The Free Cover of a Row Contraction

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Received: March 30, 2004

Communicated by Joachim Cuntz

Abstract. We establish the existence and uniqueness of finite free resolutions - and their attendant Betti numbers - for graded commuting d-tuples of Hilbert space operators. Our approach is based on the notion of *free cover* of a (perhaps noncommutative) row contraction. Free covers provide a flexible replacement for minimal dilations that is better suited for higher-dimensional operator theory.

For example, every graded d-contraction that is finitely multi-cyclic has a unique free cover of finite type - whose kernel is a Hilbert module inheriting the same properties. This contrasts sharply with what can be achieved by way of dilation theory (see Remark 2.5).

2000 Mathematics Subject Classification: 46L07, 47A99
Keywords and Phrases: Free Resolutions, Multivariable Operator Theory

1. Introduction

The central result of this paper establishes the existence and uniqueness of finite free resolutions for commuting d-tuples of operators acting on a common Hilbert space (Theorem 2.6). Commutativity is essential for that result, since finite resolutions do not exist for noncommuting d-tuples.

On the other hand, we base the existence of free resolutions on a general notion of *free cover* that is effective in a broader noncommutative context. Since free covers have applications that go beyond the immediate needs of this paper, and since we intend to take up such applications elsewhere, we present the general version below (Theorem 2.4). In the following section we give precise statements of these two results, we comment on how one passes from the larger noncommutative category to the commutative one, and we relate these results to previous work that has appeared in the literature. Section 3 concerns generators for Hilbert modules, in which we show that the examples of primary interest are properly generated. The next two sections are devoted to proofs of the main results - the existence and uniqueness of free covers and of finite free resolutions. In Section 6 we discuss examples.

In large part, this paper was composed during a visit to Kyoto University that was supported by MEXT, the Japanese Ministry of Education, Culture, Science and Technology. It is a pleasure to thank Masaki Izumi for providing warm hospitality and a stimulating intellectual environment during this period.

2. Statement of results

A *row contraction* of dimension d is a d-tuple of operators $(T_1, \dots, T_d)$ acting on a common Hilbert space H that has norm at most 1 when viewed as an operator from $H \oplus \cdots \oplus H$ to H. A *d-contraction* is a row contraction whose operators mutually commute, $T_j T_k = T_k T_j$, $1 \leq j, k \leq d$. In either case, one can view H as a module over the noncommutative polynomial algebra $\mathbb{C}\langle z_d, \dots, z_d\rangle$ by way of

$$f \cdot \xi = f(T_1, \dots, T_d)\xi, \qquad f \in \mathbb{C}\langle z_1, \dots, z_d\rangle,$$

and H becomes a *contractive* Hilbert module in the sense that

$$\|z_1 \cdot \xi_1 + \cdots + z_d \cdot \xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2, \qquad \xi_1, \dots, \xi_d \in H.$$

The maps of this category are linear operators $A \in \mathcal{B}(H_1, H_2)$ that are homomorphisms of the module structure and satisfy $\|A\| \leq 1$. It will be convenient to refer to a Hilbert space endowed with such a module structure simply as a *Hilbert module*.

Associated with every Hilbert module H there is an integer invariant that we shall call the defect, defined as follows. Let $Z \cdot H$ denote the closure of the range of the coordinate operators

$$Z \cdot H = \{z_1\xi_1 + \cdots + z_d\xi_d : \xi_1, \dots, \xi_d \in H\}^-.$$

$Z \cdot H$ is a closed submodule of H, hence the quotient $H/(Z \cdot H)$ is a Hilbert module whose row contraction is $(0, \dots, 0)$. One can identify $H/(Z \cdot H)$ more concretely as a subspace of H in terms of the ambient operators $T_1, \dots, T_d$,

$$H/(Z \cdot H) \sim H \ominus (Z \cdot H) = \ker T_1^* \cap \cdots \cap \ker T_d^*.$$

Definition 2.1. A Hilbert module H is said to be *properly generated* if $H \ominus (Z \cdot H)$ is a generator:

$$H = \overline{\operatorname{span}}\{f \cdot \zeta : f \in \mathbb{C}\langle z_1, \dots, z_d\rangle \quad \zeta \in H \ominus (Z \cdot H)\}.$$

In general, the quotient Hilbert space $H/(Z \cdot H)$ is called the *defect space* of H and its dimension $\dim(H/(Z \cdot H))$ is called the *defect*.

The defect space of a finitely generated Hilbert module must be finite-dimensional. Indeed, it is not hard to see that the defect is dominated by the smallest possible number of generators. A fuller discussion of properly generated Hilbert modules of finite defect will be found in Section 3.

The free objects of this category are defined as follows. Let Z be a Hilbert space of dimension $d = 1, 2, \dots$ and let $F^2(Z)$ be the Fock space over Z,

$$F^2(Z) = \mathbb{C} \oplus Z \oplus Z^{\otimes 2} \oplus Z^{\otimes 3} \oplus \cdots$$

where $Z^{\otimes n}$ denotes the full tensor product of n copies of Z. One can view $F^2(Z)$ as the completion of the tensor algebra over Z in a natural Hilbert space norm; in turn, if we fix an orthonormal basis $e_1, \dots, e_d$ for Z then we can define an isomorphism of the noncommutative polynomial algebra $\mathbb{C}\langle z_1, \dots, z_d\rangle$ onto the tensor algebra by sending z_k to e_k, $k = 1, \dots, d$. This allows us to realize the Fock space as a completion of $\mathbb{C}\langle z_1, \dots, z_d\rangle$, on which the multiplication operators associated with the coordinates $z_1, \dots, z_d$ act as a row contraction. We write this Hilbert module as $F^2\langle z_1, \dots, z_d\rangle$; and when there is no possibility of confusion about the dimension or choice of basis, we often use the more compact F^2.

One forms free Hilbert modules of higher multiplicity by taking direct sums of copies of F^2. More explicitly, let C be a Hilbert space of dimension $r = 1, 2, \dots, \infty$ and consider the Hilbert space $F^2 \otimes C$. There is a unique Hilbert module structure on $F^2 \otimes C$ satisfying

$$f \cdot (\xi \otimes \zeta) = (f \cdot \xi) \otimes \zeta, \qquad f \in \mathbb{C}\langle z_1, \dots, z_d\rangle, \quad \xi \in F^2, \quad \zeta \in C,$$

making $F^2 \otimes C$ into a properly generated Hilbert module of defect r.

More generally, it is apparent that any homomorphism of Hilbert modules $A : H_1 \to H_2$ induces a contraction

$$\dot{A} : H_1/(Z \cdot H_1) \to H_2/(Z \cdot H_2)$$

that maps one defect space into the other.

Definition 2.2. Let H be a Hilbert module. By a *cover* of H we mean a contractive homomorphism of Hilbert modules $A : F \to H$ that has dense range and induces a unitary operator $\dot{A} : F/(Z \cdot F) \to H/(Z \cdot H)$ from one defect space onto the other. A *free cover* of H is a cover $A : F \to H$ in which $F = F^2\langle z_1, \dots, z_d\rangle \otimes C$ is a free Hilbert module.

Remark 2.3 (The Extremal Property of Covers). In general, if one is given a contractive homomorphism with dense range $A : F \to H$, there is no way of relating the image $A(F \ominus (Z \cdot F))$ to $H \ominus (Z \cdot H)$, even when A induces a bijection $\dot{A}$ of one defect space onto the other. But since a cover is a contraction that induces a *unitary* map of defect spaces, it follows that a cover must map $F \ominus (Z \cdot F)$ isometrically onto $H \ominus (Z \cdot H)$ (see Lemma 4.1). This extremal property is critical, leading for example to the uniqueness assertion of Theorem 2.4 below.

It is not hard to give examples of finitely generated Hilbert modules H that are degenerate in the sense that $Z \cdot H = H$ (see the proof of Proposition 3.4), and in such cases, free covers $A : F \to H$ cannot exist when $H \neq \{0\}$. As we will see momentarily, the notion of free cover is effective for Hilbert modules that are *properly* generated. We emphasize that in a free cover $A : F \to H$ of a finitely generated Hilbert module H with $F = F^2 \otimes C$,

$$\dim C = \operatorname{defect}(F^2 \otimes C) = \operatorname{defect} H < \infty,$$

so that for finitely generated Hilbert modules for which a free cover exists, *the free module associated with a free cover must be of finite defect.* More generally, we say that a diagram of Hilbert modules

$$F \underset{A}{\longrightarrow} G \underset{B}{\longrightarrow} H$$

is *weakly exact* at G if $AF \subset \ker B$ and the map $A : F \to \ker B$ defines a cover of $\ker B$. This implies that AF is dense in $\ker B$, but of course it asserts somewhat more.

Any cover $A : F \to H$ of H can be converted into another one by composing it with a unitary automorphism of F on the right. Two covers $A : F_A \to H$ and $B : F_B \to H$ are said to be *equivalent* if there is a unitary isomorphism of Hilbert modules $U : F_A \to F_B$ such that $B = AU$. Notice the one-sided nature of this relation; in particular, two equivalent covers of a Hilbert module H must have identical (non-closed) ranges. When combined with Proposition 3.2 below, the following result gives an effective characterization of the existence of free covers.

Theorem 2.4. *A contractive Hilbert module H over the noncommutative polynomial algebra $\mathbb{C}\langle z_1, \dots, z_d\rangle$ has a free cover if, and only if, it is properly generated; and in that case all free covers of H are equivalent.*

Remark 2.5 (The Rigidity of Dilation Theory). Let H be a pure, finitely generated, contractive Hilbert module over $\mathbb{C}\langle z_1, \dots, z_d\rangle$ (see [Arv98]). The methods of dilation theory lead to the fact that, up to unitary equivalence of Hilbert modules, H can be realized as a quotient of a free Hilbert module

$$H = (F^2\langle z_1, \dots, z_d\rangle \otimes C)/M$$

where M is an invariant subspace of $F^2 \otimes C$. In more explicit terms, there is a contractive homomorphism $L : F^2 \otimes C \to H$ of Hilbert modules such that $LL^* = \mathbf{1}_H$. When such a realization is minimal, there is an appropriate sense in which it is unique.

The problem with this realization of H as a quotient of a free Hilbert module is that the coefficient space C is often infinite-dimensional; moreover, the connecting map L is only rarely a cover. Indeed, in order for C to be finite-dimensional it is necessary and sufficient that the "defect operator" of H, namely

$$\Delta = (\mathbf{1}_H - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}, \tag{2.1}$$

should be of finite rank. The fact is that this finiteness condition often fails, even when the underlying operators of H commute.

For example, any invariant subspace $K \subseteq H^2$ of the rank-one free commutative Hilbert module H^2, that is also invariant under the gauge group Γ_0 (see the following paragraphs), becomes a finitely generated graded Hilbert module whose operators $T_1, \dots, T_d$ are the restrictions of the d-shift to K. However, the defect operator of such a K is of infinite rank in every nontrivial case - namely, whenever K is nonzero and of infinite codimension in H^2. Thus, even though dilation theory provides a realization of K as the quotient of another free commutative Hilbert module $K \cong (H^2 \otimes C)/M$, the free Hilbert module $H^2 \otimes C$ must have *infinite defect*.

One may conclude from these observations that dilation theory is too rigid to provide an effective representation of finitely generated Hilbert modules as quotients of free modules of finite defect, and a straightforward application of dilation theory cannot lead to finite free resolutions in multivariable operator theory. Our purpose below is to initiate an approach to the existence of free resolutions that is based on free covers.

We first discuss grading in the general noncommutative context. By a *grading* on a Hilbert module H we mean a strongly continuous unitary representation of the circle group $\Gamma : \mathbb{T} \to \mathcal{B}(H)$ that is related to the ambient row contraction as follows

$$\Gamma(\lambda) T_k \Gamma(\lambda)^* = \lambda T_k, \qquad \lambda \in \mathbb{T}, \quad k = 1, \dots, d. \tag{2.2}$$

Thus we are restricting ourselves to gradings in which the given operators $T_1, \dots, T_d$ are all of degree one. The group Γ is called the *gauge group* of the Hilbert module H. While there are many gradings of H that satisfy (2.2), when we refer to H as a graded Hilbert module it is implicit that a particular gauge group has been singled out. A graded morphism $A : H_1 \to H_2$ of graded Hilbert modules is a homomorphism $A \in \hom(H_1, H_2)$ that is of degree zero in the sense that

$$A\Gamma_1(\lambda) = \Gamma_2(\lambda)A, \qquad \lambda \in \mathbb{T},$$

Γ_k denoting the gauge group of H_k.

The natural gauge group of $F^2(Z)$ is defined by

$$\Gamma_0(\lambda) = \sum_{n=0}^{\infty} \lambda^n E_n$$

where E_n is the projection onto $Z^{\otimes n}$. Thus, $F^2 = F^2\langle z_1, \dots, z_d\rangle$ *becomes a graded contractive Hilbert module over* $\mathbb{C}\langle z_1, \dots, z_d\rangle$ *of defect* 1. More generally, let $F = F^2 \otimes C$ be a free Hilbert module of higher defect. Since the ambient operators $U_1, \dots, U_d$ of F^2 generate an irreducible C^*-algebra, one readily verifies that the most general strongly continuous unitary representation Γ of the circle group on F that satisfies $\Gamma(\lambda)(U_k \otimes \mathbf{1}_C)\Gamma(\lambda)^* = \lambda U_k \otimes \mathbf{1}_C$ for $k = 1, \dots, d$ must decompose into a tensor product of representations

$$\Gamma(\lambda) = \Gamma_0(\lambda) \otimes W(\lambda), \qquad \lambda \in \mathbb{T},$$

where W is an arbitrary strongly continuous unitary representation of $\mathbb{T}$ on the coefficient space C. It will be convenient to refer to a Hilbert space C that has been endowed with such a group W as a *graded Hilbert space*.

In order to discuss free resolutions, we shift attention to the more restricted category whose objects are graded Hilbert modules over the commutative polynomial algebra $\mathbb{C}[z_1, \dots, z_d]$ and whose maps are graded morphisms. In this context, one replaces the noncommutative free module $F^2 = F^2\langle z_1, \dots, z_d\rangle$ with its commutative counterpart $H^2 = H^2[z_1, \dots, z_d]$, namely the completion of $\mathbb{C}[z_1, \dots, z_d]$ in its natural norm. While this notation differs from the notation $H^2(\mathbb{C}^d)$ used in [Arv98] and [Arv00], it is more useful for our purposes here. The commutative free Hilbert module H^2 is realized as a quotient of F^2 as follows. Consider the the operator $A \in \mathcal{B}(F^2, H^2)$ obtained by closing the map that sends a noncommutative polynomial $f \in \mathbb{C}\langle z_1, \dots, z_d\rangle$ to its commutative image $\tilde f \in \mathbb{C}[z_d, \dots, z_d]$. This operator is a graded partial isometry with range H^2, whose kernel is the closure of the commutator ideal in $\mathbb{C}\langle z_1, \dots, z_d\rangle$,

$$K = \overline{\operatorname{span}}\{f \cdot (z_j z_k - z_k z_j) \cdot g : 1 \le j, k \le d, \quad f, g \in \mathbb{C}\langle z_1, \dots, z_d\rangle\}. \tag{2.3}$$

One sees this in more concrete terms after one identifies $H^2 \subseteq F^2$ with the completion of the symmetric tensor algebra in the norm inherited from F^2. In that realization one has $H^2 = K^\perp$, and A can be taken as the projection with range $K^\perp = H^2$. The situation is similar for graded free modules having multiplicity; indeed, for any graded coefficient space C the map

$$A \otimes 1_C : F^2 \otimes C \to H^2 \otimes C$$

defines a graded *cover* of the commutative free Hilbert module $H^2 \otimes C$.

The most general graded Hilbert module over the commutative polynomial algebra $\mathbb{C}[z_1, \dots, z_d]$ is a graded Hilbert module over $\mathbb{C}\langle z_1, \dots, z_d\rangle$ whose underlying row contraction $(T_1, \dots, T_d)$ satisfies $T_j T_k = T_k T_j$ for all j, k. Any vector ζ in such a module H has a unique decomposition into a Fourier series relative to the spectral subspaces of the gauge group,

$$\zeta = \sum_{n=-\infty}^{\infty} \zeta_n,$$

where $\Gamma(\lambda)\zeta_n = \lambda^n \zeta_n$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{T}$. ζ is said to have finite Γ-spectrum if all but a finite number of the terms ζ_n of this series are zero. Finally, a graded contractive module H is said to be *finitely generated* if there is a finite set of vectors $\zeta_1, \dots, \zeta_s \in H$, each of which has finite Γ-spectrum, such that sums of the form

$$f_1 \cdot \zeta_1 + \dots + f_s \cdot \zeta_s, \qquad f_1, \dots, f_s \in \mathbb{C}\langle z_1, \dots, z_d\rangle$$

are dense in H.

Our main result is the following counterpart of Hilbert's syzygy theorem.

Theorem 2.6. *For every finitely generated graded contractive Hilbert module H over the commutative polynomial algebra $\mathbb{C}[z_1, \dots, z_d]$ there is a weakly exact finite sequence of graded Hilbert modules*

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow H \longrightarrow 0 \tag{2.4}$$

in which each $F_k = H^2 \otimes C_k$ is a free graded commutative Hilbert module of finite defect. The sequence (2.4) is unique up to a unitary isomorphism of diagrams and it terminates after at most $n = d$ steps.

Definition 2.7. The sequence (2.4) is called the *free* resolution of H.

Remark 2.8 (Betti numbers, Euler characteristic). The sequence (2.4) gives rise to a sequence of d numerical invariants of H

$$\beta_k(H) = \begin{cases} \operatorname{defect}(F_k), & 1 \leq k \leq n, \\ 0, & n < k \leq d. \end{cases}$$

and their alternating sum

$$\chi(H) = \sum_{k=1}^{d} (-1)^{k+1} \beta_k(H)$$

is called the *Euler characteristic* of H. Notice that this definition makes sense for any finitely generated (graded contractive) Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$, and generalizes the Euler characteristic of [Arv00] that was restricted to Hilbert modules of *finite rank* as in Remark 2.5.

Remark 2.9 (Curvature and Index). The curvature invariant of [Arv00] is defined only in the context of finite rank contractive Hilbert modules, hence the index formula of [Arv02] that relates the curvature invariant to the index of a Dirac operator is not meaningful in the broader context of Theorem 2.6. On the other hand, the proof of that formula included an argument showing that the Euler characteristic can be calculated in terms of the Koszul complex associated with the Dirac operator, and that part of the proof is easily adapted to this context to yield the following more general variation of the index theorem. *For any finitely generated graded Hilbert module H with Dirac operator D, both $\ker D_+$ and $\ker D_+^*$ are finite-dimensional, and*

$$(-1)^d \chi(H) = \dim \ker D_+ - \dim \ker D_+^*.$$

Remark 2.10 (Relation to Localized Dilation-Theoretic Resolutions). We have pointed out in Remark 2.5 that for pure d-contractions $(T_1, \dots, T_d)$ acting on a Hilbert space, dilation-theoretic techniques give rise to an exact sequence of contractive Hilbert modules and partially isometric maps

$$\cdots \longrightarrow H^2 \otimes C_2 \longrightarrow H^2 \otimes C_1 \longrightarrow H \longrightarrow 0,$$

in which the coefficient spaces C_k of the free Hilbert modules are typically infinite-dimensional, and which apparently fails to terminate in a finite number of steps. However, in a recent paper [Gre03], Greene studied "localizations" of the above exact sequence at various points of the unit ball, and he has shown that when one localizes at the origin of $\mathbb{C}^d$, the homology of his localized complex agrees with the homology of Taylor's Koszul complex (see [Tay70a],[Tay70b]) of the underlying operator d-tuple $(T_1, \dots, T_d)$ in all cases. Interesting as these local results are, they appear unrelated to the global methods and results of this paper.

Remark 2.11 (Resolutions of modules over function algebras). We also point out that our use of the terms *resolution* and *free resolution* differs substantially from usage of similar terms in work of Douglas, Misra and Varughese [DMV00], [DMV01], [DM03a], [DM03b]. For example, in [DM03b], the authors consider

Hilbert modules over certain algebras $\mathcal{A}(\Omega)$ of analytic functions on bounded domains $\Omega \subseteq \mathbb{C}^d$. They introduce a notion of *quasi-free* Hilbert module that is related to localization, and is characterized as follows. Consider an inner product on the algebraic tensor product $\mathcal{A}(\Omega) \otimes \ell^2$ of vector spaces with three properties: a) evaluations at points of Ω should be locally uniformly bounded, b) module multiplication from $\mathcal{A}(\Omega) \times (\mathcal{A}(\Omega) \otimes \ell^2)$ to $\mathcal{A}(\Omega) \otimes \ell^2$ should be continuous, and c) it satisfies a technical condition relating Hilbert norm convergence to pointwise convergence throughout Ω. The completion of $\mathcal{A}(\Omega) \otimes \ell^2$ in that inner product gives rise to a Hilbert module over $\mathcal{A}(\Omega)$, and such Hilbert modules are called *quasi-free*.

The main result of [DM03b] asserts that "weak" quasi-free resolutions

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow H \longrightarrow 0$$

exist for certain Hilbert modules H over $\mathcal{A}(\Omega)$, namely those that are higher-dimensional generalizations of the Hilbert modules studied by Cowen and Douglas in [CD78] for domains $\Omega \subseteq \mathbb{C}$. The modules Q_k are quasi-free in the sense above, but their ranks may be infinite and such sequences may fail to terminate in a finite number of steps.

3. Generators

Throughout this section we consider contractive Hilbert modules over the noncommutative polynomial algebra $\mathbb{C}\langle z_1, \dots, z_d\rangle$, perhaps graded.

Definition 3.1. Let H be a Hilbert module over $\mathbb{C}\langle z_1, \dots, z_d\rangle$. By a *generator* for H we mean a linear subspace $G \subseteq H$ such that

$$H = \overline{\operatorname{span}}\,\{f \cdot \zeta : f \in \mathbb{C}\langle z_1, \dots, z_d\rangle, \quad \zeta \in G\}.$$

We also say that H is *finitely generated* if it has a finite-dimensional generator, and in the category of graded Hilbert modules the term means a bit more, namely, that there is a finite-dimensional *graded* generator.

According to Definition 2.1, a finitely generated Hilbert module H is properly generated precisely when the defect subspace $H \ominus (Z \cdot H)$ is a finite-dimensional generator. In general, the defect subspace $H \ominus (Z \cdot H)$ of a finitely generated Hilbert module is necessarily finite-dimensional, but it can fail to generate and is sometimes $\{0\}$ (for examples, see the proof of Proposition 3.4). In particular, *finitely generated Hilbert modules need not be properly generated.* The purpose of this section is to show that many important examples *are* properly generated, and that many others are related to properly generated Hilbert modules in a simple way.

The following result can be viewed as a noncommutative operator theoretic counterpart of Nakayama's Lemma ([Eis04], Lemma 1.4).

Proposition 3.2. *Every finitely generated graded Hilbert module is properly generated.*

Proof. The space $G = H \ominus (Z \cdot H)$ is obviously a graded subspace of H, and it is finite-dimensional because $\dim G = \dim(H/(Z \cdot H))$ is dominated by the cardinality of any generating set. We have to show that G is a generator.

For that, we claim that the spectrum of the gauge group Γ is bounded below. Indeed, the hypothesis implies that there is a finite set of elements $\zeta_1, \dots, \zeta_s$ of H, each having finite Γ-spectrum, which generate H. By enlarging the set of generators appropriately and adjusting notation, we can assume that each ζ_k is an eigenvector of Γ,

$$\Gamma(\lambda)\zeta_k = \lambda^{n_k}\zeta_k, \qquad \lambda \in \mathbb{T}, \quad 1 \le k \le s.$$

Let n_0 be the minimum of $n_1, n_2, \dots, n_s$. For any monomial f of respective degrees $p_1, \dots, p_d$ in the noncommuting variables $z_1, \dots, z_d$ and every $k = 1, \dots, s$, $f \cdot \zeta_k$ is an eigenvector of Γ satisfying

$$\Gamma(\lambda)(f \cdot \zeta_k) = \lambda^N f \cdot \zeta_k$$

with $N = p_1 + \cdots + p_d + n_k \ge n_0$. Since elements of this form have H as their closed linear span, the spectrum of Γ is bounded below by n_0.

Setting $H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi, \lambda \in \mathbb{T}\}$ for $n \in \mathbb{Z}$, we conclude that

$$H = H_{n_0} \oplus H_{n_0+1} \oplus \cdots,$$

and one has $Z \cdot H_n \subseteq H_{n+1}$ for all $n \ge n_0$.

Since $G = H \ominus (Z \cdot H)$ is gauge-invariant it has a decomposition

$$G = G_{n_0} \oplus G_{n_0+1} \oplus \cdots,$$

in which $G_{n_0} = H_{n_0}$, $G_n = H_n \ominus (Z \cdot H_{n-1})$ for $n > n_0$, and where only a finite number of G_k are nonzero. Thus, each eigenspace H_n decomposes into a direct sum

$$H_n = G_n \oplus (Z \cdot H_{n-1}), \qquad n > n_0.$$

Setting $n = n_0 + 1$ we have $H_{n_0+1} = \operatorname{span}(G_{n_0+1} + Z \cdot G_{n_0})$ and, continuing inductively, we find that for all $n > n_0$,

$$H_n = \operatorname{span}(G_n + Z \cdot G_{n-1} + Z^{\otimes 2} \cdot G_{n-2} + \cdots + Z^{\otimes(n-n_0)} \cdot G_{n_0}),$$

where $Z^{\otimes r}$ denotes the space of homogeneous polynomials of total degree r. Since H is spanned by the subspaces H_n, it follows that G is a generator. $\square$

One obtains the most general examples of graded Hilbert submodules of the rank-one free commutative Hilbert module H^2 in explicit terms by choosing a (finite or infinite) sequence of homogeneous polynomials $\phi_1, \phi_2, \dots$ and considering the closure in H^2 of the set of all finite linear combinations $f_1 \cdot \phi_1 + \cdots + f_s \cdot \phi_s$, where $f_1, \dots, f_s$ are arbitrary polynomials and $s = 1, 2, \dots$. In Remark 2.5 above, we alluded to the fact that in all nontrivial cases, graded submodules of H^2 are Hilbert modules of infinite rank. However, Proposition 5.3 below implies that these examples are properly finitely generated, so they have free covers of finite defect by Theorem 2.4.

Remark 3.3 (Examples of Higher Degree). We now describe a class of infinite rank ungraded examples with substantially different properties. Perhaps we should point out that there is a more general notion of grading with respect to which the ambient operators $T_1, \dots, T_d$ in these examples are graded with various degrees larger than one. For brevity, we retain the simpler definition of grading (2.2) by viewing these examples as ungraded. Fix a d-tuple of positive integers $N_1, \dots, N_d$ and consider the following d-contraction acting on the Hilbert space $H = H^2(\mathbb{C}^d)$

$$(T_1, \dots, T_d) = (S_1^{N_1}, \dots, S_d^{N_d}),$$

where $(S_1, \dots, S_d)$ is the d-shift. The defect space of this Hilbert module

$$G = H \ominus (T_1H + \cdots + T_dH)^-$$

coincides with the intersection of the kernels $\ker T_1^* \cap \cdots \cap \ker T_d^*$; and in this case one can compute these kernels explicitly, with the result

$$G = \operatorname{span}\{z_1^{n_1} \cdots z_d^{n_d} : 0 \le n_k < N_k, \quad 1 \le k \le d\}.$$

Moreover, for every set of nonnegative integers $\ell_1, \dots, \ell_d$, the set of vectors $T_1^{\ell_1} \cdots T_d^{\ell_d} G$ contains all monomials of the form

$$z_1^{\ell_1 N_1 + n_1} \cdots z_d^{\ell_d N_d + n_d}, \qquad 0 \le n_k < N_k, \quad 1 \le k \le d.$$

It follows from these observations that G is a proper generator for H, and Theorem 2.4 provides a free cover of the form $A : H^2 \otimes G \to H$.

Another straightforward computation with coefficients shows that the defect operator of this Hilbert module is of infinite rank whenever at least one of the integers $N_1, \dots, N_d$ is larger than 1. In more detail, each monomial $z^n = z_1^{n_1} \cdots z_d^{n_d}$, $n_1, \dots, n_d \ge 0$, is an eigenvector of the defect operator $\Delta = (\mathbf{1} - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}$; and when $n_k \ge N_k$ for all k, a straightforward application of the formulas on pp. 178–179 of [Arv98] shows that

$$\Delta\, z_1^{n_1} \cdots z_d^{n_d} = c(n) z_1^{n_1} \cdots z_d^{n_d}$$

where the eigenvalues $c(n) = c(n_1, \dots, n_d)$ satisfy $0 < c(n) < 1$. Hence the defect operator has infinite rank. We conclude that, while dilation theory provides a coisometry $B : H^2 \otimes C \to H$ from another free Hilbert module to H, it is necessary that C be an infinite dimensional Hilbert space. Needless to say, such a B cannot define a free cover.

The preceding examples are all of infinite rank, and it is natural to ask about finite rank d-contractions – which were the focus of [Arv98], [Arv00], [Arv02]. Significantly, while the Hilbert module associated with a finite rank d-contraction is frequently not properly generated, it can always be extended to a properly generated one by way of a finite-dimensional perturbation.

Proposition 3.4. *Every pure Hilbert module H of finite rank can be extended trivially to a properly generated one in the sense that there is an exact sequence of Hilbert modules*

$$0 \longrightarrow K \longrightarrow \tilde{H} \underset{A}{\longrightarrow} H \longrightarrow 0$$

in which $\tilde{H}$ is is a properly generated pure Hilbert module of the same rank, K is a finite-dimensional Hilbert submodule of $\tilde{H}$, and A is a coisometry.

Proof. A standard dilation-theoretic technique (see [Arv98] for the commutative case, the proof of which works as well in general) shows that a pure Hilbert module of rank r is unitarily equivalent to a quotient of the form

$$H \cong (F^2 \otimes C)/M$$

where F^2 is the noncommutative free module of rank 1, C is an r-dimensional coefficient space, and M is a closed submodule of $F^2 \otimes C$. We identify H with the orthocomplement $M^\perp$ of M in $F^2 \otimes C$, with operators $T_1, \dots, T_d$ obtained by compressing to $M^\perp$ the natural operators $U_1 \otimes \mathbf{1}_C, \dots, U_d \otimes \mathbf{1}_C$ of $F^2 \otimes C$.

Consider $\tilde{H} = M^\perp + 1 \otimes C$. This is a finite-dimensional extension of $M^\perp$ that is also invariant under $U_k^* \otimes \mathbf{1}_C$, hence it defines a pure Hilbert module of rank r by compressing the natural operators in the same way to obtain $\tilde{T}_1, \dots, \tilde{T}_d \in \mathcal{B}(\tilde{H})$. Since $\tilde{H}$ contains H, the projection $P_{M^\perp}$ restricts to a homomorphism of Hilbert modules $A : \tilde{H} \to H$. A is a coisometry, and the kernel of A is finite-dimensional because $\dim(\tilde{H}/H) < \infty$.

To see that $\tilde{H}$ is properly generated, one computes the defect operator Δ of $\tilde{H}$. Indeed, $\Delta = (\mathbf{1}_{\tilde{H}} - \tilde{T}_1\tilde{T}_1^* - \cdots - \tilde{T}_d\tilde{T}_d^*)^{1/2}$ is seen to be the compression of the defect operator of $U_1 \otimes \mathbf{1}_C, \dots, U_d \otimes \mathbf{1}_C$ to $\tilde{H}$, and the latter defect operator is the projection onto $1 \otimes C$. Since $\tilde{H}$ contains $1 \otimes C$, the defect operator of $\tilde{H}$ is the projection on $1 \otimes C$.

Finally, we make use of the observation that a pure finite rank d-tuple is properly generated whenever its defect operator is a projection. Indeed, the range of the defect operator Δ is always a generator, and when Δ is a projection its range coincides with $\ker \tilde{T}_1^* \cap \cdots \cap \ker \tilde{T}_d^*$. ☐

4. Existence of Free Covers

We now establish the existence and uniqueness of free covers for properly generated Hilbert modules over $\mathbb{C}\langle z_1, \dots, z_d \rangle$. A cover $A : F \to H$ induces a unitary map of defect spaces; the following result implies that this isometry of quotients lifts to an isometry of the corresponding subspaces.

Lemma 4.1. *Let H be a Hilbert module and let $A : F \to H$ be a cover. Then A restricts to a unitary operator from $F \ominus (Z \cdot F)$ to $H \ominus (Z \cdot H)$.*

Proof. Let $Q \in \mathcal{B}(H)$ be the projection onto $H \ominus (Z \cdot H)$. The natural map of H onto the quotient Hilbert space $H/(Z \cdot H)$ is a partial isometry whose adjoint is the isometry

$$\eta + Z \cdot H \in H/(Z \cdot H) \mapsto Q\eta \in H \ominus (Z \cdot H), \qquad \eta \in H.$$

Thus we can define a unitary map $\tilde{A}$ from $F \ominus (Z \cdot F)$ onto $H \ominus (Z \cdot H)$ by composing the three unitary operators

$$\begin{aligned} \zeta \in F \ominus (Z \cdot F) &\mapsto \zeta + Z \cdot F \in F/(Z \cdot F), \\ \dot{A} : F/(Z \cdot F) &\to H/(Z \cdot H), \\ \eta + Z \cdot H \in H/(Z \cdot H) &\mapsto Q\eta \in H \ominus (Z \cdot H), \qquad \eta \in H, \end{aligned}$$

to obtain $\tilde{A}\zeta = QA\zeta$, $\zeta \in F \ominus (Z \cdot F)$. We claim now that $QA\zeta = A\zeta$ for all $\zeta \in F \ominus (Z \cdot F)$. To see that, note that $Q^\perp$ is the projection onto $Z \cdot H$, so that for all $\zeta \in F \ominus (Z \cdot F)$ one has

$$\begin{aligned} \|QA\zeta\| = \|A\zeta - Q^\perp A\zeta\| &= \inf_{\eta \in Z \cdot H} \|A\zeta - \eta\| \\ &= \|\dot{A}(\zeta + Z \cdot F)\|_{H/(Z \cdot H)} = \|\zeta + Z \cdot F\|_{F/(Z \cdot F)} = \|\zeta\|. \end{aligned}$$

Hence, $\|A\zeta - QA\zeta\|^2 = \|A\zeta\|^2 - \|QA\zeta\|^2 = \|A\zeta\|^2 - \|\zeta\|^2 \le 0$, and the claim follows. We conclude that the restriction of A to $F \ominus (Z \cdot F)$ is an isometry with range $H \ominus (Z \cdot H)$. □

Proof of Theorem 2.4. Let H be a properly generated Hilbert module over $\mathbb{C}\langle z_1, \dots, z_d\rangle$ and set $C = H \ominus (Z \cdot H)$. The hypothesis asserts that C is a generator. We will show that there is a (necessarily unique) contraction $A : F^2 \otimes C \to H$ satisfying

$$A(f \otimes \zeta) = f \cdot \zeta, \qquad f \in \mathbb{C}\langle z_1, \dots, z_d\rangle, \quad \zeta \in C, \tag{4.1}$$

and that such an operator A defines a free cover. For that, consider the completely positive map defined on $\mathcal{B}(H)$ by $\phi(X) = T_1XT_1^* + \cdots + T_dXT_d^*$, and let $\Delta = (\mathbf{1} - \phi(\mathbf{1}))^{1/2}$ be the defect operator of (2.1). Since $H \ominus (Z \cdot H)$ is the intersection of kernels $\ker T_1^* \cap \cdots \cap \ker T_d^* = \ker \phi(\mathbf{1})$, it follows that

$$C = H \ominus (Z \cdot H) = \{\zeta \in H : \Delta\zeta = \zeta\}.$$

Thus, C is a subspace of the range of Δ on which Δ restricts to the identity operator, and which generates H. We now use the "dilation telescope" to show that there is a unique contraction $L : F^2 \otimes \overline{\Delta H} \to H$ such that

$$L(f \otimes \zeta) = f \cdot \Delta\zeta, \qquad f \in \mathbb{C}\langle z_1, \dots, z_d\rangle, \quad \zeta \in \overline{\Delta H}. \tag{4.2}$$

Indeed, since the monomials $\{z_{i_1} \otimes \cdots \otimes z_{i_n} : i_1, \dots, i_n \in \{1, \dots, d\}\}$, $n = 1, 2, \dots$, together with the constant polynomial 1, form an orthonormal basis for F^2, the formal adjoint of L is easily computed and found to be

$$L^*\xi = 1 \otimes \Delta\xi + \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n = 1}^{d} z_{i_1} \otimes \cdots \otimes z_{i_n} \otimes \Delta T_{i_n}^* \cdots T_{i_1}^* \xi, \qquad \xi \in H.$$

One calculates norms in the obvious way to obtain

$$\begin{aligned}\|L^*\xi\|^2 &= \|\Delta\xi\|^2 + \sum_{n=1}^{\infty}\sum_{i_1,\dots,i_n=1}^{d}\|\Delta T_{i_n}^*\cdots T_{i_1}^*\xi\|^2\\ &= \langle(\mathbf{1}-\phi(\mathbf{1}))\xi,\xi\rangle + \sum_{n=1}^{\infty}\langle(\phi^n(\mathbf{1}-\phi(\mathbf{1}))\xi,\xi\rangle\\ &= \langle(\mathbf{1}-\phi(\mathbf{1}))\xi,\xi\rangle + \sum_{n=1}^{\infty}\langle(\phi^n(\mathbf{1})-\phi^{n+1}(\mathbf{1}))\xi,\xi\rangle\\ &= \|\xi\|^2 - \lim_{n\to\infty}\langle\phi^n(\mathbf{1})\xi,\xi\rangle \le \|\xi\|^2.\end{aligned}$$

Hence $\|L\| = \|L^*\| \le 1$. Finally, let A be the restriction of L to the submodule $F^2\otimes C \subseteq F^2\otimes\overline{\Delta H}$, where we now consider $F^2\otimes C$ as a free Hilbert module of possibly smaller defect. Since Δ restricts to the identity on C, (4.1) follows from (4.2).

By its definition, the restriction of A to $1\otimes C$ is an isometry with range $C = H\ominus(Z\cdot H)$, hence A induces a unitary operator of defect spaces

$$\dot{A} : (F^2\otimes C)/(Z\cdot(F^2\otimes C)) \cong 1\otimes C \to C = H\ominus(Z\cdot H) \cong H/(Z\cdot H).$$

The range of A is dense, since it contains

$$\{f\cdot\zeta : f\in\mathbb{C}\langle z_1,\dots,z_d\rangle,\ \zeta\in H\ominus(Z\cdot H)\}$$

and H is properly generated. It follows that $A : F^2\otimes C\to H$ is a free cover.

For uniqueness, let $B : \tilde{F} = F^2\otimes\tilde{C}\to H$ be another free cover of H. We exhibit a unitary isomorphism of Hilbert modules $V\in\mathcal{B}(F^2\otimes\tilde{C}, F^2\otimes C)$ such that $BV = A$ as follows. We have already pointed out that the defect space of $\tilde{F} = F^2\otimes\tilde{C}$ (resp. $F = F^2\otimes C$)) is identified with $1\otimes\tilde{C}$ (resp. $1\otimes C$). Similarly, the defect space of H is identified with $H\ominus(Z\cdot H)$. Since both A and B are covers of H, Lemma 4.1 implies that they restrict to unitary operators, from the respective spaces $1\otimes C$ and $1\otimes\tilde{C}$, onto the *same* subspace $H\ominus(Z\cdot H)$ of H. Thus there is a unique unitary operator $V_0 : C\to\tilde{C}$ that satisfies

$$A(1\otimes\zeta) = B(1\otimes V_0\zeta), \qquad \zeta\in C.$$

Let $V = \mathbf{1}_{F^2}\otimes V_0\in\mathcal{B}(F^2\otimes\tilde{C}, F^2\otimes C)$. Obviously V is a unitary operator, and it satisfies $BV = A$ since for every polynomial $f\in\mathbb{C}\langle z_1,\dots,z_d\rangle$

$$BV(f\otimes\zeta) = B(f\cdot(1\otimes V_0\zeta)) = f\cdot B(1\otimes V_0\zeta) = f\cdot A(1\otimes\zeta) = A(f\otimes\zeta),$$

and one can take the closed linear span on both sides. V must implement an isomorphism of modules since for any polynomials $f,g\in\mathbb{C}\langle z_1,\dots,z_d\rangle$ and every $\zeta\in\tilde{C}$ we have

$$V(f\cdot(g\otimes\zeta)) = (\mathbf{1}\otimes V_0)(f\cdot g\otimes\zeta) = f\cdot(g\otimes V_0\zeta) = f\cdot V(g\otimes\zeta).$$

Conversely, if a Hilbert module H has a free cover $A : F^2\otimes C\to H$, then since $1\otimes C$ is the orthocomplement of $Z\cdot(F^2\otimes C)$, Lemma 4.1 implies that

$A(1 \otimes C) = H \ominus (Z \cdot H))$. Since A is a module homomorphism, we see that

$$A(\mathrm{span}\{f \otimes \zeta : f \in \mathbb{C}, \quad \zeta \in C\}) = \mathrm{span}\{f \cdot \zeta : f \in \mathbb{C}, \quad \zeta \in H \ominus (Z \cdot H)\}.$$

The closure of the left side is H because A has dense range, and we conclude that $H \ominus (Z \cdot H)$ is a generator of H. □

We require the following consequence of Theorem 2.4 for finitely generated graded Hilbert modules.

Theorem 4.2. *Every finitely generated graded Hilbert module H over the non-commutative polynomial algebra has a graded free cover $A : F^2 \otimes C \to H$, and any two graded graded free covers are equivalent.*

If the underlying operators of H commute, then this free cover descends naturally to a commutative graded free cover $B : H^2 \otimes C \to H$.

Proof. Proposition 3.2 implies that the space $C = H \ominus (Z \cdot H)$ is a finite-dimensional generator. Moreover, since $Z \cdot H$ is invariant under the gauge group, so is C, and the restriction of the gauge group to C gives rise to a unitary representation $W : \mathbb{T} \to \mathcal{B}(C)$ of the circle group on C.

If we make the free Hilbert module $F^2 \otimes C$ into a graded one by introducing the gauge group

$$\Gamma(\lambda) = \Gamma_0(\lambda) \otimes W(\lambda), \qquad \lambda \in \mathbb{T},$$

then we claim that the map $A : F^2 \otimes C \to H$ defined in the proof of Theorem 2.4 must intertwine Γ and Γ_H. Indeed, this follows from the fact that for every polynomial $f \in \mathbb{C}\langle z_1, \dots, z_d\rangle$, every $\zeta \in C$, and every $\lambda \in \mathbb{T}$, one has

$$\begin{aligned}\Gamma_H(\lambda)A(f \otimes \zeta) &= \Gamma_H(\lambda)(f \cdot \zeta) = f(\lambda z_1, \dots, \lambda z_d) \cdot \Gamma_H(\lambda)\zeta \\ &= A(\Gamma_0(\lambda)f \otimes W(\lambda)\zeta) = A\Gamma(\lambda)(f \otimes \zeta).\end{aligned}$$

The proof of uniqueness in the graded context is now a straightforward variation of the uniqueness proof of Theorem 2.4. Finally, since H^2 is naturally identified with the quotient F^2/K where K is the closure of the commutator ideal in $\mathbb{C}\langle z_1, \dots, z_d\rangle$, it follows that when the underlying operators commute, the cover $A : F^2 \otimes C \to H$ factors naturally through $(F^2/K) \otimes C \sim H^2 \otimes C$ and one can promote A to a graded commutative free cover $B : H^2 \otimes C \to H$. □

5. Existence of Free Resolutions

We turn now to the proof of existence of finite resolutions for graded Hilbert modules over the commutative polynomial algebra $\mathbb{C}[z_1, \dots, z_d]$. We require some algebraic results obtained by Hilbert at the end of the century before last [Hil90], [Hil93]. While Hilbert's theorems have been extensively generalized, what we require are the most concrete versions of a) the basis theorem and b) the syzygy theorem. We now describe these classical results in a formulation that is convenient for our purposes, referring the reader to [Nor76], [Eis94] and [Ser00] for more detail on the underlying linear algebra.

Let $T_1, \dots, T_d$ be a set of commuting linear operators acting on a complex vector space M. We view M as a module over $\mathbb{C}[z_1, \dots, z_d]$ in the usual way, with $f \cdot \xi = f(T_1, \dots, T_d)\xi$, $f \in \mathbb{C}[z_1, \dots, z_d]$, $\xi \in M$. Such a module is said to be *graded* if there is a specified sequence M_n, $n \in \mathbb{Z}$, of subspaces that gives rise to an algebraic direct sum decomposition

$$M = \sum_{n=-\infty}^{\infty} M_n$$

with the property $T_k M_n \subseteq M_{n+1}$, for all $k = 1, \dots, d$, $n \in \mathbb{Z}$. Thus, every element ξ of M admits a unique decomposition $\xi = \sum_n \xi_n$, where ξ_n belongs to M_n and $\xi_n = 0$ for all but a finite number of n. We confine ourselves to the standard grading on $\mathbb{C}[z_1, \dots, z_d]$ in which the generators $z_1, \dots, z_d$ are all of degree 1. Finally, M is said to be *finitely generated* if there is a finite set $\zeta_1, \dots, \zeta_s \in M$ such that

$$M = \{f_1 \cdot \zeta_1 + \dots + f_s \cdot \zeta_s : f_1, \dots, f_d \in \mathbb{C}[z_1, \dots, z_d]\}.$$

A free module is a module of the form $F = \mathbb{C}[z_1, \dots, z_d] \otimes C$ where C is a complex vector space, the module action being defined in the usual way by $f \cdot (g \otimes \zeta) = (f \cdot g) \otimes \zeta$. The *rank* of F is the dimension of C. A free module can be graded in many ways, and for our purposes the most general grading on $F = \mathbb{C}[z_1, \dots, z_d] \otimes C$ is defined as follows. Given an arbitrary grading on the "coefficient" vector space C

$$C = \sum_{n=-\infty}^{\infty} C_n,$$

there is a corresponding grading of the tensor product $F = \sum_n F_n$ in which

$$F_n = \sum_{k=0}^{\infty} Z^k \otimes C_{n-k},$$

where Z^k denotes the space of all homogeneous polynomials of degree k in $\mathbb{C}[z_1, \dots, z_d]$, and where the sum on the right denotes the linear subspace of F spanned by $\cup\{Z^k \otimes C_{n-k} : k \in \mathbb{Z}\}$. If C is finite-dimensional, then there are integers $n_1 \leq n_2$ such that

$$C = C_{n_1} + C_{n_1+1} + \dots + C_{n_2},$$

so that

$$F_n = \sum_{k=0}^{\infty} Z^k \otimes C_{n-k} = \sum_{k=\max(n-n_2,0)}^{\max(n-n_1,0)} Z^k \otimes C_{n-k} \tag{5.1}$$

is finite-dimensional for each $n \in \mathbb{Z}$, $F_n = \{0\}$ for $n < n_1$, and F_n is spanned by $Z^{n-n_2} \cdot F_{n_2}$ for $n \geq n_2$.

Homomorphisms of graded modules $u : M \to N$ are required to be of degree zero

$$u(M_n) \subseteq N_n, \qquad n \in \mathbb{Z}.$$

It will also be convenient to adapt Serre's definition of minimality for homomorphisms of modules over local rings (page 84 of [Ser00]) to homomorphisms of graded modules over $\mathbb{C}[z_1, \dots, z_d]$, as follows. A homomorphism $u : M \to N$ of modules is said to be *minimal* if it induces an isomorphism of vector spaces

$$\dot{u} : M/(z_1 \cdot M + \cdots + z_d \cdot M) \to u(M)/u(z_1 \cdot M + \cdots + z_d \cdot M).$$

Equivalently, u is minimal iff $\ker u \subseteq z_1 \cdot M + \cdots + z_d \cdot M$.

A *free resolution* of an algebraic graded module M is a (perhaps infinite) exact sequence of graded modules

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow M \longrightarrow 0,$$

where each F_r is free or 0. Such a resolution is said to be *finite* if each F_r is of finite rank and $F_r = 0$ for sufficiently large r, and *minimal* if for every $r = 1, 2, \dots$, the arrow emanating from F_r denotes a minimal homomorphism.

Theorem 5.1 (Basis Theorem). *Every submodule of a finitely generated module over $\mathbb{C}[z_1, \dots, z_d]$ is finitely generated.*

Theorem 5.2 (Syzygy Theorem). *Every finitely generated graded module M over $\mathbb{C}[z_1, \dots, z_d]$ has a finite free resolution*

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

that is minimal with length n at most d, and any two minimal resolutions are isomorphic.

While we have stated the ungraded version of the basis theorem, all we require is the special case for graded modules. We base the proof of Theorem 2.6 on two operator-theoretic results, the first of which is a Hilbert space counterpart of the basis theorem for graded modules.

Proposition 5.3. *Let H be a finitely generated graded Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$ and let $K \subseteq H$ be a closed gauge-invariant submodule. Then K is a properly generated graded Hilbert module of finite defect.*

Proof. We first collect some structural information about H itself. Let Γ be the gauge group of H and consider the spectral subspaces of Γ

$$H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi\}, \qquad n \in \mathbb{Z}.$$

The finite-dimensional subspace $G = H \ominus (Z \cdot H)$ is invariant under the action of Γ, and Proposition 3.2 implies that G is a generator. Writing $G_n = G \cap H_n$, $n \in \mathbb{Z}$, it follows that G decomposes into a finite sum of mutually orthogonal subspaces

$$G = G_{n_1} + G_{n_1+1} + \cdots + G_{n_2},$$

where $n_1 \leq n_2$ are fixed integers. A computation similar to that of (5.1) shows that $H_n = \{0\}$ for $n < n_1$, and for $n \geq n_1$, H_n can be expressed in terms of the G_k by way of

$$H_n = \sum_{k=\max(n-n_2,0)}^{\max(n-n_1,0)} Z^k \cdot G_{n-k}. \qquad n \in \mathbb{Z}; \tag{5.2}$$

in particular, each H_n is finite-dimensional.

Consider the algebraic module

$$H^0 = \operatorname{span}\{f \cdot \zeta : f \in \mathbb{C}[z_1, \dots, z_d], \quad \zeta \in G\}$$

generated by G. Formula (5.2) shows that H^0 is linearly spanned by the spectral subspaces of H,

$$H^0 = H_{n_1} + H_{n_1+1} + \cdots . \tag{5.3}$$

Now let $K \subseteq H$ be a closed invariant subspace that is also invariant under the action of Γ. Letting $K_n = H_n \cap K$ be the corresponding spectral subspace of K, then we have a decomposition of K into mutually orthogonal finite-dimensional subspaces

$$K = K_{n_1} \oplus K_{n_1+1} \oplus \cdots ,$$

such that $z_k K_n \subseteq K_{n+1}$, for $1 \leq k \leq d$, $n \geq n_1$. Let K^0 be the (nonclosed) linear span

$$K^0 = K_{n_1} + K_{n_1+1} + \cdots .$$

Obviously, K^0 is dense in K and it is a submodule of the finitely generated algebraic module H^0. Theorem 5.1 implies that there is a finite set of vectors $\zeta_1, \dots, \zeta_s \in K^0$ such that

$$K^0 = \{f_1 \cdot \zeta_1 + \cdots + f_s \cdot \zeta_s : f_1, \dots, f_s \in \mathbb{C}[z_1, \dots, z_d]\}.$$

Choosing p large enough that $\zeta_1, \dots, \zeta_s \in K_{n_1} + \cdots + K_p$, we find that $K_{n_1} + \cdots + K_p$ is a graded finite-dimensional generator for K. An application of Proposition 3.2 now completes the proof. □

5.1. From Hilbert Modules to Algebraic Modules. A finitely generated graded Hilbert module H over $\mathbb{C}[z_1, \dots, z_d]$ has many finite-dimensional graded generators G; if one fixes such a G then there is an associated algebraic graded module $M(H, G)$ over $\mathbb{C}[z_1, \dots, z_d]$, namely

$$M(H, G) = \operatorname{span}\{f \cdot \zeta : f \in \mathbb{C}[z_1, \dots, z_d], \quad \zeta \in G\}.$$

The second result that we require is that it is possible to make appropriate choices of G so as to obtain a functor from Hilbert modules to algebraic modules. We now define this functor and collect its basic properties.

Consider the category $\mathcal{H}_d$ whose objects are graded finitely generated Hilbert modules over $\mathbb{C}[z_1, \dots, z_d]$, with covers as maps. Thus, $\hom(H, K)$ consists of graded homomorphisms $A : H \to K$ satisfying $\|A\| \leq 1$, such that AH is dense in K, and which induce unitary operators of defect spaces

$$\dot{A} : H/(Z \cdot H) \to K/(Z \cdot K).$$

Since we are requiring maps in $\hom(H, K)$ to have dense range, a straightforward argument (that we omit) shows that $\hom(\cdot, \cdot)$ is closed under composition.

The corresponding algebraic category $\mathcal{A}_d$ has objects consisting of graded finitely generated modules over $\mathbb{C}[z_1, \dots, z_d]$, in which $u \in \hom(M, N)$ means that u is a *minimal* graded homomorphism satisfying $u(M) = N$.

Proposition 5.4. *For every Hilbert module H in $\mathcal{H}_d$ let H^0 be the algebraic module over $\mathbb{C}[z_1, \dots, z_d]$ defined by*

$$H^0 = \operatorname{span}\{f \cdot \zeta : f \in \mathbb{C}[z_1, \dots, z_d], \quad \zeta \in H \ominus (Z \cdot H)\}.$$

Then H^0 belongs to $\mathcal{A}_d$. Moreover, for every $A \in \hom(H, K)$ one has $AH^0 = K^0$, and the restriction A^0 of A to H^0 defines an element of $\hom(H^0, K^0)$. The association $H \to H^0$, $A \to A^0$ is a functor satisfying:

(i) *For every $H \in \mathcal{H}_d$, $H^0 = \{0\} \implies H = \{0\}$.*
(ii) *For every $A \in \hom(H, K)$, $A^0 = 0 \implies A = 0$.*
(iii) *For every free graded Hilbert module F of defect r, F^0 is a free algebraic graded module of rank r.*

Proof. Since the defect subspace $H \ominus (Z \cdot H)$ is finite-dimensional and invariant under the action of the gauge group Γ, H^0 is a finitely generated module over the polynomial algebra that is invariant under the action of the gauge group. Thus it acquires an algebraic grading $H^0 = \sum_n H^0_n$ by setting

$$H^0_n = H^0 \cap H_n = \{\xi \in H^0 : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}\}, \qquad n \in \mathbb{Z}.$$

Let $H, K \in \mathcal{H}_d$ and let $A \in \hom(H, K)$. Lemma 4.1 implies that

$$A(H \ominus (Z \cdot H)) = K \ominus (Z \cdot K),$$

so that A restricts to a surjective graded homomorphism of modules $A^0 : H^0 \to K^0$. We claim that A^0 is minimal, i.e., $\ker A^0 \subseteq z_1 \cdot H^0 + \cdots + z_d \cdot H^0$. To see that, choose $\xi \in H^0$ such that $A\xi = 0$. Since H^0 decomposes into a sum

$$H^0 = H \ominus (Z \cdot H) + z_1 \cdot H^0 + \cdots + z_d \cdot H^0,$$

we can decompose ξ correspondingly

$$\xi = \zeta + z_1 \cdot \eta_1 + \cdots + z_d \cdot \eta_d,$$

where $\zeta \in H \ominus (Z \cdot H)$ and $\eta_j \in H^0$. Since $\dot{A}$ is an injective operator defined on $H/(Z \cdot H)$, $\ker A$ must be contained in $Z \cdot H$. It follows that $\xi \in Z \cdot H$, and therefore $\zeta = \xi - z_1 \cdot \eta_1 - \cdots - z_d \cdot \eta_d \in Z \cdot H = (H \ominus (Z \cdot H))^\perp$ is orthogonal to itself. Hence $\zeta = 0$, and we have the desired conclusion

$$\xi = z_1 \cdot \eta_1 + \cdots + z_d \cdot \eta_k \in z_1 \cdot H^0 + \cdots + z_d \cdot H^0.$$

The restriction A^0 of A to H^0 is therefore a minimal homomorphism, whence $A^0 \in \hom(H^0, K^0)$.

The composition rule $(AB)^0 = A^0 B^0$ follows immediately, so that we have defined a functor. Finally, both properties (i) and (ii) are consequences of the fact that, by Proposition 3.2, H^0 is dense in H, while (iii) is obvious. $\square$

Proof of Theorem 2.6. Given a graded finitely generated Hilbert module H, we claim that there is a weakly exact sequence

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow H \longrightarrow 0, \tag{5.4}$$

in which each F_r is a free graded Hilbert module of finite defect. Indeed, Proposition 5.3 implies that H is properly generated, and by Theorem 2.6, it has a graded free cover $A : F_1 \to H$ in which $F_1 = H^2 \otimes C_1$ is a graded free

Hilbert module with $\dim C_1 = \mathrm{defect}(F_1) = \mathrm{defect}(H) < \infty$. This gives a sequence of graded Hilbert modules

$$F_1 \underset{A}{\longrightarrow} H \longrightarrow 0 \tag{5.5}$$

that is weakly exact at H. Proposition 5.3 implies that $\ker A$ is a properly generated graded Hilbert module of finite defect, so that another application of Theorem 2.6 produces a graded free cover $B : F_2 \to \ker A$ in which F_2 is a graded free Hilbert module of finite defect. Thus we can extend (5.5) to a longer sequence

$$F_2 \longrightarrow F_1 \longrightarrow H \longrightarrow 0$$

that is weakly exact at F_1 and H. Continuing inductively, we obtain (5.4).

Another application of Theorem 2.6 implies that the sequence (5.4) is uniquely determined by H up to a unitary isomorphism of diagrams. The only issue remaining is whether its length is finite. To see that (5.4) must terminate, consider the associated sequence of graded algebraic modules provided by Proposition 5.4

$$\cdots \longrightarrow F_n^0 \longrightarrow \cdots \longrightarrow F_2^0 \longrightarrow F_1^0 \longrightarrow H^0 \longrightarrow 0.$$

Proposition 5.4 implies that this is a *minimal* free resolution of H^0 into graded free modules F_r^0 of finite rank. The uniqueness assertion of Theorem 5.2 implies that there is an integer $n \leq d$ such that $F_r^0 = 0$ for all $r > n$. By Proposition 5.4 (i), we have $F_r = 0$ for $r > n$. □

Remark 5.5 (Noncommutative Generalizations). Perhaps it is worth pointing out that there is no possibility of generalizing Theorem 2.6 to the noncommutative setting, the root cause being that Hilbert's basis theorem fails for modules over the noncommutative algebra $\mathbb{C}\langle z_1, \dots, z_d\rangle$. More precisely, there are finitely generated graded Hilbert modules H over $\mathbb{C}\langle z_1, \dots, z_d\rangle$ that do not have finite free resolutions. Indeed, while Theorem 2.4 implies that for any such Hilbert module H there is a graded free Hilbert module $F_1 = F^2 \otimes C$ with $\dim C < \infty$ and a weakly exact sequence of graded Hilbert modules

$$F_1 \underset{A}{\longrightarrow} H \longrightarrow 0,$$

and while the kernel of A is a certainly a graded submodule of $F^2 \otimes C$, the kernel of A need not be finitely generated. For such a Hilbert module H, this sequence cannot be continued beyond F_1 within the category of Hilbert modules of finite defect.

As a concrete example of this phenomenon, let $N \geq 2$ be an integer, let $Z = \mathbb{C}^d$ for some $d \geq 2$, and consider the free graded noncommutative Hilbert module

$$F^2 = \mathbb{C} \oplus Z \oplus Z^{\otimes 2} \oplus Z^{\otimes 3} \oplus \cdots.$$

We claim that there is an infinite sequence of unit vectors $\zeta_N, \zeta_{N+1}, \dots \in F^2$ such that $\zeta_n \in Z^{\otimes n}$ and, for all $n \geq N$,

$$\zeta_{n+1} \perp M_n = \{f_N \cdot \zeta_N + f_{N+1} \cdot \zeta_{N+1} + \cdots + f_n \cdot \zeta_n : f_N, \dots, f_n \in \mathbb{C}\langle z_1, \dots, z_d\rangle\}.$$

Indeed, choose a unit vector ζ_N arbitrarily in $Z^{\otimes N}$ and, assuming that $\zeta_N, \dots, \zeta_n$ have been defined with the stated properties, note that M_n is a graded submodule of F^2 such that

$$M_n \cap Z^{\otimes(n+1)} = Z^{\otimes(n+1-N)} \cdot \zeta_N + Z^{\otimes(n-N)} \cdot \zeta_{N+1} + \cdots + Z \cdot \zeta_n.$$

Recalling that $\dim Z^k = d^k$, an obvious dimension estimate implies that

$$\begin{aligned}\dim(M_n \cap Z^{\otimes(n+1)}) &\le d^{n+1-N} + \cdots + d = d\frac{d^{n-N+1}-1}{d-1} \\ &< \frac{d^{n-N+2}}{d-1} \le d^{n-N+2} < d^{n+1} = \dim(Z^{\otimes(n+1)}).\end{aligned}$$

Hence there is a unit vector $\zeta_{n+1} \in Z^{\otimes(n+1)}$ that is orthogonal to M_n. Now let M be the closure of $M_N \cup M_{N+1} \cup \cdots$. M is a graded invariant subspace of F^2 with the property that $M \ominus (Z \cdot M)$ contains the orthonormal set $\zeta_N, \zeta_{N+1}, \dots$, so that M cannot be finitely generated.

Finally, if we take H to be the Hilbert space quotient F^2/M, then H is a graded Hilbert module over $\mathbb{C}\langle z_1, \dots, z_d\rangle$ having a single gauge-invariant cyclic vector $1 + M$, such that the natural projection $A : F^2 \to H = F^2/M$ is a graded free cover of H where $\ker A = M$ is not finitely generated.

Notice that the preceding construction used the fact that the dimensions of the spaces $Z^{\otimes k}$ of noncommutative homogeneous polynomials grow exponentially in k. If one attempts to carry out this construction in the commutative setting, in which F^2 is replaced by H^2, one will find that the construction of the sequence $\zeta_N, \zeta_{N+1}, \dots$ fails at some point because the dimensions of the spaces Z^k of homogeneous polynomials grow too slowly. Indeed, as reformulated in Proposition 5.3, Hilbert's remarkable basis theorem implies that this construction *must* fail in the commutative setting, since every graded submodule of H^2 is finitely generated.

6. Examples of Free Resolutions

In this section we discuss some examples of free resolutions and their associated Betti numbers. There are two simple - and closely related - procedures for converting a free Hilbert module into one that is no longer free, by changing its ambient operators as follows.

(1) Append a number r of zero operators to the d-shift $(S_1, \dots, S_d)$ to obtain a $(d + r)$-contraction acting on $H^2[z_1, \dots, z_d]$ that is not the $(d + r)$-shift.

(2) Pass from $H^2[z_1, \dots, z_d]$ to a quotient $H^2[z_1, \dots, z_d]/K$ where K is the closed submodule generated by some of the coordinates $z_1, \dots, z_d$.

We begin by pointing out that one can understand either of these examples (1) or (2) by analyzing the other. We then calculate the Betti numbers of the Hilbert modules of (1) in the case where one appends three zero operators to the d-shift. In order to calculate the Betti numbers of a graded Hilbert module one has to calculate its free resolution, and that is the route we follow.

To see that (1) and (2) are equivalent constructions, consider the operator $(d+r)$-tuple $\bar{T} = (S_1, \dots, S_d, 0, \dots, 0)$ obtained from the d-shift $(S_1, \dots, S_d)$ acting on $H^2[z_1, \dots, z_d]$ by adjoining r zero operators. Let K be the closed invariant subspace of $H^2[z_1, \dots, z_{d+r}]$ generated by $z_{d+1}, z_{d+2}, \dots, z_{d+r}$. Recalling that $H^2[z_1, \dots, z_d]$ embeds isometrically in $H^2[z_1, \dots, z_{d+r}]$ with orthocomplement K,

$$H^2[z_1, \dots, z_{d+r}] = H^2[z_1, \dots, z_d] \oplus K,$$

one finds that the quotient Hilbert module $H^2[z_1, \dots, z_{d+r}]/K$ is identified with $H^2[z_1, \dots, z_d]$ in such a way that the natural $(d+r)$-contraction defined by the quotient is unitarily equivalent to $\bar{T}$.

Before turning to explicit computations we point out that, in order to calculate free resolutions, one has to iteratively calculate free covers. The procedure is summarized as follows.

Remark 6.1 (Free Covers and Free Resolutions). Let H be a finitely generated graded Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$. In order to calculate the free resolution of H one has to iterate the following procedure.

(1) One first calculates the free cover $A_1 : H^2[z_1, \dots, z_d] \otimes G_1 \to H$ of H, following the proof of Theorem 2.4. To carry that out, one must calculate the unique proper generator $G_1 \subseteq H$

$$G_1 = H \ominus (Z \cdot H),$$

the connecting map A_1 being the closure of the multiplication map

$$A(f \otimes \zeta) = f \cdot \zeta, \qquad f \in \mathbb{C}[z_1, \dots, z_d], \quad \zeta \in G_1,$$

where the free Hilbert module $H^2[z_1, \dots, z_d] \otimes G_1$ is endowed with the grading $\Gamma_0 \otimes W$, W being the unitary representation of the circle group on G defined by restricting the grading Γ_H of H,

$$W(\lambda) = \Gamma_H(\lambda) \restriction_G, \qquad \lambda \in \mathbb{T}.$$

Notice that in order to carry out this step, one basically has to identify $Z \cdot H$ and its orthocomplement in concrete terms.

(2) One then replaces H with the finitely generated graded Hilbert module $\ker A_1 \subseteq H^2[z_1, \dots, z_d] \otimes G_1$ and repeats the procedure. It is significant that in order to continue, one must identify the kernel of A_1 and its proper generator $G_2 = \ker A_1 \ominus (Z \cdot \ker A_1)$.

According to Theorems 2.4 and 2.6, this process will terminate in the zero Hilbert module after at most d steps, and the resulting sequence

$$0 \longrightarrow H^2[z_1, \dots, z_d] \otimes G_n \underset{A_n}{\longrightarrow} \cdots \underset{A_2}{\longrightarrow} H^2[z_1, \dots, z_d] \otimes G_1 \underset{A_1}{\longrightarrow} H \longrightarrow 0$$

is the free resolution of H. Once one has the free resolution, one can read off the Betti numbers of H as the multiplicities of the various free Hilbert modules that have appeared in the sequence, *in their order of appearance*.

We now discuss the examples of (1) for the case $r = 3$ and arbitrary d.

Proposition 6.2. *The Hilbert module associated with the* $(d+3)$*-contraction* $(S_1,\dots,S_d,0,0,0)$ *acting on* $H^2[z_1,\dots,z_d]$ *has Euler characteristic zero, and its sequence of Betti numbers is*

$$(\beta_1,\dots,\beta_{d+3})=(1,3,3,1,0,\dots,0).$$

Sketch of Proof. We show that the free resolution of H has the form

$$0\to F_4\to F_3\to F_2\to F_1\to H\to 0$$

where $F_k=H^2[z_1,\dots,z_{d+3}]\otimes G_k$, G_1,G_2,G_3,G_4 being graded coefficient spaces of respective dimensions $1,3,3,1$. We will exhibit the modules F_k and the connecting maps explicitly, but we omit the details of computations with polynomials.

We first compute the proper generator $H\ominus(Z\cdot H)$ of H. Writing

$$T_1T_1^*+\cdots+T_{d+3}T_{d+3}^*=S_1S_1^*+\cdots+S_dS_d^*,$$

one sees that the defect operator $(\mathbf{1}-\sum_k T_kT_k^*)^{1/2}$ is the one-dimensional projection [1] onto the constant polynomials. It follows that H has defect 1, and its proper generator is the one-dimensional space $\mathbb{C}\cdot 1$.

Hence the first term in the free resolution of H is given by the free cover $A_1: H^2[z_1,\dots,z_{d+3}]\to H$, where A_1 is the closure of the map defined on polynomials $f\in\mathbb{C}[z_1,\dots,z_{d+3}]$ by

$$A_1f=f(S_1,\dots,S_d,0,0,0)\cdot 1=f(z_1,\dots,z_d,0,0,0).$$

A_1 is a coisometry, and further computation with polynomials shows that its kernel is the closure $K_1=\overline{(z_{d+1},z_{d+2},z_{d+3})}$ of the ideal in $\mathbb{C}[z_1,\dots,z_{d+3}]$ generated by z_{d+1},z_{d+2},z_{d+3}. This gives a sequence of contractive homomorphisms of degree zero

$$0\longrightarrow K_1\longrightarrow H^2[z_1,\dots,z_{d+3}]\longrightarrow H\longrightarrow 0$$

that is exact at $H^2[z_1,\dots,z_{d+3}]$.

The kernel K_1 is a graded submodule of $H^2[z_1,\dots,z_{d+3}]$, but the rank of its defect operator is typically *infinite*. However, by Proposition 5.3, it has a unique finite-dimensional proper generator G, given by

$$G=K_1\ominus(Z\cdot K_1)=K_1\ominus\overline{(z_1\cdot K_1+\cdots+z_{d+3}\cdot K_1)}.$$

To compute G, note that each of the elements z_{d+1}, z_{d+2}, z_{d+3} is of degree one, while any homogeneous polynomial of $Z\cdot K_1$ is of degree at least two. It follows that $K_1=\operatorname{span}\{z_{d+1},z_{d+2},z_{d+3}\}\oplus(Z\cdot K_1)$, and this identifies G as the 3-dimensional Hilbert space

$$G=\operatorname{span}\{z_{d+1},z_{d+2},z_{d+3}\}.$$

The multiplication map $A_2: F\otimes G\to F$

$$A_2(f\otimes\zeta)=f\cdot\zeta,\qquad f\in\mathbb{C}[z_1,\dots,z_{d+3}],\quad \zeta\in G$$

is a contractive morphism that defines a free cover of K_1; and A_2 becomes a degree zero map with respect to the gauge group Γ on $H^2[z_1,\dots,z_{d+3}]\otimes G$ defined by $\Gamma=\Gamma_0\otimes W$ where W is the restriction of the gauge group of

$H^2[z_1, \dots, z_{d+3}]$ to its subspace G, namely $W(\lambda) = \lambda \mathbf{1}_G$, $\lambda \in \mathbb{T}$. It follows that the sequence

$$H^2[z_1, \dots, z_{d+3}] \otimes G \underset{A_2}{\longrightarrow} H^2[z_1, \dots, z_{d+3}] \underset{A_1}{\longrightarrow} H \longrightarrow 0$$

is weakly exact at $H^2[z_1, \dots, z_{d+3}]$ and H.

Now consider $K_2 = \ker A_2 \subseteq H^2[z_1, \dots, z_{d+3}] \otimes G$. Since every element of $H^2[z_1, \dots, z_{d+3}] \otimes G$ can be written uniquely in the form

$$\xi_1 \otimes z_{d+1} + \xi_2 \otimes z_{d+2} + \xi_3 \otimes z_{d+3}, \qquad \xi_k \in H^2[z_1, \dots, z_{d+3}]$$

we have

$$K_2 = \{\xi_1 \otimes z_{d+1} + \xi_2 \otimes z_{d+2} + \xi_3 \otimes z_{d+3} : z_{d+1} \cdot \xi_1 + z_{d+2} \cdot \xi_2 + z_{d+3} \cdot \xi_3 = 0\}.$$

A nontrivial calculation with polynomials now shows that K_2 is the closed submodule of of $H^2 \otimes G$ generated by the three "commutators" $\zeta_1, \zeta_2, \zeta_3$

$$\begin{aligned} \zeta_1 &= z_{d+2} \otimes z_{d+3} - z_{d+3} \otimes z_{d+2} = z_{d+2} \wedge z_{d+3}, \\ \zeta_2 &= z_{d+1} \otimes z_{d+3} - z_{d+3} \otimes z_{d+1} = z_{d+1} \wedge z_{d+3} \\ \zeta_3 &= z_{d+1} \otimes z_{d+2} - z_{d+2} \otimes z_{d+1} = z_{d+1} \wedge z_{d+2}. \end{aligned}$$

Note, for example, that

$$f \cdot \zeta_1 + g \cdot \zeta_2 = -g z_{d+3} \otimes z_{d+1} - f z_{d+3} \otimes z_{d+2} + (g z_{d+1} + f z_{d+2}) \otimes z_{d+3}.$$

These elements ζ_k are all homogeneous of degree two. Since any homogeneous element of $Z \cdot K_2$ has degree at most three, it must be orthogonal to $\zeta_1, \zeta_2, \zeta_3$. It follows that

$$K_2 \ominus (Z \cdot K_2) = \operatorname{span}\{\zeta_2, \zeta_2, \zeta_3\}$$

is 3-dimensional, having $2^{-1/2}\zeta_1, 2^{-1/2}\zeta_2, 2^{-1/2}\zeta_3$ as an orthonormal basis.

Set $\tilde{G} = \operatorname{span}\{\zeta_2, \zeta_2, \zeta_3\}$, with its grading (in this case homogeneous of degree 2) as inherited from the grading of $H^2[z_1, \dots, z_{d+3}] \otimes G$. The corresponding free cover $A_3 : H^2[z_1, \dots, z_{d+3}] \otimes \tilde{G} \to K_2$ is given by

$$A_3(f_1 \otimes \zeta_1 + f_2 \otimes \zeta_2 + f_3 \otimes \zeta_3) = f_1 \cdot \zeta_1 + f_2 \cdot \zeta_2 + f_2 \cdot \zeta_3,$$

for polynomials f_1, f_2, f_3, and the grading of $H^2[z_1, \dots, z_{d+3}] \otimes \tilde{G}$ is given by $\Gamma(\lambda)(f \otimes \zeta) = \lambda^2(\Gamma_0(\lambda) f \otimes \zeta)$, $\lambda \in \mathbb{T}$.

Finally, consider the submodule $K_3 = \ker A_3 \subseteq H^2[z_1, \dots, z_{d+3}] \otimes \tilde{G}$. Another computation with polynomials shows that K_3 has a single generator

$$\begin{aligned} \eta &= z_{d+1} \otimes \zeta_1 - z_{d+2} \otimes \zeta_2 + z_{d+3} \otimes \zeta_3 \\ &= z_{d+1} \otimes (z_{d+2} \wedge z_{d+3}) - z_{d+2} \otimes (z_{d+1} \wedge z_{d+3}) + z_{d+3} \otimes (z_{d+1} \wedge z_{d+2}), \end{aligned}$$

where as above, $z_j \wedge z_k$ denotes $z_j \otimes z_k - z_k \otimes z_j$. The homogeneous element η has degree 3, so that after appropriate renormalization it becomes a unit vector spanning $K_3 \ominus (Z \cdot K_3)$. Thus, we obtain a free cover $A_4 : H^2[z_1, \dots, z_{d+3}] \to K_3$ by closing the map of polynomials

$$A_4(f) = f \cdot \eta, \qquad f \in \mathbb{C}[z_1, \dots, z_{d+3}].$$

Notice that the grading that $H^2[z_1, \dots, z_{d+3}]$ acquires by this construction is not the standard grading Γ_0, but rather $\Gamma(\lambda) = \lambda^3\Gamma_0(\lambda)$, $\lambda \in \mathbb{T}$.

Since the kernel of A_4 is obviously $\{0\}$, we have obtained a free resolution

$$0 \to F \xrightarrow{A_4} F \otimes \tilde{G} \xrightarrow{A_3} F \otimes G \xrightarrow{A_2} F \xrightarrow{A_1} H \to 0$$

in which $F = H^2[z_1, \dots, z_{d+3}]$.

This shows that H is a Hilbert module over $\mathbb{C}[z_1, \dots, z_{d+3}]$ whose Betti numbers $(\beta_1, \cdots, \beta_{d+3})$ are given by a nontrivial sequence $(1, 3, 3, 1, 0, \dots, 0)$ with alternating sum zero. $\square$

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