Moduli of Prym Curves

Edoardo Ballico, Cinzia Casagrande,
and Claudio Fontanari[1]

Received: May 22, 2003
Revised: March 22, 2004

Communicated by Thomas Peternell

Abstract. Here we focus on the compactification of the moduli space of curves of genus g together with an unramified double cover, constructed by Arnaud Beauville in order to compactify the Prym mapping. We present an alternative description of it, inspired by the moduli space of spin curves of Maurizio Cornalba, and we discuss in detail its main features, both from a geometrical and a combinatorial point of view.

2000 Mathematics Subject Classification: 14H10, 14H30.
Keywords and Phrases: moduli of curves, étale and admissible double covers, spin and Prym curves.

Introduction

Let X be a smooth curve of genus g. As it is well known (see for instance [Bea96], p. 104, or [ACGH85], Appendix B, § 2, 13.), a square root of $\mathcal{O}_X$ corresponds to an unramified double cover of X.
A compactification $\overline{R}_g$ of the moduli space of curves of genus g together with an unramified double cover was constructed by Arnaud Beauville ([Bea77], Section 6; see also [DS81], Theorem 1.1) by means of admissible double covers of stable curves. This moduli space was introduced as a tool to compactify the mapping which associates to a curve plus a 2-sheeted cover the corresponding Prym variety; however, we believe that it is interesting also in its own and worthy of a closer inspection.
Here we explore some of the geometrical and combinatorial properties of $\overline{R}_g$. In order to do that, we present a description of this scheme which is different from the original one and is inspired by the construction performed by Maurizio

Cornalba in [Cor89] of the moduli space of spin curves $\overline{S}_g$. This is a natural compactification over $\overline{\mathcal{M}}_g$ of the space of pairs (X, ζ), where $\zeta \in \operatorname{Pic} X$ is a square root of the canonical bundle K_X.
In Section 1 we define a *Prym curve* to be just the analogue of a spin curve; Cornalba's approach can be easily adapted to our context and allows us to put a structure of projective variety on the set $\overline{Pr}_g$ of isomorphism classes of Prym curves of genus g. This variety has two irreducible components $\overline{Pr}_g^-$ and $\overline{Pr}_g^+$, where $\overline{Pr}_g^- \simeq \overline{\mathcal{M}}_g$ contains "trivial" Prym curves; moreover, by comparing Prym curves and admissible double covers, we give an explicit isomorphism between $\overline{Pr}_g^+$ and $\overline{R}_g$ over $\overline{\mathcal{M}}_g$.
Next, in Section 2 we reproduce the arguments in [Fon02] in order to show that $\overline{Pr}_g$ is endowed with a natural injective morphism into the compactification of the universal Picard variety constructed by Lucia Caporaso in [Cap94], just like $\overline{S}_g$.
Finally, in Section 3 we turn to the combinatorics of $\overline{Pr}_g$. Applying the same approach used in [CC03] for spin curves, we study the ramification of the morphism $\overline{Pr}_g \to \overline{\mathcal{M}}_g$ over the boundary. We describe the numerical properties of the scheme-theoretical fiber Pr_Z over a point $[Z] \in \overline{\mathcal{M}}_g$, which turn out to depend only on the dual graph Γ_Z of Z. From this combinatorial description, it follows that the morphisms $\overline{Pr}_g \to \overline{\mathcal{M}}_g$ and $\overline{S}_g \to \overline{\mathcal{M}}_g$ ramify in a different way.
The moduli space $\overline{R}_g$ of admissible double covers has been studied also by Mira Bernstein in [Ber99], where $\overline{R}_g$ is shown to be of general type for $g = 17, 19, 21, 23$ ([Ber99], Corollary 3.1.7) (for $g \geq 24$ it is obvious, since $\overline{\mathcal{M}}_g$ is).
We work over the field $\mathbb{C}$ of complex numbers.
We wish to thank Lucia Caporaso for many fruitful conversations. We are also grateful to the anonymous referee for pointing out a gap in a previous version of this paper.

1 Prym curves and admissible double covers

1.1. Defining the objects. Let X be a Deligne-Mumford semistable curve and E an irreducible component of X. One says that E is *exceptional* if it is smooth, rational, and meets the other components in exactly two points. Moreover, one says that X is *quasistable* if any two distinct exceptional components of X are disjoint. The stable model of X is the stable curve Z obtained from X by contracting each exceptional component to a point. In the sequel, $\widetilde{X}$ will denote the subcurve $\overline{X \smallsetminus \cup_i E_i}$ obtained from X by removing all exceptional components.
We fix an integer $g \geq 2$.

Definition 1. A *Prym curve* of genus g is the datum of (X, η, β) where X is a quasistable curve of genus g, $\eta \in \operatorname{Pic} X$, and $\beta \colon \eta^{\otimes 2} \to \mathcal{O}_X$ is a homomorphism of invertible sheaves satisfying the following conditions:

(i) η has total degree 0 on X and degree 1 on every exceptional component of X;

(ii) β is non zero at a general point of every non-exceptional component of X.

We say that X is the support of the Prym curve (X, η, β).
An *isomorphism* between two Prym curves (X, η, β) and (X', η', β') is an isomorphism $\sigma\colon X \to X'$ such that there exists an isomorphism $\tau\colon \sigma^*(\eta') \to \eta$ which makes the following diagram commute [2]

$$\begin{tikzcd} \sigma^*(\eta')^{\otimes 2} \arrow[r, "\tau^{\otimes 2}"] \arrow[d, "\sigma^*(\beta')"'] & \eta^{\otimes 2} \arrow[d, "\beta"] \\ \sigma^*(\mathcal{O}_{X'}) \arrow[r, "\sim"] & \mathcal{O}_X. \end{tikzcd}$$

Let (X, η, β) be a Prym curve and let $E_1, \ldots, E_r$ be the exceptional components of X. From the definition it follows that β vanishes identically on all E_i and induces an isomorphism

$$\eta^{\otimes 2}|_{\widetilde{X}} \xrightarrow{\sim} \mathcal{O}_{\widetilde{X}}(-q_1^1 - q_1^2 - \cdots - q_r^1 - q_r^2),$$

where $\{q_i^1, q_i^2\} = \widetilde{X} \cap E_i$ for $i = 1, \ldots, r$. In particular, when X is smooth, η is just a point of order two in the Picard group of X. The number of such points, as it is well-known, is exactly 2^{2g}.
We denote by $\operatorname{Aut}(X, \eta, \beta)$ the group of automorphisms of the Prym curve (X, η, β). As in [Cor89], p. 565, one can show that $\operatorname{Aut}(X, \eta, \beta)$ is finite.
We say that an isomorphism between two Prym curves (X, η, β) and (X, η', β') having the same support is *inessential* if it induces the identity on the stable model of X. We denote by $\operatorname{Aut}_0(X, \eta, \beta) \subseteq \operatorname{Aut}(X, \eta, \beta)$ the subgroup of inessential automorphisms. We have the following

Lemma 2 ([Cor89], Lemma 2.1). *There exists an inessential isomorphism between two Prym curves (X, η, β) and (X, η', β') if and only if*

$$\eta|_{\widetilde{X}} \simeq \eta'|_{\widetilde{X}}.$$

So the set of isomorphism classes of Prym curves supported on X is in bijection with the set of square roots of $\mathcal{O}_{\widetilde{X}}(-q_1^1 - q_1^2 - \cdots - q_r^1 - q_r^2)$ in $\operatorname{Pic} \widetilde{X}$, modulo the action of the group of automorphisms of $\widetilde{X}$ fixing $q_1^1, q_1^2, \ldots, q_r^1, q_r^2$.
A *family of Prym curves* is a flat family of quasistable curves $f\colon \mathcal{X} \to S$ with an invertible sheaf $\boldsymbol{\eta}$ on $\mathcal{X}$ and a homomorphism

$$\boldsymbol{\beta}\colon \boldsymbol{\eta}^{\otimes 2} \longrightarrow \mathcal{O}_{\mathcal{X}}$$

[2] Observe that we are adopting the convention that the datum of τ is not included in the definition of isomorphism, as in [Cor89]. This is different from the convention in [Cor91]; see [Cor91], end of section 2, for a discussion about this.

such that the restriction of these data to any fiber of f gives rise to a Prym curve. An isomorphism between two families of Prym curves $(\mathcal{X} \to S, \boldsymbol{\eta}, \boldsymbol{\beta})$ and $(\mathcal{X}' \to S, \boldsymbol{\eta}', \boldsymbol{\beta}')$ over S is an isomorphism $\sigma\colon \mathcal{X} \to \mathcal{X}'$ over S such that there exists an isomorphism $\tau\colon \sigma^*(\boldsymbol{\eta}') \to \boldsymbol{\eta}$ compatible with the canonical isomorphism between $\sigma^*(\mathcal{O}_{\mathcal{X}'})$ and $\mathcal{O}_{\mathcal{X}}$.

We define the moduli functor associated to Prym curves in the obvious way: $\overline{\mathcal{P}\mathrm{r}}_g$ is the contravariant functor from the category of schemes to the one of sets, which to every scheme S associates the set $\overline{\mathcal{P}\mathrm{r}}_g(S)$ of isomorphism classes of families of Prym curves of genus g over S.

1.2. The universal deformation. Fix a Prym curve (X, η, β), call Z the stable model of X and denote by $E_1, \dots, E_r$ the exceptional components of X. Let $\mathcal{Z}' \to B'$ be the universal deformation of Z, where B' is the unit policylinder in $\mathbb{C}^{3g-3}$ with coordinates $t_1, \dots, t_{3g-3}$ such that $\{t_i = 0\} \subset B'$ is the locus where the node corresponding to E_i persists for $i = 1, \dots, r$. Let B be another unit policylinder in $\mathbb{C}^{3g-3}$ with coordinates $\tau_1, \dots, \tau_{3g-3}$, and consider the map $B \to B'$ given by $t_i = \tau_i^2$ for $1 \leq i \leq r$ and $t_i = \tau_i$ for $i > r$. Call $\mathcal{Z}$ the pull-back of $\mathcal{Z}'$ to B. For $i \in \{1, \dots, r\}$ the family $\mathcal{Z}_{|\{\tau_i = 0\}} \to \{\tau_i = 0\} \subset B$ has a section V_i, corresponding to the locus of the ith node. Let $\mathcal{X} \to \mathcal{Z}$ be the blow-up of $V_1, \dots, V_r$ and call $\mathcal{E}_1, \dots, \mathcal{E}_r$ the exceptional divisors.

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{Z}' \arrow[d] \\
 & B \arrow[r] & B'
\end{tikzcd}$$

The variety $\mathcal{X}$ is smooth and $\mathcal{X} \to B$ is a family of quasistable curves, with X as central fiber. Up to an inessential automorphism, we can assume that $\eta^{\otimes 2} \simeq \mathcal{O}_{\mathcal{X}}(-\sum_i \mathcal{E}_i)_{|X}$ and that this isomorphism is induced by β. By shrinking B if necessary, we can extend η to $\boldsymbol{\eta} \in \operatorname{Pic} \mathcal{X}$ such that $\boldsymbol{\eta}^{\otimes 2} \simeq \mathcal{O}_{\mathcal{X}}(-\sum_i \mathcal{E}_i)$. Denote by $\boldsymbol{\beta}$ the composition of this isomorphism with the natural inclusion $\mathcal{O}_{\mathcal{X}}(-\sum_i \mathcal{E}_i) \hookrightarrow \mathcal{O}_{\mathcal{X}}$. Then $(\mathcal{X} \to B, \boldsymbol{\eta}, \boldsymbol{\beta})$ is a family of Prym curves, and there is a morphism $\psi\colon X \to \mathcal{X}$ which induces an isomorphism of Prym curves between (X, η, β) and the fiber of the family over $b_0 = (0, \dots, 0) \in B$. This family provides a *universal deformation* of (X, η, β):

Theorem 3. *Let $(\mathcal{X}' \to T, \boldsymbol{\eta}', \boldsymbol{\beta}')$ be a family of Prym curves and let $\varphi\colon X \to \mathcal{X}'$ be a morphism which induces an isomorphism of Prym curves between (X, η, β) and the fiber of the family over $t_0 \in T$.*

Then, possibly after shrinking T, there exists a unique morphism $\gamma\colon T \to B$ satisfying the following conditions:

(i) $\gamma(t_0) = b_0$;

(ii) there is a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "\delta"] \arrow[d] & \mathcal{X} \arrow[d] \\ T \arrow[r, "\gamma"] & B; \end{tikzcd}$$

(iii) $\boldsymbol{\eta}' \simeq \delta^*(\boldsymbol{\eta})$ *and* $\boldsymbol{\beta}' = \delta^*(\boldsymbol{\beta})$*;*

(iv) $\delta \circ \varphi = \psi$.

Since the proof of [Cor89], Proposition 4.6 applies verbatim to our case, we omit the proof of Theorem 3.

1.3. THE MODULI SCHEME. Let $\overline{Pr}_g$ be the set of isomorphism classes of Prym curves of genus g. We define a natural structure of analytic variety on $\overline{Pr}_g$ following [Cor89], § 5.
Consider a Prym curve (X, η, β) and its universal deformation $(\mathcal{X} \to B, \boldsymbol{\eta}, \boldsymbol{\beta})$ constructed in 1.2. By the universality, the group $\mathrm{Aut}(X, \eta, \beta)$ acts on B and on $\mathcal{X}$. This action has the following crucial property:

LEMMA 4 ([COR89], LEMMA 5.1). *Let $b_1, b_2 \in B$ and let $(X_{b_1}, \eta_{b_1}, \beta_{b_1})$ and $(X_{b_2}, \eta_{b_2}, \beta_{b_2})$ be the fibers of the universal family over b_1 and b_2 respectively. Then there exists $\sigma \in \mathrm{Aut}(X, \eta, \beta)$ such that $\sigma(b_1) = b_2$[3] if and only if the Prym curves $(X_{b_1}, \eta_{b_1}, \beta_{b_1})$ and $(X_{b_2}, \eta_{b_2}, \beta_{b_2})$ are isomorphic.*

Lemma 4 implies that the natural (set-theoretical) map $B \to \overline{Pr}_g$, associating to $b \in B$ the isomorphism class of the fiber over b, descends to a well-defined, injective map

$$J\colon B/\mathrm{Aut}(X, \eta, \beta) \longrightarrow \overline{Pr}_g.$$

This allows to define a complex structure on the subset $\mathrm{Im}\, J \subseteq \overline{Pr}_g$. Since $\overline{Pr}_g$ is covered by these subsets, in order to get a complex structure on $\overline{Pr}_g$ we just have to check that the complex structures are compatible on the overlaps.
This compatibility will follow from the following remark, which is an immediate consequence of the construction of the universal family in 1.2:

- *the family of Prym curves $(\mathcal{X} \to B, \boldsymbol{\eta}, \boldsymbol{\beta})$ is a universal deformation for any of its fibers.*

In fact, assume that there are two Prym curves (X_1, η_1, β_1) and (X_2, η_2, β_2) such that the images of the associated maps J_1, J_2 intersect. Choose a Prym curve (X_3, η_3, β_3) corresponding to a point in $\mathrm{Im}\, J_1 \cap \mathrm{Im}\, J_2$. Let B_i $(i = 1, 2, 3)$ be the basis of the universal deformation of (X_i, η_i, β_i). Then by the remark above, for $i = 1, 2$ there are natural open immersions $h_i\colon B_3 \hookrightarrow B_i$, equivariant with respect to the actions of the automorphism groups. Hence h_i induces an open immersion $\overline{h}_i\colon B_3/\mathrm{Aut}(X_3, \eta_3, \beta_3) \hookrightarrow B_i/\mathrm{Aut}(X_i, \eta_i, \beta_i)$, and $J_3 = J_i \circ \overline{h}_i$.

[3]Where we still denote by σ the automorphism of B induced by σ.

Observe now that the morphisms

$$B/\mathrm{Aut}(X,\eta,\beta) \longrightarrow B'/\mathrm{Aut}(Z)$$

glue together and yield a morphism $p\colon \overline{Pr}_g \to \overline{\mathcal{M}}_g$. Clearly p is finite, as a morphism between analytic varieties (see [Ray71]). Hence $\overline{Pr}_g$ is projective, because $\overline{\mathcal{M}}_g$ is. The variety $\overline{Pr}_g$ has finite quotient singularities; in particular, it is normal.

The degree of p is 2^{2g}. The fiber over a smooth curve Z is just the set of points of order two in its Picard group, modulo the action of $\mathrm{Aut}(Z)$ if non trivial. When Z is a stable curve, the set-theoretical fiber over $[Z]$ consists of isomorphism classes of Prym curves (X,η,β) such that the stable model of X is Z. In section 3 we will describe precisely the scheme-theoretical fiber over $[Z]$, following [Cor89] and [CC03]. We will show that p is étale over $\overline{\mathcal{M}}_g^0 \smallsetminus D_{irr}$, where $\overline{\mathcal{M}}_g^0$ is the locus of stable curves with trivial automorphism group and D_{irr} is the boundary component whose general member is an irreducible curve with one node.

Finally, $\overline{Pr}_g$ is a coarse moduli space for the functor $\overline{\mathcal{P}\mathrm{r}}_g$. For any family of Prym curves over a scheme T, the associated moduli morphism $T \to \overline{Pr}_g$ is locally defined by Theorem 3.

Let $\overline{Pr}_g^-$ be the closed subvariety of $\overline{Pr}_g$ consisting of classes of Prym curves (X,η,β) where $\eta \simeq \mathcal{O}_X$. Observe that when η is trivial, the curve X is stable. So $\overline{Pr}_g^-$ is the image of the obvious section of $p\colon \overline{Pr}_g \to \overline{\mathcal{M}}_g$, and it is an irreducible (and connected) component of $\overline{Pr}_g$, isomorphic to $\overline{\mathcal{M}}_g$.

Let $\overline{Pr}_g^+$ be the complement of $\overline{Pr}_g^-$ in $\overline{Pr}_g$, and denote by Pr_g^+ its open subset consisting of classes of Prym curves supported on smooth curves. Then Pr_g^+ parametrizes connected unramified double covers of smooth curves of genus g; it is well-known that this moduli space is irreducible, being a finite quotient of the moduli space of smooth curves of genus g with a level 2 structure, which is irreducible by [DM69]. So $\overline{Pr}_g^+$ is an irreducible component of $\overline{Pr}_g$.

1.4. Admissible double covers. Consider a pair (C,i) where C is a stable curve of genus $2g-1$ and i is an involution of C such that:

- the set I of fixed points of i is contained in $\mathrm{Sing}\, C$;
- for any fixed node, i does not exchange the two branches of the curve.

Then the quotient $Z := C/i$ is a stable curve of genus g, and $\pi\colon C \to Z$ is a finite morphism of degree 2, étale over $Z \smallsetminus \pi(I)$. This is called an *admissible double cover*. Remark that π is not a cover in the usual sense, since it is not flat at I.

The moduli space $\overline{R}_g$ of admissible double covers of stable curves of genus g is constructed [Bea77], Section 6 (see also [DS81, ABH01]), as the moduli space for pairs (C,i) as above.

An isomorphism of two admissible covers $\pi_1\colon C_1 \to Z_1$ and $\pi_2\colon C_2 \to Z_2$ is an isomorphism $\varphi\colon Z_1 \xrightarrow{\sim} Z_2$ such that there exists[4] an isomorphism $\widetilde{\varphi}\colon C_1 \xrightarrow{\sim} C_2$ with $\pi_2 \circ \widetilde{\varphi} = \varphi \circ \pi_1$.

We denote by $\mathrm{Aut}(C \to Z)$ the automorphism group of the admissible cover $C \to Z$, so $\mathrm{Aut}(C \to Z) \subseteq \mathrm{Aut}(Z)$. All elements of $\mathrm{Aut}(C \to Z)$ are induced by automorphisms of C, different from i, and that commute with i.

Let (C, i) be as above; we describe its universal deformation. Let $\mathcal{C}' \to W'$ be a universal deformation of C. By the universality, there are compatible involutions i' of W' and $\boldsymbol{i}'$ of $\mathcal{C}'$, extending the action of i on the central fiber. Let $W \subset W'$ be the locus fixed by i', $\mathcal{C} \to W$ the induced family and $\boldsymbol{i}$ the restriction of $\boldsymbol{i}'$ to $\mathcal{C}$. Then $(\mathcal{C}, \boldsymbol{i}) \to W$ is a universal deformation of (C, i) and the corresponding family of admissible double covers is $\mathcal{C} \to \mathcal{Q} := \mathcal{C}/\boldsymbol{i} \to W$.

We are going to show that $\overline{R}_g$ is isomorphic over $\overline{\mathcal{M}}_g$ to the irreducible component $\overline{Pr}_g^+$ of $\overline{Pr}_g$.

First of all we define a map Φ from the set of non trivial Prym curves of genus g to the set of admissible double covers of stable curves of genus g.

Let $\xi = (X, \eta, \beta)$ be a Prym curve with $\eta \not\simeq \mathcal{O}_X$; then $\Phi(\xi)$ will be an admissible double cover of the stable model Z of X, constructed as follows. The homomorphism β induces an isomorphism

$$\eta_{|\widetilde{X}}^{\otimes(-2)} \simeq \mathcal{O}_{\widetilde{X}}(q_1^1 + q_1^2 + \cdots + q_r^1 + q_r^2).$$

This determines a double cover $\widetilde{\pi}\colon \widetilde{C} \to \widetilde{X}$, ramified over $q_1^1, q_1^2, \ldots, q_r^1, q_r^2$, which are smooth points of $\widetilde{X}$. Now call C_ξ the stable curve obtained identifying $\widetilde{\pi}^{-1}(q_i^1)$ with $\widetilde{\pi}^{-1}(q_i^2)$ for all $i = 1, \ldots, r$. Then the induced map $C_\xi \to Z$ is the admissible double cover $\Phi(\xi)$.

Now consider two Prym curves ξ_1 and ξ_2 supported respectively on X_1 and X_2. Suppose that $\sigma\colon X_1 \xrightarrow{\sim} X_2$ induces an isomorphism between ξ_1 and ξ_2. Let $\overline{\sigma}\colon Z_1 \xrightarrow{\sim} Z_2$ be the induced isomorphism between the stable models. Then it is easy to see that $\overline{\sigma}$ is an isomorphism between the admissible covers $\Phi(\xi_1)$ and $\Phi(\xi_2)$. Moreover, any isomorphism between $\Phi(\xi_1)$ and $\Phi(\xi_2)$ is obtained in this way. Hence we have an exact sequence of automorphism groups:

$$1 \to \mathrm{Aut}_0(\xi) \to \mathrm{Aut}(\xi) \to \mathrm{Aut}(C_\xi \to Z) \to 1. \tag{1}$$

We show that Φ is surjective. Let $C \to Z$ be an admissible double cover, $I \subset C$ the set of fixed points of the involution and $J \subset Z$ their images. Let $\widetilde{C} \to C$ and $\nu\colon \widetilde{X} \to Z$ be the normalizations of C at I and of Z at J respectively. Then i extends to an involution on $\widetilde{C}$, whose quotient is $\widetilde{X}$, namely: $\widetilde{C} \to \widetilde{X}$ is a double cover, ramified over $q_1^1, q_1^2, \ldots, q_r^1, q_r^2$, where $r = |J|$ and $\nu(q_i^1) = \nu(q_i^2) \in J$ for $i = 1, \ldots, r$. Let $L \in \mathrm{Pic}\,\widetilde{X}$ be the associated line bundle, satisfying $L^{\otimes 2} \simeq \mathcal{O}_{\widetilde{X}}(q_1^1 + q_1^2 + \cdots + q_r^1 + q_r^2)$. Finally let X be the quasistable curve obtained by attaching to $\widetilde{X}$ r rational components $E_1, \ldots, E_r$ such that

[4] Given φ, there are exactly two choices for $\widetilde{\varphi}$; if $\widetilde{\varphi}$ is one, the other is $\widetilde{\varphi} \circ i_1$.

$E_i \cap \widetilde{X} = \{q_i^1, q_i^2\}$. Choose $\eta \in \operatorname{Pic} X$ having degree 1 on all E_i and such that $\eta|_{\widetilde{X}} = L^{\otimes(-1)}$. Let $\beta\colon \eta^{\otimes 2} \to \mathcal{O}_X$ be a homomorphism which agrees with $\eta|_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}(-q_1^1 - q_1^2 - \cdots - q_r^1 - q_r^2) \hookrightarrow \mathcal{O}_X$ on $\widetilde{X}$. Then $\xi = (X, \eta, \beta)$ is a Prym curve with $\eta \not\simeq \mathcal{O}_X$, and $C \to Z$ is $\Phi(\xi)$. For different choices of η, the corresponding Prym curves differ by an inessential isomorphism.

Proposition 5. *The map Φ just defined induces an isomorphism*

$$\widehat{\Phi}\colon \overline{Pr}_g^{\,+} \longrightarrow \overline{R}_g$$

over $\overline{\mathcal{M}}_g$.

Proof. By what precedes, Φ induces a bijection $\widehat{\Phi}\colon \overline{Pr}_g^{\,+} \to \overline{R}_g$. The statement will follow if we prove that $\widehat{\Phi}$ is a local isomorphism at every point of $\overline{Pr}_g^{\,+}$.

Fix a point $\xi = (X, \eta, \beta) \in \overline{Pr}_g^{\,+}$ and consider its universal deformation $(\mathcal{X} \to B, \boldsymbol{\eta}, \boldsymbol{\beta})$ constructed in 1.2. Keeping the notations of 1.2, the line bundle $\boldsymbol{\eta}^{\otimes(-1)}$ determines a double cover $\overline{\mathcal{P}} \to \mathcal{X}$, ramified over $\mathcal{E}_1, \ldots, \mathcal{E}_r$. The divisor $\mathcal{E}_i$ is a $\mathbb{P}^1$-bundle over $V_i \subset B$, and the restriction of its normal bundle to a non trivial fiber F is $(\mathcal{N}_{\mathcal{E}_i/\mathcal{X}})_{|F} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$. The inverse image $\overline{\mathcal{E}}_i$ of $\mathcal{E}_i$ in $\overline{\mathcal{P}}$ is again a $\mathbb{P}^1$-bundle over $V_i \subset B$, but now the restriction of its normal bundle to a non trivial fiber $\overline{F}$ is $(\mathcal{N}_{\overline{\mathcal{E}}_i/\overline{\mathcal{P}}})_{|\overline{F}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. Let $\overline{\mathcal{P}} \to \mathcal{P}$ be the blow-down of $\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_r$. We get a diagram

$$\begin{tikzcd}
\overline{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] & \\
\mathcal{X} \arrow[r] & \mathcal{Z} \arrow[r] & B
\end{tikzcd}$$

where $\mathcal{P} \to \mathcal{Z} \to B$ is a family of admissible double covers whose central fiber is $C_\xi \to Z$. Therefore, up to shrinking B, there exists a morphism $B \to W$ such that $\mathcal{P} \to \mathcal{Z} \to B$ is obtained by pull-back from the universal deformation $\mathcal{C} \to \mathcal{Q} \to W$ of $C_\xi \to Z$. Now notice that $\mathcal{Q} \to W$ is a family of stable curves of genus g, with Z as central fiber: so (again up to shrinking) it must be a pull-back of the universal deformation $\mathcal{Z}' \to B'$. In the end we get a diagram:

$$\begin{tikzcd}
\overline{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] & & \text{(genus } 2g-1\text{)} \\
\mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d] & \mathcal{Z}' \arrow[d] & \text{(genus } g\text{)} \\
& B \arrow[r, "\varphi"] \arrow[loop below, "\mathrm{Aut}(\xi)"] & W \arrow[r, "\psi"] \arrow[loop below, "\mathrm{Aut}(C_\xi \to Z)"] & B' \arrow[loop below, "\mathrm{Aut}(Z)"] & \text{(bases)}
\end{tikzcd}$$

We can assume that φ and ψ are surjective. Observe that both maps are equivariant with respect to the actions of the automorphism groups indicated in the diagram.
Clearly φ is just the restriction of Φ to the set of Prym curves parametrized by B.
Now by (1), $\varphi(b_1) = \varphi(b_2)$ if and only if there exists an inessential isomorphism between $(X_{b_1}, \eta_{b_1}, \beta_{b_1})$ and $(X_{b_2}, \eta_{b_2}, \beta_{b_2})$. Hence φ induces an equivariant isomorphism $\widehat{\varphi}$:

$$\begin{tikzcd} B/\mathrm{Aut}_0(\xi) \arrow[r, "\widehat{\varphi}"] \arrow[loop below, "\mathrm{Aut}(\xi)/\mathrm{Aut}_0(\xi)"] & W \arrow[r] \arrow[loop below, "\mathrm{Aut}(C_\xi \to Z)"] & B' \arrow[loop below, "\mathrm{Aut}(Z)"] \end{tikzcd}$$

and finally if we mod out by all the automorphism groups, we get

$$\begin{tikzcd} B/\mathrm{Aut}(\xi) \arrow[r, "\sim"] \arrow[d, hook] & W/\mathrm{Aut}(C_\xi \to Z) \arrow[r] \arrow[d, hook] & B'/\mathrm{Aut}Z \arrow[d, hook] \\ \overline{Pr}_g^+ \arrow[r, "\widehat{\Phi}"] & \overline{R}_g & \overline{\mathcal{M}}_g. \end{tikzcd}$$

This shows that $\widehat{\Phi}$ is a local isomorphism in ξ. ■

2 Embedding $\overline{Pr}_g$ in the compactified Picard variety

Let $g \geq 3$. For every integer d, there is a *universal Picard variety*

$$P_{d,g} \longrightarrow \mathcal{M}_g^0$$

whose fiber $J^d(X)$ over a point X of $\mathcal{M}_g^0$ parametrizes line bundles on X of degree d, modulo isomorphism. Denote by Pr_g^0 the inverse image of $\mathcal{M}_g^0$ under the finite morphism $\overline{Pr}_g \to \overline{\mathcal{M}}_g$; then we have a commutative diagram

$$\begin{tikzcd} Pr_g^0 \arrow[r, hook] \arrow[dr] & P_{0,g} \arrow[d] \\ & \mathcal{M}_g^0. \end{tikzcd}$$

Assume $d \geq 20(g-1)$; this is not a real restriction, since for all $t \in \mathbb{Z}_{\geq 0}$ there is a natural isomorphism $P_{d,g} \cong P_{d+t(2g-2),g}$. Then $P_{d,g}$ has a natural compactification $\overline{P}_{d,g}$, endowed with a natural morphism $\phi_d \colon \overline{P}_{d,g} \to \overline{\mathcal{M}}_g$, such that $\phi_d^{-1}(\mathcal{M}_g^0) = P_{d,g}$. It was constructed in [Cap94] as a GIT quotient

$$\pi_d \colon H_d \longrightarrow H_d /\!/ G = \overline{P}_{d,g},$$

where $G = SL(d-g+1)$ and

$$H_d := \{h \in \mathrm{Hilb}_{d-g}^{dx-g+1} \,|\, h \text{ is } G\text{-semistable} \\ \text{and the corresponding curve is connected } \}$$

(the action of G is linearized by a suitable embedding of $\mathrm{Hilb}_{d-g}^{dx-g+1}$ in a Grassmannian).
Fix now and in the sequel an integer $t \geq 10$ and define

$$K_{2t(g-1)} := \{h \in \mathrm{Hilb}_{2t(g-1)-g}^{2t(g-1)x-g+1} | \text{ there is a Prym curve } (X, \eta, \beta) \text{ and} \\ \text{an embedding } h_t \colon X \to \mathbb{P}^{2t(g-1)-g} \text{ induced by } \eta \otimes \omega_X^{\otimes t}, \\ \text{such that } h \text{ is the Hilbert point of } h_t(X)\}.$$

Our result is the following.

Theorem 6. *The set $K_{2t(g-1)}$ is contained in $H_{2t(g-1)}$; consider its projection*

$$\Pi_t := \pi_{2t(g-1)}(K_{2t(g-1)}) \subset \overline{P}_{2t(g-1),g}.$$

There is a natural injective morphism

$$f_t \colon \overline{Pr}_g \longrightarrow \overline{P}_{2t(g-1),g}$$

whose image is Π_t.

In particular, the Theorem implies that Π_t is a closed subvariety of $\overline{P}_{2t(g-1),g}$.
The proof of Theorem 6 will be achieved in several steps and will take the rest of this section. The argument is the one used in [Fon02] to show the existence of an injective morphism $\overline{S_g} \to \overline{P}_{(2t+1)(g-1),g}$ of the moduli space of spin curves in the corresponding compactified Picard variety.
One can define (see [Cap94], §8.1) the contravariant functor $\overline{\mathcal{P}}_{d,g}$ from the category of schemes to the one of sets, which to every scheme S associates the set $\overline{\mathcal{P}}_{d,g}(S)$ of equivalence classes of polarized families of quasistable curves of genus g

$$f \colon (\mathcal{X}, \mathcal{L}) \longrightarrow S$$

such that $\mathcal{L}$ is a relatively very ample line bundle of degree d whose multidegree satisfies the following Basic Inequality on each fiber.

Definition 7. Let $X = \bigcup_{i=1}^n X_i$ be a projective, nodal, connected curve of arithmetic genus g, where the X_i's are the irreducible components of X. We say that the multidegree $(d_1, \ldots, d_n)$ satisfies the *Basic Inequality* if for every complete subcurve Y of X of arithmetic genus g_Y we have

$$m_Y \leq d_Y \leq m_Y + k_Y$$

where

$$d_Y = \sum_{X_i \subseteq Y} d_i, \quad k_Y = |Y \cap \overline{X \smallsetminus Y}| \quad \text{and} \quad m_Y = \frac{d}{g-1}\left(g_Y - 1 + \frac{k_Y}{2}\right) - \frac{k_Y}{2}$$

(see [Cap94] p. 611 and p. 614).

Two families over S, $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ are equivalent if there exists an S-isomorphism $\sigma\colon \mathcal{X} \to \mathcal{X}'$ and a line bundle M on S such that $\sigma^*\mathcal{L}' \cong \mathcal{L} \otimes f^*M$. By [Cap94], Proposition 8.1, there is a morphism of functors:

$$\overline{\mathcal{P}}_{d,g} \longrightarrow \mathit{Hom}(\,\cdot\,, \overline{P}_{d,g}) \tag{2}$$

and $\overline{P}_{d,g}$ coarsely represents $\overline{\mathcal{P}}_{d,g}$ if and only if

$$(d - g + 1, 2g - 2) = 1. \tag{3}$$

Proposition 8. *For every integer $t \geq 10$ there is a natural morphism:*

$$f_t\colon \overline{Pr}_g \longrightarrow \overline{P}_{2t(g-1),g}.$$

Proof. First of all, notice that in this case (3) does not hold, so the points of $\overline{P}_{2t(g-1),g}$ are *not* in one-to-one correspondence with equivalence classes of very ample line bundles of degree $2t(g-1)$ on quasistable curves, satisfying the Basic Inequality (see [Cap94], p. 654). However, we claim that the thesis can be deduced from the existence of a morphism of functors:

$$F_t\colon \overline{\mathcal{P}\mathrm{r}}_g \longrightarrow \overline{\mathcal{P}}_{2t(g-1),g}. \tag{4}$$

Indeed, since $\overline{Pr}_g$ coarsely represents $\overline{\mathcal{P}\mathrm{r}}_g$, any morphism of functors $\overline{\mathcal{P}\mathrm{r}}_g \to \mathit{Hom}(\,\cdot\,, T)$ induces a morphism of schemes $\overline{Pr}_g \to T$, so the claim follows from (2). Now, a morphism of functors as (4) is the datum for any scheme S of a set-theoretical map

$$F_t(S)\colon \overline{\mathcal{P}\mathrm{r}}_g(S) \longrightarrow \overline{\mathcal{P}}_{2t(g-1),g}(S),$$

satisfying obvious compatibility conditions. Let us define $F_t(S)$ in the following way:

$$(f\colon \mathcal{X} \to S, \boldsymbol{\eta}, \boldsymbol{\beta}) \;\mapsto\; (f\colon (\mathcal{X}, \boldsymbol{\eta} \otimes \omega_f^{\otimes t}) \to S).$$

In order to prove that $F_t(S)$ is well-defined, the only non-trivial matter is to check that the multidegree of $\boldsymbol{\eta} \otimes \omega_f^{\otimes t}$ satisfies the Basic Inequality on each fiber, so the thesis follows from the next Lemma. ∎

Lemma 9. *Let (X, η, β) be a Prym curve. If Y is a complete subcurve of X and d_Y is the degree of $(\eta \otimes \omega_X^{\otimes t})|_Y$, then $m_Y \leq d_Y \leq m_Y + k_Y$ in the notation of the Basic Inequality. Moreover, if $d_Y = m_Y$ then $\widetilde{k}_Y := |\widetilde{Y} \cap \overline{\widetilde{X} \smallsetminus \widetilde{Y}}| = 0$.*

Proof. In the present case, the Basic Inequality simplifies as follows:

$$-\frac{k_Y}{2} \leq e_Y \leq \frac{k_Y}{2},$$

where $e_Y := \deg \eta|_Y$. By the definition of a Prym curve, the degree e_Y depends only on the exceptional components of X intersecting Y.
For any exceptional component E of X with $E \subseteq \overline{X \smallsetminus Y}$, let $m := |E \cap Y|$. The contribution of E to k_Y is m, while its contribution to e_Y is $-\frac{m}{2}$.
Next, for any exceptional component E of X with $E \subseteq Y$, let $l := |E \cap \overline{X \smallsetminus Y}|$. The contribution of E to k_Y is l, while its contribution to e_Y is $1 - \frac{2-l}{2} = \frac{l}{2}$.
Summing up, we see that the Basic Inequality holds. Finally, if $\widetilde{k}_Y \neq 0$, then there exists a non-exceptional component of X intersecting Y. Such a component contributes at least 1 to k_Y, but it does not affect e_Y; hence $-\frac{k_Y}{2} < e_Y$ and the proof is over. ■

By applying [Cap94], Proposition 6.1, from the first part of Lemma 9 we deduce

$$K_{2t(g-1)} \subset H_{2t(g-1)}.$$

Moreover, the second part of the same Lemma provides a crucial information on Hilbert points corresponding to Prym curves.

Lemma 10. *If $h \in K_{2t(g-1)}$, then the orbit of h is closed in the semistable locus.*

Proof. Let (X, η, β) be a Prym curve such that h is the Hilbert point of an embedding $h_t \colon X \to \mathbb{P}^{2t(g-1)-g}$ induced by $\eta \otimes \omega_X^{\otimes t}$. Just recall the first part of [Cap94], Lemma 6.1, which says that the orbit of h is closed in the semistable locus if and only if $\widetilde{k}_Y = 0$ for every subcurve Y of X such that $d_Y = m_Y$, so the thesis is a direct consequence of Lemma 9. ■

Proof of Theorem 6. It is easy to check that $f_t(\overline{Pr}_g) = \Pi_t$. Indeed, if $(X, \eta, \beta) \in \overline{Pr}_g$, then any choice of a base for $H^0(X, \eta \otimes \omega_X^{\otimes t})$ gives an embedding $h_t \colon X \to \mathbb{P}^{2t(g-1)-g}$ and $f_t(X, \eta, \beta) = \pi_{2t(g-1)}(h)$, where $h \in K_{2t(g-1)}$ is the Hilbert point of $h_t(X)$. Conversely, if $\pi_{2t(g-1)}(h) \in \Pi_t$, then there is a Prym curve (X, η, β) and an embedding $h_t \colon X \to \mathbb{P}^{2t(g-1)-g}$ such that h is the Hilbert point of $h_t(X)$ and $f_t(X, \eta, \beta) = \pi_{2t(g-1)}(h)$.
Next we claim that f_t is injective. Indeed, let (X, η, β) and (X', η', β') be two Prym curves and assume that $f_t(X, \eta, \beta) = f_t(X', \eta', \beta')$. Choose bases for $H^0(X, \eta \otimes \omega_X^{\otimes t})$ and $H^0(X', \eta' \otimes \omega_{X'}^{\otimes t})$ and embed X and X' in $\mathbb{P}^{2t(g-1)-g}$. If h and h' are the corresponding Hilbert points, then $\pi_{2t(g-1)}(h) = \pi_{2t(g-1)}(h')$ and the Fundamental Theorem of GIT implies that $\overline{O_G(h)}$ and $\overline{O_G(h')}$ intersect in the semistable locus. It follows from Lemma 10 that $O_G(h) \cap O_G(h') \neq \emptyset$, so $O_G(h) = O_G(h')$ and there is an isomorphism $\sigma \colon (X, \eta, \beta) \to (X', \eta', \beta')$. ■

Observe that Theorem 6 and Lemma 10 imply that $K_{2t(g-1)}$ is a constructible set in $H_{2t(g-1)}$.

3 Fiberwise description

Let Z be a stable curve of genus g. We recall that the dual graph Γ_Z of Z is the graph whose vertices are the irreducible components of Z and whose edges are the nodes of Z. The first Betti number of Γ_Z is $b_1(\Gamma_Z) = \delta - \gamma + 1 = g - g^\nu$, where δ is the number of nodes of Z, γ the number of its irreducible components and g^ν the genus of its normalization.
We denote by Pr_Z the scheme parametrizing Prym curves (X, η, β) such that the stable model of X is Z, modulo inessential isomorphisms, and by S_Z the analogue for spin curves. Since by Lemma 2 the homomorphism β is not relevant in determining the inessential isomorphism class of (X, η, β), in this section we will omit it and just write (X, η).
When $\mathrm{Aut}(Z) = \{\mathrm{Id}_Z\}$, Pr_Z is the scheme-theoretical fiber over $[Z]$ of the morphism $p\colon \overline{Pr}_g \to \overline{\mathcal{M}}_g$. Recall that p is finite of degree 2^{2g}, and étale over $\mathcal{M}_g^0$.
For any 0-dimensional scheme P we denote by $L(P)$ the set of integers occurring as multiplicities of components of P.
In this section we describe the numerical properties of Pr_Z, namely the number of irreducible components and their multiplicities, showing that they depend only on the dual graph Γ_Z of Z. Using this, we give some properties of $L(Pr_Z)$, and show that in some cases the set of multiplicities $L(Pr_Z)$ gives informations on Z. In particular, we show that the morphism $\overline{Pr}_g \to \overline{\mathcal{M}}_g$ is étale over $\overline{\mathcal{M}}_g^0 \smallsetminus D_{irr}$.
We use the techniques and results of [CC03], where the same questions about the numerics of S_Z are studied (see also [CS03], § 3). Quite surprisingly, the schemes P_Z and S_Z are not isomorphic in general.
Finally we will show with an example that, differently from the case of spin curves, the set of multiplicities $L(Pr_Z)$ appearing in Pr_Z does not always identify curves having two smooth components.
Let X be a quasistable curve having Z as stable model and consider the set

$$\Delta_X := \{z \in \mathrm{Sing}\, Z \mid z \text{ is } \textit{not} \text{ the image of an exceptional component of } X\}.$$

Given Z, the quasistable curve X is determined by Δ_X, or equivalently by

$$\Delta_X^c := \mathrm{Sing}\, Z \smallsetminus \Delta_X = \{\text{images in } Z \text{ of the exceptional components of } X\}.$$

Remark that any subset of $\mathrm{Sing}\, Z$ can be seen as a subgraph of the dual graph Γ_Z of Z.
We recall that the *valency* of a vertex of a graph is the number of edges ending in that vertex and a graph Γ is *eulerian* if it has all even valencies. Thus Γ_Z is eulerian if and only if for any irreducible component C of Z, $|C \cap \overline{Z \smallsetminus C}|$ is even. The set $\mathcal{C}_\Gamma$ of all eulerian subgraphs of Γ is called the cycle space of Γ. There is a natural identification of $\mathcal{C}_\Gamma$ with $H_1(\Gamma, \mathbb{Z}_2)$, so $|\mathcal{C}_\Gamma| = 2^{b_1(\Gamma)}$ (see [CC03]). Reasoning as in [CC03], Section 1.3, we can show the following:

Proposition 11. *Let X be a quasistable curve having Z as stable model. The curve X is the support of a Prym curve if and only if Δ_X^c is eulerian. If so, there are $2^{2g^\nu + b_1(\Delta_X)}$ different choices for $\eta \in \operatorname{Pic} X$ such that $(X, \eta) \in Pr_Z$. For each such η, the point (X, η) has multiplicity $2^{b_1(\Gamma_Z) - b_1(\Delta_X)}$ in Pr_Z.*

Hence the number of irreducible components of Pr_Z is

$$2^{2g^\nu} \cdot \sum_{\Sigma \in \mathcal{C}_{\Gamma_Z}} 2^{b_1(\Sigma^c)},$$

and its set of multiplicities is given by

$$L(Pr_Z) = \{2^{b_1(\Gamma_Z) - b_1(\Delta)} \mid \Delta^c \in \mathcal{C}_{\Gamma_Z}\}.$$

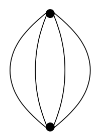
Remark that since $|\mathcal{C}_{\Gamma_Z}| = 2^{b_1(\Gamma_Z)}$, we can check immediately from the proposition that the length of Pr_Z is

$$\sum_{\Sigma \in \mathcal{C}_{\Gamma_Z}} (2^{2g^\nu + b_1(\Sigma^c)} \cdot 2^{b_1(\Gamma_Z) - b_1(\Sigma^c)}) = 2^{b_1(\Gamma_Z)} \cdot 2^{2g^\nu + b_1(\Gamma_Z)} = 2^{2g}.$$

As a consequence of Proposition 11, we see that

- *a point (X, η) in Pr_Z is non reduced if and only if X is non stable.*

Example (curves having two smooth components). Let $Z = C_1 \cup C_2$, C_i smooth irreducible, $|C_1 \cap C_2| = \delta \geq 2$.


Γ_Z

Let X be a quasistable curve having Z as stable model and let Δ_X be the corresponding subset of $\operatorname{Sing} Z$. The subgraph Δ_X^c is eulerian if and only if $|\Delta_X^c|$ is even. Therefore X is support of a Prym curve if and only if it has an even number $2r$ of exceptional components. If so, for each choice of $\eta \in \operatorname{Pic} X$ such that $(X, \eta) \in Pr_Z$, this point will have multiplicity $2^{b_1(\Gamma_Z) - b_1(\Delta_X)}$. We have $b_1(\Gamma_Z) = \delta - 1$ and $|\Delta_X| = \delta - 2r$, so

$$b_1(\Delta_X) = \begin{cases} \delta - 2r - 1 & \text{if } 2r \leq \delta - 2, \\ 0 & \text{if } \delta - 1 \leq 2r \leq \delta \end{cases}$$

and we get

$$L(Pr_Z) = \{2^{2r} \mid 0 \leq r \leq \frac{1}{2}\delta - 1\} \cup \{2^{\delta - 1}\}.$$

Proposition 12 (combinatorial properties of $L(Pr_Z)$). *The following properties hold:*

(1) $1 \in L(Pr_Z)$*;*

(2) $\max L(Pr_Z) = 2^{b_1(\Gamma_Z)}$*;*

(3) $2^g \in L(Pr_Z)$ *if and only if* Z *has only rational components;*

(4) Pr_Z *is reduced if and only if* Z *is of compact type;*

(5) if Γ_Z *is an eulerian graph, then* $L(Pr_Z) = L(S_Z)$*.*

Proof. (1) Choosing $\Delta_X = \Gamma_Z$, we get $X = Z$; since the empty set is trivially in $\mathcal{C}_{\Gamma_Z}$, there always exists $\eta \in \operatorname{Pic} Z$ such that $(Z, \eta) \in Pr_Z$. This η is a square root of $\mathcal{O}_Z$; there are $2^{2g^\nu + b_1(\Gamma_Z)}$ choices for it, and it will appear with multiplicity 1 in Pr_Z. So $1 \in L(Pr_Z)$.
(2) From Proposition 11 we get $\max L(Pr_Z) \leq 2^{b_1(\Gamma_Z)}$. Set $M = \max\{b_1(\Sigma) \,|\, \Sigma \in \mathcal{C}_{\Gamma_Z}\}$ and let $\Sigma_0 \in \mathcal{C}_{\Gamma_Z}$ be such that $b_1(\Sigma_0) = M$. By Proposition 11, we know that $2^{b_1(\Gamma_Z) - b_1(\Sigma_0^c)} \in L(Pr_Z)$. We claim that $b_1(\Sigma_0^c) = 0$. Indeed, if not, Σ_0^c contains a subgraph σ with $b_1(\sigma) = 1$ and having all valencies equal to 2. Then $\Sigma_0 \cup \sigma \in \mathcal{C}_{\Gamma_Z}$ and $b_1(\Sigma_0 \cup \sigma) > M$, a contradiction. Hence we have points of multiplicity $2^{b_1(\Gamma_Z)}$ in Pr_Z, so $\max L(Pr_Z) = 2^{b_1(\Gamma_Z)}$.
Property (3) is immediate from (2), since $b_1(\Gamma_Z) = g$ if and only $g^\nu = 0$.
Also property (4) is immediate from (2), because $L(Pr_Z) = \{1\}$ if and only if $b_1(\Gamma_Z) = 0$.
(5) Assume that Γ_Z is eulerian. Then $\Delta_X^c \in \mathcal{C}_{\Gamma_Z}$ if and only if $\Delta_X \in \mathcal{C}_{\Gamma_Z}$, so we have

$$L(Pr_Z) = L(S_Z) = \{2^{b_1(\Gamma_Z) - b_1(\Delta_X)} \,|\, \Delta_X \in \mathcal{C}_{\Gamma_Z}\}$$

(see [CC03] for the description of $L(S_Z)$). ∎

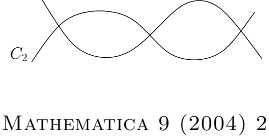
Property (4) implies the following

Corollary 13. *The morphism* $p\colon \overline{Pr}_g \to \overline{\mathcal{M}}_g$ *is étale over* $\overline{\mathcal{M}}_g^0 \smallsetminus D_{irr}$*.*

Consider now property (1) of Proposition 12. It shows, in particular, that in general Pr_Z and S_Z are not isomorphic and do not have the same set of multiplicities: indeed, for spin curves, it can very well happen that $1 \notin L(S_Z)$ (see example after Corollary 14).
The following shows that in some cases, $L(Pr_Z)$ gives informations on Z.

Corollary 14. *Let* Z *be a stable curve and* $\nu\colon Z^\nu \to Z$ *its normalization. Assume that for every irreducible component* C *of* Z*, the number* $|\nu^{-1}(C \cap \operatorname{Sing} Z)|$ *is even and at least* 4*.*

(i) If $2^{b_1(\Gamma_Z)-2} \notin L(Pr_Z)$*, then* $Z = C_1 \cup C_2$*, with* C_1 *and* C_2 *smooth and irreducible.*



(ii) If $2^{b_1(\Gamma_Z)-3} \notin L(Pr_Z)$, then either Z is irreducible with two nodes, or $Z = C_1 \cup C_2 \cup C_3$, with C_i smooth irreducible and $|C_i \cap C_j| = 2$ for $1 \leq i < j \leq 3$.

Proof. By hypothesis Γ_Z is eulerian, so property (5) says that $L(Pr_Z) = L(S_Z)$. Then (i) follow immediately from [CC03], Theorem 11. Let us show (ii). If $b_1(\Gamma_Z) \geq 4$, by property (2) we can apply [CC03], Theorem 13; then Z has three smooth components meeting each other in two points. Assume $b_1(\Gamma_Z) \leq 3$ and let δ, γ be the number of nodes and of irreducible components of Z. Since all vertices of Γ_Z have valency at least 4, we have $\delta \geq 2\gamma$, so $\gamma \leq b_1(\Gamma_Z) - 1 \leq 2$. Then by an easy check we see that the only possibility which satisfies all the hypotheses is $\gamma = 1$ and $\delta = 2$. ■

In [CC03] it is shown (Theorem 11) that $L(S_Z)$ allows to recover curves having two smooth components. Instead, when the number of nodes is odd, it is no more true that these curves are characterized by $L(Pr_Z)$. For instance, consider the graphs:

It is easy to see that if Z_1, Z_2 are stable curves with $\Gamma_{Z_i} = \Gamma_i$ for $i = 1, 2$, we have $L(P_{Z_1}) = L(P_{Z_2}) = \{1, 4, 16\}$, while $L(S_{Z_1}) = \{4, 8, 16\}$ and $L(S_{Z_2}) = \{2, 8, 16\}$.

References

[ABH01] Valery Alexeev, Christina Birkenhake, and Klaus Hulek. Degenerations of Prym varieties. *J. Reine Angew. Math.*, 553:73–116, 2001.

[ACGH85] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, and Joseph Harris. *Geometry of Algebraic Curves, I*, volume 267 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1985.

[Bea77] Arnaud Beauville. Prym varieties and the Schottky problem. *Invent. Math.*, 41:149–196, 1977.

[Bea96] Arnaud Beauville. *Complex Algebraic Surfaces*. Cambridge University Press, second edition, 1996.

[Ber99] Mira Bernstein. *Moduli of curves with level structure*. PhD thesis, Harvard University, 1999. Available at the web page `http://www.math.berkeley.edu/~mira`.

[Cap94] Lucia Caporaso. A compactification of the universal Picard variety over the moduli space of stable curves. *J. Amer. Math. Soc.*, 7(3):589–660, 1994.

[CC03] Lucia Caporaso and Cinzia Casagrande. Combinatorial properties of stable spin curves. *Comm. Algebra*, 31(8):3653–3672, 2003. Special Issue in Honor of Steven L. Kleiman.

[Cor89] Maurizio Cornalba. Moduli of curves and theta-characteristics. In *Lectures on Riemann Surfaces: Proceedings of the College on Riemann Surfaces, International Centre for Theoretical Physics, Trieste, 1987*, pages 560–589. World Scientific, 1989.

[Cor91] Maurizio Cornalba. A remark on the Picard group of spin moduli space. *Atti dell'Accademia Nazionale dei Lincei, Matematica e Applicazioni*, IX(2):211–217, 1991.

[CS03] Lucia Caporaso and Edoardo Sernesi. Characterizing curves by their odd theta-characteristics. *J. Reine Angew. Math.*, 562:101–135, 2003.

[DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publ. Math. I. H. E. S.*, 36:75–109, 1969.

[DS81] Ron Donagi and Roy Campbell Smith. The structure of the Prym map. *Acta Math.*, 146:25–102, 1981.

[Fon02] Claudio Fontanari. On the geometry of the compactification of the universal Picard variety. Preprint math.AG/0202168, 2002.

[Ray71] Michel Raynaud. Exposé XII. Géométrie algébrique et géométrie analytique. In *Revêtements Étales et Groupe Fondamental. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA1). Dirigé par Alexandre Grothendieck*, volume 224 of *Lecture Notes in Mathematics*, pages 311–343. Springer-Verlag, 1971.

Edoardo Ballico
Università degli Studi di Trento
Dipartimento di Matematica
Via Sommarive 14
38050 Povo (Trento), Italy
ballico@science.unitn.it

Claudio Fontanari
Università degli Studi di Trento
Dipartimento di Matematica
Via Sommarive 14
38050 Povo (Trento), Italy
fontanar@science.unitn.it

Cinzia Casagrande
Università degli Studi Roma Tre
Dipartimento di Matematica
Largo S. L. Murialdo 1
00146 Roma, Italy
casagran@mat.uniroma3.it