Enriched Functors and Stable Homotopy Theory

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Abstract. In this paper we employ enriched category theory to construct a convenient model for several stable homotopy categories. This is achieved in a three-step process by introducing the pointwise, homotopy functor and stable model category structures for enriched functors. The general setup is shown to describe equivariant stable homotopy theory, and we recover Lydakis' model category of simplicial functors as a special case. Other examples – including motivic homotopy theory – will be treated in subsequent papers.

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1 Introduction

An appropriate setting to study stable phenomena in algebraic topology is the stable homotopy theory of spectra described in [1]. More recently, much research has been focused on rebuilding the foundation of stable homotopy theory. There are now several equivalent model categories to study structured ring spectra and their modules. These frameworks are important in many aspects and make powerful tools from algebra applicable to "brave new rings". For the purpose of this paper, the relevant constructions are those of symmetric spectra [9] and simplicial functors [11].

In [8], Hovey considers the notion of spectra for general model categories. This level of generality allows one to use techniques from stable homotopy theory in traditionally unrelated subjects. Of particular interest, where further applications are expected, is algebraic geometry and Voevodsky's motivic stable homotopy category [16]. We are interested in an approach to this subject where all coherence problems which arise when one tries to make a smash product are encoded in the underlying category. This is different from the popular means of attack through symmetric spectra, where the controlling categories are much more restricted. Our point of view is analogous to Lydakis' work [11] on simplicial functors as a model for ordinary spectra. But the theory we develop here is complicated by the fact that we do not assume properties which are particular to simplicial sets. Let us state a tentative version of the main theorem in this paper. Due to their technical nature, we defer on listing all the required assumptions. The basic input is a monoidal model category $\mathcal{V}$ and a finitely presentable cofibrant object T in $\mathcal{V}$, the 1-sphere. See Sections 6 and 7 for precise statements.

Theorem 1. *There is a monoidal model category $(\mathcal{F}, \wedge, \mathbb{I})$ which satisfies the monoid axiom, and a right Quillen equivalence from $\mathcal{F}$ to the stable model category of T-spectra.*

A T-spectrum is a sequence $(E_0, E_1, \ldots)$ of objects in $\mathcal{V}$ together with structure maps $T \otimes E_n \longrightarrow E_{n+1}$. An object in $\mathcal{F}$ is a functor X from a category of finitely presentable objects in $\mathcal{V}$ to $\mathcal{V}$, which is "continuous" or *enriched* in the sense that for finitely presentable objects v and w there is a natural map

$$v \otimes X(w) \longrightarrow X(v \otimes w).$$

Using this map, it follows that any enriched functor yields a T-spectrum by evaluating at spheres $T^{\otimes n}$. We show that the induced functor from $\mathcal{F}$ to the category of T-spectra is a right Quillen equivalence. The monoidal structure is a special case of a result due to Day [4]. By construction, the sphere spectrum or unit $\mathbb{I}$ is the inclusion of the subcategory of finitely presentable objects. For $\mathcal{V}$ the category of simplicial sets, Lydakis [11] has shown that $\mathcal{F}$ models the classical stable homotopy category. Our theorem extends this result to a wide range of model categories. In the sequel [5] we construct a model for Voevodsky's motivic stable homotopy category. Motivic cohomology has a natural description as an algebra in this model. The monoid axiom implies that also categories of algebras and modules in $\mathcal{F}$ have model structures [15].

As a guide to this paper, it seems appropriate to summarize the content of each section. In Section 2 we recall categorical precursors and Day's smash product for enriched functors. This material is included to make the paper reasonably self-contained and to set notation. Next we record a general isomorphism between enriched functor categories build from spheres and symmetric spectra. Moreover, under this isomorphism the corresponding smash products are shown to agree. Section 3 recalls some frequently used notions in homotopical algebra. An expert could skip most of this part. We introduce a class of model categories dubbed *weakly finitely generated* and show that weak equivalences and fibrant objects are closed under filtered colimits. Such a model structure is cofibrantly generated, with additional finiteness conditions on the generating cofibrations and acyclic cofibrations which are satisfied in many cases of interest. This introductory part ends with a discussion of fibrant replacement functors. Quillen's small object argument is the usual device for replacing objects in a model category by fibrant objects. Some modifications are necessary in the enriched setting. If the monoid axiom holds and the model category is weakly finitely generated, we construct enriched fibrant replacements. Our constructions are primarily of a technical interest and might be omitted on a first cursory reading. However, we should remark that much of the following relies on this input.

In the remaining sections we study homotopical algebra for enriched functor categories. First we construct the *pointwise model structure* where the fibrations and weak equivalences are defined pointwise. This gives an example of a weakly finitely generated model structure provided some weak assumptions are satisfied. In many cases of interest, we prove the monoid axiom and that smashing with a cofibrant object preserves weak equivalences. The latter result requires further assumptions on $\mathcal{F}$, and is similar to the algebraic fact that tensoring with a projective module preserves short exact sequences. These two results are important for the model structures we construct later on.

One drawback with the pointwise model structure is that it has far too many homotopy types. For example, a weak equivalence $v \xrightarrow{\sim} w$ does not necessarily induce a pointwise weak equivalence $\mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ on the level of representable functors. However, for all fibrant objects u in $\mathcal{V}$ the map $\mathcal{V}(w,u) \longrightarrow \mathcal{V}(v,u)$ is a weak equivalence. We therefore enlarge the class of pointwise weak equivalences by looking at fibrant objects as input only. The result is the *homotopy functor model structure* which has the same cofibrations as the pointwise model structure. The fibrant functors are precisely the pointwise fibrant functors which preserve weak equivalences, thus any enriched functor is weakly equivalent in the homotopy functor model structure to a homotopy functor. It seems to be of considerable interest to discuss a motivic version of Goodwillie's calculus of functors. Let us remark that the homotopy functor model structure is a first step in this direction.

In Section 6, the *stable model structure* is constructed by means of a general stabilization process. Theorem 6.26 lists conditions for the stable model structure to exist. The stable fibrations are characterized by pointwise homotopy pullback squares, and stable acyclic fibrations are precisely the pointwise acyclic fibrations. To prove these results we compare with spectra [8]. The stabilization we use does not coincide with the usual stabilization for spectra, and it requires some cruel details to compare them. These can be found in Appendix A. We note that the monoid axiom holds under an additional assumption on the source of the functor category. For a particular choice of the source category, which is explained in Section 7, the evaluation functor is the right adjoint in a Quillen equivalence. It follows that the highly structured category of en-

riched functors describes the same homotopy theory as spectra in many cases of interest. In Section 8, we give a short summary of the important algebraic consequences of the previous sections.
In the last section we discuss equivariant homotopy theory for finite groups and we prove the following theorem. The general machinery gives deeper structure than stated, but we refer the reader to Section 9 for more details.

Theorem 2. *Let G be a finite group. Then there is a monoidal model category $(G\mathcal{F}, \wedge, \mathbb{S}^G)$ satisfying the monoid axiom and a right Quillen equivalence from $G\mathcal{F}$ to the category of G-spectra.*

The general framework may seem abstract, but we obtain a common footing for applications. A project in progress suggests that the approach anticipated in the present paper is relevant for the theory of motives. We hope the reader finds results herein which he or she can prove to have further applications.

2 Enriched Categories

This section contains an introduction to enriched categories, Day's work on enriched functor categories [4], and simplicial homotopies in categories enriched over simplicial sets. In the last part we show that spectra and symmetric spectra are isomorphic to enriched functor categories build from spheres.

2.1 Introduction

Entry points to the literature on enriched category theory include [2] and [12]. A *monoidal* category consists of a category $\mathcal{V}$, and

- a functor $\otimes: \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and natural associativity isomorphisms

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

  subject to the coherence law [2, 6.1],

- an object e of $\mathcal{V}$ called the *unit*, and natural unit isomorphisms

$$l_A: e \otimes A \longrightarrow A \quad \text{and} \quad r_A: A \otimes e \longrightarrow A$$

  such that [2, 6.2] holds.

The functor $\otimes$ is the *tensor* or *monoidal* product of $\mathcal{V}$. A monoidal category is *symmetric monoidal* if there is a natural isomorphism $\sigma_{A,B}: A \otimes B \longrightarrow B \otimes A$ subject to the coherence laws [2, 6.3, 6.4, 6.5]. A symmetric monoidal category $(\mathcal{V}, \otimes, e)$ is *closed* if there exists a right adjoint $\mathrm{Hom}_{\mathcal{V}}(A, -): \mathcal{V} \longrightarrow \mathcal{V}$ to the endofunctor $- \otimes A$ for every object A of $\mathcal{V}$. The categories of sets **Set** and pointed sets $\mathbf{Set}_*$ are both closed symmetric monoidal categories. Let Δ denote the simplicial category. Its objects are the finite ordered sets $[n] =$

$\{0 < 1 < \cdots < n\}$ for $n \geq 0$, and morphisms are order-preserving maps. Consider $\mathbf{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, the category of simplicial sets. Its monoidal product is the categorical product, formed degree-wise in $\mathbf{Set}$. If $K, L \in \mathbf{sSet}$, the simplicial set of maps $\mathrm{Hom}_{\mathbf{sSet}}(K, L)$ has n-simplices the set of maps from $K \times \Delta^n$ to L. Here Δ^n is the simplicial set represented by $[n]$. The unit of the product is the terminal object Δ^0.

Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category. Then a *$\mathcal{V}$-category $\mathcal{C}$*, or a category *enriched over* $\mathcal{V}$, consists of a class $\mathrm{Ob}\,\mathcal{C}$ of objects and

- for any pair (a, b) of objects in $\mathcal{C}$, an object $\mathcal{V}_{\mathcal{C}}(a, b)$ of $\mathcal{V}$ called the *$\mathcal{V}$-object of maps in $\mathcal{C}$*,
- a composition $\mathcal{V}_{\mathcal{C}}(b, c) \otimes \mathcal{V}_{\mathcal{C}}(a, b) \longrightarrow \mathcal{V}_{\mathcal{C}}(a, c)$, an identity or unit map $e \longrightarrow \mathcal{V}_{\mathcal{C}}(a, a)$ subject to the associativity and unit coherence laws listed in [2, 6.9 and 6.10].

Categories in the usual sense are the $\mathbf{Set}$-categories. If $\mathcal{C}$ is a category, let $\mathbf{Set}_{\mathcal{C}}(a, b)$ denote the set of maps in $\mathcal{C}$ from a to b. A closed symmetric monoidal category $\mathcal{V}$ is a $\mathcal{V}$-category due to its internal Hom objects [2, 6.2.6]. Let $\mathcal{V}(A, B)$ denote the $\mathcal{V}$-object $\mathrm{Hom}_{\mathcal{V}}(A, B)$ of maps in $\mathcal{V}$. Any $\mathcal{V}$-category $\mathcal{C}$ defines a $\mathbf{Set}$-category $\mathcal{U}\mathcal{C}$. Its class of objects is $\mathrm{Ob}\,\mathcal{C}$, the morphism sets are $\mathbf{Set}_{\mathcal{U}\mathcal{C}}(a, b) = \mathbf{Set}_{\mathcal{V}}(e, \mathcal{V}_{\mathcal{C}}(a, b))$. For example, the $\mathbf{Set}$-category obtained from a $\mathbf{sSet}$-category $\mathcal{C}$ has morphism sets $\mathbf{Set}_{\mathbf{sSet}}(\Delta^0, \mathbf{sSet}_{\mathcal{C}}(a, b)) = \mathbf{sSet}_{\mathcal{C}}(a, b)_0$ the zero-simplices of the simplicial sets of maps.

A *$\mathcal{V}$-functor* F from $\mathcal{C}$ to $\mathcal{D}$ is an assignment from $\mathrm{Ob}\,\mathcal{C}$ to $\mathrm{Ob}\,\mathcal{D}$ together with morphisms $\mathrm{hom}^F_{a,b}\colon \mathcal{V}_{\mathcal{C}}(a, b) \longrightarrow \mathcal{V}_{\mathcal{D}}(F(a), F(b))$ in $\mathcal{V}$ which preserve composition and identities. A small $\mathbf{sSet}$-category defines a simplicial object in the category $\mathbf{Cat}$ of small categories. With this description, a $\mathbf{sSet}$-functor is a natural transformation of functors from Δ^{op} to $\mathbf{Cat}$ [11, 3.2].

If F and G are $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{D}$, a *$\mathcal{V}$-natural transformation* $t\colon F \longrightarrow G$ consists of the following data: There is a morphism $t(a)\colon F(a) \longrightarrow G(a)$ in $\mathcal{U}\mathcal{D}$ for every $a \in \mathrm{Ob}\,\mathcal{C}$, and all the diagrams of the following form commute.

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{C}}(a,b) \arrow[r, "\mathrm{hom}^F_{a,b}"] \arrow[d, "\mathrm{hom}^G_{a,b}"'] & \mathcal{V}_{\mathcal{D}}(F(a),F(b)) \arrow[d, "\mathcal{V}_{\mathcal{D}}(F(a),t(b))"] \\
\mathcal{V}_{\mathcal{D}}(G(a),G(b)) \arrow[r, "\mathcal{V}_{\mathcal{D}}(t(a),G(b))"] & \mathcal{V}_{\mathcal{D}}(F(a),G(b))
\end{tikzcd}$$

The $\mathcal{V}$-natural isomorphisms and $\mathcal{V}$-adjoint pairs of $\mathcal{V}$-functors are defined as for $\mathcal{V} = \mathbf{Set}$. The adjoint pair of endofunctors $(-\otimes A, \mathcal{V}(A, -))$ on $\mathcal{V}$ explicate a $\mathcal{V}$-adjoint pair. Denote the unit of the adjunction by $\eta_A\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow \mathcal{V}(A, -\otimes A)$, and the counit by $\epsilon_A\colon \mathcal{V}(A, -) \otimes A \longrightarrow \mathrm{Id}_{\mathcal{V}}$. Details concerning $\mathrm{hom}^{-\otimes A}$ and $\mathrm{hom}^{\mathcal{V}(A,-)}$ can be found in Appendix A.

Note that any $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ gives a functor $\mathcal{U}F\colon \mathcal{U}\mathcal{C} \longrightarrow \mathcal{U}\mathcal{D}$ with the same effect on objects as F, and similarly for $\mathcal{V}$-natural transformations. That is, one can consider $\mathcal{U}$ as a 2-functor from the 2-category of small $\mathcal{V}$-categories,

$\mathcal{V}$-functors and $\mathcal{V}$-natural transformations to the 2-category of small categories, functors and natural transformations. A $\mathcal{V}$-category $\mathcal{C}$ is called small provided $\mathcal{UC}$ is small. In fact, $\mathcal{U}$ is the base change along the lax symmetric monoidal functor $\mathbf{Set}_{\mathcal{V}}(e,-)\colon \mathcal{V} \longrightarrow \mathbf{Set}$, see [2, 6.4]. If no confusion can arise, we will omit $\mathcal{U}$ from the notation.

The *monoidal product* $\mathcal{C} \otimes \mathcal{D}$ of two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathcal{V}$-category where $\mathrm{Ob}\,(\mathcal{C}\otimes\mathcal{D}) := \mathrm{Ob}\,\mathcal{C}\times\mathrm{Ob}\,\mathcal{D}$ and $\mathcal{V}_{\mathcal{C}\otimes\mathcal{D}}((a,x),(b,y)) := \mathcal{V}_{\mathcal{C}}(a,b)\otimes\mathcal{V}_{\mathcal{D}}(x,y)$. Note that the monoidal product in $\mathcal{V}$ induces a $\mathcal{V}$-functor $\mathrm{mon}\colon \mathcal{V}\otimes\mathcal{V} \longrightarrow \mathcal{V}$. A $\mathcal{V}$-category $\mathcal{C}$ is a *right* $\mathcal{V}$*-module* if there is a $\mathcal{V}$-functor $\mathrm{act}\colon \mathcal{C}\otimes\mathcal{V} \longrightarrow \mathcal{C}$, denoted $(c,A) \longmapsto c\oslash A$ and a $\mathcal{V}$-natural unit isomorphism $r_c\colon \mathrm{act}(c,e) \longrightarrow c$ subject to the following conditions.

- There are natural coherent associativity isomorphisms

$$\mathrm{act}(c, A\otimes B) \longrightarrow \mathrm{act}(\mathrm{act}(c,A),B).$$

- The isomorphisms $\mathrm{act}(c, e\otimes A) \rightrightarrows \mathrm{act}(c,A)$ coincide.

A right $\mathcal{V}$-module $(\mathcal{C},\mathrm{act},r)$ is *closed* if there is a $\mathcal{V}$-functor

$$\mathrm{coact}\colon \mathcal{V}^{\mathrm{op}}\otimes\mathcal{C} \longrightarrow \mathcal{C}$$

such that for all $A\in\mathrm{Ob}\,\mathcal{V}$ and $c\in\mathrm{Ob}\,\mathcal{C}$, the $\mathcal{V}$-functor $\mathrm{act}(-,A)\colon\mathcal{C}\longrightarrow\mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathrm{coact}(A,-)$ and $\mathrm{act}(c,-)\colon\mathcal{V}\longrightarrow\mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathcal{V}_{\mathcal{C}}(c,-)$.

A *monoidal* $\mathcal{V}$-category consists of a $\mathcal{V}$-category $\mathcal{C}$ equipped with a $\mathcal{V}$-functor $\diamond\colon\mathcal{C}\otimes\mathcal{C}\longrightarrow\mathcal{C}$, a unit $u\in\mathrm{Ob}\,\mathcal{C}$, a $\mathcal{V}$-natural associativity isomorphism and two $\mathcal{V}$-natural unit isomorphisms satisfying the conditions mentioned for $\mathcal{V}=\mathbf{Set}$. Symmetric monoidal and closed symmetric monoidal $\mathcal{V}$-categories are defined similarly.

2.2 Categories of enriched functors

If $\mathcal{C}$ is a small $\mathcal{V}$-category, $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ and their $\mathcal{V}$-natural transformations form the category $[\mathcal{C},\mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. If $\mathcal{V}$ is complete, then $[\mathcal{C},\mathcal{V}]$ is also a $\mathcal{V}$-category. Denote this $\mathcal{V}$-category by $\mathcal{F}(\mathcal{C})$, or $\mathcal{F}$ if no confusion can arise. The morphism $\mathcal{V}$-object $\mathcal{V}_{\mathcal{F}}(X,Y)$ is the end

$$\int_{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(X(c),Y(c)).$$

See [2, 6.3.1] for details. Note that $\mathcal{UF}$ is $[\mathcal{C},\mathcal{V}]$. One can compare $\mathcal{F}$ with $\mathcal{C}$ and $\mathcal{V}$ as follows: Given $c\in\mathrm{Ob}\,\mathcal{C}$, $X\longmapsto X(c)$ defines the $\mathcal{V}$-functor $\mathrm{Ev}_c\colon\mathcal{F}\longrightarrow\mathcal{V}$ called "evaluation at c". The assignment $c\longmapsto\mathcal{V}_{\mathcal{C}}(c,-)$ from $\mathcal{C}$ to $\mathcal{F}$ is again a $\mathcal{V}$-functor $\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{F}$, called the $\mathcal{V}$*-Yoneda embedding* [2, 6.3.6]. $\mathcal{V}_{\mathcal{C}}(c,-)$ is a representable functor, represented by c.

Lemma 2.1 (Enriched Yoneda lemma). *Let $\mathcal{V}$ be a complete closed symmetric monoidal category and $\mathcal{C}$ a small $\mathcal{V}$-category. For every $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ and every $c \in \mathrm{Ob}\,\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism $X(c) \cong \mathcal{V}_{\mathcal{F}}(\mathcal{V}_{\mathcal{C}}(c,-),X)$.*

The isomorphism in 2.1 is called the *Yoneda* isomorphism [2, 6.3.5]. It follows from 2.1 that every $\mathcal{V}$-functor can be expressed as a colimit of representable functors [2, 6.6.13, 6.6.17]:

Lemma 2.2. *If $\mathcal{V}$ is a bicomplete closed symmetric monoidal category and $\mathcal{C}$ is a small $\mathcal{V}$-category, then $[\mathcal{C},\mathcal{V}]$ is bicomplete. (Co)limits are formed pointwise.*

Corollary 2.3. *Assume $\mathcal{V}$ is bicomplete, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then any $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ is $\mathcal{V}$-naturally isomorphic to the coend*

$$\int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}_{\mathcal{C}}(c,-)\otimes X(c).$$

See [2, 6.6.18] for a proof of 2.3.

Proposition 2.4. *Let $\mathcal{V}$ be a closed symmetric monoidal category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then $\mathcal{F}$ is a closed $\mathcal{V}$-module.*

Proof. There is an obvious "pointwise" closed $\mathcal{V}$-module structure. The $\mathcal{V}$-functor $\mathcal{F}\otimes\mathcal{V} \longrightarrow \mathcal{F}$ defined by $(X,A) \longmapsto (-\otimes A)\circ X$ gives the action of $\mathcal{V}$ on $\mathcal{F}$. Next, the assignment $(A,X) \longmapsto \mathcal{V}(A,-)\circ X$ defines the coaction. There are $\mathcal{V}$-natural isomorphisms $\mathcal{V}_{\mathcal{F}}((-\otimes A)\circ X, Y) \cong \mathcal{V}_{\mathcal{F}}(X,\mathcal{V}(A,-)\circ Y) \cong \mathcal{V}(A,\mathcal{V}_{\mathcal{F}}(X,Y))$ induced from the natural closed $\mathcal{V}$-module structure on $\mathcal{V}$. From this, a routine check finishes the proof. $\square$

Recall the notion of *left Kan extensions*:

Proposition 2.5. *Fix a bicomplete closed symmetric monoidal category $\mathcal{V}$, and a $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ of small $\mathcal{V}$-categories. For any $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$, there exists a $\mathcal{V}$-functor $F_*(X)\colon \mathcal{D} \longrightarrow \mathcal{V}$ and a $\mathcal{V}$-natural isomorphism*

$$\mathcal{V}_{\mathcal{F}(\mathcal{D})}(F_*X,Y) \cong \mathcal{V}_{\mathcal{F}(\mathcal{C})}(X, Y\circ F).$$

In other words, there exists a $\mathcal{V}$-adjoint pair of $\mathcal{V}$-functors

$$F_*\colon \mathcal{F}(\mathcal{C}) \rightleftarrows \mathcal{F}(\mathcal{D})\colon F^*$$

where F^* denotes pre-composition with F.

See [2, 6.7.7] for a proof of 2.5. The $\mathcal{V}$-functor F_*X is the left Kan extension of X along F. An explicit expression is given by the coend

$$F_*X = \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}_{\mathcal{D}}(F(c),-)\otimes X(c).$$

2.3 Smash product of enriched functors

Let $(\mathcal{C}, \diamond, u)$ be a small symmetric monoidal $\mathcal{V}$-category where $\mathcal{V}$ is bicomplete. In [4], B. Day constructed a closed symmetric monoidal product $\wedge$ on the category $[\mathcal{C}, \mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. For $X, Y \in \mathrm{Ob}\,[\mathcal{C}, \mathcal{V}]$, there is the $\mathcal{V}$-functor

$$X \overline{\wedge} Y : \mathcal{C} \otimes \mathcal{C} \xrightarrow{X \otimes Y} \mathcal{V} \otimes \mathcal{V} \xrightarrow{\text{mon}} \mathcal{V}.$$

The smash product $X \wedge Y \in \mathrm{Ob}\,[\mathcal{C}, \mathcal{V}]$ is the left Kan extension

$$\diamond_*(X \overline{\wedge} Y) = \int^{\mathrm{Ob}\,(\mathcal{C}\otimes\mathcal{C})} \mathcal{V}_{\mathcal{C}}(c \diamond d, -) \otimes (X(c) \otimes Y(d)) : \mathcal{C} \longrightarrow \mathcal{V}.$$

The next result is a special case of [4, 3.3], cf. [4, 3.6, 4.1].

Theorem 2.6 (Day). *Let $(\mathcal{V}, \otimes, e)$ be a bicomplete closed symmetric monoidal category and $(\mathcal{C}, \diamond, u)$ a small symmetric monoidal $\mathcal{V}$-category. Then the category $([\mathcal{C}, \mathcal{V}], \wedge, \mathcal{V}_{\mathcal{C}}(u, -))$ is closed symmetric monoidal.*

The $\mathcal{V}$-category $\mathcal{F}$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ is also a closed symmetric monoidal $\mathcal{V}$-category. The internal Hom functor, right adjoint to $- \wedge X$, is given by

$$\mathcal{F}(X, Y)(c) = \mathcal{V}_{\mathcal{F}}(X, Y(c \diamond -)) = \int_{d \in \mathrm{Ob}\,\mathcal{C}} \mathcal{V}(X(d), Y(c \diamond d)).$$

Concerning smash products of representable functors, one has the following result.

Lemma 2.7. *The smash product of representable functors is again representable. There is a natural isomorphism $\mathcal{V}_{\mathcal{C}}(c, -) \wedge \mathcal{V}_{\mathcal{C}}(d, -) \cong \mathcal{V}_{\mathcal{C}}(c \diamond d, -)$.*

In the following, a sub-$\mathcal{V}$-category means a sub-$\mathcal{V}$-category of $\mathcal{V}$. We use 2.7 to define assembly maps if $\mathcal{C}$ is a full sub-$\mathcal{V}$-category containing the unit and closed under the monoidal product. In this case, the inclusion $\mathbb{I} : \mathcal{C} \hookrightarrow \mathcal{V}$ can be chosen as the unit of $[\mathcal{C}, \mathcal{V}]$. The *composition* $X \circ Y$ of two $\mathcal{V}$-functors $X, Y : \mathcal{C} \longrightarrow \mathcal{V}$ is given by $\mathbb{I}_* X \circ Y$. Up to coherent natural isomorphisms, the composition is associative with unit $\mathbb{I}$.

Corollary 2.8. *Given functors X and Y in $[\mathcal{C}, \mathcal{V}]$, there exists a natural assembly map $X \wedge Y \longrightarrow X \circ Y = \mathbb{I}_* X \circ Y$ which is an isomorphism if Y is representable.*

Proof. One can define the assembly map objectwise via the composition

$$\mathbb{I}_* X(c) \otimes Y(d) \xrightarrow{\mathrm{sw}^{\mathbb{I}_* X}_{Y(d)}(c)} \mathbb{I}_* X(Y(d) \otimes c) \xrightarrow{\mathbb{I}_* X(\mathrm{sw}^Y_c(d))} \mathbb{I}_* X(Y(c \otimes d))$$

where $\mathrm{sw}^Z_c : Z \otimes c \longrightarrow Z(c \otimes -)$ is 'the' natural map described in Appendix A. Here is another description via representable functors. Suppose $X = \mathcal{V}(c, -)$ and $Y = \mathcal{V}(d, -)$, for $c, d \in \mathrm{Ob}\,\mathcal{C}$. By 2.7, $X \wedge Y$ is naturally isomorphic to

$\mathcal{V}(c \otimes d, -)$, i.e. to $\mathcal{V}(c, -) \circ \mathcal{V}(d, -)$. If X is arbitrary, it follows from 2.3 that $X \wedge \mathcal{V}(d, -)$ is naturally isomorphic to $\mathbb{I}_* X \circ \mathcal{V}(d, -)$. If also Y is arbitrary, apply 2.3 and consider the natural map

$$\int^{\mathrm{Ob}\,\mathcal{C}} (\mathbb{I}_* X \circ \mathcal{V}(c, -)) \otimes Y(c) \longrightarrow \mathbb{I}_* X \circ \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, -) \otimes Y(c).$$

□

2.4 Categories enriched over simplicial sets

A functor $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ of monoidal categories $(\mathcal{V}, \otimes, e)$ and $(\mathcal{W}, \otimes', e')$ is *lax monoidal* if there is a natural transformation $t_{A,B}\colon F(A) \otimes' F(B) \longrightarrow F(A \otimes B)$ and a morphism $e' \longrightarrow F(e)$, such that the diagrams [2, 6.27, 6.28] commute. The word "lax" is replaced by "*strict*" if $t_{A,B}$ is a natural isomorphism and $e' \longrightarrow F(e)$ is an isomorphism. F is *lax symmetric monoidal* if $t_{B,A} \circ \sigma_{FA,FB} = F(\sigma_{A,B}) \circ t_{A,B}$. In this case, every $\mathcal{W}$-category is a $\mathcal{V}$-category by [2, 6.4.3]. We used this fact for the forgetful 2-functor $\mathcal{U}$ induced by $\mathbf{Set}_{\mathcal{V}}(e, -)$. The assembly map makes $\mathrm{Id}_{[\mathcal{C},\mathcal{V}]}$ into a lax monoidal functor from the monoidal category $([\mathcal{C}, \mathcal{V}], \circ, \mathbb{I})$ to the closed symmetric monoidal category $([\mathcal{C}, \mathcal{V}], \wedge, \mathbb{I})$. Suppose that $F\colon \mathbf{sSet} \longrightarrow \mathcal{V}$ is a lax symmetric monoidal functor. One can then lift the notion of *simplicial homotopy equivalence* from $\mathbf{sSet}$-categories to $\mathcal{V}$-categories.

Definition 2.9. Let $\mathcal{C}$ be a $\mathbf{sSet}$-category and $f, f'\colon \Delta^0 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c, d)$ maps in $\mathcal{C}$. Then $H\colon \Delta^1 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c, d)$ is a *simplicial homotopy* from f to f' if the following diagram commutes, where i_0 and i_1 are the canonical inclusions.

$$\begin{tikzcd} \Delta^0 \arrow[r, "i_0"] \arrow[dr, "f"'] & \Delta^1 \arrow[d, "H"] & \Delta^0 \arrow[l, "i_1"'] \arrow[dl, "f'"] \\ & \mathbf{sSet}_{\mathcal{C}}(c, d) & \end{tikzcd}$$

The map f is called a *simplicial homotopy equivalence* if there exists a map $g\colon \Delta^0 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(d, c)$, and simplicial homotopies from $g \circ f$ to id_c and from $f \circ g$ to id_d. Let the symbol $\simeq$ denote simplicial homotopy equivalences.

Simplicial homotopy equivalence is in general not an equivalence relation. If $\mathcal{C}$ is a closed $\mathbf{sSet}$-module with action $(c, K) \longmapsto c \oslash K$ and coaction $(c, K) \longmapsto c^K$, a simplicial homotopy may also be described by maps $c \oslash \Delta^1 \longrightarrow d$ or $c \longrightarrow d^{\Delta^1}$. If $\mathcal{C}$ has pushouts, the *simplicial mapping cylinder* factors any map f as follows: Let C_f denote the pushout of the diagram

$$c \oslash \Delta^1 \xleftarrow{c \oslash i_1} c \oslash \Delta^0 \xleftarrow{\cong} c \xrightarrow{f} d.$$

The maps $c \oslash \Delta^1 \xrightarrow{c \oslash s} c \oslash \Delta^0 \xrightarrow{\cong} c \xrightarrow{f} d$ and $\mathrm{id}_d : d \longrightarrow d$ induce the simplicial homotopy equivalence $p_f\colon C_f \longrightarrow d$. Its homotopy inverse is the canonical map. Denote the composition $c \longrightarrow c \oslash \Delta^0 \xrightarrow{c \oslash i_0} c \oslash \Delta^1 \longrightarrow C_f$ by $i_f\colon c \longrightarrow C_f$. Note the factorization $p_f \circ i_f = f$. The relevance of this will become clear in the context of simplicial model categories. In this case, i_f is a cofibration provided c is cofibrant. It is easy to prove the next result.

Lemma 2.10. *A* **sSet***-functor preserves simplicial homotopies, and therefore simplicial homotopy equivalences.*

Corollary 2.11. *Assume $F\colon$* **sSet** *$\longrightarrow \mathcal{V}$ is a lax monoidal functor. Then any $\mathcal{V}$-functor preserves simplicial homotopy equivalences.*

2.5 Spectra as enriched functors

Let $(\mathcal{V}, \otimes, e)$ denote a bicomplete closed symmetric monoidal category with initial object $\emptyset$. For $T \in \operatorname{Ob}\mathcal{V}$, one can consider T-spectra in $\mathcal{V}$, see [8, 1.1]. A *T-spectrum* E is a sequence $E_0, E_1, \cdots$ of objects in $\mathcal{V}$, together with structure maps $e_n\colon E_n \otimes T \longrightarrow E_{n+1}$ for all n. If E and F are T-spectra, a *map of T-spectra* $g\colon E \longrightarrow F$ is a collection of maps $g_n\colon E_n \longrightarrow F_n$ such that

$$\begin{tikzcd} E_n \otimes T \arrow[r, "e_n"] \arrow[d, "g_n \otimes T"'] & E_{n+1} \arrow[d, "g_{n+1}"] \\ F_n \otimes T \arrow[r, "f_n"] & F_{n+1} \end{tikzcd}$$

commutes for all n. Thus T-spectra in $\mathcal{V}$ form a category $\operatorname{Sp}(\mathcal{V}, T)$, see [8, 1.3]. We claim $\operatorname{Sp}(\mathcal{V}, T)$ can be viewed as an enriched functor category, cf. [11, 4.3]. Its domain category is the $\mathcal{V}$-category $T\mathrm{Sph}$. The objects in $T\mathrm{Sph}$ are the objects T^n for $n \geq 0$, where $T^0 = e$ and $T^n := T \otimes T^{n-1}$ for $n > 0$. The $\mathcal{V}$-objects of morphisms are $\mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) := T^{n-m}$ for $n \geq m \geq 0$ and $\mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) := \emptyset$ for $n < m$. Note that there are canonical unit maps $\mathrm{id}_{T^0}\colon T^0 \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^n, T^n)$ for all $n \geq 0$. It remains to describe the composition. For $k, l, m \geq 0$, the map

$$\mathcal{V}_{T\mathrm{Sph}}(T^{l+m}, T^{k+l+m}) \otimes \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{l+m}) \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{k+l+m})$$

is the associativity isomorphism $\alpha_{k,l}\colon T^k \otimes T^l \longrightarrow T^{k+l}$. In all other cases, the composition is uniquely determined. It follows that $T\mathrm{Sph}$ is a $\mathcal{V}$-category, using the associativity and unit coherence laws in $\mathcal{V}$.

To elaborate on this definition, let us describe a $\mathcal{V}$-functor $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$. Define π to be the identity on objects. Concerning morphisms, it suffices to give $\hom^{\pi}_{T^m, T^{k+m}}\colon \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{k+m}) \longrightarrow \mathcal{V}(T^m, T^{k+m})$ for $k, m \geq 0$, which we choose as

$$i_{k,m}\colon T^k \xrightarrow{\eta_{T^m} T^k} \mathcal{V}(T^m, T^k \otimes T^m) \xrightarrow{\mathcal{V}(T^m, \alpha_{k,m})} \mathcal{V}(T^m, T^{k+m}).$$

Associativity coherence and a calculation with adjoints imply that the composition $T^k \otimes T^l \xrightarrow{\alpha_{k,l}} T^{k+l} \xrightarrow{i_{k+l,m}} \mathcal{V}(T^m, T^{k+l+m})$ is the same as the composition

$T^k \otimes T^l \xrightarrow{i_{k,l+m}\otimes i_{l,m}} \mathcal{V}(T^{l+m}, T^{k+l+m}) \otimes \mathcal{V}(T^m, T^{l+m}) \xrightarrow{\text{comp}} \mathcal{V}(T^m, T^{k+l+m})$. Hence π preserves composition, and it clearly preserves identites. In our cases of interest, the maps $\eta_{T^m} T^k$ are monomorphisms so that $T\mathrm{Sph}$ can be regarded as a sub-$\mathcal{V}$-category.

Proposition 2.12. *The categories* $\mathrm{Sp}(\mathcal{V}, T)$ *and* $[T\mathrm{Sph}, \mathcal{V}]$ *are isomorphic.*

Proof. Let $X \colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$-functor. Define $\Psi(X)$ to be the spectrum with $\Psi(X)_n := X(T^n)$ and structure maps $\Psi(X)_n \otimes T \longrightarrow \Psi(X)_{n+1}$ adjoint to $T = \mathcal{V}_{T\mathrm{Sph}}(T^n, T^{n+1}) \longrightarrow \mathcal{V}(X(T^n), X(T^{n+1}))$. If $f \colon X \longrightarrow Y$ is a $\mathcal{V}$-natural transformation, let $\Psi(f)_n := f(T^n) \colon X(T^n) \longrightarrow Y(T^n)$. The diagram

$$\begin{tikzcd} X(T^n) \otimes T \arrow[r] \arrow[d, "f(T^n)\otimes T"'] & X(T^{n+1}) \arrow[d, "f(T^{n+1})"] \\ Y(T^n) \otimes T \arrow[r] & Y(T^{n+1}) \end{tikzcd}$$

commutes by $\mathcal{V}$-naturality, and Ψ respects identities and composition.
Define $\Phi \colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ by $\Phi(E)(T^n) := E_n$. If $n = m + k$, $m \geq 0$ and $k \geq 1$, $\hom^{\Phi(E)}_{T^m,T^n} \colon T^k = \mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) \longrightarrow \mathcal{V}(E_m, E_n)$ is the adjoint of the composition $E_m \otimes T^k \xrightarrow{\alpha^{-1}_{E_m,T,T^{k-1}}} (E_m \otimes T) \otimes T^{k-1} \xrightarrow{e_m \otimes T^{k-1}} E_{m+1} \otimes T^{k-1} \longrightarrow \cdots \longrightarrow E_n$. The maps $\hom^{\Phi(E)}_{T^n,T^n}$ are determined by the property that $\Phi(E)$ has to preserve identities. To prove that $\Phi(E)$ is a $\mathcal{V}$-functor, it remains to note that

$$\begin{tikzcd} T^k \otimes T^l \arrow[rr, "\alpha_{k,l}"] \arrow[d, "\hom^{\Phi(E)}_{T^{l+m},T^{k+l+m}} \otimes \hom^{\Phi(E)}_{T^m,T^{l+m}}"'] & & T^{k+l} \arrow[d, "\hom^{\Phi(E)}_{T^m,T^{k+l+m}}"] \\ \mathcal{V}(E_{l+m}, E_{k+l+m}) \otimes \mathcal{V}(E_m, E_{l+m}) \arrow[rr, "\text{comp}"] & & \mathcal{V}(E_m, E_{k+l+m}) \end{tikzcd}$$

commutes for $m \geq 0$ and $k, l \geq 1$. This uses $\epsilon_{E_m}(-\otimes E_m) \circ \eta_{E_m} \otimes E_m = \mathrm{id}_{-\otimes E_m}$, associativity coherence, and associativity of composition in $\mathcal{V}$. If g is a map, let $\Phi(g)(T^n)$ be g_n. Then $\Phi(g)$ is $\mathcal{V}$-natural, and functoriality of Φ follows.
Note that $\Phi(\Psi(X)) = X$ on objects, i.e. for all powers of T. The structure maps of X and $\Phi(\Psi(X))$ coincide, since the adjointness isomorphism that defines the structure maps of $\Phi(\Psi(X))$ is inverse to the adjointness isomorphism that defines the structure maps of $\Psi(X)$. The equality $\Phi(\Psi(f)) = f$ is obvious, hence $\Phi \circ \Psi$ is the identity functor. Likewise, one finds $\Psi \circ \Phi = \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$. □

Remark 2.13. In the following, we will identify $\mathrm{Sp}(\mathcal{V}, T)$ with $[T\mathrm{Sph}, \mathcal{V}]$ via 2.12. Then $\mathrm{Sp}(\mathcal{V}, T)$ is a closed $\mathcal{V}$-module by 2.4. A consequence of A.1 is that the functor "suspension with T" obtained from the action of $\mathcal{V}$ on $\mathrm{Sp}(\mathcal{V}, T)$ can also be defined as the prolongation [8, 1.5] of the functor $-\otimes T \colon \mathcal{V} \longrightarrow \mathcal{V}$ using the natural transformation $t \colon (-\otimes T) \circ (-\otimes T) \longrightarrow (-\otimes T) \circ (-\otimes T)$, which twists the factors. In detail, the latter is defined by the coherence isomorphism

$(A\otimes T)\otimes T \xrightarrow{\alpha_{A,T,T}} A\otimes(T\otimes T) \xrightarrow{A\otimes\sigma_{T,T}} A\otimes(T\otimes T) \xrightarrow{\alpha^{-1}_{A,T,T}} (A\otimes T)\otimes T$. Another functor "suspension with T" – which we denote by Σ_T – is obtained as the prolongation of $-\otimes T$ using the identity natural transformation $\mathrm{id}_{(-\otimes T)\circ(-\otimes T)}$. If $X\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor, then the nth structure map in the associated spectrum of $X\otimes T$ is the left hand side composition in the following diagram.

$$\begin{tikzcd}
 & (X_n\otimes T)\otimes T \arrow[dl, "\sigma_{X_n\otimes T,T}"'] \arrow[dr, "\sigma_{X_n,T\otimes T}"] & \\
T\otimes(X_n\otimes T) \arrow[rr, "\alpha^{-1}_{T,X_n,T}"] \arrow[d, "\mathrm{hom}^X_{n,n+1}\otimes(X_n\otimes T)"'] & & (T\otimes X_n)\otimes T \arrow[d, "(\mathrm{hom}^X_{n,n+1}\otimes X_n)\otimes T"] \\
\mathcal{V}(X_n,X_{n+1})\otimes(X_n\otimes T) \arrow[rr, "\alpha^{-1}_{\mathcal{V}(X_n,X_{n+1}),X_n,T}"] \arrow[d, "\mathrm{hom}^{-\otimes T}_{X_n,X_{n+1}}"'] & & (\mathcal{V}(X_n,X_{n+1})\otimes X_n)\otimes T \arrow[d, "(\epsilon_{X_n}X_{n+1})\otimes T"] \\
\mathcal{V}(X_n\otimes T,X_{n+1}\otimes T)\otimes(X_n\otimes T) \arrow[rr, "\epsilon_{X_n\otimes T}(X_{n+1}\otimes T)"] & & X_{n+1}\otimes T
\end{tikzcd}$$

The right hand side composition is the structure map of the spectrum $\Sigma_T X$. The lower square commutes by A.1, the middle square commutes by naturality, but the triangle does not commute in general. This will cause some complications in our comparison of stable model categories, cp. Section 6.

The monoidal product $\otimes$ defines a $\mathcal{V}$-functor $\mathrm{mon}\colon\mathcal{V}\otimes\mathcal{V}\longrightarrow\mathcal{V}$ where

$$\mathrm{hom}^{\mathrm{mon}}_{(A_1,A_2)(B_1,B_2)}\colon \mathcal{V}(A_1,B_1)\otimes\mathcal{V}(A_2,B_2) \longrightarrow \mathcal{V}(A_1\otimes A_2, B_1\otimes B_2)$$

is the adjoint of the composition

$$\begin{tikzcd}
\mathcal{V}(A_1,B_1)\otimes\mathcal{V}(A_2,B_2)\otimes A_1\otimes A_2 \arrow[d, "\mathcal{V}(A_1,B_1)\otimes"', "\sigma_{\mathcal{V}(A_2,B_2),A_1}\otimes A_2"] \\
\mathcal{V}(A_1,B_1)\otimes A_1\otimes\mathcal{V}(A_2,B_2)\otimes A_2 \arrow[d, "\epsilon_{A_1}B_1\otimes"', "\epsilon_{A_2}B_2"] \\
B_1\otimes B_2.
\end{tikzcd}$$

Now suppose that the symmetric monoidal product in $\mathcal{V}$ induces a $\mathcal{V}$-functor $\mathrm{mon}\colon T\mathrm{Sph}\otimes T\mathrm{Sph} \longrightarrow T\mathrm{Sph}$. On objects we have that $\mathrm{mon}(T^k,T^l) = T^k\otimes T^l = T^{k+l}$, while for $\mathcal{V}$-objects of morphisms there is a map f from $\mathcal{V}_{T\mathrm{Sph}}(T^m,T^k\otimes T^m)\otimes\mathcal{V}_{T\mathrm{Sph}}(T^n,T^l\otimes T^n)$ to $\mathcal{V}_{T\mathrm{Sph}}(T^m\otimes T^n,T^k\otimes T^m\otimes T^l\otimes T^n)$ rendering the following diagram commutative.

$$\begin{tikzcd}
T^k\otimes T^l \arrow[r, "f"] \arrow[d, "\eta_{T^m}T^k\otimes\eta_{T^n}T^l"'] & T^k\otimes T^l \arrow[d, "\eta_{T^m\otimes T^n}(T^k\otimes T^l)"] \\
\mathcal{V}(T^m,T^k\otimes T^m)\otimes\mathcal{V}(T^n,T^l\otimes T^n) \arrow[r, "\mathrm{hom}^{\mathrm{mon}}"] & \mathcal{V}(T^m\otimes T^n,T^k\otimes T^m\otimes T^l\otimes T^n)
\end{tikzcd}$$

Reverting to adjoints, a straightforward calculation shows that the maps

$$T^{k+l+m+n} = T^k \otimes T^l \otimes T^m \otimes T^n \underset{T^k \otimes \sigma_{T^l,T^m} \otimes T^n}{\overset{f \otimes T^m \otimes T^n}{\rightrightarrows}} T^k \otimes T^m \otimes T^l \otimes T^n = T^{k+l+n+m}$$

must coincide. This is only possible if $\sigma_{T,T} = \mathrm{id}_{T^2}$ and $f = \mathrm{id}$. In other words, the product $\otimes$ does not necessarily restrict to a monoidal product of $T\mathrm{Sph}$ via $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$. In the next section, we show – following [11, 5.15] – how to remedy this by enlarging $T\mathrm{Sph}$.

2.6 Symmetric spectra as enriched functors

For ease of notation, we will leave out associativity and unit isomorphisms throughout this section. If $n \geq 1$, let $\overline{n}$ be short for $\{1, \cdots, n\}$ and let $\overline{0}$ denote the empty set. Let Inj be the category with objects the sets $\overline{n}$ for all $n \geq 0$, and injective maps as morphisms. If $m \leq n$, define

$$\mathrm{Inj}(m,n) := \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{m},\overline{n})} T^0$$

where T^0 is the unit of $\mathcal{V}$. Note that $\mathrm{Inj}(n,n)$ is a group object in $\mathcal{V}$, the symmetric group on n letters. By regarding $\mathrm{Inj}(n,n)$ as a $\mathcal{V}$-category with a single object, a left $\mathrm{Inj}(n,n)$-action on $A \in \mathrm{Ob}\,\mathcal{V}$ is a $\mathcal{V}$-functor $\mathrm{Inj}(n,n) \longrightarrow \mathcal{V}$ with value A. One gets a left $\mathrm{Inj}(n,n)$-action on $T^n = T_n \otimes \cdots \otimes T_2 \otimes T_1$ using iterations of the commutativity isomorphism $\sigma_{T,T}$.

A *symmetric T-spectrum X in $\mathcal{V}$* as defined in [8, 7.2] consists of a sequence $X_0, X_1, \cdots, X_n, \cdots$, where X_n is an object of $\mathcal{V}$ with a left $\mathrm{Inj}(n,n)$-action, and with structure maps $X_n \otimes T \longrightarrow X_{n+1}$ such that the following composition $X_n \otimes T^m \longrightarrow X_{n+1} \otimes T^{m-1} \longrightarrow \cdots \longrightarrow X_{n+m}$ is $\mathrm{Inj}(n,n) \otimes \mathrm{Inj}(m,m)$-equivariant. A map of symmetric T-spectra consists of maps $X_n \longrightarrow Y_n$ that are compatible with the $\mathrm{Inj}(n,n)$-action and the structure maps. We will show that the category $\mathrm{Sp}^\Sigma(\mathcal{V},T)$ of symmetric T-spectra in $\mathcal{V}$ is isomorphic to a category of $\mathcal{V}$-functors with codomain $\mathcal{V}$ and domain $T\mathrm{Sph}^\Sigma$.

The objects in $T\mathrm{Sph}^\Sigma$ are the objects in $T\mathrm{Sph}$, but the morphism objects are different. If $n = k+m$ with $m \geq 0, k \geq 0$, define the $\mathcal{V}$-object $\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m,T^n)$ to be $\mathrm{Inj}(m,n) \otimes T^k$. If $n < m$, define $\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m,T^n)$ to be the initial object. The unit map $T^0 \longrightarrow \mathrm{Inj}(n,n) = \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n,T^n)$ is the canonical map to the summand corresponding to $\mathrm{id}_{\overline{n}}$. Next, to describe the composition, identify $\mathrm{Inj}(m,k+m) \otimes T^k$ indexed by $\beta\colon \overline{m} \longrightarrow \overline{k+m}$ with $T_{i_1^\beta} \otimes \cdots \otimes T_{i_k^\beta} = T^k$. Here $\{i_1^\beta, i_2^\beta, \cdots, i_k^\beta\}$ is the reordering of $\overline{k+m} \setminus \beta(\overline{m})$ which satisfies that $i_1^\beta > i_2^\beta > \cdots > i_k^\beta$. If $n = k+l+m$ with $k,l,m \geq 0$, we define the map

$$\mathrm{Inj}(l+m,n) \otimes T^k \otimes \mathrm{Inj}(m,l+m) \otimes T^l \longrightarrow \mathrm{Inj}(m,n) \otimes T^{k+l}$$

in two steps. First, we identify the source of the map with the coproduct $\coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{l+m},\overline{n}) \times \mathbf{Set}_{\mathrm{Inj}}(\overline{m},\overline{l+m})} T^{k+l}$. For the second step, consider the unique isomorphism $T^{k+l} \longrightarrow T^{k+l}$ induced by the permutation that reorders the set $\{i_1^\gamma, i_2^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \gamma(i_2^\beta), \cdots, \gamma(i_l^\beta)\}$, i.e. the set $\overline{k+l+m} \setminus (\gamma \circ \beta)(\overline{m})$, as $\{i_1^{\gamma\circ\beta}, i_2^{\gamma\circ\beta}, \cdots, i_{k+l}^{\gamma\circ\beta}\}$. This isomorphism maps the summand $T_{i_1^\gamma} \otimes T_{i_2^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes T_{i_1^\beta} \otimes T_{i_2^\beta} \otimes \cdots \otimes T_{i_l^\beta} = T^{k+l}$ indexed by (γ, β) to the summand $T_{i_1^{\gamma\circ\beta}} \otimes T_{i_2^{\gamma\circ\beta}} \otimes \cdots \otimes T_{i_{k+l}^{\gamma\circ\beta}} = T^{k+l}$ indexed by $\gamma \circ \beta$.

Lemma 2.14. *$T\mathrm{Sph}^\Sigma$ is a symmetric monoidal $\mathcal{V}$-category.*

Proof. The composition in $T\mathrm{Sph}^\Sigma$ is clearly unital. Associativity follows, since the permutation reordering the set $\{i_1^\gamma, i_2^\gamma, \ldots, i_k^\gamma, \gamma(i_1^\beta), \gamma(i_2^\beta), \ldots, \gamma(i_l^\beta)\}$ is unique.

On objects, $(T^l, T^k) \longmapsto T^{k+l}$ defines the monoidal product in $T\mathrm{Sph}^\Sigma$. For morphisms, consider injections $\beta\colon \overline{n} \longrightarrow \overline{l+n}$ and $\gamma\colon \overline{m} \longrightarrow \overline{k+m}$. Define $\beta \triangleright \gamma\colon \overline{m+n} \longrightarrow \overline{k+l+m+n}$ by concatenation. That is, $\beta \triangleright \gamma(i)$ is defined as $\gamma(i)$ for $i \leq m$ and as $\beta(i-m)+k+m$ for $i > m$. Note that the ordered set $i_1^\beta + k + m > \cdots > i_l^\beta + k + m > i_1^\gamma > \cdots > i_k^\gamma$ coincides with $i_1^{\beta\triangleright\gamma} > \cdots > i_{k+l}^{\beta\triangleright\gamma}$.

On $\mathcal{V}$-objects of morphisms, the map

$$\mathrm{Inj}(n, l+n) \otimes T^l \otimes \mathrm{Inj}(m, k+m) \otimes T^k \longrightarrow \mathrm{Inj}(m+n, k+l+m+n) \otimes T^{k+l}$$

sends the summand $T^l \otimes T^k = T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$ indexed by $(\overline{n} \xrightarrow{\beta} \overline{l+n}, \overline{m} \xrightarrow{\gamma} \overline{k+m})$ via the identity onto the summand T^{k+l} indexed by $\beta \triangleright \gamma$. To see this, one can rewrite the indices according to the above equality of ordered sets, that is, changing $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$ into $T_{i_1^\beta + k+m} \otimes \cdots \otimes T_{i_l^\beta + k+m} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$.

Let us check that this defines a $\mathcal{V}$-functor $\mathrm{mon}^\Sigma\colon T\mathrm{Sph}^\Sigma \otimes T\mathrm{Sph}^\Sigma \longrightarrow T\mathrm{Sph}^\Sigma$. The reason is essentially that concatenation $\triangleright\colon \mathrm{Inj} \times \mathrm{Inj} \longrightarrow \mathrm{Inj}$ is a functor. Since $\mathrm{id}_{\overline{n}} \triangleright \mathrm{id}_{\overline{m}} = \mathrm{id}_{\overline{m+n}}$, if follows that mon^Σ preserves identities. To check compatibility with composition, fix four injective maps $\alpha\colon \overline{m} \longrightarrow \overline{l+m}$, $\beta\colon \overline{l+m} \longrightarrow \overline{k+l+m =: n}$, $\gamma\colon \overline{r} \longrightarrow \overline{q+r}$ and $\delta\colon \overline{q+r} \longrightarrow \overline{p+q+r}$. The equality $(\delta \circ \gamma) \triangleright (\beta \circ \alpha) = (\delta \triangleright \beta) \circ (\gamma \triangleright \alpha)$ implies that it suffices to consider only one summand, say $M = \{i_1^\delta, \cdots, i_p^\delta, i_1^\beta, \cdots, i_k^\beta, i_1^\gamma, \cdots, i_q^\gamma, i_1^\alpha, \cdots, i_l^\alpha\}$. First, by rewriting M we find $\{i_1^\delta, \cdots, i_p^\delta, \delta(i_1^\gamma), \cdots, \delta(i_q^\gamma), i_1^\beta, \cdots, i_k^\beta, \beta(i_1^\alpha), \cdots, \beta(i_l^\alpha)\}$, and next by reordering we get $\{i_1^{\delta\circ\gamma}, \cdots, i_{p+q}^{\delta\circ\gamma}, i_1^{\beta\circ\alpha}, \cdots, i_{k+l}^{\beta\circ\alpha}\}$. The monoidal product rewrites the latter as $M' = \{i_1^{\delta\circ\gamma} + n, \cdots, i_{p+q}^{\delta\circ\gamma} + n, i_1^{\beta\circ\alpha}, \cdots, i_{k+l}^{\beta\circ\alpha}\}$ and there is a bijection $M \xrightarrow{\cong} M'$. On the other hand, the monoidal product rewrites M as $\{i_1^\delta + n, \cdots, i_p^\delta + n, i_1^\beta, \cdots, i_k^\beta, i_1^\gamma + l + m, \cdots, i_q^\gamma + l + m, i_1^\alpha, \cdots, i_l^\alpha\}$, and composition rewrites this set as $\{i_1^\delta + n, \cdots, i_p^\delta + n, i_1^\beta, \cdots, i_k^\beta, \delta \triangleright \beta(i_1^\gamma + l + m), \cdots, \delta \triangleright \beta(i_q^\gamma + l + m), \delta \triangleright \beta(i_1^\alpha), \cdots, \delta \triangleright \beta(i_l^\alpha).\}$ By definition of $\triangleright$, this set coincides with

$$\{i_1^\delta + n, \cdots, i_p^\delta + n, i_1^\beta, \cdots, i_k^\beta, \delta(i_1^\gamma) + n, \cdots, \delta(i_q^\gamma) + n, \beta(i_1^\alpha), \cdots, \beta(i_l^\alpha)\}$$

which is then reordered as $M' = \{i_1^{(\delta\triangleright\beta)\circ(\gamma\triangleright\alpha)}, \cdots, i_{k+l+p+q}^{(\delta\triangleright\beta)\circ(\gamma\triangleright\alpha)}\}$. The corresponding bijection $M \xrightarrow{\cong} M'$ is thus the same as above. Hence mon^Σ is a $\mathcal{V}$-functor.
By definition, $\mathrm{mon}^\Sigma(T^0, T^m) = T^m = \mathrm{mon}^\Sigma(T^m, T^0)$. So T^0 is a strict unit in $T\mathrm{Sph}^\Sigma$. Similarly, strict associativity holds. The commutativity isomorphism $T^0 \longrightarrow \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^{m+n}, T^{m+n})$ is the canonical map on the summand indexed by the permutation $\overline{m+n} \longrightarrow \overline{n+m}$ which interchanges m and n. Next, the coherence conditions [2, 6.3,6.4,6.5] follow by a straightforward calculation with permutations of $\overline{l+m+n}$. This ends the proof. $\square$

To explain the composition in $T\mathrm{Sph}^\Sigma$ more throughly, we will define $\mathcal{V}$-functors $\nu\colon T\mathrm{Sph} \longrightarrow T\mathrm{Sph}^\Sigma$ and $\sigma\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ such that $\sigma \circ \nu = \pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$. Both ν and σ are the respectively identities on objects. The map

$$\mathrm{hom}^\nu_{T^m, T^{l+m}}\colon T^l \longrightarrow \mathrm{Inj}(m, l+m) \otimes T^l \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{m}, \overline{l+m})} T^l$$

hits the summand indexed by the inclusion $\overline{m} \hookrightarrow \overline{l+m}$. It is then clear that ν preserves identities. For the composition, consider

$$\begin{tikzcd}
T^k \otimes T^l \arrow[rr, "\mathrm{id}"] \arrow[d, "\mathrm{hom}^\nu_{T^{l+m}, T^{k+l+m}} \otimes \mathrm{hom}^\nu_{T^m, T^{l+m}}"'] & & T^{k+l} \arrow[d, "\mathrm{hom}^\nu_{T^m, T^{k+l+m}}"] \\
\mathrm{Inj}(l+m, k+l+m) \otimes T^k \otimes \mathrm{Inj}(m, l+m) \otimes T^l \arrow[rr, "\mathrm{comp}"] & & \mathrm{Inj}(m, k+l+m) \otimes T^{k+l}
\end{tikzcd}$$

and observe that the left vertical map hits the summand

$$(T_{k+l+m} \otimes T_{k+l+m-1} \otimes \cdots \otimes T_{l+m+1}) \otimes (T_{l+m} \otimes T_{l+m-1} \otimes \cdots \otimes T_{m+1})$$

indexed by the inclusions $(\overline{l+m} \hookrightarrow \overline{k+l+m}, \overline{m} \hookrightarrow \overline{l+m})$. Composition in $T\mathrm{Sph}^\Sigma$ maps this summand by the identity to the summand $T_{k+l+m} \otimes \cdots \otimes T_{m+1}$ indexed by the inclusion $\overline{m} \hookrightarrow \overline{l+m} \hookrightarrow \overline{k+l+m}$, since the indices are ordered in the prescribed way. So the diagram commutes and ν is a $\mathcal{V}$-functor. This also explains the ordering $i_1^\beta > i_2^\beta > \cdots > i_k^\beta$ of the set $\overline{k+m} \setminus \beta(\overline{k})$.
The adjoint $\mathrm{Inj}(m, l+m) \otimes T^l \otimes T^m \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{m}, \overline{l+m})} T^l \otimes T^m \longrightarrow T^{l+m}$ of $\mathrm{hom}^\sigma_{T^m, T^{l+m}}\colon \mathrm{Inj}(m, l+m) \otimes T^l \longrightarrow \mathcal{V}(T^m, T^{l+m})$ is defined as follows: Consider the summand $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_m \otimes \cdots \otimes T_1$ indexed by β. First rewrite the indices as $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{\beta(m)} \otimes \cdots \otimes T_{\beta(1)}$, and then map to $T_{l+m} \otimes \cdots \otimes T_2 \otimes T_1$ by the unique permutation which reorders the set $\{i_1^\beta, \cdots, i_l^\beta, \beta(m), \cdots, \beta(1)\}$. To conclude that σ is a $\mathcal{V}$-functor, one has to check that reordering the set $\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta), \gamma(\beta(m)), \cdots, \gamma(\beta(1))\}$ is the same as first reordering $\{i_1^\gamma, \cdots, i_l^\gamma, i_1^\beta, \cdots, i_k^\beta, \beta(m), \cdots, \beta(1)\}$ as $\{i_1^\gamma, \cdots, i_l^\gamma, l+m, \cdots, 2, 1\}$, and then reordering $\{i_1^\gamma, \cdots, i_l^\gamma, \gamma(l+m), \cdots, \gamma(2), \gamma(1)\}$. However,

the monoidal product mon^Σ on $T\mathrm{Sph}^\Sigma$ is such that the diagram

$$\begin{tikzcd} T\mathrm{Sph}^\Sigma \otimes T\mathrm{Sph}^\Sigma \arrow[r, "\sigma\otimes\sigma"] \arrow[d, "\mathrm{mon}^\Sigma"'] & \mathcal{V}\otimes\mathcal{V} \arrow[d, "\mathrm{mon}"] \\ T\mathrm{Sph}^\Sigma \arrow[r, "\sigma"] & \mathcal{V} \end{tikzcd}$$

commutes, so σ is in fact a strict symmetric monoidal $\mathcal{V}$-functor.

Proposition 2.15. *The category* $[T\mathrm{Sph}^\Sigma,\mathcal{V}]$ *is isomorphic to the category of symmetric T-spectra in $\mathcal{V}$, and ν induces the forgetful functor from symmetric T-spectra to T-spectra. The smash product on* $[T\mathrm{Sph}^\Sigma,\mathcal{V}]$ *corresponds to the smash product on* $\mathrm{Sp}^\Sigma(\mathcal{V},T)$.

Proof. This is similar to 2.12, and some details will be left out in the proof. The functor $\Phi\colon[T\mathrm{Sph}^\Sigma,\mathcal{V}]\longrightarrow \mathrm{Sp}^\Sigma(\mathcal{V},T)$ maps $X\colon T\mathrm{Sph}^\Sigma\longrightarrow\mathcal{V}$ to the sequence $XT^0, XT^1,\cdots$. It is clear that XT^n has a left $\mathrm{Inj}(n,n)$-action. The adjoint of the structure map $XT^n\otimes T\longrightarrow XT^{n+1}$ is given by $\hom^X_{T^n,T^{n+1}}\circ\hom^\nu_{T^n,T^{n+1}}$. More generally, the composition of structure maps $XT^n\otimes T^k\longrightarrow XT^{n+k}$ is the adjoint of $T^k\xrightarrow{\hom^\nu_{T^n,T^{k+n}}}\mathrm{Inj}(n,k+n)\otimes T^k\xrightarrow{\hom^X_{T^n,T^{k+n}}}\mathcal{V}(XT^n,XT^{k+n})$. This proves the required equivariance. The definition of Φ on $\mathcal{V}$-natural transformations is clear, and also functoriality.

The inverse $\Psi\colon\mathrm{Sp}^\Sigma(\mathcal{V},T)\longrightarrow[T\mathrm{Sph}^\Sigma,\mathcal{V}]$ is harder to define. If $X_0,X_1,\cdots$ is a symmetric T-spectrum with structure maps $x^1_n\colon X_n\otimes T\longrightarrow X_{n+1}$, define $\Psi(X)\colon T\mathrm{Sph}^\Sigma\longrightarrow\mathcal{V}$ on objects by $T^n\longmapsto X_n$. Since X_n has a left $\mathrm{Inj}(n,n)$-action, there is the map $\hom^{\Psi(X)}_{T^n,T^n}\colon\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n,T^n)=\mathrm{Inj}(n,n)\longrightarrow\mathcal{V}(X_n,X_n)$. Choose an injection $\beta\colon\overline{n}\longrightarrow\overline{k+n}$, and define

$$\hom^{\Psi(X)}_{T^n,T^{k+n}}\colon\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n,T^{k+n})=\mathrm{Inj}(n,k+n)\longrightarrow\mathcal{V}(X_n,X_{k+n})$$

on the summand $T_{i^\beta_1}\otimes\cdots\otimes T_{i^\beta_k}$ as the adjoint map of the following composition $T_{i^\beta_1}\otimes\cdots\otimes T_{i^\beta_k}\otimes X_n\xrightarrow{\sigma_{T^k,X_n}}X_n\otimes T_{i^\beta_1}\otimes\cdots\otimes T_{i^\beta_k}\xrightarrow{x^k_n}X_{k+n}\longrightarrow X_{k+n}$ where x^k_n is defined by the structure maps. The right hand map is the isomorphism associated to the permutation of the set $\overline{k+n}$ that changes $\{k+n,\cdots,2,1\}$ to $\{i^\beta_1,\cdots,i^\beta_k,\beta(n),\cdots,\beta(1)\}$ and reorders this as $\{k+n,\cdots,2,1\}$. For the latter we use the left $\mathrm{Inj}(k+n,k+n)$-action on X_{k+n}.

The $\mathrm{Inj}(n,n)$-action on X_n is unital, so $\Psi(X)$ preserves identities. To prove that $\Psi(X)\colon T\mathrm{Sph}^\Sigma\longrightarrow\mathcal{V}$ is a $\mathcal{V}$-functor, pick injective maps $\beta\colon\overline{m}\longrightarrow\overline{l+m}$,

$\gamma\colon \overline{l+m} \longrightarrow \overline{k+l+m}$ and consider the following diagram.

$$\begin{tikzcd}
T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes X_m \arrow[rr, "T^k\otimes(x^l_m\circ\sigma_{T^l,X_m})"] \arrow[d, "f\otimes X_m"'] \arrow[dr, "x^{k+l}_m\circ\sigma_{T^{k+l},X_m}"] & & T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes X_{l+m} \arrow[d, "T^k\otimes b"] \arrow[dl, "x^k_{l+m}\circ\sigma_{T^k,X_{l+m}}"'] \\
T_{i_1^{\gamma\circ\beta}} \otimes \cdots \otimes T_{i_{k+l}^{\gamma\circ\beta}} \otimes X_m \arrow[d, "x^{k+l}_m\circ\sigma_{T^{k+l},X_m}"'] & X_{k+l+m} \arrow[d, "f'"] & T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes X_{l+m} \arrow[d, "x^k_{l+m}\circ\sigma_{T^k,X_{l+m}}"] \\
X_{k+l+m} \arrow[dr, "cb"] & X_{k+l+m} \arrow[d, "cb"] & X_{k+l+m} \arrow[dl, "c"'] \\
& X_{k+l+m} &
\end{tikzcd}$$

Here $b\colon X_{l+m} \longrightarrow X_{l+m}$ is the isomorphism obtained from reordering the set $\{i_1^\beta, \cdots, i_k^\beta, \beta(m), \cdots, \beta(1)\}$ as $\{l+m, \cdots, 2, 1\}$, and similarly for c and cb. The isomorphism $f\colon T^{k+l} \longrightarrow T^{k+l}$ is induced by the permutation

$$\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta)\} \xrightarrow{\cong} \{i_1^{\gamma\circ\beta}, \cdots, i_{k+l}^{\gamma\circ\beta}\}.$$

Similarly, $f'\colon X_{k+l+m} \longrightarrow X_{k+l+m}$ is induced by the permutation

$$\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta), m, \cdots, 1\} \xrightarrow{\cong} \{i_1^{\gamma\circ\beta}, \cdots, i_{k+l}^{\gamma\circ\beta}, m, \cdots, 1\}.$$

Since x_m^{k+l} is $\mathrm{Inj}(k+l, k+l) \otimes \mathrm{Inj}(m,m)$-equivariant, the left parallelogram commutes. The upper triangle commutes by commutativity coherence. Finally, the right parallelogram commutes, because x_{l+m}^k is $\mathrm{Inj}(k,k) \otimes \mathrm{Inj}(l+m, l+m)$-equivariant, the $\mathrm{Inj}(k+l+m, k+l+m)$-action on X_{k+l+m} is associative and the permutation obtained from reordering $\{i_1^\gamma, \cdots, i_k^\gamma, i_1^\beta, \cdots, i_l^\beta, \beta(m), \cdots, \beta(1)\}$ as $\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(l+m), \cdots, \gamma(1)\}$ and then as $\{k+l+m, \cdots, 2, 1\}$ equals the permutation obtained from $\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta), \gamma(\beta(m)), \cdots, \gamma(\beta(1))\}$. The definition of Ψ on maps is clear and functoriality follows easily. The proof of 2.12 shows that $\Phi \circ \Psi = \mathrm{Id}_{\mathrm{Sp}^\Sigma(\mathcal{V},T)}$, but an extra argument is required to prove the equality $\Psi \circ \Phi = \mathrm{Id}_{[T\mathrm{Sph}^\Sigma, \mathcal{V}]}$. The only point which is not obvious is whether the maps $\hom^X_{T^m, T^{k+m}}\colon \mathrm{Inj}(m, k+m) \otimes T^k \longrightarrow \mathcal{V}(XT^m, XT^{m+k})$ and $\hom^{\Psi(\Phi(X))}_{T^m, T^{k+m}}\colon \mathrm{Inj}(m, k+m) \otimes T^k \longrightarrow \mathcal{V}(XT^m, XT^{m+k})$ coincide. To prove this, we fix an injection $\beta\colon \overline{m} \longrightarrow \overline{k+m}$ and let $\iota\colon \overline{m} \hookrightarrow \overline{k+m}$ denote the inclusion. The permutation $\gamma\colon \overline{k+m} \longrightarrow \overline{k+m}$ obtained from rewriting the set $\{k+m, \cdots, 1+m, m, \cdots, 1\}$ as $\{i_1^\beta, \cdots, i_k^\beta, \beta(m), \cdots, \beta(1)\}$ and reordering this set as $\{k+m, \cdots, 2, 1\}$ has the property that $\gamma \circ \iota = \beta$. Since X is a $\mathcal{V}$-functor, the map $\hom^X_{T^m, T^{k+m}}$ is determined by its restriction to the summand indexed by $\overline{m} \hookrightarrow \overline{k+m}$. This shows that $\hom^X_{T^m, T^{k+m}} = \hom^{\Psi(\Phi(X))}_{T^m, T^{k+m}}$.

The claim concerning $[\nu, \mathcal{V}]\colon [T\mathrm{Sph}^\Sigma, \mathcal{V}] \longrightarrow [T\mathrm{Sph}, \mathcal{V}]$ is clear by the above. It remains to prove compatibility of the smash products. The smash product $\wedge'$

of symmetric T-spectra satisfies $F_mT^0 \wedge' F_nT^0 \cong F_{m+n}T^0$ by the remark below [8, 7.3]. On the other hand, the monoidal product in $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is determined by representable functors, and from 2.7 there is the natural isomorphism

$$\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m, -) \wedge \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, -) \cong \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^{m+n}, -).$$

This completes the proof. □

Corollary 2.16. *Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category. Assume $\mathcal{C}$ is closed under the monoidal product, and contains the unit and T. Then $\sigma\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ factors over $\mathcal{C} \hookrightarrow \mathcal{V}$, and the induced $\mathcal{V}$-functor $T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{C}$ induces a lax symmetric monoidal functor $[\mathcal{C}, \mathcal{V}] \longrightarrow \mathrm{Sp}^\Sigma(\mathcal{V}, T)$ which has a strict symmetric monoidal left adjoint.*

Proof. The factorization of σ is clear. Under the isomorphism in 2.15, the closed symmetric monoidal product on $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ coincides with the closed symmetric monoidal product on symmetric T-spectra. For formal reasons, the $\mathcal{V}$-functor induced by the first factor of σ is lax symmetric monoidal. By checking on representable functors, it follows that its left $\mathcal{V}$-adjoint obtained by an enriched Kan extension 2.5 is strict symmetric monoidal. □

3 Model Categories

The term *model category* is to be understood in the sense of [7, 1.1.4]. We denote weak equivalences by $\xrightarrow{\sim}$, fibrations by $\twoheadrightarrow$ and cofibrations by $\rightarrowtail$.

3.1 Types of model categories

The model structures we will consider on enriched functor categories require a *cofibrantly generated* model category $(\mathcal{C}, I, J)$ as input. For a definition of this type of model categories and related terminology, consider [7, 2.1]. Maps in $I = \{i\colon si \rightarrowtail ti\}_{i\in I}$ are called *generating cofibrations*, and maps in $J = \{j\colon sj \overset{\sim}{\rightarrowtail} tj\}_{j\in J}$ are called *generating acyclic cofibrations*. The (co)domains of I and J may have additional properties.

Definition 3.1. An object $A \in \mathrm{Ob}\,\mathcal{C}$ is *finitely presentable* if the set-valued Hom-functor $\mathbf{Set}_\mathcal{C}(A, -)$ commutes with all filtered colimits. If $\mathcal{C}$ is a $\mathcal{V}$-category, $A \in \mathrm{Ob}\,\mathcal{C}$ is *$\mathcal{V}$-finitely presentable* if the $\mathcal{V}$-valued Hom-functor $\mathcal{V}_\mathcal{C}(A, -)$ commutes with all filtered colimits.

A set is finitely presentable in the category of sets if and only if it is a finite set. If the unit e in $\mathcal{V}$ is finitely presentable, any $\mathcal{V}$-finitely presentable object of $\mathcal{V}$ is finitely presentable. See [8, 4.1] for 3.2 and 3.3.

Definition 3.2. A cofibrantly generated model category $\mathcal{C}$ is *finitely generated* if I and J can be chosen such that the (co)domains of the maps in I and J are κ-small for a finite cardinal κ.

Finitely generated model categories are not necessarily closed under Bousfield localization, cf. [8, §4]. The following definition was suggested by Voevodsky.

Definition 3.3. The cofibrantly generated model category $\mathcal{C}$ is *almost finitely generated* if I and J can be chosen such that the (co)domains of the maps in I are κ-small for a finite cardinal κ, and there exists a subset J' of J for which

- the domains and the codomains of the maps in J' are κ-small for a finite cardinal κ,
- a map $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that B is fibrant is a fibration if and only if it is contained in J'-inj.

The left Bousfield localization with respect to a set with **sSet**-small domains and codomains preserves the structure of almost finitely generated cellular left proper simplicial model categories [6, Chapters 3,4]. For almost finitely generated model categories, the classes of weak equivalences and fibrant objects are closed under sequential colimits. We require these classes to be closed under filtered colimits, which holds for model categories of the following type.

Definition 3.4. A cofibrantly generated model category $\mathcal{V}$ is *weakly finitely generated* if I and J can be chosen such that the following conditions hold.

- The domains and the codomains of the maps in I are finitely presentable.
- The domains of the maps in J are small.
- There exists a subset J' of J of maps with finitely presentable domains and codomains, such that a map $f\colon A \longrightarrow B$ in $\mathcal{V}$ with fibrant codomain B is a fibration if and only if it is contained in J'-inj.

The choices of the sets I, J and J' will often be left implicit in the following. A weakly finitely generated model category is almost finitely generated. Examples include simplicial sets, simplicial sets with an action of a finite group, cp. 9.5, and the category of pointed simplicial presheaves on the smooth Nisnevich site. The latter will be discussed in [5].

Lemma 3.5. *Assume $\mathcal{V}$ is a weakly finitely generated model category. Then the following classes are closed under filtered colimits: weak equivalences, acyclic fibrations, fibrations with fibrant codomain, and fibrant objects.*

Proof. Since $\mathcal{V}$ is cofibrantly generated, [6, 11.6.1] shows that $\mathrm{Fun}(\mathcal{I},\mathcal{V})$ supports a cofibrantly generated model structure for any small category $\mathcal{I}$. Fibrations and weak equivalences are defined pointwise. Any weak equivalence in $\mathrm{Fun}(\mathcal{I},\mathcal{V})$ factors as an acyclic cofibration g composed with an acyclic fibration $p\colon T \twoheadrightarrow B$. Consider the induced factorization $\mathrm{colim}\, p \circ \mathrm{colim}\, g$. Note that $\mathrm{colim}\, g$ is an acyclic cofibration, since colim is a left Quillen functor. Therefore, the second claim will imply the first claim. If $\mathcal{I}$ is filtered, then colim$p$ is an acyclic fibration: Let

$$\begin{tikzcd} si \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathrm{colim}T \arrow[d, "\mathrm{colim}p"] \\ ti \arrow[r, "\beta"] & \mathrm{colim}B \end{tikzcd}$$

be a lifting problem, where $i \in I$. The existence of a lift in this diagram for all choices of α and β is equivalent to surjectivity of the canonical map

$$\phi\colon \mathbf{Set}_{\mathcal{V}}(ti, \mathrm{colim}T) \longrightarrow \mathbf{Set}_{\mathcal{V}}(si, \mathrm{colim}T) \times_{\mathbf{Set}_{\mathcal{V}}(si,\mathrm{colim}B)} \mathbf{Set}_{\mathcal{V}}(ti, \mathrm{colim}B).$$

Since si and ti are finitely presentable by assumption and filtered colimits commute with pullbacks in **Set**, the canonical map ϕ is the filtered colimit of the canonical maps

$$\phi_d\colon \mathbf{Set}_{\mathcal{V}}(ti, T(d)) \longrightarrow \mathbf{Set}_{\mathcal{V}}(si, T(d)) \times_{\mathbf{Set}_{\mathcal{V}}(si,B(d))} \mathbf{Set}_{\mathcal{V}}(ti, B(d))$$

induced by composition with $p(d)$ and pre-composition with i. Note that ϕ_d is surjective since i is a cofibration and p is a pointwise acyclic fibration. It follows that ϕ is surjective and colim$p$ is an acyclic fibration. The proof of the third claim – including the last claim as a special case – is analogous. □

Definition 3.6. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category, and $\mathcal{C}$ a right $\mathcal{V}$-module with action $(C, A) \longmapsto C \oslash A$. Consider, for $f\colon C \longrightarrow D$ a map in $\mathcal{C}$ and $g\colon A \longrightarrow B$ a map in $\mathcal{V}$, the diagram:

$$\begin{tikzcd} C \oslash A \arrow[r, "C\oslash g"] \arrow[d, "f\oslash A"'] & C \oslash B \arrow[d, "f\oslash B"] \\ D \oslash A \arrow[r, "D\oslash f"] & D \oslash B \end{tikzcd}$$

If $\mathcal{C}$ has pushouts, denote the induced map from the pushout of the diagram to the terminal corner by $f\Box g\colon D \oslash A \cup_{C\oslash A} C \oslash B \longrightarrow D \oslash B$. This is the *pushout product map* of f and g.

Note that 3.6 applies to $(\mathcal{V}, \otimes, e)$ considered as a right $\mathcal{V}$-module. Recall the pushout product axiom [15, 3.1].

Definition 3.7. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category and a model category. It is a *monoidal model category* if the pushout product $f\Box g$ of two cofibrations f and g is a cofibration, which is acyclic if either f or g is acyclic.

Monoidal model categories with a cofibrant unit are symmetric monoidal model categories in the sense of [7, 4.2.6]. See [15, 3.3] for the following definition.

Definition 3.8. Let $\mathcal{V}$ be a monoidal model category. For a class $\mathcal{K}$ of maps in $\mathcal{V}$, define $\mathcal{K} \otimes \mathcal{V}$ as the class of maps $f \otimes A$, where f is a map in $\mathcal{K}$ and $A \in \operatorname{Ob} \mathcal{V}$. Let $\operatorname{aCof}(\mathcal{V})$ be the class of acyclic cofibrations in $\mathcal{V}$. The *monoid axiom* holds if every map in $\operatorname{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell is a weak equivalence.

In the proof of 4.2, we will use the monoid axiom to construct the pointwise model structure on enriched functor categories. Let us end this section with the definition of two other types of model categories.

Definition 3.9. Let $\mathcal{V}$ be a monoidal model category, $\mathcal{C}$ a closed $\mathcal{V}$-module and a model category. The action of $\mathcal{V}$ on $\mathcal{C}$ allows us to consider the pushout product of a map in $\mathcal{C}$ and a map in $\mathcal{V}$. Then $\mathcal{C}$ is a *$\mathcal{V}$-model category* if the pushout product of a cofibration f in $\mathcal{C}$ and a cofibration g in $\mathcal{V}$ is a cofibration in $\mathcal{C}$, which is acyclic if either f or g is acyclic.

Note that a simplicial model category is a **sSet**-model category.

Lemma 3.10. *A simplicial homotopy equivalence in a* **sSet**-*model category is a weak equivalence.*

Proof. This follows from [6, 9.5.16]. Simplicial homotopy equivalences in 2.9 are also simplicial homotopy equivalences as defined in [6, 9.5.8]. □

Definition 3.11. Let $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ be a strict symmetric monoidal functor of monoidal model categories. If F is a left Quillen functor, $\mathcal{W}$ is called a *monoidal $\mathcal{V}$-model category*.

A monoidal $\mathcal{V}$-model category is clearly a $\mathcal{V}$-model category.

3.2 Homotopy pullback squares

Homotopy pullback squares will be used to characterize fibrations. Definition 3.12 is equivalent to [7, 7.1.12].

Definition 3.12. Let $\mathcal{C}$ be a model category. A commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

in $\mathcal{C}$ is a *homotopy pullback square* if for any commutative diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\sim"'] & D \arrow[d, "\sim"'] & B \arrow[l, "f"'] \arrow[d, "\sim"'] \\ C' \arrow[r] & D' & B' \arrow[l, two heads] \end{tikzcd}$$

where C' and D' are fibrant and $B' \twoheadrightarrow D'$ is a fibration, the canonical map $A \longrightarrow C' \times_{D'} B'$ is a weak equivalence.

This definition seems to by asymmetric, but one of the squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "i"] \\ B \arrow[r, "h"] & D \end{tikzcd}$$

is a homotopy pullback square if and only if the other is a homotopy pullback square. We list some elementary properties.

Lemma 3.13. *All diagrams below are commutative diagrams in $\mathcal{C}$.*

1. *The diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\sim"] & D \end{tikzcd}$$

is a homotopy pullback square if and only if f is a weak equivalence.

2. *Consider a natural transformation $f\colon A \longrightarrow B$ of diagrams*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & A_3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_3 \end{tikzcd}$$

which is a pointwise weak equivalence. That is, $f_i\colon A_i \longrightarrow B_i$ is a weak equivalence for all $i \in \{0,1,2,3\}$. Then A is a homotopy pullback square if and only if B is a homotopy pullback square.

3. *Let*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & A_2 \arrow[d] \\ B_0 \arrow[r] & B_1 \arrow[r] & B_2 \end{tikzcd}$$

be a diagram where (2) is a homotopy pullback square. Then the composed square (12) is a homotopy pullback square if and only (1) is a homotopy pullback square.

Lemma 3.14. *Assume $\mathcal{C}$ is right proper. Then*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pullback square if and only if for some factorization of f as a weak equivalence $B \xrightarrow{\sim} E$ followed by a fibration $E \twoheadrightarrow D$, the induced map $A \longrightarrow C \times_D E$ is a weak equivalence.

Proof. Since $\mathcal{C}$ is right proper, the dual of the gluing lemma holds. The statement follows easily. □

3.3 Fibrant replacement functors

In every model category, any object may be replaced by a fibrant object in a functorial way up to an acyclic cofibration. It is often desirable to explicate fibrant replacement functors. Quillen's small object argument is the classical method 3.3.1. We place emphasis on enriched fibrant replacement functors 3.3.2. Another fibrant replacement functor is constructed in 3.3.3 as a certain filtered colimit.

3.3.1 Classical

Fix a cocomplete category $\mathcal{V}$ and a set $K = \{sk \longrightarrow tk\}_{k\in K}$ of maps in $\mathcal{V}$ with finitely presentable domains and finitely presentable codomains. The set K gives rise to a natural transformation of endofunctors on $\mathcal{V}$, namely

$$\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} sk \xrightarrow{\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} k} \coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} tk.$$

Consider also

$$\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} sk \xrightarrow{\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} f} \coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} \mathrm{Id}_{\mathcal{V}} \xrightarrow{\text{codiagonal}} \mathrm{Id}_{\mathcal{V}},$$

and take the pushout of the two natural transformations. Let $\iota_1^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F_1^K$ denote the canonically induced map. This construction can be iterated. Suppose there is a natural transformation $\iota_n^K\colon F_{n-1}^K \longrightarrow F_n^K$ of endofunctors of $\mathcal{V}$. Next, define F_{n+1}^K as the pushout of

$$\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} tk \longleftarrow \coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} sk \longrightarrow F_n.$$

The colimit of the sequence $\mathrm{Id}_{\mathcal{V}} = F_0 \xrightarrow{\iota_1^K} F_1^K \xrightarrow{\iota_2^K} \cdots$ is denoted as $F^K\colon \mathcal{V} \longrightarrow \mathcal{V}$, and $\iota^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F^K$ is the canonical natural transformation. The following statement is a special case of [7, 2.1.14].

Lemma 3.15. *For every object A in $\mathcal{V}$, the map $\iota^K(A)\colon A \longrightarrow F^K(A)$ is in K-cell, and the map $F^K(A) \longrightarrow *$ is in K-inj.*

Corollary 3.16. *Suppose that $\mathcal{V}$ is a weakly finitely generated model category. Then $\iota^{J'}: \mathrm{Id}_{\mathcal{V}} \longrightarrow F^{J'}$ is a fibrant replacement functor, i.e. $F^{J'}(A)$ is fibrant and the natural map $\iota^{J'}(A): A \longrightarrow F^{J'}(A)$ is an acyclic cofibration for all $A \in \mathrm{Ob}\,\mathcal{V}$.*

This fibrant replacement functor yields "big" objects. We will make this more precise after recalling some definitions from [6, Chapter 10].

Definition 3.17. A relative K-cell complex $f: A \longrightarrow B$ is called *finite* if f is a composition $A = B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} B_n = B$ where each f_m is a cobase change of a map in K, i.e. f is obtained by attaching finitely many cells from K.

Definition 3.18. A relative K-cell complex $f: A \longrightarrow B$ is called *presented* if there is an explicit choice of the data [6, 10.6.3]. In detail, one chooses a limit ordinal λ, a λ-sequence $A = B_0 \xrightarrow{f_1} B_1 \longrightarrow \cdots \xrightarrow{f_{\beta+1}} B_{\beta+1} \longrightarrow \cdots$ whose sequential composition is f, and $f_{\beta+1}: B_\beta \longrightarrow B_{\beta+1}$ is gotten from the pushout of the following diagram for every $\beta < \lambda$:

$$\coprod_{m \in M_\beta} ti_m \xleftarrow{\coprod i_m} \coprod_{m \in M_\beta} si_m \longrightarrow B_\beta$$

We omit the choice from the notation. Let $f: A \longrightarrow B$ be a presented relative K-cell complex. A *subcomplex* of f is a presented K-cell complex $g: A \longrightarrow C$ relative to A such that the explicit choice relevant for g is a subset of the explicit choice for f, see [6, 10.6.7]. In particular, there exists a map $h: C \longrightarrow B$ in K-cell such that $h \circ g = f$. A subcomplex is called *finite* if it is a finite K-cell complex relative to A, using the explicit choice.

Any relative K-cell complex can be turned into a presented one. If we consider $\iota^K(A)$ as a presented relative K-cell complex, we will use the explicit choices appearing in its construction.

Lemma 3.19. *Let $f: A \longrightarrow B$ be a finite relative K-cell complex. Then f has the structure of a finite subcomplex of $\iota^K(K): A \longrightarrow F^K(A)$.*

Proof. We will prove this by induction on the number of cells. By convention, a map obtained by attaching no cells is an identity map. Suppose the following is true. If $f: A \longrightarrow B$ is obtained by attaching n cells from K, with $n \geq 0$, then f has the structure of a finite subcomplex of the presented relative K-cell complex $A \xrightarrow{\iota_1^K(A)} \cdots \xrightarrow{\iota_n^K(A)} F_n^K(A)$. Assume $f: A \longrightarrow B$ is obtained by attaching $n+1$ cells from K. By definition, there is a factorization $A \xrightarrow{g} C \xrightarrow{h} B$ where g is obtained by attaching n cells from K and h is the cobase change of some $k \in K$ along a map $\alpha: sk \longrightarrow C$. From the induction hypothesis, α induces a map $sk \longrightarrow C \longrightarrow F_n A$, hence an element in $\mathbf{Set}_{\mathcal{V}}(sk, F_n A)$. It follows that there is a map $\beta\colon tk \longrightarrow F_{n+1}A$ rendering the diagram

$$\begin{tikzcd} sk \arrow[r, "\alpha"] \arrow[d, "k"', "\sim"] & C \arrow[r] & F_n^K(A) \arrow[d, "\iota_{n+1}^K(A)"] \\ tk \arrow[rr, "\beta"] & & F_{n+1}^K(A) \end{tikzcd}$$

commutative. Since h is the cobase change of k along α, there is a unique induced map $B \longrightarrow F_{n+1}^K(A)$ which gives f the structure of a subcomplex of the complex $A \xrightarrow{\iota_1^K(A)} \cdots \xrightarrow{\iota_{n+1}^K(A)} F_{n+1}^K(A)$. □

3.3.2 Enriched

Suppose that $\mathcal{V}$ is a cocomplete and closed symmetric monoidal category, and that K is a set of maps with finitely presentable domains and codomains. The fibrant replacement functor defined in 3.3.1 is a priori not a $\mathcal{V}$-functor, but one can remedy this as follows. Each k in K induces a $\mathcal{V}$-natural transformation $\mathcal{V}(sk,-)\otimes k\colon \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathcal{V}(sk,-)\otimes tk$ of endo-$\mathcal{V}$-functors. On the other hand, the counit $\epsilon_{sk}\colon \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathrm{Id}_{\mathcal{V}}$ is also a $\mathcal{V}$-natural transformation. By taking the coproduct over all $k\in K$, one gets the diagram of $\mathcal{V}$-functors

$$\coprod_{k\in K}\mathcal{V}(sk,-)\otimes tk \xleftarrow{\coprod \mathcal{V}(sk,-)\otimes k} \coprod_{k\in K}\mathcal{V}(sk,-)\otimes sk \longrightarrow \mathrm{Id}_{\mathcal{V}}.$$

Denote the pushout by R_1^K. It is clear that one can iterate this construction. Given a $\mathcal{V}$-natural transformation $\rho_n^K\colon R_{n-1}^K \longrightarrow R_n^K$ of endo-$\mathcal{V}$-functors of $\mathcal{V}$, let R_{n+1}^K be the pushout of

$$\coprod_{k\in K}\mathcal{V}(sk,R_n^K(-))\otimes tk \xleftarrow{\coprod_{k\in K}\mathcal{V}(sk,R_n^K(-))\otimes k} \coprod_{k\in K}\mathcal{V}(sk,R_n^K(-))\otimes sk \longrightarrow R_n^K.$$

The colimit of the diagram $\mathrm{Id}_{\mathcal{V}} = R_0 \xrightarrow{\rho_1^K} R_1^K \xrightarrow{\rho_2^K} \cdots$ is called $R^K\colon \mathcal{V} \longrightarrow \mathcal{V}$. Let $\rho^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R^K$ be the canonical $\mathcal{V}$-natural transformation.

Lemma 3.20. *Given any object A in $\mathcal{V}$, the map $\rho^K(A)\colon A \longrightarrow R^K(A)$ is in $K\otimes\mathcal{V}$-cell, and the map $R^K(A) \longrightarrow *$ is in K-inj.*

Proof. The first statement is obvious. To prove that $R^K(A) \longrightarrow *$ is in K-inj, consider a lifting problem for $k\in K$:

$$\begin{tikzcd} sk \arrow[r, "f"] \arrow[d, "k"', "\sim"] & R^K(A) \arrow[d] \\ tk \arrow[r] & * \end{tikzcd}$$

Note that f factors as $sk \xrightarrow{g} R^K_n(A) \longrightarrow R^K(A)$ for some n, since sk is finitely presentable and R is a sequential colimit. The adjoint of g, tensored with sk, induces the map $h\colon sk \longrightarrow \mathcal{V}(sk, R^K_n(A)) \otimes sk$ such that the following diagram commutes.

$$\begin{tikzcd} & \mathcal{V}(sk, R^K_n(A)) \otimes sk \arrow[dr, "\epsilon_{sk}(R^K_n(A))"] & \\ sk \arrow[ur, "h"] \arrow[rr, "g"] & & R^K_n(A) \end{tikzcd}$$

Let h' denote the canonical map to the coproduct given by h. Then the diagram

$$\begin{tikzcd} sk \arrow[r, "h'"] \arrow[d, "k"', "\sim"] & \coprod_{k\in K} \mathcal{V}(sk, R^K_n(A)) \otimes sk \arrow[r] \arrow[d] & R^K_n(A) \arrow[d] \\ tk \arrow[r] & \coprod_{k\in K} \mathcal{V}(sk, R^K_n(A)) \otimes tk \arrow[r] & R^K_{n+1}(A) \end{tikzcd}$$

shows that $tk \longrightarrow \coprod_{k\in K} \mathcal{V}(sk, R^K_n(A)) \longrightarrow R^K_{n+1}(A) \longrightarrow R^K(A)$ solves the lifting problem above. This proves that $R^K(A) \longrightarrow *$ is in K-inj. $\square$

Corollary 3.21. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category satisfying the monoid axiom. Then $\rho^{J'}\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R^{J'}$ is a fibrant replacement $\mathcal{V}$-functor, i.e. $R^{J'}(A)$ is fibrant and $\rho^{J'}(A)$ is a weak equivalence for all A.*

Proof. It remains to prove that the natural map $\rho^{J'}_A\colon A \longrightarrow R^{J'}(A)$ is a weak equivalence for every $A \in \mathrm{Ob}\,\mathcal{V}$. Note that $\rho^{J'}_A$ is contained in $\mathrm{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell. The monoid axiom for $\mathcal{V}$ implies that $\rho^{J'}_A$ is a weak equivalence. $\square$

Remark 3.22. The map $\rho^{J'}(A)$ is not a cofibration in general, even if A is cofibrant. But if all objects in $\mathcal{V}$ are cofibrant, then $\rho^{J'}(A)$ is a cofibration.

3.3.3 Filtered

Let $\mathcal{V}$ denote a cocomplete closed symmetric monoidal category, and K a set of maps with finitely presentable domains and codomains. We will define for each object A of $\mathcal{V}$ three categories $\mathrm{ac}^K(A)$, $\mathrm{ac}^K(A,R)$ and $\mathrm{ac}^K(A,F)$, and functors $U_R\colon \mathrm{ac}^K(A,R) \longrightarrow \mathrm{ac}^K(A)$, $U_F\colon \mathrm{ac}^K(A,F) \longrightarrow \mathrm{ac}^K(A)$.
Objects in $\mathrm{ac}^K(A)$ are finite K-cell complexes $\beta\colon A \longrightarrow B$ relative to A. The morphisms in $\mathrm{ac}^K(A)$ from $\beta\colon A \longrightarrow B$ to $\gamma\colon A \longrightarrow C$ are finite K-cell complexes $\tau\colon B \longrightarrow C$ for which $\tau \circ \beta = \gamma$. The identity id_A is the initial object of $\mathrm{ac}^K(A)$. Consider the functor $\Psi_A\colon \mathrm{ac}^K(A) \longrightarrow \mathcal{V}$ which sends $\beta\colon A \longrightarrow B$ to B and $\tau\colon B \longrightarrow C$ to τ. The colimit of Ψ_A will define the desired fibrant replacement of A, up to isomorphism.
Objects in $\mathrm{ac}^K(A,R)$ are pairs $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow R^K(A))$ such that β is a finite K-cell complex relative to A, and $\beta' \circ \beta = \rho^K(A)$ holds. A map

from $(\beta: A \longrightarrow B, \beta': B \longrightarrow R^K(A))$ to $(\gamma: A \overset{\sim}{\rightarrowtail} C, \gamma': C \xrightarrow{\sim} R^K(A))$ is a finite K-cell complex $\tau: B \rightarrowtail C$ such that

$$\begin{tikzcd} & B \arrow[dd, "\tau"] \arrow[dr, "\beta'"] & \\ A \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & R^K(A) \\ & C \arrow[ur, "\gamma'"'] & \end{tikzcd}$$

commutes. The initial object of $\mathrm{ac}^K(A, R)$ is (id_A, ρ_A). Denote the forgetful functor $\mathrm{ac}^K(A, R) \longrightarrow \mathrm{ac}^K(A)$ which maps the pair (β, β') to β by U_R.
The category $\mathrm{ac}^K(A, F)$ has objects finite subcomplexes of the relative K-cell complex $\iota^K(A): A \longrightarrow F^K(A)$. Such a subcomplex $\beta: A \longrightarrow B$ comes with a map $\beta': B \longrightarrow F^K(A)$ in K-cell such that $\beta' \circ \beta = \iota^K(A)$, whence the objects in $\mathrm{ac}^K(A, F)$ are also denoted (β, β'). Maps are defined as for $\mathrm{ac}^K(A, R)$. Here $(\mathrm{id}_A, \iota(A))$ is the initial object. Let $U_F: \mathrm{ac}^K(A, F) \longrightarrow \mathrm{ac}^K(A)$ be the forgetful functor. Recall from [12, IX.3] the notion of a final functor.

Lemma 3.23. *If maps in K-cell are monomorphisms, then U_R and U_F are final functors, and $\mathrm{ac}^K(A, F)$ is a small filtered preorder for all $A \in \mathrm{Ob}\,\mathcal{V}$.*

Proof. To prove that U_R is final, let $\beta: A \longrightarrow B$ be an object of $\mathrm{ac}^K(A)$. Since $R^K(A) \longrightarrow *$ is in K-inj, there exists a lift in the following diagram.

$$\begin{tikzcd} A \arrow[r, "\rho^K(A)"] \arrow[d, "\beta"'] & R^K(A) \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

Hence the comma category $\beta \downarrow U_R$ is nonempty. Consider objects (γ, γ') and (δ, δ') in $\mathrm{ac}^K(A, R)$, and maps $\sigma: \beta \longrightarrow \gamma$, $\tau: \beta \longrightarrow \delta$ in $\mathrm{ac}^K(A)$. The pushout

$$\begin{tikzcd} B \arrow[r, "\tau"] \arrow[d, "\sigma"'] & D \arrow[d] \\ C \arrow[r] & C \cup_B D \end{tikzcd}$$

yields a finite K-cell complex $\alpha: A \longrightarrow C \cup_B D$. The maps γ' and δ' induce a map $\alpha': C \cup_B D \longrightarrow R^K(A)$ turning (α, α') into an object of $\mathrm{ac}^K(A, R)$. The cobase changes of σ and τ are maps $(\gamma, \gamma') \longrightarrow (\alpha, \alpha')$ and $(\delta, \delta') \longrightarrow (\alpha, \alpha')$ in $\mathrm{ac}^K(A, R)$. This implies that $\beta \downarrow U_R$ is connected. Hence U_R is final. Next we consider U_F. Let $\beta: A \longrightarrow B$ be an object of $\mathrm{ac}^K(A)$. By 3.19, $\beta \downarrow U_F$ is nonempty. Since maps in K-cell are monomorphisms, the union of two finite subcomplexes is again a finite subcomplex [6, 12.2.1]. Connectness follows.
A category is a preorder if there is at most one map between any two objects. Suppose that there exist two different maps $\sigma: B \longrightarrow C$ and $\tau: B \longrightarrow C$

from $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow F^K(A))$ to $(\gamma\colon A \longrightarrow C, \gamma'\colon C \longrightarrow F^K(A))$. By definition, we have $\gamma' \circ \sigma = \beta' = \gamma' \circ \tau$. Since γ' is a monomorphism by assumption, it follows that $\sigma = \tau$. Note that $\mathrm{ac}^K(A, F)$ is nonempty, since it contains the initial object (id_A, ι_A). Hence to prove that $\mathrm{ac}^K(A, F)$ is filtered, it remains to observe that any two objects have a common upper bound, given by the union of subcomplexes, cf. [6, 12.2.1] □

The colimits of Ψ_A, $\Psi_A \circ U_R$ and $\Psi_A \circ U_F$ are isomorphic via the canonical maps $\mathrm{colim}\Psi_A \circ U_R \longrightarrow \mathrm{colim}\Psi_A \longleftarrow \mathrm{colim}\Psi_A \circ U_F$ by Theorem 1 of [12, IX.3]. To have a natural comparison with the $\mathcal{V}$-functor R^K, we let $\Phi^K(A)$ be the colimit of the functor $\Psi_A \circ U_R\colon \mathrm{ac}^K(A, R) \longrightarrow \mathcal{V}$. There is a canonical map $\varphi_A\colon A \longrightarrow \Phi^K A$ induced by the object $(\mathrm{id}_A, \rho^K(A))$ of $\mathrm{ac}^K(A, R)$. Likewise there is a canonical map $\omega_A\colon \Phi^K(A) \longrightarrow R^K(A)$ induced by the maps β'.

Proposition 3.24. *Suppose that relative K-cell complexes are monomorphisms. Then $\Phi^K(A) \longrightarrow *$ is in K-inj, and $\varphi_A\colon A \longrightarrow \Phi^K(A)$ is a filtered colimit of finite relative K-cell complexes. Moreover, the assignment $A \longmapsto \Phi^K(A)$ defines a functor Φ^K, and the maps $A \xrightarrow{\varphi_A} \Phi^K(A)$ and $\Phi^K(A) \xrightarrow{\omega_A} R^K(A)$ are natural transformations $\mathrm{Id}_{\mathcal{V}} \xrightarrow{\varphi} \Phi^K \xrightarrow{\omega} R^K$ such that $\omega \circ \varphi = \rho^K$.*

Proof. The object $\Phi^K(A)$ is isomorphic to $\tilde{A} := \mathrm{colim}\Psi_A \circ U_F$. Hence to prove the first claim, it suffices to consider a lifting problem

$$\begin{tikzcd} sk \arrow[r, "f"] \arrow[d, "k"', "\sim"] & \tilde{A} \arrow[d] \\ tk \arrow[r] & * \end{tikzcd}$$

The object sk is finitely presentable, and $\tilde{A}$ is the colimit of a filtered diagram by 3.23. Hence f factors as $sk \xrightarrow{g} a \longrightarrow \tilde{A}$ for some finite subcomplex $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow F^K(A))$ in $\mathrm{ac}^K(A, F)$. Take the pushout of

$$\begin{tikzcd} sk \arrow[r, "g"] \arrow[d, "k"'] & B \arrow[d, "\tau"] \\ tk \arrow[r, "h"] & C \end{tikzcd}$$

and define $\gamma := \tau \circ \beta\colon A \longrightarrow C$. Since τ is the cobase change of $k \in K$, 3.19 implies that $\gamma\colon A \longrightarrow C$ is a finite subcomplex of ι_A such that $\tau\colon \beta \longrightarrow \gamma$ is a map in $\mathrm{ac}^K(A, F)$. The canonical map $C \longrightarrow \tilde{A}$ belongs to γ, and the map $tk \xrightarrow{h} C \longrightarrow \tilde{A}$ solves the lifting problem. Hence $\tilde{A} \longrightarrow *$ is in K-inj.
To prove the second assertion, let $\mathrm{ac}^K(A, F) \longrightarrow \pi_0\mathrm{ac}^K(A, F)$ be the canonical projection onto the set of connected components of $\mathrm{ac}^K(A, F)$. This functor is final, and the constant diagram $c_A\colon \mathrm{ac}^K(A, F) \longrightarrow \mathcal{V}$ with value A factors it. Since $\mathrm{ac}^K(A, F)$ is connected, it follows that $\mathrm{colim} c_A \cong A$. There

is a natural transformation of diagrams $c_A \longrightarrow \Psi_A \circ U_F$ with value β at (β, β'). The induced map $\operatorname{colim} c_A \longrightarrow \Phi^K(A)$ coincides with the canonical map $\varphi_A\colon A \longrightarrow \Phi^K(A)$ up to the isomorphism above. The proof of the remaining claim is clear and will be left to the reader. □

Corollary 3.25. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category satisfying the monoid axiom. If relative J'-cell complexes are monomorphisms, then*

$$A \xrightarrow{\varphi_A} \Phi^{J'}(A) \xrightarrow{\omega_A} R^{J'}(A)$$

are weak equivalences with fibrant codomains.

If $\mathcal{V}$ is pointed by $*$, any endo-$\mathcal{V}$-functor on $\mathcal{V}$ maps the point to the point. On the other hand, ι_* is not an isomorphism and there may be non-trivial finite subcomplexes of ι_*. So Φ^K is not a $\mathcal{V}$-functor in general. To define the stable model structure we need Φ to be "enriched with respect to spheres". This requires a natural map $T \otimes \Phi(A) \longrightarrow \Phi(T \otimes A)$ for some finitely presentable object T, such that

$$\begin{tikzcd} T \otimes \Phi^K(A) \arrow[r] \arrow[d, "T\otimes\omega_A"'] & \Phi^K(T \otimes A) \arrow[d, "\omega_{T\otimes A}"] \\ T \otimes R^K(A) \arrow[r, "\mathrm{sw}^R_T(A)"] & R^K(T \otimes A) \end{tikzcd}$$

is commutative. Here the lower horizontal map is the adjoint of

$$T \xrightarrow{\eta_A(T)} \mathcal{V}(A, T \otimes A) \xrightarrow{\hom^{R^K}_{A,T\otimes A}} \mathcal{V}(R^K(A), R^K(T \otimes A)).$$

More details can be found in Appendix A. Suppose $T \otimes -$ maps finite K-cell complexes relative to A to finite K-cell complexes relative to $T \otimes A$, for all $A \in \operatorname{Ob}\mathcal{V}$. Then tensoring with T defines a functor $\Theta_A\colon \mathrm{ac}^K(A, R) \longrightarrow \mathrm{ac}^K(T \otimes A, R)$ by sending $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow R^K(A))$ to

$$(T \otimes A \xrightarrow{T\otimes\beta} T \otimes B, T \otimes B \xrightarrow{T\otimes\beta'} T \otimes R^K(A) \xrightarrow{r(T,A)} R^K(T \otimes A)).$$

This functor induces a map

$$\begin{aligned} T \otimes \Phi^K(A) &= T \otimes \operatorname{colim}_{\mathrm{ac}^K(A,R)} \Psi_A \circ U_R \\ &\cong \operatorname{colim}_{\mathrm{ac}^K(A,R)} (T \otimes -) \circ \Psi_A \circ U_R \\ &= \operatorname{colim}_{\mathrm{ac}^K(A,R)} \Psi_{T\otimes A} \circ U_R \circ \Theta_A \\ &\longrightarrow \operatorname{colim}_{\mathrm{ac}(T\otimes A,R)} \Psi_{T\otimes A} \circ U_R \\ &= \Phi^K(T \otimes A) \end{aligned}$$

via the identity natural transformation $(T \otimes -) \circ \Psi_A \circ U_R \longrightarrow \Psi_{T\otimes A} \circ U_R \circ \Theta_A$.

Lemma 3.26. *Let $\mathcal{V}$ be a monoidal model category satisfying the monoid axiom. Suppose that $\mathcal{V}$ is weakly finitely generated, and that relative J'-cell complexes are monomorphisms. If A is an object of $\mathcal{V}$, the map $\theta_A: T \otimes \Phi^{J'}(A) \longrightarrow \Phi^{J'}(T \otimes A)$ is a weak equivalence making the diagram*

$$\begin{tikzcd} T \otimes \Phi^{J'}(A) \arrow[r, "\theta_A"] \arrow[d, "T\otimes\omega_A"'] & \Phi^{J'}(T \otimes A) \arrow[d, "\omega_{T\otimes A}"] \\ T \otimes R^{J'}(A) \arrow[r, "r(T,A)"] & R^{J'}(T \otimes A) \end{tikzcd}$$

commutative. Furthermore, θ_A is natural in A.

Proof. Commutativity and naturality follow by the construction of θ_A. Note that $\rho_{T\otimes A}$ and $T \otimes \rho_A^{J'} \in \mathrm{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell are weak equivalences. It follows that $r(T,A)$ is a weak equivalence since $r(T,A) \circ T \otimes \rho_A^{J'} = \rho_{T\otimes A}^{J'}$. The map $\varphi_{T\otimes A}$ is a weak equivalence, and $T \otimes \varphi_A$ is a filtered colimit of weak equivalences. Hence the vertical maps in the diagram are weak equivalences. □

Finally, we relate $\mathcal{V}$-functors and Φ^K in the case where $\mathbb{I}: \mathcal{C} \hookrightarrow \mathcal{V}$ is a full sub-$\mathcal{V}$-category and all objects in $\mathcal{C}$ are $\mathcal{V}$-finitely presentable.

Lemma 3.27. *Suppose that relative K-cell complexes are monomorphisms. Let $X: \mathcal{C} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$-functor and let $A \in \mathrm{Ob}\,\mathcal{C}$. There is an isomorphism*

$$\mathbb{I}_* X(\Phi^K(A)) \cong \mathrm{colim}(X \circ \Psi_A \circ U_R : \mathrm{ac}^K(A,R) \longrightarrow \mathcal{V})$$

which is natural in X.

Proof. To prove this, use the canonical expression of a $\mathcal{V}$-functor as a coend of representables 2.3, 3.23 and $\mathcal{V}$-finiteness of the objects in $\mathcal{C}$. Then

$$\mathbb{I}_* X(\Phi^K(A)) = \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, \Phi^K(A)) \otimes X(c) \cong \mathrm{colim} \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, \Psi_A \circ U_R) \otimes X(c),$$

and the claim follows. □

4 The pointwise model structure

Let $\mathcal{V}$ be a weakly finitely generated monoidal model category. If $\mathcal{C}$ is a small $\mathcal{V}$-category and the monoid axiom holds in $\mathcal{V}$, we introduce the pointwise model structure on $[\mathcal{C}, \mathcal{V}]$. Of particular interest are the cases where $\mathcal{C}$ is a full sub-$\mathcal{V}$-category, and $\mathcal{C}$ satisfies the following properties.

f0 Every object of $\mathcal{V}$ is a filtered colimit of objects in $\mathcal{C}$.

f1 Every object of $\mathcal{C}$ is $\mathcal{V}$-finitely presentable.

f2 The unit e is in $\mathcal{C}$, and $\mathcal{C}$ is closed under the monoidal product in $\mathcal{V}$.

4.1 The general case

Our pointwise notions of weak equivalences, fibrations and cofibrations are as follows.

Definition 4.1. A morphism f in $[\mathcal{C}, \mathcal{V}]$ is a

- *pointwise weak equivalence* if $f(c)$ is a weak equivalence in $\mathcal{V}$ for all $c \in \operatorname{Ob} \mathcal{C}$,
- *pointwise fibration* if $f(c)$ is a fibration in $\mathcal{V}$ for all $c \in \operatorname{Ob} \mathcal{C}$,
- *cofibration* if f has the left lifting property with respect to all pointwise acyclic fibrations.

Theorem 4.2. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Suppose the monoid axiom holds in $\mathcal{V}$. Then $[\mathcal{C}, \mathcal{V}]$, with the classes of maps in 4.1, is a weakly finitely generated model category.*

Proof. We will use [7, 2.1.19]. The category $[\mathcal{C}, \mathcal{V}]$ is bicomplete by 2.2. The class of pointwise weak equivalences is closed under retracts and satisfies the "two out of three" or saturation axiom. Let I be the generating cofibrations in $\mathcal{V}$, and J the generating acyclic cofibrations in $\mathcal{V}$. Let $\mathcal{P}_I$ be the set of maps

$$\{\mathcal{V}_{\mathcal{C}}(c,-) \otimes si \xrightarrow{\mathcal{V}_{\mathcal{C}}(c,-)\otimes i} \mathcal{V}_{\mathcal{C}}(c,-) \otimes ti \mid i \in I, c \in \operatorname{Ob} \mathcal{C}\}.$$

Likewise, let $\mathcal{P}_J$ denote the set of maps

$$\{\mathcal{V}_{\mathcal{C}}(c,-) \otimes sj \xrightarrow{\mathcal{V}_{\mathcal{C}}(c,-)\otimes j} \mathcal{V}_{\mathcal{C}}(c,-) \otimes tj \mid j \in J, c \in \operatorname{Ob} \mathcal{C}\}.$$

Since $\mathcal{V}$ is cofibrantly generated, it follows from adjointness that $\mathcal{P}_J$-inj coincides with the class of pointwise fibrations, and $\mathcal{P}_I$-inj coincides with the class of pointwise acyclic fibrations. If A is finitely presentable (small) in $\mathcal{V}$, then $\mathcal{V}_{\mathcal{C}}(c,-) \otimes A$ is finitely presentable (small) in $[\mathcal{C}, \mathcal{V}]$ for any $c \in \operatorname{Ob} \mathcal{C}$, since colimits are computed pointwise according to 2.2. Hence the smallness conditions listed in 3.4 are satisfied. It remains to show that maps in $\mathcal{P}_J$-cell are pointwise weak equivalences. Every map in $\mathcal{P}_J$-cell is pointwise a map in $J \otimes \mathcal{V}$-cell, and the latter class consists of weak equivalences by the monoid axiom. $\square$

We refer to the model structure in 4.2 as the *pointwise* model structure. Note that the evaluation functor preserves fibrations and acyclic fibrations.

Lemma 4.3. *Suppose the pointwise model structure exists. Then the functor from $\mathcal{V}$ to $[\mathcal{C}, \mathcal{V}]$ which maps A to $\mathcal{V}_{\mathcal{C}}(c,-) \otimes A$ is a left Quillen functor, with right adjoint Ev_c for all $c \in \operatorname{Ob} \mathcal{C}$.*

If the unit in $\mathcal{V}$ is cofibrant, then the representable functors are cofibrant in the pointwise model structure.

Theorem 4.4. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.2. Then the pointwise model structure gives $[\mathcal{C},\mathcal{V}]$ the structure of a $\mathcal{V}$-model category. Likewise, $[\mathcal{C},\mathcal{V}]$ is a monoidal $\mathcal{V}$-model category provided $\mathcal{C}$ is a symmetric monoidal $\mathcal{V}$-category, and the monoid axiom holds.*

Proof. Recall from 2.4 that $[\mathcal{C},\mathcal{V}]$ is a closed $\mathcal{V}$-module. By [7, 4.2.5] it suffices to check the following conditions.

- Let $\mathcal{V}_{\mathcal{C}}(c,-)\otimes i\colon \mathcal{V}_{\mathcal{C}}(c,-)\otimes si \rightarrowtail \mathcal{V}_{\mathcal{C}}(c,-)\otimes ti$ be a map in $\mathcal{P}_I$, and let $j\colon sj \rightarrowtail tj$ be a map in I. Then the pushout product $(\mathcal{V}_{\mathcal{C}}(c,-)\otimes i)\Box j$ is a cofibration.
- If either i or j in the above sentence are generating acyclic cofibrations, then $(\mathcal{V}_{\mathcal{C}}(c,-)\otimes i)\Box j$ is a pointwise acyclic cofibration.

Since $\mathcal{V}_{\mathcal{C}}(c,-)\otimes -$ is a left adjoint, the pushout product map in question is of the form $\mathcal{V}_{\mathcal{C}}(c,-)\otimes(i\Box j)$. Hence the conditions hold, because $\mathcal{V}$ is a monoidal model category and 4.3 holds.
The monoidality statement is proven similarly using [7, 4.2.5]. Note from 2.7 and the compatibility of $\wedge$ and $\otimes$, that the pushout product map of $\mathcal{V}_{\mathcal{C}}(c,-)\otimes i$ and $\mathcal{V}_{\mathcal{C}}(d,-)\otimes j$ is isomorphic to $\mathcal{V}_{\mathcal{C}}(c\diamond d,-)\otimes(i\Box j)$ where $\diamond$ denotes the monoidal product in $\mathcal{C}$. Let u be the unit of $\mathcal{C}$. Then $\mathcal{V}_{\mathcal{C}}(u,-)\otimes -$ is a strict symmetric monoidal functor and a left Quillen functor by 4.3.
It remains to prove the monoid axiom. Abbreviate $[\mathcal{C},\mathcal{V}]$ by $\mathcal{F}$. Since $\mathcal{F}$ is cofibrantly generated, it suffices to check that every map in the class $\mathcal{P}_J\wedge\mathcal{F}$-cell is a pointwise weak equivalence. Let $c\in\operatorname{Ob}\mathcal{C}$, $j\in J$ and $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ a $\mathcal{V}$-functor. Then $(\mathcal{V}(c,-)\otimes j)\wedge X$ coincides up to isomorphism with the map $(\mathcal{V}(c,-)\wedge X)\otimes j$. In particular, $((\mathcal{V}(c,-)\wedge X)\otimes j)(d)=(\mathcal{V}(c,-)\wedge X)(d)\otimes j$ is contained in $J\otimes\mathcal{V}$ for every $d\in\operatorname{Ob}\mathcal{C}$, $X\in\operatorname{Ob}\mathcal{F}$. For a map f in $\mathcal{P}_J\otimes\mathcal{F}$-cell, $f(d)$ belongs to $J\otimes\mathcal{V}$-cell because colimits are formed pointwise. Since the monoid axiom holds in $\mathcal{V}$, f is a pointwise weak equivalence. □

Remark 4.5. Via 2.12, the pointwise model structure on T-spectra corresponds to the pointwise model structure on $[T\mathrm{Sph},\mathcal{V}]$.

For a discussion of properness of the pointwise model structure, we introduce the following definition. Let $\operatorname{Cof}(\mathcal{V})$ denote the class of cofibrations in $\mathcal{V}$.

Definition 4.6. A monoidal model category $\mathcal{V}$ is *strongly left proper* if the cobase change of a weak equivalence along any map in $\operatorname{Cof}(\mathcal{V})\otimes\mathcal{V}$-cell is again a weak equivalence.

Strongly left proper monoidal model categories are left proper. If a model category has only cofibrant objects, it is left proper. If a monoidal model category has only cofibrant objects, it is strongly left proper. The relevance of 4.6 is explained by the following lemma.

Lemma 4.7. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.2. If f is a cofibration in $[\mathcal{C},\mathcal{V}]$, then $f(c)$ is a retract of a map in $\operatorname{Cof}(\mathcal{V})\otimes\mathcal{V}$-cell for every $c\in\operatorname{Ob}\mathcal{C}$.*

Proof. Any cofibration in $[\mathcal{C},\mathcal{V}]$ is a retract of a relative $\mathcal{P}_I$-cell complex, and $\mathcal{V}_{\mathcal{C}}(d,c)\otimes i$ is a map in $I\otimes\mathcal{V}$-cell for every $c\in \mathrm{Ob}\,\mathcal{C}$. The claim follows. □

Corollary 4.8. *The pointwise model structure on $[\mathcal{C},\mathcal{V}]$ is right proper if $\mathcal{V}$ is right proper, and left proper if $\mathcal{V}$ is strongly left proper.*

4.2 The subcategory case

The goal in this section is to develop conditions under which smashing with cofibrant $\mathcal{V}$-functors preserves pointwise weak equivalences. This fact will be used to prove the monoid axiom for the stable model structure. Recall that if $\mathcal{C}$ is a full sub-$\mathcal{V}$-category of $\mathcal{V}$, the left Kan extension along the inclusion functor $\mathbb{I}\colon \mathcal{C}\hookrightarrow \mathcal{V}$ is

$$\mathbb{I}_*X = \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c,-)\otimes X(c).$$

Lemma 4.9. *Assume $\mathcal{V}$ is a weakly finitely generated monoidal model category, and $\mathcal{C}$ is a small and full sub-$\mathcal{V}$-category satisfying* **f0** *and* **f1**. *If f is a pointwise weak equivalence in $[\mathcal{C},\mathcal{V}]$, then so is $\mathbb{I}_*f$.*

Proof. We have to check that, for any $A\in\mathrm{Ob}\,\mathcal{V}$, $\mathbb{I}_*f(A)$ is a pointwise weak equivalence. Write A as the colimit of $C\colon\mathcal{I}\longrightarrow\mathcal{C}$ for $\mathcal{I}$ filtering by **f0**. Since coends commute with colimits and **f1** holds, it follows that

$$\begin{aligned}
\mathbb{I}_*f(A) &= \int^{\mathrm{Ob}\,(\mathcal{C})} \mathcal{V}(B,A)\otimes f(B)\\
&\cong \int^{\mathrm{Ob}\,(\mathcal{C})} \mathcal{V}(B,\mathrm{colim}_{i\in\mathcal{I}}C)\otimes f(B)\\
&\cong \mathrm{colim}_{i\in\mathcal{I}}\int^{\mathrm{Ob}\,(\mathcal{C})} \mathcal{V}(B,C(i))\otimes f(B)\\
&\cong \mathrm{colim}_{i\in\mathcal{I}}\; f\circ C.
\end{aligned}$$

Note that, since f is a pointwise weak equivalence, $\mathbb{I}_*f(A)$ is a filtered colimit of weak equivalences and hence a weak equivalence by 3.5. □

Corollary 4.10. *Let $\mathcal{V}$ and $\mathcal{C}$ be as in 4.9. If $\mathcal{C}$ satisfies* **f2** *and f is a pointwise weak equivalence in $[\mathcal{C},\mathcal{V}]$, then $f\wedge\mathcal{V}(c,-)$ is a pointwise weak equivalence for all $c\in\mathrm{Ob}\,\mathcal{C}$.*

Proof. Axiom **f2** implies that the smash product of two enriched functors exists. By 2.8, $f\wedge\mathcal{V}(c,-)$ is isomorphic to $\mathbb{I}_*f\circ\mathcal{V}(c,-)$. Since $\mathbb{I}_*f$ is a pointwise weak equivalence by 4.9, $\mathbb{I}_*f(\mathcal{V}(c,d))$ is a weak equivalence for all $d\in\mathrm{Ob}\,\mathcal{C}$. □

Theorem 4.11. *Let $\mathcal{V}$ and $\mathcal{C}$ be as in 4.10. Assume $\mathcal{V}$ satisfies the monoid axiom and is strongly left proper, and tensoring with the domains and the codomains of the generating cofibrations in $\mathcal{V}$ preserves weak equivalences. Then smashing with a cofibrant object in $[\mathcal{C},\mathcal{V}]$ preserves pointwise weak equivalences.*

Proof. Let $f\colon X \longrightarrow Y$ be a pointwise weak equivalence, and let Z be a cofibrant $\mathcal{V}$-functor. Since Z is cofibrant, it is a retract of some $\mathcal{V}$-functor Z', such that $* \longrightarrow Z'$ is the sequential composition of a sequence

$$* = Z'_0 \xrightarrow{g_0} Z'_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} Z'_n \xrightarrow{g_n} \cdots,$$

where g_n is the cobase change of a coproduct of maps in $\mathcal{P}_I$. It suffices to consider sequences indexed by the natural numbers, since the domains of the maps in $\mathcal{P}_I$ are finitely presentable. The map $f \wedge Z$ is a retract of $f \wedge Z'$, hence it remains to prove that the latter is a pointwise weak equivalence. We will prove this by induction on n. Consider the following diagram.

$$\begin{tikzcd}
X \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes ti_m \arrow[d, "f\wedge \coprod \mathcal{V}(c_m,-)\otimes ti_m"] & X \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes si_m \arrow[l, "X\wedge\coprod \mathcal{V}(c_m,-)\otimes i_m"'] \arrow[r] \arrow[d, "f\wedge \coprod \mathcal{V}(c_m,-)\otimes si_m"] & X \wedge Z_n \arrow[d, "f\wedge Z_n"] \\
Y \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes ti_m & Y \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes si_m \arrow[l, "X\wedge\coprod \mathcal{V}(c_m,-)\otimes i_m"'] \arrow[r] & Y \wedge Z_n.
\end{tikzcd}$$

The map induced on the pushouts of the upper and lower row is $f \wedge Z_{n+1}$. Suppose $f \wedge Z'_n$ is a pointwise weak equivalence. By 4.10 and the hypothesis on I, it follows that both of the other vertical maps are pointwise weak equivalences. The horizontal maps on the left hand side are not necessarily cofibrations. However, evaluation at any object gives maps in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell. Since $\mathcal{V}$ is strongly left proper, this implies that the map induced on the pushouts – which is computed pointwise – is a pointwise weak equivalence. □

Remark 4.12. Let $\mathcal{V}$ be a weakly finitely generated monoidal model category. Suppose that $-\otimes si$ and $-\otimes ti$ preserve weak equivalences for every $i \in I$, and $\mathcal{V}$ is strongly left proper. Then $-\otimes A$ preserves weak equivalences for any cofibrant object A in $\mathcal{V}$, cp. 4.11. We say a strongly left proper monoidal model category is *strongly monoidal* if $-\otimes A$ preserves weak equivalences for A either cofibrant or a domain or codomain of the generating cofibrations, if they exist. A monoidal model category in which every object is cofibrant satisfies this condition and also the monoid axiom.

5 The homotopy functor model structure

Suppose $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor of categories with chosen subclasses of weak equivalences. If F maps weak equivalences to weak equivalences, then F is called a *homotopy functor*. As a first step towards the stable model structure on enriched functors, we define a model structure in which every enriched functor is weakly equivalent to a homotopy functor.

Let $\mathcal{V}$ be a weakly finitely generated strongly monoidal **sSet**-model category. Additionally, assume the following for $\mathcal{V}$: the monoid axiom holds, the unit is cofibrant, Δ^1 is finitely presentable in $\mathcal{V}$, filtered colimits commute with pullbacks, and cofibrations are monomorphisms. The simplicial structure is used for the simplicial mapping cylinder construction. Let $\mathbb{I}: \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ be a small full sub-$\mathcal{V}$-category such that the following axioms hold.

f1 Every object of $\mathbf{f}\mathcal{V}$ is $\mathcal{V}$-finitely presentable.

f2 The unit e is in $\mathbf{f}\mathcal{V}$, and $\mathbf{f}\mathcal{V}$ is closed under the monoidal product.

f3 If $tj \xleftarrow{\sim} sj \longrightarrow v$ is a diagram in $\mathcal{V}$ where $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$ and $j \in J'$, then the pushout $tj \cup_{sj} v$ is in $\mathbf{f}\mathcal{V}$.

Consider also the following additional axioms.

f0 Every object of $\mathcal{V}$ is a filtered colimit of objects in $\mathbf{f}\mathcal{V}$.

f4 All objects in $\mathbf{f}\mathcal{V}$ are cofibrant.

f5 The simplicial mapping cylinder exists in $\mathbf{f}\mathcal{V}$.

Objects in $\mathbf{f}\mathcal{V}$ will usually be denoted by small letters, since $\mathbf{f}\mathcal{V}$ should be thought of as a category of small objects. Let $\mathcal{F}$ be short for $[\mathbf{f}\mathcal{V}, \mathcal{V}]$. Recall the left Kan extension $\mathbb{I}_* X$ of $X \in \operatorname{Ob} \mathcal{F}$ along $\mathbb{I}: \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$, cp. 2.5.

5.1 Equivalences of homotopy functors

Let $\Phi: \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$ denote the functor induced by the fibrant replacement functor $\Phi^{J'}$ from 3.3.3. In general it is not a $\mathcal{V}$-functor. Denote by $\hbar(X): \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$ the composition $\mathbb{I}_* X \circ \Phi$, and by $\hbar: \mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ the induced functor. There is a natural transformation $X \longrightarrow \hbar(X)$ induced by the canonical maps $\varphi_v: v \longrightarrow \Phi(v)$ where v varies through the set of objects in $\mathbf{f}\mathcal{V}$.

Lemma 5.1. *The functor $\hbar$ commutes with colimits and the action of $\mathcal{V}$. The natural transformations $X \longrightarrow \hbar(X)$ define a natural transformation from the forgetful functor $\mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ to $\hbar$.*

Definition 5.2. A map f is an *hf-equivalence* if $\hbar(f)(v)$ is a weak equivalence in $\mathcal{V}$ for all $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.

Lemma 5.3. *Any pointwise weak equivalences is an hf-equivalence. The class of hf-equivalences is saturated.*

Proof. The first statement follows as in 4.9, since $\hbar(f)(v) = \mathbb{I}_*(f)(\Phi(v))$ is a filtered colimit of weak equivalences provided f is a pointwise weak equivalence. The second statement follows from 5.1 and the analogous fact in $\mathcal{V}$. □

Lemma 5.4. *Let f be a cofibration in $\mathcal{F}$. Then $\hbar(f)(v)$ is a retract of a map in $\operatorname{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell for every $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.*

Proof. If $\mathcal{V}(v, -) \otimes i$ is a generating cofibration, then $\hbar(\mathcal{V}(v, w) \otimes i)$ coincides with $\mathcal{V}(v, \Phi(w)) \otimes i$, which is in $\operatorname{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell. The general case follows since $\hbar$ commutes with colimits. □

5.2 Fibrations of homotopy functors

If $\phi\colon v \longrightarrow w$ is an acyclic cofibration in $\mathbf{f}\mathcal{V}$, the simplicial mapping cylinder factors $\mathcal{V}(\phi,-)\colon \mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ as a cofibration $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ followed by a simplicial homotopy equivalence. This uses that $\mathcal{V}(w,-)$ is a cofibrant functor (since the unit in $\mathcal{V}$ is cofibrant), and that $\mathcal{V}$ (and hence $\mathcal{F}$ by 4.4) is a **sSet**-model category. Take a generating cofibration $i\colon si \longrightarrow ti$ in $\mathcal{V}$ and form the pushout product

$$c_\phi \square i\colon \mathcal{V}(w,-)\otimes ti \cup_{\mathcal{V}(w,-)\otimes si} C_\phi \otimes si \longrightarrow C_\phi \otimes ti.$$

Let $\mathcal{H}$ denote the set $\{c_\phi \square i\}$, where ϕ runs through the set of acyclic cofibrations in $\mathbf{f}\mathcal{V}$ and i runs through the set I of generating cofibrations in $\mathcal{V}$.

Definition 5.5. A map is an *hf-fibration* if it is a pointwise fibration having the right lifting property with respect to $\mathcal{H}$.

Lemma 5.6. *Let $f\colon X \longrightarrow Y$ be a pointwise fibration. Then f is an hf-fibration if and only if the following diagram is a homotopy pullback square in $\mathcal{V}$ for every acyclic cofibration $\phi\colon v \overset{\sim}{\rightarrowtail} w$ in $\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} X(v) \arrow[r, "X(\phi)"] \arrow[d, "f(v)"'] & X(w) \arrow[d, "f(w)"] \\ Y(v) \arrow[r, "Y(\phi)"] & Y(w) \end{tikzcd}$$

Proof. Let $\mathcal{V}_\mathcal{F}(X,Y)$ denote the $\mathcal{V}$-object of maps in $\mathcal{F}$ from X to Y. For a map of $\mathcal{V}$-functors $f\colon X \longrightarrow Y$, the square in the statement of the lemma is naturally isomorphic, by the Yoneda lemma 2.1, to the square

$$\begin{tikzcd} \mathcal{V}_\mathcal{F}(\mathcal{V}(v,-),X) \arrow[r, "\mathcal{V}_\mathcal{F}(\mathcal{V}(\phi,-){,}X)"] \arrow[d, "\mathcal{V}_\mathcal{F}(\mathcal{V}(v,-){,}f)"'] & \mathcal{V}_\mathcal{F}(\mathcal{V}(w,-),X) \arrow[d, "\mathcal{V}_\mathcal{F}(\mathcal{V}(w,-){,}f)"] \\ \mathcal{V}_\mathcal{F}(\mathcal{V}(v,-),Y) \arrow[r, "\mathcal{V}_\mathcal{F}(\mathcal{V}(\phi,-){,}Y)"] & \mathcal{V}_\mathcal{F}(\mathcal{V}(w,-),Y). \end{tikzcd}$$

The factorization of $\mathcal{V}(\phi,-)$ as a cofibration $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ followed by a simplicial homotopy equivalence $C_\phi \longrightarrow \mathcal{V}(v,-)$ induces a factorization of the square above into two squares. Since $\mathcal{V}_\mathcal{F}(-,X)$ preserves simplicial homotopy equivalences by 2.11, which are pointwise weak equivalences by 3.10, the square above is a homotopy pullback square if and only if

$$\begin{tikzcd} \mathcal{V}_\mathcal{F}(C_\phi,X) \arrow[r, "\mathcal{V}_\mathcal{F}(c_\phi{,}X)"] \arrow[d, "\mathcal{V}_\mathcal{F}(C_\phi{,}f)"'] & \mathcal{V}_\mathcal{F}(\mathcal{V}(w,-),X) \arrow[d, "\mathcal{V}_\mathcal{F}(\mathcal{V}(w,-){,}f)"] \\ \mathcal{V}_\mathcal{F}(C_\phi,Y) \arrow[r, "\mathcal{V}_\mathcal{F}(c_\phi{,}Y)"] & \mathcal{V}_\mathcal{F}(\mathcal{V}(w,-),Y) \end{tikzcd}$$

is a homotopy pullback square. If $f: X \longrightarrow Y$ is a pointwise fibration, the induced map $g: \mathcal{V}_{\mathcal{F}}(C_\phi, X) \longrightarrow \mathcal{V}_{\mathcal{F}}(C_\phi, Y) \times_{\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y)} \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),X)$ is a fibration in $\mathcal{V}$. Here we use **f2**, so $\mathcal{F}$ is a monoidal model category when equipped with the pointwise model structure, see 4.4. The square in 5.6 is therefore a homotopy pullback square if and only if g has the right lifting property with respect to the generating cofibrations in $\mathcal{V}$. By adjointness, this holds if and only if f has the right lifting property with respect to $\mathcal{H}$. ☐

Lemma 5.7. *Let $f: X \longrightarrow Y$ be an hf-fibration. Then*

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \hbar(X)(v) \arrow[d, "\hbar(f)(v)"] \\ Y(v) \arrow[r] & \hbar(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Proof. Let $f: X \twoheadrightarrow Y$ be a pointwise fibration of pointwise fibrant functors. From 3.27, $\hbar(f)(v)$ is a filtered colimit of fibrations of fibrant objects in $\mathcal{V}$, and therefore a fibration of fibrant objects by 3.5. This uses properties **f1** and **f3**. The square in the lemma is therefore a homotopy pullback square if and only if $X(v) \longrightarrow Y(v) \times_{\mathrm{colim} Y(a)} \mathrm{colim} X(a)$ is a weak equivalence in $\mathcal{V}$. Up to isomorphism, the colimit is taken over the category of finite subcomplexes $(\alpha: v \overset{\sim}{\rightarrowtail} a, \alpha': a \overset{\sim}{\rightarrowtail} Fv)$ in $\mathrm{Ob}\,\mathrm{ac}^{J'}(v,F)$. Note that the colimit is filtered by 3.23. Filtered colimits commute with pullbacks in $\mathcal{V}$ by assumption, so the map in question is a filtered colimit of maps $X(v) \longrightarrow Y(v) \times_{Y(a)} X(a)$ for acyclic cofibrations $\alpha: v \overset{\sim}{\rightarrowtail} a$ in $\mathbf{f}\mathcal{V}$. If f is an hf-fibration, then by 5.6 the map in question is a filtered colimit of weak equivalences in $\mathcal{V}$, and hence a weak equivalence.

If $f: X \twoheadrightarrow Y$ is any pointwise fibration, use the factorizations in the pointwise model structure to construct a commutative square

$$\begin{tikzcd} X \arrow[r, tail, "\sim", "g"'] \arrow[d, two heads, "f"'] & X' \arrow[d, two heads, "f'"] \\ Y \arrow[r, tail, "\sim", "h"'] & Y' \end{tikzcd}$$

for f' a pointwise fibration of pointwise fibrant functors, and g and f pointwise acyclic cofibrations. Note that f is an hf-fibration if and only f' is, see 3.13 and 5.6. The maps g and h are pointwise weak equivalences, hence $\hbar(g)(v)$ and $\hbar(h)(v)$ are weak equivalences in $\mathcal{V}$ for every v. The square in question is therefore a homotopy pullback square by the previous case. ☐

Corollary 5.8. *A map is an hf-fibration and an hf-equivalence if and only if it is a pointwise acyclic fibration.*

Proof. If f is an hf-fibration, it is a pointwise fibration by definition. If f is also an hf-equivalence, it is a pointwise weak equivalence by 3.13 and 5.7. A pointwise acyclic fibration is an hf-fibration by 3.13 and 5.6. $\square$

5.3 The homotopy functor theorem

Before we prove the existence of the homotopy functor model structure, let us first consider the maps in $\mathcal{H}$-cell.

Lemma 5.9. *The maps in $\mathcal{H}$-cell are hf-equivalences.*

Proof. Let $\phi\colon v \xrightarrow{\sim} w$ be an acyclic cofibration in $\mathbf{f}\mathcal{V}$. Then the induced map $\mathcal{V}(\phi,-)\colon \mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ is an hf-equivalence, because $\hbar(\mathcal{V}(\phi,-))(v)$ is naturally isomorphic to $\mathcal{V}(\phi,\Phi(v))$, ϕ is an acyclic cofibration and $\Phi(v)$ is fibrant. Pointwise weak equivalences are hf-equivalences by 5.3, so the map $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ is an hf-equivalence.
Let $i\colon si \rightarrowtail ti$ be a generating cofibration in $\mathcal{V}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{V}(w,-)\otimes si \arrow[r, "c_\phi\otimes si"] \arrow[d, "\mathcal{V}(w,-)\otimes i"'] & C_\phi\otimes si \arrow[d, "C_\phi\otimes i"] \\
\mathcal{V}(w,-)\otimes ti \arrow[r, "c_\phi\otimes ti"] & C_\phi\otimes ti
\end{tikzcd}$$

and the pushout product map $c_\phi\Box i$. The functor $\hbar$ commutes with pushouts and the action of $\mathcal{V}$ by 5.1, so $\hbar(c_\phi\Box i)(u)$ is the pushout product map obtained from $\hbar(c_\phi)(u)$ and i. Since $\mathcal{V}$ is strongly monoidal, it follows that $\hbar(c_\phi)(u)\Box i$ is a weak equivalence. Hence the maps in $\mathcal{H}$ are hf-equivalences. The general case of a map in $\mathcal{H}$-cell follows using similar arguments and 5.4. $\square$

Theorem 5.10. *Let $\mathcal{V}$ be a weakly finitely generated monoidal* **sSet***-model category, and let* $\mathbf{f}\mathcal{V}$ *be a full sub-$\mathcal{V}$-category satisfying* **f1**, **f2** *and* **f3**. *Suppose the monoid axiom holds in $\mathcal{V}$, pullbacks commute with filtered colimits in $\mathcal{V}$, and Δ^1 is finitely presentable in $\mathcal{V}$. Suppose also that $\mathcal{V}$ is strongly monoidal, and that cofibrations in $\mathcal{V}$ are monomorphisms. Then $\mathcal{F}$ is a weakly finitely generated model category, with hf-equivalences as weak equivalences, hf-fibrations as fibrations, and cofibrations as cofibrations.*

Proof. Again we use [7, 2.1.19]. The set of generating cofibrations is $\mathcal{P}_I$, and the set of generating acyclic cofibrations is the union $\mathcal{P}_J\cup\mathcal{H}$. It is clear that the class of hf-equivalences is saturated and closed under retracts. It is also clear that the domains of the maps in $\mathcal{H}$ are finitely presentable, because finitely presentable objects are closed under pushouts and tensoring with finitely presentable objects. Here we use that Δ^1 is finitely presentable in $\mathcal{V}$. The other properties which have to be checked are either obvious or follow from 5.8, 5.9 and the corresponding fact for the pointwise model structure 4.2. $\square$

The model structure in 5.10 is called the *homotopy functor* model structure. To emphasize the model structure, we use the notation $\mathcal{F}_{\mathrm{hf}}$. An hf-equivalence is denoted by $\xrightarrow{\sim \mathrm{hf}}$ and an hf-fibration by $\xrightarrow{\mathrm{hf}}\!\!\!\!\rightarrow$. Likewise, we use the notations $\mathcal{F}_{\mathrm{pt}}$, $\xrightarrow{\sim \mathrm{pt}}$ and $\xrightarrow{\mathrm{pt}}\!\!\!\!\rightarrow$ for the pointwise model structure.

LEMMA 5.11. *The identity induces a left Quillen functor* $\mathrm{Id}_{\mathcal{F}}\colon \mathcal{F}_{\mathrm{pt}} \longrightarrow \mathcal{F}_{\mathrm{hf}}$.

LEMMA 5.12. *Let $\mathcal{V}$ and $\mathbf{f}\mathcal{V}$ be as in 5.10, and assume $\mathbf{f}\mathcal{V}$ satisfies* **f4**. *Then $\mathcal{F}_{\mathrm{hf}}$ is a monoidal $\mathcal{F}_{\mathrm{pt}}$-model category.*

Proof. Condition **f2** is used to construct the smash product on $\mathcal{F}$, and 4.4 holds. To prove that the homotopy functor model structure is monoidal, it suffices to show that the pushout product map of a map $c_\phi \square i$ in $\mathcal{H}$ (where $\phi\colon v \overset{\sim}{\rightarrowtail} w$) and a generating cofibration $\mathcal{V}(u,-)\otimes j$ is an hf-equivalence. It is straightforward to check that this pushout product map coincides with the pushout product map $c_{\phi\otimes u}\square f$ where f is the pushout product map in $\mathcal{V}$ of i and j. Since $\mathcal{V}$ is a monoidal model category and $\mathbf{f}\mathcal{V}$ satisfies **f2** and **f4**, $\phi\otimes u$ is an acyclic cofibration in $\mathbf{f}\mathcal{V}$. Hence the map in question is an hf-equivalence. $\square$

LEMMA 5.13. *The homotopy functor model structure is left proper. If $\mathcal{V}$ is right proper, then the homotopy functor model structure is right proper.*

Proof. For left properness, let $i\colon Y \longrightarrow Z\cup_X Y$ be the cobase change of an hf-equivalence $g\colon X \xrightarrow{\sim \mathrm{hf}} Z$ along a cofibration $f\colon X \rightarrowtail Y$. Factor g as a cofibration $h\colon X \rightarrowtail T$, followed by a pointwise acyclic fibration $p\colon T \overset{\sim}{\twoheadrightarrow} Z$. Then g is an hf-equivalence, hence an acyclic cofibration in the homotopy functor model structure. These maps are closed under cobase changes, so i factors as an acyclic cofibration, followed by the cobase change of p. The latter is a pointwise weak equivalence, since the cobase change of f along g is a cofibration and the pointwise model structure is left proper by 4.8 provided $\mathcal{V}$ is strongly left proper.

A slightly stronger property than right properness holds. Consider the maps $f\colon X \overset{\mathrm{pt}}{\twoheadrightarrow} Z$ and $g\colon Y \xrightarrow{\sim \mathrm{hf}} Z$. We claim the base change i of g along f is an hf-equivalence. Let us shorten the notation by setting $R = R^{J'}$. To prove that i is an hf-equivalence, factor $Rf\colon RX \longrightarrow RZ$ as $h\colon RX \overset{\sim \mathrm{pt}}{\rightarrowtail} T$ followed by $p\colon T \overset{\mathrm{pt}}{\twoheadrightarrow} RZ$. Then $\hbar(p)(v)$ is a fibration of fibrant objects for any v by 3.5. Moreover, $\hbar(h)(v)$ and $\hbar(\rho_{Z(v)})\colon \hbar(Z)(v) \longrightarrow \hbar(RZ)(v)$ are weak equivalences for any v. Hence the base change of the weak equivalence $\hbar(\rho_{Z(v)})\circ \hbar(g)(v)$ along $\hbar(b)(v)$ is a weak equivalence, using that $\mathcal{V}$ is right proper. Note that the base change map factors as $\hbar(h)(v)$ composed with the base change of $\hbar(g)(v)$ along $\hbar(f)(v)$, i.e. $\hbar(i)(v)$ since pullbacks commute with filtered colimits. It follows that i is an hf-equivalence. $\square$

5.4 Homotopy functors

We end this section with a discussion of homotopy functors. If **f4** and **f5** hold, then a fibrant $\mathcal{V}$-functor X in $\mathcal{F}_{\mathrm{hf}}$ is a homotopy functor for the following reasons: By 5.6, X maps acyclic cofibrations to weak equivalences. Using **f4** and **f5**, every weak equivalence in $\mathbf{f}\mathcal{V}$ can be factored as an acyclic cofibration in $\mathbf{f}\mathcal{V}$, followed by a simplicial homotopy equivalence in $\mathbf{f}\mathcal{V}$. It follows from 2.11 that X preserves arbitrary weak equivalences in $\mathbf{f}\mathcal{V}$. Conversely, any $\mathcal{V}$-functor which is pointwise fibrant and a homotopy functor is fibrant in $\mathcal{F}_{\mathrm{hf}}$. Therefore we regard the expressions "pointwise fibrant homotopy functor" and "fibrant in $\mathcal{F}_{\mathrm{hf}}$" as synonymous. Next we define a fibrant replacement functor in $\mathcal{F}_{\mathrm{hf}}$ which allows to replace our definition of hf-equivalence by a better one.

Definition 5.14. For $X \in \mathrm{Ob}\,\mathcal{F}$, define X^h as the composition $\mathbb{I}_* X \circ R$, where $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ is the inclusion and $R := R^{J'}$ is the fibrant replacement $\mathcal{V}$-functor constructed in 3.3.2.

For pointed simplicial sets, Lydakis [11, 8.6] uses the singular complex applied to the geometric realization as an enriched fibrant replacement functor. This functor preserves fibrations, weak equivalences and finite limits.

Lemma 5.15. *The map $X \longrightarrow X^h$ is a $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors, and extends to a natural transformation $\mathrm{Id}_{\mathcal{F}} \longrightarrow (-)^h$. The functor $(-)^h$ commutes with colimits and the action of $\mathcal{V}$.*

Proof. The first two statements follow from 3.25 and properties of enriched Kan extension. For the last statement, use that coends commute with colimits and the action of $\mathcal{V}$, which are pointwise constructions. □

Lemma 5.16. *Assume* **f4** *and let $X \in \mathrm{Ob}\,\mathcal{F}$ be cofibrant. For every object v in $\mathbf{f}\mathcal{V}$, the weak equivalence $\omega_v\colon \Phi(v) \xrightarrow{\sim} R(v)$ induces a weak equivalence $\hbar(X)(v) \longrightarrow X^h(v)$.*

Proof. Since $\mathcal{V}$ is monoidal, **f4** implies that $\mathcal{V}(w, \Phi v) \longrightarrow \mathcal{V}(w, Rv)$ induced by the canonical weak equivalence $\omega_v\colon \Phi v \xrightarrow{\sim} R(v)$ is a weak equivalence for every $w \in \mathbf{f}\mathcal{V}$. Now express X as a retract of a $\mathcal{P}_I$-cell complex. The lemma follows then by induction, because $\mathcal{V}$ is strongly monoidal. □

Corollary 5.17. *Assume* **f4** *holds and X is cofibrant. Then $R \circ X^h$ is a pointwise fibrant homotopy functor and $X \longrightarrow R \circ X^h$ is an hf-equivalence. A map f of cofibrant functors is an hf-equivalence if and only if f^h is a pointwise weak equivalence.*

Proof. The second claim follows directly from 5.16, while the first claim requires just a slight variation of the proof of 5.16. □

Lemma 5.18. *Suppose $\mathcal{C}$ satisfies* **f0** *and* **f4**. *For any $X \in \mathrm{Ob}\,\mathcal{V}$, the canonical map $\hbar(X)(v) \longrightarrow X^h(v)$ induced by the weak equivalence $\omega_v\colon \Phi(v) \xrightarrow{\sim} R(v)$ is then a weak equivalence for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Note that 4.9 holds. Hence, by a cofibrant replacement in $\mathcal{F}_{\mathrm{pt}}$, it suffices to consider cofibrant functors. This case follows from 5.16. □

Corollary 5.19. *Assume* **f0** *and* **f4** *hold. A map f is then an hf-equivalence if and only if f^h is a pointwise weak equivalence. Furthermore, for any $X \in \mathrm{Ob}\,\mathcal{F}$, the functor $R \circ X^h$ is a pointwise fibrant homotopy functor and the natural map $X \longrightarrow R \circ X^h$ is an hf-equivalence.*

Proof. As in 5.17, using 4.9. □

Remark 5.20. If $\rho\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R$ has the property that its restriction to $\mathbf{f}\mathcal{V}$ takes values in cofibrant objects, then X^h is always a homotopy functor. The reason is that in a **sSet**-model category, any weak equivalence of fibrant and cofibrant objects is a simplicial homotopy equivalence, and $\mathcal{V}$-functors preserve them.

6 The stable model structure

We will construct the stable model structure on the category $\mathcal{F} = [\mathbf{f}\mathcal{V}, \mathcal{V}]$ with respect to some cofibrant object T of $\mathbf{f}\mathcal{V}$. For this, assume $\mathcal{V}$ and $\mathbf{f}\mathcal{V}$ are as in 5.10. In addition, $\mathcal{V}$ has to be right proper and cellular. We also require that $\mathbf{f}\mathcal{V}$ satisfies **f4**, in order to have a well-behaved fibrant replacement in $\mathcal{F}_{\mathrm{hf}}$. Finally, we assume the adjoint pair $(-\otimes T, \mathcal{V}(T,-))$ is a Quillen equivalence on the stable model structure on spectra described in 6.16. Since T is contained in $\mathbf{f}\mathcal{V}$ and **f2** holds, the canonical functor $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ factors over the inclusion as $i\colon T\mathrm{Sph} \longrightarrow \mathbf{f}\mathcal{V}$. Let (i_*, ev) denote the corresponding adjoint pair of functors.

6.1 Stable equivalences

We start by describing the stabilization process. For every object v in $\mathbf{f}\mathcal{V}$, the composition of the counit $\epsilon_T\mathcal{V}(v,-)\colon \mathcal{V}(T,\mathcal{V}(v,-))\otimes T \longrightarrow \mathcal{V}(v,-)$ and the natural isomorphism $\mathcal{V}(T,\mathcal{V}(v,-)) \cong \mathcal{V}(T\otimes v,-)$ define a morphism $\tau_v\colon \mathcal{V}(T\otimes v,-)\otimes T \longrightarrow \mathcal{V}(v,-)$ which is natural in v. If X is a $\mathcal{V}$-functor, then the induced map $\mathcal{V}_{\mathcal{F}}(\tau_v, X)\colon \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),X) \longrightarrow \mathcal{V}_{\mathcal{F}}(\mathcal{V}(T\otimes v,-)\otimes T, X)$ is natural in v and X. Using the enriched Yoneda lemma 2.1, one obtains a map $t_X(v)\colon X(v) \longrightarrow \mathcal{V}(T, X(T\otimes v))$. Let $\mathrm{Sh}\colon \mathcal{F} \longrightarrow \mathcal{F}$ denote the 'shift' functor obtained by pre-composing with the $\mathcal{V}$-functor $T\otimes -\colon \mathbf{f}\mathcal{V} \longrightarrow \mathbf{f}\mathcal{V}$. Define $\mathbb{T}\colon \mathcal{F} \longrightarrow \mathcal{F}$ to be the composition $\mathcal{V}(T,-)\circ \mathrm{Sh}$, so that $\mathbb{T}(X)(v) = \mathcal{V}(T, X(T\otimes v))$. The collection of the maps $t_X(v)$ is a $\mathcal{V}$-natural transformation $t_X\colon X \longrightarrow \mathbb{T}(X)$. Let $\mathbb{T}^\infty(X)$ denote the colimit of the sequence $X \xrightarrow{t(X)} \mathbb{T}(X) \xrightarrow{\mathbb{T}(t(X))} \mathbb{T}(\mathbb{T}(X)) \longrightarrow \cdots$. The canonical map $t_X^\infty\colon X \longrightarrow \mathbb{T}^\infty(X)$ yields a natural transformation $t^\infty\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \mathbb{T}^\infty$.
The definition of stable weak equivalences uses the fibrant replacement functor $\Phi^{J'}$ considered in 3.3.3. Let Φ be short notation for $\Phi^{J'}$, and similarly for the other fibrant replacement functors R and F. Recall that $\hbar(X)$ is not necessarily

a $\mathcal{V}$-functor for every $X \in \operatorname{Ob}\mathcal{F}$. But it can be stabilized, since there are natural weak equivalences $\theta_v: T \otimes \Phi(v) \longrightarrow \Phi(T \otimes v)$ according to 3.26. Let $\mathbb{T}': \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V}) \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ be the functor that maps X to the composition $\mathcal{V}(T, -) \circ X \circ (T \otimes -): \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$. Define the map $t'_{\hbar(X)}: \hbar(X) \longrightarrow \mathbb{T}'(\hbar(X))$ pointwise as the adjoint of

$$\mathbb{I}_* X(\Phi(v)) \otimes T \longrightarrow \mathbb{I}_* X(T \otimes \Phi(v)) \longrightarrow \mathbb{I}_* X(\Phi(T \otimes v)).$$

The map on the left hand side in this composition is adjoint to the composition $T \xrightarrow{\eta_{\Phi(v)}(T)} \mathcal{V}(\Phi(v), T \otimes \Phi(v)) \xrightarrow{\hom^{\mathbb{I}_* X}_{\Phi(v), T\otimes\Phi(v)}} \mathcal{V}(\mathbb{I}_* X(\Phi(v)), \mathbb{I}_* X(T \otimes \Phi(v)))$, and the map on the right hand side is $\mathbb{I}_* X(\theta_v)$

Lemma 6.1. *There is a natural transformation $t'_\hbar: \hbar \longrightarrow \mathbb{T}' \circ \hbar$. The natural transformation $u: U \longrightarrow \hbar$, where $U: \mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ is the forgetful functor, makes the following diagram commutative.*

$$\begin{tikzcd} U \circ \mathrm{Id}_{\mathcal{F}} \arrow[r, "U \circ t"] \arrow[d, "u"'] & U \circ T = \mathbb{T}' \circ U \arrow[d, "\mathbb{T}' \circ u"] \\ \hbar \arrow[r, "t'_\hbar"] & \mathbb{T}' \circ \hbar. \end{tikzcd}$$

Proof. The claim follows since $t_X(v): X(v) \longrightarrow \mathbb{T}(X)(v)$ can be defined as the adjoint (under tensoring with T) of the adjoint (under tensoring with $X(v)$) of $\hom^X_{v, T\otimes v} \circ \eta_v(T): T \longrightarrow \mathcal{V}(v, T \otimes v) \longrightarrow \mathcal{V}(X(v), X(T \otimes v))$, cp. A.8. $\square$

Denote the colimit of $\hbar(X) \xrightarrow{t'_{\hbar(X)}} \mathbb{T}'(\hbar(X)) \xrightarrow{\mathbb{T}'(t'_{\hbar(X)})} \mathbb{T}'(\mathbb{T}'(\hbar(X))) \longrightarrow \cdots$ by $\mathbb{T}'^\infty(\hbar(X))$, and let $t'^\infty_{\hbar(X)}$ be the canonical map $\hbar(X) \longrightarrow \mathbb{T}'^\infty(\hbar(X))$.

Definition 6.2. A map f in $\mathcal{F}$ is a *stable equivalence* if $\mathbb{T}'^\infty(\hbar(Rf))(v)$ is a weak equivalence in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Lemma 6.3. *Every hf-equivalence is a stable equivalence. The class of stable equivalences is saturated.*

There are canonical maps $X(v) \longrightarrow RX(v) \longrightarrow \hbar(RX)(v)$ for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$. Consider the induced map $\mathbb{T}^\infty(X) \longrightarrow \mathbb{T}'^\infty(\hbar(RX))$. The latter is sometimes a pointwise weak equivalence.

Lemma 6.4. *Assume $X \in \operatorname{Ob}\mathcal{F}$ is a pointwise fibrant homotopy functor. Then $\mathbb{T}^\infty(X)(v) \longrightarrow \mathbb{T}'^\infty(\hbar(RX))(v)$ is a weak equivalence in $\mathcal{V}$ for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$.*

Proof. Note that $X(v) \longrightarrow \hbar(RX)(v)$ is a weak equivalence of fibrant objects. $\mathcal{V}(T, -)$ preserves weak equivalences of fibrant objects since T is cofibrant and $\mathcal{V}$ is a monoidal model category. The map is question is hence a sequential colimit of weak equivalences, so 3.5 concludes the proof. $\square$

Corollary 6.5. *A map f between pointwise fibrant homotopy functors is a stable equivalence if and only if $\mathbb{T}^\infty(f)$ is a pointwise weak equivalence.*

6.2 Stable fibrations

Let $\tau_v\colon \mathcal{V}(T\otimes v,-)\otimes T \longrightarrow \mathcal{V}(v,-)$ be the canonical map of $\mathcal{V}$-functors described in section 6.1. The simplicial mapping cylinder factors τ_v as a cofibration $d_v\colon \mathcal{V}(T\otimes v,-)\otimes T \rightarrowtail D_v$ followed by a simplicial homotopy equivalence. Take a generating cofibration $i\colon si \rightarrowtail ti \in I$ in $\mathcal{V}$ and form the pushout product $d_v\square i$. The set $\mathcal{D}$ of generating acyclic cofibrations for the class of stable equivalences is $\{d_v\square i\}$, where $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$ and $i \in I$.

Definition 6.6. A map is called a *stable fibration* if it is an hf-fibration having the right lifting property with respect to the set $\mathcal{D}$.

Lemma 6.7. *An hf-fibration $f\colon X \longrightarrow Y$ is a stable fibration if and only if*

$$\begin{tikzcd} X(v) \arrow[r, "t_X(v)"] \arrow[d, "f(v)"'] & \mathbb{T}(X)(v) \arrow[d, "\mathbb{T}(f)(v)"] \\ Y(v) \arrow[r, "t_Y(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Proof. The proof is formally the same as for 5.6. $\square$

The rest of this section is devoted to prove that a stable fibration which is also a stable equivalence is a pointwise weak equivalence.

Lemma 6.8. *Assume $\mathcal{V}$ is right proper, and that filtered colimits commute with pullbacks in $\mathcal{V}$. Let $f\colon X \longrightarrow Y$ be a stable fibration. Then*

$$\begin{tikzcd} \hbar(RX)(v) \arrow[r, "t'_{\hbar(RX)}(v)"] \arrow[d, "\hbar(Rf)(v)"'] & \mathbb{T}'(\hbar(RX))(v) \arrow[d, "\mathbb{T}'(\hbar(Rf))(v)"] \\ \hbar(RY)(v) \arrow[r, "t'_{\hbar(RY)}(v)"] & \mathbb{T}'(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$.

The proof of 6.8 uses 6.10, 6.11 and 6.12. We start with a general fact about model categories.

Lemma 6.9. *Let $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between right proper model categories which preserves pullbacks, fibrations and acyclic fibrations. Suppose*

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, "f"'] & B \arrow[d] \\ C \arrow[r, "\sim"] & D \end{tikzcd}$$

is a commutative diagram in $\mathcal{C}$, such that the horizontal maps are weak equivalences with fibrant targets, and such that $f\colon A \twoheadrightarrow C$ is a fibration. Then the image of this square under G is a homotopy pullback square.

Proof. By Ken Brown's lemma, we can assume that the horizontal maps are in fact acyclic cofibrations. Factor the composition $A \longrightarrow D$ as an acyclic cofibration $i\colon A \overset{\sim}{\rightarrowtail} E$ followed by a fibration $p\colon E \twoheadrightarrow D$. Then the map $B \longrightarrow D$ factors as $B \xrightarrow{\sim} E \overset{p}{\twoheadrightarrow} D$, by choosing a lift in the diagram

$$\begin{tikzcd} A \arrow[r, tail, "\sim", "i"'] \arrow[d, tail, "\sim"'] & E \arrow[d, two heads, "p"] \\ B \arrow[r] & D. \end{tikzcd}$$

Define P to be the pullback of $C \xrightarrow{\sim} D \overset{p}{\twoheadleftarrow} E$ and call the map induced by f and i $h\colon A \longrightarrow P$. Since $C \xrightarrow{\sim} D$ and i are weak equivalences and $\mathcal{C}$ is right proper, h is a weak equivalence. Using the assumptions on G and right properness of $\mathcal{D}$ (3.14), we have to prove that $G(h)$ is a weak equivalence.

Let Q be the pullback of $A \overset{f}{\twoheadrightarrow} C \overset{p'}{\twoheadleftarrow} P$, where p' is the base change of p. The maps id_A and h induce a map $A \longrightarrow Q$ which can be factored as $A \underset{\sim}{\overset{s}{\rightarrowtail}} W \overset{q}{\twoheadrightarrow} Q$. After all, we have a factorization of h as $A \underset{\sim}{\overset{s}{\rightarrowtail}} W \overset{f'\circ q}{\twoheadrightarrow} P$, where f' is the base change of the fibration f. The map h is a weak equivalence, thus $f' \circ q$ is an acyclic fibration. In particular, $G(f' \circ q)$ is an acyclic fibration. The map $p'' \circ q\colon W \longrightarrow A$ is a fibration (where p'' is the base change of the fibration p') and has $s\colon A \overset{\sim}{\rightarrowtail} W$ as a section. Hence $p'' \circ q$ is an acyclic fibration, and so is $G(p'' \circ q)$. Now $G(s)$ is a section of $G(p'' \circ q)$, implying that $G(s)$, and therefore $G(h = (f' \circ q) \circ s)$, is a weak equivalence. □

Corollary 6.10. *Let $f\colon X \longrightarrow Y$ be a pointwise fibration, and assume $\mathcal{V}$ is right proper. Then the following is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} \mathcal{V}(T, X(v)) \arrow[r, "\mathcal{V}(T,\rho_{X(v)})"] \arrow[d, "\mathcal{V}(T,f(v))"'] & \mathcal{V}(T, RX(v)) \arrow[d, "\mathcal{V}(T,Rf(v))"] \\ \mathcal{V}(T, Y(v)) \arrow[r, "\mathcal{V}(T,\rho_{Y(v)})"] & \mathcal{V}(T, RY(v)) \end{tikzcd}$$

Proof. Follows from 6.9, since $\mathcal{V}(T, -)\colon \mathcal{V} \longrightarrow \mathcal{V}$ is a right Quillen functor. □

Corollary 6.11. *Suppose $\mathcal{V}$ is right proper and $f\colon X \longrightarrow Y$ is a stable fibration. Then the following is a homotopy pullback square for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} RX(v) \arrow[r, "t_{RX(v)}"] \arrow[d, "Rf(v)"'] & \mathbb{T}(RX)(v) \arrow[d, "\mathbb{T}(Rf)(v)"] \\ RY(v) \arrow[r, "t_{RY(v)}"] & \mathbb{T}(RY)(v) \end{tikzcd}$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
X(v) \arrow[rr, "t_X(v)"] \arrow[dr, "\rho_{X(v)}"] \arrow[dd, "f(v)"'] & & \mathbb{T}(X)(v) \arrow[dr, "\mathbb{T}(\rho_{X(v)})"] \arrow[dd, "\mathbb{T}(f)(v)"' near start] & \\
& RX(v) \arrow[rr, "t_{RX}(v)" near start] \arrow[dd, "Rf(v)"' near start] & & \mathbb{T}(RX)(v) \arrow[dd, "\mathbb{T}(Rf)(v)"] \\
Y(v) \arrow[rr, "t_Y(v)" near end] \arrow[dr, "\rho_{Y(v)}"'] & & \mathbb{T}(Y)(v) \arrow[dr, "\mathbb{T}(\rho_{Y(v)})"] & \\
& RY(v) \arrow[rr, "t_{RY}(v)"] & & \mathbb{T}(RY)(v)
\end{tikzcd}$$

The right hand square of the cube is a homotopy pullback square by 6.10. Likewise for the square in the back using the assumption on f. In the left hand square the horizontal maps are weak equivalences. Hence the square in question is a homotopy pullback square by 3.13. ☐

Lemma 6.12. *Assume $f\colon X \longrightarrow Y$ is a stable fibration of pointwise fibrant functors, and filtered colimits commute with pullbacks in $\mathcal{V}$. Then*

$$\begin{tikzcd}
\hbar(X)(v) \arrow[r, "t'_{\hbar(X)}(v)"] \arrow[d, "\hbar(f)(v)"'] & \mathbb{T}'(\hbar(X))(v) \arrow[d, "\mathbb{T}'(\hbar(f))(v)"] \\
\hbar(Y)(v) \arrow[r, "t'_{\hbar(Y)}(v)"] & \mathbb{T}'(\hbar(Y))(v)
\end{tikzcd}$$

is a homotopy pullback square for all $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.

Proof. Up to isomorphism, the square above decomposes into two squares:

$$\begin{tikzcd}
\operatorname{colim}_{\mathrm{ac}(v,R)} X(a) \arrow[r, "\operatorname{colim} t_X(a)"] \arrow[d, "\operatorname{colim} f(a)"'] & \operatorname{colim}_{\mathrm{ac}(v,R)} \mathbb{T}(X)(a) \arrow[r, "c_X"] \arrow[d, "\operatorname{colim} \mathbb{T}(f)(a)"'] & \operatorname{colim}_{\mathrm{ac}(T\otimes v,R)} \mathcal{V}(T, X(b)) \arrow[d, "\operatorname{colim}"', "\mathcal{V}(T,f(b))"] \\
\operatorname{colim}_{\mathrm{ac}(v,R)} Y(a) \arrow[r, "\operatorname{colim} t_Y(a)"] & \operatorname{colim}_{\mathrm{ac}(v,R)} \mathbb{T}(Y)(a) \arrow[r, "c_Y"] & \operatorname{colim}_{\mathrm{ac}(T\otimes v,R)} \mathcal{V}(T, Y(b))
\end{tikzcd}$$

Recall from 3.3.3 that the objects in $\mathrm{ac}(T\otimes v, R)$ are $T\otimes v \overset{\sim}{\rightarrowtail} b \longrightarrow R(T\otimes v)$. The maps c_X and c_Y are obtained from $\Theta_v\colon \mathrm{ac}(v,R) \longrightarrow \mathrm{ac}(T\otimes v, R)$ which maps $v \overset{\sim}{\rightarrowtail} a \longrightarrow R(v)$ to $T\otimes v \overset{\sim}{\rightarrowtail} T\otimes a \longrightarrow T\otimes R(v) \longrightarrow R(T\otimes v)$, and the natural transformation $\mathcal{V}(T, X(T\otimes \Phi_v)) \longrightarrow \mathcal{V}(T, X(\Phi_{T\otimes v} \circ \Theta_v))$ which consists of identity maps. From 3.23, one can replace the indexing categories $\mathrm{ac}(v,R)$ and $\mathrm{ac}(T\otimes v, R)$ by filtered ones, namely $\mathrm{ac}(v,F)$ and $\mathrm{ac}(T\otimes v, F)$. Then all the vertical maps are fibrations of fibrant objects, because f is a pointwise fibration of pointwise fibrant functors and 3.5 holds. It follows that the left hand square is a homotopy pullback square if and only if the canonical map g from $\mathrm{colim}_{\mathrm{ac}(v,F)} X(a)$ to the pullback is a weak equivalence. Filtered colimits commute with pullbacks, so g is the filtered colimit of the canonical maps induced by the squares

$$\begin{tikzcd} X(a) \arrow[r, "t_X(a)"] \arrow[d, "f(a)"'] & \mathbb{T}(X)(a) \arrow[d, "\mathbb{T}(f)(a)"] \\ Y(a) \arrow[r, "t_Y(a)"] & \mathbb{T}(Y)(a) \end{tikzcd}$$

for finite sub-complexes $v \xrightarrow{\sim} a$ of $v \xrightarrow{\iota_v} Fv$. These squares are all homotopy pullback squares: f is a stable fibration, and the vertical maps are fibrations of fibrant objects. It follows that g is a filtered colimit of weak equivalences, so the left hand square is a homotopy pullback square.
That the right hand square is a homotopy pullback depends on whether f is an hf-fibration. We claim that

$$\begin{tikzcd} X(T \otimes a) \arrow[r] \arrow[d, "f(T\otimes a)"'] & \mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} X(b) \arrow[d, "\mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} f(b)"] \\ Y(T \otimes a) \arrow[r] & \mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} Y(b) \end{tikzcd}$$

is a homotopy pullback square for all $v \xrightarrow[\alpha]{\sim} a \longrightarrow R(v)$ in $\mathrm{ac}(v,R)$. Denote the full subcategory of $\mathrm{ac}(T\otimes v, R)$ consisting of $T\otimes v \xrightarrow[\beta]{\sim} b \longrightarrow R(T\otimes v)$, where β factors as $T\otimes v \xrightarrow{T\otimes\alpha} T\otimes a \longrightarrow b$, by $\mathrm{ac}(T\otimes v,R)_a$. This category is a final subcategory of $\mathrm{ac}(T\otimes v,R)$. Hence we may assume the colimit in the square above is indexed by $\mathrm{ac}(T\otimes v,R)_a$. As in the proof of 5.7, it follows that the square above is a homotopy pullback square for all $v \xrightarrow[\alpha]{\sim} a \longrightarrow R(v)$ in $\mathrm{ac}(v,R)$. Here we use that f is an hf-fibration. The right hand square in the main diagram is then a homotopy pullback square, using the by now standard argument for filtered colimits of homotopy pullback squares. □

A proof of 6.8 follows:

Proof. By 6.11, we may assume f is a stable fibration of pointwise fibrant functors. The result is therefore a consequence of 6.12. □

Lemma 6.13. *Suppose that $\mathcal{V}$ is right proper and filtered colimits commute with pullbacks in $\mathcal{V}$. Let $f\colon X \longrightarrow Y$ be a stable fibration. Then*

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \mathbb{T}'^{\infty}(\hbar(RX))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(Rf))(v)"] \\ Y(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.

Proof. Let $f\colon X \longrightarrow Y$ be a stable fibration. Factor Rf as $RX \xrightarrow{\sim\,\mathrm{pt}} Z$ followed by $g\colon Z \xrightarrow{\mathrm{pt}} RY$. Here g is a stable fibration of pointwise fibrant functors by 5.6 and 6.11. The claim in 6.13 is equivalent to the statement that

$$\begin{tikzcd} Z(v) \arrow[r] \arrow[d, "g(v)"'] & \mathbb{T}'^{\infty}(\hbar(Z))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(g))(v)"] \\ RY(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square. The vertical maps are fibrations of fibrant objects, and there is the decomposition

$$\begin{tikzcd} Z(v) \arrow[r] \arrow[d, "g(v)"'] & \hbar(Z)(v) \arrow[r] \arrow[d, "\hbar(g)(v)"] & \mathbb{T}'^{\infty}(\hbar(Z))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(g))(v)"] \\ RY(v) \arrow[r] & \hbar(RY)(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

The left hand square is a homotopy pullback square by 5.7. The functor $\mathcal{V}(T,-)$ preserves homotopy pullback squares provided the vertical maps are fibrations of fibrant objects. It follows, using 6.8, that the right hand square is a homotopy pullback square. $\square$

Corollary 6.14. *A map is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. Let $f\colon X \longrightarrow Y$ be a stable fibration and a stable equivalence. Stable fibrations are in particular pointwise fibrations, so it remains to prove that f is a pointwise weak equivalence. By 6.13, the following diagram is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \mathbb{T}'^{\infty}(\hbar(RX))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(Rf))(v)"] \\ Y(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

Since f is a stable equivalence, the right hand vertical map is a weak equivalence. It follows that f is a pointwise weak equivalence. Consider the other implication.

A pointwise acyclic fibration $f\colon X \longrightarrow Y$ is a stable equivalence according to 6.3. So 5.8 implies that f is an hf-fibration. By 6.7, it remains to prove that

$$\begin{tikzcd} X(v) \arrow[r, "t(X)(v)"] \arrow[d, "f(v)"'] & \mathbb{T}(X)(v) \arrow[d, "\mathbb{T}(f)(v)"] \\ Y(v) \arrow[r, "t(Y)(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathbf{f}\mathcal{V}$. The maps $f(v)$ and $f(T\otimes v)$ are acyclic fibrations and $\mathcal{V}(T,-)$ preserves acyclic fibrations. This implies that $\mathbb{T}(f)(v)$ is an acyclic fibration. $\square$

6.3 Comparison with T-spectra

To proceed with the stable model structure we will compare the stabilizations of enriched functors 6.1 and spectra [8, §4]. Recall the "suspension with T" functors $-\otimes T$ and Σ_T on $\mathrm{Sp}(\mathcal{V},T)$ from 2.13. Let $\mathrm{Ev}_n := \mathrm{Ev}_{T^n}$ denote the functor evaluating a spectrum on T^n. If $n \geq 0$, there is the commutative diagram:

$$\begin{tikzcd}
\mathrm{Sp}(\mathcal{V},T) \arrow[r, "-\otimes T"] \arrow[d, "\Sigma_T"'] & \mathrm{Sp}(\mathcal{V},T) \arrow[d, "\mathrm{Ev}_n"] \\
\mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{Ev}_n"] & \mathcal{V}
\end{tikzcd}$$

Hence, for any spectrum E, $E\otimes T$ and $\Sigma_T E$ differ only in their structure maps. This statement carries over to the adjoints $\mathcal{V}(T,-)\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$ and $\Omega_T\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$. The composition $T\mathrm{Sph} \hookrightarrow \mathcal{V} \xrightarrow{T\otimes -} \mathcal{V}$ does not factor over the inclusion $T\mathrm{Sph} \hookrightarrow \mathcal{V}$. Hence the shift functor $\mathrm{Sh}\colon \mathcal{F} \longrightarrow \mathcal{F}$ does not have a compatible analog in the category of spectra. But there is the shift $\mathrm{sh}\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$ where $(\mathrm{sh}(E))_n := E_{n+1}$. The nth structure map of $\mathrm{sh}(E)$ is e_{n+1}, and the following diagram commutes for all $n \geq 0$.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathrm{Sh}"] \arrow[d, "\mathrm{ev}"'] & \mathcal{F} \arrow[r, "\mathrm{ev}"] & \mathrm{Sp}(\mathcal{V},T) \arrow[d, "\mathrm{Ev}_n"] \\
\mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{sh}"] & \mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{Ev}_n"] & \mathcal{V}
\end{tikzcd}$$

The stabilization for spectra uses that the structure maps of a spectrum E define a natural map $s(E)\colon E \longrightarrow \mathrm{sh}(\Omega_T E)$. Let us abbreviate the composition $\mathrm{sh}\circ\Omega_T$ by $S\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$. Then $s(E)_n\colon E_n \longrightarrow S(E)_n = \Omega_T E_{n+1}$ is the adjoint of e_n. The stabilization $S^\infty(E)$ of a spectrum E is the colimit of the diagram $E \xrightarrow{s(E)} S(E) \xrightarrow{S(s(E))} S(S(E)) \longrightarrow \cdots$. Let $s^\infty\colon E \longrightarrow S^\infty(E)$ be the canonical map. In [8, §4], the notation $\iota\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow \Theta$ is used instead of $s\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow S$, and $j\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow \Theta^\infty$ instead of $s^\infty\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow S^\infty$.

Definition 6.15. A map $f\colon E \longrightarrow F$ of spectra is a *stable equivalence* if $S^\infty(R\circ f)$ is a pointwise weak equivalence, and a *stable fibration* if f is a pointwise fibration and

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "f"'] & R\circ E \arrow[r, "s^\infty(R\circ E)"] \arrow[d, "R\circ f"'] & S^\infty(R\circ E) \arrow[d, "S^\infty(R\circ f)"] \\
F \arrow[r] & R\circ F \arrow[r, "s^\infty(R\circ F)"] & S^\infty(R\circ F)
\end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure.

We summarize the results [8, 4.12, 4.14, 6.5].

Theorem 6.16 (Hovey). *Let $\mathcal{V}$ be an almost finitely generated, pointed, proper, and cellular monoidal model category. Let T be some cofibrant and $\mathcal{V}$-finitely presentable object of $\mathcal{V}$. Assume sequential colimits commute with pullbacks in $\mathcal{V}$. Then $\mathrm{Sp}(\mathcal{V},T)$ is an almost finitely generated proper $\mathcal{V}$-model category with stable equivalences as weak equivalences, stable fibrations as fibrations and cofibrations as cofibrations.*

The condition that $\mathcal{V}$ be cellular might be weakened according to the remark after [8, 4.12]. An important input in the proof of 6.16 is the following lemma.

Lemma 6.17. *If E is pointwise fibrant, then $s^\infty(E)$ is a stable equivalence with a stably fibrant codomain.*

The natural maps $\mathrm{ev}(t^\infty(X))$ and $s^\infty(\mathrm{ev}(X))$ do not coincide. Compatibility of the two stabilization processes is therefore an issue, see 6.18 and 6.19 below.

Lemma 6.18. *If v is an object of $\mathbf{f}\mathcal{V}$, let $v^*\colon \mathcal{F} \longrightarrow \mathcal{F}$ be the functor where $X \longmapsto X \circ (v \otimes -)$ and let $\gamma_{v,n}$ be short for the coherence isomorphism*

$$T \otimes (v \otimes T^n) \xrightarrow{\alpha_{T,V,T^n}} (T \otimes v) \otimes T^n \xrightarrow{\sigma_{T,v\otimes T^n}} (v \otimes T) \otimes T^n \xrightarrow{\alpha^{-1}_{v,T,T^n}} v \otimes T^{n+1}.$$

Then the next diagram is commutative and natural in X.

$$\begin{tikzcd}
X(v\otimes T^n) \arrow[r, "t(X)(v\otimes T^n)"] \arrow[dr, "s(\mathrm{ev}(v^*(X)))_n"'] & \mathcal{V}(T, X(T\otimes(v\otimes T^n))) = \mathbb{T}(X)(v\otimes T^n) \arrow[d, "\mathcal{V}(T,X(\gamma_{v,n}))"] \\
 & \mathcal{V}(T, X(v\otimes T^{n+1}))
\end{tikzcd}$$

Proof. By definition, $t(X)(v \otimes T^n)$ is the adjoint of

$$T \xrightarrow{\eta_{v\otimes T^n}(T)} \mathcal{V}(v\otimes T^n, T\otimes(v\otimes T^n)) \xrightarrow{\hom^X} \mathcal{V}(X(v\otimes T^n), X(T\otimes(v\otimes T^n))).$$

Likewise, the map $s(\mathrm{ev}(v^*(X)))_n$ is the adjoint of

$$T \xrightarrow{\eta_{T^n}(T)} \mathcal{V}(T^n, T^{n+1}) \xrightarrow{\hom^{v^*(X)}_{T^n,T^{n+1}}} \mathcal{V}(X(v\otimes T^n), X(v\otimes T^{n+1})).$$

Note that $\hom^{v^*(X)}_{T^n,T^{n+1}} = \hom^{X\circ(v\otimes -)}_{T^n,T^{n+1}} = \hom^X_{v\otimes T^n, v\otimes T^{n+1}} \circ \hom^{v\otimes -}_{T^n,T^{n+1}}$. The claim follows from A.2. $\square$

Corollary 6.19. *For all $X \in \mathrm{Ob}\,\mathcal{F}$ and $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$, there exists an isomorphism $\gamma\colon \mathbb{T}^\infty(X)(v\otimes T^n) \longrightarrow S^\infty(\mathrm{ev}(v^*(X)))_n$ and the diagram below is commutative and natural in X.*

$$\begin{tikzcd}
X(v\otimes T^n) \arrow[r, "t^\infty(X)(v\otimes T^n)"] \arrow[dr, "s^\infty(\mathrm{ev}(v^*(X)))_n"'] & \mathbb{T}^\infty(X)(v\otimes T^n) \arrow[d, "\gamma"] \\
 & S^\infty(\mathrm{ev}(v^*(X)))_n
\end{tikzcd}$$

Proof. The maps γ and $s^\infty(\mathrm{ev}(v^*(X)))_n$ are the vertical sequential compositions in the following commutative diagram.

$$\begin{tikzcd}
X(v\otimes T^n) \arrow[r, "\mathrm{id}_{X(v\otimes T^n)}"] \arrow[d, "t(X)(v\otimes T^n)"'] & X(v\otimes T^n) \arrow[d, "s(\mathrm{ev}(v^*(X)))_n"] \\
\mathbb{T}(X)(v\otimes T^n) \arrow[r, "\mathcal{V}(T,X(\gamma^1))"] \arrow[d, "\mathbb{T}(t(X))(v\otimes T^n)"'] & \mathcal{V}(T, X(v\otimes T^{n+1})) \arrow[d, "\mathcal{V}(T,s(\mathrm{ev}(v^*(X)))_{n+1})"] \\
\mathbb{T}(\mathbb{T}(X))(v\otimes T^n) \arrow[r, "\mathcal{V}(T,\mathcal{V}(T,X(\gamma^2)))"] \arrow[d] & \mathcal{V}(T,\mathcal{V}(T,X(v\otimes T^{n+2}))) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Here γ^1 is the map $\gamma_{v,n}$ in 6.18. Note that the second square above is $\mathcal{V}(T,-)$ applied to the diagram

$$\begin{tikzcd}
X(T\otimes(v\otimes T^n)) \arrow[r, "t(X)(t\otimes(v\otimes T^n))"] \arrow[d, "\mathcal{V}(T,X(\gamma_1))"'] & \mathbb{T}(X)(T\otimes(v\otimes T^n)) \arrow[d, "\mathcal{V}(T,X(\gamma_2))"] \\
X(v\otimes T^{n+1}) \arrow[r, "s(\mathrm{ev}(v^*(X)))_{n+1}"] & \mathcal{V}(T,X(v\otimes T^{n+1})).
\end{tikzcd}$$

Let γ^2 be defined as $T\otimes(T\otimes(v\otimes T^n)) \xrightarrow{T\otimes\gamma_{v,n}} T\otimes(v\otimes T^{n+1}) \xrightarrow{\gamma_{v,n+1}} v\otimes T^{n+1}$. Then the lower square commutes because of naturality and 6.18. Likewise one constructs γ^n inductively, and puts γ to be $\mathrm{colim}_n\gamma^n$. The result follows. □

If $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$, the composition $\mathbb{I}_*X\circ\Phi\circ(v\otimes -) = \hbar(X)\circ(v\otimes -)$ determines a T-spectrum $\overline{v^*X}$ for every $\mathcal{V}$-functor X. The nth term of $\overline{v^*X}$ is $\mathbb{I}_*X(\Phi(v\otimes T^n))$ and the structure map $\overline{v^*X}_n\otimes T \longrightarrow \overline{v^*X}_{n+1}$ is the composition

$$\mathbb{I}_*X(\Phi(v\otimes T^n))\otimes T \xrightarrow{\mathrm{sw}_T^{\mathbb{I}_*X}} \mathbb{I}_*X(\Phi(v\otimes T^n)\otimes T) \xrightarrow{\mathbb{I}_*X(\theta_{v\otimes T^n})} \mathbb{I}_*X(\Phi(v\otimes T^{n+1}))$$

up to an associativity isomorphism. This construction is functorial and commutes with colimits and the closed $\mathcal{V}$-module structures.

Lemma 6.20. *A map $f\colon X \longrightarrow Y$ in $\mathcal{F}$ is a stable equivalence if and only if $\overline{v^*f}\colon \overline{v^*X} \longrightarrow \overline{v^*Y}$ is a stable equivalence of T-spectra for all $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Let $f\colon X \longrightarrow Y$ be a stable equivalence, and pick $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$. The map $\mathbb{T}'^\infty(\hbar(R\circ f))(v\otimes T^n)$ is a weak equivalence in $\mathcal{V}$ by definition. The isomorphism mentioned in 6.19 implies that this map is isomorphic to $S^\infty(v^*(R\circ f)')_n$. Thus $\overline{v^*(R\circ f)}$ is a stable equivalence of T-spectra. Since this is a pointwise fibrant replacement of $\overline{v^*f}$ in $\mathrm{Sp}(\mathcal{V},T)$, it follows that $\overline{v^*f}$ is a stable equivalence. The converse holds by running the argument backwards. □

If additional conditions are satisfied, the characterization of stable equivalences can be improved in that $\hbar$ becomes redundant. Note that the last characterization uses the axiom **f4**.

Corollary 6.21. *A map $f\colon X \longrightarrow Y$ of pointwise fibrant homotopy functors is a stable equivalence if and only if* $\mathrm{ev}(v^*(f))$ *is a stable equivalence of spectra for every object v of $\mathbf{f}\mathcal{V}$.*

Proof. By 6.4, f is a stable equivalence if and only if $\mathbb{T}^\infty(f)(w)$ is a weak equivalence in $\mathcal{V}$ for every w. The proof proceeds as in 6.20. □

Corollary 6.22. *A map f of cofibrant functors is a stable equivalence if and only if* $\mathrm{ev}(v^*(f^h))$ *is a stable equivalence of spectra for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\sim\mathrm{hf}"] \arrow[d, "f"'] & RX^h \arrow[d, "Rf^h"] \\ Y \arrow[r, "\sim\mathrm{hf}"] & RY^h. \end{tikzcd}$$

¿From 5.17 – which uses **f4** – and 6.3 we have that f is a stable equivalence if and only if Rf^h is a stable equivalence. Corollary 6.21 shows that f is a stable equivalence if and only if $\mathrm{ev}(v^*(Rf^h))$ is a stable equivalence in $\mathrm{Sp}(\mathcal{V}, T)$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. Since $\mathrm{ev}(v^*(Rf^h)) = R\,\mathrm{ev}(v^*(f^h))$ and pointwise weak equivalences of spectra are stable equivalences, it follows that f is a stable equivalence if and only if $\mathrm{ev}(v^*(f^h))$ is a stable equivalence of spectra. □

6.4 The generating stable equivalences

Recall the stable model structure on $\mathrm{Sp}(\mathcal{V}, T)$ from 6.16. In this structure, (Σ_T, Ω_T) is a Quillen equivalence by [8, 3.9]. In general, it is not clear whether $(-\otimes T, \mathcal{V}(T, -))$ – which is more natural to consider when viewing spectra as $\mathcal{V}$-functors – is a Quillen equivalence. As explained in [8, 10.3], this holds if T is *symmetric*, which roughly means that the cyclic permutation on $T \otimes T \otimes T$ is homotopic to the identity. We formulate a working hypothesis.
Hypothesis: The adjoint functor pair $(-\otimes T, \mathcal{V}(T, -))$ is a Quillen equivalence for the model category $\mathrm{Sp}(\mathcal{V}, T)$ described in 6.16.

Lemma 6.23. *The maps in $\mathcal{D}$ are stable equivalences.*

Proof. A map $d_v \Box i$ in $\mathcal{D}$ is a cofibration of cofibrant functors, and by 6.22 a stable equivalence if and only if $\mathrm{ev}(w^*(d_v \Box i)^h) = \mathrm{ev}(w^*(d_v)^h)\Box i$ is a stable equivalence of spectra for all $w \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. We claim the latter holds if $f := \mathrm{ev}(w^*(d_v)^h)$ is a stable equivalence.
Factor f as a cofibration g followed by a pointwise acyclic fibration p. The stable model structure on spectra is a $\mathcal{V}$-model structure by 6.16, hence $f\Box i$ factors as a cobase change of the stable acyclic cofibration $g\Box i$, followed by the

map $p\Box i$. Since $\mathcal{V}$ is assumed to be strongly monoidal, $p\Box i$ is a pointwise weak equivalence.
It remains to prove that d_v is a stable equivalence. This condition is equivalent to τ_v being a stable equivalence. The latter factors by definition as

$$\mathcal{V}(T\otimes v,-)\otimes T \xrightarrow{\cong} \mathcal{V}(T,\mathcal{V}(v,-))\otimes T \xrightarrow{\epsilon_T V(v,-)} \mathcal{V}(v,-)$$

where ϵ_T is the counit $(-\otimes T)\circ\mathcal{V}(T,-) \longrightarrow \mathrm{Id}_{\mathcal{F}}$. We are reduced to prove that $\epsilon_T V(v,-)$ is a stable equivalence. By 6.22, this map is a stable equivalence if and only if $\mathrm{ev}(w^*(\epsilon_T\mathcal{V}(v,-))^h)$ is a stable equivalence of spectra for every w. Since ev commutes with the action and coaction of $\mathcal{V}$, $\mathrm{ev}(w^*(\epsilon_T\mathcal{V}(v,-))^h)$ coincides with $\epsilon_T\mathrm{ev}(w^*(\mathcal{V}(v,-))^h)$.
Let $q\colon Q \longrightarrow \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$ be a cofibrant replacement functor in the category of spectra, so $q(E)_n$ is an acyclic fibration in $\mathcal{V}$ for every spectrum E and $n\geq 0$. Consider the following diagram, where the notation is simplified.

$$\begin{tikzcd}
(Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))\otimes T \arrow[r, "q\otimes T"] \arrow[d, "Q\epsilon_T"'] & \mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))\otimes T \arrow[d, "\epsilon_T"] \\
Q\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))) \arrow[r, "q"] & \mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))
\end{tikzcd}$$

The composition

$$s^\infty\circ q\circ Q\epsilon_T\colon (Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))\otimes T \longrightarrow S^\infty(\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))$$

is a stable equivalence of spectra by the hypothesis. The target of $s^\infty\circ q\circ Q\epsilon_T$ is stably fibrant, and its domain is cofibrant. Recall that $\mathcal{V}(T,-)$ commutes with filtered colimits. Up to an isomorphism, the stable weak equivalence

$$s^\infty\circ q\colon Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))) \longrightarrow S^\infty(\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))$$

is an adjoint of $s^\infty\circ q\circ Q\epsilon_T$. Thus $q\circ Q\epsilon_T$ is a stable equivalence. The map $q\otimes T$ is a pointwise weak equivalence since $\mathcal{V}$ is strongly monoidal. Hence ϵ_T is a stable equivalence of spectra. This ends the proof. $\square$

To consider maps in $\mathcal{D}$-cell, we need to record a property of the stable model structure of spectra.

Lemma 6.24. *Let $f\colon E \longrightarrow F$ be a stable equivalence of spectra such that f_n is a retract of a map in* $\mathrm{Cof}(\mathcal{V})\otimes\mathcal{V}$*-cell for every $n\geq 0$. Then any cobase change of f is a stable equivalence.*

Proof. Let $g\colon E \longrightarrow G$ be a map of spectra. Factor g as $i\colon E \rightarrowtail T$ followed by $p\colon T \xrightarrow{\sim\mathrm{pt}} G$, and consider the diagram:

$$\begin{tikzcd}
E \arrow[r, tail, "i"] \arrow[d, "f"'] & T \arrow[r, two heads, "p", "\sim\mathrm{pt}"'] \arrow[d, "f'"] & G \arrow[d, "f''"] \\
F \arrow[r, tail] & F\cup_E T \arrow[r, "p'"] & F\cup_E G
\end{tikzcd}$$

The stable model structure on spectra is left proper by 6.16, hence f' is a stable equivalence. Pushouts are formed pointwise, so f'_n is a retract of a map belonging to $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell for every $n \geq 0$. By the assumption that $\mathcal{V}$ is strongly left proper, the cobase change p' of the pointwise weak equivalence p along f' is again a pointwise weak equivalence. Hence f'' is a stable equivalence. □

Lemma 6.25. *The maps in $\mathcal{D}$-cell are stable equivalences.*

Proof. Let $f\colon X \longrightarrow Y$ be a map in $\mathcal{D}$-cell. First suppose that X is cofibrant. Then Y is automatically cofibrant. By 6.22, it suffices to prove that $\mathrm{ev}(v^* f^h)$ is a stable equivalence of spectra for every v. The functors ev, v^* and $(-)^h$ preserve colimits, hence $\mathrm{ev}(v^* f^h)$ is in $\mathrm{ev}(v^*(\mathcal{D}^h))$-cell. Every map in $\mathrm{ev}(v^*(\mathcal{D}^h))$ is of the form considered in 6.24, so cobase changes of these are stable equivalences of spectra. The stable model structure on spectra is almost finitely generated, which implies that stable equivalences of spectra are closed under sequential compositions. This proves the lemma for maps in $\mathcal{D}$-cell with cofibrant domain.

For f arbitrary, we will construct a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\sim\mathrm{pt}"'] & Y' \arrow[d, "\sim\mathrm{pt}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where X' is cofibrant and f' is a map in $\mathcal{P}_J \cup \mathcal{D}$-cell. It allows to finish the proof using the special case treated above. Without loss of generality, f is the sequential composition of $X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$, where f_n is the cobase change of a coproduct of maps in $\mathcal{D}$. We construct f' as a sequential composition. Consider a cofibrant replacement $g_0\colon X'_0 = X' \overset{\sim\mathrm{pt}}{\twoheadrightarrow} X$. Assume f_0 is the cobase change of $z_0\colon sZ_0 \longrightarrow tZ_0$, and let $a_0\colon sZ_0 \longrightarrow X$ be the attaching map. The functor sZ_0 is cofibrant, so a_0 lifts to a map $a'_0\colon sZ_0 \longrightarrow X'_0$. Taking pushouts in the commutative diagram

$$\begin{tikzcd} tZ_0 \arrow[d, "\mathrm{id}"'] & sZ_0 \arrow[l, "z_0"'] \arrow[r, "a'_0"] \arrow[d, "\mathrm{Id}"] & X'_0 \arrow[d, "g_0"] \\ tZ_0 & sZ_0 \arrow[l, "z_0"'] \arrow[r, "a_0"] & X_0 \end{tikzcd}$$

gives a pointwise weak equivalence $tZ_0 \cup_{sZ_0} X'_0 \xrightarrow{\sim\mathrm{pt}} X_1$. It factors as a map in $\mathcal{P}_J$-cell followed by say $X'_1 \overset{\sim\mathrm{pt}}{\twoheadrightarrow} X_1$. By iterating this construction one finds pointwise acyclic fibrations $g_n\colon X'_n \overset{\sim\mathrm{pt}}{\twoheadrightarrow} X_n$ for all $n \geq 0$. Taking the colimit gives $f'\colon X' \longrightarrow Y := \mathrm{colim}_n X'_n$. □

6.5 The main theorem

Before stating our main theorem, we summarize the list of assumptions. First, $\mathcal{V}$ is a weakly finitely generated monoidal **sSet**-model category for which the monoid axiom holds. Moreover, $\mathcal{V}$ is strongly monoidal as defined in 4.12, right proper and cellular. Assume that filtered colimits commute with pullbacks in $\mathcal{V}$, that Δ^1 is finitely presentable in $\mathcal{V}$, and that cofibrations are monomorphisms. We require $\mathbf{f}\mathcal{V}$ to satisfy

f1 Every object of $\mathbf{f}\mathcal{V}$ is $\mathcal{V}$-finitely presentable.

f2 The unit e is in $\mathbf{f}\mathcal{V}$, and $\mathbf{f}\mathcal{V}$ is closed under the monoidal product.

f3 If $tj \xleftarrow{\sim} sj \longrightarrow v$ is a diagram in $\mathcal{V}$ where $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$ and $j \in J'$, then the pushout $tj \cup_{sj} v$ is in $\mathbf{f}\mathcal{V}$.

f4 All objects in $\mathbf{f}\mathcal{V}$ are cofibrant.

In what follows, T is a cofibrant object of $\mathbf{f}\mathcal{V}$ with the property that $(- \otimes T, \mathcal{V}(T, -))$ is a Quillen equivalence in the stable model structure on $\mathrm{Sp}(\mathcal{V}, T)$.

Theorem 6.26. *Under the assumptions above, the classes of stable equivalences, stable fibrations and cofibrations give $\mathcal{F} = [\mathbf{f}\mathcal{V}, \mathcal{V}]$ the structure of a weakly finitely generated model category.*

Proof. The proof is analogous to the proof of 5.10, using 6.14 and 6.25. Nevertheless we give some details. The set of additional generating acyclic cofibrations is the set $\mathcal{D}$, and the domains and codomains of the maps in $\mathcal{D}$ are finitely presentable. Lemma 6.25 then shows that relative cell complexes built from the generating acyclic cofibrations are stable equivalences. By 6.14 and 4.2, the stable acyclic fibrations are detected by the generating cofibrations. With 6.3 we get that all criteria of [7, 2.1.19] are satisfied. $\square$

We refer to the model structure in 6.26 as the *stable* model structure. If $\mathcal{F}$ is equipped with the stable model structure, we indicate this by the subscript "st".

Lemma 6.27. *The model category $\mathcal{F}_{\mathrm{st}}$ is a monoidal $\mathcal{F}_{\mathrm{hf}}$-model category.*

Proof. Axiom **f4** implies that the homotopy functor structure is monoidal. Let $d_v \square i$ be a map in $\mathcal{D}$, and let $\mathcal{V}(w, -) \otimes j$ be a map in $\mathcal{P}_I$. Then the pushout product map in $\mathcal{F}$ is isomorphic to $d_{v \otimes w} \square (i \square j)$, i.e. a stable equivalence. $\square$

Lemma 6.28. *The stable model structure is proper.*

Proof. Left properness follows since the pointwise model structure is left proper. To prove right properness, it remains by 5.13 to check that the base change of a stable equivalence of pointwise fibrant homotopy functors along an hf-fibration of pointwise fibrant homotopy functors is a stable equivalence. This follows from 6.5 since $\mathbb{T}^\infty$ preserves pullbacks and pointwise fibrations of pointwise fibrant functors, and $\mathcal{F}_{\mathrm{pt}}$ is right proper. $\square$

In the stable structure we have the following important result, analogous to and in fact an easy a consequence of Theorem 4.11.

Theorem 6.29. *If* $\mathbf{f}\mathcal{V}$ *satisfies* **f0**, *then smashing with a cofibrant* $\mathcal{V}$*-functor in* $\mathcal{F}$ *preserves stable equivalences.*

Proof. Factor a stable equivalence f as a stable acyclic cofibration followed by a stable acyclic fibration, i.e. a pointwise acyclic fibration. By 4.11, we may assume f is a stable acyclic cofibration. The claim follows since $\mathcal{F}_{\mathrm{st}}$ is monoidal. □

Lemma 6.30. *Suppose* $\mathbf{f}\mathcal{V}$ *satisfies* **f0**. *Then the monoid axiom holds in* $\mathcal{F}_{\mathrm{st}}$.

Proof. The domains of the generating acyclic cofibrations $\mathcal{D}' := \mathcal{P}_J \cup \mathcal{H} \cup \mathcal{D}$ for the stable model structure on $\mathcal{F}$ are cofibrant. Because **f0** holds, 4.11 implies that every map in $\mathcal{D}' \wedge \mathcal{F}$ is a stable equivalence. The case of a map in $\mathcal{D}' \wedge \mathcal{F}$-cell follows similarly as in the proof of 6.25. □

7 A Quillen equivalence

In this section, we will discuss two natural choices for the domain category $\mathbf{f}\mathcal{V}$. One of the choices gives a Quillen equivalence between the stable model structure $\mathrm{Sp}(\mathcal{V}, T)_{\mathrm{st}}$ on spectra and $\mathcal{F}_{\mathrm{st}}$.

7.1 The choices

Let $\mathbf{f}\mathcal{V}_{\max}$ be the category of all cofibrant $\mathcal{V}$-finitely presentable objects, and $\mathbf{f}\mathcal{V}_{\min}$ the full subcategory of $\mathbf{f}\mathcal{V}_{\max}$ given by the objects v for which there exists an acyclic cofibration $T^n \overset{\sim}{\rightarrowtail} v$ for some $n \geq 0$. In the applications, the category of $\mathcal{V}$-finitely presentable objects is equivalent to a small category, hence its subcategories are valid domain categories. Axioms **f1**, **f2** and **f4** hold in both cases. If the domains and codomains of the maps in J' are $\mathcal{V}$-finite, then **f3** holds. The minimal choice satisfies a property which does not hold for the maximal choice in general.

Lemma 7.1. *A map* f *of pointwise fibrant homotopy functors in* $[\mathbf{f}\mathcal{V}_{\min}, \mathcal{V}]$ *is a stable equivalence if and only if* $\mathrm{ev}(f)$ *is a stable equivalence of spectra.*

Proof. This follows by definition of $\mathbf{f}\mathcal{V}_{\min}$. □

The evaluation functor is a right Quillen functor for both choices.

Lemma 7.2. *Evaluation* $\mathrm{ev}\colon \mathcal{F}_{\mathrm{st}} \longrightarrow \mathrm{Sp}(\mathcal{V}, T)_{\mathrm{st}}$ *is a right Quillen functor.*

Proof. Pointwise fibrations and pointwise acyclic fibrations are preserved by ev. The characterizations of stable fibrations using homotopy pullback squares can be compared using 6.19, which implies that ev preserves stable fibrations. □

To deduce that ev is the right adjoint of a Quillen equivalence for the minimal choice, we prove a property of the stable model structure of spectra which is independent of the choice of $\mathbf{f}\mathcal{V}$.

7.2 The unit of the adjunction

The following lemma is a crucial observation which depends on the hypotheses on T and the stabilization functor of spectra 6.17. Recall that i_* is the left adjoint of ev.

Lemma 7.3. *The canonical map*

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow (\mathcal{V}(T^n,-))^h = \mathrm{ev}((i_*\mathcal{V}_{T\mathrm{Sph}}(T^n,-))^h)$$

is a stable equivalence of T-spectra for all $n \geq 0$.

Proof. By 6.17, this follows if $\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow S^\infty((\mathcal{V}(T^n,-))^h)$ is a stable equivalence. The canonical map $\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \otimes T^n \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^0,-)$ consists of isomorphisms in degree n and on, so it is a stable equivalence. Note that the map $\mathcal{V}_{T\mathrm{Sph}}(T^0,-) \longrightarrow \mathcal{V}(T^0,-)$ is the identity, and that $\mathcal{V}(T^0,-) \longrightarrow (\mathcal{V}(T^0,-))^h$ is even a pointwise weak equivalence. 6.17 shows that $\mathcal{V}(T^0,-)^h \longrightarrow S^\infty((\mathcal{V}(T^0,-))^h)$ is a stable equivalence whose codomain is a stably fibrant T-spectrum. This uses that $(\mathcal{V}(T^0,-))^h$ is pointwise fibrant: T^0 is cofibrant and the input is fibrant. Hence the composition

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \otimes T^n \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^0,-) \longrightarrow \mathcal{V}(T^0,-)^h \longrightarrow S^\infty(\mathcal{V}(T^0,-)^h)$$

is a stable equivalence from the n-fold T-suspension of a cofibrant T-spectrum to a stably fibrant T-spectrum. The functor $-\otimes T$ is assumed to be a Quillen equivalence. Thus its adjoint $\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow \mathcal{V}(T^n, S^\infty((\mathcal{V}(T^0,-))^h))$ is a stable equivalence. Since T is $\mathcal{V}$-finite, the latter T-spectrum is isomorphic to $S^\infty((\mathcal{V}(T^n,-))^h)$. It is straightforward to check that these observations imply the claim. $\square$

Corollary 7.4. *The canonical map $c_E\colon E \longrightarrow \mathrm{ev}((i_*E)^h)$ is a stable equivalence for every cofibrant T-spectrum E.*

Proof. For any T-spectrum E and $A \in \mathrm{Ob}\,\mathcal{V}$, the map $c_{E\otimes A}$ is isomorphic to $c_E \otimes A$. Tensoring with the domains and codomains of the generating cofibrations preserves stable equivalences of spectra since the analogues statement holds for $\mathcal{V}$. The cofibrant T-spectra are precisely the retracts of $\mathrm{Sph}(I)$-cell complexes, where $\mathrm{Sph}(I)$ denotes the set of generating cofibrations $\{\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes i\}_{n\geq 0, i\in I}$ from [8, 1.8]. So it suffices to consider $\mathrm{Sph}(I)$-cell complexes. Recall that ev, i_* and $(-)^h$ preserve colimits, and stable equivalences of T-spectra are closed under sequential compositions. This allows to use transfinite induction. The induction step follows from the diagram:

$$\begin{tikzcd}
\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes ti \arrow[d, "\sim"'] & \mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes si \arrow[l] \arrow[r] \arrow[d, "\sim"'] & E \arrow[d, "\sim"'] \\
\mathcal{V}(T^n,-)^h\otimes ti & \mathcal{V}(T^n,-)^h\otimes si \arrow[l, "f"'] \arrow[r] & \mathrm{ev}((i_*E)^h)
\end{tikzcd}$$

The right hand vertical map is a stable equivalence by the induction hypothesis, and likewise for the other vertical maps by 7.3 and the argument given above. Note that f is not necessarily a cofibration of spectra, but it is pointwise in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$. Finally, strong left properness of $\mathcal{V}$ implies that the map induced on the pushouts of the rows in the diagram is a stable equivalence. $\square$

Corollary 7.5. *The functor* $\mathrm{ev}\colon [\mathbf{f}\mathcal{V}_{\min}, \mathcal{V}] \longrightarrow \mathrm{Sp}(\mathcal{V}, T)$ *is the right adjoint in a Quillen equivalence.*

Proof. Use 7.1 and 7.4. $\square$

Let $\mathcal{V}$ be the usual model category $\mathbf{sSet}_* = \mathcal{S}$ of pointed simplicial sets, and let T be the circle $S^1 = \Delta^1/\partial\Delta^1$. The n-sphere S^n is the n-fold smash product of S^1. Then $\mathbf{f}\mathcal{S}_{\max}$ is the full subcategory given by the finitely presentable pointed simplicial sets, and $\mathbf{f}\mathcal{S}_{\min}$ is the full subcategory of pointed simplicial sets K for which there exists an acyclic cofibration $S^n \overset{\sim}{\rightarrowtail} K$ for some $n \geq 0$. By [11] implies that the canonical functor $[\mathbf{f}\mathcal{S}_{\max}, \mathcal{S}] \longrightarrow [\mathbf{f}\mathcal{S}_{\min}, \mathcal{S}]$ is the right adjoint in a Quillen equivalence of stable model categories. This uses that all pointed simplicial sets are generated by spheres.

In general, one needs to distinguish between using $\mathbf{f}\mathcal{V}_{\min}$ and $\mathbf{f}\mathcal{V}_{\max}$. For example, if $\mathcal{V} = \mathcal{S}$ and T is the coproduct $S^0 \vee S^0$, we claim the corresponding stable model categories are different. In this case, $\mathbf{f}\mathcal{S}_{\max}$ is as above, while $\mathbf{f}\mathcal{S}_{\min}$ is the full subcategory of finitely presentable pointed simplicial sets which are weakly equivalent to a discrete pointed simplicial set with $1+2^n$ points for some $n \geq 0$. To show that the resulting stable model structures are not Quillen equivalent via the restriction functor, we will describe a map $f\colon X \longrightarrow Y$ of $S^0 \vee S^0$-stably fibrant functors in $\mathbf{SF} := [\mathbf{f}\mathcal{S}_{\max}, \mathcal{S}]$ that is not a weak equivalence, although $f(K)$ is a weak equivalence for every discrete pointed simplicial set. Let Y be the constant functor with value $*$. Let X be the stably fibrant replacement of X', which maps K to the connected component of K containing the basepoint. Since $X'(K \wedge \Delta^n_+) = X'(K) \wedge \Delta^n_+$, X' is enriched over $\mathcal{S}$. Clearly, X' is a homotopy functor, so using an enriched fibrant replacement functor $R\colon \mathcal{S} \longrightarrow \mathcal{S}$, the $S^0 \vee S^0$-stably fibrant replacement X of X' maps K to

$$X(K) = \mathrm{colim}_n \mathcal{S}((S^0 \vee S^0)^n, RX'((S^0 \vee S^0)^n \wedge K)).$$

If K is discrete, $X(K) = *$, hence the map $X \longrightarrow *$ is a weak equivalence in $[\mathbf{f}\mathcal{S}_{\min}, \mathcal{S}]$. However, $X(S^1)$ is weakly equivalent to a countable product of a countable coproduct of S^1 with itself, and hence not contractible.

8 Algebraic structure

This section recalls the important algebraic structures which $\mathcal{F}_{\mathrm{st}}$ supports if the monoid axiom holds. Recall that $\mathcal{F}_{\mathrm{st}}$ satisfies the monoid axiom if $\mathbf{f}\mathcal{V}$ satisfies **f0**, cp. 6.30. Fix a $\mathcal{V}$ which satisfies the conditions listed in the beginning of Section 5, and a small full sub-$\mathcal{V}$-category $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ which satisfies the axioms **f1**–**f4**.

8.1 Recollections

In a symmetric monoidal category like $(\mathcal{F}, \wedge, \mathbb{I})$ there are notions of algebras and modules over the algebras. Recall that an $\mathbb{I}$-algebra is a monoid in $(\mathcal{F}, \wedge, \mathbb{I})$ or just an $\mathcal{F}$-category with only one object.
If A is an $\mathbb{I}$-algebra, a (left) A-module M is an object in $\mathcal{F}$ with an appropriate action of A. It can alternatively be described as an $\mathcal{F}$-functor from A to $\mathcal{F}$. The category of A-modules mod_A is then an $\mathcal{F}$-category. There is a smash product $\wedge_A\colon \mathrm{mod}_{A^{\mathrm{op}}} \wedge \mathrm{mod}_A \longrightarrow \mathcal{F}$. If $M \in \mathrm{mod}_{A^{\mathrm{op}}}$ and $N \in \mathrm{mod}_A$, then $M \wedge_A N$ is the coequalizer of $M \wedge A \wedge N \rightrightarrows M \wedge N$. Likewise, for A-modules M and N the function object $\mathrm{mod}_A(M,N)$ in $\mathcal{F}$ is the equalizer of $\mathcal{F}(M,N) \rightrightarrows \mathcal{F}(A \wedge M, N)$.
If k is a *commutative* $\mathbb{I}$-algebra, recall that $\mathrm{mod}_k \cong \mathrm{mod}_{k^{\mathrm{op}}}$ and mod_k is a closed symmetric monoidal category under $\wedge_k$ with internal morphism object $\mathrm{mod}_k(M,N)$. A k-algebra is a monoid in $(\mathrm{mod}_k, \wedge_k, k)$ or a mod_k-category with one object. With this notation, notice that $\mathrm{mod}_{\mathbb{I}}$ is $\mathcal{F}$.

8.2 The model structures

Definition 8.1. If A is an $\mathbb{I}$-algebra and k is a commutative $\mathbb{I}$-algebra, then a map in mod_A or alg_k is called a *weak equivalence* (resp. *fibration*) if it is so when considered in $\mathcal{F}_{\mathrm{st}}$. Cofibrations are defined by the left lifting property.

Remark 8.2. Note that we chose the stable model structure as our basis. This is fixed in the following (at least on the top level), so the missing prefix "stable" from fibrations and weak equivalences should not be a source of confusion.

The next result is due to Schwede and Shipley [15, 4.1].

Theorem 8.3. *Suppose that $\mathcal{F}_{\mathrm{st}}$ satisfies the monoid axiom. With the structures described above, the following is true.*

- *Let $A \in \mathcal{F}$ be an $\mathbb{I}$-algebra. Then the category mod_A of (left) A-modules is a cofibrantly generated model category.*
- *Let $k \in \mathcal{F}$ be a commutative $\mathbb{I}$-algebra. Then the category of k-modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*
- *Let $k \in \mathcal{F}$ be a commutative $\mathbb{I}$-algebra. Then the category alg_k of k-algebras is a cofibrantly generated model category.*

Note that we did not state the hypothesis that all objects in $\mathcal{F}$ are small. Since $\mathcal{V}$ is weakly finitely generated, $\mathcal{F}_{\mathrm{st}}$ is so too. The smallness of the domains and codomains of the generating cofibrations and generating acyclic cofibrations in $\mathcal{F}_{\mathrm{st}}$ carries over to the relevant smallness conditions needed to prove the theorem. See also [15, 2.4].

Lemma 8.4. *Suppose that* $\mathbf{f}\mathcal{V}$ *satisfies* $\mathbf{f0}$. *Let* A *be an* $\mathbb{I}$*-algebra. Then for any cofibrant* A*-module* N*, the functor* $-\wedge_A N$ *takes weak equivalences in* $\mathrm{mod}_{A^{\mathrm{op}}}$ *to weak equivalences in* $\mathcal{F}_{\mathrm{st}}$.

Proof. Given a weak equivalence in mod_A, factor it as an acyclic cofibration followed by an acyclic fibration. The only trouble is with the acyclic fibration, but this is a pointwise acyclic fibration, and the argument can be phrased in the analogous theory for A-modules built on the pointwise structure $\mathcal{F}_{\mathrm{pt}}$. In this case, the generating cofibrations in mod_A are of the form $A\wedge S \xrightarrow{A\wedge i} A\wedge T$ where $S \overset{i}{\rightarrowtail} T$ is a generating cofibration in $\mathcal{F}_{\mathrm{pt}}$. The argument of 4.11 goes through verbatim: smashing commutes with colimits and $A\wedge_A S \cong S$. $\square$

Lemma 8.4 and [15, 4.3] imply:

Corollary 8.5. *Suppose that* $\mathbf{f}\mathcal{V}$ *satisfies the axiom* $\mathbf{f0}$. *Let* $f\colon A \xrightarrow{\sim} B$ *be a weak equivalence of* $\mathbb{I}$*-algebras. Then extension and restriction of scalars define the Quillen equivalence*

$$\mathrm{mod}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{mod}_B.$$

If A *and* B *are commutative, there is the Quillen equivalence*

$$\mathrm{alg}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{alg}_B.$$

9 Equivariant Stable Homotopy Theory

Let $\mathcal{S}$ be the category of pointed simplicial sets, or *spaces* for short. The finitely presentable spaces are the ones with only finitely many non-degenerate simplices, thus we may call these *finite*. A simplicial functor in the sense of [11, 4.5] is an $\mathcal{S}$-functor from the category of finite spaces to the category of all spaces. In [11], Lydakis showed how simplicial functors give rise to a monoidal model category which is Quillen equivalent to the model category of spectra. Thus simplicial functors model the stable homotopy category. The purpose of this section is to use the machinery developed in the main part of the paper to give a functor model for the equivariant stable homotopy category. For technical reasons we will only consider finite groups. Fix a finite group G with multiplication $\mu\colon G\times G \longrightarrow G$.

9.1 Equivariant spaces

The category $G\mathcal{S}$ of G*-spaces* consists of pointed simplicial sets with a basepoint preserving left G-action. Note that G_+ is a $\mathcal{S}$-category with only one object and composition $G_+\wedge G_+ \longrightarrow G_+$ induced by μ. One can identify $G\mathcal{S}$ with the category $[G_+,\mathcal{S}]$ of $\mathcal{S}$-functors $K : G_+ \longrightarrow \mathcal{S}$. We will often write (uK, a_K) for K to stress the underlying space $uK \in \mathrm{Ob}\,\mathcal{S}$, i.e. the value of K at the single object, and the left G-action $a_K: G_+ \wedge uK \longrightarrow uK$. Note that a_K is adjoint to $\hom^K: G_+ \longrightarrow \mathcal{S}(uK, uK)$, where $g \longmapsto g: uK \longrightarrow uK$. According to 2.4, $G\mathcal{S}$ is a closed $\mathcal{S}$-module. The functor $u: G\mathcal{S} \longrightarrow \mathcal{S}$ has a left $\mathcal{S}$-adjoint $G_+ \wedge -: \mathcal{S} \longrightarrow G\mathcal{S}$ by 2.5. Consider the G-space $G_+ \wedge K: G_+ \longrightarrow \mathcal{S}$. Its underlying space is $G_+ \wedge K$, with left G-action

$$G_+ \wedge (G_+ \wedge K) \xrightarrow{\cong} (G \times G)_+ \wedge K \xrightarrow{\mu_+ \wedge K} G_+ \wedge K.$$

Similarly, the right $\mathcal{S}$-adjoint of u is given by $K \longmapsto \times_{g \in G} K_g$, the G-fold product of K where $h \in G$ sends K_g to K_{hg} via the identity. Let $(-)^{\wedge G}: \mathcal{S} \longrightarrow G\mathcal{S}$ be the functor whose value on K is the G-fold smash product $K^{\wedge G} = \wedge_{g \in G} K_g$ of K, where G acts by permuting the factors as above. Another functor we consider is $\mathrm{ct}: \mathcal{S} \longrightarrow G\mathcal{S}$. The G-space $\mathrm{ct}K$ is constant, i.e. the underlying space is K and $\hom^{\mathrm{ct}K}: G_+ \longrightarrow \mathcal{S}(K, K)$ sends g to either the identity map or the trivial map.

Let $\Delta G: G \longrightarrow G \times G$ be the diagonal map. The *smash product* $K \wedge L$ of two G-spaces $K, L : G_+ \longrightarrow \mathcal{S}$ is given by the composition

$$G_+ \xrightarrow{\Delta G_+} (G \times G)_+ \cong G_+ \wedge G_+ \xrightarrow{K \wedge L} \mathcal{S} \wedge \mathcal{S} \xrightarrow{\wedge} \mathcal{S}.$$

The right hand smash product uses [2, 6.2.9]. In other words, the smash product of G-spaces is defined on the underlying spaces and G acts diagonally. For this reason we denote the smash product of G-spaces by $\wedge$. If G is commutative, another closed symmetric monoidal product of G-spaces exists by 2.6.

Proposition 9.1. *The category $(G\mathcal{S}, \wedge, \mathrm{ct}S^0)$ is closed symmetric monoidal. The functors u, ct and $(-)^{\wedge G}$ are strict symmetric monoidal, and $G_+ \wedge -$ is lax symmetric monoidal.*

Lemma 9.2. *Let $K: G_+ \longrightarrow \mathcal{S}$ be a G-space. The following are equivalent.*

1. *K is $G\mathcal{S}$-finitely presentable.*
2. *K is finitely presentable.*
3. *uK is finite.*

Proof. Let $\mathrm{Fix}(G, -)$ be the $\mathcal{S}$-functor that maps a G-space (uK, a_K) to the subspace $\mathrm{Fix}(G, K) = \{x \in uK \mid a_K\overline{(g, x)} = x \text{ for all } g \in G\}$ fixed under the action of G. Equivalently, $\mathrm{Fix}(G, K)$ is $\lim(K: G_+ \longrightarrow \mathcal{S})$. Note that $\mathrm{Fix}(G, -)$ commutes with filtered colimits since G_+ is a finite index category. $\mathrm{Fix}(G, -)$ is the right $\mathcal{S}$-adjoint of $\mathrm{ct}: \mathcal{S} \longrightarrow G\mathcal{S}$. In particular, the G-space $\mathrm{ct}S^0$ is finitely presentable. This proves the implication $1 \Rightarrow 2$. Likewise, the right adjoint of u commutes with filtered colimits, thus $2 \Rightarrow 3$.

It remains to prove $3 \Rightarrow 1$. Let $D: \mathcal{I} \longrightarrow G\mathcal{S}$ be a functor where $\mathcal{I}$ is filtered, and consider the canonical map $f_K: \mathrm{colim}_{\mathcal{I}} G\mathcal{S}(K, D) \longrightarrow G\mathcal{S}(K, \mathrm{colim}_{\mathcal{I}} D)$. Since colimits in $G\mathcal{S}$ are formed on underlying spaces, $u(f_K)$ is the canonical map $\mathrm{colim}_{\mathcal{I}} \mathcal{S}(uK, u \circ D) \longrightarrow \mathcal{S}(uK, \mathrm{colim}_{\mathcal{I}} u \circ D)$. If uK is finite, $u(f_K)$ is an isomorphism, which implies that f_K is an isomorphism since the G-action on the domain coincides with the G-action on the codomain. We are done. $\square$

The full subcategory of finitely presentable G-spaces is equivalent to a small category, which can be chosen to be closed under the smash product of 9.1. Often we will refer to *finite* G-spaces instead of finitely presentable ones.

Lemma 9.3. *Any G-space is the filtered colimit of its finite sub-G-spaces.*

9.2 Unstable equivariant homotopy theory

Theorems 4.2 and 4.4 give $G\mathcal{S}$ the *coarse* model structure, with weak equivalences and fibrations defined on underlying spaces. A cofibration is an injective map $f\colon K \longrightarrow L$ where G acts freely on the complement of $f(K)$ in L. Hence, the cofibrant G-spaces are the G-spaces with a free G-action away from the basepoint. In the following, we will consider another model structure on $G\mathcal{S}$. If H is a subgroup of G, let G/H_+ be the pointed G-space with action $g \cdot g'H := (gg')H$. Consider the $\mathcal{S}$-functor $G/H_+ \wedge\colon \mathcal{S}_+ \longrightarrow G\mathcal{S}_+$, $K \longmapsto G/H_+ \wedge K$, with trivial action on K. Its right $\mathcal{S}$-adjoint is $\mathrm{Fix}(H,-)\colon G\mathcal{S}_+ \longrightarrow \mathcal{S}_+$ which maps L to the space of fixed points under the action of H on L. Note that $\mathrm{Fix}(H,-)$ coincides with $\mathcal{S}_{G\mathcal{S}}(G/H_+,-)$. If H and H' are two subgroups of G, there is a natural isomorphism $\mathrm{Fix}(H', G/H_+ \wedge K) \cong \mathrm{Fix}(H', G/H_+) \wedge K$.

Definition 9.4. A map f in $G\mathcal{S}$ is a *G-weak equivalence* if $\mathrm{Fix}(H,f)$ is a weak equivalence in $\mathcal{S}$ for every subgroup H of G. Likewise for *G-fibrations*.

Theorem 9.5. *There is a proper monoidal model structure on $G\mathcal{S}$ with G-weak equivalences as weak equivalences and G-fibrations as fibrations. Cofibrations are the injective maps. One can choose generating acyclic cofibrations and generating cofibrations with finitely presentable domains and codomains.*

Proof. This result is well-known. A proof is included for completeness. To prove the existence of the model structure, we will apply [7, 2.1.19]. Let

$$I_G := \{G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n\geq 0,\, H \text{ subgroup of } G}$$

and

$$J_G := \{G/H_+ \wedge (\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{n\geq 1,\, 0\leq i\leq n,\, H \text{ subgroup of } G}.$$

It is clear by adjointness that a map is a G-fibration if and only if it is in J_G-inj, or a G-fibration and a G-weak equivalence if and only if it is in I_G-inj. From 9.2, the domains and codomains of the maps in I_G and J_G are finite. The natural isomorphism $\mathrm{Fix}(H', G/H_+ \wedge K) \cong \mathrm{Fix}(H', G/H_+) \wedge K$ for subgroups H and H' of G, shows that maps in J_G are G-weak equivalences. The existence of the model structure follows, if every map in J_G-cell is a G-weak equivalence. Since $\mathrm{Fix}(H',-)$ commutes with sequential colimits, it suffices to check that the cobase change of a map in J_G is a G-weak equivalence. Fix a subgroup H' and consider the pushout diagram:

$$\begin{tikzcd}
A := (G/H \times \Lambda^n_i)_+ \arrow[r] \arrow[d, hook] & K \arrow[d, hook] \\
B := (G/H \times \Delta^n)_+ \arrow[r] & L
\end{tikzcd}$$

The induced map $\mathrm{Fix}(H',B)\cup_{\mathrm{Fix}(H',A)}\mathrm{Fix}(H',K) \longrightarrow \mathrm{Fix}(H',L)$ is injective. Surjectivity follows since $\mathrm{Fix}(H',-)$ preserves injective maps. Thus any cobase change of a map in J_G is a G-weak equivalence, and the model structure exists. Any map in I_G-cell is clearly injective. Conversely, by considering fixed point spaces it follows that any injective map is contained in I_G-cell. The statement about the cofibrations follows, and also left properness. Right properness holds since $\mathrm{Fix}(H,-)$ commutes with pullbacks and $\mathcal{S}$ is right proper.
The pushout product map of injective maps is again injective, so consider the pushout product map

$$G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+)\Box G/H'_+ \wedge (\Lambda^m_i \hookrightarrow \Delta^m)_+ \cong (G/H \times G/H')_+ \wedge i,$$

where i is a weak equivalence of spaces. Since there is an isomorphism of spaces $\mathrm{Fix}(H'',(G/H \times G/H')_+ \wedge i) \cong \mathrm{Fix}(H'',(G/H \times G/H')_+) \wedge i$, the pushout product map of a generating cofibration and a generating acyclic cofibration is again acyclic. Hence the model structure is monoidal. The monoid axiom then holds, since all G-spaces are cofibrant. □

We will refer to the model structure in 9.5 as the *fine* model structure. The regular representation $S^{\wedge G}$ is the G-fold smash product of $S^1 = \Delta^1/\partial\Delta^1$ where G acts by permuting the factors. Its geometric realization is homeomorphic – as a G-space – to the one-point compactification of the real vector space $\mathbb{R}^G$ of maps $G \longrightarrow \mathbb{R}$. The G-space $S^{\wedge G}$ is finite, since G is a finite group.

9.3 Stable equivariant homotopy theory

Let $\mathbf{f}G\mathcal{S}$ denote the full sub-$G\mathcal{S}$-category given by the finite G-spaces. It is equivalent to a small $G\mathcal{S}$-category. Objects of the enriched functor category $G\mathcal{F} = [\mathbf{f}G\mathcal{S}, G\mathcal{S}]$ will be called *G-simplicial functors*. If G is the trivial group, then $G\mathcal{F}$ is Lydakis' category of simplicial functors [11, 4.4]. Let $\wedge$ denote the smash product of G-simplicial functors. The unit of $\wedge$ is the inclusion $\mathbb{S}^G = \mathbb{I}\colon \mathbf{f}G\mathcal{S} \hookrightarrow G\mathcal{S}$. All G-spaces are cofibrant in the fine model structure.

Definition 9.6. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is a

- *pointwise weak equivalence* if $f(K)$ is a G-weak equivalence for all finite G-spaces K,
- *pointwise fibration* if $f(K)$ is a G-fibration for all finite G-spaces K,
- *cofibration* if f has the left lifting property with respect to all pointwise acyclic fibrations.

Theorem 9.7. *The category $G\mathcal{F}$, equipped with the classes described in 9.6, is a monoidal proper model category satisfying the monoid axiom. Generating cofibrations and generating acyclic cofibrations can be chosen with finitely presentable domains and codomains. Finally, smashing with a cofibrant G-simplicial functor preserves pointwise weak equivalences.*

Proof. ¿From 4.2, $\mathcal{P}_{I_G} = \{G\mathcal{S}(K,-) \wedge i\}_{i \in I_G,\, K \text{ finite}}$ are the generating cofibrations, and $\mathcal{P}_{J_G} = \{G\mathcal{S}(K,-) \wedge j\}_{j \in J_G,\, K \text{ finite}}$ are the generating acyclic cofibrations. The model structure is monoidal and satisfies the monoid axiom by 4.4. Properness holds by 4.8. The functor $- \wedge X$ preserves pointwise weak equivalences for cofibrant X since 4.11 holds, cp. 9.3. $\square$

Let us write $G\mathcal{F}_{\mathrm{pt}}$ for the *pointwise* model structure. To define the homotopy functor model structure on $G\mathcal{F}$, let $\rho\colon \mathrm{Id}_{G\mathcal{S}} \longrightarrow R$ denote the enriched fibrant replacement functor from 3.3.2 applied to the set J_G. Denote by $\mathbb{I}_*X$ the enriched left Kan extension of $X\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$ along $\mathbb{I}\colon \mathbf{f}G\mathcal{S} \hookrightarrow G\mathcal{S}$, and by X^h the composition $\mathbb{I}_*X \circ R \circ \mathbb{I}$. Then X^h defines an endofunctor of $G\mathcal{F}$ and there is a natural transformation $\mathrm{Id}_{G\mathcal{F}} \longrightarrow (-)^h$.

Definition 9.8. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is an

- *hf-equivalence* if f^h is a pointwise weak equivalence.
- *hf-fibration* if f is a pointwise fibration and there is a homotopy pullback square in $G\mathcal{S}$

$$\begin{tikzcd} XK \arrow[r] \arrow[d, "f(K)"'] & XL \arrow[d, "f(L)"] \\ YK \arrow[r] & YL \end{tikzcd}$$

  for every G-weak equivalence $K \xrightarrow{\sim} L$ of finitely presentable G-spaces.

Lemma 9.9. *A map of G-simplicial functors is a pointwise acyclic fibration if and only if it is an hf-fibration and an hf-equivalence.*

Proof. The definition of hf-equivalences in 5.2 uses the filtered fibrant replacement functor Φ^{J_G} from 3.3.3. All G-spaces are cofibrant, so the canonical map $\omega_K\colon \Phi^{J_G}K \longrightarrow RK$ is a G-weak equivalence of fibrant G-spaces, hence a simplicial homotopy equivalence. Recall that $\hbar(X) = \mathbb{I}_*X \circ \Phi^{J_G}$. If K is finite, the induced map $\mathbb{I}_*X(\omega_K)\colon \hbar(X)(K) \longrightarrow X^hK$ is a simplicial homotopy equivalence by 2.11, in particular a G-weak equivalence. It follows that f^h is a pointwise weak equivalence if and only if $\hbar(f)(K)$ is a G-weak equivalence for every finite G-space K. The arguments in 5.4 show that the hf-fibrations in 9.8 allow the same characterization as general hf-fibrations, cp. 5.6. Since every G-space is cofibrant, any G-weak equivalence of finite G-spaces can be factored as an acyclic cofibration of finite G-spaces and a simplicial homotopy equivalence. The lemma follows from 5.8. $\square$

Theorem 9.10. *The category $G\mathcal{F}$, equipped with the classes of hf-equivalences, hf-fibrations and cofibrations, is a proper monoidal model category satisfying the monoid axiom. Smashing with a cofibrant G-simplicial functor preserves hf-equivalences. One can choose generating cofibrations and generating acyclic cofibrations with finitely presentable domains and codomains.*

Proof. The model structure exists according to 5.10, is monoidal by 5.12 and proper by 5.13. A factorization argument and 9.7 imply the claim concerning cofibrant G-simplicial functors. The monoid axiom follows easily.
The generating acyclic cofibrations are $\mathcal{H}_G \cup \mathcal{P}_{J_G}$. It remains to define $\mathcal{H}_G$. A G-weak equivalence of finite G-spaces $w\colon K \longrightarrow L$ induces an hf-equivalence $G\mathcal{S}(w,-)\colon G\mathcal{S}(L,-) \longrightarrow G\mathcal{S}(K,-)$. The simplicial mapping cylinder gives a cofibration $c_w : G\mathcal{S}(L,-) \rightarrowtail C_w$. Let $i \in I_G$. Then $\mathcal{H}_G$ is the set of pushout product maps $\{c_w \square i\}$, cp. Subsection 5.2. It is also possible, without changing the model structure, to consider for w only acyclic cofibrations of finite G-spaces. □

Denote the model category in 9.10 by $G\mathcal{F}_{\mathrm{hf}}$. In this category, $X \longrightarrow R \circ X^h$ is a fibrant replacement of X. Recall the functor $\mathbf{S}\colon G\mathcal{F} \longrightarrow G\mathcal{F}$ mapping X to $G\mathcal{S}(S^{\wedge G}, X(S^{\wedge G} \wedge -))$ and the natural transformation $s\colon \mathrm{Id}_{G\mathcal{F}} \longrightarrow \mathbf{S}$ obtained pointwise as the adjoint of $\mathrm{sw}^X_{S^{\wedge G}}(K)\colon XK \wedge S^{\wedge G} \longrightarrow X(S^{\wedge G} \wedge K)$. This map is the adjoint of

$$S^{\wedge G} \xrightarrow{\eta_K S^{\wedge G}} G\mathcal{S}(K, S^{\wedge G} \wedge K) \xrightarrow{\mathrm{hom}^X_{K, S^{\wedge G} \wedge K}} G\mathcal{S}(XK, X(S^{\wedge G} \wedge K)).$$

Let $\mathbf{S}^\infty(X)$ denote the colimit of $X \xrightarrow{s(X)} \mathbf{S}(X) \xrightarrow{\mathbf{S}(s(X))} \mathbf{S}(\mathbf{S}(X)) \longrightarrow \cdots$, and write $s\colon \mathrm{Id}_{G\mathcal{F}} \longrightarrow \mathbf{S}^\infty$ for the canonically induced natural transformation.

Definition 9.11. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is a

- *stable equivalence* if $\mathbf{S}^\infty(R \circ f^h)$ is a pointwise weak equivalence,
- *stable fibration* if f is an hf-fibration and

$$\begin{tikzcd} X \arrow[r, "s(X)"] \arrow[d, "f"'] & \mathbf{S}(X) \arrow[d, "\mathbf{S}(f)"] \\ X \arrow[r, "s(X)"] & \mathbf{S}(X) \end{tikzcd}$$

  is a homotopy pullback square in $G\mathcal{F}_{\mathrm{pt}}$.

Lemma 9.12. *A map of G-simplicial functors is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. ¿From the proof of 9.9, one sees that the definition of stable equivalences in 9.11 agrees with 6.2. The result follows from 6.14. □

The definition of stable fibrations leads to a set of generating stable acyclic cofibrations as in section 6.4. Recall the definition of $S^{\wedge G}$-spectra in $G\mathcal{S}$.

Definition 9.13. A *G-spectrum* E consists of a sequence $E_0, E_1, \cdots$ of G-spaces, together with structure maps $E_n \wedge S^{\wedge G} \longrightarrow E_{n+1}$. A *map* $f\colon E \longrightarrow F$

of G-spectra is a sequence of maps $f_n\colon E_n \longrightarrow F_n$ making the diagram

$$\begin{tikzcd} E_n \wedge S^{\wedge G} \arrow[r] \arrow[d, "f_n \wedge S^{\wedge G}"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ F_n \wedge S^{\wedge G} \arrow[r] & F_{n+1} \end{tikzcd}$$

commutative for every $n \geq 0$.

By 2.12, $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ is isomorphic to the enriched category $[S^{\wedge G}\mathrm{Sph}, G\mathcal{S}]$ of G-simplicial functors from the category of $S^{\wedge G}$-spheres $S^{\wedge G}\mathrm{Sph}$ to $G\mathcal{S}$. Thus 4.2 gives $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ a pointwise model structure. A fibrant replacement functor in this model structure is $E \longmapsto R \circ E$, where R is defined in 3.3.2.
Recall that the adjoints of the structure maps of a G-spectrum E can be viewed as a natural map $E \longrightarrow \Omega_G\mathrm{sh}E$, where the nth structure map of the G-spectrum $\Omega_G\mathrm{sh}E$ is $G\mathcal{S}(S^{\wedge G}, E_{n+1} \longrightarrow G\mathcal{S}(S^{\wedge G}, E_{n+2}))$, the $S^{\wedge G}$-loops of the $n+1$-th structure map of E. Denote this natural transformation by $\mathrm{st}\colon \mathrm{Id} \longrightarrow \mathrm{St} = \Omega_G \circ \mathrm{sh}$, and let $\mathrm{st}^\infty\colon \mathrm{Id} \longrightarrow \mathrm{St}^\infty$ be the colimit of $\mathrm{Id} \xrightarrow{\mathrm{st}} \mathrm{St} \xrightarrow{\mathrm{St(st)}} \mathrm{St}^2 \longrightarrow \cdots$.
The stable model structure on G-spectra, which has the same cofibrations as the pointwise model structure, is defined as follows.

Definition 9.14. A map $f\colon E \longrightarrow F$ of G-spectra is a *stable equivalence* if $\mathrm{St}^\infty(R \circ f)$ is a pointwise weak equivalence, and a *stable fibration* if f is a pointwise fibration such that

$$\begin{tikzcd} E \arrow[r, "\mathrm{st}"] \arrow[d, "f"'] & \mathrm{St}(E) \arrow[d, "\mathrm{St}(f)"] \\ F \arrow[r, "\mathrm{st}"] & \mathrm{St}(F) \end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure.

¿From now on, we consider G-spectra with the stable model structure. Note that geometric realization induces a functor from $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ to the category of G-prespectra, cf. [3, 3.2]. It is plausible that this functor induces an equivalence of homotopy categories. Hence the homotopy category of $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ is the G-equivariant stable homotopy category. Let us illustrate this by introducing spectra which are indexed on more general representations.
A *G-representation* is a finite-dimensional euclidean vector space on which G acts via linear isometries. A G-representation V is *irreducible* if zero and V are the only sub-G-representations. Let $\mathrm{Irr}_G = \{W_1, \ldots, W_r\}$ be a complete set of pairwise non-isomorphic irreducible G-representations. Every G-representation is isomorphic to a direct sum of representations in Irr_G. That is, given a G-representation V, there exist unique natural numbers $(n_1, \ldots, n_r)$ such that V is isomorphic to $W_1^{\oplus n_1} \oplus \cdots \oplus W_r^{\oplus n_r}$.

If V is a G-representation, let S^V_{top} denote the one-point compactification of V, with ∞ as the (G-fixed) basepoint. This is a finite G-CW-complex. One has $S^{V\oplus W}_{\mathrm{top}} \cong S^V_{\mathrm{top}} \wedge S^W_{\mathrm{top}}$. Furthermore, one can choose a finite G-space S^V such that the geometric realization $|S^V|$ is homeomorphic to S^V_{top}. Let Rep be the $G\mathcal{S}$-category with objects smash products

$$S^{n_1,\ldots,n_r} := S^{W_1} \wedge \cdots \wedge S^{W_1} \wedge S^{W_2} \wedge \cdots \wedge S^{W_r}$$

and morphisms $G\mathcal{S}_{\mathrm{Rep}}(S^{n_1,\ldots,n_r}, S^{n_1+k_1,\ldots,n_r+k_r}) := S^{k_1,\ldots,k_r}$. Hence Rep contains essentially all G-representations. The G-sphere $S^{\wedge G}$ does not reside in this category, but it contains a G-space $\tilde{S}^{\wedge G}$ with homeomorphic realization. Consequently, the stable model categories $[S^{\wedge G}, G\mathcal{S}]$ and $[\tilde{S}^{\wedge G}, G\mathcal{S}]$ are Quillen equivalent, and there are inclusions $\tilde{S}^{\wedge G} \xhookrightarrow{j} \mathrm{Rep} \hookrightarrow \mathbf{f}G\mathcal{S}$ inducing functors

$$G\mathcal{F} = [\mathbf{f}G\mathcal{S}, G\mathcal{S}] \longrightarrow [\mathrm{Rep}, G\mathcal{S}] \xrightarrow{j^*} [\tilde{S}^{\wedge G}, G\mathcal{S}].$$

Note that j is a full inclusion.

It is straightforward to define the stabilization $\mathrm{St}^\infty_{\mathrm{Rep}} X$ of a pointwise fibrant $G\mathcal{S}$-functor $X\colon \mathrm{Rep} \longrightarrow G\mathcal{S}$. Since every G-representation is a direct summand of a direct sum of copies of the regular representation $\mathbb{R}^G$, $j^*(\mathrm{St}^\infty_{\mathrm{Rep}} X)$ is equivalent to $\mathrm{St}^\infty j^* X$. This shows that the stable model structure on $[\mathrm{Rep}, G\mathcal{S}]$ is Quillen equivalent to the stable model structure on G-spectra. In particular, smashing with S^V is a Quillen equivalence of G-spectra for every G-representation V.

Let us turn to the last ingredient needed in the proof of 9.16.

Proposition 9.15. *The functor* $- \wedge S^{\wedge G}\colon \mathrm{Sp}(G\mathcal{S}, S^{\wedge G}) \longrightarrow \mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ *is a Quillen equivalence.*

Proof. We will show that $S^{\wedge G}$ is G-weakly equivalent to a symmetric G-space. The result follows then by [8, 10.3]. A G-space K is symmetric if there exists a map H such that the following diagram commutes, where $\mathrm{cyc}\colon K^{\wedge 3} \longrightarrow K^{\wedge 3}$ is the cyclic permutation map.

$$\begin{tikzcd}
K^{\wedge 3} \wedge \mathrm{ct}S^0 \arrow[r, "K^{\wedge 3}\wedge \mathrm{ct}i_0"] \arrow[dr, "\mathrm{id}_{K^{\wedge 3}\wedge \mathrm{ct}S^0}"'] & K^{\wedge 3} \wedge \mathrm{ct}\Delta^1_+ \arrow[d, "H"] & K^{\wedge 3} \wedge \mathrm{ct}S^0 \arrow[l, "K^{\wedge 3}\wedge \mathrm{ct}i_1"'] \arrow[dl, "\mathrm{cyc}\wedge \mathrm{ct}S^0"] \\
& K^{\wedge 3} \wedge \mathrm{ct}S^0 &
\end{tikzcd}$$

Consider first the trivial group. The cyclic permutation map on the geometric realization $|S^1|^{\wedge 3}$ is homotopic to the identity. Thus the singular complex $K := \mathrm{sing}|S^1|$ is a symmetric space and there is a weak equivalence $S^1 \longrightarrow K$. We claim the induced map $S^{\wedge G} \longrightarrow K^{\wedge G}$ of G-fold smash products is a G-weak equivalence. To see this, choose a subgroup H of G and write the underlying set of G as the union of the cosets gH. Fix a space L, and an element $x = x_{g_1} \wedge \cdots \wedge x_{g_n}$ of $L^{\wedge G}$ where n is the order of G and $x_{g_k} \in L$ for

every k. Note that x is invariant under the action of all $h \in H$ if and only if $x_{g_k} = x_{g_j}$ for all g_k, g_j in the same coset of H. Thus $\mathrm{Fix}(H, L^{\wedge G})$ is – up to a natural isomorphism – the G/H-fold smash product of L. In particular, we get an expression of the "diagonal" $d_L\colon \mathrm{ct}L \longrightarrow L^{\wedge G}$ as the adjoint of the isomorphism $L \longrightarrow \mathrm{Fix}(G, L^{\wedge G})$.
We will use essentially three maps to define a homotopy from the identity map to the cyclic permutation map on $K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G}$: (1) the natural isomorphism, cp. 9.1, of G-spaces $f_{K,L}\colon K^{\wedge G} \wedge L^{\wedge G} \longrightarrow (K \wedge L)^{\wedge G}$ which rearranges the factors, (2) the diagonal $d = d_{\Delta^1_+}\colon \mathrm{ct}\Delta^1_+ \longrightarrow (\Delta^1_+)^{\wedge G}$, and (3) the homotopy $F : K \wedge K \wedge K \wedge \Delta^1_+ \longrightarrow K \wedge K \wedge K$ from the cyclic permutation map to the identity map. Consider now the composition

$$\begin{tikzcd}
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G} \wedge \mathrm{ct}\Delta^1_+ \arrow[d, "\mathrm{id}\wedge d"'] \\
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G} \wedge (\Delta^1_+)^{\wedge G} \arrow[d, "\cong"'] \\
(K \wedge K \wedge K)^{\wedge G} \arrow[d, "F^{\wedge G}"'] \\
(K \wedge K \wedge K \wedge \Delta^1_+)^{\wedge G} \arrow[d, "\cong"'] \\
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G}.
\end{tikzcd}$$

This is the homotopy which shows that $K^{\wedge G}$ is a symmetric G-space. □

Theorem 9.16. *The category $G\mathcal{F}$ and the classes of stable equivalences, stable fibrations and cofibrations, is a proper monoidal model category satisfying the monoid axiom. One can choose generating cofibrations and generating acyclic cofibrations with finite domains and codomains. Smashing with a cofibrant G-simplicial functor preserves stable equivalences.*

Proof. The results above allow us to apply 6.26, 6.27, 6.28, 6.29 and 6.30. □

Let $G\mathcal{F}_{\mathrm{st}}$ refer to the *stable* model category in 9.16. We end this section by comparing $G\mathcal{F}_{\mathrm{st}}$ with the stable model category of G-spectra. By Section 2.5, $S^{\wedge G}\mathrm{Sph}$ is a sub-$G\mathcal{S}$-category of $\mathbf{f}G\mathcal{S}$. Let i_* be the enriched left Kan extension along the corresponding inclusion i. It is left adjoint to pre-composition with i, which we denote by ev. The next result follows from 7.2.

Lemma 9.17. $\mathrm{ev}\colon G\mathcal{F}_{\mathrm{st}} \longrightarrow \mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ *is a right Quillen functor.*

Lemma 7.4 implies that the unit of the adjunction has the following property.

Lemma 9.18. *The canonical map* $E \longrightarrow \mathrm{ev}(i_*E)^h$ *is a stable equivalence of* G*-spectra for every cofibrant* G*-spectrum* E.

If G is the trivial group, Lydakis proved that a map of homotopy functors is a stable equivalence if and only its evaluation is a stable equivalence of spectra. The proof uses the Blakers-Massey theorem. We will extend this result to any finite group using Spanier-Whitehead duality, cp. [14, 17.6].

Proposition 9.19. *Let* K *and* L *be finitely presentable* G*-spaces. The canonical map*

$$\mathrm{ev}(G\mathcal{S}(K, R(-)) \wedge L) \longrightarrow \mathrm{ev}(G\mathcal{S}(K, R(L \wedge -)))$$

of G*-spectra is a stable equivalence.*

Proof. Let $E \star F$ denote the (closed) symmetric monoidal product in the equivariant stable homotopy category $\mathrm{SH}(G)$, with unit $\mathbb{S}$, and let $\mathrm{Hom}(E, -)$ denote the right adjoint of $- \star E$. A G-spectrum D is *dualizable* if the canonical map $\mathrm{Hom}(D, \mathbb{S}) \star D \longrightarrow \mathrm{Hom}(D, D)$ is an isomorphism in $\mathrm{SH}(G)$. It follows that the canonical map $\mathrm{Hom}(D, E) \star F \longrightarrow \mathrm{Hom}(D, E \star F)$ is an isomorphism for all $E, F \in \mathrm{Ob}\,\mathrm{SH}(G)$ if D is dualizable [10, II, Section 1],
Suspension G-spectra of finite G-spaces are dualizable in $\mathrm{SH}(G)$ [3, 2.C], [10, II, 2.7]. In particular, given finite G-spaces K and L, the canonical map

$$\mathrm{Hom}(\mathrm{ev}(-\wedge K), \mathrm{ev}R(-)) \star \mathrm{ev}(-\wedge L) \longrightarrow \mathrm{Hom}(\mathrm{ev}(-\wedge K), \mathrm{ev}R(-) \star \mathrm{ev}(-\wedge L))$$

is an isomorphism in $\mathrm{SH}(G)$. In this special situation, a map of G-spectra lifting this isomorphism can be given as

$$\mathrm{St}^\infty(R(G\mathcal{S}(K, \mathrm{ev}R(-)) \wedge L \longrightarrow G\mathcal{S}(K, \mathrm{ev}R(L \wedge -)))).$$

In particular, this map is a stable equivalence. This finishes the proof, because the above is a stably fibrant replacement of the map in question. □

Corollary 9.20. *Let* X *be a* G*-simplicial functor and* L *be a finite* G*-space. The canonical map* $\mathrm{ev}X^h \wedge L \longrightarrow \mathrm{ev}X^h(L \wedge -)$ *is a stable equivalence of* G*-spectra. In particular,* ev *reflects stable equivalences of homotopy functors.*

Proof. We sketch a proof, following the script for the trivial group [11, 11.7]. Consider the first statement. If X is a cofibrant G-simplicial functor, use 9.19 by attaching cells. If X is arbitrary, use a cofibrant replacement X^c. The second statement then follows from 6.21. □

Corollary 9.21. *The stable model structure on* G*-simplicial functors is Quillen equivalent to the stable model structure on* G*-spectra via the right Quillen functor* ev.

The Quillen equivalence from 9.21 factors through a Quillen equivalence to the category of symmetric G-spectra. In fact, the work [8] of Hovey shows that G-spectra and symmetric G-spectra are Quillen equivalent via a zig-zag of Quillen equivalences, but not necessarily via the canonical forgetful functor. In this case, however, it is possible to conclude this by extending results of [9] to G-spaces. Here are the details.

Theorem 9.22. *The forgetful functor from symmetric G-spectra to G-spectra is the right adjoint of a Quillen equivalence.*

Proof. In both the categories of G-spectra and symmetric G-spectra, a map of fibrant objects is a weak equivalence if and only if it is a pointwise weak equivalence. Hence the forgetful functor U preserves and reflects weak equivalence of fibrant objects. Its left adjoint V preserves cofibrations and pointwise acyclic cofibrations. Further, if L resp. L^Σ is the set of maps of G-spectra resp. symmetric G-spectra that Hovey uses to localize the pointwise model structures, then V maps L to L^Σ (up to isomorphism). Hence V also preserves stable acyclic cofibrations by properties of Bousfield localization [6] and is thus a left Quillen functor.

It remains to prove that the canonical map

$$E \longrightarrow U(V(E)^f)$$

is a stable equivalence of G-spectra for E cofibrant. Here $(-)^f$ denotes a fibrant replacement in the stable model structure of symmetric G-spectra. In fact, since both U and V preserve colimits and homotopy pushouts, it suffices to prove this for E varying through the domains and codomains of the generating cofibrations. To do so, we use the functors $\mathrm{Fix}(H,-)$ on (symmetric) spectrum level. This means the following. If E is a G-spectrum, the sequence $(\mathrm{Fix}(H,E_0),\mathrm{Fix}(H,E_1),\ldots)$ is a $Fix(H,S^{\wedge G}) = S^{|G/H|}$-spectrum of spaces. That is, it has structure maps

$$\mathrm{Fix}(H,S^{\wedge G}) \wedge \mathrm{Fix}(H,E_n) \cong \mathrm{Fix}(H,S^{\wedge G}\wedge E_n) \longrightarrow \mathrm{Fix}(H,E_{n+1}).$$

The same construction works on the level of symmetric G-spectra, so we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}^\Sigma(G\mathcal{S},S^{\wedge G}) \arrow[r, "\mathrm{Fix}(H,-)"] \arrow[d, "U"'] & \mathrm{Sp}^\Sigma(\mathcal{S},S^{|G/H|}) \arrow[d, "U_H"] \\
\mathrm{Sp}(G\mathcal{S},S^{\wedge G}) \arrow[r, "\mathrm{Fix}(H,-)"] & \mathrm{Sp}(\mathcal{S},S^{|G/H|}).
\end{tikzcd}$$

The domains and codomains of the generating cofibrations of G-spectra are of the form $\mathrm{Fr}_n(K)$ (representable $\wedge K$) for K in a certain set of G-spaces, and similarly for the $S^{|G/H|}$-spectra of spaces. Since $\mathrm{Fix}(H,-)\colon G\mathcal{S} \longrightarrow \mathcal{S}$ commutes with the smash product (up to natural isomorphism), we get a natural isomorphism

$$V_H(\mathrm{Fr}_n(\mathrm{Fix}(H,K))) \cong V_H(\mathrm{Fix}(H,\mathrm{Fr}_n(K))) \cong \mathrm{Fix}(H,V(\mathrm{Fr}_n(K))).$$

This isomorphism is compatible with the units of the adjunctions (V,U) and (V_H,U_H), so that

$$\mathrm{Fix}(H,\mathrm{Fr}_n(K) \longrightarrow UV\mathrm{Fr}_n(K)) \cong \mathrm{Fr}_n\mathrm{Fix}(H,K) \longrightarrow U_HV_H(\mathrm{Fr}_n\mathrm{Fix}(H,K)).$$

The categories of (symmetric) $S^{|G/H|}$-spectra of spaces are just slight variations of the categories of (symmetric) spectra of spaces, which implies that (V_H,U_H) is a Quillen equivalence. To conclude the same for (V,U), it is sufficient, by the above, to prove the following two facts.

- A map f of G-spectra is a stable equivalence if so is $\mathrm{Fix}(H,f)$ for every subgroup H.
- If $j_E\colon E \xrightarrow{\sim} E^f$ is a stably fibrant replacement of the symmetric G-spectrum E, then $\mathrm{Fix}(H,j_E\colon E \xrightarrow{\sim} E^f)$ is a stably fibrant replacement of $\mathrm{Fix}(H,E)$.

Concerning the first fact: a map f of G-spectra is clearly a pointwise weak equivalence if and only if so is $\mathrm{Fix}(H,f)$ for every subgroup H. Since a stable equivalence of stably fibrant G-spectra is a pointwise weak equivalence, it suffices to prove that $\mathrm{Fix}(H,-)$ preserves stably fibrant replacements for G-spectra. For this purpose, we apply the small object argument to the following set. Let $J_{\mathrm{pt}} := \{\mathrm{Fr}_n(\wedge(G/H \times (\Lambda^m_i \hookrightarrow \Delta^m))_+\}_{n,H,m,i}$ be the set of generating pointwise acyclic cofibrations. Obtain $\tilde{J}_{\mathrm{st}}$ from $L = \{\mathrm{Fr}_{n+1}S^{\wedge G} \longrightarrow \mathrm{Fr}_n(S^0)\}_n$ by applying the simplicial mapping cylinder, and let J_{st} be the set

$$\tilde{J}_{\mathrm{st}}\square\{(G/H \times (\partial\Delta^m \hookrightarrow \Delta^m))\}_{H,m}$$

of pushout product maps. Finally, $J = J_{\mathrm{pt}} \cup J_{\mathrm{st}}$ is the set we may use for a fibrant replacement[1]. Note first that $\mathrm{Fix}(H,\mathrm{Fr}_{n+1}S^{\wedge G} \longrightarrow \mathrm{Fr}_n(S^0)) \cong \mathrm{Fr}_{n+1}S^{|G/H|} \longrightarrow \mathrm{Fr}_n(S^0)$. Further, $\mathrm{Fix}(H,-)$ is compatible with the simplicial mapping cylinder construction, since it commutes with the smash product and with pushouts of diagrams containing a monomorphism. The latter fact was already used in the proof of 9.5. It follows that $\mathrm{Fix}(H,-)$ maps J to the corresponding set J_H in the category $\mathrm{Sp}(\mathcal{S},S^{|G/H|})$. In particular, $\mathrm{Fix}(H,-)$ maps sequential compositions of cobase changes of maps in J to stable equivalences.

To conclude that $\mathrm{Fix}(H,-)$ preserves the fibrant replacement, we have to show that it preserves stably fibrant objects. One can see this by arguing that it is in fact a right Quillen functor, whose left adjoint l_H is determined by the requirement that $l_H\mathrm{Fr}_n(K) = \mathrm{Fr}_n(G/H_+ \wedge K)$ for any n and any space K. Using this description, one can see that l_H preserves generating (acyclic) cofibrations, hence is a left Quillen functor. It follows that $\mathrm{Fix}(H,-)$ preserves the fibrant replacement. This proof translates to the category of symmetric

[1] The set J is in fact a set of generating acyclic cofibrations.

G-spectra, which justifies the second fact in the list above. This finishes the proof. □

Since the resulting Quillen equivalence (j_*, j^*) between G-simplicial functors and symmetric G-spectra has nice monoidal properties according to 2.16, the closed symmetric monoidal structure induced by the smash product of G-simplicial functors is the correct one. Given the above, comparisons of modules and algebras along the lines of [14] are possible. Note, however, that at present it is not clear how to compare commutative algebras [14, 0.9].

Corollary 9.23. *The model categories of symmetric ring G-spectra and of $\mathbb{I}$-algebras in $G\mathcal{F}$ are Quillen equivalent via the canonical adjoint pair (ι_*, ι^*). If R is a cofibrant $\mathbb{I}$-algebra, the model categories of R-modules and of $\iota^* R$-modules are Quillen equivalent. If Q is a cofibrant symmetric ring G-spectrum, the model categories of Q-modules and of $j_* Q$-modules are Quillen equivalent.*

Proof. First we observe that the model category of symmetric G-spectra satisfies the monoid axiom. Here are some details. A pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r] & Z \cup_X Y \end{tikzcd}$$

of symmetric G-spectra in which f is a pointwise cofibration is a homotopy pushout square. This, the fact that stable equivalences of symmetric G-spectra are closed under sequential colimits (see 3.5) and general arguments from [15] show that it suffices to prove the following. Let X be a symmetric G-spectrum and $j\colon sj \overset{\sim}{\rightarrowtail} tj$ a generating acyclic cofibration, then $X \wedge j$ is a stable equivalence and a pointwise cofibration. To see that the latter holds, note that the pushout product map of a pointwise cofibration and a cofibration is a pointwise cofibration, by comparing with the smash product of *symmetric sequences of G-spaces* as in [9, section 5.3]. Now one can use that the stable model structure on symmetric G-spectra is *stable* in the sense that suspension with S^1 is a Quillen equivalence. So $X \wedge j$ will be a stable equivalence if and only if $X \wedge (tj/sj)$ is stably equivalent to a point. Note that tj/sj is cofibrant. Hence it suffices to prove that the smash product of a cofibrant symmetric G-spectrum and a pointwise weak equivalence is a pointwise weak equivalence. By arguments which already appeared in this proof, one can reduce to the case of the domains and codomains of the generating cofibrations (which are cofibrant). These, however, are gotten directly from symmetric sequences of G-spaces. A comparison like [9, proof of 5.3.7] of the smash products of symmetric G-spectra and these sequences concludes the proof of the monoid axiom.

The hard work is done. Let $\mathbb{S}$ denote the unit in the category of symmetric G-spectra, and recall that the unit in $G\mathcal{F}$ is the inclusion $\mathbb{I}$. By 2.16, the canonical adjoint pair (j_*, j^*) induced by the inclusion $j\colon S^{\wedge G}\mathrm{Sph}^{\Sigma} \hookrightarrow \mathbf{f}G\mathcal{S}$

induces an adjoint pair

$$\iota_*\colon \mathrm{alg}_{\mathbb{S}} \rightleftarrows \mathrm{alg}_{\mathbb{I}}\colon \iota^*.$$

Since the forgetful functors with domain $\mathrm{alg}_{\mathbb{S}}$ resp. $\mathrm{alg}_{\mathbb{I}}$ detect weak equivalences and fibrations, (ι_*, ι^*) is a Quillen adjunction. Moreover, since j^* preserves and detects weak equivalences of fibrant G-simplicial functors, ι^* detects weak equivalences of fibrant $\mathbb{I}$-algebras. To conclude that (ι_*, ι^*) is a Quillen equivalence, it suffices to note that a cofibrant $\mathbb{S}$-algebra is in particular a cofibrant symmetric G-spectrum [15, 4.1].
The other two cases are similar, modulo an application of 8.4. □

It is desirable to compare the stable model category of G-simplicial functors also with the stable model category of orthogonal G-spectra [13]. As an intermediate step, we will use symmetric spectra of topological G-spaces. Let $\mathbf{T}$ denote the (closed symmetric monoidal) model category of pointed compactly generated topological spaces [7, 2.4.21], and let $G\mathbf{T}$ be the category of G-objects in $\mathbf{T}$. The latter is closed symmetric monoidal by an analog of 9.1, and it is a monoidal model category by transferring the model structure from 9.5 to the topological situation (see [13, II.1.8 and II.1.22]). The (strict symmetric monoidal) Quillen equivalence $|-|\colon \mathcal{S} \longrightarrow \mathbf{T}$ given by geometric realization extends to a (strict symmetric monoidal) Quillen equivalence $|-|\colon G\mathcal{S} \longrightarrow G\mathbf{T}$. In particular, we can regard $G\mathbf{T}$ as a $G\mathcal{S}$-model category. As one can check using results from [13, II.1], the model structure on $G\mathbf{T}$ is cellular, so the stable model category of symmetric spectra in $G\mathbf{T}$ with respect to $|S^{\wedge G}|$ exists by [8]. Further, we can apply [8, 9.3] to conclude that

$$|-|\colon \mathrm{Sp}^{\Sigma}(G\mathcal{S}, S^{\wedge G}) \longrightarrow \mathrm{Sp}^{\Sigma}(G\mathbf{T}, |S^{\wedge G}|)$$

is a Quillen equivalence. By inspection, this Quillen functor is strict symmetric monoidal, and its right adjoint is lax symmetric monoidal. Thus to obtain a variant of 9.23, it suffices to note that the stable model category $\mathrm{Sp}^{\Sigma}(G\mathbf{T}, |S^{\wedge G}|)$ satisfies the monoid axiom. A proof can be obtained by translating the proof of the monoid axiom for $\mathrm{Sp}^{\Sigma}(G\mathcal{S}, S^{\wedge G})$ to the topological situation. It remains to relate orthogonal G-spectra to symmetric G-spectra of topological spaces. For the definition of an orthogonal G-spectrum, which we take to be indexed on all G-representations, consider [13]. Any orthogonal G-spectrum X gives rise to a symmetric G-spectrum of topological spaces uX by neglect of structure, or rather – since both objects are simply enriched functors on certain domain $G\mathbf{T}$-categories – restriction. The restriction takes place both on objects (from all G-representations to direct sums of the regular representation $\mathbb{R}^G$) and morphisms (from orthogonal groups to symmetric groups). See [14, 4.4] (for the non-equivariant case) and [13, II.4]. Viewed as a restriction, u has a left adjoint v by enriched Kan extension.

Theorem 9.24. *The adjoint pair (v, u) is a Quillen equivalence. It induces Quillen equivalences between orthogonal ring G-spectra and symmetric ring G-spectra of topological spaces. If R is a cofibrant symmetric ring G-spectrum and*

P is a cofibrant orthogonal ring G-spectrum, (v, u) induces Quillen equivalences

$$\mathrm{mod}_R \rightleftarrows \mathrm{mod}_{vR} \qquad \mathrm{mod}_{uP} \rightleftarrows \mathrm{mod}_P.$$

Proof. The forgetful functor $U_{\mathbf{T}}\colon \mathrm{Sp}^\Sigma(G\mathbf{T}, |S^{\wedge G}|) \longrightarrow \mathrm{Sp}(G\mathbf{T}, |S^{\wedge G}|)$ has a left adjoint $V_{\mathbf{T}}$ fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(G\mathcal{S}, S^{\wedge G}) \arrow[r, "|-|"] \arrow[d, "V"'] & \mathrm{Sp}(G\mathbf{T}, |S^{\wedge G}|) \arrow[d, "V_{\mathbf{T}}"] \\ \mathrm{Sp}^\Sigma(G\mathcal{S}, S^{\wedge G}) \arrow[r, "|-|"] & Spe^\Sigma(G\mathbf{T}, |S^{\wedge G}|) \end{tikzcd}$$

in which the left vertical functor resp. the lower horizontal functor are Quillen equivalences by 9.22 resp. [8, 9.3]. The upper horizontal functor is a Quillen equivalence by [8, 5.7], hence the right vertical functor is a Quillen equivalence. Thus to conclude that (v, u) is a Quillen equivalence, it suffices to prove that the forgetful functor from orthogonal G-spectra to $\mathrm{Sp}(G\mathbf{T}, |S^{\wedge G}|)$ is a Quillen equivalence.
By [13, III.4.16], the forgetful functor from orthogonal G-spectra to G-*prespectra* as defined in [13, II.1.2] is a Quillen equivalence. The category of G-prespectra so far is indexed on all G-representations. However, as observed in [13, II.2.2 and V.1.10], one can index both orthogonal G-spectra and G-prespectra on a collection of G-representations with is both closed under direct sum and cofinal in the collection of all G-representations without changing the homotopy theory. An acceptable candidate is the collection of direct sums of the regular representation. Hence the restriction from orthogonal G-spectra to $\mathrm{Sp}(G\mathbf{T}, |S^{\wedge G}|)$ is a Quillen equivalence. This proves the first statement. The other statements then follow as in the proof of 9.23, since the monoid axiom holds for orthogonal G-spectra [13, III.7.4]. □

Hence for the purpose of studying the homotopy theory of algebras and modules, the category of G-simplicial functors is as good as the category of orthogonal G-spectra for a finite group G. Another comparison functor can be obtained as in [14, 19.11] by passing from $G\mathcal{F}$ to $G\mathbf{T}$-functors from an appropriate domain category (say, finite G-CW-complexes) to $G\mathbf{T}$ via geometric realization, and then restricting to orthogonal G-spectra. Up to geometric realization, this functor amounts to a neglect of structure.

A Calculations

This appendix looks into the proofs of the remaining claims in the main part of the paper. The structure map $\mathrm{hom}_{A,B}^{-\otimes T}\colon \mathcal{V}(A, B) \longrightarrow \mathcal{V}(A \otimes T, B \otimes T)$ of the $\mathcal{V}$-functor $- \otimes T\colon \mathcal{V} \longrightarrow \mathcal{V}$ is defined as the adjoint of the composition

$$\mathcal{V}(A, B) \otimes (A \otimes T) \xrightarrow[\cong]{\alpha^{-1}_{\mathcal{V}(A,B),A,T}} (\mathcal{V}(A, B) \otimes A) \otimes T \xrightarrow{(\epsilon_A B) \otimes T} T \otimes A$$

where $\alpha^{-1}_{\mathcal{V}(B,T),B,A}$ is the associativity isomorphism. The next lemma shows that there are in general two different suspension functors for T-spectra.

Lemma A.1. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B)\otimes(A\otimes T) \arrow[r, "\hom^{-\otimes T}_{A,B}\otimes(A\otimes T)"] \arrow[d, "\alpha^{-1}_{\mathcal{V}(A,B),A,T}"'] & \mathcal{V}(A\otimes T,B\otimes T)\otimes(A\otimes T) \arrow[d, "\epsilon_{A\otimes T}(B\otimes T)"] \\
(\mathcal{V}(A,B)\otimes A)\otimes T \arrow[r, "(\epsilon_A B)\otimes T"] & B\otimes T
\end{tikzcd}$$

Proof. Use naturality and the triangular identity

$$\epsilon_{A\otimes T}(\mathcal{V}(A,B)\otimes(A\otimes T))\circ(\eta_{A\otimes T}\mathcal{V}(A,B))\otimes(A\otimes T)=\mathrm{id}_{\mathcal{V}(A,B)\otimes(A\otimes T)}.$$

□

A similar statement is used to show that the stabilization of enriched functors and the stabilization of spectra can be compared.

Lemma A.2. *The following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta_B A"] \arrow[d, "\eta_{B\otimes T}A"'] & \mathcal{V}(B,A\otimes B) \arrow[d, "\hom^{-\otimes T}_{B,A\otimes B}"] \\
\mathcal{V}(B\otimes T,A\otimes(B\otimes T)) \arrow[r, "\mathcal{V}(B\otimes T,\alpha^{-1}_{A,B,T})"] & \mathcal{V}(B\otimes T,(A\otimes B)\otimes T)
\end{tikzcd}$$

Proof. Similar to A.1, using $\epsilon_B(A\otimes B)\circ(\eta_B A)\otimes B=\mathrm{id}_{A\otimes B}$. □

Next we start the proof of: the two natural stabilization maps $X\longrightarrow T(X)$ described in 6.1 coincide. It is lengthy and perhaps not very illuminating. The map $\hom^{\mathcal{V}(T,-)}_{A,B}:\mathcal{V}(A,B)\longrightarrow\mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B))$ is given as the adjoint of comp: $\mathcal{V}(A,B)\otimes\mathcal{V}(T,A)\longrightarrow\mathcal{V}(T,B)$ which, up to an associativity isomorphism, is adjoint to $\mathcal{V}(A,B)\otimes(\mathcal{V}(T,A)\otimes T)\xrightarrow{\mathcal{V}(A,B)\otimes\epsilon_T A}\mathcal{V}(A,B)\otimes A\xrightarrow{\epsilon_A B}B$.

Lemma A.3. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B) \arrow[rr, "\hom^{\mathcal{V}(T,-)}_{A,B}"] \arrow[dr, "\mathcal{V}(\epsilon_T A,B)"'] & & \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B)) \\
& \mathcal{V}(\mathcal{V}(T,A)\otimes T,B) \arrow[ur, "\cong"', "f"] &
\end{tikzcd}$$

Proof. Here, f is $\mathcal{V}(\eta_T\mathcal{V}(T,A),\mathcal{V}(T,B))\circ\hom^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}$. The diagram

$$\begin{tikzcd}
\mathcal{V}(A,B) \arrow[r, "\hom^{\mathcal{V}(T,-)}_{A,B}"] \arrow[d, "\mathcal{V}(\epsilon_T A,B)"'] & \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B)) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\epsilon_T A),\mathcal{V}(T,B))"] \\
\mathcal{V}(\mathcal{V}(T,A)\otimes T,B) \arrow[r, "\hom^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}"] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(T,A)\otimes T),\mathcal{V}(T,B))
\end{tikzcd}$$

commutes, because $\mathcal{V}(T,-)$ is a $\mathcal{V}$-functor. Therefore, $f\circ\mathcal{V}(\epsilon_T(A),B)$ coincides with $\mathcal{V}(\eta_T\mathcal{V}(T,A),\mathcal{V}(T,B))\circ\mathcal{V}(\mathcal{V}(T,\epsilon_T(A)),\mathcal{V}(T,B))\circ\mathrm{hom}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}$ and the triangular identity $\mathcal{V}(T,\epsilon_T(A))\circ\eta_T\mathcal{V}(T,A)=\mathrm{id}_{\mathcal{V}(T,A)}$ completes the proof. $\square$

Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category closed under $\otimes$. If $v\in\mathrm{Ob}\,\mathcal{C}$ and $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ is a $\mathcal{V}$-functor, one can consider the adjoint $X(c)\otimes v\longrightarrow X(v\otimes c)$ of the map $v\xrightarrow{\eta_c v}\mathcal{V}(c,v\otimes c)\xrightarrow{\mathrm{hom}^X_{c,v\otimes c}}\mathcal{V}(X(c),X(v\otimes c))$. It defines a $\mathcal{V}$-natural transformation $\mathrm{sw}^X_v\colon X\otimes v\longrightarrow X\circ(v\otimes -)$ (X *swallows* v). One of the maps $X\longrightarrow\mathbb{T}(X)$ is defined using the map sw^X_T. Using a commutativity isomorphism, one can define a map $v\otimes X\longrightarrow X\circ(v\otimes -)$ which will also be denoted sw^X_v. An interesting case is $\mathcal{V}(T,-)\colon\mathcal{V}\longrightarrow\mathcal{V}$. Then $\mathrm{sw}^{\mathcal{V}(T,-)}_A(B)\colon A\otimes\mathcal{V}(T,B)\longrightarrow\mathcal{V}(T,A\otimes B)$ is the adjoint of $(A\otimes\mathcal{V}(T,B))\otimes T\xrightarrow{\alpha_{A,\mathcal{V}(T,B)}}A\otimes(\mathcal{V}(T,B)\otimes T)\xrightarrow{A\otimes\epsilon_T B}A\otimes B$.

Lemma A.4. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta_{\mathcal{V}(T,B)}A"] \arrow[d, "\eta_B A"'] & \mathcal{V}(\mathcal{V}(T,B),A\otimes\mathcal{V}(T,B)) \arrow[d, "\mathcal{V}(\mathcal{V}(T,B),\mathrm{sw}^{\mathcal{V}(T,-)}_A(B))"] \\
\mathcal{V}(B,A\otimes B) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}_{B,A\otimes B}"] & \mathcal{V}(\mathcal{V}(T,B),V(T,A\otimes B))
\end{tikzcd}$$

Proof. Using the definition of $\mathrm{hom}^{\mathcal{V}(T,-)}$ and the description of $\mathrm{sw}^{\mathcal{V}(T,-)}_A$ from above, one gets a large diagram which commutes by naturality and the triangular identity $\epsilon_B(A\otimes B)\circ(\eta_B A)\otimes B=\mathrm{id}_{A\otimes B}$. $\square$

Lemma A.5. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B)\otimes\mathcal{V}(T,A) \arrow[r, "\mathrm{sw}^{\mathcal{V}(T,-)}_{\mathcal{V}(A,B)}(A)"] \arrow[d, "\mathrm{hom}^{\mathcal{V}(T,-)}_{A,B}\otimes\mathcal{V}(T,A)"'] & \mathcal{V}(T,\mathcal{V}(A,B)\otimes A) \arrow[d, "\mathcal{V}(T,\epsilon_A B)"] \\
\mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B))\otimes\mathcal{V}(T,A) \arrow[r, "\epsilon_{\mathcal{V}(T,A)}\mathcal{V}(T,B)"] & \mathcal{V}(T,B)
\end{tikzcd}$$

Proof. This is similar to the proof of A.4; the relevant triangular identity is $\epsilon_{\mathcal{V}(T,A)}(\mathcal{V}(A,B)\otimes\mathcal{V}(T,A))\circ(\eta_{\mathcal{V}(T,A)}\mathcal{V}(A,B))\otimes\mathcal{V}(T,A)=\mathrm{id}_{\mathcal{V}(A,B)\otimes\mathcal{V}(T,A)}$. $\square$

Lemma A.6. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
A\otimes\mathcal{V}(T,B) \arrow[r, "\eta_T A\otimes\mathcal{V}(T,B)"] \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_A(B)"'] & \mathcal{V}(T,A\otimes T)\otimes\mathcal{V}(T,B) \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,B)}(A\otimes T)"] \\
\mathcal{V}(T,A\otimes B) \arrow[d, "\mathcal{V}(T,\sigma_{A,B})"'] & \mathcal{V}(T,\mathcal{V}(T,B)\otimes(A\otimes T)) \arrow[d, "\mathcal{V}(T,c)"] \\
\mathcal{V}(T,B\otimes A) & \mathcal{V}(T,(\mathcal{V}(T,B)\otimes T)\otimes A) \arrow[l, "\mathcal{V}(T,(\epsilon_T B)\otimes A)"']
\end{tikzcd}$$

Here c is the composition of an associativity and a commutativity isomorphism.

Proof. Insert the description of $\mathrm{sw}^{\mathcal{V}(T,-)}$ and use naturality, associativity and commutativity coherence, and $\epsilon_T(A \otimes T) \circ (\eta_T A) \otimes T = \mathrm{id}_{A\otimes T}$. $\square$

Lemma A.7. *Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category closed under $\otimes$, $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ a $\mathcal{V}$-functor, and $f\colon \mathcal{V}(T, \mathcal{V}(v,w)) \longrightarrow \mathcal{V}(T\otimes v, w)$ the adjointness isomorphism, with $T, v, w \in \mathrm{Ob}\,\mathcal{C}$. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes (T \otimes Xv) \arrow[rr, "\alpha^{-1}"] \arrow[d, "f\otimes \mathrm{sw}^X_T(v)"'] & & (\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T) \otimes Xv \arrow[d, "(\epsilon_T \mathcal{V}(v,w))\otimes Xv"] \\
\mathcal{V}(T\otimes v, w) \otimes X(T\otimes v) \arrow[d, "\mathrm{hom}^X_{T\otimes v,w}\otimes X(T\otimes v)"'] & & \mathcal{V}(v,w)\otimes Xv \arrow[d, "\mathrm{hom}^X_{v,w}\otimes Xv"] \\
\mathcal{V}(X(T\otimes v), Xw) \otimes X(T\otimes v) \arrow[dr, "\epsilon_{X(T\otimes v)}Xw"'] & & \mathcal{V}(Xv,Xw)\otimes Xv \arrow[dl, "\epsilon_{Xv}Xw"] \\
& Xw &
\end{tikzcd}$$

Proof. The proof is divided into two steps. First we note that

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes (T \otimes Xv) \arrow[rr, "\alpha^{-1}"] \arrow[d, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes(T\otimes Xv)"'] & & (\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T) \otimes Xv \arrow[d, "(\epsilon_T \mathcal{V}(v,w))\otimes Xv"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes (T\otimes Xv) \arrow[d, "g\otimes(T\otimes Xv)"'] & & \mathcal{V}(v,w)\otimes Xv \arrow[d, "\mathrm{hom}^X_{v,w}\otimes Xv"] \\
\mathcal{V}(T\otimes Xv, Xw) \otimes (T\otimes Xv) \arrow[dr, "\epsilon_{T\otimes Xv}Xw"'] & & \mathcal{V}(Xv,Xw)\otimes Xv \arrow[dl, "\epsilon_{Xv}Xw"] \\
& Xw &
\end{tikzcd}$$

commutes, where g is the adjointness isomorphism

$$\mathcal{V}(T\otimes Xv, \epsilon_{Xv}Xw) \circ \mathrm{hom}^{-\otimes Xv}_{T,\mathcal{V}(Xv,Xw)}.$$

Commutativity of the diagram follows from the definition of $\mathrm{hom}^{-\otimes Xv}$, naturality and the triangular identity $\epsilon_{T\otimes Xv}(-\otimes(T\otimes Xv))\circ(\eta_{T\otimes Xv})\otimes(T\otimes Xv) = \mathrm{id}$ applied to $\mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes (T\otimes Xv)$. In the second step, we prove that

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T \otimes Xv \arrow[r, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes T\otimes Xv"] \arrow[d, "f\otimes \mathrm{sw}^X_T"'] & \mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes T \otimes Xv \arrow[d, "g\otimes T\otimes Xv"] \\
\mathcal{V}(T\otimes v, w) \otimes X(T\otimes v) \arrow[d, "\mathrm{hom}^X_{T\otimes v,w}"', "\otimes X(T\otimes v)"] & \mathcal{V}(T\otimes Xv, Xw)\otimes T\otimes Xv \arrow[d, "\epsilon_{T\otimes Xv}Xw"] \\
\mathcal{V}(X(T\otimes v), Xw) \otimes X(T\otimes v) \arrow[r, "\epsilon_{X(T\otimes v)}Xw"] & Xw
\end{tikzcd}$$

commutes. The adjoint of $\epsilon_{X(T\otimes v)}Xw \circ (\mathrm{hom}^X_{T\otimes v,w} \otimes \mathrm{sw}^X_T)$ coincides with the composition $\mathrm{comp} \circ (\mathrm{hom}^X_{T\otimes v,w} \otimes \mathrm{hom}^X_{v,T\otimes v}) \circ \mathcal{V}(T\otimes v, w) \otimes \eta_v T$. Because X is a $\mathcal{V}$-functor, this map is the same as $\mathrm{hom}^X_{v,w} \circ \mathrm{comp} \circ \mathcal{V}(T\otimes v, w) \otimes \eta_v T$. Hence the diagram above commutes if and only if

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w)) \otimes T \otimes Xv \arrow[rr, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes T\otimes Xv"] \arrow[d, "f\otimes\eta_v T\otimes Xv"'] & & \mathcal{V}(T,\mathcal{V}(Xv,Xw)) \otimes T \otimes Xv \arrow[d, "g\otimes T\otimes Xv"] \\
\mathcal{V}(T\otimes v, w) \otimes \mathcal{V}(v, T\otimes v) \otimes Xv \arrow[d, "\mathrm{comp}\otimes Xv"'] & & \mathcal{V}(T\otimes Xv, Xw) \otimes T \otimes Xv \arrow[d, "\epsilon_{T\otimes Xv}Xw"] \\
\mathcal{V}(v,w) \otimes Xv \arrow[dr, "\mathrm{hom}^X_{v,w}\otimes Xv"'] & & Xw \\
& \mathcal{V}(Xv,Xw) \otimes Xv \arrow[ur, "\epsilon_{Xv}Xw"'] &
\end{tikzcd}$$

commutes. The shortest composition is $\mathrm{hom}^X_{v,w} \otimes Xw \circ (\epsilon_T \mathcal{V}(v,w)) \otimes Xw$, as the triangular identity $\epsilon_{T\otimes Xv}(-\otimes(T\otimes Xv))\circ(\eta_{T\otimes Xv})\otimes(T\otimes Xv) = \mathrm{id}_{-\otimes(T\otimes Xv)}$ evaluated at $\mathcal{V}(T,\mathcal{V}(Xv,Xw))$ shows. Therefore it remains to prove that the map $\mathrm{comp} \circ (f \otimes \eta_v T)$ coincides with the map $\epsilon_T \mathcal{V}(v,w)$. It is equivalent to switch to the adjoints (under tensoring with v), and here naturality and the triangular identity $\epsilon_{T\otimes v}(-\otimes(T\otimes v))\circ(\eta_{T\otimes v})\otimes(T\otimes v) = \mathrm{id}_{-\otimes(T\otimes v)}$ evaluated at $\mathcal{V}(T,\mathcal{V}(v,w))$ give the desired identification. □

In the proof of 6.1, we used the next result.

Proposition A.8. *The two maps $X \longrightarrow \mathbb{T}(X)$ coincide.*

Proof. Most for notational convenience, we will often leave out associativity and commutativity constraints. The two maps in question are determined by $Xv \xrightarrow{\eta_T Xv} V(T, Xv\otimes T) \xrightarrow{V(T,\mathrm{sw}^X_T(v))} \mathcal{V}(T, X(T\otimes v))$ and (up to Yoneda isomorphism) $\tau_v(w)\colon \mathcal{V}(T\otimes v, w)\otimes T \xrightarrow{f\otimes T} \mathcal{V}(T,\mathcal{V}(v,w))\otimes T \xrightarrow{\epsilon_T\mathcal{V}(v,w)} \mathcal{V}(v,w)$. The Yoneda isomorphism

$$Xv \xrightarrow{\cong} \int_{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(\mathcal{V}(v,w), Xw)$$

is induced by the natural transformation $y^X_v(w)\colon Xv \longrightarrow \mathcal{V}(\mathcal{V}(v,w),Xw)$, that is, the composition

$$\begin{aligned}
Xv &\xrightarrow{\eta_{\mathcal{V}(v,w)}Xv} \mathcal{V}(\mathcal{V}(v,w), Xv\otimes\mathcal{V}(v,w)) \\
&\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),Xv\otimes\mathrm{hom}^X_{v,w})} \mathcal{V}(\mathcal{V}(v,w), Xv\otimes\mathcal{V}(Xv,Xw)) \\
&\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),\sigma_{Xv,\mathcal{V}(Xv,Xw)})} \mathcal{V}(\mathcal{V}(v,w), \mathcal{V}(Xv,Xw)\otimes Xv) \\
&\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),\epsilon_{Xv}Xw)} \mathcal{V}(\mathcal{V}(v,w), Xw).
\end{aligned}$$

Hence it suffices to prove that, for every $w \in \mathrm{Ob}\,\mathcal{C}$, the composition

$$a(w) := y^{\mathcal{V}(T,X)}_{T\otimes v}(w) \circ \mathcal{V}(T, \mathrm{sw}^X_T(v)) \circ \eta_T Xv$$

coincides with the composition $b(w) := h \circ \mathcal{V}(\tau_v, Xw) \circ y^X_v(w)$. Here h denotes the adjointness isomorphism $\mathcal{V}(\mathcal{V}(T\otimes v, w)\otimes T, Xw) \longrightarrow \mathcal{V}(\mathcal{V}(T\otimes v, w), \mathcal{V}(T, Xw))$. The isomorphism $f\colon \mathcal{V}(T, \mathcal{V}(v,w)) \longrightarrow \mathcal{V}(T\otimes v, w)$ will be used in the proof. The diagram

$$\begin{tikzcd}
\mathcal{V}(\mathcal{V}(v,w), Xw) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}"] \arrow[d, "\mathcal{V}(\tau_v, Xw)"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), \mathcal{V}(T,Xw)) \\
\mathcal{V}(\mathcal{V}(T\otimes v, w))\otimes T, Xw) \arrow[r, "\mathcal{V}(f\otimes T, Xw)"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w))\otimes T, Xw) \arrow[u, "\cong"']
\end{tikzcd}$$

commutes by A.3, where the vertical map on the right hand side is the adjointness isomorphism. Then by naturality and A.4, $b(w)$ coincides with the composition $\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), \mathcal{V}(T, \epsilon_{Xv}Xw \circ \mathrm{hom}^X_{v,w}) \circ \mathrm{sw}^{\mathcal{V}(T,-)}_{Xv}) \circ \eta_{\mathcal{V}(T,\mathcal{V}(v,w))} Xv$. The diagram

$$\begin{tikzcd}
Xv \arrow[r, "\eta_{\mathcal{V}(T,\mathcal{V}(v,w))}Xv"] \arrow[d, "\eta_T Xv"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), Xv \otimes \mathcal{V}(T,\mathcal{V}(v,w))) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),(\eta_T Xv)\otimes\mathcal{V}(T,\mathcal{V}(v,w)))"] \\
\mathcal{V}(T, Xv\otimes T) \arrow[d, "\mathcal{V}(T,\mathrm{sw}^X_T(v))"'] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), \mathcal{V}(T, Xv\otimes T)\otimes \mathcal{V}(T,\mathcal{V}(v,w))) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)),\mathcal{V}(T,\mathrm{sw}^X_T(v))\otimes\mathcal{V}(T,\mathcal{V}(v,w)))"] \\
\mathcal{V}(T, X(T\otimes v)) \arrow[r, "\eta_{\mathcal{V}(T,\mathcal{V}(v,w))}"] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), \mathcal{V}(T, X(T\otimes v))\otimes \mathcal{V}(T,\mathcal{V}(v,w)))
\end{tikzcd}$$

commutes by naturality. Hence the maps $a(w)$ and $b(w)$ coincide if

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes Xv \arrow[r, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\eta_T Xv"] \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{Xv}"'] & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T, Xv\otimes T) \arrow[d, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,\mathrm{sw}^X_T(v))"] \\
\mathcal{V}(T, Xv\otimes\mathcal{V}(v,w)) \arrow[d, "\mathcal{V}(T,Xv\otimes\mathrm{hom}^X_{v,w})"'] & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,X(T\otimes v)) \arrow[d, "f\otimes\mathcal{V}(T,X(T\otimes v))"] \\
\mathcal{V}(T, Xv\otimes\mathcal{V}(Xv,Xw)) \arrow[d, "\mathcal{V}(T,\sigma)"'] & \mathcal{V}(T\otimes v, w)\otimes\mathcal{V}(T,X(T\otimes v)) \arrow[d, "\mathrm{hom}^X\otimes\mathcal{V}(T,X(T\otimes v))"] \\
\mathcal{V}(T, \mathcal{V}(Xv,Xw)\otimes Xv) \arrow[d, "\mathcal{V}(T,\epsilon_{Xv}Xw)"'] & \mathcal{V}(X(T\otimes v), Xw)\otimes\mathcal{V}(T,X(T\otimes v)) \arrow[d, "\mathrm{hom}^{\mathcal{V}(T,-)}\otimes\mathcal{V}(T,X(T\otimes v))"] \\
\mathcal{V}(T,Xw) & \mathcal{V}(\mathcal{V}(T,X(T\otimes v)), V(T,Xw))\otimes V(T,X(T\otimes v)) \arrow[l, "\epsilon_{\mathcal{V}(T,X(T\otimes v))}"']
\end{tikzcd}$$

commutes for all w. Now use A.5 (with $A = X(T\otimes v)$ and $B = Xw$), naturality of the map $\mathrm{sw}^{\mathcal{V}(T,-)}$ and the isomorphism f to replace the composition from the upper corner on the right hand side to the lower corner on the left hand side. The result is the diagram

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes Xv \arrow[rr, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\eta_T Xv"] \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{Xv}"'] & & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,Xv\otimes T) \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,\mathcal{V}(v,w))}"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(v,w)) \arrow[d, "\mathcal{V}(T,Xv\otimes\mathrm{hom}^X_{v,w})"] & & \mathcal{V}(T,\mathcal{V}(T,\mathcal{V}(v,w))\otimes X(T\otimes v)) \arrow[ll, "\mathcal{V}(T,Xv\otimes\epsilon_T\mathcal{V}(v,w))"'] \arrow[d, "\mathcal{V}(T,f\otimes X(T\otimes v))"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(Xv,Xw)) \arrow[d, "\mathcal{V}(T,\sigma)"'] & & \mathcal{V}(T,\mathcal{V}(T\otimes v,w)\otimes X(T\otimes v)) \arrow[d, "\mathcal{V}(T,\mathrm{hom}^X\otimes X(T\otimes v))"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw)\otimes Xv) \arrow[dr, "\mathcal{V}(T,\epsilon_{Xv}Xw)"'] & & \mathcal{V}(T,\mathcal{V}(X(T\otimes v),Xw)\otimes X(T\otimes v)) \arrow[dl, "\mathcal{V}(T,\epsilon_{X(T\otimes v)}Xw)"] \\
& \mathcal{V}(T,Xw). &
\end{tikzcd}$$

The upper part commutes by A.6 (with $A = Xv$ and $B = \mathcal{V}(v,w)$), the lower part is $\mathcal{V}(T,-)$ applied to a diagram which commutes by A.7. This completes the proof. $\square$

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