Nodal Domain Theorems à la Courant

A. Ancona, B. Helffer, and T. Hoffmann-Ostenhof*

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Abstract. Let $H(\Omega_0) = -\Delta + V$ be a Schrödinger operator on a bounded domain $\Omega_0 \subset \mathbb{R}^d$ $(d \geq 2)$ with Dirichlet boundary condition. Suppose that Ω_ℓ $(\ell \in \{1, \dots, k\})$ are some pairwise disjoint subsets of Ω_0 and that $H(\Omega_\ell)$ are the corresponding Schrödinger operators again with Dirichlet boundary condition. We investigate the relations between the spectrum of $H(\Omega_0)$ and the spectra of the $H(\Omega_\ell)$. In particular, we derive some inequalities for the associated spectral counting functions which can be interpreted as generalizations of Courant's nodal theorem. For the case where equality is achieved we prove converse results. In particular, we use potential theoretic methods to relate the Ω_ℓ to the nodal domains of some eigenfunction of $H(\Omega_0)$.

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1 Introduction

Consider a Schrödinger operator

$$H = -\Delta + V \tag{1.1}$$

on a bounded domain $\Omega_0 \subset \mathbb{R}^d$ with Dirichlet boundary condition. Further we assume that V is real valued and satisfies $V \in L^\infty(\Omega_0)$. (We could relax this condition and extend our results to the case $V \in L^\beta(\Omega_0)$ for some $\beta > d/2$ using [11].)
The operator H is selfadjoint if viewed as the Friedrichs extension of the quadratic form of H with form domain $W_0^{1,2}(\Omega_0)$ and form core $C_0^\infty(\Omega_0)$ and

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we denote it by $H(\Omega_0)$. Further $H(\Omega_0)$ has compact resolvent. So the spectrum of $H(\Omega_0)$, $\sigma\big(H(\Omega_0)\big)$, can be described by an increasing sequence of eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \dots \tag{1.2}$$

tending to $+\infty$, such that the associated eigenfunctions u_j form an orthonormal basis of $L^2(\Omega_0)$. We can assume that these eigenfunctions u_j are real valued and by elliptic regularity, [9] (Corollary 8.36), u_j belongs to $C^{1,\alpha}(\Omega_0)$ for every $\alpha < 1$. Moreover λ_1 is simple and the corresponding eigenfunction u_1 can be chosen to satisfy, see e.g. [17],

$$u_1(x) > 0\ , \ \text{ for all } x \in \Omega_0\ . \tag{1.3}$$

For a bounded domain D we let $H(D)$ be the corresponding selfadjoint operator, with Dirichlet boundary condition on ∂D. Its lowest eigenvalue will be denoted by $\lambda(D)$.
We denote the zero set of an eigenfunction u by

$$N(u) = \overline{\{x \in \Omega_0 \mid u(x) = 0\}}. \tag{1.4}$$

The nodal domains of u, which are by definition the connected components of $\Omega_0 \setminus N(u)$, will be denoted by $D_j, j = 1, \dots, \mu(u)$, where $\mu(u)$ denotes the number of nodal domains of u.
Suppose that Ω_ℓ $(\ell = 1, 2, \dots, k)$ are k open pairwise disjoint subsets of Ω_0. In this paper we shall investigate relations between the spectrum of $H(\Omega_0)$ and the spectra of the $H(\Omega_\ell)$. Roughly speaking, we shall derive an inequality between the counting function of $H(\Omega_0)$ and those of the $H(\Omega_\ell)$. This inequality can be interpreted as a generalization of Courant's classical nodal domain theorem. For the case where equality is achieved this will lead to a partial characterization of the Ω_ℓ which will turn out to be related to the nodal domains of one of the eigenfunctions of $H(\Omega_0)$.
These results will be given in sections 2 and 3. From these results some natural questions of potential theoretic nature arise which will be analyzed and answered in section 7.
The proofs of the results stated in sections 2 and 3 are given in sections 4 and 5. In section 6 some illustrative explicit examples are given.

2 Main results

We start with a result which will turn out to be a generalization of Courant's nodal theorem. We consider again (1.1) on a bounded domain Ω_0 and the corresponding eigenfunctions and eigenvalues. We first introduce

$$\overline{n}(\lambda, \Omega_0) = \#\{j \mid \lambda_j(\Omega_0) \leq \lambda\}, \tag{2.1}$$

where $\lambda_j(\Omega_0)$ is the j-th eigenvalue of $H(\Omega_0)$.
We also define

$$\underline{n}(\lambda, \Omega_0) = \#\{j \mid \lambda_j(\Omega_0) < \lambda\}, \tag{2.2}$$

and

$$n(\lambda,\Omega_0)=\begin{cases}\underline{n}(\lambda,\Omega_0) & \text{if } \lambda\notin\sigma\big(H(\Omega_0)\big)\\ \underline{n}(\lambda,\Omega_0)+1 & \text{if } \lambda\in\sigma\big(H(\Omega_0)\big).\end{cases} \tag{2.3}$$

So we always have :

$$\underline{n}(\lambda,\Omega_0)\le n(\lambda,\Omega_0)\le \overline{n}(\lambda,\Omega_0), \tag{2.4}$$

with equality when λ is not an eigenvalue. Note that $\overline{n}(\lambda,\Omega_0)-\underline{n}(\lambda,\Omega_0)$ is the multiplicity of λ when λ is an eigenvalue of $H(\Omega_0)$, i.e. the dimension of the eigenspace associated to λ. We shall consider a family of k open sets Ω_ℓ ($\ell=1,\dots,k$) contained in Ω_0 and the corresponding Dirichlet realizations $H(\Omega_\ell)$. For each $H(\Omega_\ell)$ the corresponding eigenvalues counted with multiplicity are denoted by $(\lambda_j^\ell)_{j\in\mathbb{N}\setminus\{0\}}$ (with $\lambda_j^\ell\le\lambda_{j+1}^\ell$). When counting the eigenvalues less than some given λ , we shall for simplicity write

$$n_\ell=n_\ell(\lambda)=n(\lambda,\Omega_\ell), \tag{2.5}$$

and analogously for the quantities with over-, respectively, underbars.

Theorem 2.1
Suppose $\Omega_0\subset\mathbb{R}^d$ is a bounded domain and that $\lambda\in\sigma\big(H(\Omega_0)\big)$. Suppose that the sets Ω_ℓ ($\ell=1,\dots,k$) are pairwise disjoint open subsets of Ω_0. Then

$$\sum_{\ell=1}^{k}\overline{n}_\ell\le n_0+\min_{\ell\ge 0}\left(\overline{n}_\ell-n_\ell\right) . \tag{2.6}$$

A direct weaker consequence of (2.6) is the more standard

Corollary 2.2
Under the assumptions of Theorem 2.1, we have

$$\sum_{\ell=1}^{k}\overline{n}_\ell\le\overline{n}_0 . \tag{2.7}$$

This corollary is actually present in the proofs of the asymptotics of the counting function (see for example the Dirichlet-Neumann bracketing in Lieb-Simon [14]).

Remark 2.3
Inequality (2.6) is also true if $\lambda\notin\sigma\big(H(\Omega_0)\big)$. The statement becomes

$$\sum_{\ell=1}^{k}\overline{n}_\ell\le n_0 ,$$

and is proved essentially in the same way.

Remark 2.4
The assumption that Ω_0 is connected is necessary. Indeed, suppose Ω_1 and Ω_2 are connected and assume that $\Omega_0 = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and that $\lambda = \lambda_1(\Omega_1) = \lambda_1(\Omega_2)$. Then $\lambda_1(\Omega_0) = \lambda_2(\Omega_0)$ and we deduce $n(\lambda, \Omega_0) = 1$. If we no longer assume the connectedness of Ω_0, we in general just have Corollary 2.2.

Finally we show that Courant's nodal theorem is an easy corollary of Theorem 2.1.

Corollary 2.5 : Courant's nodal theorem
If Ω_0 is connected and if u is an eigenfunction of $H(\Omega_0)$ associated to some eigenvalue λ, then

$$\mu(u) \leq n(\lambda, \Omega_0) .$$

Proof.
We now simply apply Theorem 2.1 by taking $\Omega_1, \ldots, \Omega_{\mu(u)}$ as the nodal domains associated to u. We just have to use (1.3) for each Ω_ℓ, $\ell = 1, \ldots, \mu(u)$, which gives $\overline{n}_\ell = n_\ell = 1$. □

Remark 2.6
Courant's nodal theorem is one of the basic results in spectral theory of Schrödinger-type operators. It is the natural generalization of Sturm's oscillation theorem for second order ODE's. For recent investigations see for instance [1] and [4].

3 Converse results.

In this section we consider some results that are converse to Theorem 2.1.

Theorem 3.1
Suppose that the Ω_ℓ, $1 \leq \ell \leq k$, are pairwise disjoint open subsets of Ω_0. If $\lambda \in \sigma\big(H(\Omega_0)\big)$ and

$$\sum_{\ell=1}^{k} \overline{n}_\ell \geq n_0 , \tag{3.1}$$

then $\lambda \in \sigma(H(\Omega_\ell))$ for each Ω_ℓ. If $U_\ell(\lambda)$ denotes the eigenspace of $H(\Omega_\ell)$ associated to the eigenvalue λ, then there is an eigenfunction u of $H(\Omega_0)$ with eigenvalue λ such that

$$u = \sum_{\ell=1}^{k} \varphi_\ell \text{ in } W_0^{1,2}(\Omega_0) , \tag{3.2}$$

where each φ_ℓ belongs to $U_\ell(\lambda) \setminus \{0\}$ and is identified with its extension by 0 outside Ω_ℓ.

Remark 3.2
One can naturally think that formula (3.2) *has immediate consequences on the family* Ω_ℓ*, which should for example have some covering property. The question is a bit more subtle because we do not a priori want to assume strong regularity properties for the boundaries of the* Ω_ℓ*. We shall discuss this point in detail in the last section.*

Another consequence of equalities in Theorems 2.1 or 3.1 is given by the following result.

Theorem 3.3
Suppose that, for some bounded domain Ω_0 *in* $\mathbb{R}^d$*, some* $\lambda \in \sigma(H(\Omega_0))$ *and some family of pairwise disjoint open sets* $\Omega_\ell \subset \Omega_0$*,* $0 < \ell \leq k$*, we have*

$$\sum_{\ell=1}^{k} \overline{n}_\ell = n_0 + \min_{\ell \geq 0} \left(\overline{n}_\ell - n_\ell\right) . \tag{3.3}$$

Then, for any subset $L \subset \{1, 2, \ldots, k\}$ *such that* $\Omega_L^* = \mathrm{Int}\left(\cup_{\ell \in L} \overline{\Omega_\ell}\right) \setminus \partial\Omega_0$ *is connected, we have*

$$\sum_{\ell \in L} \overline{n}_\ell = n(\lambda, \Omega_L^*) + \min\left(\min_{\ell \in L}\left(\overline{n}_\ell - n_\ell\right),\, \overline{n}(\lambda, \Omega_L^*) - n(\lambda, \Omega_L^*)\right) . \tag{3.4}$$

A simpler variant is the following :

Theorem 3.4
Suppose (3.1) *holds and that* Ω_L^* *is defined as above. Then we have the inequality :*

$$\sum_{\ell \in L} \overline{n}_\ell \geq n(\lambda, \Omega_L^*) . \tag{3.5}$$

On the sharpness of Courant's nodal theorem
It is well known that Courant's nodal theorem is sharp only for finitely many k's [15].
Let Ω_0 be connected. We will say that an eigenfunction u associated to an eigenvalue λ of $H(\Omega_0)$ is Courant-sharp if $\mu(u) = n(\lambda, \Omega_0)$. Theorem 3.3 now implies :

Corollary 3.5
i) Let u *be a* Courant-sharp *eigenfunction of* $H(\Omega_0)$ *with* $\mu(u) = k$*. Let* $\{D_i\}_{i=1,\ldots,k}$ *be the family of the nodal domains associated to* u*, let* L *be a subset of* $\{1, \ldots, k\}$ *with* $\#L = \ell$ *and let* $\Omega_L^* = \mathrm{Int}\,(\overline{\cup_{i \in L} D_i}) \setminus \partial\Omega_0$*. Then*

$$\lambda_\ell(\Omega_L^*) = \lambda_k , \tag{3.6}$$

where $\lambda_j(\Omega_L^*)$ *are the eigenvalues of* $H(\Omega_L^*)$*.*
ii) Moreover, if Ω_L^* *is connected, and if* $\ell < k$*, then* $u\big|_{\Omega_L^*}$ *is* Courant-sharp *and* $\lambda_\ell(\Omega_L^*)$ *is simple.*

4 Basic tools

Let us first recall some basic tools (see e.g. [17]) which were already vital for the proof of Courant's classical result.

4.1 Variational characterization

Let us first recall the variational characterization of eigenvalues.

Proposition 4.1
Let Ω be a bounded open set in $\mathbb{R}^d$ and let $V \in L^\infty(\Omega)$ be real-valued. Suppose $\lambda \in \sigma\big(H(\Omega)\big)$ and let $\mathcal{U}_\pm = span\ \langle u_1, \dots, u_{k_\pm}\rangle$ where

$$k_- = \underline{n}(\lambda, \Omega)\ ,\ k_+ = \overline{n}(\lambda, \Omega)\ , \tag{4.1}$$

and $(u_j)_{j\geq 1}$ is as before an orthonormal basis of eigenfunctions of $H(\Omega)$ associated to $(\lambda_j)_{j\geq 1}$. Then

$$\lambda = \inf_{\varphi \perp \mathcal{U}_-,\, \varphi \in W_0^{1,2}(\Omega)} \frac{\langle \varphi,\ H(\Omega)\varphi\rangle}{\|\varphi\|^2} \tag{4.2}$$

and

$$\lambda < \lambda_{\overline{n}(\lambda,\, \Omega)+1} = \inf_{\varphi \perp \mathcal{U}_+,\, \varphi\, \in W_0^{1,2}(\Omega)} \frac{\langle \varphi,\ H(\Omega)\varphi\rangle}{\|\varphi\|^2}. \tag{4.3}$$

If equality is achieved in (4.2) for some $\varphi \not\equiv 0$, then φ is an eigenfunction in the eigenspace of λ.

Note that (4.2) and (4.3) are actually the same statement. We just stated them separately for later reference. Note that we have not assumed that Ω is connected.

4.2 Unique continuation

Next we restate a weak form of the unique continuation property:

Theorem 4.2
Let Ω be an open set in $\mathbb{R}^d$ and let $V \in L^\infty_{loc}(\Omega)$ be real-valued. Then any distributional solution in Ω to $(-\Delta + V)u = \lambda u$ which vanishes on an open subset ω of Ω is identically zero in the connected component of Ω containing ω.

There are stronger results of this type under weaker assumptions on the potential, see [11].

4.3 A consequence of Harnack's inequality

The standard Harnack's inequality (see e.g. Theorem 8.20 in [9]), together with the unique continuation theorem leads to the following theorem :

Theorem 4.3
If Ω is a bounded domain in $\mathbb{R}^d$ and u is an eigenfunction of $H(\Omega)$, then for any x in $N(u) \cap \Omega$ and any ball $B(x,r)$ $(r>0)$, there exist $y_\pm \in B(x,r)\cap\Omega$ such that $\pm u(y_\pm) > 0$.

5 Proof of the main theorems

5.1 Proof of Theorem 2.1

Assume first for contradiction that

$$\sum_{\ell\geq 1} \overline{n}_\ell > n_0 + \min_{\ell\geq 0}\left(\overline{n}_\ell - n_\ell\right) , \tag{5.1}$$

and recall that we assume that $\lambda \in \sigma\big(H(\Omega_0)\big)$. Pick some ℓ_0 such that

$$\overline{n}_{\ell_0} - n_{\ell_0} = \min_{\ell\geq 0}\left(\overline{n}_\ell - n_\ell\right) .$$

Suppose first that $\ell_0 \geq 1$.
We can rewrite (5.1) to obtain

$$\sum_{\ell\neq\ell_0,\ \ell\geq 1} \overline{n}_\ell + n_{\ell_0} > n_0 . \tag{5.2}$$

Let $\varphi_i^{\ell_0}$, $i = 1,\dots,\underline{n}(\lambda,\Omega_{\ell_0})$, denote the first $\underline{n}_{\ell_0}$ eigenfunctions of $H(\Omega_{\ell_0})$. The corresponding eigenvalues are strictly smaller than λ. The functions $\varphi_i^{\ell_0}$ and the remaining $\sum_{\ell\neq\ell_0}\overline{n}_\ell$ eigenfunctions associated to the other $H(\Omega_\ell)$ span a space of dimension at least n_0. We can pick a linear combination $\Phi \not\equiv 0$ of these functions which is orthogonal to the $\underline{n}_0$ eigenfunctions of $H(\Omega_0)$. By assumption

$$\frac{\langle \Phi, H(\Omega_0)\Phi\rangle}{\|\Phi\|^2} \leq \lambda, \tag{5.3}$$

hence Φ must by the variational principle be an eigenfunction and there must be equality in (5.3).
There are two possibilities: either some $\varphi_i^{\ell_0}, i < n_{\ell_0}$ contributes to the linear combination which makes up Φ or not. In the first case this means that the left hand side of (5.3) is strictly smaller than λ, contradicting the variational characterization of λ. In the other case we obtain a contradiction to unique continuation, since then $\Phi \equiv 0$ in Ω_{ℓ_0} and hence Φ vanishes identically in all of Ω_0.

Consider now the case when $\ell_0 = 0$.
We have to show that the assumption

$$\sum_{\ell \geq 1} \overline{n}_\ell > \overline{n}_0 \ , \tag{5.4}$$

leads to a contradiction. To this end it suffices to apply (4.3). Indeed, we can find a linear combination Φ of the eigenfunctions φ_j^ℓ, $j \leq \overline{n}_\ell$, corresponding to the different $H(\Omega_\ell)$ such that $\Phi \perp \mathcal{U}_+$, $\Phi \not\equiv 0$, but Φ satisfies

$$\frac{\langle \Phi,\ H(\Omega_0)\ \Phi \rangle}{\|\Phi\|^2} \leq \lambda = \lambda_{\overline{n}_0} \ ,$$

and this contradicts (4.3). This proves (2.6).

5.2 Proof of Theorem 3.1

The inequality (3.1) implies that we can find a non zero $u \perp \mathcal{U}_-$ in the span of the eigenfunctions φ_j^ℓ, $j = 1, \dots \overline{n}_\ell$, of the different $H(\Omega_\ell)$. Again by the variational characterization, (4.2) and (5.3) hold and hence u must be an eigenfunction. $\square$

5.3 Proof of Theorem 3.3

We assume (3.3). Without loss we might assume that we have labeled the Ω_ℓ such that $L = \{1, \dots, K\}$, with $K \leq k$. Let $n_* = n(\lambda, \Omega_L^*)$. We apply Theorem 2.1 to the family Ω_ℓ ($\ell \in L$) and replace Ω_0 by Ω_L^* and obtain :

$$\sum_{1 \leq \ell \leq K} \overline{n}_\ell \leq n_* + \min \left(\overline{n}_* - n_*,\ \min_{1 \leq \ell \leq K} (\overline{n}_\ell - n_\ell) \right) . \tag{5.5}$$

We assume for contradiction that

$$\sum_{1 \leq \ell \leq K} \overline{n}_\ell < n_* + \min \left(\overline{n}_* - n_*,\ \min_{1 \leq \ell \leq K} (\overline{n}_\ell - n_\ell) \right) . \tag{5.6}$$

This implies

$$\sum_{1 \leq \ell \leq K} \overline{n}_\ell < \overline{n}_* \ , \tag{5.7}$$

and

$$\sum_{1 \leq \ell \leq K} \overline{n}_\ell < n_* + \min_{1 \leq \ell \leq K} (\overline{n}_\ell - n_\ell) \ . \tag{5.8}$$

Theorem 2.1, applied to the family Ω_L^*, Ω_ℓ ($\ell > K$), implies that

$$\overline{n}_* + \sum_{K < \ell \leq k} \overline{n}_\ell \leq n_0 + \min \left(\overline{n}_0 - n_0,\ \min_{K < \ell \leq k} (\overline{n}_\ell - n_\ell) \right) , \tag{5.9}$$

and

$$n_* + \sum_{K<\ell\leq k} \overline{n}_\ell \leq n_0 \, . \tag{5.10}$$

By adding (5.7) and (5.9), we get :

$$\sum_{1\leq\ell\leq k} \overline{n}_\ell < n_0 + \min\left(\overline{n}_0 - n_0,\ \min_{K<\ell\leq k}(\overline{n}_\ell - n_\ell)\right) . \tag{5.11}$$

By adding (5.8) and (5.10), we obtain

$$\sum_{1\leq\ell\leq k} \overline{n}_\ell < n_0 + \min_{1\leq\ell\leq K}(\overline{n}_\ell - n_\ell) \, . \tag{5.12}$$

The combination of (5.11) and (5.12) is in contradiction with (3.3).

5.4 Proof of Theorem 3.4

For the case that (3.1) holds, (3.5) can be shown similarly. (3.1) reads

$$\sum_{1\leq\ell\leq k} \overline{n}_\ell \geq n_0 \, .$$

We assume for contradiction that

$$\sum_{1\leq\ell\leq K} \overline{n}_\ell < n_* \, , \tag{5.13}$$

where n_* is defined as above. The addition of (5.10) and (5.13) leads to a contradiction. □

6 Illustrative examples

6.1 Examples for a rectangle

We illustrate Theorem 2.1 by the analysis of various examples in rectangles. Pick a rectangle $\Omega_0 = (0, 2\pi) \times (0, \pi)$ and take $\Omega_1 = (0, \pi) \times (0, \pi)$ and consequently $\Omega_2 = (\pi, 2\pi) \times (0, \pi)$. The eigenvalues corresponding to Ω_0 for $-\Delta$ with Dirichlet boundary condition are given by

$$\sigma\big(H(\Omega_0)\big) = \left\{\lambda \in \mathbb{R} \,\middle|\, \lambda = m^2/4 + n^2,\ (m,n) \in \mathbb{Z}^2,\ m,n > 0\right\} , \tag{6.1}$$

while those for Ω_1, and hence for Ω_2 which can be obtained by a translation of Ω_1, are given by

$$\sigma\big(H(\Omega_1)\big) = \sigma\big(H(\Omega_2)\big) = \left\{\lambda \in \mathbb{R} \middle| \lambda = m^2 + n^2,\ (m,n) \in \mathbb{Z}^2, m,n > 0\right\}. \tag{6.2}$$

Denote the eigenvalues associated to Ω_0 by $\{\lambda_i\}$ and those to Ω_1 by $\{\nu_i\}$. We easily check that $\lambda_5 = \lambda_6 = \nu_2 = \nu_3 = 5$, $\lambda_{11} = \lambda_{12} = \nu_5 = \nu_6 = 10$ so that Theorem 2.1 is sharp for these cases.
One could ask whether there are arbitrarily high eigenvalues cases for which we have equality in (2.6). This is not the case, as can be seen from the following standard number theoretical considerations. We have (see [18] and for more recent contributions [16] and [2]) the following asymptotic estimate for the number of lattice points in an ellipse. Let $a, b > 0$, then

$$A(\lambda) := \#\left\{(m,n) \in \mathbb{Z}^2 \,\middle|\, am^2 + bn^2 \leq \lambda\right\} \tag{6.3}$$

has the following asymptotics as λ tends to infinity:

$$A(\lambda) = \frac{\pi}{\sqrt{ab}}\lambda + \mathcal{O}(\lambda^{1/3}). \tag{6.4}$$

We have not to consider $A(\lambda)$ but rather

$$A^+ = \#\left\{(m,n) \in \mathbb{Z}^2, m,n > 0 \,\middle|\, am^2 + bn^2 \leq \lambda\right\}. \tag{6.5}$$

Hence we get

$$\begin{aligned} A(\lambda) = 4A^+(\lambda) + 2\#\left\{m \in \mathbb{N},\ m > 0 \,\middle|\, m \leq \left[(\lambda/a)^{1/2}\right]\right\} \\ +2\#\left\{n \in \mathbb{N},\ n > 0 \,\middle|\, n \leq \left[(\lambda/b)^{1/2}\right]\right\} + 1\,. \end{aligned} \tag{6.6}$$

If we apply this to A^+ with $a = 1/4, b = 1$ (in this case denoted by A_0^+) and to A^+ with $a = 1, b = 1$ (in this case denoted by A_1^+), we get asymptotically

$$A_0^+(\lambda) - 2A_1^+(\lambda) = \frac{1}{2}\sqrt{\lambda} + o\,(\sqrt{\lambda})\,. \tag{6.7}$$

Note that

$$\overline{n}_i(\lambda) = A_i^+(\lambda),\ i = 0, 1\,.$$

In order to control $n_i(\lambda)$, we observe that, for any $\epsilon > 0$:

$$\overline{n}_i(\lambda - \epsilon) \leq n_i(\lambda) \leq \overline{n}_i(\lambda)\,.$$

This implies

$$\overline{n}_i(\lambda) - n_i(\lambda) = \mathcal{O}(\lambda^{\frac{1}{3}})\,. \tag{6.8}$$

The asymptotic formula (6.4) implies

$$\overline{n}_i(\lambda) - n_i(\lambda) = o(\sqrt{\lambda})\,, \tag{6.9}$$

and this shows that (2.6) is never sharp for large λ.

6.2 About Corollary 3.5

One can ask whether there is a converse to Corollary 3.5 in the following sense. Suppose we have an eigenfunction u with k nodal domains and eigenvalue λ. For each pair of neighboring nodal domains of u, say, D_i and D_j, let $\Omega_{i,j} = \operatorname{Int}(\overline{D_i \cup D_j})$ and suppose that $\lambda = \lambda_2(\Omega_{i,j})$. Does this imply that $\lambda = \lambda_k$? The answer to the question is negative, as the following easy example shows :
Consider the rectangle $Q = (0, a) \times (0, 1) \subset \mathbb{R}^2$ and consider $H_0(Q)$. We can work out the eigenvalues explicitly as

$$\{\pi^2(\frac{m^2}{a^2} + n^2)\}, \text{ for } m, n \in \mathbb{N} \setminus 0, \tag{6.10}$$

with corresponding eigenfunctions $(x, y) \mapsto \sin(\pi m \frac{x}{a})(\sin \pi n y)$. If

$$a^2 \in (\frac{9}{4}, \frac{8}{3}) , \tag{6.11}$$

then

$$\lambda_3(Q) = \pi^2(\frac{1}{a^2} + 4) < \lambda_4(Q) = \pi^2(\frac{9}{a^2} + 1) ,$$

and the zeroset of u_4 is given by $\{(x, y) \in Q \mid x = a/3,\ x = 2a/3\}$. For u_4 we have $\Omega_{1,2} = Q \cap \{0 < x < 2a/3\}$. If $2a/3 > 1$ (which is the case under assumption (6.11)), then $\lambda_2(\Omega_{1,2}) = \lambda_4(Q)$. We have consequently an example with $k = 3$.

7 Converse theorems in the case of regular open sets

7.1 Preliminaries

As a consequence of Theorem 3.1 and using (1.3), we get that each nodal domain $D_{k\ell}$ of φ_ℓ is included in a nodal domain D_{j0} of u. Using a result of Gesztesy and Zhao ([8], Theorem 1), this implies also that the capacity (see next subsection) of $D_{j0} \setminus D_{k\ell}$ (hence the Lebesgue-measure) is 0.
We now would like to show that under some extra condition the nodal domains of u are those of the φ_ℓ. This is easy when it is assumed that the boundaries of the Ω_ℓ are $C^{1,\alpha}$. However, this regularity assumption is rather strong. A natural weaker regularity condition involving the notion of capacity will be given in this section.

7.2 Capacity

There are various equivalent definitions of polar sets and capacity (see e.g. [5], [7], [10], [13]). If U is a bounded open subset of $\mathbb{R}^d$, we denote by $\|.\|_{W_0^{1,2}(U)}$ the Hilbert norm on $W_0^{1,2}(U)$:

$$\|u\|_{W_0^{1,2}(U)} := (\int_U |\nabla u|^2\, dx)^{\frac{1}{2}} .$$

The capacity in U of $A \subset U$ is defined[†] as

$$\begin{aligned}\mathrm{Cap}_U(A) \quad &:= \inf\{\|s\|^2_{W_0^{1,2}(U)} \,;\, s \in W_0^{1,2}(U) \\ &\qquad \text{and } s \geq 1 \text{ a.e. in some neighborhood of } A\}\,.\end{aligned}$$

It is easily checked that if K is compact and $K \subset U \cap V$, where V is also open and bounded in $\mathbb{R}^d$, then there is a $c = c(K,U,V)$ such that $\mathrm{Cap}_U(A) \leq c\ \mathrm{Cap}_V(A)$ for $A \subset K$. So $\mathrm{Cap}_U(A) = 0$ for some bounded open $U \supset A$ iff for each $a \in A$ there exists an $r > 0$ and a bounded domain V such that $V \supset B(a,r)$ and $\mathrm{Cap}_V(B(a,r) \cap A) = 0$. In this case we may simply write $\mathrm{Cap}(A) = 0$ without referring to U.

7.3 Converse theorem

We are now able to formulate our definition of a regular point.

Definition 7.1
Let D be an open set in $\mathbb{R}^d$. We shall say that a point $x \in \partial D$ is (capacity)-regular (for D) if, for any $r > 0$, the capacity of $B(x,r) \cap \complement D$ is strictly positive.

Theorem 7.2
Under the assumptions of Theorem 3.1, any point $x \in \partial\Omega_\ell \cap \Omega_0$ which is (capacity)-regular with respect to Ω_ℓ (for some ℓ) is in the nodal set of u.

This theorem admits the following corollary :

Corollary 7.3
Under the assumptions of Theorem 3.1 and if, for all ℓ, every point in $(\partial\Omega_\ell) \cap \Omega_0$ is (capacity)-regular for Ω_ℓ, then the family of the nodal domains of u coincides with the union over ℓ of the family of the nodal domains of the φ_ℓ, where u and φ_ℓ are introduced in (3.2).

Proof of corollary
It is clear that any nodal domain of φ_ℓ is contained in a unique nodal domain of u.
Conversely, let D be a nodal domain of u and let $\ell \in \{1, \ldots, k\}$. Then, by combining the assumption on $\partial\Omega_\ell$, Proposition 7.4 and (3.2), we obtain the property :

$$\partial\Omega_\ell \cap D = \emptyset\,.$$

Now, D being connected, either $\Omega_\ell \cap D = \emptyset$ or $D \subset \Omega_\ell$. Moreover the second case should occur for at least one ℓ, say $\ell = \ell_0$. Coming back to the definition of a nodal set and (3.2), we observe that D is necessarily contained in a nodal domain $D_j^{\ell_0}$ of φ_{ℓ_0}.
Combining the two parts of the proof gives that any nodal set of u is a nodal set of φ_ℓ and vice-versa.

[†] For $d \geq 3$ the restriction that U is bounded can be removed and one may take $U = \mathbb{R}^d$.

7.4 Proof of Theorem 7.2

The proof is a consequence of (3.2) and of the following proposition :

Proposition 7.4
Let $D, \Omega \subset \mathbb{R}^d$ be open sets such that $D \subset \Omega$, and let $x_0 \in \partial D \cap \Omega$. Assume that, for some given $r_0 > 0$ such that $B(x_0, r_0) \subset \Omega$, there exists $u \in W_0^{1,2}(D)$ and $v \in C^0(B(x_0, r_0))$ such that :

$$u_{|D\cap B(x_0,r_0)} = v_{|D\cap B(x_0,r_0)} \quad \text{a.e. in } D \cap B(x_0, r_0)\,.$$

Then if $v(x_0) \neq 0$, there exists a ball $B(x_0, r_1)$ ($r_1 > 0$), such that $B(x_0, r_1) \setminus D$ is polar, that is, of capacity 0.

Remark 7.5
Using some standard potential theoretic arguments, Proposition 7.4 can be deduced from Théorème 5.1 in [6] which characterizes, in the case where $d \geq 3$, those $u \in W^{1,2}(\Omega)$ that belong to $W_0^{1,2}(\Omega)$. The proof below should be more elementary in character.

Remark 7.6
Given an open subset $D \subset \mathbb{R}^d$ and a ball $B = B(x, r)$, $x \in \partial D$, the difference set $B \setminus D$ is polar if and only if $B \cap \partial D$ is polar. This follows from the fact that a polar subset of B does not disconnect B [3].

Remark 7.7
If D is a nodal domain of an eigenfunction u of $H(\Omega)$, then any point of $\partial D \cap \Omega$ is capacity-regular for D. This is an immediate consequence of Theorem 4.3 (it also follows from the preceding remark). Indeed, if x is in $\partial D \cap \Omega$, then for any $r > 0$, one can find a ball $B(y, r')$ in $\complement D \cap B(x, r)$.

To prove Proposition 7.4 we require some well-known facts stated in the next three lemmas.

Lemma 7.8
Let U be a bounded convex domain in $\mathbb{R}^d$ and let $B(a, \rho)$, $\rho > 0$ be a ball such that $\overline{B}(a, \rho) \subset U$. There exists a positive constant $c = c(a, \rho, U)$ such that, for every $f \in W^{1,2}(U)$ vanishing a.e. in $B(a, \rho)$,

$$\|f\|_{W^{1,2}(U)} \leq c\, \|\nabla f\|_{L^2(U)}\,.$$

Proof of Lemma 7.8

We can assume without loss of generality that $a = 0$ and let $U' = U \setminus B(0,\rho)$. Fix R so large that $U \subset B(0,R)$. By approximating f by smooth functions (e.g. regularize the function $x \mapsto f((1-\delta)x)$ for $\delta > 0$ and small to get $f_1 \in C^\infty(\overline{U})$), we may restrict to functions $f \in C^\infty(U)$ vanishing in $B(0,\rho)$. Then, since

$$|f(x)|^2 = |\int_0^1 x \cdot \nabla f(sx)\, ds|^2 \leq R^2 \int_{\frac{\rho}{|x|}}^1 |\nabla f(sx)|^2\, ds \text{ for } x \in U' ,$$

we have

$$\begin{aligned} \int_{U'} |f(x)|^2\, dx &\leq R^2 \iint_{x\in U',\, \frac{\rho}{|x|}\leq s \leq 1} |\nabla f(sx)|^2\, dx\, ds \\ &\leq R^2 \iint_{z\in sU',\, \rho\leq |z|,\, s\leq 1} |\nabla f(z)|^2\, dz\, \frac{ds}{s} \qquad (7.1) \\ &\leq \frac{R^3}{\rho} \int_{U'} |\nabla f(x)|^2\, dx, \end{aligned}$$

and the lemma follows.

Lemma 7.9

Let U be a domain in $\mathbb{R}^d$. For every real-valued $f \in W^{1,2}(U)$ the function $g = f_+$ is also in $W^{1,2}(U)$, with $\|g\|_{W^{1,2}(U)} \leq \|f\|_{W^{1,2}(U)}$. Moreover the map $f \mapsto g$ from $W^{1,2}(U)$ into itself is continuous (in the norm topology).

Remark 7.10

Since $\inf\{f_n, 1\} = 1 - (1-f_n)_+$, *it follows from the lemma that* $\inf\{f_n,1\} \to \inf\{f,1\}$ *in* $W^{1,2}(U)$ *whenever* $f_n \to f$ *in* $W^{1,2}(U)$.

Proof of Lemma 7.9

For the first two facts we refer to [12] or [13], where it is moreover shown that the weak partial derivatives $\partial_j f_+$ and $\partial_j f$ satisfy

$$\partial_j f_+ = 1_{\{f>0\}}\, \partial_j f = 1_{\{f\geq 0\}}\, \partial_j f \quad \text{a.e. in} \quad U.$$

Therefore, for any $\delta > 0$, we have :

$$\begin{aligned} &\|\nabla [f_n]_+ - \nabla f_+\|_{L^2} \\ &\qquad = \|1_{\{f_n>0\}} \nabla f_n - 1_{\{f>0\}} \nabla f\|_{L^2} \\ &\qquad \leq \|1_{\{f_n>0\}}(\nabla f_n - \nabla f)\|_{L^2} + \|(1_{\{f>0\}} - 1_{\{f_n>0\}})\nabla f\|_{L^2} \qquad (7.2) \\ &\qquad \leq \|\nabla f_n - \nabla f\|_{L^2} + \|(1_{\{f>0; f_n\leq 0\}} + 1_{\{f\leq 0; f_n>0\}})\nabla f\|_{L^2} \\ &\qquad \leq \|\nabla f_n - \nabla f\|_{L^2} + \|1_{\{0\leq |f|\leq \delta\}} \nabla f\|_{L^2} + 2\|1_{\{|f_n - f|\geq \delta\}} \nabla f\|_{L^2}. \end{aligned}$$

Given $\varepsilon > 0$, fix $\delta > 0$ so that $\|1_{\{0\leq|f|\leq\delta\}}\nabla f\|_{L^2} \leq \varepsilon$ (recall that $\nabla f = 0$ a.e. in $\{f = 0\}$). Since $\nabla f \in L^2(U)$ and $\|1_{\{|f-f_n|\geq\delta\}}\|_{L^1} \leq \frac{\|f_n - f\|^2_{L^2}}{\delta^2}$, it follows that $\lim_{n\to\infty} \|(1_{\{|f-f_n|\geq\delta\}})\nabla f\|_{L^2} = 0$. Therefore $\limsup_{n\to\infty} \|\nabla[f_n]_+ - \nabla f_+\|_{L^2} \leq \varepsilon$, which proves that $[f_n]_+ \to f_+$ in $W^{1,2}(U)$, if $f_n \to f$ in $W^{1,2}(U)$.

Lemma 7.11
Let ω be open in $\mathbb{R}^d$ and let $\{f_n\}$ be a sequence of functions continuous in ω such that $f_n \in W^{1,2}(\omega)$ for each $n \geq 1$ and $\lim_{n\to\infty} \|f_n\|_{W^{1,2}(\omega)} = 0$.
Then the set $F = \{x \in \omega\,;\, \liminf_{n\to\infty} |f_n(x)| > 0\}$ is polar.

Proof of Lemma 7.11
It suffices to show that $\mathrm{cap}_\omega(F \cap K) = 0$ for any compact subset K of ω. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^d$, $\varphi = 1$ in K and $\mathrm{supp}(\varphi) \subset \omega$. Then $g_n = f_n\varphi \to 0$ in $W_0^{1,2}(\omega)$ and $g_n = f_n$ in K.
Set $F_\nu = \{x \in \omega\,;\, |g_n(x)| \geq 2^{-\nu}$ for all $n \geq \nu\}$. By the definition of the capacity, we have $\mathrm{Cap}_\omega(F_\nu) \leq 2^{2\nu}\|\nabla g_n\|^2_{L^2}$ for all $n \geq \nu$ and $\mathrm{cap}(F_\nu) = 0$. Therefore $\mathrm{cap}_\omega(\bigcup_{\nu\geq 1} F_\nu) = 0$ and $\mathrm{cap}_\omega(F \bigcap K) = 0$, since $F \bigcap K \subset \bigcup_{\nu \geq 1} F_\nu$.

Proposition 7.12
Let U be a non-empty open subset of the ball $B = B(a,r)$ in $\mathbb{R}^d$. Suppose there exist a function f continuous in U and a sequence $\{f_n\}$ of functions continuous in B such that
(i) $f \geq 1$ in U and $f \in W^{1,2}(U)$,
(ii) $f_n = 0$ in a neighborhood of $B \setminus U$ and $f_n \in W^{1,2}(U)$ for each $n \geq 1$,
(iii) $\lim_{n\to\infty} \|f - f_n\|_{W^{1,2}(U)} = 0$.
Then the set $F := B \setminus U$ is polar.

Proof of Proposition 7.12
Replacing f by $\inf\{f,1\}$ and f_n by $\inf\{f_n,1\}$, we see[‡] from Lemma 7.9 that we may assume that $f = 1$ in U. So

$$\lim_{n\to\infty} \|\nabla f_n\|_{L^2(U)} = 0 \text{ and } \lim_{n\to\infty} \|1 - f_n\|_{L^2(U)} = 0\,.$$

Fix a ball $\overline{B}(z_0, 2\rho) \subset U$, $\rho > 0$, and a cut-off function $\alpha \in C^\infty(\mathbb{R}^d)$ such that $\alpha = 1$ in $B(z_0,\rho)$, $\alpha = 0$ in $\mathbb{R}^d \setminus B(z_0, 2\rho)$. Set $g = 1 - \alpha$, $g_n = (1-\alpha)f_n$. Then g, g_n belong to $W^{1,2}(B)$, $\nabla g = \nabla g_n = 0$ a.e. in F and

$$\lim_{n\to\infty} \|\nabla(g - g_n)\|_{L^2(B)} = \lim_{n\to\infty} \|\nabla(g-g_n)\|_{L^2(U)} = 0.$$

So, by Lemma 7.8, $\lim_{n\to\infty} \|g - g_n\|_{W^{1,2}(B)} = 0$. But $g - g_n \geq 1$ in F and it follows from Lemma 7.11 that F is polar.

[‡]The weak convergence $\inf\{f_n,1\} \xrightarrow{w} \inf\{f,1\}$ suffices here. It allows the approximation of $1 = \inf\{f,1\}$ in the norm topology in $W^{1,2}(U)$ by finite convex combination of the $\inf\{f_n,1\}$. So we are again left with the case when $f = 1$ in U.

Proof of Proposition 7.4

Without loss of generality, we can assume that $v(x_0) > 0$. Choose $r_1 > 0$ so small that $v \geq c_0 := \frac{1}{2}v(x_0)$ in $B(x_0, r_1)$. Since $u \in W_0^{1,2}(D)$, there is a sequence $\{u_n\}$ in $C_0^\infty(\mathbb{R}^d)$ such that $\mathrm{supp}(u_n) \subset D$ and $u_n \to u$ in $W^{1,2}(\mathbb{R}^d)$. Applying Proposition 7.12 to the ball $B(x_0, r_1)$ and the functions $f = c_0^{-1} u_{|B(x_0,r_1)}$, $f_n = c_0^{-1} u_{n|B(x_0,r_1)}$, we see that $B(x_0, r_1) \setminus D$ is polar.

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A. Ancona
Département de Mathématiques
Bat. 425
Université Paris-Sud
91 405 Orsay Cedex, France
Alano.Ancona@math.u-psud.fr

B. Helffer
Département de Mathématiques
Bat. 425
Université Paris-Sud
91 405 Orsay Cedex, France
Bernard.Helffer@math.u-psud.fr

T. Hoffmann-Ostenhof
Institut für Theoretische Chemie
Universität Wien
Währinger Strasse 17
A-1090 Wien, Austria
and
International Erwin Schrödinger
Institute for Mathematical Physics
Boltzmanngasse 9, A-1090
Wien, Austria
thoffman@esi.ac.at