Bounds for the Anticanonical Bundle of a Homogeneous Projective Rational Manifold

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Abstract. The following bounds for the anticanonical bundle $K_X^* = \det T_X$ of a complex homogeneous projective rational manifold X of dimension n are established:

$$3^n \leq \dim H^0(X, K_X^*) \leq \binom{2n+1}{n} \quad \text{and} \quad 2^n n! \leq \deg K_X^* \leq (n+1)^n$$

with equality in the lower bounds if and only if X is a flag manifold and equality in the upper bounds if and only if X is complex projective space. None of these bounds holds for general Fano manifolds.

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The homogeneous compact complex manifolds X that admit an equivariant embedding in projective space are precisely the quotients $X = G/P$ where G is a semisimple complex Lie group and P is a parabolic subgroup. Moreover, any such quotient is rational and has a very ample anticanonical bundle, $K_X^* = \det T_X$. In particular, X is a Fano manifold.

Various bounds have been established for the numerical invariants of K_X^* when X is a general Fano manifold, see [6, 8, 9, 10]. For example, there exists a constant $c(n)$ that depends only on $n = \dim X$ such that $\deg K_X^* \leq c(n)^n$. In this article we establish the following bounds when $X = G/P$:

$$3^n \leq \dim H^0(X, K_X^*) \leq \binom{2n+1}{n} \quad \text{and} \quad 2^n n! \leq \deg K_X^* \leq (n+1)^n$$

with equality in the lower bounds if and only if X is a flag manifold (i.e., P is a Borel subgroup of G), and equality in the upper bounds if and only if X is complex projective space, $\mathbf{P}^n$.

These bounds do not hold for general Fano manifolds. For example, let $X = \mathbf{P}(\mathcal{O}_{\mathbf{P}^{n-1}} \oplus \mathcal{O}_{\mathbf{P}^{n-1}}(n-1))$. Then X is a $\mathbf{P}^1$-bundle over $\mathbf{P}^{n-1}$, $\pi : X \to \mathbf{P}^{n-1}$, and $K_X^* = \pi^*\mathcal{O}_{\mathbf{P}^{n-1}}(1) \otimes \xi^2$ where ξ is the tautological line bundle on X whose restriction to any fiber $\mathbf{P}^1$ of π gives $\xi|_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(1)$. It follows that X is a Fano manifold with $\dim H^0(X, K_X^*) = n + \binom{2n-1}{n-1} + \binom{3n-2}{n-1}$ and $\deg K_X^* = ((2n-1)^n - 1)/(n-1)$. An example where the lower bounds do not hold is given by $X = S \times (\mathbf{P}^1)^{n-2}$ where S is a del Pezzo surface.

In the homogeneous case there are well-known formulas from representation theory that can be used to calculate $\dim H^0(X, K_X^*)$ and $\deg K_X^*$ exactly. However, these formulas, which are products of rational numbers indexed by the roots of the group, do not easily lend themselves to comparison with expressions in $n = \dim X$. The point of this paper is to overcome this difficulty. The bounds are proved by first reducing to the case of simple Lie groups and then showing for each classical type that the known formulas can be broken up into subproducts of certain simple sequences. These subproducts are shown to satisfy inequalities that can be combined to yield the desired inequalities for the full product. The bounds for the exceptional types are verified through exhaustive calculations.

The above upper bounds can be trivially extended to any homogeneous compact complex manifold $X = G/H$. The sections of K_X^* define an equivariant map of X to projective space that coincides with the normalizer fibration $G/H \to G/N$, $N = N_G(H^0)$, [1, p.79]. Since the base $Y = G/N$ is a homogeneous projective rational manifold, the upper bounds hold for Y and hence for X.

For a homogeneous projective rational manifold X, the dimension of the holomorphic automorphism group, $\dim \mathrm{Aut}(X) = \dim H^0(X, T_X)$, never exceeds $n(n+2)$. In fact, this bound holds when X is any homogeneous compact Kähler manifold [5]. However, there are homogeneous compact complex manifolds for which $\dim \mathrm{Aut}(X)$ grows exponentially in n, see [12]. In [13], the above estimate for $\dim H^0(X, K_X^*)$ plays an important role in establishing the following bound for the non-Kähler case: $\dim \mathrm{Aut}(X) \le n^2 - 1 + \binom{2n-1}{n-1} \sim O(2^{2n-1}/\sqrt{(n-1)\pi})$.

1 Roots and Weights

In this section we introduce some notation and well-known facts about semisimple Lie groups [2, 7], and recall a formula for finding the weight μ_X associated to the line bundle K_X^* when $X = G/P$ [4, 11].

Let G be a semisimple complex Lie group and let T be a maximal torus of G. Let $\mathrm{Lie}(G)$ and $\mathrm{Lie}(T)$ be the corresponding Lie algebras. Let $\Phi \subset \mathrm{Lie}(T)^*$ denote the roots of G with respect to T and let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a system of simple roots. Let Φ^+ denote the subset of positive roots—those that are positive integral combinations of the simple roots. For any root $\alpha \in \Phi$, let $e_\alpha \in \mathrm{Lie}(G)$ be the corresponding root vector: $[x, e_\alpha] = \alpha(x)e_\alpha$ for all $x \in \mathrm{Lie}(T)$.

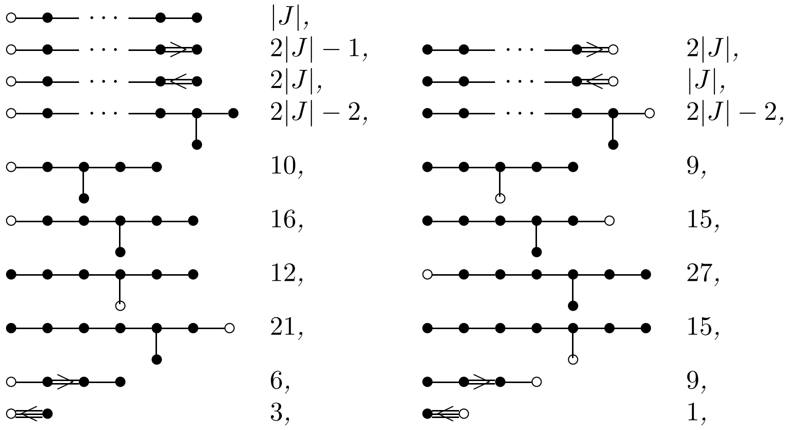
Let $\lambda_1, \ldots, \lambda_\ell$ be the fundamental dominant weights of G—those weights defined by $\langle \lambda_i, \alpha_j \rangle = 2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ where (,) denotes the Killing form.

Any weight $\mu \in \mathrm{Lie}(T)^*$ can be written $\mu = \sum_{i=1}^{\ell} \langle \mu, \alpha_i \rangle \lambda_i$.
A Borel subgroup is a maximal solvable subgroup of G, and any such subgroup is conjugate to the subgroup B generated by T and the root groups $\exp \mathbf{C} e_\alpha$, for all $\alpha \in -\Phi^+$. Let P be a parabolic subgroup of G, that is, a subgroup containing a Borel subgroup. We may assume that P contains B. Let $P = R \cdot S$ be a Levi decomposition of P where R is a maximal solvable normal subgroup of P and S is semisimple. We let Φ_P denote the subsystem of roots of S and let $\Phi_P^+ = \Phi_P \cap \Phi^+$. Let I denote the subset of indexes, $I \subset \{1, \ldots, \ell\}$, such that $\Phi_P^+ \cap \{\alpha_1, \ldots, \alpha_\ell\} = \{\alpha_i\}_{i \in I}$. The conjugacy class of P is uniquely determined by I and any such choice of indexes is associated to a parabolic subgroup of G.
Let $X = G/P$, and define $\Phi_X^+ = \Phi^+ \setminus \Phi_P^+$. Since T_X is generated at the identity coset by the root vectors e_α for $\alpha \in \Phi_X^+$, the anticanonical bundle $K_X^* = \det T_X$, $n = \dim X$, is the homogeneous line bundle associated to the weight

$$\mu_X = \sum_{\alpha \in \Phi_X^+} \alpha$$

The weight μ_X is dominant: $\langle \mu_X, \alpha_i \rangle > 0$ for $i \notin I$, and $\langle \mu_X, \alpha_i \rangle = 0$ for $i \in I$. In particular, K_X^* is a very ample line bundle and μ_X is orthogonal to the roots Φ_P^+. If $P = B$, X is called a flag manifold.
We now recall a simple formula for calculating the coefficients $\langle \mu_X, \alpha_i \rangle$ of μ_X, see [11]: A set of indexes J is called connected if the subdiagram of the Dynkin diagram of G corresponding to the simple roots α_j, $j \in J$, is connected. An index i is said to be adjacent to J if $i \notin J$ and $J_0 \cup \{i\}$ is connected for some connected component J_0 of J. The set of indexes adjacent to J is denoted by ∂J. The number of elements in J is denoted $|J|$.

Definition 1 *Let J be a connected set of indexes. For $i \notin \partial J$ define $\nu_i(J) = 0$. For $i \in \partial J$ define $\nu_i(J)$ to be the number next to the appropriate diagram below. The black nodes correspond to J and the white node corresponds to i. Symmetry of Dynkin diagrams is tacitly assumed.*



For an arbitrary set of indexes I define $\nu_i(I) = \nu_i(I_1) + \cdots + \nu_i(I_p)$ where $I_1, \ldots, I_p$ are the connected components of I.

Proposition 1 ([11]) *Let $X = G/P$ where G is a semisimple complex Lie group and P is a parabolic subgroup defined by a set of indexes I. Let μ_X be the weight of the anticanonical bundle K_X^* of X. Then*

$$\mu_X = \sum_{i \notin I} (2 + \nu_i(I))\lambda_i$$

2 Estimating Products

We now prove some estimates for various products that appear in the proof of the main theorem.

Lemma 1 *For any non-negative integers s and t,*

$$\binom{2t+1}{t}\binom{2s+1}{s} \leq \binom{2(t+s)+1}{t+s} \tag{1}$$

$$\frac{(t+1)^t}{t!} \cdot \frac{(s+1)^s}{s!} \leq \frac{(t+s+1)^{t+s}}{(t+s)!} \tag{2}$$

with equality if and only if t or s is 0.

Proof. The inequalities are obviously equalities when s or t is 0. So we assume $t, s > 0$ and show strict inequalities hold for (1) and (2) by fixing s and applying induction on t. They are easily seen to hold for $t = 1$. Let $g(t) = \binom{2t+1}{t}$ (resp., $(t+1)^t/t!$), and let $f(t) = g(t+1)/g(t) = 4 - 2/(t+2)$ (resp., $[1+1/(t+1)]^{t+1}$), an increasing function of $t > 0$. By the induction hypothesis,

$$g(t+1)g(s) = f(t)g(t)g(s) < f(t)g(t+s) < f(t+s)g(t+s) = g(t+s+1)$$

$\square$

Definition 2 *Let t and s be positive integers. A* simple sequence *(of length s) is a set S of rational numbers of the form*

$$S = S(t,s) = \left\{ \frac{3t+s-1+i}{t+i} \,\middle|\, 0 \leq i \leq s-1 \right\}$$

The shifted sequence *of $S(t,s)$ is*

$$S' = S'(t,s) = \left\{ \frac{3t+s-1+i}{t+i} - 1 \,\middle|\, 0 \leq i \leq s-1 \right\}$$

The products of the numbers in S and S' are denoted by

$$\Pi S = \prod_{i=0}^{s-1} \frac{3t+s-1+i}{t+i} = \binom{3t+2s-2}{s} \Big/ \binom{t+s-1}{s}$$

$$\Pi S' = \prod_{i=0}^{s-1} \frac{2t+s-1}{t+i} = \frac{(2t+s-1)^s (t-1)!}{(t+s-1)!}$$

Lemma 2 *Let $S(t,s)$ be a simple sequence and let $S'(t,s)$ be the shifted sequence of $S(t,s)$.*

1. *If the first and last elements of $S(t,s)$ are removed, the remaining set is the simple sequence $S(t+1,s-2)$.*
2. *$\Pi S(t,s)$ and $\Pi S'(t,s)$ are decreasing in t. In particular,*

$$3^s = \lim_{t\to\infty} \Pi S(t,s) \le \Pi S(t,s) \;\le\; \Pi S(1,s) = \binom{2s+1}{s}$$
$$2^s = \lim_{t\to\infty} \Pi S'(t,s) \le \Pi S'(t,s) \;\le\; \Pi S'(1,s) = \frac{(s+1)^s}{s!}$$

Proof. The first assertion is immediate from the definition. To prove the second assertion, let $f(t) = \Pi S(t,s)$ and let $m = [(s-1)/2]$ be the least integer $\le (s-1)/2$. Then, for $t > 0$,

$$\begin{aligned}\frac{d}{dt}\log f(t) &= \sum_{i=0}^{s-1} \frac{2i-(s-1)}{(3t+s-1+i)(t+i)} \\ &= \sum_{i=0}^{m} -\frac{s-1-2i}{(3t+s-1+i)(t+i)} + \frac{s-1-2i}{(3t+2s-2-i)(t+s-1-i)} \le 0\end{aligned}$$

and hence f is decreasing.
Now let $g(t) = \Pi S'(t,s)$ and define $h(t) = g(t+1)/g(t) = [1+2/(2t+s-1)]^s t/(t+s)$. Then

$$\frac{d}{dt}\log h(t) = \frac{s(s^2-1)}{t(t+s)((2t+s)^2-1)} \ge 0$$

so h is increasing and approaches 1 as $t\to\infty$. Therefore, g is decreasing. $\square$

3 Bounds for K_X^*

Theorem 1 *Let X be a homogeneous projective rational manifold of dimension n. Then*

$$3^n \le \dim H^0(X, K_X^*) \le \binom{2n+1}{n} \quad \text{and} \quad 2^n n! \le \deg K_X^* \le (n+1)^n$$

with equality in the lower bounds if and only if X is a flag manifold and equality in the upper bounds if and only if $X = \mathbf{P}^n$.

Proof. Write $X = G/P$ where G is a semisimple Lie group and P is a parabolic subgroup. Let I be the subset of indexes that defines P, and let $I_1, \ldots, I_m$ be the connected components of I. Let μ_X be the weight of the anticanonical bundle as given in Proposition 1 so that $H^0(X, K_X^*)$ is the irreducible representation of G with highest weight $\mu_X = \sum_{i \notin I}(2+\nu_i(I))\lambda_i$. Set $\delta = (1/2)\sum_{\alpha>0}\alpha = \lambda_1 + \cdots + \lambda_\ell$. By the Weyl dimension formula [7],

$$h = \dim H^0(X, K_X^*) = \prod_{\alpha \in \Phi_X^+} \frac{(\mu_X + \delta, \alpha)}{(\delta, \alpha)} \tag{3}$$

and the degree of K_X^* is given by [4]

$$d = \deg K_X^* = n! \prod_{\alpha \in \Phi_X^+} \frac{(\mu_X, \alpha)}{(\delta, \alpha)} \tag{4}$$

Let $G_1, \ldots, G_r$ be the simple factors of G. Then $X = X_1 \times \cdots \times X_r$ where $X_i = G_i/G_i \cap P$. Let $n = \dim X$, $n_i = \dim X_i$, $h_i = \dim H^0(X_i, K_{X_i}^*)$ and $d_i = \deg K_{X_i}^*$, $1 \le i \le r$. If $3^{n_i} \le h_i \le \binom{2n_i+1}{n_i}$ and $2^{n_i} n_i! \le d_i \le (n_i+1)^{n_i}$, $1 \le i \le r$, then the above formulas along with Lemma 1 imply $3^n \le h = h_1 \cdots h_r \le \prod_{i=1}^r \binom{2n_i+1}{n_i} \le \binom{2n+1}{n}$ and $2^n n! \le d = n!\,(d_1/n_1!) \cdots (d_r/n_r!) \le n! \prod_{i=1}^r (n_i+1)^{n_i}/n_i! \le (n+1)^n$, since $n = n_1 + \cdots + n_r$. We may therefore assume that G is simple.

The theorem can be verified by direct calculation for each of the exceptional simple Lie groups and their finite number of conjugacy classes of parabolic subgroups. While the details are too lengthy to include in this article, the results can be summarized as follows. The minimum of h is 3^n and is achieved only for Borel subgroups. The maximum of h is always strictly less than $\binom{2n+1}{n}$. In fact, the minimum of $\binom{2n+1}{n}/h$ over all parabolic subgroups for each type is greater than 3.11 for E_6, 9.96 for E_7, 758.2 for E_8, 3.24 for F_4, and 1.22 for G_2, and this minimum is achieved for the maximal parabolic subgroups defined by $I = \{2, \ldots, \ell\}$, or $\{1, 2, 3\}$ for F_4 (the simple roots are indexed from left to right in the diagrams shown in Definition 1).

The proof for the classical types A_ℓ, B_ℓ, C_ℓ and D_ℓ is accomplished by showing that the product (3) can be written as a product of simple sequences $S_1, \ldots, S_\sigma$. For, if we know that $h = \Pi S_1 \cdots \Pi S_\sigma$, it follows from (4) that $d = n!\,\Pi S_1' \cdots \Pi S_\sigma'$, and by Lemmas 1 and 2, we obtain $3^n \le h \le \prod_{i=1}^\sigma \binom{2|S_i|+1}{|S_i|} \le \binom{2n+1}{n}$ and $2^n n! \le d \le n! \prod_{i=1}^\sigma (|S_i|+1)^{|S_i|}/|S_i|! \le (n+1)^n$, since $n = |S_1| + \cdots + |S_\sigma|$. We now prove that such a decomposition of (3) is possible for each simple classical type.

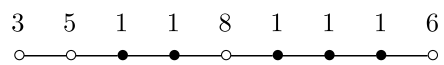
Type A_ℓ: Let $\mu_X + \delta = m_1\lambda_1 + \cdots + m_\ell\lambda_\ell$, where $m_i = \langle \mu_X, \alpha_i \rangle + 1$, $1 \le i \le \ell$, and let $\epsilon_1, \ldots, \epsilon_{\ell+1}$ denote the standard orthonormal basis of $\mathbf{R}^{\ell+1}$. The simple roots for type A_ℓ are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le \ell$ and the positive roots are $\alpha_i + \cdots + \alpha_{j-1} = \epsilon_i - \epsilon_j$, $1 \le i < j \le \ell+1$. The dimension formula (3) becomes $h = \prod a_{ij}$ where $a_{ij} = (m_i + \cdots + m_{j-1})/(j-i)$ and the product is taken over all indexes $i < j$ that are not both in same connected component of I.

According to Proposition 1, the coefficients $m_1, \ldots, m_\ell$ are given by

$$m_i = \begin{cases} 1 & \text{if } i \in I \\ 3 + |I_\nu| & \text{if } i \in \partial I_\nu \text{ for some } \nu \\ 3 + |I_\nu| + |I_{\nu+1}| & \text{if } i \in \partial I_\nu \cap \partial I_{\nu+1} \text{ for some } \nu \\ 3 & \text{otherwise} \end{cases} \tag{5}$$

An example is given by the list of numbers at the top of Figure 1 (the indexes in I correspond to black nodes).
The numbers in the product $h = \prod a_{ij}$ can be arranged into rectangular arrays as follows. Let $i_1 < \cdots < i_k$ be an ordered list of those indexes i not in I and set $i_0 = 0$, $i_{k+1} = \ell + 1$. For $1 \le p \le q \le k$, define $R_{pq} = \{a_{ij} \,|\, i_{p-1} < i \le i_p,\ i_q < j \le i_{q+1}\}$, as illustrated in Figure 1.

Figure 1: Type A_ℓ decomposition



$\frac{3}{1}$	$\frac{8}{2}$	$\frac{9}{3}$	$\frac{10}{4}$	$\frac{18}{5}$	$\frac{19}{6}$	$\frac{20}{7}$	$\frac{21}{8}$	$\frac{27}{9}$
	$\frac{5}{1}$	$\frac{6}{2}$	$\frac{7}{3}$	$\frac{15}{4}$	$\frac{16}{5}$	$\frac{17}{6}$	$\frac{18}{7}$	$\frac{24}{8}$
				$\frac{10}{3}$	$\frac{11}{4}$	$\frac{12}{5}$	$\frac{13}{6}$	$\frac{19}{7}$
				$\frac{9}{2}$	$\frac{10}{3}$	$\frac{11}{4}$	$\frac{12}{5}$	$\frac{18}{6}$
				$\frac{8}{1}$	$\frac{9}{2}$	$\frac{10}{3}$	$\frac{11}{4}$	$\frac{17}{5}$
								$\frac{9}{4}$
								$\frac{8}{3}$
								$\frac{7}{2}$
								$\frac{6}{1}$

Then h is the product of the numbers in all the rectangular arrays R_{pq}, $1 \le p \le q \le k$. Each R_{pq} consists of rational numbers whose numerators and denominators both increase by 1 in each row and column, starting in the lower left corner, $a_{i_p(i_q+1)}$. From (5) it follows that $a_{i_p(i_q+1)}$ has the form $(3t+s-1)/t$ where $t = i_q - i_p + 1$ and s is the number of rows + columns $- 1 = (i_p - i_{p-1}) + (i_{q+1} - i_q) - 1$. Therefore, R_{pq} may be decomposed into simple sequences, $R_{pq} = S_0 \cup \ldots \cup S_\sigma$ where $S_0 = \{a_{ij} \,|\, i = i_{p-1}+1 \text{ or } j = i_q+1\} = S(t,s)$ is the set of numbers in the left column and the top row of R_{pq}, and $S_i = S(t+i, s-2i)$ is obtained by removing the lower left and top right number from S_{i-1}, $1 \le i \le \sigma$, as illustrated in Figure 2.

Figure 2: Type A_ℓ rectangular array

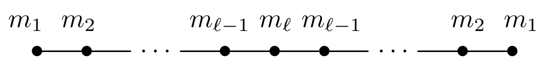
$\frac{10}{3}$	$\frac{11}{4}$	$\frac{12}{5}$	$\frac{13}{6}$
$\frac{9}{2}$	$\frac{10}{3}$	$\frac{11}{4}$	$\frac{12}{5}$
$\frac{8}{1}$	$\frac{9}{2}$	$\frac{10}{3}$	$\frac{11}{4}$

Repeating this procedure for each rectangular array R_{pq}, $1 \leq p \leq q \leq k$, shows that for type A_ℓ the product (3) can be decomposed into a product of simple sequences.

Type B_ℓ: We now show the same type of decomposition is possible for type B_ℓ by embedding the appropriate numbers into a diagram for type $A_{2\ell-1}$. We again write $\mu_X + \delta = m_1\lambda_1 + \cdots + m_\ell\lambda_\ell$, $m_i = \langle \mu_X, \alpha_i \rangle + 1$, $1 \leq i \leq \ell$, and let $\epsilon_1, \ldots, \epsilon_\ell$ denote the standard orthonormal basis of $\mathbf{R}^\ell$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq \ell - 1$, and $\alpha_\ell = \epsilon_\ell$. The positive roots are $\alpha_i + \cdots + \alpha_{j-1} = \epsilon_i - \epsilon_j$, $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_\ell = \epsilon_i + \epsilon_j$, $1 \leq i < j \leq \ell$, and $\alpha_i + \cdots + \alpha_\ell = \epsilon_i$, $1 \leq i \leq \ell$. The dimension formula (3) becomes $h = \prod a_{ij} \times \prod b_{ij}$ where $a_{ij} = (m_i + \cdots + m_{j-1})/(j-i)$, $1 \leq i < j \leq \ell$, $b_{ij} = (m_i + \cdots + m_{j-1} + 2m_j + \cdots + 2m_{\ell-1} + m_\ell)/(2\ell - i - j + 1)$, $1 \leq i \leq j \leq \ell$. To avoid trivial factors, these products should be taken over i, j not in the same connected component of I, although in the following arguments it is convenient to include all terms.

Define $\hat{I} = \{i \,|\, i \in I \text{ or } 2\ell - i \in I\}$. Then $\hat{I}$ defines a parabolic subgroup $\hat{P}$ of a simple group $\hat{G}$ of type $A_{2\ell-1}$. Let $\hat{X} = \hat{G}/\hat{P}$. By Proposition 1, the coefficients of $\mu_X + \delta$ appear as the first half of the coefficients of $\mu_{\hat{X}} + \delta$, see Figure 3.

Figure 3: Conversion of type B_ℓ to type $A_{2\ell-1}$

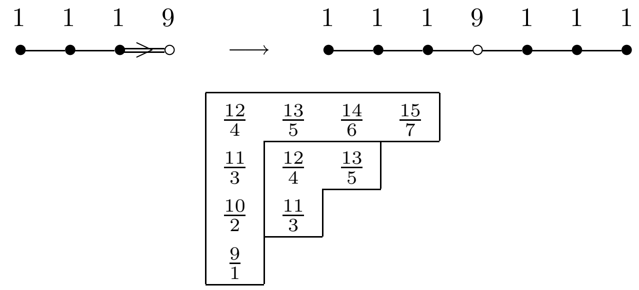


For a fixed i the product $h_i = \prod a_{ij} \times \prod b_{ij}$ can be arranged as

$$\frac{m_i}{1} \cdot \frac{m_i + m_{i+1}}{2} \cdots \frac{m_i + \cdots + m_\ell}{\ell - i + 1} \cdot \frac{s_i + m_{\ell-1}}{\ell - i + 2} \cdots \frac{s_i + m_{\ell-1} + \cdots + m_i}{2(\ell - i) + 1}$$

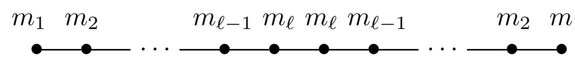
where $s_i = m_i + \cdots + m_\ell$. Therefore, the non-trivial terms in h correspond to the numbers in the upper left half of the rectangular arrays R_{pq} defined for type $A_{2\ell-1}$. These triangular arrays can clearly be broken up into simple sequences, see Figure 4, showing that h is a product of simple sequences.

Figure 4: Type B_ℓ decomposition



Type C_ℓ: The proof for this case is almost identical to type B_ℓ. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le \ell - 1$, and $\alpha_\ell = 2\epsilon_\ell$. The positive roots are $\alpha_i + \cdots + \alpha_{j-1} = \epsilon_i - \epsilon_j$, $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = \epsilon_i + \epsilon_j$, $1 \le i < j \le \ell$, and $2\alpha_i + \cdots + 2\alpha_\ell + \alpha_\ell = 2\epsilon_i$, $1 \le i \le \ell$. The dimension formula (3) becomes $h = \prod a_{ij} \times \prod b_{ij}$ where $a_{ij} = (m_i + \cdots + m_{j-1})/(j-i)$, $1 \le i < j \le \ell$, $b_{ij} = (m_i + \cdots + m_{j-1} + 2m_j + \cdots + 2m_\ell)/(2\ell - i - j + 2)$, $1 \le i \le j \le \ell$.
Define $\hat{I} = \{i \,|\, i \in I \text{ or } 2\ell - i + 1 \in I\}$. Then $\hat{I}$ defines a parabolic subgroup $\hat{P}$ of a simple group $\hat{G}$ of type $A_{2\ell}$. Let $\hat{X} = \hat{G}/\hat{P}$. By Proposition 1, the coefficients of $\mu_X + \delta$ appear as the first half of the coefficients of $\mu_{\hat{X}} + \delta$, see Figure 5.

Figure 5: Conversion of type C_ℓ to type $A_{2\ell}$

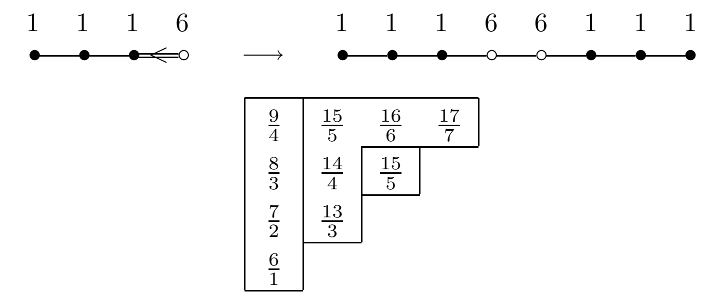


For a fixed i the product $h_i = \prod a_{ij} \times \prod b_{ij}$ can be arranged as

$$\frac{m_i}{1} \cdot \frac{m_i + m_{i+1}}{2} \cdots \frac{m_i + \cdots + m_\ell}{\ell - i + 1} \cdot \frac{s_i + m_\ell}{\ell - i + 2} \cdots \frac{s_i + m_\ell + \cdots + m_{i+1}}{2(\ell - i) + 1}$$

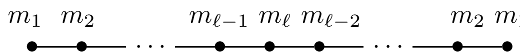
where $s_i = m_i + \cdots + m_\ell$. Therefore, the non-trivial terms in h correspond to the numbers in the upper left half (above the diagonal) of the rectangular arrays R_{pq} defined for type $A_{2\ell}$. These triangular arrays can be broken up into simple sequences as before, see Figure 6, showing that h is a product of simple sequences.
Type D_ℓ: The proof for this case must be handled somewhat differently than the previous two cases. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le \ell - 1$, and $\alpha_\ell = \epsilon_{\ell-1} + \epsilon_\ell$. The positive roots are $\alpha_i + \cdots + \alpha_{j-1} = \epsilon_i - \epsilon_j$, $1 \le i < j \le \ell$,

Figure 6: Type C_ℓ decomposition



$\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell = \epsilon_i + \epsilon_j$, $1 \le i < j \le \ell - 1$, and $\alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell = \epsilon_i + \epsilon_\ell$, $1 \le i \le \ell - 2$. The dimension formula (3) becomes $h = \prod a_{ij} \times \prod b_{ij} \times \prod c_i$ where $a_{ij} = (m_i + \cdots + m_{j-1})/(j-i)$, $1 \le i < j \le \ell$, $b_{ij} = (m_i + \cdots + m_{j-1} + 2m_j + \cdots + 2m_{\ell-2} + m_{\ell-1} + m_\ell)/(2\ell - i - j)$, $1 \le i < j \le \ell - 1$, $c_i = (m_i + \cdots + m_{\ell-2} + m_\ell)/(\ell - i)$, $1 \le i \le \ell - 2$, and $c_{\ell-1} = m_\ell$.
By symmetry of the Dynkin diagram, we may assume $m_{\ell-1} \le m_\ell$. We first assume $m_{\ell-1} = m_\ell$. Define $\hat{I} = \{i \,|\, i \in I \text{ or } 2\ell - i - 1 \in I \text{ (and } i > \ell)\}$. Then $\hat{I}$ defines a parabolic subgroup $\hat{P}$ of a simple group $\hat{G}$ of type $A_{2\ell-2}$. Let $\hat{X} = \hat{G}/\hat{P}$. By Proposition 1, the coefficients of $\mu_X + \delta$ appear as the first half of the coefficients of $\mu_{\hat{X}} + \delta$, see Figure 7.

Figure 7: Conversion of type D_ℓ to type $A_{2\ell-2}$

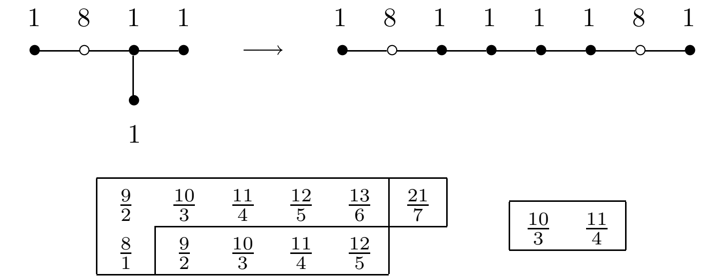


For a fixed i the product $h_i = \prod a_{ij} \times \prod b_{ij}$ can be arranged as

$$\frac{m_i}{1} \cdot \frac{m_i + m_{i+1}}{2} \cdots \frac{m_i + \cdots + m_\ell}{\ell - i + 1} \cdot \frac{s_i + m_{\ell-2}}{\ell - i + 2} \cdots \frac{s_i + m_{\ell-2} + \cdots + m_{i+1}}{2(\ell - i) - 1}$$

where $s_i = m_i + \cdots + m_\ell$. Therefore, the non-trivial terms in $\prod h_i$ correspond to the numbers in the upper left half (above the diagonal) of the rectangular arrays R_{pq} defined for type $A_{2\ell-2}$. These triangular arrays can be broken up into simple sequences as before, see Figure 8. The numbers in the remaining product, $\prod c_i$, are easily seen to form a product of simple sequences by Proposition 1. Therefore, the full product h is a product of simple sequences.
We now assume $m_{\ell-1} < m_\ell$. In this case, the product h is organized in a slightly different way. For fixed i, the previous product h_i is split into two

Figure 8: Type D_ℓ decomposition, $m_{\ell-1} = m_\ell$



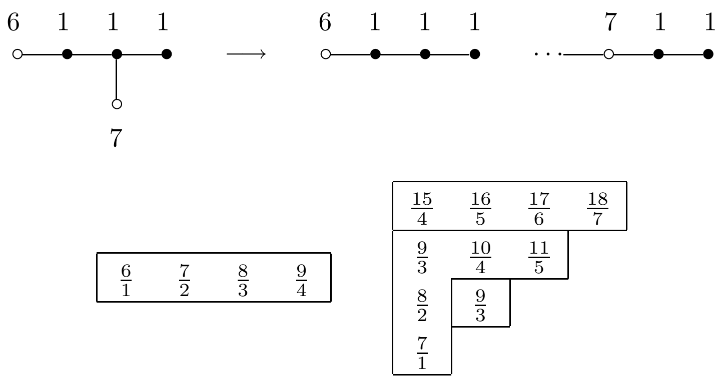
terms with c_i inserted at the beginning of the second term:

$$\frac{m_i}{1} \cdot \frac{m_i + m_{i+1}}{2} \cdots \frac{m_i + \cdots + m_{\ell-1}}{\ell - i}$$

$$\frac{m_i + \cdots + m_{\ell-2} + m_\ell}{\ell - i} \cdot \frac{s_i}{\ell - i + 1} \cdot \frac{s_i + m_{\ell-2}}{\ell - i + 2} \cdots \frac{s_i + m_{\ell-2} + \cdots + m_{i+1}}{2(\ell - i) - 1}$$

Therefore, the non-trivial terms in the product h come from two arrays, the first corresponding to the numbers in the rectangular arrays R_{pq} defined for type $A_{\ell-1}$ and the second corresponding to the numbers in the upper half of certain rectangular arrays R_{pq} defined for type $A_{2\ell-3}$. As before, these rectangular and triangular arrays can be broken up into simple sequences, see Figure 9, and hence the product h is a product of simple sequences.

Figure 9: Type D_ℓ decomposition, $m_{\ell-1} < m_\ell$



It remains to show that equality is obtained only in the designated cases. From Lemmas 1 and 2 is is clear that if $h = 3^n$ then all the simple sequences making up h must have length one and each consists of the number 3. Consequently, $m_i = 3$ for $1 \le i \le \ell$, so that $\mu_X = 2\delta$, and, by Proposition 1, X is a flag manifold. Likewise, if $h = \binom{2n+1}{n}$, then h must be the product of just one simple sequence, $h = S(1, n)$. By Proposition 1, this situation occurs either in type A_n when $m_1 = n+2$ and $m_i = 1$, $2 \le i \le n$ (or $m_n = n+2$ and $m_i = 1$, $1 \le i \le n-1$), or in type C_ℓ when $n = 2\ell - 1$, $m_1 = n+2 = 2\ell+1$, and $m_i = 1$, $2 \le i \le \ell$. In both of these cases the underlying manifold is projective space, $\mathbf{P}^n$. If the degree is $d = 2^n n!$ (resp. $(n+1)^n$), then from (3) and (4), $h = 3^n$ (resp. $\binom{2n+1}{n}$), and the same argument applies. □

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