*_{\mathbb{Q}_\ell[G]}(1-\theta : H^i(P^{\cdot})_{\mathbb{Q}_\ell})),$$

and so the claimed result will follow if we can show that

$$\chi_\ell(D,\tau_\theta) = \chi_\ell(C,\tau_{\theta^n[-n]}) + \chi_\ell(E,\tau_\phi). \tag{6}$$

Before discussing the proof of this equality we introduce some convenient notation: for any $\mathbb{Z}_\ell$-module A we set $\overline{A} := A\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$, and we use similar abbreviations for both complexes and morphisms of $\mathbb{Z}_\ell$-modules. For any complex A we also set $H^+_A := H^+(A)$ and $H^-_A := H^-(A)$.
The key to proving (6) is the observation (which is itself straightforward to verify directly) that one can choose elements κ_1, κ_2 and κ_3 of $\tau_{\theta^n[-n]}, \tau_\theta$ and τ_ϕ respectively which together lie in a commutative diagram of short exact sequences of the form

$$\begin{tikzcd} 0 \arrow[r] & H^+_{\overline{C}} \arrow[r, "H^+(\overline{\alpha})"] \arrow[d, "\kappa_1"] & H^+_{\overline{D}} \arrow[r, "H^+(\overline{\beta})"] \arrow[d, "\kappa_2"] & H^+_{\overline{E}} \arrow[r] \arrow[d, "\kappa_3"] & 0 \\ 0 \arrow[r] & H^-_{\overline{C}} \arrow[r, "H^-(\overline{\alpha})"] & H^-_{\overline{D}} \arrow[r, "H^-(\overline{\beta})"] & H^-_{\overline{E}} \arrow[r] & 0. \end{tikzcd} \tag{7}$$

Indeed, the equality (6) follows directly upon combining such a diagram with the exact sequence (5) and the result of [9, Th. 2.8]. However, for the convenience of the reader, we also now indicate a more direct argument.

After taking account of the construction of $\chi_\ell(\cdot,\cdot)$ given in the Appendix (the notation of which we now assume) and the definitions of $\tau_{\theta^n[-n]}(C), \tau_\theta(D)$ and $\tau_\phi(E)$ it is enough to prove the existence of a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{C}^+ & \xrightarrow{\overline{\alpha}^+} & \overline{D}^+ & \xrightarrow{\overline{\beta}^+} & \overline{E}^+ & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{\overline{C}} \oplus H^+_{\overline{C}} & \xrightarrow{(\alpha', H^+(\overline{\alpha}))} & B_{\overline{D}} \oplus H^+_{\overline{D}} & \xrightarrow{(\beta', H^+(\overline{\beta}))} & B_{\overline{E}} \oplus H^+_{\overline{E}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle(\mathrm{id},\kappa_1)} & & \downarrow{\scriptstyle(\mathrm{id},\kappa_2)} & & \downarrow{\scriptstyle(\mathrm{id},\kappa_3)} & & \\
0 & \longrightarrow & B_{\overline{C}} \oplus H^-_{\overline{C}} & \xrightarrow{(\alpha', H^-(\overline{\alpha}))} & B_{\overline{D}} \oplus H^-_{\overline{D}} & \xrightarrow{(\beta', H^-(\overline{\beta}))} & B_{\overline{E}} \oplus H^-_{\overline{E}} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{C}^- & \xrightarrow{\overline{\alpha}^-} & \overline{D}^- & \xrightarrow{\overline{\beta}^-} & \overline{E}^- & \longrightarrow & 0,
\end{array}$$

where $B_{\overline{C}}$ denotes $B^{\mathrm{all}}(\overline{C})$, and similarly for $B_{\overline{D}}$ and $B_{\overline{E}}$, $\alpha' : B_{\overline{C}} \to B_{\overline{D}}$ and $\beta' : B_{\overline{D}} \to B_{\overline{E}}$ are the natural homomorphisms that are induced by α and β respectively, κ_1, κ_2 and κ_3 are as in (7) and all unlabelled vertical maps are the isomorphisms induced by a choice of sections to each of the natural homomorphisms $\overline{C}^i \to B^{i+1}(\overline{C}), Z^i(\overline{C}) \to H^i(\overline{C}), \overline{D}^i \to B^{i+1}(\overline{D}), Z^i(\overline{D}) \to H^i(\overline{D}), \overline{E}^i \to B^{i+1}(\overline{E})$ and $Z^i(\overline{E}) \to H^i(\overline{E})$. Indeed, if such a diagram exists, then the composite vertical isomorphisms belong to $\tau_{\theta^n[-n]}(C), \tau_\theta(D)$ and $\tau_\phi(E)$ respectively, and so the commutativity of the diagram formed by the first and fourth rows combines with the exactness of the sequences $0 \to C^+ \xrightarrow{\alpha^+} D^+ \xrightarrow{\beta^+} E^+ \to 0$ and $0 \to C^- \xrightarrow{\alpha^-} D^- \xrightarrow{\beta^-} E^- \to 0$ and the defining relations of $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell)$ [46, p. 415] to imply the required equality (6). Thus, upon noting that the rows of this diagram are all exact (the second and third as a consequence of the exactness of the rows of (7)), it is enough to prove that sections of the above form can be chosen in such a way that the top and bottom two squares of the diagram commute, and this in turn can be proved by a straightforward and explicit exercise using the following facts. After choosing $\mathbb{Q}_\ell[G]$-equivariant direct sum decompositions $\overline{P^{n-1}} = \overline{\mathrm{im}(1-\theta^{n-1})} \oplus S^{n-1}$ and $\overline{P^n} = \overline{\ker(1-\theta^n)} \oplus S^n$, one obtains a direct sum decomposition $\overline{D}^n = \overline{P^{n-1}} \oplus \overline{P^n} = B^n(\overline{D}) \oplus S^{n-1,*} \oplus (0, \overline{\ker(1-\theta^n)}) \oplus (0, S^n)$ where $S^{n-1,*}$ denotes the set of elements (π, π') where π runs over S^{n-1} and π' denotes the unique element of $B^n(\overline{P^\cdot})$ which is such that $(\pi, \pi') \in Z^n(\overline{D})$; one has $Z^n(\overline{D}) = B^n(\overline{D}) \oplus S^{n-1,*} \oplus (0, \overline{\ker(1-\theta^n)})$; the natural projection maps induce isomorphisms $B^n(\overline{D}) \cong B^n(\overline{E}), Z^{n-1}(\overline{D}) \cong Z^{n-1}(\overline{E})$ and $B^{n-1}(\overline{D}) \cong B^{n-1}(\overline{E})$. $\square$

Remark 6. There are two special cases in which the formula of Proposition 3.1 has already been proved: if $C(\theta)^{\cdot}_{\mathbb{Q}_\ell}$ is acyclic, then $H^i(1-\theta)$ is automatically semi-simple at 0 in each degree i and the given formula has been proved by Greither and the present author in [16, proof of Prop. 4.1]; if G is abelian, then Proposition 3.1 can be reinterpreted in terms of graded determinants and in this case the given formula has been proved to within a sign ambiguity by Kato in [35, Lem. 3.5.8]. (This sign ambiguity arises because Kato uses ungraded determinants - for more details in this regard see [loc cit., Rem. 3.2.3(3) and 3.2.6(3),(5)] and [13, Rem. 9].)

3.2. Zeta functions of varieties. In this subsection we fix a prime ℓ that is distinct from p. We also fix an algebraic closure $\mathbb{F}_p^c$ of $\mathbb{F}_p$, we set $\Gamma := \mathrm{Gal}(\mathbb{F}_p^c/\mathbb{F}_p)$ and we write σ for the (arithmetic) Frobenius element in Γ. For any scheme X over $\mathbb{F}_p$ we write X^c for the associated scheme $\mathbb{F}_p^c \times_{\mathbb{F}_p} X$ over $\mathbb{F}_p^c$.

We let J be a finite group, X and Y separated schemes of finite type over $\mathbb{F}_p$ and $\pi : X \to Y$ an étale morphism that is Galois of group J. For each ℓ-adic sheaf $\mathcal{G}$ on $Y_{\text{ét}}$ we follow the approach of Deligne [27, Rem. 2.12] to define a J-equivariant Zeta function by setting

$$Z_J(Y, \pi_*\pi^*\mathcal{G} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, t) := \\ \prod_y \mathrm{detred}_{\mathbb{Q}_\ell[J]}(1 - f_y^{-1} \cdot t^{\deg(y)} \mid (\pi_*\pi^*\mathcal{G} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)_{\overline{y}})^{-1} \in \zeta(\mathbb{Q}_\ell[J])[[t]],$$

where y runs over the set of closed points of Y, f_y denotes the arithmetic Frobenius of y, $\deg(y)$ the degree of y and subscript $\overline{y}$ denotes taking stalk at a geometric point over y.

We now combine the algebraic approach of the previous subsection with a well known result of Grothendieck from [32] to describe, for each integer r, the image of the leading term $Z_J^*(Y, \pi_*\pi^*\mathbb{Z}_\ell(r) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, 1)^\#$ under the homomorphism $\delta_{J,\ell} : \zeta(\mathbb{Q}_\ell[J])^\times \to K_0(\mathbb{Z}_\ell[J], \mathbb{Q}_\ell)$.

To this end we observe that π_* is exact and hence that, for each sheaf $\mathcal{G}$ as above, there is a natural isomorphism $R\Gamma(Y, \pi_*\pi^*\mathcal{G}) \cong R\Gamma(X, \pi^*\mathcal{G})$ in $\mathcal{D}(\mathbb{Z}_\ell[J])$. This implies that if $\mathcal{G}$ is any étale (pro-)sheaf of finitely generated $\mathbb{Z}_\ell$-modules on Y and we set $\mathcal{F} := \pi^*\mathcal{G}$, then the complexes $R\Gamma(X, \mathcal{F})$ and $R\Gamma(X^c, \mathcal{F})$ both belong to $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[J])$ (cf. [29, Th. 5.1]). We may therefore fix a bounded complex of finitely generated projective $\mathbb{Z}_\ell[J]$-modules $C^\cdot$ for which there exists an isomorphism $\alpha : C^\cdot \xrightarrow{\sim} R\Gamma(X^c, \mathcal{F})$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[J])$ and a $\mathbb{Z}_\ell[J]$-endomorphism θ of $C^\cdot$ that induces the action of σ on $R\Gamma(X^c, \mathcal{F})$ (the existence of such a θ follows from [41, Chap. VI, Lem. 8.17] - but note that the map ψ in loc. cit. need not, in general, be a quasi-isomorphism). In this way we obtain a commutative diagram in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[J])$ of the form

$$\begin{tikzcd} & C^\cdot \arrow[r, "1-\theta"] \arrow[d, "\alpha"'] & C^\cdot \arrow[d, "\alpha"'] & \\ R\Gamma(X,\mathcal{F}) \arrow[r] & R\Gamma(X^c,\mathcal{F}) \arrow[r, "1-\sigma"] & R\Gamma(X^c,\mathcal{F}) \arrow[r] & R\Gamma(X,\mathcal{F})[1], \end{tikzcd} \tag{8}$$

where the lower row denotes the natural distinguished triangle. Taken in conjunction with the Octahedral axiom, this diagram implies the existence of an isomorphism $\alpha' : C(\theta)^\cdot \xrightarrow{\sim} R\Gamma(X,\mathcal{F})$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[J])$. Further, the hypothesis that the composite (4) with $\psi = H^i(1-\theta)$ and $M = H^i(C^\cdot)$ has both finite kernel and finite cokernel is equivalent to the hypothesis that σ acts 'semi-simply' on the space $H^i(X^c,\mathcal{F})\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ and is therefore expected to be true under some very general conditions [35, Rem. 3.5.4]. In this context, and in terms of the notation of Lemma A2, we write $\tau_{X,\mathcal{F},\sigma}$ for the $\mathbb{Q}_\ell$-trivialisation of $R\Gamma(X,\mathcal{F})$ which is equal to $(\tau_\theta)_{\alpha'}$ where τ_θ is the $\mathbb{Q}_\ell$-trivialisation of $C(\theta)^\cdot$ that is defined just prior to Proposition 3.1 (with $P^\cdot = C^\cdot$).

Remark 7. The trivialisation $\tau_{X,\mathcal{F},\sigma}$ defined above has an alternative description. To explain this we let $\mathcal{C}(\mathcal{F})^\cdot$ denote the complex

$$H^0(X,\mathcal{F}) \xrightarrow{\kappa} H^1(X,\mathcal{F}) \xrightarrow{\kappa} H^2(X,\mathcal{F}) \xrightarrow{\kappa} \cdots$$

where $H^0(X,\mathcal{F})$ occurs in degree 0 and κ denotes cup-product with the element of $H^1(X,\mathbb{Z}_\ell)$ obtained by pulling back the element ϕ_p of $H^1(\mathrm{Spec}(\mathbb{F}_p),\mathbb{Z}_\ell) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma,\mathbb{Z}_\ell)$ which sends σ to 1. Then the complex $\mathcal{C}(\mathcal{F})^\cdot\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ is acyclic if and only if σ acts semi-simply on each space $H^i(X^c,\mathcal{F})\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ [35, Lem. 3.5.3]. Further, in each degree i the homomorphism $H^i(X,\mathcal{F}) \xrightarrow{\kappa} H^{i+1}(X,\mathcal{F})$ is equal to the 'Bockstein homomorphism'

$$\beta^i_{X,\mathcal{F},\sigma} : H^i(X,\mathcal{F}) \to H^{i+1}(X,\mathcal{F})$$

that is obtained as the composite

$$H^i(X,\mathcal{F}) \to H^i(X^c,\mathcal{F})^\Gamma \to H^i(X^c,\mathcal{F})_\Gamma \to H^{i+1}(X,\mathcal{F})$$

where the first and third maps are induced by the long exact sequence of cohomology associated to the lower row of (8) and the second map is as in (4). Indeed, this equality is a consequence of the description of κ on the level of complexes that is given by Rapaport and Zink in [45, 1.2] (cf. [43, Prop. 6.5] and [35, §3.5.2] in this regard). These equalities imply in turn that $\tau_{X,\mathcal{F},\sigma}$ coincides with the $\mathbb{Q}_\ell$-trivialisation of $R\Gamma(X,\mathcal{F})$ that is induced by the acyclicity of $\mathcal{C}(\mathcal{F})^\cdot\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ together with the assignment $\tau \mapsto \tau(\mathcal{C}(\mathcal{F})^\cdot\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)$ which is described just prior to Lemma A1.

We now state the main result of this subsection.

Theorem 3.2. *Let $\pi : X \to Y$ be a finite étale morphism of separated schemes of dimension d over $\mathbb{F}_p$. If π is Galois of group J and r is any integer for* *which σ acts semi-simply on $H^i(X^c,\mathbb{Z}_\ell(r))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ in all degrees i, then in $K_0(\mathbb{Z}_\ell[J],\mathbb{Q}_\ell)$ one has*

$$\delta_{J,\ell}(Z_J^*(Y,\pi_*\pi^*\mathbb{Z}_\ell(r)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,1)^\#) = -\chi_{J,\ell}(R\Gamma(X,\mathbb{Z}_\ell(d-r)),\tau_{X,\mathbb{Z}_\ell(d-r),\sigma}).$$

Proof. We set $r' := d-r$ and make a choice of morphisms θ and α as in diagram (8) with $\mathcal{F}=\mathbb{Z}_\ell(r')$. Upon applying Lemma A2 to the induced isomorphism $\alpha' : C(\theta)^\cdot \xrightarrow{\sim} R\Gamma(X,\mathbb{Z}_\ell(r'))$ and then Proposition 3.1 with $P^\cdot = C^\cdot$ and $G=J$, we find that

$$\begin{aligned}
&\chi_{J,\ell}(R\Gamma(X,\mathbb{Z}_\ell(r')),\tau_{X,\mathbb{Z}_\ell(r'),\sigma})\\
=&\chi_{J,\ell}(C(\theta)^\cdot,\tau_\theta)\\
=&\sum_{i\in\mathbb{Z}}(-1)^i\delta_{J,\ell}(\mathrm{detred}^*_{\mathbb{Q}_\ell[J]}(1-\theta : H^i(C^\cdot)_{\mathbb{Q}_\ell}))\\
=&\sum_{i\in\mathbb{Z}}(-1)^i\delta_{J,\ell}(\mathrm{detred}^*_{\mathbb{Q}_\ell[J]}(1-\sigma : H^i(X^c,\mathbb{Q}_\ell(r')))).
\end{aligned}$$

For each integer i we set $V^i := H^i_c(Y^c,\pi_*\pi^*\mathbb{Z}_\ell(r)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)\cong H^i_c(X^c,\mathbb{Q}_\ell(r))$, where subscript '$c$' denotes cohomology with compact support. Then, by Poincaré Duality (cf. [41, Chap. VI, Cor. 11.2]), in each degree i one has an isomorphism of $\mathbb{Q}_\ell[J]$-modules $H^i(X^c,\mathbb{Q}_\ell(r'))\cong \mathrm{Hom}_{\mathbb{Q}_\ell}(V^{2d-i},\mathbb{Q}_\ell)$. This isomorphism respects the action of Frobenius in the sense that the action of σ on $H^i(X^c,\mathbb{Q}_\ell(r'))$ corresponds to the inverse of the action of σ that is induced on $\mathrm{Hom}_{\mathbb{Q}_\ell}(V^{2d-i},\mathbb{Q}_\ell)$ by its natural action on V^{2d-i} (since the linear duality functor is contravariant). Hence one has

$$\begin{aligned}
&\mathrm{detred}_{\mathbb{Q}_\ell[J]}(1-\sigma\cdot t : H^i(X^c,\mathbb{Q}_\ell(r')))\\
=&\mathrm{detred}_{\mathbb{Q}_\ell[J]}(1-\sigma^{-1}\cdot t : \mathrm{Hom}_{\mathbb{Q}_\ell}(V^{2d-i},\mathbb{Q}_\ell))\\
=&\mathrm{detred}_{\mathbb{Q}_\ell[J]}(1-\sigma^{-1}\cdot t : V^{2d-i})^\#,
\end{aligned}$$

where the involution $x\mapsto x^\#$ acts coefficient-wise on elements of $\zeta(\mathbb{Q}_\ell[J])[[t]]$ and the second equality is valid because J acts contragrediently on $\mathrm{Hom}_{\mathbb{Q}_\ell}(V^{2d-i},\mathbb{Q}_\ell)$ (cf. [7, (2.0.5)]). From the above formula one therefore has

$$\begin{aligned}
&\chi_{J,\ell}(R\Gamma(X,\mathbb{Z}_\ell(r')),\tau_{X,\mathbb{Z}_\ell(r'),\sigma})\\
=&\delta_{J,\ell}(\prod_{i\in\mathbb{Z}}(\mathrm{detred}^*_{\mathbb{Q}_\ell[J]}(1-\sigma^{-1} : V^i))^{\#,(-1)^i})\\
=&-\delta_{J,\ell}((\prod_{i\in\mathbb{Z}}\mathrm{detred}^*_{\mathbb{Q}_\ell[J]}(1-\sigma^{-1} : V^i)^{(-1)^{i+1}})^\#).
\end{aligned}$$

To complete the proof it is thus sufficient to observe that, by Grothendieck [32], one has an equality of functions of the complex variable t

$$\prod_{i\in\mathbb{Z}}\mathrm{detred}_{\mathbb{Q}_\ell[J]}(1-\sigma^{-1}\cdot t : V^i)^{(-1)^{i+1}} = Z_J(Y,\pi_*\pi^*\mathbb{Z}_\ell(r)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,t).$$

Indeed, the exposition of [41, Chap. VI, proof of Th. 13.3] proves just such an equality with $\mathbb{Q}_\ell[J]$ replaced by an arbitrary finite degree field extension Ω of $\mathbb{Q}_\ell$ and $\pi_*\pi^*\mathbb{Z}_\ell(r)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ by any constructible sheaf of vector spaces over Ω, and the last displayed equality can be verified by reduction to such cases since both sides are defined via Galois descent (cf. [27, Rem. 2.12]). $\square$

3.3. The case $\ell \neq p$. In this subsection we deduce Theorem 3.1i) from a special case of Theorem 3.2.

To this end we first reinterpret $\mathrm{C}_\ell(K/k)$ in the style of Theorem 3.2. We note that the isomorphism ι_ℓ constructed in the following result is as predicted by [30, Conj. 7.2] (with $X = U_{K,S}$ and $n = 1$).

Lemma 3. *There exists a natural isomorphism in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[G])$ of the form*

$$\iota_\ell : R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)\otimes\mathbb{Z}_\ell \xrightarrow{\sim} R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))[1]\,.$$

Set $\mathrm{D}_{K,S,\ell} := H^1(\iota_\ell)\circ(\mathrm{D}_{K,S}\otimes\mathbb{Z}_\ell)\circ H^0(\iota_\ell)^{-1}$. Then the inverse of $\mathrm{D}_{K,S,\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ induces a $\mathbb{Q}_\ell$-trivialisation of $R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))$ and $\mathrm{C}_\ell(K/k)$ is valid if and only if in $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell)$ one has

$$\delta_\ell(Z^*_{K/k,S}(1)^{\#}) = -\chi_\ell(R\Gamma(U_{K,S},\mathbb{Z}_\ell(1)),(-\mathrm{D}_{K,S,\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)^{-1}). \tag{9}$$

Proof. Following Lemma 1iii) we fix a bounded complex of finitely generated projective $\mathbb{Z}[G]$-modules $P^\cdot$ that is isomorphic in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ to $R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$. Since $R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))$ is an object of $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[G])$ we may also fix a bounded complex of finitely generated projective $\mathbb{Z}_\ell[G]$-modules $Q^\cdot$ that is isomorphic in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[G])$ to $R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))$.

For each natural number n we consider the following diagram

$$\begin{tikzcd}
X_{K,S}\otimes\mathbb{Q}[-2] \arrow[r] \arrow[d, "\ell^n"'] & R\Gamma(U_{K,S},\mathbb{G}_m) \arrow[r] \arrow[d, "\ell^n"'] & P^\cdot \arrow[r] \arrow[d, "\ell^n"'] & X_{K,S}\otimes\mathbb{Q}[-1] \\
X_{K,S}\otimes\mathbb{Q}[-2] \arrow[r] \arrow[d] & R\Gamma(U_{K,S},\mathbb{G}_m) \arrow[r] \arrow[d] & P^\cdot \arrow[r] \arrow[d] & X_{K,S}\otimes\mathbb{Q}[-1] \\
0 & Q^\cdot/\ell^n[1] & P^\cdot/\ell^n. &
\end{tikzcd}$$

The first two rows of this diagram are the distinguished triangles that are induced by Lemma 1ii) and the isomorphism $P^\cdot \cong R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$. In addition, all columns of the diagram are distinguished triangles: the first is obviously so, the second is the triangle which is induced by the exact sequence of étale sheaves $1\to\mu_{\ell^n}\to\mathbb{G}_m\xrightarrow{\ell^n}\mathbb{G}_m\to 1$, the exact sequence of étale pro-sheaves $0\to\mathbb{Z}_\ell(1)\xrightarrow{\ell^n}\mathbb{Z}_\ell(1)\to\mu_{\ell^n}\to 1$, the isomorphism $Q^\cdot\cong R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))$ and the exact sequence of modules $0\to Q^i\xrightarrow{\ell^n}Q^i\to Q^i/\ell^n\to 0$ in each degree i, and the third column is the distinguished triangle which is induced by the exact sequence of modules $0\to P^i\xrightarrow{\ell^n}P^i\to P^i/\ell^n\to 0$ in each degree i. Since the diagram commutes in $\mathcal{D}(\mathbb{Z}[G])$ and all rows and columns are distinguished triangles, one can deduce the existence of an isomorphism $\alpha_n : Q^\cdot/\ell^n[1] \cong P^\cdot/\ell^n$ in $\mathcal{D}^{\rm perf}(\mathbb{Z}/\ell^n[G])$. Further, as n varies, the isomorphisms α_n may be chosen to be compatible with the natural transition morphisms (cf. [12, the proof of Prop. 3.3]). The inverse limit of such a compatible system of isomorphisms $\{\alpha_n\}_n$ then gives an isomorphism in $\mathcal{D}^{\rm perf}(\mathbb{Z}_\ell[G])$ of the form $R\Gamma(U_{K,S}, \mathbb{Z}_\ell(1))[1] \cong Q^\cdot[1] \cong \varprojlim_n Q^\cdot/\ell^n[1] \cong \varprojlim_n P^\cdot/\ell^n \cong P^\cdot \otimes \mathbb{Z}_\ell \cong R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \otimes \mathbb{Z}_\ell$, as required.

Taken in conjunction with Lemma A2 the quasi-isomorphism ι_ℓ implies that

$$\begin{aligned}&\rho_\ell(\chi_{\mathbb{Q}}(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{D}_{K,S} \otimes \mathbb{Q}))\\ =&\chi_\ell(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \otimes \mathbb{Z}_\ell, \mathrm{D}_{K,S} \otimes \mathbb{Q}_\ell)\\ =&\chi_\ell(R\Gamma(U_{K,S}, \mathbb{Z}_\ell(1))[1], \mathrm{D}_{K,S,\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)\\ =&-\chi_\ell(R\Gamma(U_{K,S}, \mathbb{Z}_\ell(1)), (-\mathrm{D}_{K,S,\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^{-1}),\end{aligned}$$

where the last equality follows from [9, Th. 2.1(3)]. To prove the final assertion of the lemma we need therefore only observe that $\rho_\ell(\delta(Z^*_{K/k,S}(1)^\#)) = \delta_\ell(Z^*_{K/k,S}(1)^\#)$. Indeed, this equality follows from the fact that on $\zeta(\mathbb{Q}[G])^\times$ one has $\rho_\ell \circ \delta = \delta_\ell \circ i_\ell$ where i_ℓ denotes the natural inclusion $\zeta(\mathbb{Q}[G])^\times \to \zeta(\mathbb{Q}_\ell[G])^\times$. □

To prove $\mathrm{C}_\ell(K/k)$ we need only show that (9) coincides with the formula of Theorem 3.2 in the case $X = U_{K,S}$, $Y = U_{k,S}$ (so that $d = 1$), $\pi : U_{K,S} \to U_{k,S}$ is the natural morphism of spectra, $J = G$ and $r = 0$.

We first compare the left hand sides of the respective formulas. If y is any closed point of $U_{k,S}$, then, after fixing a $\overline{y}$ point x of $U_{K,S}$ and writing G_x for the decomposition subgroup of x in G, the stalk of $\pi_*\pi^*\mathbb{Z}_\ell(0) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ at $\overline{y}$ identifies as a (left) $G \times G_x$-module with $\mathbb{Q}_\ell[G]$ where elements of the form $(g, \mathrm{id}) \in G \times G_x$ act via left multiplication by g and elements of the form $(\mathrm{id}, g_x) \in G \times G_x$ act via right multiplication by g_x^{-1} (in this regard compare the discussion of [7, beginning of §2]). By using this identification one computes that $Z_G(U_{k,S}, \pi_*\pi^*\mathbb{Z}_\ell(0) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, t)$ has the same Euler factor at y as does $Z_{K/k,S}(t)$. Since this is true for all closed points y it follows that there is an equality of functions of the complex variable t

$$Z_{K/k,S}(t) = Z_G(U_{k,S}, \pi_*\pi^*\mathbb{Z}_\ell(0) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, t).$$

This implies that the left hand side of (9) is equal to the left hand side of the relevant special case of the formula in Theorem 3.2. Hence our proof of (9) will be complete if we can verify the relevant semi-simplicity hypothesis (in order to apply Theorem 3.2) and then prove that the trivialisation $\tau_{U_{K,S},\mathbb{Z}_\ell(1),\sigma}$ is induced by the isomorphism $(-D_{K,S,\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^{-1}$. Our proof is therefore completed by combining the description of $\tau_{U_{K,S},\mathbb{Z}_\ell(1),\sigma}$ in Remark 7 together with the following result.

Lemma 4. i) *σ acts semi-simply on $H^i(U^c_{K,S}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ in all degrees i.*
ii) $\beta^1_{U_{K,S},\mathbb{Z}_\ell(1),\sigma} = -\mathrm{D}_{K,S,\ell}$.

Proof. Lemma 3 combines with Lemma 1iii) to imply that $R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))$ is acyclic outside degrees 1 and 2. Remark 7 therefore implies claim i) is equivalent to asserting that the map $\beta^1_{U_{K,S},\mathbb{Z}_\ell(1),\sigma}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ is bijective and this is an immediate consequence of the explicit description given in claim ii).

We now fix an arbitrary place v in S and write $c_v : Y_{K,S}\otimes\mathbb{Z}_\ell \to \bigoplus_{w|v}\mathbb{Z}_\ell$ for the homomorphism induced by projecting each element of $Y_{K,S}$ to its respective coefficient at each place w of K above v. Then claim ii) will follow if we show that the composite homomorphism

$$(10)\quad \begin{aligned}\mathcal{O}^\times_{K,S}\otimes\mathbb{Z}_\ell \xrightarrow{H^0(\iota_\ell)} H^1(U_{K,S},\mathbb{Z}_\ell(1)) &\xrightarrow{\beta^1_{U_{K,S},\mathbb{Z}_\ell(1),\sigma}} H^2(U_{K,S},\mathbb{Z}_\ell(1))\\ &\xrightarrow{H^1(\iota_\ell)^{-1}} X_{K,S}\otimes\mathbb{Z}_\ell \xrightarrow{\subseteq} Y_{K,S}\otimes\mathbb{Z}_\ell \xrightarrow{c_v} \bigoplus_{w|v}\mathbb{Z}_\ell\end{aligned}$$

is equal to $(-\deg(w)\cdot\mathrm{val}_w(-))_{w|v}$. To prove this we set $S' := S\setminus\{v\}$, let Z denote the complement of $U_{K,S}$ in $U_{K,S'}$ and write $j : U_{K,S}\to U_{K,S'}$, resp. $i : Z\to U_{K,S'}$, for the natural open, resp. closed, immersion. Then there exists a natural morphism of étale sheaves $j_*\mathbb{G}_m \to i_*i^*\mathbb{Z}$ on $U_{K,S'}$ that is induced by taking valuations. In turn this gives rise to a morphism $R\Gamma(U_{K,S},\mathbb{G}_m)\to R\Gamma(Z,\mathbb{Z})$ in $\mathcal{D}(\mathbb{Z}[G])$ and hence, for each non-negative integer n, to a morphism $R\Gamma(U_{K,S},\mu_{\ell^n})\to R\Gamma(Z,\mathbb{Z}/\ell^n)[-1]$ in $\mathcal{D}(\mathbb{Z}/\ell^n[G])$. These morphisms are compatible with the natural transition maps as n varies and therefore induce, upon passage to the inverse limit, a morphism in $\mathcal{D}(\mathbb{Z}_\ell[G])$ of the form $\lambda : R\Gamma(U_{K,S},\mathbb{Z}_\ell(1))\to R\Gamma(Z,\mathbb{Z}_\ell)[-1]$.

Now $H^0(Z,\mathbb{Z}_\ell) = \bigoplus_{w|v}\mathbb{Z}_\ell$ and each w-component of $H^1(\lambda)\circ H^0(\iota_\ell)$ is induced by the respective valuation map val_w. In addition, if we identify $H^1(Z,\mathbb{Z}_\ell) = \bigoplus_{w|v}\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\mathbb{F}^c_p/k(w)),\mathbb{Z}_\ell)$ with $\bigoplus_{w|v}\mathbb{Z}_\ell$ by evaluating each homomorphism at the topological generator $\sigma^{\deg(w)}$ of $\mathrm{Gal}(\mathbb{F}^c_p/k(w))$, then $H^2(\lambda)\circ H^1(\iota_\ell)$ is induced by projection of an element of $X_{K,S}$ to its respective coefficients at each place w above v.

Upon replacing $U_{K,S}$ and Z by $U^c_{K,S}$ and Z^c one obtains in a similar manner a morphism $\lambda^c : R\Gamma(U^c_{K,S},\mathbb{Z}_\ell(1))\to R\Gamma(Z^c,\mathbb{Z}_\ell)[-1]$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_\ell[G])$ that induces a morphism of distinguished triangles of the form

$$\begin{array}{ccccc} R\Gamma(U_{K,S},\mathbb{Z}_\ell(1)) & \longrightarrow & R\Gamma(U^c_{K,S},\mathbb{Z}_\ell(1)) & \xrightarrow{1-\sigma} & R\Gamma(U^c_{K,S},\mathbb{Z}_\ell(1))\\ {\scriptstyle\lambda}\big\downarrow & & {\scriptstyle\lambda^c}\big\downarrow & & {\scriptstyle\lambda^c}\big\downarrow\\ R\Gamma(Z,\mathbb{Z}_\ell)[-1] & \longrightarrow & R\Gamma(Z^c,\mathbb{Z}_\ell)[-1] & \xrightarrow{1-\sigma} & R\Gamma(Z^c,\mathbb{Z}_\ell)[-1]. \end{array}$$

After passing to cohomology this diagram induces a commutative diagram

$$\begin{array}{ccc} H^1(U_{K,S},\mathbb{Z}_\ell(1)) & \xrightarrow{\beta^1_{U_{K,S},\mathbb{Z}_\ell(1),\sigma}} & H^2(U_{K,S},\mathbb{Z}_\ell(1))\\ {\scriptstyle H^1(\lambda)}\big\downarrow & & {\scriptstyle H^2(\lambda)}\big\downarrow\\ H^0(Z,\mathbb{Z}_\ell) & \xrightarrow{-\beta^0_{Z,\mathbb{Z}_\ell,\sigma}} & H^1(Z,\mathbb{Z}_\ell), \end{array}$$

where the minus sign in the lower row occurs because of the -1-shift in the lower row of the previous diagram. Now the pull-back to $H^1(Z,\mathbb{Z}_\ell)$ of ϕ_p is the element $(\phi_w)_{w|v}$ where $\phi_w(\sigma^{\deg(w)}) = \deg(w)$ for each w dividing v. After identifying both $H^0(Z,\mathbb{Z}_\ell)$ and $H^1(Z,\mathbb{Z}_\ell)$ with $\bigoplus_{w|v}\mathbb{Z}_\ell$ in the manner prescribed above, the description of Remark 7 (with $X = Z$ and $\mathcal{F} = \mathbb{Z}_\ell$) therefore implies that $\beta^0_{Z,\mathbb{Z}_\ell,\sigma}$ is given by component-wise multiplication with the element $(\deg(w))_{w|v}$. Upon combining the commutativity of this diagram with the explicit descriptions of $H^1(\lambda)$ and $H^2(\lambda)$ given above, it follows that the composite homomorphism (10) is indeed equal to $(-\deg(w)\cdot \mathrm{val}_w(-))_{w|v}$, as required. $\square$

3.4. THE CASE $\ell = p$. In this subsection we prove Theorem 3.1ii).
For each subgroup H of G we let $\rho_H^{G,*}$ denote the natural restriction of scalars homomorphism $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p) \to K_0(\mathbb{Z}_p[H],\mathbb{Q}_p)$. For each abelian group H and each subgroup J of H we also let $q^H_{H/J,*}$ denote the natural coinflation homomorphism $K_0(\mathbb{Z}_p[H],\mathbb{Q}_p) \to K_0(\mathbb{Z}_p[H/J],\mathbb{Q}_p)$. Then one has

$$K_0(\mathbb{Z}_p[G],\mathbb{Q}_p)_{\mathrm{tors}} = \bigcap \ker(q^H_{H/J,*}\circ \rho_H^{G,*})$$

where the intersection runs over all cyclic subgroups H of G and over all subgroups J of H which are such that $p \nmid |H/J|$ [9, Th. 4.1].
Taken in conjunction with the functorial properties of $\mathrm{C}_p(K/k)$ under change of group (Remark 4), the above displayed equality implies that $\mathrm{C}_p(K/k)$ is valid modulo $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p)_{\mathrm{tors}}$ if and only if $\mathrm{C}_p(F/E)$ is valid for each cyclic extension F/E with $k \subseteq E \subseteq F \subseteq K$ and $p \nmid [F:E]$. But, for each such extension F/E, the argument of [7, Lem. 2.2.7] shows that $\mathrm{C}_p(F/E)$ is implied by the Strong-Stark Conjecture for F/E, as formulated by Chinburg (cf. [3, §3.1]). The required result therefore follows directly from Bae's proof of the Strong-Stark Conjecture in this case [3, Th. 3.5.4].
This completes our proof of Theorem 3.1.

4. THE CONJECTURES OF CHINBURG

4.1. CANONICAL 2-EXTENSIONS. In the sequel we shall say that two complexes of G-modules $C^\cdot$ and $D^\cdot$ are 'equivalent' if $H^i(C) = H^i(D)$ in each degree i and there exists an isomorphism in $\mathcal{D}(\mathbb{Z}[G])$ from $C^\cdot$ to $D^\cdot$ which induces the identity map in all degrees of cohomology.
If now $C^\cdot$ is any complex of G-modules which is acyclic outside degrees 0 and 1, then $C^\cdot$ is naturally isomorphic in $\mathcal{D}(\mathbb{Z}[G])$ to its double truncation $\tau_{\ge 0}\tau_{\le 1}C^\cdot$. In addition, the tautological exact sequence

$$0 \to H^0(C^\cdot) \to (\tau_{\ge 0}\tau_{\le 1}C^\cdot)^0 \to (\tau_{\ge 0}\tau_{\le 1}C^\cdot)^1 \to H^1(C^\cdot) \to 0$$

determines a unique Yoneda extension class $e(C^\cdot) \in \mathrm{Ext}^2_G(H^1(C^\cdot), H^0(C^\cdot))$.

LEMMA 5. *Let $C^\cdot$ and $D^\cdot$ be any complexes of G-modules which are acyclic outside degrees* 0 *and* 1 *and are also such that $H^i(C^\cdot) = H^i(D^\cdot)$ for $i = 0, 1$. Then $C^\cdot$ and $D^\cdot$ are equivalent if and only if one has $e(C^\cdot) = e(D^\cdot)$.*

Proof. An easy consequence of the definition of equivalence of Yoneda extensions. □

This result implies that $R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$ corresponds to a unique element $c_{\mathcal{W},S}(K/k)$ of $\mathrm{Ext}^2_G(H^1_{\mathcal{W}}(U_{K,S},\mathbb{G}_m),H^0_{\mathcal{W}}(U_{K,S},\mathbb{G}_m))$. In this subsection we relate $c_{\mathcal{W},S}(K/k)$ to the canonical extension class which is defined in terms of class field theory by Tate in [49].

To make such a connection we assume that the G-module $A_{K,S}$ is c-t. In this case the displayed short exact sequence in Lemma 1iii) splits (since $\mathrm{Ext}^1_G(X_{K,S},A_{K,S})=0$) and also $\mathrm{Ext}^2_G(A_{K,S},\mathcal{O}^\times_{K,S})=0$ and so there exists a natural isomorphism

$$\iota_S:\mathrm{Ext}^2_G(X_{K,S},\mathcal{O}^\times_{K,S})\xrightarrow{\sim}\mathrm{Ext}^2_G(H^1_{\mathcal{W}}(U_{K,S},\mathbb{G}_m),H^0_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)).$$

We choose a finite set of places W of k which do not belong to S, are each totally split in K/k and are such that $A_{K,S}$ is generated by the classes of places in $W(K)$. We set $S':=S\cup W$ (so that $A_{K,S'}$ is trivial) and we observe that there are natural exact sequences of G-modules of the form

$$0\to X_{K,S}\xrightarrow{\subseteq}X_{K,S'}\to Y_{K,W}\to 0$$

$$0\to\mathcal{O}^\times_{K,S}\xrightarrow{\subseteq}\mathcal{O}^\times_{K,S'}\to Y_{K,W}\to A_{K,S}\to 0.$$

Since $Y_{K,W}$ is a free $\mathbb{Z}[G]$-module these sequences combine to induce an isomorphism of extension groups

$$\iota_{S',S}:\mathrm{Ext}^2_G(X_{K,S'},\mathcal{O}^\times_{K,S'})\xrightarrow{\sim}\mathrm{Ext}^2_G(X_{K,S},\mathcal{O}^\times_{K,S}).$$

In the sequel we shall identify Yoneda-Ext-groups with derived functor Ext-groups by means of a projective resolution of the first variable (this convention differs from that used in [12] - see in particular [loc. cit., Lem. 3]). We also write $c_{S'}(K/k)$ for the canonical element of $\mathrm{Ext}^2_G(X_{K,S'},\mathcal{O}^\times_{K,S'})$ which is defined in [49].

PROPOSITION 4.1. *If the G-module $A_{K,S}$ is c-t, then one has $c_{\mathcal{W},S}(K/k)=\iota_S\circ\iota_{S',S}(-c_{S'}(K/k))$.*

Proof. For each $w\in S'(K)$ we set $V_w:=\mathrm{Spec}(K_w)$. We also let j' denote the natural open immersion $U_{K,S'}\to C_K$ and we consider the following diagram in $\mathcal{D}(\mathbb{Z}[G])$

$$\begin{array}{ccccc}
X_{K,S'}\otimes\mathbb{Q}[-2] & \longrightarrow & R\Gamma(U_{K,S'},\mathbb{G}_m) & \longrightarrow & R\Gamma_{\mathcal{W}}(U_{K,S'},\mathbb{G}_m)\\
\downarrow & & \downarrow & & \\
Y_{K,S'}\otimes\mathbb{Q}[-2] & \xrightarrow{\alpha} & \oplus_{w\in S'(K)}R\Gamma(V_w,\mathbb{G}_m) & & \\
\downarrow & & \downarrow & & \\
\mathbb{Q}[-2] & & R\Gamma(C_K,j'_!\mathbb{G}_m)[1]. & &
\end{array}$$

The top row of this diagram is the distinguished triangle from Lemma 1ii) (with S replaced by S'), the first column is the distinguished triangle induced by the tautological exact sequence $0 \to X_{K,S'} \xrightarrow{\subseteq} Y_{K,S'} \to \mathbb{Z} \to 0$ and the second column is the distinguished triangle from [42, Chap. II, Prop. 2.3]. Further, under the isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}[G])}(Y_{K,S'} \otimes \mathbb{Q}[-2], \oplus_{w\in S'(K)} R\Gamma(V_w, \mathbb{G}_m)) \cong \mathrm{Hom}_G(Y_{K,S'} \otimes \mathbb{Q}, \oplus_{w\in S'(K)} H^2(V_w, \mathbb{G}_m))$$

that is induced by [12, Lem. 7(b)], the morphism α corresponds to the composite of the projection $Y_{K,S'} \otimes \mathbb{Q} \to Y_{K,S'} \otimes \mathbb{Q}/\mathbb{Z}$ and the natural identification $Y_{K,S'} \otimes \mathbb{Q}/\mathbb{Z} \cong \oplus_{w\in S'(K)} H^2(V_w, \mathbb{G}_m)$.

It is straightforward to show that the square in the above diagram commutes (for example, by using [12, Lem. 7(b)] to reduce to cohomological considerations). By comparing this diagram to the diagrams (85) and (88) from loc. cit., and then using the Octahedral axiom, one may therefore conclude that $R\Gamma_{\mathcal{W}}(U_{K,S'}, \mathbb{G}_m)$ is equivalent to the complex $\Psi_{S'}$ which is defined in [12, Prop. 3.1]. From the proof of [12, Prop. 3.5] we may thus deduce that $c_{\mathcal{W},S'}(K/k) = -c_{S'}(K/k)$. (We remark that whilst the results of [12] are phrased solely in terms of number fields, all of the constructions and arguments of loc. cit. extend directly to the case of global function fields. In addition, we obtain $-c_{S'}(K/k)$ rather than $c_{S'}(K/k)$ in the present context because we have changed conventions regarding Yoneda-Ext-groups.)

To conclude that $c_{\mathcal{W},S}(K/k) = \iota_S \circ \iota_{S',S}(-c_{S'}(K/k))$ it suffices to prove that there exists a morphism $R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \to R\Gamma_{\mathcal{W}}(U_{K,S'}, \mathbb{G}_m)$ in $\mathcal{D}(\mathbb{Z}[G])$ which induces upon cohomology the natural maps $\mathcal{O}_{K,S}^\times \xrightarrow{\subseteq} \mathcal{O}_{K,S'}^\times$ and $H^1_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \twoheadrightarrow X_{K,S} \xrightarrow{\subseteq} X_{K,S'}$. But, following [40, the proof of Th. 7.1], the existence of such a morphism can be seen to be a consequence of the morphism of étale sheaves $\mathbb{G}_m \to j_*\mathbb{G}_m$ on $U_{K,S}$ where $j : U_{K,S'} \to U_{K,S}$ denotes the natural open immersion. □

4.2. Galois module theory. In this subsection we relate $\mathrm{C}(K/k)$ to the conjectures formulated by Chinburg in [18, §4.2]. We recall that the conjectures of loc. cit. are natural function field analogues of the central conjectures of Galois module theory which had earlier been formulated by Chinburg in [19, 21]. We write $\Omega(K/k, 1), \Omega(K/k, 2)$ and $\Omega(K/k, 3)$ for the Galois structure invariants defined by Chinburg in [18, the end of §4.1] and $W_{K/k}$ for the so-called 'Cassou-Noguès-Fröhlich Root Number Class' (cf. [loc. cit., Rem. 4.18]).

Conjecture $\mathrm{Ch}(K/k)$ (Chinburg, [18, §4.2, Conj. 3]): *In* $K_0(\mathbb{Z}[G])$ *one has*

i) $\Omega(K/k, 1) = 0$,
ii) $\Omega(K/k, 2) = W_{K/k}$,
iii) $\Omega(K/k, 3) = W_{K/k}$.

We now state the main results of this section.

Theorem 4.1. *The image under* $\partial^0_{\mathbb{Z}[G],\mathbb{R}}$ *of the equality of* $\mathrm{C}(K/k)$ *is equivalent to the equality of* $\mathrm{Ch}(K/k)$iii).

Proof. Following Remark 4, we may consider $\mathrm{C}(K/k)$ with respect to a set S which is large enough to ensure that $A_{K,S}$ is trivial, and in this case Proposition 4.1 (with $S = S'$) implies that $c_{\mathcal{W},S} = \iota_S(-c_S(K/k))$.
Let now $C^\cdot$ and $D^\cdot$ be any objects of $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ which are acyclic outside degrees 0 and 1 and are such that $H^i(C^\cdot) = H^i(D^\cdot)$ for $i = 0, 1$. It is easily shown that if $e(C^\cdot) = -e(D^\cdot)$, then $C^\cdot$ and $D^\cdot$ have the same Euler characteristic in $K_0(\mathbb{Z}[G])$. This observation combines with the equality $c_{\mathcal{W},S} = \iota_S(-c_S(K/k))$ and the very definition of $\Omega(K/k, 3)$ to imply that the latter element can be computed as the Euler characteristic of $R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$ in $K_0(\mathbb{Z}[G])$. It therefore follows that $\Omega(K/k, 3) = \partial^0_{\mathbb{Z}[G],\mathbb{R}}(\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S}))$.
On the other hand, the same argument as used to prove [7, Lem. 2.3.7] shows that $\partial^0_{\mathbb{Z}[G],\mathbb{R}}(\delta(\theta^*_{K/k,S}(0)^{\#})) = W_{K/k}$. The claimed result is now clear. □

COROLLARY 2. i) $\mathrm{Ch}(K/k)$iii) *is valid modulo* $\partial^0_{\mathbb{Z}[G],\mathbb{R}}(K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\mathrm{tors}})$.
ii) *If* $p \nmid |G|$, *then* $\mathrm{Ch}(K/k)$ *is valid.*

Proof. Claim i) follows directly from Theorem 4.1 and Corollary 1.
We now assume that $p \nmid |G|$. In this case $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ is torsion-free [13, proof of Lem. 11c)] and hence claim i) implies $\Omega(K/k, 3) = W_{K/k}$. In addition, K/k is tamely ramified and so $\mathrm{Ch}(K/k)$ii) has been proved by Chinburg. Indeed, the equality $\Omega(K/k, 2) = W_{K/k}$ follows directly upon combining [18, §4.2, Th. 4] with [23, Cor. 4.10]. Finally, we observe that the validity of $\mathrm{Ch}(K/k)$i) now follows immediately from the fact that $\Omega(K/k, 1) = \Omega(K/k, 2) - \Omega(K/k, 3)$ [18, §4.1, Th. 2 and the remarks which follow it]. □

Remark 8. The image of $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\mathrm{tors}}$ under $\partial^0_{\mathbb{Z}[G],\mathbb{R}}$ is equal to the group $D^p(\mathbb{Z}[G])$ that arises in [24, Th. 6.13]. We recall that the arguments of Chinburg in loc. cit., and of Bae in [3] (the results of which provided the key ingredient in our proof of Theorem 3.1ii) in §3.4), rely crucially upon results of Milne and Illusie concerning p-adic cohomology. In particular, in both cases the occurrence of the term $D^p(\mathbb{Z}[G])$ reflects difficulties involved in formulating and proving suitable equivariant refinements of the results of [43].

5. THE CONJECTURE OF GROSS

In this section we assume unless explicitly stated otherwise that G is abelian. We set $G^* := \mathrm{Hom}(G, \mathbb{C}^\times)$ and for each $\chi \in G^*$ we let e_χ denote the associated idempotent $|G|^{-1}\sum_{g\in G}\chi(g)g^{-1}$ of $\mathbb{C}[G]$. In terms of this notation one has $\theta_{K/k,S}(s) = \sum_{\chi\in G^*} e_\chi L_S(\chi, s)$.
We let I_G denote the kernel of the homomorphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ which sends each element of G to 1.

5.1. STATEMENT OF THE CONJECTURE. We set $n := |S| - 1$ and let $|n|$, resp. $|n|^*$, denote the set of integers j which satisfy $1 \le j \le n$, resp. $0 \le j \le n$. We henceforth label (and thereby order) the places in S as $\{v_i : i \in |n|^*\}$. For each $j \in |n|^*$ we fix a place w_i of K which restricts to v_i on k. For any place v of k which is unramified in K/k we write σ_v for its frobenius automorphism in G and $\mathrm{N}v$ for the cardinality of the associated residue field. We also fix a finite non-empty set T of places of k which is disjoint from S and then set $\Delta_T := \prod_{v\in T}(1-\sigma_v\cdot \mathrm{N}v) \in \mathbb{Q}[G]^\times$ and

$$\theta_{K/k,S,T}(s) = \Delta_T \cdot \theta_{K/k,S}(s).$$

This $\mathbb{C}[G]$-valued function is holomorphic at $s=0$ and, by using results of Weil, Gross has shown that $\theta_{K/k,S,T}(0)$ belongs to $\mathbb{Z}[G]$ [31, Prop. 3.7].

For any intermediate field F of K/k and any place w of K we let w' denote the restriction of w to F and then write $f_{K/F,w}$ for the homomorphism $F^\times \to G$ which is obtained as the composite of the natural inclusion $F^\times \to F^\times_{w'}$, the reciprocity map $F^\times_{w'} \to \mathrm{Gal}(K_w/F_{w'})$ and the natural injection $\mathrm{Gal}(K_w/F_{w'}) \to G$. We also write $\mathcal{O}^\times_{F,S,T}$ for the subgroup of $\mathcal{O}^\times_{F,S}$ consisting of those $S(F)$-units which are congruent to 1 modulo all places in $T(F)$. It is known that each such group $\mathcal{O}^\times_{F,S,T}$ is torsion-free. In particular, after choosing an ordered $\mathbb{Z}$-basis $\{u_j : j\in |n|\}$ of $\mathcal{O}^\times_{k,S,T}$, we may define an element of $\mathbb{Z}[G]$ by setting

$$\mathrm{Reg}_{G,S,T} := \det((f_{K/k,w_i}(u_j)-1)_{1\le i,j\le n}).$$

At the same time we also define a rational integer $m_{k,S,T}$ by means of the following equality in $\wedge^n X_{k,S}\otimes\mathbb{R}$

$$(\lim_{s\to 0} s^{-n}\theta_{k/k,S,T}(s))\cdot \wedge_{j\in|n|}(v_j - v_0) = m_{k,S,T}\cdot \lambda_{k,S}(\wedge_{j\in|n|}u_j), \tag{11}$$

where $\lambda_{k,S}$ denotes the isomorphism

$$\wedge^n\mathcal{O}^\times_{k,S,T}\otimes\mathbb{R} \to \wedge^n X_{k,S}\otimes\mathbb{R}$$

induced by the n-th exterior power of the map $-\mathrm{R}_{k,S}$ as defined in (2) (cf. [31, (1.7)]).

CONJECTURE $\mathrm{Gr}(K/k)$ (Gross, [31, Conj. 4.1]): *One has*

$$\theta_{K/k,S,T}(0) \equiv m_{k,S,T}\cdot \mathrm{Reg}_{G,S,T} \pmod{I_G^{n+1}}.$$

Remark 9. The term $m_{k,S,T}\cdot\mathrm{Reg}_{G,S,T}$ belongs to I_G^n and is, when considered modulo I_G^{n+1}, independent of the chosen ordering of S and of the precise choice of the places $\{w_i : i\in|n|^*\}$ and of the ordered basis $\{u_j : j\in|n|\}$.

5.2. STATEMENT OF THE MAIN RESULTS. At the present time, the best results concerning $\mathrm{Gr}(K/k)$ are due to Tan and to Lee. Specifically, it is known that $\mathrm{Gr}(K/k)$ is valid if either $|G|$ is a power of p [47] or if $|G|$ is coprime to both $|\mu_K|$ and the order of the group of divisors of degree 0 of the curve C_k [39]. However, these results are proved either by reduction to special cases or by induction on $|G|$ and so do not provide an insight into why $\mathrm{Gr}(K/k)$ should be true in general. In contrast, in this section we shall show that Gross's integral regulator mapping $\mathcal{O}^\times_{k,S}\to X_{k,S}\otimes G$ [31, (2.1)] arises as a natural Bockstein homomorphism in Weil-étale cohomology and we shall use this observation to prove the following result.

Theorem 5.1. *If the G-module μ_K is c-t, then* $\mathrm{C}(K/k)$ *implies* $\mathrm{Gr}(K/k)$.

Corollary 3. *If the G-module μ_K is c-t, then* $\mathrm{Gr}(K/k)$ *is valid.*

Proof of Corollary 3. It is easily seen to be enough to prove $\mathrm{Gr}(K/k)$ in the case that $|G|$ is a prime power. The aforementioned result of Tan therefore allows us to assume that $p \nmid |G|$ (so that $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ is torsion-free). But since the G-module μ_K is assumed to be c-t, in this case the validity of $\mathrm{Gr}(K/k)$ follows directly from Theorem 5.1 and Corollary 1. □

Remark 10. The G-module μ_K is c-t if and only if for each prime divisor ℓ of $|G|$ one has either $\ell \nmid |\mu_K|$ or $\ell \nmid [K : k(\mu_K^{(\ell)})]$ where $\mu_K^{(\ell)}$ is the maximal subgroup of μ_K of ℓ-power order. It seems likely that a further development of the method we use to prove Theorem 5.1 will allow the removal of any such hypothesis on μ_K. Indeed, in certain special cases this is already achieved in the present manuscript (cf. Corollary 5).

The proof of Theorem 5.1 will be the subject of the next three subsections.

5.3. The computation of $\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S})$. In this subsection we assume that the G-module μ_K is c-t, but we do not assume that G is abelian. We set $\mathrm{Tr}_G := \sum_{g\in G} g \in \mathbb{Z}[G]$. For any abelian group A we write $\overline{A}$ in place of A/A_{tors} and for any extension field E of $\mathbb{Q}$ we set $A_E := A \otimes E$. For any homomorphism of abelian groups $\phi : A \to A'$ we also let ϕ_E denote the induced homomorphism $\phi \otimes \mathrm{id}_E : A_E \to A'_E$.
In the following result we let $\mathrm{Cone}(\alpha)$ denote the 'mapping cone' of a particular morphism α in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ - our application of this construction can be made rigorous by the same observation as used in [15, Rem. 5.2].

Lemma 6. *There exists an endomorphism ϕ of a finitely generated free $\mathbb{Z}[G]$-module F which satisfies both of the following conditions.*
Let $F^\cdot$ denote the complex $F \xrightarrow{\phi} F$, where the first term is placed in degree 0.

i) *There exists a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ of the form*
$$F^\cdot \xrightarrow{\beta} \mathrm{Cone}(\alpha) \to Q[0] \to F^\cdot[1]$$
where α is the morphism $\mu_K[0] \to R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ that is induced by the inclusion $\mu_K \subset \mathcal{O}_{K,S}^\times$ and Q is a finite G-module of order coprime to $|G|$.

ii) *The endomorphism ϕ^G of F^G is semi-simple at* 0*. Indeed, there exists an integer d with $d \geq n$ and an ordered $\mathbb{Z}[G]$-basis $\{b_i : 1 \leq i \leq d\}$ of F which satisfies both of the following conditions.*

   a) *The $\mathbb{Z}[G]$-module F_1 which is generated by $\{b_i : i \in |n|\}$ satisfies $F_1^G = \ker(\phi^G)$ and, for each $i \in |n|$, the element $\mathrm{Tr}_G(b_i)$ is a pre-image of $v_i - v_0$ under the composite map*

$$F_1^G \subseteq F^G \twoheadrightarrow \mathrm{cok}(\phi^G) \to H^1(R\,\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m))) \\ \cong H^1(R\Gamma_{\mathcal{W}}(U_{k,S}, \mathbb{G}_m)) \twoheadrightarrow X_{k,S},$$

where the third, fourth and fifth maps are induced by $H^1(\beta)$, the isomorphism of Lemma 1iv) (with $J = G$) and the short exact sequence of Lemma 1iii) (with $K = k$) respectively.

b) *The $\mathbb{Z}[G]$-module F_2 which is generated by $\{b_i : n < i \le d\}$ is such that $\phi^G(F_2^G) \subseteq F_2^G$.*

Proof. We set $C^\cdot := \mathrm{Cone}(\alpha)$. Then, since μ_K is c-t, Lemma 1iii) implies that $C^\cdot$ is an object of $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ which is acyclic outside degrees 0 and 1 and that $H^0(C^\cdot) = \overline{\mathcal{O}_{K,S}^\times}$ and $H^1(C^\cdot) = H^1_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$. It is therefore clear that $C^\cdot$ is equivalent to a complex $\hat{F}^\cdot$ of the form $P \xrightarrow{\psi} F$ where F, resp. P, is a finitely generated free $\mathbb{Z}[G]$-module, resp. a finitely generated $\mathbb{Z}[G]$-module which is both c-t and $\mathbb{Z}$-free, and P is placed in degree 0. Now any such $\mathbb{Z}[G]$-module P is projective [2, Th. 8]. In addition, since the $\mathbb{Q}[G]$-modules $H^0(C^\cdot)_\mathbb{Q}$ and $H^1(C^\cdot)_\mathbb{Q}$ are isomorphic, Wedderburn's Theorem implies that the $\mathbb{Q}[G]$-modules $P_\mathbb{Q}$ and $F_\mathbb{Q}$ are also isomorphic. From Swan's Theorem [26, Th. (32.1)] we may therefore deduce that, for each prime q, the $\mathbb{Z}_q[G]$-modules $P \otimes \mathbb{Z}_q$ and $F \otimes \mathbb{Z}_q$ are isomorphic. We may thus apply Roiter's Lemma [26, (31.6)] to deduce the existence of a $\mathbb{Z}[G]$-submodule P' of P for which the quotient P/P' is finite and of order coprime to $|G|$ and one has an isomorphism of $\mathbb{Z}[G]$-modules $\iota : F \xrightarrow{\sim} P'$. We set $\lambda := \psi \circ \iota \in \mathrm{End}_{\mathbb{Z}[G]}(F)$.

The $\mathbb{Z}$-module $\mathrm{im}(\lambda^G)$ is free and so the exact sequence $0 \to \ker(\lambda^G) \xrightarrow{\subseteq} F^G \xrightarrow{\lambda^G} \mathrm{im}(\lambda^G) \to 0$ splits. Hence we may choose a submodule D of F^G which λ^G maps isomorphically to $\mathrm{im}(\lambda^G)$. We next let T denote the pre-image under the tautological surjection $F^G \to \mathrm{cok}(\lambda^G)$ of the subgroup $\mathrm{cok}(\lambda^G)_{\mathrm{tors}}$. Then the exact sequence $0 \to T \to F^G \to \overline{\mathrm{cok}(\lambda^G)} \to 0$ is also split and so we may choose a submodule D' of F^G which is mapped isomorphically to $\overline{\mathrm{cok}(\lambda^G)}$ under the natural surjection. Now D' and $\ker(\lambda^G)$ have the same $\mathbb{Z}$-rank since $D'_\mathbb{Q} \cong \mathrm{cok}(\lambda^G)_\mathbb{Q} \cong \mathrm{cok}(\lambda)^G_\mathbb{Q} \cong \ker(\lambda)^G_\mathbb{Q} \cong \ker(\lambda^G)_\mathbb{Q}$. The direct sum decompositions $\ker(\lambda^G) \oplus D = F^G = T \oplus D'$ therefore imply that there exists an automorphism ψ' of F^G such that both $\psi'(T) = D$ and $\psi'(D') = \ker(\lambda^G)$. It is then easily checked that $\psi' \circ \lambda^G(D) \subseteq D$ and that $\ker(\psi' \circ \lambda^G) = \ker(\lambda^G)$ is mapped bijectively to $\overline{\mathrm{cok}(\psi' \circ \lambda^G)}$ under the composite of the tautological surjections $F^G \to \mathrm{cok}(\psi' \circ \lambda^G)$ and $\mathrm{cok}(\psi' \circ \lambda^G) \to \overline{\mathrm{cok}(\psi' \circ \lambda^G)}$.

Since F is a free $\mathbb{Z}[G]$-module we may choose an element $\tilde{\psi}$ of $\mathrm{Aut}_{\mathbb{Z}[G]}(F)$ such that $\tilde{\psi}^G = \psi'$. We now set $\phi := \tilde{\psi} \circ \lambda \in \mathrm{End}_{\mathbb{Z}[G]}(F)$ and we let β denote the morphism in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ which corresponds to the morphism from the complex $F^\cdot$ (as described in the statement of the Lemma) to $\hat{F}^\cdot$ that is induced by ι in degree 0 and is equal to $\tilde{\psi}^{-1}$ in degree 1. It is then easily checked that this gives rise to a distinguished triangle of the form stated in i) in which $Q := P/P'$.

Now $\phi^G = \psi' \circ \lambda^G$ and so the above remarks imply both that $\phi^G(D) \subseteq D$ and that the natural map $\ker(\phi^G) \to \overline{\mathrm{cok}(\phi^G)}$ is bijective. We next observe that the decomposition $F^G = \ker(\phi^G) \oplus D$ can be lifted to a direct sum decomposition $F = F_1 \oplus F_2$ in which both F_1 and F_2 are free $\mathbb{Z}[G]$-modules (of ranks n and

$d-n$ respectively), $F_1^G = \ker(\phi^G)$ and $F_2^G = D$. We let κ denote the composite homomorphism described in claim ii)a). Our earlier observations imply that κ is bijective, and so $\{\kappa^{-1}(v_i - v_0) : i \in |n|\}$ is a $\mathbb{Z}$-basis of $F_1^G = \mathrm{Tr}_G(F_1)$. It is then easily shown that there exists a $\mathbb{Z}[G]$-basis $\{b_i : i \in |n|\}$ of F_1 such that $\mathrm{Tr}_G(b_i) = \kappa^{-1}(v_i - v_0)$ for each $i \in |n|$. To complete the proof of claim ii) we simply let $\{b_i : n < i \leq d\}$ denote any choice of (ordered) $\mathbb{Z}[G]$-basis of F_2. $\square$

If M is any finite G-module which is c-t, then $M[0]$ is an object of $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ and we set $\chi(M) := \chi(M[0], \mathrm{id}_0) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ where id_0 denotes the identity map on the zero space. We also set $\mathrm{R}^{\beta}_{K,S} := H^1(\beta)_{\mathbb{R}}^{-1} \circ \mathrm{R}_{K,S} \circ H^0(\beta)_{\mathbb{R}}$.
Then upon applying Lemma A2 firstly to the distinguished triangle

$$\mu_K[0] \xrightarrow{\alpha} R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \to \mathrm{Cone}(\alpha) \to \mu_K[1]$$

and then to the distinguished triangle in Lemma 6i) we obtain equalities

$$\begin{aligned} \chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S}) &= \chi(\mathrm{Cone}(\alpha), \mathrm{R}_{K,S}) + \chi(\mu_K) \\ &= \chi(F^{\cdot}, \mathrm{R}^{\beta}_{K,S}) + \chi(Q) + \chi(\mu_K) \\ &= \delta(\mathrm{detred}_{\mathbb{R}[G]}(\langle \mathrm{R}^{\beta}_{K,S}, \phi\rangle_{\iota_1,\iota_2})) + \chi(Q) + \chi(\mu_K), \end{aligned} \tag{12}$$

where ι_1 and ι_2 are any choices of $\mathbb{R}[G]$-equivariant sections to the tautological surjections $F_{\mathbb{R}} \to \mathrm{im}(\phi)_{\mathbb{R}}$ and $F_{\mathbb{R}} \to \mathrm{cok}(\phi)_{\mathbb{R}}$ and the last equality follows from Lemma A1.

5.4. The connection to $\mathrm{Gr}(K/k)$. In this subsection we assume that G is abelian and identify $K_0(\mathbb{Z}[G], \mathbb{R})$ with the multiplicative group of invertible $\mathbb{Z}[G]$-lattices in $\mathbb{R}[G]$ (see Remark A1). In particular, we note that if M is any finite G-module which is c-t, then its (initial) Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}[G]}(M)$ is an invertible ideal of $\mathbb{Z}[G]$ and under the stated identification one has $\chi(M) = \mathrm{Fitt}_{\mathbb{Z}[G]}(M)^{-1}$ in $\mathbb{R}[G]$.
Now $\theta^*_{K/k,S,T}(0)^{\#} = \Delta_T^{\#} \cdot \theta^*_{K/k,S}(0)^{\#}$ and $\Delta_T^{\#} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) = \mathrm{Fitt}_{\mathbb{Z}[G]}(\mu_K)$. Hence, in this case, (12) implies that the validity of $\mathrm{C}(K/k)$ is equivalent to the existence of an element x_T of $\mathbb{Q}[G]^{\times}$ which satisfies both

$$\theta^*_{K/k,S,T}(0)^{\#} = x_T \cdot \det_{\mathbb{R}[G]}(\langle \mathrm{R}^{\beta}_{K,S}, \phi\rangle_{\iota_1,\iota_2}) \in \mathbb{R}[G]^{\times} \tag{13}$$

and

$$\mathbb{Z}[G] \cdot x_T = \Delta_T^{\#} \cdot \mathrm{Fitt}_{\mathbb{Z}[G]}(\mu_K)^{-1}\, \mathrm{Fitt}_{\mathbb{Z}[G]}(Q)^{-1} \subseteq \mathrm{Fitt}_{\mathbb{Z}[G]}(Q)^{-1}. \tag{14}$$

We let $G^*_{(0)}$ denote the set of characters $\chi \in G^*$ at which $L_S(\chi, 0) \neq 0$, and we set $e_0 := \sum_{\chi \in G^*_{(0)}} e_\chi$. Then the criterion of [50, Chap. I, Prop. 3.4] implies that $e_0 \in \mathbb{Q}[G]$, that $e_0 \cdot \ker(\phi)_{\mathbb{Q}} = 0$ and hence $e_0 \mathrm{det}_{\mathbb{R}[G]}(\langle \mathrm{R}^{\beta}_{K,S}, \phi\rangle_{\iota_1,\iota_2}) = e_0 \mathrm{det}_{\mathbb{Z}[G]}(\phi)$, and also that for any $\chi \in G^* \setminus G^*_{(0)}$ one has $e_\chi \cdot \ker(\phi)_{\mathbb{C}} \neq 0$ and so $e_0 \mathrm{det}_{\mathbb{Z}[G]}(\phi) = \mathrm{det}_{\mathbb{Z}[G]}(\phi)$. Since $\theta_{K/k,S,T}(0)^{\#} = e_0 \theta^*_{K/k,S,T}(0)^{\#}$ we therefore deduce from (13) that

$$\theta_{K/k,S,T}(0)^{\#} = x_T \mathrm{det}_{\mathbb{Z}[G]}(\phi).$$

Now $|Q|$ is coprime to $|G|$ and I_G^n/I_G^{n+1} is annihilated by a power of $|G|$, and so (14) implies that x_T acts naturally on I_G^n/I_G^{n+1}. In addition, Lemma 6ii) implies that the matrix of ϕ with respect to the ordered $\mathbb{Z}[G]$-basis $\{b_i : 1 \le i \le d\}$ of F is a block matrix of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{15}$$

where $A := (A_{ij})_{1\le i,j\le n} \in \mathrm{M}_n(I_G)$, $D \in M_{d-n}(\mathbb{Z}[G])$ and all entries of both B and C belong to I_G. Since $\det(A) \in I_G^n$ one has

$$\det(A)^{\#} \equiv (-1)^n \det(A) \pmod{I_G^{n+1}}$$

and so the above matrix representation combines with the previous displayed equality to imply that

$$\theta_{K/k,S,T}(0) \equiv (-1)^n \epsilon(x_T)\epsilon(\det(D)) \cdot \det(A) \pmod{I_G^{n+1}}. \tag{16}$$

To compute the term $(-1)^n\epsilon(x_T)\epsilon(\det(D))$ we first multiply (13) by Tr_G and obtain an equality

$$\lim_{s\to 0} s^{-n}\theta_{k/k,S,T}(s) = \epsilon(x_T)\mathrm{det}_{\mathbb{R}}(\langle \mathrm{R}^{\beta}_{K,S}, \phi\rangle_{\iota_1,\iota_2})^G.$$

For convenience we fix the sections ι_1 and ι_2 so that ι_1^G is equal to the inverse of the automorphism of $F^G_{2,\mathbb{R}}$ induced by ϕ^G and ι_2^G is the inverse of the composite map $F^G_{1,\mathbb{R}} \subseteq F^G_{\mathbb{R}} \twoheadrightarrow \mathrm{cok}(\phi^G)_{\mathbb{R}}$. Then $(\langle \mathrm{R}^{\beta}_{K,S}, \phi\rangle_{\iota_1,\iota_2})^G = \psi_1 \oplus \psi_2$ where ψ_2 is equal to the restriction of ϕ to $F^G_{2,\mathbb{R}}$ and ψ_1 is the automorphism of $F^G_{1,\mathbb{R}}$ that is obtained as the composite

$$F^G_{1,\mathbb{R}} = \ker(\phi^G)_{\mathbb{R}} \xrightarrow{H^0(\beta)^G_{\mathbb{R}}} (\mathcal{O}^{\times}_{K,S})^G_{\mathbb{R}} \xrightarrow{\mathrm{R}_{K,S}} X^G_{K,S,\mathbb{R}} \xrightarrow{\sigma} X_{k,S,\mathbb{R}} \to F^G_{1,\mathbb{R}}$$

where σ is the bijection induced by the injection $X_{k,S} \to X_{K,S}$ described in Lemma 1iv) (with $J = G$), and the final arrow denotes the inverse of the isomorphism induced by the displayed map in Lemma 6ii)a). Now, with respect to the ordered $\mathbb{Z}$-basis $\{\mathrm{Tr}_G(b_i) : n < i \le d\}$ of F_2^G, each component of the matrix of ψ_2 is the image under ϵ of the corresponding component of D and so

$$\lim_{s\to 0} s^{-n}\theta_{k/k,S,T}(s) = \epsilon(x_T)\cdot \mathrm{det}_{\mathbb{R}}(\psi_1)\cdot\epsilon(\det(D)).$$

On the other hand, the commutative diagram

$$\begin{array}{ccc} (\mathcal{O}^{\times}_{K,S})^G_{\mathbb{R}} & \xrightarrow{\mathrm{R}_{K,S}} & X^G_{K,S,\mathbb{R}} \\ {\scriptstyle\subseteq}\uparrow & & \downarrow{\scriptstyle\sigma} \\ (\mathcal{O}^{\times}_{k,S})_{\mathbb{R}} & \xrightarrow{\mathrm{R}_{k,S}} & X_{k,S,\mathbb{R}} \end{array}$$

(cf. [50, Chap. I, §6.5]) combines with the above description of ψ_1 to imply that $\mathrm{det}_{\mathbb{R}}(\psi_1)$ is equal to the determinant of the map $\wedge^n(H^0(\beta)(F_1^G))_{\mathbb{R}} \to (\wedge^n X_{k,S})_{\mathbb{R}}$ induced by $\wedge^n_{\mathbb{R}}\mathrm{R}_{k,S} = (-1)^n\lambda_{k,S}$, as computed with respect to the $\mathbb{R}$-bases $\wedge_{i\in|n|}H^0(\beta)(\mathrm{Tr}_G(b_i))$ and $\wedge_{i\in|n|}(v_i - v_0)$. Hence, if we fix an ordered $\mathbb{Z}$-basis $\{d_i : i \in |n|\}$ of $\overline{\mathcal{O}^{\times}_{k,S}}$, regard $\mathcal{O}^{\times}_{k,S,T}$ as a subgroup of $\overline{\mathcal{O}^{\times}_{k,S}}$ in the natural way, and define elements $a := (a_{ij})_{1\le i,j\le n}$ and $b := (b_{ij})_{1\le i,j\le n}$ of $\mathrm{M}_n(\mathbb{Z})$ by the equalities $u_i = \sum_{j\in|n|} a_{ij}d_j$ and $\mathrm{Tr}_G(H^0(\beta)(b_i)) = \sum_{j\in|n|} b_{ij}d_j$ for each $i\in|n|$, then the last displayed formula implies that

$$\begin{aligned}&(\lim_{s\to 0} s^{-n}\theta_{k/k,S,T}(s))\cdot\wedge_{j\in|n|}(v_j-v_0)\\ &=\epsilon(x_T)\epsilon(\det(D))\mathrm{det}_{\mathbb{R}}(\psi_1)\cdot\wedge_{j\in|n|}(v_j-v_0)\\ &=(-1)^n\epsilon(x_T)\epsilon(\det(D))\cdot\lambda_{k,S}(\wedge_{j\in|n|}H^0(\beta)(\mathrm{Tr}_G(b_j)))\\ &=(-1)^n\epsilon(x_T)\epsilon(\det(D))\det(b)\det(a)^{-1}\cdot\lambda_{k,S}(\wedge_{j\in|n|}u_j).\end{aligned}$$

Comparing this equality with (11) implies that

$$(-1)^n\epsilon(x_T)\epsilon(\det(D))\det(b)\det(a)^{-1} = m_{k,S,T}$$

and hence that

$$(-1)^n\epsilon(x_T)\epsilon(\det(D)) = m_{k,S,T}\det(a)\det(b)^{-1}.$$

In turn, upon substituting this equality into (16) we obtain a congruence

$$\theta_{K/k,S,T}(0)\equiv m_{k,S,T}\det(a)\det(b)^{-1}\cdot\det(A)\ (\mathrm{mod}\ I_G^{n+1}). \tag{17}$$

5.5. Bockstein homomorphisms. In this subsection we complete our proof of Theorem 5.1 by showing that the factor $\det(a)\det(b)^{-1}\cdot\det(A)$ which occurs in (17) is equal to $\mathrm{Reg}_{G,S,T}$. The key to our proof of this equality will be the observation that the 'regulator map' $\mathcal{O}_{k,S}^\times\to X_{k,S}\otimes G$ introduced by Gross in [31, (2.1)] arises as a natural Bockstein homomorphism in Weil-étale cohomology (this is Lemma 8). The material in this subsection is strongly influenced by the general philosophy of algebraic height pairings that is developed by Nekovář in [44, §11].

At the outset we let Γ be any finite abelian group and $C^\cdot$ any object of $\mathcal{D}^{\mathrm{fpd}}(\mathbb{Z}[\Gamma])$. Then, upon tensoring $C^\cdot$ with the tautological exact sequence $0\to I_\Gamma\to\mathbb{Z}[\Gamma]\to\mathbb{Z}\to 0$ we obtain a distinguished triangle in $\mathcal{D}(\mathbb{Z})$ of the form

$$C^\cdot\to C^\cdot_\Gamma\to C\otimes^{\mathbb{L}}_{\mathbb{Z}[\Gamma]} I_\Gamma[1]\to C^\cdot[1],$$

where $C^\cdot_\Gamma := C^\cdot\otimes^{\mathbb{L}}_{\mathbb{Z}[\Gamma]}\mathbb{Z}$. In addition, if $C^\cdot$ is acyclic outside degrees 0 and 1, then there are natural identifications $H^0(C^\cdot_\Gamma)\cong H^0(C^\cdot)^\Gamma$ (induced by the action of Tr_Γ), $H^1(C^\cdot_\Gamma)\cong H^1(C^\cdot)_\Gamma$ and $H^1(C^\cdot\otimes^{\mathbb{L}}_{\mathbb{Z}[\Gamma]} I_\Gamma)\cong H^1(C^\cdot)\otimes_{\mathbb{Z}[\Gamma]} I_\Gamma$. In this case the canonical identification $I_\Gamma/I_\Gamma^2\cong\Gamma$ therefore combines with the cohomology sequence of the above triangle to induce a 'Bockstein homomorphism'

$$\begin{aligned}\beta_{C^\cdot,\Gamma}:\ H^0(C^\cdot)^\Gamma &\to H^1(C^\cdot\otimes^{\mathbb{L}}_{\mathbb{Z}[\Gamma]} I_\Gamma)\cong H^1(C^\cdot)\otimes_{\mathbb{Z}[\Gamma]} I_\Gamma\\ &\to\overline{H^1(C^\cdot)_\Gamma}\otimes_{\mathbb{Z}[\Gamma]}(I_\Gamma/I_\Gamma^2)\cong\overline{H^1(C^\cdot_\Gamma)}\otimes\Gamma\end{aligned}$$

and also an associated pairing

$$\rho_{C^\cdot,\Gamma}: H^0(C^\cdot)^\Gamma\times\mathrm{Hom}_{\mathbb{Z}}(\overline{H^1(C^\cdot_\Gamma)},\mathbb{Z})\to I_\Gamma/I_\Gamma^2.$$

In the remainder of this subsection we shall use these constructions in the cases that $\Gamma = G$ and $C^\cdot$ is equal to both $F^\cdot$ (as described in Lemma 6) and

$R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$, and also in the case that Γ is equal to a given decomposition subgroup of G and $C^\cdot$ is a local analogue of $R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$. In the course of so doing we shall always use the $\mathbb{Z}$-basis $\{v_i - v_0 : i \in |n|\}$ to identify $X_{k,S}$ with $\mathrm{Hom}_{\mathbb{Z}}(X_{k,S}, \mathbb{Z})$.

Before stating our first result we observe that the action of Tr_G (in each degree) induces an isomorphism in $\mathcal{D}(\mathbb{Z})$ between $F^\cdot_G = F^\cdot \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ and the complex $F^G \xrightarrow{\phi^G} F^G$ in which the first term is placed in degree 0. We shall use this isomorphism to identify $\overline{H^1(F^\cdot_G)}$ with $X_{k,S}$ by means of the map $F^G \to X_{k,S}$ described in Lemma 6ii)a).

Lemma 7. *With respect to the ordered $\mathbb{Z}$-bases $\{\mathrm{Tr}_G(b_i) : i \in |n|\}$ and $\{v_i - v_0 : i \in |n|\}$ of $H^0(F^\cdot)^G$ and $\mathrm{Hom}_{\mathbb{Z}}(\overline{H^1(F^\cdot_G)}, \mathbb{Z})$ respectively, the matrix of $\rho_{F^\cdot,G}$ is equal to A (mod $M_n(I_G^2)$).*

Proof. The homomorphism $\beta_{F^\cdot,G}$ can be computed as the composite of the connecting homomorphism in the following commutative diagram

$$\begin{tikzcd}
 & & & H^0(F^\cdot)^G \arrow[d] & \\
0 \arrow[r] & F \otimes_{\mathbb{Z}[G]} I_G \arrow[r, "\subseteq"] \arrow[d, "\phi\otimes_{\mathbb{Z}[G]}\mathrm{id}"] & F \arrow[r, "\cdot\,\mathrm{Tr}_G"] \arrow[d, "\phi"] & F^G \arrow[r] \arrow[d, "\phi^G"] & 0 \\
0 \arrow[r] & F \otimes_{\mathbb{Z}[G]} I_G \arrow[r, "\subseteq"] \arrow[d] & F \arrow[r, "\cdot\,\mathrm{Tr}_G"] & F^G \arrow[r] & 0 \\
 & H^1(F^\cdot) \otimes_{\mathbb{Z}[G]} I_G & & &
\end{tikzcd}$$

with the natural surjection $H^1(F^\cdot) \otimes_{\mathbb{Z}[G]} I_G \to \overline{H^1(F^\cdot_G)} \otimes I_G/I_G^2$. Upon computing the above connecting homomorphism by using the matrix representation of ϕ given in (15), and observing Lemma 6ii) implies that the tautological surjection $F^G \to \overline{\mathrm{cok}(\phi^G)} \cong \overline{H^1(F^\cdot_G)}$ factors through the projection $F^G \to F_1^G$, one finds that the required composite sends each element $\mathrm{Tr}_G(b_i)$ to $\sum_{j\in|n|}(v_j - v_0) \otimes A_{ij}$ (mod I_G^2). This implies the stated result. $\square$

The construction of the pairing $\rho_{C^\cdot,G}$ is natural in $C^\cdot$ in the following sense: if $\mu : C^\cdot \to D^\cdot$ is any morphism in $\mathcal{D}^{\mathrm{fpd}}(\mathbb{Z}[G])$ which induces a bijection $\overline{H^1(\mu_G)}$ from $\overline{H^1(C^\cdot_G)}$ to $\overline{H^1(D^\cdot_G)}$, then there is a commutative diagram

$$\begin{tikzcd}
H^0(C^\cdot)^G \times \mathrm{Hom}_{\mathbb{Z}}(\overline{H^1(C^\cdot_G)}, \mathbb{Z}) \arrow[r, "\rho_{C^\cdot,G}"] \arrow[d, "{(H^0(\mu)^G, \mathrm{Hom}_{\mathbb{Z}}(\overline{H^1(\mu_G)}, \mathbb{Z})^{-1})}"'] & I_G/I_G^2 \arrow[d, equal] \\
H^0(D^\cdot)^G \times \mathrm{Hom}_{\mathbb{Z}}(\overline{H^1(D^\cdot_G)}, \mathbb{Z}) \arrow[r, "\rho_{D^\cdot,G}"] & I_G/I_G^2.
\end{tikzcd}$$

When taken in conjunction with the computation of Lemma 7 and the fact that multiplication by $\det(b)$ is invertible on I_G^n/I_G^{n+1} (since $|Q^G|$ is coprime to

$|G|$), this observation implies that the term $\det(a)\det(b)^{-1} \cdot \det(A) \pmod{I_G^{n+1}}$ in (17) is equal to the discriminant of the restriction of $\rho_{R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m),G}$ to $\mathcal{O}_{k,S,T}^{\times} \times X_{k,S}$ as computed with respect to the ordered $\mathbb{Z}$-bases $\{u_i : i \in |n|\}$ and $\{v_i - v_0 : i \in |n|\}$.

To prove Theorem 5.1 we therefore need only show that the homomorphism $\beta_{R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m),G}$ coincides with the regulator mapping $\mathcal{O}_{k,S}^{\times} \to X_{k,S} \otimes G$ defined by Gross in [31, (2.1)]. In turn, this is achieved by the following result (which, we observe, does not assume that the G-module μ_K is c-t).

Lemma 8. *Set $C^{\cdot} := R\Gamma_{\mathcal{W}}(U_{K,S},\mathbb{G}_m)$. Then for each $u \in \mathcal{O}_{k,S,T}^{\times}$ one has $\beta_{C^{\cdot},G}(u) = \sum_{i\in|n|}(v_i - v_0) \otimes f_{K/k,w_i}(u)$.*

Proof. We fix an index $i \in |n|$ and set $v := v_i$, $w := w_i$ and $D := \mathrm{Gal}(K_w/k_v)$. We let $\beta_{C^{\cdot},G,v}$ denote the composite of $\beta_{C^{\cdot},G}$ with the inclusion $X_{k,S} \otimes G \subset Y_{k,S} \otimes G$ and the homomorphism $Y_{k,S} \otimes G \to G$ which is induced by mapping each element of $Y_{k,S}$ to its coefficient at v. Then we need to show that $\beta_{C^{\cdot},G,v} = f_{K/k,w}$.

We set $V_w := \mathrm{Spec}(K_w)$. Then the result of [12, Lem. 7(b)] combines with the fact that $H^1(V_w,\mathbb{G}_m) = 0$ to imply that there exists a unique morphism α_w from $\mathbb{Q}[-2]$ to $R\Gamma(V_w,\mathbb{G}_m)$ in $\mathcal{D}(\mathbb{Z}[D])$ for which $H^2(\alpha_w)$ is equal to the composite of the natural projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and the canonical identification $\mathbb{Q}/\mathbb{Z} \cong H^2(V_w,\mathbb{G}_m)$. We set $C_w^{\cdot} := \mathrm{Cone}(\alpha_w)$ (cf. the remark just prior to Lemma 6). Then, by an argument similar to that used in the proof of Lemma 1iii), one shows that $C_w^{\cdot}$ is an object of $\mathcal{D}^{\mathrm{fpd}}(\mathbb{Z}[D])$ which is acyclic outside degrees 0 and 1 and is such that $H^0(C_w^{\cdot})$ and $H^1(C_w^{\cdot})$ identify canonically with $K_w^{\times}$ and $\mathbb{Z}$ respectively. Further, in the notation of §4.1, the result of [12, Prop. 3.5(a)] implies that the associated Yoneda extension class $e(C_w^{\cdot})$ is equal to the element $-e_w$ of $\mathrm{Ext}_D^2(\mathbb{Z},K_w^{\times}) \cong H^2(D,K_w^{\times})$ where $\mathrm{inv}_{k_v}(e_w) = \frac{1}{|D|}$ (recall that, following the approach of §4.1, we are here using a different convention regarding Yoneda-Ext-groups than that used in [12], and hence $e(C_w^{\cdot})$ is equal to $-e_w$ rather than e_w.)

The natural localisation morphism $R\Gamma(U_{K,S},\mathbb{G}_m) \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[D]} R\Gamma(V_w,\mathbb{G}_m)$ in $\mathcal{D}(\mathbb{Z}[G])$ induces a morphism $C^{\cdot} \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[D]} C_w^{\cdot}$ and by consideration of this morphism one finds that $\beta_{C^{\cdot},G,v}$ is equal to the composite of the embedding $\mathcal{O}_{k,S}^{\times} \to k_v^{\times}$, the homomorphism $\beta_{C_w^{\cdot},D}$ and the natural injection $D \subseteq G$. It is therefore enough for us to prove that $\beta_{C_w^{\cdot},D}$ is equal to the reciprocity map $\mathrm{rec}_w : k_v^{\times} \to D$ of the extension K_w/k_v.

To this end we first recall that rec_w is defined to be the map induced by the inverse of the isomorphism $D \cong \hat{H}^0(D,K_w^{\times})$ which results from the canonical identifications $D \cong I_D/I_D^2 = \hat{H}^{-1}(D,I_D)$, the isomorphism $\hat{H}^{-1}(D,I_D) \cong \hat{H}^{-2}(D,\mathbb{Z})$ which is induced by the connecting homomorphism associated to the tautological exact sequence $0 \to I_D \to \mathbb{Z}[D] \to \mathbb{Z} \to 0$ and the isomorphism $\hat{H}^{-2}(D,\mathbb{Z}) \cong \hat{H}^0(D,K_w^{\times})$ which is given by cup-product with e_w.

To proceed we choose an extension of D-modules

$$0 \to K_w^{\times} \xrightarrow{\iota} A \xrightarrow{\psi} \mathbb{Z}[D] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

of Yoneda extension class $-e_w$. Then $C_w^{\cdot}$ is equivalent to the complex $A^{\cdot}$ which is given by $A \xrightarrow{\psi} \mathbb{Z}[D]$, where the modules are placed in degrees 0 and 1 and the cohomology is identified with $K_w^\times$ and $\mathbb{Z}$ by means of the given maps. Taken in conjunction with the description of rec_w in the preceding paragraph and the compatibility of cup-products with connecting homomorphisms in Tate cohomology (cf. [2, Th. 3 and Th. 4(iii),(iv)]), this observation implies that rec_w is induced by the canonical isomorphism $D \cong I_D/I_D^2 = (I_D)_D$ together with the inverse of the connecting homomorphism in the following commutative diagram

$$\begin{tikzcd}
 & & & (I_D)_D \arrow[d, equal] & \\
 & (K_w^\times)_D \arrow[r, "\iota_D"] \arrow[d, "\mathrm{Tr}_w"] & A_D \arrow[r, "\psi_D"] \arrow[d, "\mathrm{Tr}_w"] & (I_D)_D \arrow[r] \arrow[d, "\mathrm{Tr}_w"] & 0 \\
0 \arrow[r] & (K_w^\times)^D \arrow[r, "\iota^D"] \arrow[d] & A^D \arrow[r, "\psi^D"] & (I_D)^D = 0 & \\
 & \hat{H}^0(D, K_w^\times) & & &
\end{tikzcd}$$

where $\mathrm{Tr}_w := \sum_{d\in D} d \in \mathbb{Z}[D]$. On the other hand, the fact that $C_w^{\cdot}$ is equivalent to $A^{\cdot}$ combines with the definition of $\beta_{C_w^{\cdot},D}$ to imply that the latter homomorphism can be computed as the composite of the natural identification $D \cong (I_D)_D$ and the connecting homomorphism in the following commutative diagram

$$\begin{tikzcd}
 & & & k_v^\times \arrow[d, "\iota^D"] & \\
 & A \otimes_{\mathbb{Z}[D]} I_D \arrow[r] \arrow[d, "\psi\otimes_{\mathbb{Z}[G]}\mathrm{id}"] & A \arrow[r, "\mathrm{Tr}_w"] \arrow[d, "\psi"] & A^D \arrow[r] \arrow[d, "\psi^D"] & 0 \\
0 \arrow[r] & I_D \arrow[r, "\subset"] \arrow[d] & \mathbb{Z}[D] \arrow[r, "\mathrm{Tr}_w"] & \mathbb{Z}\cdot\mathrm{Tr}_w \arrow[r] & 0 \\
 & (I_D)_D. & & &
\end{tikzcd}$$

We remark that the upper row of this diagram is exact since the D-module A is c-t. Our proof now concludes by means of an explicit diagram chase showing that the connecting homomorphism in the second of these diagrams induces the inverse of the connecting homomorphism in the first diagram. $\square$

6. The conjecture of Tate

In this section we provide evidence for Tate's refinement of $\mathrm{Gr}(K/k)$. To do so we continue to use the notation of §5.1. In addition, we fix a prime number ℓ and assume henceforth that G has order ℓ^m with $m \geq 1$. For each index j in $|n|^*$ we let G_j denote the decomposition subgroup of w_j in G and we define an integer m_j by the equality $|G_j| = \ell^{m-m_j}$.

6.1. Statement of the conjecture.

In this subsection we assume S to be ordered so that $m_0 \leq m_1 \leq \cdots \leq m_n$.

Conjecture $\mathrm{Ta}(K/k, S, T)$ (Tate, [51]): *If G is cyclic of order ℓ^m, $m_0 = 0$ and $m_n = m - 1$, then one has*

$$\theta_{K/k,S,T}(0) \equiv m_{k,S,T} \cdot \mathrm{Reg}_{G,S,T} \pmod{I_G^{(\sum_{i=0}^{n-1} \ell^{m_i})+1}}.$$

For a further discussion of this conjecture see, for example, [38, §4].

6.2. Statement of the main results.

We recall from §4.1 that if the G-module $A_{K,S} := \mathrm{Pic}(\mathcal{O}_{K,S})$ is c-t, then one can define a canonical element $c_S(K/k) := \iota_{S',S}(c_{S'}(K/k))$ of $\mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S})$, where S' is any set as described in §4.1 (and $c_S(K/k)$ is indeed independent of the choice of S').
For each index j in $|n|$ we write I_j for the kernel of the natural projection map $\mathbb{Z}[G] \to \mathbb{Z}[G/G_j]$. We consider the following hypothesis on K/k.

Hypothesis (S,T): There exist finite non-empty sets S and T of places of k which satisfy each of the following conditions:

i) S contains all places which ramify in K/k,
ii) the G-module $A_{K,S}$ is c-t,
iii) $G_0 = G$, $n > 0$ and G_j is cyclic for each $j \in |n|$,
iv) T is disjoint from S and $c_S(K/k)$ lies in the image of the map

$$\mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S,T}) \to \mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S})$$

induced by the inclusion $\mathcal{O}^\times_{K,S,T} \subset \mathcal{O}^\times_{K,S}$.

Remark 11. If K/k is cyclic, then there always exists a set of places S which satisfies conditions i), ii) and iii) above. In general however, for a given field K there are restrictions on the abstract structure of the decomposition group G_0 and therefore (under condition iii)) also on G. Nevertheless, the validity of Hypothesis (S,T) does not itself imply, for example, that G is abelian. If $\ell \nmid |\mu_k|$, then (since $|G|$ is a power of ℓ) one has $\ell \nmid |\mu_K|$ and so [50, Chap. IV, Lem. 1.1] implies that there exists a set T which is disjoint from S and satisfies $\ell \nmid [\mathcal{O}^\times_{K,S} : \mathcal{O}^\times_{K,S,T}]$ and hence also condition iv). In fact, condition iv) can be shown to be satisfied under reasonably general conditions even if $\ell \mid |\mu_k|$ (cf. [17, Lem. 2]).

The following result will be proved in §6.4.

Theorem 6.1. *If S and T are as in Hypothesis (S,T) and G is abelian, then* $\mathrm{C}(K/k)$ *implies that*

$$\theta_{K/k,S,T}(0) \equiv m_{k,S,T} \cdot \mathrm{Reg}_{G,S,T} \pmod{I_G \cdot \prod_{j \in |n|} I_j}.$$

Corollary 4. *Assume the notation and hypotheses of* $\mathrm{Ta}(K/k, S, T)$*. If the G-module* $\mathrm{Pic}(\mathcal{O}_{K,S})$ *is c-t and $c_S(K/k)$ lies in the image of the map*

$$\mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S,T}) \to \mathrm{Ext}^2_G(X_{K,S}, \mathcal{O}^\times_{K,S})$$

induced by the inclusion $\mathcal{O}^\times_{K,S,T} \subset \mathcal{O}^\times_{K,S}$, then $\mathrm{C}(K/k)$ *implies that*

$$\theta_{K/k,S,T}(0) \equiv m_{k,S,T} \cdot \mathrm{Reg}_{G,S,T} \pmod{I_G \cdot \prod_{j \in |n|} I_j}.$$

In particular, in this case $\mathrm{Ta}(K/k, S, T)$ *is valid.*

Proof. Since, by assumption, $m_0 = 0$ the sets S and T satisfy all parts of Hypothesis (S,T). The first assertion thus follows directly from Theorem 6.1. To prove the second assertion we recall that if $\ell = p$, then $\mathrm{Ta}(K/k, S, T)$ has been proved by Tan [48]. We may therefore assume that $\ell \neq p$ so that $\mathrm{C}(K/k)$ is valid by Corollary 1. It thus suffices to deduce the validity of $\mathrm{Ta}(K/k, S, T)$ from the stated congruence for $\theta_{K/k,S,T}(0)$ and this is true because $\prod_{j \in |n|} I_j \subseteq I_G^{\sum_{i=0}^{n-1} \ell^{m_i}}$. Indeed, since $m_n = m - 1$, the required inclusion follows directly from the criterion of [8, Lem. 5.11]. □

The next result improves upon Corollary 3 and also the main result of Lee in [37].

Corollary 5. *If G has prime exponent, then* $\mathrm{Gr}(K/k)$ *is valid.*

Proof. In this case, the functorial properties of $\theta_{K/k,S,T}(0)$ and $\mathrm{Reg}_{G,S,T}$ under change of K/k combine with results on the structure of I_G/I_G^{n+1} to show that it is enough to prove $\mathrm{Gr}(L/k)$ for each sub-extension L/k of K/k which is of prime degree. The theorem of Tan [47] also allows us to assume that $[L : k]$ is a prime number different from p, and in this case the required congruence can be proved by combining the result of Corollary 4 (with $K = L$) together with arguments of Gross from [31, §6]. The precise details of this argument are presented in joint work of the author with Lee [17]. □

6.3. $\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S})$ revisited. In this subsection we prepare for the proof of Theorem 6.1 by using Hypothesis (S,T) to refine the computation of $\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S})$ given in §5.3. We do not assume here that G is abelian or that the G-module μ_K is c-t.

At the outset we fix sets S and T as in Hypothesis (S,T). Since S is fixed we abbreviate $\mathcal{O}^\times_{K,S,T}, X_{K,S}$ and $\mathcal{O}^\times_{K,S}$ to $\mathcal{O}^\times_{K,T}, X_K$ and $\mathcal{O}^\times_K$ respectively. We also set $A_K := \mathrm{Pic}(\mathcal{O}_{K,S})$ and write $A_{K,T}$ for the quotient of the group of fractional ideals of $\mathcal{O}_{K,S}$ that are prime to T by the subgroup of principal ideals with a generator congruent to 1 modulo all places in $T(K)$.

For each $j \in |n|$ we fix a generator g_j of G_j and a set of representatives $S(j)$ of the orbits of G_j on the set of places of K lying above $\{v_i : i \in |n|\}$. We assume that $S(j)$ contains w_i for each $i \in |n|$. For each place w in $S(j)$ we define δ_{jw} to be 1 if $w = w_j$ and to be 0 otherwise. For each $j \in |n|$ we also set $\mathrm{Tr}_j := \sum_{g \in G_j} g \in \mathbb{Z}[G]$ and $K_j := K^{G_j}$.

If d is any strictly positive integer, then in the sequel we shall use the canonical basis of $\mathbb{R}[G]^d$ to identify the groups $\mathrm{GL}_d(\mathbb{R}[G])$ and $\mathrm{Aut}_{\mathbb{R}[G]}(\mathbb{R}[G]^d)$.

PROPOSITION 6.1. *Let S and T be as in Hypothesis (S,T). Assume also that G_j is not trivial for any $j \in |n|$. Then, for each $j \in |n|$ there exists an element ϵ_j of $\mathcal{O}^\times_{K_j,T}$ which satisfies all of the following conditions.*

i) *For each $w \in S(j)$ one has $f_{K/K_j,w}(\epsilon_j) = g_j^{\delta_{jw}}$.*

ii) *For each pair of integers i, j in $|n|$ let y_{ji} denote the (unique) element of $\mathbb{R}[G] \cdot \mathrm{Tr}_i$ which satisfies*

$$\frac{1}{|G_j|}\mathrm{R}_{K,S}(\epsilon_j) = \sum_{i \in |n|} y_{ji}(w_i - w_0).$$

Then the matrix $M_T := (\delta_{ij}(g_i - 1) + y_{ij})_{1 \le i,j \le n}$ belongs to $\mathrm{GL}_n(\mathbb{R}[G])$.

iii) *The G-module $\mathcal{E}$ that is generated by the set $\{\epsilon_j : j \in |n|\}$ has finite index in $\mathcal{O}^\times_{K,T}$. The G-modules $\mathcal{O}^\times_{K,S}/\mathcal{E}$ and $A_{K,T}$ are both c-t and in $K_0(\mathbb{Z}[G], \mathbb{R})$ one has*

$$\begin{aligned}\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S}) = {}& \chi(\mathcal{O}^\times_{K,T}/\mathcal{E}) - \chi(A_{K,T}) \\ & + \delta(\mathrm{detred}_{\mathbb{R}[G]}(M_T)) - \delta(\mathrm{detred}_{\mathbb{R}[G]}(\Delta_T^{\#})).\end{aligned}$$

To prove this result we let $\hat{\Psi}^\cdot$ denote any complex of G-modules of the form $\hat{\Psi}^0 \xrightarrow{d} \hat{\Psi}^1$ where $\hat{\Psi}^0$ occurs in degree 0 and $e(\hat{\Psi}^\cdot) = c_{\mathcal{W},S}(K/k)$ in the notation of §4.1. We write Ψ^1 for the pullback of the natural surjection $\hat{\Psi}^1 \to H^1_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m)$ and a choice of section γ to the surjection $H^1_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \to X_K$ provided by Lemma 1iii) (such a section always exists under Hypothesis (S,T)ii)). In this way we obtain a complex $\Psi^\cdot$ of the form $\hat{\Psi}^0 \xrightarrow{\hat{d}} \Psi^1$ which satisfies $e(\Psi^\cdot) = \iota_S^{-1}(c_{\mathcal{W},S}(K/k)) \in \mathrm{Ext}^2_G(X_K, \mathcal{O}^\times_K)$ and lies in a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ of the form

$$\Psi^\cdot \xrightarrow{\alpha} R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m) \to A_K[-1] \to \Psi^\cdot[1],$$

where $H^0(\alpha)$ is the identity map and $H^1(\alpha) = \gamma$. Upon applying Lemma A2 to this triangle we obtain an equality

$$\chi(R\Gamma_{\mathcal{W}}(U_{K,S}, \mathbb{G}_m), \mathrm{R}_{K,S}) = \chi(\Psi^\cdot, \mathrm{R}_{K,S}) - \chi(A_K). \tag{18}$$

To compute $\chi(\Psi^\cdot, \mathrm{R}_{K,S})$ we shall first be more explicit about the computation of the group $\mathrm{Ext}^2_G(X_K, \mathcal{O}^\times_K)$. For each $j \in |n|$ one has an exact sequence

$$0 \to \mathbb{Z}[G] \cdot \mathrm{Tr}_j \xrightarrow{\subseteq} \mathbb{Z}[G] \xrightarrow{d_j} \mathbb{Z}[G] \xrightarrow{\theta_j} \mathbb{Z}[G](w_j - w_0) \to 0$$

where $d_j(x) = (g_j - 1)x$ and $\theta_j(x) = x(w_j - w_0)$ for each $x \in \mathbb{Z}[G]$. By taking the direct sum of these sequences over j in $|n|$ we obtain a resolution of $X_K = \bigoplus_{j\in|n|} \mathbb{Z}[G](w_j - w_0)$ of the form $0 \to \Sigma_K \xrightarrow{\subseteq} F \xrightarrow{d} F \xrightarrow{\theta} X_K \to 0$ in which $\Sigma_K := \bigoplus_{j\in|n|} \mathbb{Z}[G] \cdot \mathrm{Tr}_j, F = \bigoplus_{j\in|n|} \mathbb{Z}[G], d = \bigoplus_{j\in|n|} d_j$ and $\theta = \bigoplus_{j\in|n|} \theta_j$. When computing $\mathrm{Ext}^2_G(X_K, \mathcal{O}_K^\times)$ with respect to this resolution, we may choose an injective G-homomorphism $\phi : \Sigma_K \to \mathcal{O}_K^\times$ which represents $\iota_S^{-1}(c_{\mathcal{W},S}(K/k))$ [4, Lem. 2.4]. In addition, from Proposition 4.1, one has $\iota_S^{-1}(c_{\mathcal{W},S}(K/k)) = -c_S(K/k)$ and so Hypothesis (S,T)iv) allows us to assume that ϕ factors through a homomorphism $\phi_T : \Sigma_K \to \mathcal{O}_{K,T}^\times$. In this case one has $\phi_T = \bigoplus_{i\in|n|} \phi_j$ with $\phi_j \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \cdot \mathrm{Tr}_j, \mathcal{O}_{K,T}^\times)$ and so we set $\epsilon_j := \phi_j(\mathrm{Tr}_j) \in (\mathcal{O}_{K,T}^\times)^{G_j} = \mathcal{O}_{K_j,T}^\times$. Then, since ϕ represents $-c_S(K/k)$, the descriptions of [22, Prop. 3.2.1, Prop. 4.5.2] imply that condition i) is satisfied. (Note that the result of [22, Prop. 4.5.2] should state that $\psi_w(t_v(c))$ is equal to g^{-1} rather than g. Indeed, the compatibility of cup-product with connecting homomorphisms in Tate cohomology implies that (in the notation of the proof given in loc. cit.) cup-product with β is equal to the *negative* of the composite of the connecting homomorphisms $H^{-2}(G_v, \mathbb{Z}) \to H^{-1}(G_v, I_\Gamma a)$ and $H^{-1}(G_v, I_\Gamma a) \to H^0(G_v, \mathbb{Z}c)$ described there. See also the proof of [20, Cor. 2.1] in this regard.)

We next let $\hat{F}$ denote the push-out of ϕ and the inclusion map $\Sigma_K \xrightarrow{\subseteq} F$, and we write $F^\cdot$ and $\hat{F}^\cdot$ for the complexes $F \xrightarrow{d} F$ and $\hat{F} \xrightarrow{\hat{d}} F$ where (in both cases) the modules are placed in degrees 0 and 1 and $\hat{d}$ denotes the morphism induced by d. Then Lemma 5 combines with our choice of ϕ to imply that the complexes $\hat{F}^\cdot$ and $\Psi^\cdot$ are equivalent and hence there exists a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form

$$F^\cdot \xrightarrow{\beta} \Psi^\cdot \to \mathrm{cok}(\phi)[0] \to F^\cdot[1]$$

in which $H^0(\beta) = \phi$ and $H^1(\beta)$ is the identity map. Note that since both $F^\cdot$ and $\Psi^\cdot$ belong to $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ this triangle implies that $\mathrm{cok}(\phi)[0]$ (and hence also the triangle itself) belongs to $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$. In particular, it follows that the G-module $\mathrm{cok}(\phi)$ is both finite (since ϕ is injective) and c-t. In addition, we may apply Lemma A2 to the triangle to deduce that

$$\chi(\Psi^\cdot, \mathrm{R}_{K,S}) = \chi(F^\cdot, \mathrm{R}_{K,S} \circ \phi) + \chi(\mathrm{cok}(\phi)). \tag{19}$$

To compute $\chi(F^\cdot, \mathrm{R}_{K,S} \circ \phi)$ we observe that the differential of $F^\cdot$ is semisimple at 0, when considered as an endomorphism of F. Indeed, the submodule $D := \bigoplus_{j\in|n|} \mathbb{Q}[G]\cdot(g_j - 1)$ is a $\mathbb{Q}[G][d]$-equivariant direct complement to $\Sigma_K \otimes \mathbb{Q}$ in $F \otimes \mathbb{Q}$. We may therefore apply Lemma A1 with $P = F, R = \mathbb{Z}, E = \mathbb{R}, \phi = d, \lambda = \mathrm{R}_{K,S} \circ \phi$ and with ι_1, ι_2 equal to the sections which are induced by D. In this context, the definition of the elements y_{ij} in the statement of claim ii) implies that the restriction of the automorphism $\langle \lambda, \phi \rangle_{\iota_1,\iota_2}$ which occurs in Lemma A1 to the direct summand $\Sigma_K \otimes \mathbb{R}$, resp. $D \otimes_{\mathbb{Q}} \mathbb{R}$, of $F \otimes \mathbb{R}$ is the map which sends each Tr_j to $\sum_{i\in|n|} y_{ji}\,\mathrm{Tr}_i$, resp. is the map which is induced by multiplication by (g_j-1) on each summand $\mathbb{R}[G]\cdot(g_j-1)$. It follows that, with respect to a suitable ordered $\mathbb{R}[G]$-basis of $F\otimes\mathbb{R}$, the matrix of $\langle\lambda,\phi\rangle_{\iota_1,\iota_2}$ is equal to M_T and hence that M_T is invertible, as required by claim ii). In addition, in this case Lemma A1 implies that $\chi(F^{\cdot},\mathrm{R}_{K,S}\circ\phi)=\delta(\mathrm{detred}_{\mathbb{R}[G]}(M_T))$. Our proof of Proposition 6.1 is thus completed by combining this equality with (18), (19) and the following two results.

Lemma 9. *If the G-module $\mathrm{cok}(\phi_T)=\mathcal{O}_{K,T}^\times/\mathcal{E}$ is c-t, then so also are $\mathrm{cok}(\phi)=\mathcal{O}_K^\times/\mathcal{E}$, A_K and $A_{K,T}$, and in $K_0(\mathbb{Z}[G],\mathbb{R})$ one has*

$$\chi(\mathrm{cok}(\phi))-\chi(A_K)=\chi(\mathrm{cok}(\phi_T))-\chi(A_{K,T})-\delta(\mathrm{detred}_{\mathbb{R}[G]}(\Delta_T^{\#})).$$

Proof. We use the natural exact sequence of finite G-modules

$$0\to\mathrm{cok}(\phi_T)\xrightarrow{\subseteq}\mathrm{cok}(\phi)\to\bigoplus_{v\in T}\mathbb{F}_{(v)}^\times\to A_{K,T}\to A_K\to 0 \tag{20}$$

where $\mathbb{F}_{(v)}$ denotes the direct sum of the residue fields $\mathbb{F}_w$ of each place w of K which lies above v [31, (1.5)]. Let G_w denote the decomposition group of w in G. Then, if η is any generator of the cyclic group $\mathbb{F}_w^\times$, there exists a G_w-equivariant surjection $\mathbb{Z}[G_w]\to\mathbb{F}_w^\times$ which sends 1 to η. In this way one obtains an exact sequence $0\to\mathbb{Z}[G]\xrightarrow{1-\sigma_v^{-1}\mathrm{N}v}\mathbb{Z}[G]\to\mathbb{F}_{(v)}^\times\to 0$ of G-modules. These sequences combine to imply that the G-module $\bigoplus_{v\in T}\mathbb{F}_{(v)}^\times$ is c-t and moreover that $\chi(\bigoplus_{v\in T}\mathbb{F}_{(v)}^\times)=\sum_{v\in T}\chi(\mathbb{F}_{(v)}^\times)=-\sum_{v\in T}\delta(\mathrm{detred}_{\mathbb{R}[G]}(1-\sigma_v^{-1}\mathrm{N}v))=-\delta(\mathrm{detred}_{\mathbb{R}[G]}(\Delta_T^{\#}))$.
At this stage we know that all of the modules which occur in (20) are c-t, except possibly for $A_{K,T}$. The exactness of this sequence therefore implies that $A_{K,T}$ is also c-t. Finally, the claimed equality follows upon decomposing (20) into short exact sequences and then using Lemma A2 (repeatedly). □

Lemma 10. *The G-module $\mathrm{cok}(\phi_T)$ is c-t. Indeed, one has $\ell\nmid|\mathrm{cok}(\phi_T)|$.*

Proof. It suffices to prove that $\ell\nmid|\mathrm{cok}(\phi_T)^G|$. Now $\mathcal{E}\cong\Sigma_K$ so $H^1(G,\mathcal{E})\cong H^1(G,\Sigma_K)=0$. This implies $\mathrm{cok}(\phi_T)^G=(\mathcal{O}_{K,T}^\times/\mathcal{E})^G\cong\mathcal{O}_{k,T}^\times/\mathcal{E}^G$ and also that $\mathcal{E}^G$ is generated by $\{\mathrm{N}_j(\epsilon_j):j\in|n|\}$ where, for each $j\in|n|$, we write N_j for the field theoretic norm map $K_j^\times\to k^\times$.
We fix an ordered $\mathbb{Z}$-basis $\{u_i:i\in|n|\}$ of $U_{k,T}$ and define an element $b:=(b_{ij})$ of $\mathrm{M}_n(\mathbb{Z})$ by the equalities $\mathrm{N}_i(\epsilon_i)=\prod_{j=1}^n u_j^{b_{ij}}$ for each i,j in $|n|$. Then $|\mathcal{O}_{k,T}^\times/\mathcal{E}^G|=\pm\det(b)$ and so we must show that $\ell\nmid\det(b)$. To prove this we choose for each i,j in $|n|$, an integer a_{ij} such that $f_{K/k,w_i}(u_j)=g_i^{a_{ji}}$, we set $a:=(a_{ij})\in\mathrm{M}_n(\mathbb{Z})$ and we show that $b\cdot a\equiv I_n\ (\mathrm{mod}\ \ell\cdot\mathrm{M}_n(\mathbb{Z}))$.
For each intermediate field F of K/k we write J_F for the idele group of F and $f_F:J_F\to\mathrm{Gal}(K/F)$ for the global reciprocity map. For each $j\in|n|$ we write $f_{F,j}:F^\times\to\mathrm{Gal}(K/F)$ for the composite of f_F and the natural inclusion of $F^\times$ into $\prod_s F_s^\times\subset J_F$ where the product is taken over the set of places s of F which lie above v_j. We note that if $F=k$, then $f_{F,j}=f_{K/k,w_j}$.

For each pair of elements i,j of $|n|$ we set $S(ij) := \{w \in S(i) : w \mid v_j\}$. Then property i) in the statement of the Proposition implies $f_{K_i,j}(\epsilon_i) = \prod_{w\in S(ij)} f_{K/K_i,w}(\epsilon_i) = \prod_{w\in S(ij)} g_i^{\delta_{iw}} = g_i^{\delta_{ij}}$. After taking account of the functorial behaviour of Artin maps, this implies $g_i^{\delta_{ij}} = f_{K/k,w_j}(\mathrm{N}_i(\epsilon_i)) = \prod_{s\in|n|} f_{K/k,w_j}(u_s)^{b_{is}} = \prod_{s\in|n|} g_j^{b_{is}a_{sj}} = g_j^{\sum_{s\in|n|} b_{is}a_{sj}}$ and hence, since by assumption no element g_j is trivial, that $\sum_{s\in|n|} b_{is}a_{sj} \equiv \delta_{ij} \pmod{\ell}$. It follows that $b\cdot a \equiv I_n \pmod{\ell\cdot M_n(\mathbb{Z})}$, as required. □

6.4. The proof of Theorem 6.1. In this subsection we use Proposition 6.1 to prove Theorem 6.1. We assume throughout that G is abelian. Our argument is similar to that used in §5.4 and so we continue to use the notation $G^*_{(0)}$ and e_0 introduced in that subsection.

At the outset we observe that if G_j is trivial for any $j\in|n|$, then $\theta_{K/k,S,T}(0)$ and $\mathrm{Reg}_{G,S,T}$ are both equal to 0 and so the congruence of Theorem 6.1 is valid trivially. In the sequel we shall therefore assume that G_j is not trivial for any $j\in|n|$, as is required by Proposition 6.1.

Now, since G is abelian, Proposition 6.1iii) shows that $\mathrm{C}(K/k)$ implies the existence of an element x_T of $\mathbb{Q}[G]^\times$ which satisfies both

$$\theta^*_{K/k,S,T}(0)^{\#} = x_T\cdot\det(M_T)\in\mathbb{R}[G]^\times \tag{21}$$

and

$$\mathbb{Z}[G]\cdot x_T = \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,T}/\mathcal{E})^{-1}\,\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,T}) \subseteq \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{K,T}/\mathcal{E})^{-1}. \tag{22}$$

For all i,j in $|n|$ one has $e_0\cdot\mathrm{Tr}_j = 0$ so that $e_0y_{ij}=0$ and hence $(M_Te_0)_{ij} = \delta_{ij}(g_i-1)e_0$. Also, for each $\chi\in G^*\setminus G^*_{(0)}$ there exists $j\in|n|$ such that $\chi(g_j)=1$ and so $\prod_{i\in|n|}(g_i-1)(1-e_0)=0$. It follows that

$$\begin{aligned} e_0\det(M_T) &= \det(M_Te_0)\\ &= \prod_{i\in|n|}(g_i-1)e_0\\ &= \prod_{i\in|n|}(g_i-1). \end{aligned}$$

This combines with (21) to imply that $\theta_{K/k,S,T}(0)^{\#} = x_T\cdot e_0\det(M_T) = x_T\prod_{i\in|n|}(g_i-1)$. In addition, (22) combines with Lemma 10 to imply that $x_T\in\mathbb{Z}_\ell[G]$ and hence one has

$$\theta_{K/k,S,T}(0)^{\#} \equiv \epsilon(x_T)\prod_{i\in|n|}(g_i-1) \pmod{I_G\cdot\prod_{i\in|n|}I_i}.$$

Since $\mathrm{Reg}_{G,S,T}\in\prod_{i\in|n|}I_i$ one also has

$$(\mathrm{Reg}_{G,S,T})^{\#} \equiv (-1)^n\mathrm{Reg}_{G,S,T} \pmod{I_G\cdot\prod_{i\in|n|}I_i}.$$

To deduce the congruence of Theorem 6.1 from the previous displayed congruence we therefore need only show that

$$\epsilon(x_T)\prod_{i\in|n|}(g_i-1)\equiv(-1)^n m_{k,S,T}\cdot\mathrm{Reg}_{G,S,T}\pmod{I_G\cdot\prod_{i\in|n|}I_i}$$

where $m_{k,S,T}$ is as defined in (11). In addition, with the matrix b as defined in the proof of Lemma 10, one has

$$\begin{aligned}\prod_{i\in|n|}(g_i-1) &\equiv \det((f_{K/k,w_j}(\mathrm{N}_i(\epsilon_i))-1)_{1\le i,j\le n})\\ &\equiv \det(b)\cdot\det((f_{K/k,w_j}(u_i)-1)_{1\le i,j\le n})\\ &\equiv \det(b)\cdot\mathrm{Reg}_{G,S,T}\ (\mathrm{mod}\ I_G\cdot\prod_{j\in|n|}I_j),\end{aligned}$$

and so it suffices to show that $\epsilon(x_T)\cdot\det(b)=(-1)^n m_{k,S,T}$. But, just as in the deduction of (17) from (13), this can be proved by first multiplying (21) by Tr_G and then comparing the resulting equality to (11).
This completes our proof of Theorem 6.1.

Appendix

We recall some relevant properties of the refined Euler characteristic construction discussed in §2.1 (the notation of which we continue to use). For further details we refer the reader to [9] (or to [7, §1] for a fuller review than that given here).
We let R denote either $\mathbb{Z}$ or $\mathbb{Z}_\ell$ for some prime ℓ and E an extension of the field of fractions of R. For any $R[G]$-module M, resp. homomorphism of R-modules ϕ, we set $M_E:=M\otimes_R E$, resp. $\phi_E:=\phi\otimes_R \mathrm{id}_E$.
Let $P^\cdot$ be a bounded complex of finitely generated projective $R[G]$-modules. For each integer i we let B^i, resp. Z^i, denote the submodules of coboundaries, resp. cocycles, of $P^\cdot_E$ in degree i. After choosing $E[G]$-equivariant splittings of the tautological exact sequences $0\to Z^i\to P^i_E\to B^{i+1}\to 0$ and $0\to B^i\to Z^i\to H^i(P^\cdot_E)\to 0$ one obtains non-canonical isomorphisms

$$\begin{aligned}P^+_E &\cong B^{\mathrm{all}}\oplus H^+(P^\cdot)_E\\ P^-_E &\cong B^{\mathrm{all}}\oplus H^-(P^\cdot)_E.\end{aligned}$$

By using the identity map on B^{all} one can therefore extend each element ϕ of $\mathrm{Is}_{E[G]}(H^+(P^\cdot)_E,H^-(P^\cdot)_E)$ to give an element $\phi(P^\cdot_E)$ of $\mathrm{Is}_{E[G]}(P^+_E,P^-_E)$. This construction clearly depends upon the above choice of splittings but nevertheless induces a well-defined map from $\mathrm{Is}_{E[G]}(H^+(P^\cdot)_E,H^-(P^\cdot)_E)/\sim$ to $\mathrm{Is}_{E[G]}(P^+_E,P^-_E)/\sim$ which is independent of all such choices. We denote this map by $\tau\mapsto\tau(P^\cdot_E)$ and we obtain a well-defined element of $K_0(R[G],E)$ by setting $\chi_{R[G],E}(P^\cdot,\tau):=(P^+,\phi,P^-)$ for any (and therefore every) $\phi\in\tau(P^\cdot_E)$.
In the following result we record this construction in a special case.

Lemma A1. *Let P be a finitely generated projective $R[G]$-module, ϕ an $R[G]$-endomorphism of P and $\lambda:\ker(\phi)_E\to\mathrm{cok}(\phi)_E$ an $E[G]$-isomorphism. Choose*

$E[G]$-equivariant sections ι_1 and ι_2 to the tautological surjections $P_E \to \mathrm{im}(\phi)_E$ and $P_E \to \mathrm{cok}(\phi)_E$, and let $\langle\lambda,\phi\rangle_{\iota_1,\iota_2}$ denote the automorphism of P_E which is equal to $\iota_2\circ\lambda$ on $\ker(\phi)_E$ and to ϕ_E on $\iota_1(\mathrm{im}(\phi)_E)$. If $P^\cdot$ denotes the complex $P \xrightarrow{\phi} P$, where the first term is placed in degree 0, then in $K_0(R[G],E)$ one has $\chi_{R[G],E}(P^\cdot,\lambda) = \partial^1_{R[G],E}([\langle\lambda,\phi\rangle_{\iota_1,\iota_2}])$.

For each $i \in \{1,2,3\}$ let $P_i^\cdot$ be a bounded complex of finitely generated projective $R[G]$-modules. We assume that there exists a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(R[G])$ of the form

$$P_1^\cdot \xrightarrow{\alpha} P_2^\cdot \to P_3^\cdot \to P_1^\cdot[1]$$

and that $P_{3,E}^\cdot$ is acyclic (so that $H^i(\alpha)_E$ is bijective in each degree i). For any E-trivialisation τ of $P_1^\cdot$ we let τ_α denote the unique E-trivialisation of $P_2^\cdot$ that contains $H^-(\alpha)_E\circ\phi\circ H^+(\alpha)_E^{-1}$ for any (and therefore every) $\phi\in\tau$. The following result is a special case of [9, Th. 2.8].

Lemma A2. *If $P_{3,E}^\cdot$ is acyclic, then for any E-trivialisation τ of $P_1^\cdot$ one has*

$$\chi_{R[G],E}(P_2^\cdot,\tau_\alpha) = \chi_{R[G],E}(P_1^\cdot,\tau) + \chi_{R[G],E}(P_3^\cdot,\mathrm{id}_0),$$

where id_0 *denotes the identity map on the zero space.*

Note that if $P_3^\cdot$ is acyclic, then $\chi_{R[G],E}(P_3^\cdot,\mathrm{id}_0) = 0$ and so Lemma A2 implies $\chi_{R[G],E}(\cdot,\cdot)$ is well-defined on pairs of the form (X,τ) where X is an object of $\mathcal{D}^{\mathrm{perf}}(R[G])$ and τ an element of $\mathrm{Is}_{E[G]}(H^+(X)_E, H^-(X)_E)/\sim$.

Remark A1. The element $\chi_{R[G],E}(X,\tau)$ of $K_0(R[G],E)$ constructed above can be naturally reinterpreted as an isomorphism class of objects in a fibre product category involving a suitable category of virtual objects as introduced by Deligne in [28]. (Indeed, this more conceptual approach has important technical advantages and is used systematically in [13]). As a result, if G is abelian, then $\chi_{R[G],E}(X,\tau)$ can also be described by using the graded determinant functor of Grothendieck, Knudsen and Mumford (that is described in [36]). In fact, if G is abelian, then $K_0(R[G],E)$ identifies naturally with the multiplicative group of invertible $R[G]$-lattices in $E[G]$ (cf. [4, Lem. 2.6]), the reduced norm map $\mathrm{nr}_{E[G]} : K_1(E[G]) \to E[G]^\times$ is bijective and, with respect to the stated identification, for each $x\in E[G]^\times$ one has

$$\partial^1_{R[G],E}(\mathrm{nr}^{-1}_{E[G]}(x)) = R[G]\cdot x \subset E[G].$$

This shows in particular that, if K/k is abelian, then the equality of $\mathrm{C}(K/k)$ is equivalent to a formula for the sublattice $\mathbb{Z}[G]\cdot\theta^*_{K/k,S}(0)^\#$ of $\mathbb{R}[G]$.

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David Burns
Dept. of Mathematics
King's College London
London WC2R 2LS
United Kingdom
david.burns@kcl.ac.uk