Adding Tails to C^*-Correspondences

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Abstract. We describe a method of adding tails to C^*-correspondences which generalizes the process used in the study of graph C^*-algebras. We show how this technique can be used to extend results for augmented Cuntz-Pimsner algebras to C^*-algebras associated to general C^*-correspondences, and as an application we prove a gauge-invariant uniqueness theorem for these algebras. We also define a notion of relative graph C^*-algebras and show that properties of these C^*-algebras can provide insight and motivation for results about relative Cuntz-Pimsner algebras.

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1 Introduction

In [18] Pimsner introduced a way to construct a C^*-algebra $\mathcal{O}_X$ from a pair (A, X), where A is a C^*-algebra and X is a C^*-correspondence (sometimes called a Hilbert bimodule) over A. Throughout his analysis Pimsner assumed that his correspondence was full and that the left action of A on X was injective. These Cuntz-Pimsner algebras have been found to compose a class of C^*-algebras that is extraordinarily rich and includes numerous C^*-algebras found in the literature: crossed products by automorphisms, crossed products by endomorphisms, partial crossed products, Cuntz-Krieger algebras, C^*-algebras of graphs with no sinks, Exel-Laca algebras, and many more. Consequently, the study of Cuntz-Pimsner algebras has received a fair amount of attention by the operator algebra community in recent years, and because information about $\mathcal{O}_X$ is very densely codified in (A, X), determining how to extract it has been the focus of much current effort.

One interesting consequence of this effort has been the introduction of the so-called *relative Cuntz-Pimsner algebras*, denoted $\mathcal{O}(K, X)$, that have Cuntz-Pimsner algebras as quotients. Very roughly speaking, a relative Cuntz-Pimsner algebra arises by relaxing some of the relations that must hold among the generators of a Cuntz-Pimsner algebra. These relations are codified in an ideal K of A. (The precise definition will be given shortly.) Relative Cuntz-Pimsner algebras arise quite naturally, particularly when trying to understand the ideal structure of a Cuntz-Pimsner algebra (See, e.g., [15, 6]). It turns out, in fact, that not only are Cuntz-Pimsner algebras quotients of relative Cuntz-Pimsner algebras, but quotients of Cuntz-Pimsner algebras are often relative Cuntz-Pimsner algebras [6, Theorem 3.1].

Although in his initial work Pimsner assumed that his C^*-correspondences were full and had injective left action, in recent years there have been efforts to remove these restrictions. Pimsner himself described how to deal with the case when X was not full, defining the so-called *augmented Cuntz-Pimsner algebras* [18, Remark 1.2(3)]. However, the case when the left action is not injective has been more elusive. In [6] it was shown that for any C^*-correspondence X and for any ideal K of A consisting of elements that act as compact operators on the left of X, one may define $\mathcal{O}(K, X)$ to be a C^*-algebra which satisfies a certain universal property [6, Proposition 1.3]. In the case that X is full with injective left action, this definition agrees with previously defined notions of relative Cuntz-Pimsner algebras, and the Cuntz-Pimsner algebra $\mathcal{O}_X$ is equal to $\mathcal{O}(J(X), X)$, where $J(X)$ denotes the ideal consisting of all elements of A which act on the left of X as compact operators.

In [6] it was proposed that for a general C^*-correspondence X, the C^*-algebra $\mathcal{O}(J(X), X)$ is the proper analogue of the Cuntz-Pimsner algebra. However, upon further analysis it seems that this is not exactly correct. To see why, consider the case of graph C^*-algebras. If $E = (E^0, E^1, r, s)$ is a graph, then there is a natural C^*-correspondence $X(E)$ over $C_0(E^0)$ associated to E (see [7, Example 1.2]). If E has no sinks, then the C^*-algebra $\mathcal{O}(J(X(E)), X(E))$ is isomorphic to the graph C^*-algebra $C^*(E)$. However, when E has sinks this will not necessarily be the case.

It is worth mentioning that graphs with sinks play an important role in the study of graph C^*-algebras. Even if one begins with a graph E containing no sinks, an analysis of $C^*(E)$ will often necessitate considering C^*-algebras of graphs with sinks. For example, quotients of $C^*(E)$ will often be isomorphic to C^*-algebras of graphs with sinks even when E has no sinks. Consequently, one needs a theory that incorporates these objects.

This deficiency in the generalization of Cuntz-Pimsner algebras was addressed by Katsura in [10] and [11]. If X is a C^*-correspondence over a C^*-algebra A with left action $\phi : A \to \mathcal{L}(X)$, then Katsura proposed that the appropriate analogue of the Cuntz-Pimsner algebra is $\mathcal{O}_X := \mathcal{O}(J_X, X)$, where

$$J_X := \{a \in J(X) : ab = 0 \text{ for all } b \in \ker \phi\}.$$

(Note that when ϕ is injective $J_X = J(X)$.) It turns out that when ϕ is injective, $\mathcal{O}_X$ is equal to the augmented Cuntz-Pimsner algebra of X, and when X is also full $\mathcal{O}_X$ coincides with the Cuntz-Pimsner algebra of X. Furthermore, if E is a graph (possibly containing sinks), then $\mathcal{O}_{X(E)}$ is isomorphic to $C^*(E)$. In addition, as with graph algebras, the class of $\mathcal{O}_X$'s is closed under quotients by gauge-invariant ideals. These facts, together with the analysis described in [11] and [12], provide strong arguments for using $\mathcal{O}_X := \mathcal{O}(J_X, X)$ as the analogue of the Cuntz-Pimsner algebra. We shall adopt this viewpoint here, and for a general C^*-correspondence X we define $\mathcal{O}_X := \mathcal{O}(J_X, X)$ to be the C^*-algebra associated to X.

In this paper we shall describe a method which will allow one to "bootstrap" many results for augmented Cuntz-Pimsner algebras to C^*-algebras associated to general correspondences. This method is inspired by a technique from the theory of graph C^*-algebras, where one can often reduce to the sinkless case by the process of "adding tails to sinks". Specifically, if E is a graph and v is a vertex of E, then by *adding a tail to* v we mean attaching a graph of the form

$$v \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

to E. It is well known that if F is the graph formed by adding a tail to every sink of E, then F is a graph with no sinks and $C^*(E)$ is canonically isomorphic to a full corner of $C^*(F)$. Thus in the proofs of many theorems about graph C^*-algebras, one can reduce to the case of no sinks.

In this paper we describe a generalization of this process for C^*-correspondences. More specifically, if X is a C^*-correspondence over a C^*-algebra A, then we describe how to construct a C^*-algebra B and a C^*-correspondence Y over B with the property that the left action of Y is injective and $\mathcal{O}_X$ is canonically isomorphic to a full corner of $\mathcal{O}_Y$. Thus many questions about C^*-algebras associated to correspondences can be reduced to questions about augmented Cuntz-Pimsner algebras, and many results characterizing properties of augmented Cuntz-Pimsner algebras may be easily generalized to C^*-algebras associated to general correspondences. As an application of this technique, we use it in the proof of Theorem 5.1 to extend the Gauge-Invariant Uniqueness Theorem for augmented Cuntz-Pimsner algebras to C^*-algebras of general correspondences.

This paper is organized as follows. We begin in Section 2 with some preliminaries. In Section 3 we analyze graph C^*-algebras in the context of Cuntz-Pimsner and relative Cuntz-Pimsner algebras, and describe a notion of a *relative graph C^*-algebra*. Since graph algebras provide much of the impetus for our analysis of C^*-correspondences, we examine these objects carefully in order to provide a framework which will motivate and illuminate the results of subsequent sections. In Section 4 we describe our main result — a process of "adding tails" to general C^*-correspondences. We also prove that this process preserves the Morita equivalence class of the associated C^*-algebra. In Section 5 we provide an application of our technique of "adding tails" by using it to extend the Gauge-Invariant Uniqueness Theorem for augmented Cuntz-Pimsner algebras to C^*-algebras associated to general correspondences. We also interpret this theorem in the context of relative Cuntz-Pimsner algebras, and in Section 6 we use it to classify the gauge-invariant ideals in C^*-algebras associated to certain correspondences. Finally, we conclude in Section 7 by discussing other possible applications of our technique.

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2 Preliminaries

For the most part we will use the notation and conventions of [6], augmenting them when necessary with the innovations of [10] and [11].

Definition 2.1. If A is a C^*-algebra, then a *right Hilbert A-module* is a Banach space X together with a right action of A on X and an A-valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying

(i) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$

(ii) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$

(iii) $\langle \xi, \xi \rangle_A \geq 0$ and $\|\xi\| = \langle \xi, \xi \rangle_A^{1/2}$

for all $\xi, \eta \in X$ and $a \in A$. For a Hilbert A-module X we let $\mathcal{L}(X)$ denote the C^*-algebra of adjointable operators on X, and we let $\mathcal{K}(X)$ denote the closed two-sided ideal of compact operators given by

$$\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta^X_{\xi,\eta} : \xi, \eta \in X\}$$

where $\Theta^X_{\xi,\eta}$ is defined by $\Theta^X_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle_A$. When no confusion arises we shall often omit the superscript and write $\Theta_{\xi,\eta}$ in place of $\Theta^X_{\xi,\eta}$.

Definition 2.2. If A is a C^*-algebra, then a *C^*-correspondence* is a right Hilbert A-module X together with a $*$-homomorphism $\phi : A \to \mathcal{L}(X)$. We consider ϕ as giving a left action of A on X by setting $a \cdot x := \phi(a)x$.

Definition 2.3. If X is a C^*-correspondence over A, then a *representation* of X into a C^*-algebra B is a pair (π, t) consisting of a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ satisfying

(i) $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle_A)$

(ii) $t(\phi(a)\xi) = \pi(a)t(\xi)$

(iii) $t(\xi a) = t(\xi)\pi(a)$

for all $\xi, \eta \in X$ and $a \in A$.

Note that Condition (iii) follows from Condition (i) due to the equation

$$\|t(\xi)\pi(a) - t(\xi a)\|^2 = \|(t(\xi)\pi(a) - t(\xi a))^*(t(\xi)\pi(a) - t(\xi a))\| = 0.$$

If (π, t) is a representation of X into a C^*-algebra B, we let $C^*(\pi, t)$ denote the C^*-subalgebra of B generated by $\pi(A) \cup t(X)$.
A representation (π, t) is said to be *injective* if π is injective. Note that in this case t will also be isometric since

$$\|t(\xi)\|^2 = \|t(\xi)^* t(\xi)\| = \|\pi(\langle \xi, \xi \rangle_A)\| = \|\langle \xi, \xi \rangle_A\| = \|\xi\|^2.$$

When (π, t) is a representation of X into $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, we say that (π, t) is *a representation of X on $\mathcal{H}$*.

In the literature a representation (π, t) is sometimes referred to as a *Toeplitz representation* (See, e.g., [7] and [6].), and as an *isometric representation* [15]. However, here, all representations considered will be at least Toeplitz or isometric and so we drop the additional adjective. We note that in [7] the authors show that given a correspondence X over a C^*-algebra A, there is a C^*-algebra, denoted $\mathcal{T}_X$ and a representation (π_X, t_X) of X in $\mathcal{T}_X$ that is universal in the following sense: $\mathcal{T}_X$ is generated as a C^*-algebra by the ranges of π_X and t_X, and given any representation (π, t) in a C^*-algebra B, then there is a C^*-homomorphism of $\mathcal{T}_X$ into B, denoted $\rho_{(\pi,t)}$, that is unique up to an inner automorphism of B, such that $\pi = \rho_{(\pi,t)} \circ \pi_X$ and $t = \rho_{(\pi,t)} \circ t_X$. The C^*-algebra $\mathcal{T}_X$ and the representation (π_X, t_X) are unique up to an obvious notion of isomorphism. We call $\mathcal{T}_X$ *the* Toeplitz algebra of the correspondence X, but we call (π_X, t_X) *a universal representation* of X in $\mathcal{T}_X$, with emphasis on the indefinite article, because at times we want to consider more than one.

Definition 2.4. For a representation (π, t) of a C^*-correspondence X on B there exists a $*$-homomorphism $\pi^{(1)} : \mathcal{K}(X) \to B$ with the property that

$$\pi^{(1)}(\Theta_{\xi,\eta}) = t(\xi) t(\eta)^*.$$

See [18, p. 202], [9, Lemma 2.2], and [7, Remark 1.7] for details on the existence of this $*$-homomorphism. Also note that if (π, t) is an injective representation, then $\pi^{(1)}$ will be injective as well.

Definition 2.5. For an ideal I in a C^*-algebra A we define

$$I^\perp := \{a \in A : ab = 0 \text{ for all } b \in I\}.$$

If X is a C^*-correspondence over A, we define an ideal $J(X)$ of A by $J(X) := \phi^{-1}(\mathcal{K}(X))$. We also define an ideal J_X of A by

$$J_X := J(X) \cap (\ker \phi)^\perp.$$

Note that $J_X = J(X)$ when ϕ is injective, and that J_X is the maximal ideal on which the restriction of ϕ is an injection into $\mathcal{K}(X)$.

Definition 2.6. If X is a C^*-correspondence over A and K is an ideal in $J(X)$, then we say that a representation (π, t) is *coisometric on K*, or is *K-coisometric* if

$$\pi^{(1)}(\phi(a)) = \pi(a) \qquad \text{for all } a \in K.$$

In [6, Proposition 1.3] the authors show that given a correspondence X over a C^*-algebra A, and an ideal K of A contained in $J(X)$, there is a C^*-algebra, denoted $\mathcal{O}(K,X)$, and a representation (π_X, t_X) of X in $\mathcal{O}(K,X)$ that is coisometric on K and is universal with this property, in the following sense: $\mathcal{O}(K,X)$ is generated as a C^*-algebra by the ranges of π_X and t_X, and given any representation (π, t) of X in a C^*-algebra B that is K-coisometric, then there is a C^*-homomorphism of $\mathcal{O}(K,X)$ into B, denoted $\rho_{(\pi,t)}$, that is unique up to an inner automorphism of B, such that $\pi = \rho_{(\pi,t)} \circ \pi_X$ and $t = \rho_{(\pi,t)} \circ t_X$.

Definition 2.7. The algebra $\mathcal{O}(K,X)$, associated with an ideal K in $J(X)$, is called the *relative Cuntz-Pimsner algebra* determined by X and the ideal K. Further, a representation (π_X, t_X) that is coisometric on K and has the universal property just described is called a *universal* K-coisometric representation of X.

Remark 2.8. When the ideal K is the zero ideal in $J(X)$, then the algebra $\mathcal{O}(K,X)$ becomes $\mathcal{T}_X$ and a universal 0-coisometric representation of X is simply a representation of X. Furthermore, if X is a C^*-correspondence in which ϕ is injective, then $\mathcal{O}_X := \mathcal{O}(J_X, X)$ is precisely the augmented Cuntz-Pimsner algebra of X defined in [18]. If X is full, i.e., if $\overline{\operatorname{span}}\{\langle \xi, \eta\rangle_A : \xi, \eta \in X\} = A$, then the augmented Cuntz-Pimsner algebra of X and the Cuntz-Pimsner algebra of X coincide. Thus $\mathcal{O}_X$ coincides with the Cuntz-Pimsner algebra of [18] when ϕ is injective and X is full. Whether or not ϕ is injective, a universal $J(X)$-coisometric representation is sometimes called a universal Cuntz-Pimsner covariant representation [6, Definition 1.1].

Remark 2.9. If $\mathcal{O}(K,X)$ is a relative Cuntz-Pimsner algebra associated to a C^*-correspondence X, and if (π, t) is a universal K-coisometric representation of X, then for any $z \in \mathbb{T}$ (π, zt) is also a universal K-coisometric representation. Hence by the universal property, there exists a homomorphism $\gamma_z : \mathcal{O}(K,X) \to \mathcal{O}(K,X)$ such that $\gamma_z(\pi(a)) = \pi(a)$ for all $a \in A$ and $\gamma_z(t(\xi)) = zt(\xi)$ for all $\xi \in X$. Since $\gamma_{z^{-1}}$ is an inverse for this homomorphism, we see that γ_z is an automorphism. Thus we have an action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}(K,X)$ with the property that $\gamma_z(\pi(a)) = \pi(a)$ and $\gamma_z(t(\xi)) = zt(\xi)$. Furthermore, a routine $\epsilon/3$ argument shows that γ is strongly continuous. We call γ the *gauge action on* $\mathcal{O}(K,X)$.

3 Viewing graph C^*-algebras as Cuntz-Pimsner algebras

Let $E := (E^0, E^1, r, s)$ be a directed graph with countable vertex set E^0, countable edge set E^1, and range and source maps $r, s : E^1 \to E^0$. A *Cuntz-Krieger E-family* is a collection of partial isometries $\{s_e : e \in E^1\}$ with commuting range projections together with a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ that satisfy

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$

3. $p_v = \sum_{\{e:s(e)=v\}} s_e s_e^*$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$

The *graph algebra* $C^*(E)$ is the C^*-algebra generated by a universal Cuntz-Krieger E-family (see [14, 13, 2, 5, 1]).

Example 3.1 *(The Graph C^*-correspondence).* If $E = (E^0, E^1, r, s)$ is a graph, we define $A := C_0(E^0)$ and

$$X(E) := \{x : E^1 \to \mathbb{C} : \text{ the function } v \mapsto \sum_{\{f \in E^1 : r(f)=v\}} |x(f)|^2 \text{ is in } C_0(E^0)\ \}.$$

Then $X(E)$ is a C^*-correspondence over A with the operations

$$(x \cdot a)(f) := x(f)a(r(f)) \text{ for } f \in E^1$$
$$\langle x, y\rangle_A(v) := \sum_{\{f \in E^1 : r(f)=v\}} \overline{x(f)}y(f) \text{ for } f \in E^1$$
$$(a \cdot x)(f) := a(s(f))x(f) \text{ for } f \in E^1$$

and we call $X(E)$ the *graph C^*-correspondence* associated to E. Note that we could write $X(E) = \bigoplus^0_{v \in E^0} \ell^2(r^{-1}(v))$ where this denotes the C_0 direct sum (sometimes called the restricted sum) of the $\ell^2(r^{-1}(v))$'s. Also note that $X(E)$ and A are spanned by the point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$, respectively.

Theorem 3.2 ([5, Proposition 12]). *If E is a graph with no sinks, and $X(E)$ is the associated graph C^*-correspondence, then $\mathcal{O}(J(X(E)), X(E)) \cong C^*(E)$. Furthermore, if (π_X, t_X) is a universal $J(X(E))$-coisometric representation, then $\{t_X(\delta_e), \pi_X(\delta_v)\}$ is a universal Cuntz-Krieger E-family in $\mathcal{O}(J(X(E)), X(E))$.*

It was shown in [7, Proposition 4.4] that

$$J(X(E)) = \overline{\operatorname{span}}\{\delta_v : |s^{-1}(v)| < \infty\}$$

and if v emits finitely many edges, then

$$\phi(\delta_v) = \sum_{\{f \in E^1 : s(f)=v\}} \Theta_{\delta_f, \delta_f} \quad \text{and} \quad \pi_X(\phi(\delta_v)) = \sum_{\{f \in E^1 : s(f)=v\}} t_X(\delta_f)t_X(\delta_f)^*.$$

Furthermore, one can see that $\delta_v \in \ker \phi$ if and only if v is a sink in E. Also $\delta_v \in \overline{\operatorname{span}}\{\langle x, y\rangle_A\}$ if and only if v is a source, and since $\delta_{s(f)} \cdot \delta_f = \delta_f$ we see that $\overline{\operatorname{span}}\, A \cdot X = X$ and $X(E)$ is essential. These observations show that we have the following correspondences between the properties of the graph E and the properties of the graph C^*-correspondence $X(E)$.

Property of $X(E)$	Property of E
$\phi(\delta_v) \in \mathcal{K}(X(E))$	v emits a finite number of edges
$\phi(A) \subseteq \mathcal{K}(X(E))$	E is row-finite
ϕ is injective	E has no sinks
$X(E)$ is full	E has no sources
$X(E)$ is essential	always

Remark 3.3. If E is a graph with no sinks, then $\mathcal{O}(J(X(E)), X(E))$ is canonically isomorphic to $C^*(E)$. When E has sinks, this will not be the case. If (π, t) is the universal $J(X(E))$-coisometric representation of $X(E)$, then it will be the case that $\{t(\delta_e), \pi(\delta_v)\}$ is a Cuntz-Krieger E-family. However, when v is a sink in E, $\phi(\delta_v) = 0$ and thus $\pi(\delta_v) = \pi^{(1)}(\phi(\delta_v)) = 0$. Consequently, $\{t(\delta_e), \pi(\delta_v)\}$ will not be a universal Cuntz-Krieger E-family when E has sinks. However, if E is a graph with sinks, then we see that $\phi(\delta_v) = 0$ if and only if v is a sink, and $\delta_v \in (\ker \phi)^\perp$ if and only if v is not a sink. Thus

$$J_{X(E)} = \overline{\operatorname{span}}\{\delta_v : 0 < |s^{-1}(v)| < \infty\}$$

and a proof similar to that in [7, Proposition 4.4] shows that $\mathcal{O}_{X(E)} := \mathcal{O}(J_{X(E)}, X(E))$ is isomorphic to $C^*(E)$. Furthermore, if (π_X, t_X) is a universal $J(X(E))$-coisometric representation of $X(E)$, then $\{t_X(\delta_e), \pi_X(\delta_v)\}$ is a universal Cuntz-Krieger E-family in $\mathcal{O}_{X(E)}$.

3.1 Relative Graph Algebras

We shall now examine relative Cuntz-Pimsner algebras in the context of graph algebras. If E is a graph and $X(E)$ is the associated graph C^*-correspondence, then $J_{X(E)} := \overline{\operatorname{span}}\{\delta_v : 0 < |s^{-1}(v)| < \infty\}$. If K is an ideal in $J_{X(E)}$, then $K = \overline{\operatorname{span}}\{\delta_v : v \in V\}$ for some subset V of vertices which emit a finite and nonzero number of edges. If $(\mathcal{O}(K, X(E)), t_X, \pi_X)$ is the relative Cuntz-Pimsner algebra determined by K, then the relation $\pi_X(\delta_v) = \sum_{s(e)=v} t_X(\delta_e) t_X(\delta_e)^*$ will hold only for vertices $v \in V$. This motivates the following definition.

Definition 3.4. Let $E = (E^0, E^1, r, s)$ be a graph and define $R(E) := \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$. For any $V \subseteq R(E)$ we define a *Cuntz-Krieger* (E, V)-*family* to be a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ together with a collection of partial isometries $\{s_e : e \in E^1\}$ that satisfy

1. $s_e^* s_e = p_{r(e)}$ for $e \in E^1$
2. $s_e s_e^* < p_{s(e)}$ for $e \in E^1$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in V$

We refer to a Cuntz-Krieger $(E, R(E))$-family as simply a Cuntz-Krieger E-family, and we refer to a Cuntz-Krieger $(E, \emptyset)$-family as a Toeplitz-Cuntz-Krieger family.

Definition 3.5. If E is a graph and $V \subseteq R(E)$, then we define the *relative graph algebra* $C^*(E, V)$ to be the C^*-algebra generated by a universal Cuntz-Krieger (E, V)-family.

The existence of $C^*(E, V)$ can be proven by adapting the argument for the existence of graph algebras in [13], or by realizing $C^*(E, V)$ as a relative Cuntz-Pimsner algebra.

Note that $C^*(E, R(E))$ is the graph algebra $C^*(E)$, and $C^*(E, \emptyset)$ is the Toeplitz algebra defined in [7, Theorem 4.1] (but different from the Toeplitz algebra defined in [4]). It is also the case that if $\{s_e, p_v\}$ is a universal Cuntz-Krieger (E, V)-family, then whenever $v \in R(E) \backslash V$ we have $p_v > \sum_{s(e)=v} s_e s_e^*$.

Definition 3.6. Let $E = (E^0, E^1, r, s)$ be a graph and $V \subseteq R(E)$. We define the graph E_V to be the graph with vertex set $E_V^0 := E^0 \cup \{v' : v \in R(E) \backslash V\}$, edge set $E^1 \cup \{e' : e \in E^1 \text{ and } r(e) \in R(E) \backslash V\}$, and r and s extended to E_V^1 by defining $s(e') := s(e)$ and $r(e') := r(e)'$.

Roughly speaking, when forming E_V one takes E and adds a sink for each element $v \in R(E) \backslash V$ as well as edges to this sink from each vertex that feeds into v.

Theorem 3.7. *If E is a graph and $V \subseteq R(E)$, then the relative graph algebra $C^*(E, V)$ is canonically isomorphic to the graph algebra $C^*(E_V)$.*

Proof. Let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be a generating Cuntz-Krieger (E, V)-family in $C^*(E, V)$. For $w \in E_V^0$ and $f \in E_V^1$ define

$$q_w := \begin{cases} p_v & \text{if } w \notin R(E) \backslash V \\ \sum_{\{e \in E^1 : s(e) = w\}} s_e s_e^* & \text{if } w \in R(E) \backslash V \\ p_v - \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^* & \text{if } w = v' \text{ for some } v \in R(E) \backslash V. \end{cases}$$

$$t_f := \begin{cases} s_f q_{r(f)} & \text{if } f \in E^1 \\ s_e q_{r(e)'} & \text{if } f = e' \text{ for some } e \in E^1. \end{cases}$$

It is straightforward to check that $\{t_f, q_w : f \in E_V^1, w \in E_V^0\}$ is a Cuntz-Krieger E_V-family in $C^*(E, V)$. Thus by the universal property there exists a homomorphism $\alpha : C^*(E_V) \to C^*(E, V)$ taking the generators of $C^*(E_V)$ to $\{t_f, q_w\}$. By the gauge-invariant uniqueness theorem [1, Theorem 2.1] α is injective. Furthermore, whenever $v \in R(E) \backslash V$ we see that $p_v = q_v + q_{v'}$ and whenever $r(e) \in R(E) \backslash V$ we see that $s_e = t_e + t_{e'}$. Thus $\{q_w, t_f\}$ generates $C^*(E, V)$ and α is surjective. Consequently α is an isomorphism. $\square$

This theorem shows that the class of relative graph algebras is the same as the class of graph algebras. Thus we gain no new C^*-algebras by considering relative graph algebras in place of graph algebras. However, we maintain that relative graph algebras are still useful and arise naturally in the study of graph algebras. In particular, we give three examples of common situations in which relative graph algebras prove convenient.

Example 3.8 *(Subalgebras of Graph Algebras).* Let $E = (E^0, E^1, r, s)$ be a graph and let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be a generating Cuntz-Krieger E-family in $C^*(E)$. If $F = (F^0, F^1, r_F, s_F)$ is a subgraph of E, and A denotes the C^*-subalgebra of $C^*(E)$ generated by $\{s_e, p_v : e \in F^1, v \in F^0\}$, then it is well-known that A is a graph algebra (but not necessarily the C^*-algebra associated to F). In fact, we see that for any $v \in F^0$, the sum $\sum_{\{e \in F^1 : s_F(e) = v\}} s_e s_e^*$ may

not add up to p_v because some of the edges in $s^{-1}(v)$ may not be in F. However, if we let $V := \{v \in R(F) : s_F^{-1}(v) = s^{-1}(v)\}$. Then $\{s_e, p_v : e \in F^1, v \in F^0\}$ is a Cuntz-Krieger (F, V)-family and $A \cong C^*(F, V)$.
These subalgebras arise often in the study of graph algebras. In [8, Lemma 2.4] they were realized as graph algebras by the method shown in the proof of Theorem 3.7, and in [19, Lemma 1.2] these subalgebras were realized as graph algebras by using the notion of a dual graph. In both of these instances it would have been convenient to have used relative graph algebras. Realizing the subalgebra as $C^*(F, V)$ would have provided an economy of notation as well as a more direct analysis of the subalgebras under consideration.

Example 3.9 *(Spielberg's Toeplitz Graph Algebras).* In [21] Spielberg introduced a notion of a Toeplitz graph groupoid and a Toeplitz graph algebra. The Toeplitz graph algebras defined in [21, Definition 2.17] are relative graph algebras as defined in Definition 3.5 (see [21, Theorem 2.9]). Spielberg also made use of his Toeplitz graph algebras in [22] to construct graph algebras with a specified K-theory.

Example 3.10 *(Quotients of Graph Algebras).* If $E = (E^0, E^1, r, s)$ is a row-finite graph and H is a saturated hereditary subset of vertices of E, then it follows from [2, Theorem 4.1(b)] that $C^*(E)/I_H \cong C^*(F)$ where F is the subgraph defined by

$$F^0 := E^0 \backslash H \qquad\qquad F^1 := \{e \in E^1 : r(e) \notin H\}.$$

If E is not row-finite, then this is not necessarily the case. The obstruction is due to the vertices in the set

$$B_H = \{v \in E^0 \,|\, v \text{ is an infinite emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

In fact, if $\{s_e, p_v\}$ is a generating Cuntz-Krieger E-family in $C^*(E)$, then the cosets $\{s_e + I_H, p_v + I_H : v \notin H, r(e) \notin H\}$ will have the property that $p_v + I_H \geq \sum_{e \in E \backslash H : s(e) = v\}} (s_e + I_H)(s_e + I_H)^*$ with equality occurring if and only if $v \in R(F) \backslash B_H$. Thus it turns out that $\{s_e + I_H, p_v + I_H : v \notin H, r(e) \notin H\}$ will be a Cuntz-Krieger $(F, R(F)\backslash B_H)$-family and $C^*(E)/I_H \cong C^*(F, R(F)\backslash B_H)$. The quotient $C^*(E)/I_H$ was realized as a graph algebra in [1, Proposition 3.4] by a technique similar to that used in the proof of Theorem 3.7. However, relative graph algebras provide a more natural context for describing these quotients.

In addition to their applications in the situations mentioned above, relative graph algebras can be useful for another reason. Since any relative graph algebra is canonically isomorphic to a graph algebra, we see that for every theorem about graph algebras there will be a corresponding theorem for relative graph algebras. Thus the relative graph algebras provide a class of relative Cuntz-Pimsner algebras that are well understood. With this in mind, we shall now state a version of the Gauge-Invariant Uniqueness Theorem for relative graph algebras.

THEOREM 3.11 (GAUGE-INVARIANT UNIQUENESS FOR RELATIVE GRAPH ALGEBRAS). *Let $E = (E^0, E^1, r, s)$ be a graph and $V \subseteq R(E)$. Also let $\{s_e, p_v : e \in E^1, v \in E^0\}$ and let $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E,V)$ denote the gauge action on $C^*(E,V)$. If $\rho : C^*(E,V) \to A$ is a $*$-homomorphism between C^*-algebras that satisfies*

1. *$\rho(p_v) \neq 0$ for all $v \in E^0$*
2. *$\rho(p_v - \sum_{s(e)=v} s_e s_e^*) \neq 0$ for all $v \in V$*
3. *there exists a strongly continuous action $\beta : \mathbb{T} \to \operatorname{Aut} A$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$.*

then ρ is injective.

Proof. By Theorem 3.7 there exists an isomorphism $\alpha : C^*(E_V) \to C^*(E,V)$ and a generating Cuntz-Krieger E_V-family $\{t_e, q_w\}$ for which

$$\alpha(q_w) := \begin{cases} p_v & \text{if } w \notin R(E)\backslash V \\ \sum_{\{e \in E^1 : s(e)=w\}} s_e s_e^* & \text{if } w \in R(E)\backslash V \\ p_v - \sum_{\{e \in E^1 : s(e)=v\}} s_e s_e^* & \text{if } w = v' \text{ for some } v \in R(E)\backslash V. \end{cases}$$

$$\alpha(t_f) := \begin{cases} s_f q_{r(f)} & \text{if } f \in E^1 \\ s_e q_{r(e)'} & \text{if } f = e' \text{ for some } e \in E^1. \end{cases}$$

To show that ρ is injective, it suffices to show that $\rho \circ \alpha$ is injective. We shall do this by applying the gauge-invariant uniqueness theorem for graph algebras [1, Theorem 2.1] to $\rho \circ \alpha$. Now clearly if $w \notin R(E)\backslash V$, then $\rho \circ \alpha(q_w) \neq 0$ by (1). If $w = v'$, then $\rho \circ \alpha(q_w) \neq 0$ by (2). Furthermore, if $w \in R(E)\backslash V$ then $\rho \circ \alpha(q_w) = 0$ implies that $\rho(\sum_{s(e)=w} s_e s_e^*) = 0$ and thus for any $f \in s^{-1}(v)$ we have

$$\rho(s_f) = \rho(\sum_{s(e)=w} s_e s_e^*)\rho(s_f) = 0.$$

But then $\rho(p_{r(f)}) = \rho(s_f^* s_f) = 0$ which contradicts (1). Hence we must have $\rho \circ \alpha(q_w) \neq 0$. Finally, if γ' denotes the gauge action on $C^*(E_V)$, then by checking on generators we see that $\beta_z \circ (\rho \circ \alpha) = (\rho \circ \alpha) \circ \gamma'_z$. Therefore, $\rho \circ \alpha$ is injective by the gauge invariant uniqueness theorem for graph algebras, and consequently ρ is injective. □

We have shown in Theorem 3.7 that every relative graph algebra is isomorphic to a graph algebra. More generally, Katsura has shown in [12] that every relative Cuntz-Pimsner algebra is isomorphic to the C^*-algebra associated to a correspondence; that is, if $\mathcal{O}(K, X)$ is a relative Cuntz-Pimsner algebra, then there exists a C^*-correspondence X' such that $\mathcal{O}_{X'} := \mathcal{O}(J_{X'}, X')$ is isomorphic to $\mathcal{O}(K, X)$. In Theorem 5.1 we shall prove a gauge-invariant uniqueness theorem for C^*-algebras associated to correspondences. Afterwards, in Remark 5.3, we shall use Katsura's analysis in [12] to give an interpretation of Theorem 3.11 in the context of relative Cuntz-Pimsner algebras.

4 Adding Tails to C^*-correspondences

If E is a graph and v is a vertex of E, then by *adding a tail to* v we mean attaching a graph of the form

$$v \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \cdots$$

to E. It was shown in [2, §1] that if F is the graph formed by adding a tail to every sink of E, then F is a graph with no sinks and $C^*(E)$ is canonically isomorphic to a full corner of $C^*(F)$. The technique of adding tails to sinks is a simple but powerful tool in the analysis of graph algebras. In the proofs of many results it allows one to reduce to the case in which the graph has no sinks and thereby avoid certain complications and technicalities.

Our goal in this section is to develop a process of "adding tails to sinks" for C^*-correspondences, so that given any C^*-correspondence X we may form a C^*-correspondence Y with the property that the left action of Y is injective and $\mathcal{O}_X$ is canonically isomorphic to a full corner in $\mathcal{O}_Y$.

Definition 4.1. Let X be a C^*-correspondence over A with left action $\phi : A \to \mathcal{L}(X)$, and let I be an ideal in A. We define *the tail determined by* I to be the C^*-algebra

$$T := I^{(\mathbb{N})}$$

where $I^{(\mathbb{N})}$ denotes the c_0-direct sum of countably many copies of the ideal I. We shall denote the elements of T by

$$\vec{f} := (f_1, f_2, f_3, \ldots)$$

where each f_i is an element of I. We shall consider T as a right Hilbert C^*-module over itself (see [20, Example 2.10]). We define $Y := X \oplus T$ and $B := A \oplus T$. Then Y is a right Hilbert B-module in the usual way; that is, the right action is given by

$$(\xi, \vec{f}) \cdot (a, \vec{g}) := (\xi \cdot a, \vec{f}\vec{g}) \qquad \text{for } \xi \in X,\ a \in A, \text{ and } \vec{f}, \vec{g} \in T$$

and the inner product is given by

$$\langle (\xi, \vec{f}), (\nu, \vec{g}) \rangle_B := (\langle \xi, \nu \rangle_A, \vec{f}^* \vec{g}) \qquad \text{for } \xi, \nu \in X \text{ and } \vec{f}, \vec{g} \in T.$$

Furthermore, we shall make Y into a C^*-correspondence over B by defining a left action $\phi_B : B \to \mathcal{L}(Y)$ as

$$\phi_B(a, \vec{f})(\xi, \vec{g}) := (\phi(a)(\xi), (ag_1, f_1 g_2, f_2 g_3, \ldots)) \text{ for } a \in A,\ \xi \in X, \text{ and } \vec{f}, \vec{g} \in T.$$

We call Y *the C^*-correspondence formed by adding the tail T to X.*

Lemma 4.2. *Let X be a C^*-correspondence over A, and let $T := (\ker \phi)^{(\mathbb{N})}$ be the tail determined by $\ker \phi$. If $Y := X \oplus T$ is the C^*-correspondence over $B := A \oplus T$ formed by adding the tail T to X, then the left action $\phi_B : B \to \mathcal{L}(Y)$ is injective. Consequently, $J_Y = J(Y)$ and $\mathcal{O}_Y = \mathcal{O}(J(Y), Y)$ is equal to the C^*-algebra defined by Pimsner in [18].*

Proof. If $(a, \vec{f}) \in \ker \phi_B$, then for all $\xi \in X$ we have

$$(\phi(a)\xi, \vec{0}) = \phi_B(a, \vec{f})(\xi, \vec{0}) = (0, \vec{0})$$

so that $\phi(a)\xi = 0$ and $a \in \ker \phi$. Thus $(0, (a, f_1, f_2, \ldots)) \in X \oplus T$ and

$$(0, (aa^*, f_1 f_1^*, f_2 f_2^*, \ldots)) = \phi_B(a, \vec{f})(0, (a^*, f_1, f_2, \ldots)) = (0, \vec{0})$$

so that $\|a\|^2 = \|aa^*\| = 0$ and $\|f_i\|^2 = \|f_i f_i^*\| = 0$ for all $i \in \mathbb{N}$. Consequently, $a = 0$ and $\vec{f} = \vec{0}$ so that ϕ_B is injective. □

Theorem 4.3. *Let X be a C^*-correspondence over A, and let $T := (\ker \phi)^{(\mathbb{N})}$ be the tail determined by $\ker \phi$. Also let $Y := X \oplus T$ be the C^*-correspondence over $B := A \oplus T$ formed by adding the tail T to X.*

(a) If (π, t) is a J_X-coisometric representation of X on a Hilbert space $\mathcal{H}_X$, then there is a Hilbert space $\mathcal{H}_Y = \mathcal{H}_X \oplus \mathcal{H}_T$ and a $J(Y)$-coisometric representation $(\tilde{\pi}, \tilde{t})$ of Y on $\mathcal{H}_Y$ with the property that $\tilde{\pi}|_X = \pi$ and $\tilde{t}|_A = t$.

(b) If $(\tilde{\pi}, \tilde{t})$ is a $J(Y)$-coisometric representation of Y into a C^*-algebra C, then $(\tilde{\pi}|_A, \tilde{t}|_X)$ is a J_X-coisometric representation of X into C. Furthermore, if $\tilde{\pi}|_A$ is injective, then $\tilde{\pi}$ is injective.

(c) Let (π_Y, t_Y) be a universal $J(Y)$-coisometric representation of Y. Then $(\pi, t) := (\pi_Y|_A, t_Y|_X)$ is a J_X-coisometric representation of X in $C^*(\pi_Y, t_Y)$. Furthermore, $\rho_{(\pi,t)} : \mathcal{O}_X \to C^*(\pi_X, t_X) \subseteq \mathcal{O}_Y$ is an isomorphism onto the C^*-subalgebra of $\mathcal{O}_Y$ generated by

$$\{\pi_Y(a, \vec{0}), t_Y(\xi, \vec{0}) : a \in A \text{ and } \xi \in X\}$$

and this C^*-subalgebra is a full corner of $\mathcal{O}_Y$. Consequently, $\mathcal{O}_X$ is naturally isomorphic to a full corner of $\mathcal{O}_Y$.

Corollary 4.4. *If X is a C^*-correspondence and (π_X, t_X) is a universal $J(X)$-coisometric representation of X, then (π_X, t_X) is injective.*

Proof. By the theorem (π_X, t_X) extends to a universal $J(Y)$-coisometric representation (π_Y, t_Y) of Y. Since ϕ_B is injective by Lemma 4.2 it follows from [6, Corollary 6.2] that (π_Y, t_Y) is injective. Consequently, $\pi_X = \pi_Y|_A$ is injective. □

To prove this theorem we shall need a number of lemmas.

Lemma 4.5. *Let X be a C^*-correspondence and let $T := (\ker \phi)^{(\mathbb{N})}$ be the tail determined by $\ker \phi$. Also let $Y := X \oplus T$ be the C^*-correspondence over $B := A \oplus T$ formed by adding the tail T to X. Then for any $(a, \vec{f}) \in Y$ we have that $(a, \vec{f}) \in J(Y)$ if and only if $a = a_1 + a_2$ with $a_1 \in J_X$ and $a_2 \in \ker \phi$.*

Proof. Suppose $a = a_1 + a_2$ with $a_1 \in J_X$ and $a_2 \in \ker \phi$. Then we may write $\phi(a_1) = \lim_n \sum_{k=1}^{N_n} \Theta^X_{\xi_{n,k},\eta_{n,k}}$ for some $\xi_{n,k}, \eta_{n,k} \in X$. But then

$$\phi_B(a_1, \vec{0}) = \lim_n \sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{0}),(\eta_{n,k},\vec{0})} \in \mathcal{K}(Y).$$

In addition, since $a_2 \in \ker \phi$ we see that if we let $\{\vec{e}_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for T with $\vec{e}_\lambda = (e^1_\lambda, e^2_\lambda, \ldots)$ for each λ, then

$$\phi_B(a_2, \vec{f}) = \lim_\lambda \Theta^Y_{(0,(a_2,f_1,f_2,\ldots)),(0,(e^1_\lambda,e^2_\lambda,\ldots))} \in \mathcal{K}(Y).$$

Thus $\phi_B(a, \vec{f}) = \phi_B(a_1, \vec{0}) + \phi_B(a_2, \vec{f})) \in \mathcal{K}(Y)$.

Conversely, suppose that $\phi_B(a, \vec{f}) \in \mathcal{K}(Y)$. Then we may write

$$\phi_B(a, \vec{f}) = \lim_n \sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})}.$$

If we write $\vec{f}_{n,k} = (f^1_{n,k}, f^2_{n,k}, \ldots)$ and $\vec{g}_{n,k} = (g^1_{n,k}, g^2_{n,k}, \ldots)$ then for any $(\xi, \vec{g}) \in X \oplus T$ we have that

$$\begin{aligned}
(\phi(a)\xi, (ag_1, f_1g_2, \ldots)) &= \phi_B(a, \vec{f})(\xi, \vec{g}) \\
&= \lim_n \sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})}(\xi, \vec{g}) \\
&= \lim_n \sum_{k=1}^{N_n} ((\xi_{n,k}\langle \eta_{n,k}, \xi\rangle_A, (f^1_{n,k}{g^1_{n,k}}^* g_1, f^2_{n,k}{g^2_{n,k}}^* g_2, \ldots)) \\
&= \lim_n \sum_{k=1}^{N_n} ((\Theta^X_{\xi_{n,k},\eta_{n,k}}\xi, (f^1_{n,k}{g^1_{n,k}}^* g_1, f^2_{n,k}{g^2_{n,k}}^* g_2, \ldots)).
\end{aligned} \tag{1}$$

Now since the operator norm on $\mathcal{L}(Y)$ dominates the operator norm on $\mathcal{L}(X)$, we see that $\lim_n \sum_{k=1}^{N_n} \Theta^X_{\xi_{n,k},\eta_{n,k}}$ converges and $\phi(a) = \lim_n \sum_{k=1}^{N_n} \Theta^X_{\xi_{n,k},\eta_{n,k}}$. Thus $a \in \mathcal{K}(X)$.

Furthermore, if $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for $\ker \phi$, then for any $n, m \in \mathbb{N}$

we have

$$
\begin{aligned}
&\Big\|\big(\sum_{k=1}^{N_n} f^1_{n,k}{g^1_{n,k}}^* - \sum_{k=1}^{N_m} f^1_{m,k}{g^1_{m,k}}^*\big)e_\lambda\Big\| \\
&= \Big\|\big(\sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})} - \sum_{k=1}^{N_m} \Theta^Y_{(\xi_{m,k},\vec{f}_{m,k}),(\eta_{m,k},\vec{g}_{m,k})}\big)(0,(e_\lambda,0,0,\ldots))\Big\| \\
&\leq \Big\|\sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})} - \sum_{k=1}^{N_m} \Theta^Y_{(\xi_{m,k},\vec{f}_{m,k}),(\eta_{m,k},\vec{g}_{m,k})}\Big\| \ \Big\|(0,(e_\lambda,0,0,\ldots))\Big\| \\
&= \Big\|\sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})} - \sum_{k=1}^{N_m} \Theta^Y_{(\xi_{m,k},\vec{f}_{m,k}),(\eta_{m,k},\vec{g}_{m,k})}\Big\|
\end{aligned}
$$

for all $\lambda \in \Lambda$. Taking the limit with respect to λ shows that

$$
\begin{aligned}
&\|\sum_{k=1}^{N_n} f^1_{n,k}{g^1_{n,k}}^* - \sum_{k=1}^{N_m} f^1_{m,k}{g^1_{m,k}}^*\| \\
&\qquad\qquad \leq \|\sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})} - \sum_{k=1}^{N_m} \Theta^Y_{(\xi_{m,k},\vec{f}_{m,k}),(\eta_{m,k},\vec{g}_{m,k})})\|.
\end{aligned}
$$

Since the $\sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{f}_{n,k}),(\eta_{n,k},\vec{g}_{n,k})}$'s converge in the operator norm on $\mathcal{L}(Y)$, this inequality implies that $\sum_{k=1}^{N_n} f^1_{n,k}{g^1_{n,k}}^*$ converges to an element in $\ker\phi$. If we let $a_2 = \lim_n \sum_{k=1}^{N_n} f^1_{n,k}{g^1_{n,k}}^* \in \ker\phi$, then Eq.(1) shows that $ag = a_2 g$ for all $g \in \ker\phi$. But then $a_1 := a - a_2 \in (\ker\phi)^\perp$, and consequently $a_1 \in J_X$. Since $a = a_1 + a_2$ the proof is complete. □

Lemma 4.6. *Let $(\tilde{\pi}, \tilde{t})$ be a representation of Y which is coisometric on* $\ker\phi \oplus T$*, and suppose that $\tilde{\pi}|_A$ is injective. For any $f \in \ker\phi$ we define $\epsilon_i(f) := (0,\ldots,0,f,0,\ldots) \in T$ where f appears in the i^{th} position. Then for every $i \in \mathbb{N}$ and for every $f \in \ker\phi$, the equation $\tilde{\pi}(0,\epsilon_i(f)) = 0$ implies that $f = 0$.*

Proof. First note that it suffices to prove the lemma for $f \geq 0$, because if $\tilde{\pi}(0,\epsilon_i(f)) = 0$ then $\tilde{\pi}(0,\epsilon_i(ff^*)) = \tilde{\pi}(0,\epsilon_i(f))\tilde{\pi}(0,\epsilon_i(f))^* = 0$, and $ff^* = 0$ if and only if $f = 0$.
If $\tilde{\pi}(0,\epsilon_i(f)) = 0$ and $f \geq 0$, then

$$
\begin{aligned}
\|\tilde{t}(0,\epsilon_i(\sqrt{f}))\|^2 &= \|\tilde{t}(0,\epsilon_i(\sqrt{f}))^*\tilde{t}(0,\epsilon_i(\sqrt{f}))\| \\
&= \|\tilde{\pi}(\langle(0,\epsilon_i(\sqrt{f})),(0,\epsilon_i(\sqrt{f}))\rangle_B\| \\
&= \|\tilde{\pi}(0,\epsilon_i(f))\| \\
&= 0
\end{aligned}
$$

so that $\tilde{t}(0,\epsilon_i(\sqrt{f}))=0$ and consequently

$$\begin{aligned}0&=\tilde{t}(0,\epsilon_i(\sqrt{f}))\tilde{t}(0,\epsilon_i(\sqrt{f}))^*=\tilde{\pi}^{(1)}(\Theta^Y_{(0,\epsilon_i(\sqrt{f})),(0,\epsilon_i(\sqrt{f}))})\\&=\begin{cases}\tilde{\pi}^{(1)}(\phi_B(f,\vec{0})) & \text{if } i=1\\ \tilde{\pi}^{(1)}(\phi_B(0,\epsilon_{i-1}(f))) & \text{if } i\geq 2\end{cases}=\begin{cases}\tilde{\pi}(f,\vec{0}) & \text{if } i=1\\ \tilde{\pi}(0,\epsilon_{i-1}(f)) & \text{if } i\geq 2.\end{cases}\end{aligned}$$

If $i=1$, the fact that $\tilde{\pi}|_A$ is injective implies that $f=0$. Furthermore, an inductive argument combined with the above equality shows that for all $i\in\mathbb{N}$ we have $f=0$. □

LEMMA 4.7. *Let $(\tilde{\pi},\tilde{t})$ be a representation of Y which is coisometric on $J_Y=J(Y)$. If $\vec{f}=(f_1,f_2,\ldots)\in T$ and $\vec{g}=(g_1,g_2,\ldots)\in T$, then*

$$\tilde{t}(0,\vec{f})\tilde{t}(0,\vec{g})^*=\tilde{\pi}(f_1g_1^*,(f_2g_2^*,f_3g_3^*,\ldots)).$$

Proof. For any $(\xi,\vec{h})\in Y=X\oplus T$ we have

$$\begin{aligned}\Theta_{(0,\vec{f}),(0,\vec{g})}(\xi,\vec{h})&=(0,\vec{f})\langle(0,\vec{g}),(\xi,\vec{h})\rangle_B=(0,\vec{f}\vec{g}^*\vec{h})\\&=\phi_B(f_1g_1^*,((f_2g_2^*,\ldots))(\xi,\vec{h})\end{aligned}$$

so that $\Theta_{(0,\vec{f}),(0,\vec{g})}=\phi_B(f_1g_1^*,(f_2g_2^*,\ldots))$. Thus

$$\begin{aligned}\tilde{t}(0,\vec{f})\tilde{t}(0,\vec{g})^*&=\tilde{\pi}^{(1)}(\Theta_{(0,\vec{f}),(0,\vec{g})})=\tilde{\pi}^{(1)}(\phi_B(f_1g_1^*,(f_2g_2^*,\ldots))\\&=\tilde{\pi}(f_1g_1^*,(f_2g_2^*,f_3g_3^*,\ldots)).\end{aligned}$$

□

LEMMA 4.8. *Let $(\tilde{\pi},\tilde{t})$ be a representation of Y. If $\xi\in X$, $a\in A$, and $\vec{f}\in T$, then the following relations hold:*

$$\begin{aligned}&(1)\quad \tilde{t}(0,\vec{f})\tilde{\pi}(a,\vec{0})=0\\&(2)\quad \tilde{t}(0,\vec{f})\tilde{t}(\xi,\vec{0})=0\\&(3)\quad \tilde{t}(0,\vec{f})\tilde{t}(\xi,\vec{0})^*=0\end{aligned}$$

Proof. To see (1) we note that $\tilde{t}(0,\vec{f})\tilde{\pi}(a,\vec{0})=\tilde{t}((0,\vec{f})(a,\vec{0}))=\tilde{t}(0,0)=0$. To see (2) and (3) let $\{\vec{e}_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for T. Then

$$\tilde{t}(0,\vec{f})\tilde{t}(\xi,\vec{0})=\lim_\lambda\tilde{t}(0,\vec{f}\vec{e}_\lambda)\tilde{t}(\xi,\vec{0})=\lim_\lambda\tilde{t}(0,\vec{f})\tilde{\pi}(0,\vec{e}_\lambda)\tilde{t}(\xi,\vec{0})=0$$

which shows that (2) holds, and

$$\begin{aligned}\tilde{t}(0,\vec{f})\tilde{t}(\xi,\vec{0})^*&=\lim_\lambda\tilde{t}(0,\vec{f}\vec{e}_\lambda)\tilde{t}(\xi,\vec{0})^*=\lim_\lambda\tilde{t}(0,\vec{f})\tilde{\pi}(0,\vec{e}_\lambda)\tilde{t}(\xi,\vec{0})^*\\&=\lim_\lambda\tilde{t}(0,\vec{f})(\tilde{t}(\xi,\vec{0})\tilde{\pi}(0,\vec{e}_\lambda))^*=0\end{aligned}$$

which shows that (3) holds. □

Lemma 4.9. *Let $(\tilde{\pi}, \tilde{t})$ be a representation of Y, and define $(\pi, t) := (\tilde{\pi}|_A, \tilde{t}|_X)$. If $c \in C^*(\pi, t)$ and $\vec{f} \in T$, then*

$$\tilde{t}(0, \vec{f})c = 0.$$

Proof. Since $C^*(\pi, t)$ is generated by elements of the form $\tilde{\pi}(a, \vec{0})$ and $\tilde{t}(\xi, \vec{0})$, the result follows from the relations in Lemma 4.8. ☐

Lemma 4.10. *Let $(\tilde{\pi}, \tilde{t})$ be a representation of Y which is coisometric on $J_Y = J(Y)$, and define $(\pi, t) := (\tilde{\pi}|_A, \tilde{t}|_X)$. If $n \in \{0, 1, 2, \dots\}$, then any element of the form*

$$\tilde{t}(\xi_1, \vec{f_1}) \dots \tilde{t}(\xi_n, \vec{f_n}) \tilde{\pi}(a, \vec{h}) \tilde{t}(\eta_n, \vec{g_n})^* \dots \tilde{t}(\eta_1, \vec{g_1})^*$$

will be equal to $c + \tilde{\pi}(0, \vec{k})$ for some $c \in C^*(\pi, t)$ and some $\vec{k} \in T$.

Proof. We shall prove this by induction on n.
Base Case: $n = 0$. Then the term above is equal to $\tilde{\pi}(a, \vec{h}) = \tilde{\pi}(a, \vec{0}) + \tilde{\pi}(0, \vec{h})$ and the claim holds trivially.
Inductive Step: Assume the claim holds for n. Given an element

$$\tilde{t}(\xi_1, \vec{f_1}) \dots \tilde{t}(\xi_{n+1}, \vec{f}_{n+1}) \tilde{\pi}(a, \vec{h}) \tilde{t}(\eta_{n+1}, \vec{g}_{n+1})^* \dots \tilde{t}(\eta_1, \vec{g_1})^*$$

it follows from the inductive hypothesis that

$$\tilde{t}(\xi_2, \vec{f_2}) \dots \tilde{t}(\xi_{n+1}, \vec{f}_{n+1}) \tilde{\pi}(a, \vec{h}) \tilde{t}(\eta_{n+1}, \vec{g}_{n+1})^* \dots \tilde{t}(\eta_2, \vec{g_2})^*$$

has the form $c + \tilde{\pi}(0, \vec{k})$ for $c \in C^*(\pi, t)$ and $\vec{k} \in T$. Thus using Lemma 4.9 gives

$$\begin{aligned}
&\tilde{t}(\xi_1, \vec{f_1}) \dots \tilde{t}(\xi_{n+1}, \vec{f}_{n+1}) \tilde{\pi}(a, \vec{h}) \tilde{t}(\eta_{n+1}, \vec{g}_{n+1})^* \dots \tilde{t}(\eta_1, \vec{g_1})^* \\
&= \tilde{t}(\xi_1, \vec{f_1})(c + \tilde{\pi}(0, \vec{k})) \tilde{t}(\eta_1, \vec{g_1})^* \\
&= (\tilde{t}(\xi_1, \vec{0}) + \tilde{t}(0, \vec{f_1}))(c + \tilde{\pi}(0, \vec{k}))(\tilde{t}(\eta_1, \vec{0}) + \tilde{t}(0, \vec{g_1})^*) \\
&= \tilde{t}(\xi_1, \vec{0}) c \tilde{t}(\eta_1, \vec{0})^* + \tilde{t}(0, \vec{f_1}) \tilde{\pi}(0, \vec{k}) \tilde{t}(0, \vec{g_1})^* \\
&= \tilde{t}(\xi_1, \vec{0}) c \tilde{t}(\eta_1, \vec{0})^* + \tilde{t}(0, \vec{f_1}\vec{k}) \tilde{t}(0, \vec{g_1})^*.
\end{aligned}$$

It follows from Lemma 4.7 that $\tilde{t}(0, \vec{f_1}\vec{k}) \tilde{t}(0, \vec{g_1})^*$ is of the form $c' + \tilde{\pi}(0, \vec{k'})$ with $c' \in \operatorname{im} \pi \subseteq C^*(\pi, t)$. Since $\tilde{t}(\xi_1, \vec{0}) c \tilde{t}(\eta_1, \vec{0})^*$ is also in $C^*(\pi, t)$ the proof is complete. ☐

We wish to show that if $(\tilde{\pi}, \tilde{t})$ is a representation of Y and if we restrict to obtain $(\pi, t) := (\tilde{\pi}|_X, \tilde{t}|_A)$, then $C^*(\pi, t)$ is a corner of $C^*(\tilde{\pi}, \tilde{t})$. If A is unital and X is left essential, then this corner will be determined by the projection $\pi(1, \vec{0})$. However, in the following lemma we wish to consider the general case and must make use of approximate units to define the projection that determines the corner.

LEMMA 4.11. *Let X be a C^*-correspondence over A and let $T := (\ker \phi)^{\mathbb{N}}$ be the tail determined by $\ker \phi$. If $Y := X \oplus T$ is the C^*-correspondence over $B := A \oplus T$ formed by adding the tail T to X, and if $(\tilde{\pi}, \tilde{t})$ is a representation of Y, then there exists a projection $p \in \mathcal{M}(C^*(\tilde{\pi}, \tilde{t}))$ with the property that for all $a \in A$, $\xi \in X$, and $\vec{f} \in T$ the following relations hold:*

$$(1) \quad p\tilde{t}(\xi, \vec{f}) = \tilde{t}(\xi, (f_1, 0, 0, \ldots))$$
$$(2) \quad \tilde{t}(\xi, \vec{f})p = \tilde{t}(\xi, \vec{0})$$
$$(3) \quad p\tilde{\pi}(a, \vec{f}) = \tilde{\pi}(a, \vec{f})p = \tilde{\pi}(a, \vec{0})$$

Proof. Let $\{\vec{e}_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for T, and for each $\lambda \in \Lambda$ let $\vec{e}_\lambda = (e^1_\lambda, e^2_\lambda, \ldots)$. Consider $\{\tilde{\pi}(0, \vec{e}_\lambda)\}_{\lambda \in \Lambda}$. For any element

$$\tilde{t}(\xi_1, \vec{f_1}) \ldots \tilde{t}(\xi_n, \vec{f_n})\tilde{\pi}(a, \vec{h})\tilde{t}(\eta_m, \vec{g_m})^* \ldots \tilde{t}(\eta_1, \vec{g_1})^* \tag{2}$$

we have

$$\begin{aligned} & \lim_\lambda \tilde{\pi}(0, \vec{e}_\lambda)\tilde{t}(\xi_1, \vec{f_1}) \ldots \tilde{t}(\xi_n, \vec{f_n})\tilde{\pi}(a, \vec{h})\tilde{t}(\eta_m, \vec{g_m})^* \ldots \tilde{t}(\eta_1, \vec{g_1})^* \\ = & \lim_\lambda \tilde{t}(0, (0, e^1_\lambda f_{12}, e^2_\lambda f_{13}, \ldots) \ldots \tilde{t}(\xi_n, \vec{f_n})\tilde{\pi}(a, \vec{h})\tilde{t}(\eta_m, \vec{g_m})^* \ldots \tilde{t}(\eta_1, \vec{g_1})^* \\ = & \, \tilde{t}(0, (0, f_{12}, f_{13}, \ldots)) \ldots \tilde{t}(\xi_n, \vec{f_n})\tilde{\pi}(a, \vec{h})\tilde{t}(\eta_m, \vec{g_m})^* \ldots \tilde{t}(\eta_1, \vec{g_1})^* \end{aligned}$$

so this limit exists.
Now since any $c \in C^*(\tilde{\pi}, \tilde{t})$ can be approximated by a finite sum of elements of the form shown in (2), it follows that $\lim_\lambda \tilde{\pi}(0, \vec{e}_\lambda)c$ exists for all $c \in C^*(\tilde{\pi}, \tilde{t})$. Let us view $C^*(\tilde{\pi}, \tilde{t})$ as a C^*-correspondence over itself (see [20, Example 2.10]). If we define $q : C^*(\tilde{\pi}, \tilde{t}) \to C^*(\tilde{\pi}, \tilde{t})$ by $q(c) = \lim_\lambda \tilde{\pi}(0, \vec{e}_\lambda)c$ then we see that for any $c, d \in C^*(\tilde{\pi}, \tilde{t})$ we have

$$d^*q(c) = \lim_\lambda d^*\tilde{\pi}(0, \vec{e}_\lambda)c = \lim_\lambda (\tilde{\pi}(0, \vec{e}_\lambda)d)^*c = q(d)^*c$$

and hence q is an adjointable operator on $C^*(\tilde{\pi}, \tilde{t})$. Therefore q defines (left multiplication by) an element in the multiplier algebra $\mathcal{M}(C^*(\tilde{\pi}, \tilde{t}))$ [20, Theorem 2.47]. It is easy to check that $q^2 = q^* = q$ so that q is a projection. Now if we let $p := 1 - q$ in $\mathcal{M}(C^*(\tilde{\pi}, \tilde{t}))$, then it is easy to check that relations (1), (2), and (3) follow from the definition of q. ☐

Proof of Theorem 4.3. (a) Let $I := \ker \phi$, set $\mathcal{H}_0 := \pi(I)\mathcal{H}_X$, and define $\mathcal{H}_T := \bigoplus_{i=1}^\infty \mathcal{H}_i$ where $\mathcal{H}_i = \mathcal{H}_0$ for all $i = 1, 2, \ldots$. We define $\tilde{t} : Y \to \mathcal{B}(\mathcal{H}_X \oplus \mathcal{H}_T)$ and $\tilde{\pi} : B \to \mathcal{B}(\mathcal{H}_X \oplus \mathcal{H}_T)$ as follows: Viewing Y as $Y = X \oplus T$ and B as $B = A \oplus T$, for any $(h, (h_1, h_2, \ldots)) \in \mathcal{H}_\mathcal{Q} \oplus \mathcal{H}_T$ we define

$$\tilde{t}(\xi, (f_1, f_2, \ldots))(h, (h_1, h_2, \ldots)) = (t(\xi)h + \pi(f_1)h_1, (\pi(f_2)h_2, \pi(f_3)h_3, \ldots))$$

and

$$\tilde{\pi}(a, (f_1, f_2, \ldots))(h, (h_1, h_2, \ldots)) = (\pi(a)h, (\pi(f_1)h_1, \pi(f_2)h_2, \ldots)).$$

Then it is straightforward to show that $(\tilde{\pi}, \tilde{t})$ is a representation of Y on $\mathcal{H}_Q \oplus \mathcal{H}_T$. To see that $(\tilde{\pi}, t)$ is coisometric on $J(Y)$, choose an element $(a,(f_1,f_2,\ldots)) \in J(Y)$. By Lemma 4.5 we know that $a = a_1 + a_2$ for $a_1 \in J_X$ and $a_2 \in \ker\phi$. Furthermore, since $a_1 \in J(X)$ we may write $\phi(a_1) = \lim_n \sum_{k=1}^{N_n} \Theta^X_{\xi_{n,k},\eta_{n,k}}$ for some $\xi_{n,k}, \eta_{n,k} \in X$. It follows that

$$\phi_B(a_1, \vec{0}) = \lim_n \sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{0}),(\eta_{n,k},\vec{0})} \in \mathcal{K}(Y).$$

In addition, since $a_2 \in \ker\phi$ we see that if we let $\{\vec{e}_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for T with $\vec{e}_\lambda = (e^1_\lambda, e^2_\lambda, \ldots)$ for each λ, then

$$\phi_B(a_2, \vec{f}) = \lim_\lambda \Theta^Y_{(0,(a,f_1,f_2,\ldots)),(0,(e^1_\lambda,e^2_\lambda,\ldots))} \in \mathcal{K}(Y).$$

Now for any $n \in \mathbb{N}$ we see that $\{e^n_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for $\ker\phi$. Furthermore, we see that for all $(\xi,\vec{f}), (\eta,\vec{g}) \in Y = X \oplus T$ we have

$$\tilde{t}(\xi,\vec{f})\tilde{t}(\eta,\vec{g})^* = (t(\xi)t(\eta)^* + \pi(f_1g_1^*), (\pi(f_2g_2^*), \pi(f_3g_3^*), \ldots))$$

and thus

$$\begin{aligned}
&\tilde{\pi}^{(1)}(\phi_B(a,\vec{f}))\\
&= \tilde{\pi}^{(1)}(\phi_B(a_1,\vec{0})) + \tilde{\pi}^{(1)}(\phi_B(a_2,\vec{f}))\\
&= \lim_n \sum_{k=1}^{N_n} \tilde{t}(\xi_{n,k},\vec{0})\tilde{t}(\eta_{n,k},\vec{0})^* + \lim_\lambda \tilde{t}(0,(a_2,f_1,f_2,\ldots))\tilde{t}(0,(e^1_\lambda,e^2_\lambda,\ldots))^*\\
&= \lim_n \sum_{k=1}^{N_n} (t(\xi_{n,k})t(\eta_{n,k})^*, \vec{0}) + \lim_\lambda (\pi(a_2e^1_\lambda), (\pi(f_1e^2_\lambda), \pi(f_2e^3_\lambda), \ldots))\\
&= (\pi^{(1)}(\phi(a_1)), \vec{0}) + (\pi(a_2), (\pi(f_1), \pi(f_2), \ldots))\\
&= (\pi(a_1), \vec{0}) + (\pi(a_2), (\pi(f_1), \pi(f_2), \ldots))\\
&= \tilde{\pi}(a,\vec{f})
\end{aligned}$$

so $(\tilde{\pi}, \tilde{t})$ is coisometric on $J(Y)$.

(b) If $(\tilde{\pi}, \tilde{t})$ is a representation of Y in a C^*-algebra C which is coisometric on $J(Y)$, then it is straightforward to see that the restriction $(\tilde{\pi}|_A, \tilde{t}|_X)$ is a representation. To see that $(\tilde{\pi}|_A, \tilde{t}|_X)$ is coisometric on J_X, choose an element $a \in J_X$. Since $J_X \subseteq J(X)$ we may write $\phi(a) = \lim_n \sum_{k=1}^{N_n} \Theta^X_{\xi_{n,k},\eta_{n,k}}$ for some $\xi_{n,k}, \eta_{n,k} \in X$. In addition, since $a \in (\ker\phi)^\perp \subseteq J_X$ we have that

$$\phi_B(a, \vec{0}) = \lim_n \sum_{k=1}^{N_n} \Theta^Y_{(\xi_{n,k},\vec{0}),(\eta_{n,k},\vec{0})} \in \mathcal{K}(Y)$$

and we have

$$\begin{aligned}\tilde{\pi}|_A^{(1)}(\phi_A(a)) &= \lim_n \sum_{k=1}^{N_n} \tilde{t}|_X(\xi_{n,k})\tilde{t}|_X(\eta_{n,k})^* = \lim_n \sum_{k=1}^{N_n} \tilde{t}(\xi_{n,k},\vec{0})\tilde{t}(\eta_{n,k},\vec{0})^* \\ &= \tilde{\pi}^{(1)}(\phi_B(a,\vec{0})) = \tilde{\pi}(a,\vec{0}) = \tilde{\pi}|_A(a)\end{aligned}$$

so $(\tilde{\pi}|_A, \tilde{t}|_X)$ is coisometric on J_X.

Furthermore, suppose that the restriction $\tilde{\pi}|_A$ is injective. If $(a,\vec{f}) \in B := A \oplus T$ and $\tilde{\pi}(a,\vec{f}) = 0$, let $\{g_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for $\ker\phi$, and for any $f \in \ker\phi$ and $i \in \mathbb{N}$ let $\epsilon_i(f) := (0,\ldots,0,f,0,\ldots)$ where f is in the i^{th} position. Since $\tilde{\pi}(a,\vec{f}) = 0$ we see that if we write $\vec{f} = (f_1, f_2, \ldots)$, then for all $i \in \mathbb{N}$ we have

$$\tilde{\pi}(0,\epsilon_i(g_\lambda f_i)) = \tilde{\pi}(0,\epsilon_i(g_\lambda))\tilde{\pi}(a,\vec{f}) = 0,$$

and taking limits with respect to λ shows that $\tilde{\pi}(0,\epsilon_i(f_i)) = 0$ for all $i \in \mathbb{N}$. From Lemma 4.6 it follows that $f_i = 0$ for all $i \in \mathbb{N}$. Thus $\vec{f} = 0$, and since $\tilde{\pi}|_A$ is injective we also have that $a = 0$. Hence $\tilde{\pi}$ is injective.

(c) The fact that $(\pi,t) := (\pi_Y|_A, t_Y|_X)$ is a representation which is coisometric on J_X follows from Part (b). Furthermore, the fact that $\rho_{(\pi,t)}$ is injective follows from Part (a) which shows that any $*$-representation of $\mathcal{O}_X$ factors through a $*$-representation of $\mathcal{O}_Y$. All that remains is to show that $\operatorname{im}\rho_{(\pi,t)} = C^*(\pi,t)$ is a full corner of $\mathcal{O}_Y$.

Let $p \in \mathcal{M}(\mathcal{O}_Y)$ be the projection described in Lemma 4.11. We shall first show that $C^*(\pi,t) = p\mathcal{O}_Y p$. To begin, we see from the relations in Lemma 4.11 that for all $a \in A$ we have $p\pi(a)p = p\pi_Y(a,\vec{0})p = \pi_Y(a,\vec{0}) = \pi(a)$ and for all $\xi \in X$ we have $pt(\xi)p = p(t_Y(\xi,\vec{0}))p = t_Y(\xi,\vec{0}) = t(\xi)$. Thus $C^*(\pi,t) \subseteq p\mathcal{O}_Y p$.

To see the reverse inclusion, note that any element in $\mathcal{O}_Y$ is the limit of sums of elements of the form

$$t_Y(\xi_1,\vec{f_1})\ldots t_Y(\xi_n,\vec{f_n})\pi_Y(a,\vec{h})t_Y(\eta_m,\vec{g_m})^*\ldots t_Y(\eta_1,\vec{g_1})^*$$

and thus any element of $p\mathcal{O}_Y p$ is the limit of sums of elements of the form

$$pt_Y(\xi_1,\vec{f_1})\ldots t_Y(\xi_n,\vec{f_n})\pi_Y(a,\vec{h})t_Y(\eta_m,\vec{g_m})^*\ldots t_Y(\eta_1,\vec{g_1})^*p$$

Therefore, it suffices to show that each of these elements is in $C^*(\pi,t)$. Now if $n \geq m$, then we may use Lemma 4.10 to write

$$t_Y(\xi_{n-m+1},\vec{f}_{n-m+1})\ldots t_Y(\xi_n,\vec{f_n})\pi_Y(a,\vec{h})t_Y(\eta_m,\vec{g_m})^*\ldots t_Y(\eta_1,\vec{g_1})^*$$

as $c + \pi_Y(0, \vec{k})$ for $c \in C^*(\pi, t)$ and $\vec{k} \in T$. Then

$$\begin{aligned}
&pt_Y(\xi_1, \vec{f_1}) \dots t_Y(\xi_n, \vec{f_n})\pi_Y(a, \vec{h})t_Y(\eta_m, \vec{g_m})^* \dots t_Y(\eta_1, \vec{g_1})^* p \\
&= pt_Y(\xi_1, \vec{f_1}) \dots t_Y(\xi_{n-m}, \vec{f}_{n-m})(c + \pi_Y(0, \vec{k}))p \\
&= pt_Y(\xi_1, \vec{f_1}) \dots t_Y(\xi_{n-m}, \vec{f}_{n-m})cp \\
&= pt_Y(\xi_1, \vec{f_1}) \dots t_Y(\xi_{n-m}, \vec{f}_{n-m})pcp \\
&= pt_Y(\xi_1, \vec{f_1}) \dots t_Y(\xi_{n-m-1}, \vec{f}_{n-m-1})pt_Y(\xi_{n-m}, \vec{0})pcp \\
&\quad \vdots \\
&= pt_Y(\xi_1, \vec{0})p \dots pt_Y(\xi_{n-m}, \vec{0})pcp \\
&= t_Y(\xi_1, \vec{0}) \dots t_Y(\xi_{n-m}, \vec{0})c \\
&= t(\xi_1) \dots t(\xi_{n-m})c
\end{aligned}$$

which is in $C^*(\pi, t)$. The case when $n \leq m$ is similar. Hence $p\mathcal{O}_Y p \subseteq C^*(\pi, t)$. To see that the corner $C^*(\pi, t) = p\mathcal{O}_Y p$ is full, suppose that $\mathcal{I}$ is an ideal in $\mathcal{O}_Y$ that contains $C^*(\pi, t)$. For $f \in \ker \phi$ and $n \in N$ define $\epsilon_n(f) := (0, \dots, 0, f, 0, \dots) \in T$, where the term f is in the n^{th} position. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit for $\ker \phi$. Now $t_Y(\xi, \vec{0}), \pi_Y(a, \vec{0}) \in C^*(\pi, t) \subseteq \mathcal{I}$ for all $a \in A$ and $\xi \in X$, and since T is the c_0-direct sum of countably many copies of $\ker \phi$ in order to show that $\mathcal{I}$ is all of $\mathcal{O}_Y$ it suffices to prove that for all $n \in N$ and $\lambda \in \Lambda$ we have $t_Y(0, \epsilon_n(e_\lambda)) \in \mathcal{I}$ and $\pi_Y(0, \epsilon_n(e_\lambda)) \in \mathcal{I}$. We shall prove this by induction on n.

Base Case: For any $\beta, \lambda \in \Lambda$ we have from Lemma 4.7 that

$$t_Y(0, \epsilon_1(e_\lambda))t_Y(0, \epsilon_1(e_\beta))^* = \pi_Y(e_\lambda e_\beta, \vec{0}) \in \mathcal{I}.$$

Also for any $\alpha \in \Lambda$ we have

$$\begin{aligned}
t_Y(0, (\epsilon_1(e_\lambda e_\beta e_\alpha)) &= t_Y(0, \epsilon_1(e_\lambda))\pi_Y(0, \epsilon_1(e_\beta^* e_\alpha)) \\
&= t_Y(0, \epsilon_1(e_\lambda))t_Y(0, \epsilon_1(e_\beta))^* t_Y(0, \epsilon_1(e_\alpha))
\end{aligned}$$

which is in $\mathcal{I}$. Taking limits with respect to α and β gives

$$t_Y(0, \epsilon_1(e_\lambda)) = \lim_\beta \lim_\alpha t_Y(0, \epsilon_1(e_\lambda e_\beta e_\alpha)) \in \mathcal{I}.$$

Furthermore, since $t_Y(0, \epsilon_1(e_\lambda)) \in \mathcal{I}$ for all $\lambda \in \Lambda$, we see that

$$\pi_Y(0, \epsilon_1(e_\lambda)) = \lim_\beta \pi_Y(0, \epsilon_1(e_\lambda e_\beta)) = \lim_\beta t_Y(0, \epsilon_1(e_\lambda))^* t_Y(0, \epsilon_1(e_\beta)) \in \mathcal{I}.$$

Inductive step: Suppose that $t_Y(0, \epsilon_n(e_\lambda)), \pi_Y(0, \epsilon_n(e_\lambda)) \in \mathcal{I}$ for any $\lambda \in \Lambda$. Then for all $\lambda, \beta \in \Lambda$ we have

$$\begin{aligned}
t_Y(0, \epsilon_{n+1}(e_\lambda))t_Y(0, \epsilon_{n+1}(e_\beta))^* &= \pi_Y^{(1)}(\Theta^Y_{(0,\epsilon_{n+1}(e_\lambda)),(0,\epsilon_{n+1}(e_\beta))}) \\
&= \pi_Y^{(1)}(\phi_B(0, \epsilon_n(e_\beta e_\lambda))) \\
&= \pi_Y(0, \epsilon_n(e_\beta e_\lambda)) \\
&= \pi_Y(0, \epsilon_n(e_\beta))\pi_Y(0, \epsilon_n(e_\lambda))
\end{aligned}$$

which is in $\mathcal{I}$. Thus for any $\alpha \in \Lambda$ we have that

$$\begin{aligned} t_Y(0, \epsilon_{n+1}(e_\lambda e_\beta e_\alpha)) &= t_Y(0, \epsilon_{n+1}(e_\lambda))\pi_Y(0, \epsilon_{n+1}(e_\beta e_\alpha)) \\ &= t_Y(o, \epsilon_{n+1}(e_\lambda))t_Y(0, \epsilon_{n+1}(e_\beta))^* t_Y(0, \epsilon_{n+1}(e_\alpha)) \end{aligned}$$

is in $\mathcal{I}$. Taking limits with respect to α and β gives

$$t_Y(0, \epsilon_{n+1}(e_\lambda)) = \lim_\beta \lim_\alpha t_Y(0, \epsilon_{n+1}(e_\lambda e_\beta e_\alpha)) \in \mathcal{I}.$$

Furthermore, since $t_Y(0, \epsilon_{n+1}(e_\lambda)) \in \mathcal{I}$ for all $\lambda \in \Lambda$, we have

$$\pi_Y(0, \epsilon_{n+1}(e_\lambda)) = \lim_\beta \pi_Y(0, \epsilon_{n+1}(e_\beta e_\lambda)) = \lim_\beta t_Y(0, \epsilon_{n+1}(e_\beta))^* t_Y(0, \epsilon_{n+1}(e_\lambda))$$

which is in the ideal $\mathcal{I}$. □

5 Gauge-Invariant Uniqueness

Recall that we let γ denote the gauge action of $\mathbb{T}$ on $\mathcal{O}_X$. A gauge-invariant uniqueness was proven in [6, Theorem 4.1] for (augmented) Cuntz-Pimsner algebras. Our method of adding tails, together with Theorem 4.3, will allow us to extend this theorem to the case when ϕ is not injective, and ultimately to all relative Cuntz-Pimsner algebras.

The following Gauge-Invariant Uniqueness Theorem was proven by Katsura using direct methods in [11, Theorem 6.4]. We shall now give an alternate proof, showing how the method of adding tails can be used to bootstrap [6, Theorem 4.1] to the general case.

Theorem 5.1 (Gauge-Invariant Uniqueness). *Let X be a C^*-correspondence over A, and let (π_X, t_X) be a universal $J(X)$-coisometric representation of X. If $\rho : \mathcal{O}_X \to C$ is a homomorphism between C^*-algebras which satisfies the following two conditions:*

1. *the restriction of ρ to $\pi_X(A)$ is injective*
2. *there is a strongly continuous action $\beta : \mathbb{T} \to \operatorname{Aut}(\rho(\mathcal{O}_X))$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$*

then ρ is injective.

Remark 5.2. When ϕ is injective, the statement above is actually an equivalent reformulation of [6, Theorem 4.1]. The equivalence relies on the fact that for any C^*-correspondence X, the universal $J(X)$-coisometric representation (i_A, i_X) has the property that i_A is injective if and only if the left action ϕ is injective.

Proof of Theorem 5.1. Let $T := (\ker \phi)^{\mathbb{N}}$ be the tail determined by $\ker \phi$, and let $Y := X \oplus T$ be the C^*-correspondence over $B := A \oplus T$ formed by adding the tail T to X. By Theorem 4.3(c) we may identify $(\mathcal{O}_X, \pi_X, t_X)$ with $(S, \pi_Y|_A, t_Y|_X)$ where S is the C^*-subalgebra of $\mathcal{O}_Y$ generated by

$$\{\pi_Y(a, \vec{0}), t_Y(\xi, \vec{0}) : a \in A \text{ and } \xi \in X\}.$$

Since $\beta : \mathbb{T} \to \mathrm{Aut}(\mathrm{im}\,\rho)$ is an action of $\mathbb{T}$ on $\mathrm{im}\,\rho$, there exists a Hilbert space $\mathcal{H}_X$, a faithful representation $\kappa : \mathrm{im}\,\rho \to \mathcal{B}(\mathcal{H}_X)$, and a unitary representation $U : \mathbb{T} \to \mathcal{U}(\mathcal{H}_X)$ such that

$$\kappa(\beta_z(x)) = U_z \kappa(x) U_z^* \qquad \text{for all } x \in \mathrm{im}\,\rho \text{ and } z \in \mathbb{T}.$$

In addition, since $\tau := \kappa \circ \rho$ is a $*$-homomorphism from S into $\mathcal{B}(\mathcal{H}_X)$ which is faithful on $\pi_X(A)$, it follows from Theorem 4.3(a) that τ may be extended to a $*$-homomorphism $\tilde{\tau} : \mathcal{O}_Y \to \mathcal{B}(\mathcal{H}_X \oplus \mathcal{H}_T)$ with $\tilde{\tau}$ faithful on $\pi_Y(B)$.
We shall now define a unitary representation $W : \mathbb{T} \to \mathcal{B}(\mathcal{H}_X \oplus \mathcal{H}_T)$ as follows. We see from the proof of Theorem 4.3(a) that $\mathcal{H}_T := \bigoplus_{i=1}^\infty \mathcal{H}_i$. Thus for $(h, (h_1, h_2, \ldots)) \in \mathcal{H}_\mathcal{Q} \oplus \mathcal{H}_T$ we define

$$W_z(h, (h_1, h_2, \ldots)) := (U_z h, (z^{-1} h_1, z^{-2} h_2, \ldots)) \qquad \text{for } z \in \mathbb{T}.$$

We may then define $\tilde{\beta} : \mathbb{T} \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}_X \oplus \mathcal{H}_T))$ by $\tilde{\beta}_z(T_0) := W_z T_0 W_z^*$, and we see that $\tilde{\beta}$ is a strongly continuous gauge action. Furthermore, if γ' denotes the gauge action of $\mathbb{T}$ on $\mathcal{O}_Y$, then $\tilde{\beta}_z \circ \tilde{\tau} = \tilde{\tau} \circ \gamma'_z$ (to see this recall how the extension $\tilde{\tau}$ is defined in the proof of Theorem 4.3(a) and then simply check on the generators $\{t_Y(\xi, \vec{f}), \pi_Y(a, \vec{g}) : \xi \in X, a \in A, \text{ and } \vec{f}, \vec{g} \in T\}$). Thus by [6, Theorem 4.1] we have that $\tilde{\tau}$ is injective. Hence $\tilde{\tau}|_S = \tau = \kappa \circ \rho$ is injective, and ρ is injective. ☐

To conclude this section we shall interpret our result in the relative Cuntz-Pimsner setting.

Remark 5.3. Katsura has shown in [12] that if $\mathcal{O}(K, X)$ is a relative Cuntz-Pimsner algebra, then there exists a C^*-correspondence X' with the property that $\mathcal{O}_{X'}$ is naturally isomorphic to $\mathcal{O}(K, X)$. Using this analysis one can obtain the following interpretation of Theorem 5.1 for relative Cuntz-Pimsner algebras.

Interpretation of Theorem 5.1 for Relative Cuntz-Pimsner Algebras: *Let X be a C^*-correspondence with left action $\phi : X \to \mathcal{L}(X)$, let K be an ideal in $J(X) := \phi^{-1}(\mathcal{K}(X))$, and let (π_X, t_X) be a universal K-coisometric representation of X. If $\rho : \mathcal{O}_X \to C$ is a homomorphism between C^*-algebras which satisfies the following three conditions:*

(1) the restriction of ρ to $\pi_X(A)$ is injective

(2) if $\rho(\pi_X(a)) \in \rho(\pi_X^{(1)}(\mathcal{K}(X)))$, then $\pi_X(a) \in \pi_X(K)$

(3) there is a strongly continuous action $\beta : \mathbb{T} \to \mathrm{Aut}(\rho(\mathcal{O}_X))$ such that $\beta_z \circ \rho = \rho \circ \gamma_z$ for all $z \in \mathbb{T}$

then ρ is injective.

Finally, we mention that if we define a map $T_K : J(X) \to \mathcal{O}(K, X)$ by

$$T_K(a) := \pi_X(a) - \pi_X^{(1)}(\phi(a))$$

then the equation

$$\begin{aligned} T_K(a)T_K(b) &= (\pi_X(a) - \pi_X^{(1)}(\phi(a)))(\pi_X(b) - \pi_X^{(1)}(\phi(b))) \\ &= \pi_X(ab) - \pi_X^{(1)}(\phi(a))\pi_X(b) - \pi_X(a)\pi_X^{(1)}(\phi(b)) + \pi_X^{(1)}(\phi(ab)) \\ &= \pi_X(ab) - \pi_X^{(1)}(\phi(ab)) \\ &= T_K(ab) \end{aligned}$$

shows that this map is a homomorphism. If π_X is injective (which by [15, Proposition 2.21] occurs if and only if $K \cap \ker \phi = \emptyset$), then we may replace Condition (2) in the above statement by the condition

(2′) the restriction of ρ to $T_K(J(X))$ is injective.

6 Gauge-Invariant Ideals

In this section we use Theorem 5.1 to characterize the gauge-invariant ideals in C^*-algebras associated to certain correspondences.

Definition 6.1. Let X be a C^*-correspondence over A. We say that an ideal $I \lhd A$ is *X-invariant* if $\phi(I)X \subseteq XI$. We say that an X-invariant ideal $I \lhd A$ is *X-saturated* if

$$a \in J_X \text{ and } \phi(a)X \subseteq XI \implies a \in I.$$

Remark 6.2. In [9] the authors only considered Hilbert bimodules (i.e. C^*-correspondences) for which ϕ is injective and $\phi(A) \subseteq \mathcal{K}(X)$, and thus the definition of X-saturated that they gave was that $a \in A$ and $\phi(a)X \subseteq XI$ implies $a \in I$. Since $J_X = A$ throughout their paper, this notion is equivalent to the one defined in Definition 6.1. In [6, Remark 3.11] it was suggested that the definition of X-saturated for general C^*-correspondences should also be that $a \in A$ and $\phi(a)X \subseteq XI$ implies $a \in I$. However, after considering how the definition of *saturated* was extended to (or rather modified for) non-row-finite graphs in [1, §3] and [3, §3] we believe that Definition 6.1 is the appropriate generalization.

Recall that if I is an ideal of A, then

$$X_I := \{x \in X : \langle x, y\rangle_A \in I \text{ for all } y \in X\}$$

is a right Hilbert A-module, and by the Hewitt-Cohen Factorization Theorem $X_I = XI := \{x \cdot i : x \in X \text{ and } i \in I\}$ (see [6, §2]). Furthermore, X/XI is a right Hilbert A/I-module in the obvious way [6, Lemma 2.1]. In order for

X/XI to be a C^*-correspondence, we need the ideal I to be X-invariant. Let $q^I : A \to A/I$ and $q^{XI} : X \to X/XI$ be the appropriate quotient maps. If I is X-invariant, then one may define $\phi_{A/I} : A/I \to \mathcal{L}(X/XI)$ by

$$\phi_{A/I}(q^I(a))(q^{XI}(x)) := q^{XI}(\phi(a)(x))$$

and with this action X/XI is a C^*-correspondence over A/I [6, Lemma 3.2].

Lemma 6.3. *Let X be a C^*-correspondence over a C^*-algebra A, and let I be an X-saturated X-invariant ideal in A. If $q^I : A \to A/I$ denotes the quotient map, then*

$$q^I(J_X) \subseteq J_{X/XI}.$$

Furthermore, if X has the following two properties:

1. $\phi(A) \subseteq \mathcal{K}(X)$
2. $\ker\phi$ *is complemented in A (i.e. there exists an ideal J of A with the property that $A = J \oplus \ker\phi$),*

then

$$q^I(J_X) = J_{X/XI}.$$

Proof. Let $a \in J_X$. Then $a \in J(X)$, and it follows from [6, Lemma 2.7] that $q^I(a) \in J(X/XI)$. Also, if $q^I(b) \in \ker\phi_{A/I}$, then $q^I(ab) \in \ker\phi_{A/I}$ and for all $x \in X$ we have

$$q^{XI}(\phi(ab)(x)) = \phi_{A/I}(ab)q^{XI}(x) = 0$$

and thus

$$\phi(ab)XI \subseteq XI. \tag{3}$$

Since $a \in J_X$ and J_X is an ideal, we see that $ab \in J_X$. Now since I is X-saturated, (3) implies that $ab \in I$ and $q^I(a)q^I(b) = q^I(ab) = 0$. Thus $q^I(a) \in (\ker\phi_{A/I})^\perp$ and $q^I(a) \in J_{X/XI}$.

Now suppose that Conditions (1) and (2) in the statement of the lemma hold. Since $\phi(A) \subseteq \mathcal{K}(X)$ it follows that $J(X) = A$. In addition, [6, Lemma 2.7] shows that $q^I(J(X)) = J(X/XI)$. From Condition (2) we know that $A = J \oplus \ker\phi$ for some ideal J of A. However, the definition of J_X then implies that $J = J_X$. Thus if $a \in A$ and $q^I(a) \in J_{X/XI}$, then we may write $a = b + c$ for $b \in J_X$ and $c \in \ker\phi$. But then $q^I(b) \in J_{X/XI}$ by the first part of the lemma, and $q^I(c) = q^I(a) - q^I(b) \in J_{X/XI}$. Since $c \in \ker\phi$ it follows that for all $x \in X$ we have

$$\phi_{A/I}(q^I(c))q^{XI}(x) = q^{XI}(\phi(c)(x)) = 0$$

and thus $q^I(c) \in \ker\phi_{A/I}$. Thus $q^I(c) \in J_{X/XI} \cap \ker\phi_{A/I} = \{0\}$ so $q^I(c) = 0$ and $q^I(a) = q^I(b) \in q^I(J_X)$. Thus $J_{X/XI} \subseteq q^I(J_X)$. ☐

The following theorem was proven in [9, Theorem 4.3] under the hypotheses that ϕ is injective, A is unital, and X is full and finite projective as a right A-module (so in particular, $\phi(A) \subseteq \mathcal{K}(X)$). However, Theorem 5.1 allows us to give a fairly simple proof of the result for much more general C^*-correspondences.

Theorem 6.4. *Let X be a C^*-correspondence with the following two properties:*

1. $\phi(A) \subseteq \mathcal{K}(X)$
2. $\ker \phi$ *is complemented in A (i.e. there exists an ideal J of A with the property that $A = J \oplus \ker \phi$),*

and let (π_X, t_X) be a universal $J(X)$-coisometric representation of X. Then there is a lattice isomorphism from the X-saturated X-invariant ideals of A onto the gauge-invariant ideals of $\mathcal{O}_X$ given by

$$I \mapsto \mathcal{I}(I) := \text{the ideal in } \mathcal{O}_X \text{ generated by } \pi_X(I)$$

Proof. To begin we see that $\mathcal{I}(I)$ is in fact gauge invariant since

$$\begin{aligned}\mathcal{I}(I) = \overline{\operatorname{span}}\{ & t_X(x_1)\ldots t_X(x_n)\pi_X(a)t_X(y_1)^* \ldots t_X(y_m)^* \\ & : a \in I, x_1 \ldots x_n \in X, y_1 \ldots y_m \in X, \text{and } n, m \geq 0\}.\end{aligned}$$

In addition, the map $I \mapsto \mathcal{I}(I)$ is certainly inclusion preserving.
To see that the map is surjective, let $\mathcal{I}$ be a gauge-invariant ideal in $\mathcal{O}_X$. If we define $I := \pi_X^{-1}(\mathcal{I})$, then it is straightforward to show that I is X-invariant and X-saturated. Now clearly $\mathcal{I}(I) \subseteq \mathcal{I}$ so there exists a quotient map $q : \mathcal{O}_X/\mathcal{I}(I) \to \mathcal{O}_X/\mathcal{I}$. Furthermore, by [6, Theorem 3.1] we have that $\mathcal{O}_X/\mathcal{I}(I)$ is canonically isomorphic to $\mathcal{O}(q^I(J_X), X/XI)$, which by Lemma 6.3 is equal to $\mathcal{O}_{X/XI} := \mathcal{O}(J_{X/XI}, X/XI)$. If we identify $\mathcal{O}_X/\mathcal{I}(I)$ with $\mathcal{O}_{X/XI}$, then we see that $q(\pi_{X/XI}(q^I(a))) = 0$ implies that $\pi_X(a) \in \mathcal{I}$ so that $a \in I$ and $q^I(a) = 0$. Thus q is faithful on $\pi_{X/XI}(A/I)$. Furthermore, since $\mathcal{I}$ is gauge invariant, the gauge action on $\mathcal{O}_X$ descends to an action on the quotient $\mathcal{O}_X/\mathcal{I}$, and q intertwines this action and the action on $\mathcal{O}_{X/XI}$. Therefore Theorem 5.1 implies that q is injective and consequently $\mathcal{I}(I) = \mathcal{I}$.
To see that the above map is injective it suffices to prove that $\pi_X(a) \in \mathcal{I}(I)$ if and only if $a \in I$. Now $\mathcal{O}_X/\mathcal{I}(I)$ is canonically isomorphic to $\mathcal{O}_{X/XI}$ as in the previous paragraph. Hence $\pi_X(a) \in \mathcal{I}(I)$ implies $\pi_{A/I}(q^I(a)) = 0$, but since $\pi_{X/XI}$ is injective by Corollary 4.4 it follows that $q^I(a) = 0$ and $a \in I$. ☐

Remark 6.5. We mention that in [17] we have constructed examples which show that the above theorem does not hold if either of the hypotheses (1) or (2) are removed. We also mention that Katsura [12] has given a description of the gauge-invariant ideals in C^*-algebras associated to general C^*-correspondences in terms certain pairs of ideals in A.

7 Concluding Remarks

In Section 4 we gave a method for "adding tails to sinks" in C^*-correspondences; that is, given a C^*-correspondence X we described how to form a C^*-correspondence Y with the property that the left action of Y is injective and $\mathcal{O}_X$ is canonically isomorphic to a full corner in $\mathcal{O}_Y$. The process of adding tails to

C^*-correspondences provides a useful tool for extending results for augmented Cuntz-Pimsner algebras (i.e. C^*-algebras associated to C^*-correspondences in which ϕ is injective) to C^*-algebras associated to general C^*-correspondences. We used this idea in Section 5 to extend the Gauge-Invariant Uniqueness Theorem for augmented Cuntz-Pimsner algebras to the general case. More generally, however, we see that many questions about C^*-algebras associated to correspondences may be reduced to the corresponding questions for augmented Cuntz-Pimsner algebras. For example, we see that for any property that is preserved by Morita equivalence (e.g. simplicity, AF-ness, pure infiniteness), one need only characterize when augmented Cuntz-Pimsner algebras will have this property, and then by adding tails one may easily deduce a theorem for C^*-algebras associated to general C^*-correspondences.

In addition, if $p \in \mathcal{M}(\mathcal{O}_Y)$ is the projection that determines $\mathcal{O}_X$ as a full corner of $\mathcal{O}_Y$ (so that $\mathcal{O}_X \cong p\mathcal{O}_Y p$), then the Rieffel correspondence from the lattice of ideals of $\mathcal{O}_Y$ to the lattice of ideals of $\mathcal{O}_X$ takes the form $I \mapsto pIp$. Furthermore, we see from Lemma 4.11 that p is gauge invariant, and consequently the Rieffel correspondence preserves gauge invariance of ideals. Thus questions about the ideal structure of $\mathcal{O}_X$, or about gauge-invariant ideals of $\mathcal{O}_X$, may be reduced to the corresponding questions for ideals in the augmented Cuntz-Pimsner algebra $\mathcal{O}_Y$.

Finally, we mention that in [17, §4] the method of adding tails has proven very useful in the analysis of topological quivers. Topological quivers, which were first introduced in [16, Example 5.4], are generalizations of graphs in which the sets of vertices and edges are replaced by topological spaces. By adding tails to topological quivers in [17] the authors are able to reduce their analyses to the case when there are no sinks, or equivalently, to the case when the left action of the associated C^*-correspondence is injective. This simplifies the proofs of many results for topological quivers and allows one to avoid a number of technicalities.

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