The Reciprocity Obstruction for Rational Points on Compactifications of Torsors under Tori over Fields with Global Duality

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Abstract. This paper studies the reciprocity obstruction to the local–global principle for compactifications of torsors under tori over a generalised global field of characteristic zero. Under a non-divisibility condition on the second Tate–Shafarevich group for tori, it is shown that the reciprocity obstruction is the only obstruction to the local–global principle. This gives in particular an elegant unified proof of Sansuc's result on the Brauer–Manin obstruction for compactifications of torsors under tori over number fields, and Scheiderer's result on the reciprocity obstruction for compactifications of torsors under tori over p-adic function fields.

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Let K be a field of characteristic zero that has $(n{+}2)$-dimensional global duality in étale cohomology with respect to a collection of n-local fields $K \subset K_v \subset \overline{K}$ indexed by $v \in \Omega_K$. Examples of such fields are totally imaginary number fields (then $n = 1$) and function fields over n-local fields. See Section 1 for details. Let X be a nonsingular complete variety over K. Writing $X(\mathbf{A}_K) := \prod_{v\in\Omega} X(K_v)$, we have a *reciprocity pairing*

$$X(\mathbf{A}_K) \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n)) \to \mathbf{Q}/\mathbf{Z}.$$

Writing $X(\mathbf{A}_K)^{\mathrm{rcpr}}$ for the collections of points that pair to zero with every $\omega \in H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$, we have that $X(K) \hookrightarrow X(\mathbf{A}_K)^{\mathrm{rcpr}}$. In particular, when $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ then $X(K) = \emptyset$.

Hence the reciprocity pairing gives an obstruction to the local–global principle. When K is a number field, this obstruction is easily seen to be equivalent to the obstruction coming from the well-known *Brauer–Manin pairing*

$$X(\mathbf{A}_K) \times H^2(X, \mathbf{G}_{\mathrm{m}}) \to \mathbf{Q}/\mathbf{Z}$$

introduced by Yu. Manin in [Ma].

Main result

In this paper we will show that under a technical assumption on Galois cohomology of tori the reciprocity obstruction is the only obstruction to the local–global principle for any smooth compactification of a torsor under a torus over K (i.e., any nonsingular complete variety containing a principal homogeneous space under a torus over K as a Zariski-dense open subvariety).

Theorem 1. *Let K be a field of characteristic zero with global duality. Assume that $\text{Ш}^2(K,T)$ is of finite exponent for every torus T over K.*
Then for any smooth compactification X of a torsor under a torus over K we have that $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ if and only if $X(K) = \emptyset$.

Proof. This follows immediately from Corollary 3.3 and Corollary 4.3. □

This generalises (and simplifies the proof of) the original result of Sansuc that for a smooth compactification of a torsor under a torus over a number field the Brauer–Manin obstruction is the only obstruction against the Hasse principle (see [San] and also [Sk]).
The condition on $\text{Ш}^2(K,T)$ is not only known to hold for number fields, but also for p-adic function fields (this follows from the duality theorems in [SvH]). In particular, we get a proof of the following unconditional result, due to Scheiderer (private communication), that has not appeared in the literature before.

Corollary. *Let p be a prime and let K be a p-adic function field (i.e., a finite extension of $\mathbf{Q}_p(t)$). Then for any smooth compactification X of a torsor under a torus over K we have that $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ if and only if $X(K) = \emptyset$.*

I do not know any other examples of fields of characteristic zero with global duality and finite exponent for Ш^2 of tori, nor do I know any examples of tori over fields of characteristic zero with global duality where Ш^2 has infinite exponent.

Method of proof

The proof uses *pseudo-motivic* homology

$${}^1H_*(X, \mathbf{Z}) := \mathrm{Ext}^{-*}_{k_{\mathrm{sm}}}(R\Gamma(X/k, \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}})$$

as defined in [vH1] for nonsingular complete varieties over a field k of characteristic zero (see Section 2 for some more information).

This homology theory (covariant in X) can be considered to be in between motivic homology and étale homology with coefficients in $\hat{\mathbf{Z}}$ (see [vH1], [vH2]). It is more tractable than motivic homology, but it still contains some important geometric/arithmetic data. In particular, in certain cases ${}^1H_0(X, \mathbf{Z})$ can decide whether X has k-rational points.

Theorem 2. *Let X be a smooth compactification of a torsor under a torus over a field k of characteristic zero. Then the degree map*

$$^1H_0(X, \mathbf{Z}) \to {}^1H_0(\operatorname{Spec} k, \mathbf{Z}) = \mathbf{Z}$$

is surjective if and only if $X(k) \neq \emptyset$.

Proof. If $X(k) \neq \emptyset$, then functoriality of ${}^1H_0(-, \mathbf{Z})$ implies the surjectivity of the degree map. The converse follows from Corollary 4.3. □

This is the key result in the paper and in fact an easy consequence of Hilbert's Theorem 90 and Rosenlicht's result that the invertible functions on a torus are characters up to translation. Theorem 1 is then essentially a purely formal consequence of global duality. However, to avoid any unnecessary technical subtleties we will actually derive Theorem 1 from the slightly stronger Corollary 4.3.

As we will see in Section 5, the approach taken here is strongly related to the approach of Colliot-Thélène and Sansuc in the case of number fields: Corollary 4.3 is equivalent to their result that a smooth compactification of a torsor under a torus has rational points if and only if the so-called *elementary obstruction* vanishes. However, the proofs in the present paper are simpler, and extend easily to higher cohomological dimension. This can be explained by the fact that for the varieties under consideration the homological formalism of pseudo-motivic homology happens to be more natural than the dual *co*homological formalism of descent.

Structure of the paper

Most of this paper is devoted to setting up the conceptual framework and establishing its formal properties. In Section 1 we recall the concept of an n-local field, originally due to Parshin, and we will introduce a cohomological global analogue: $(n+2)$-dimensional global duality in étale cohomology. We will introduce the reciprocity pairing in this framework and establish some basic properties. In Section 2 we will recall the definition and basic properties of pseudo-motivic homology. In Section 3 we define a cap-product between pseudo-motivic homology and étale cohomology and we establish a partial duality.

After setting up the proper framework in the first three sections, we show in Section 4 that a principal homogeneous space under a torus actually coincides with the degree 1 part of its zero-dimensional homology. This is essentially a

rephrasing of Rosenlicht's result on the invertible functions on a torus. The main results then follow immediately.
Finally, in Section 5 we will compare the methods used here to other methods in the literature.

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1 HIGHER DIMENSIONAL LOCAL AND GLOBAL DUALITY

1.1 HIGHER DIMENSIONAL LOCAL DUALITY

In this paper, an *n-local field* (for $n \geq 1$) will be a field k that admits a sequence of fields

$$k_0, k_1, \ldots, k_n = k$$

such that:

- k_0 is a finite field
- For each $i > 0$ the field k_i is the quotient field of an excellent henselian discrete valuation ring $\mathscr{O}_{k_i}$ with residue field k_{i-1}.

A *generalised n-local field* will be a field satisfying the same hypotheses, except that k_0 is only required to be *quasi-finite*, i.e., a perfect field with absolute Galois group isomorphic to $\hat{\mathbf{Z}}$.
A generalised n-local field k with k_1 of characteristic zero satisfies *n-dimensional local duality in étale cohomology*:

- There is a canonical isomorphism $H^{n+1}_{\text{ét}}(k, \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$
- For any finite $\operatorname{Gal}(\bar{k}/k)$-module M and any $i \in \mathbf{Z}$ the Yoneda pairing

  $$H^i_{\text{ét}}(k, M) \times \operatorname{Ext}^{n+1-i}_{\text{ét}}(M, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+1}_{\text{ét}}(k, \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

  is a perfect pairing of finite groups.
- For a finite unramified $\operatorname{Gal}(\bar{k}/k)$-module M of order prime to the characteristic of k_{n-1}, the unramified cohomology of M is precisely the annihilator of the unramified cohomology of $\mathscr{H}om(M, \mathbf{Q}/\mathbf{Z}(n))$ in the duality pairing.

Here an *unramified* $\operatorname{Gal}(\bar{k}/k)$-module M is a Galois module on which the inertia group I acts trivially, and the unramified cohomology of M is the image in $H^i_{\text{ét}}(k, M) = H^i(\operatorname{Gal}(\bar{k}/k), M)$ of the Galois cohomology group $H^i(\operatorname{Gal}(\bar{k}/k)/I, M)$ under the restriction map (see for example [Mi, I, p.36]). In

terms of étale cohomology it is the image of the restriction map $H^i_{\text{ét}}(\mathscr{O}_k, M) \to H^i_{\text{ét}}(k, M)$.
For ordinary local fields (the case $n = 1$ and finite k_0), the duality in étale cohomology is due to Tate (see for example [Mi, §I.4]). For higher dimensional local fields (with finite k_0) this is [DW, Th. 1.1, Prop. 1.2]. Since the proofs only rely on the cohomological properties of k_0, they easily generalise to the case of quasi-finite k_0.

1.2 Higher dimensional global duality

Let K be a field of characteristic zero and suppose we have

- A collection Ω of discrete valuations $v\colon K \to \mathbf{Z}$.
- An $n \geq 1$ such that for every $v \in \Omega$ the quotient field K_v^{h} of the henselisation $\mathscr{O}_v^{\mathrm{h}}$ of the discrete valuation ring $\mathscr{O}_v := \{x \in K\colon v(k) \geq 0\}$ is an n-local field.
- A noetherian ring $\mathscr{O}_K \subset K$ such that K is the quotient field of $\mathscr{O}_K$ and such that for all but finitely many $v \in \mathscr{O}$ we have that $\mathscr{O}_K \subset \mathscr{O}_v$.

We will use the notation $\mathbf{A}_K$ (or simply $\mathbf{A}$) for the ring of (henselian) *adèles* corresponding to (K, Ω), i.e., the subring of $\prod_{v\in\Omega} K_v^{\mathrm{h}}$ consisting of the $\{x_v\}_{v\in\Omega}$ with $x_v \in \mathscr{O}_v$ for all but finitely many $v \in \Omega$. Since for every finite $\mathrm{Gal}(\overline{K}/K)$-module M we have that M extends to a locally constant étale sheaf over an affine open subscheme $U \subset \operatorname{Spec} \mathscr{O}_K$, we may define the *adèlic étale cohomology group*

$$H^*_{\text{ét}}(\mathbf{A}_K, M) := \varinjlim_{\substack{V\subset U\\ \text{open affine}}} \left(\prod_{\substack{v\in\Omega\\ v\in V}} H^*_{\text{ét}}(\mathscr{O}_v^{\mathrm{h}}, M) \times \prod_{\substack{v\in\Omega\\ v\notin V}} H^*_{\text{ét}}(K_v^{\mathrm{h}}, M) \right).$$

By abuse of notation we write $v \in \operatorname{Spec} \mathscr{O}_K$ if $\mathscr{O}_K \subset \mathscr{O}_v$ and similarly for every affine open subscheme $U \subset \operatorname{Spec} \mathscr{O}_K$.
As an example, observe that the canonical isomorphisms $H^{n+1}(K_v^{\mathrm{h}}, \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$ (with the unramified part being zero) induce an isomorphism

$$H^{n+1}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \simeq \bigoplus_{v\in\Omega} \mathbf{Q}/\mathbf{Z}.$$

We write $Ш^i(K, M)$ for the kernel of the map

$$H^i_{\text{ét}}(K, M) \to H^i_{\text{ét}}(\mathbf{A}, M).$$

Similarly we define the complex of abelian groups $R\Gamma_{\text{ét}}(\mathbf{A}, M)$ for any étale sheaf (or complex of étale sheaves) M over some open subscheme $U \subset \operatorname{Spec} \mathscr{O}_K$. We have a map

$$R\Gamma_{\text{ét}}(K, M) \to R\Gamma_{\text{ét}}(\mathbf{A}, M)$$

and we define the complex $R\Gamma(K, \mathbf{A}; M)$ to be the complex of abelian groups that makes a triangle

$$R\Gamma_{\text{ét}}(K, \mathbf{A}; M) \to R\Gamma_{\text{ét}}(K, M) \to R\Gamma_{\text{ét}}(\mathbf{A}, M).$$

As the notation indicates, the corresponding cohomology groups $H^i_{\text{ét}}(K, \mathbf{A}; M) := H^i(R\Gamma(K, \mathbf{A}; M))$ should be thought of as relative cohomology groups. By definition we have a long exact sequence

$$\cdots \to H^i_{\text{ét}}(K, \mathbf{A}; M) \to H^i_{\text{ét}}(K, M) \to H^i_{\text{ét}}(\mathbf{A}, M) \to H^{i+1}_{\text{ét}}(K, \mathbf{A}; M) \to \dots$$

Observe that, even if the henselian adèles used here are different from the usual adèles (defined using completions), their cohomology with finite coefficients is the same, since their Galois groups are isomorphic (see for example [Mi, App. I.A]).

Remark 1.1. The relative cohomology groups $H^*_{\text{ét}}(K, \mathbf{A}; -)$ can be thought of as the cohomology with compact supports of $\operatorname{Spec} K$ regarded as something very open in a compactification of $\operatorname{Spec} \mathscr{O}_K$ (compare [Mi, §II.2]). This way of seeing it is more in line with the Grothendieck–Verdier approach to cohomology and duality. However, a notation H^*_{c} can lead to confusion when studying the cohomology of varieties over K, so the 'Eilenberg–MacLane'-style of notation as relative cohomology seems more convenient.

For any finite $\operatorname{Gal}(\overline{K}/K)$-module M, any i, $j \in \mathbf{Z}$ we have that an $\omega \in \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j))$ induces maps

$$\begin{gathered}
H^q_{\text{ét}}(K, \mathbf{A}; M) \to H^{q+i}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)) \\
H^q_{\text{ét}}(K, M) \to H^{q+i}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(j)) \\
H^q_{\text{ét}}(\mathbf{A}, M) \to H^{q+i}_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))
\end{gathered}$$

which are compatible with the long exact sequences of the pair $(K, \mathbf{A})$. Allowing ω to vary we get the *Yoneda pairings*

$$\begin{gathered}
H^q_{\text{ét}}(K, \mathbf{A}; M) \otimes \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)), \\
H^q_{\text{ét}}(K, M) \otimes \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(j)), \\
H^q_{\text{ét}}(\mathbf{A}, M) \otimes \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)).
\end{gathered}$$

We say that K has $(n+2)$-dimensional global duality in étale cohomology if:

- We have an isomorphism $H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$ such that the boundary map $H^{n+1}_{\text{ét}}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))$ corresponds to the summation map $\bigoplus_{v \in \Omega} \mathbf{Q}/\mathbf{Z} \xrightarrow{\sum} \mathbf{Q}/\mathbf{Z}$.
- For every finite $\operatorname{Gal}(\overline{K}/K)$-module M and any $i \in \mathbf{Z}$ the Yoneda pairing

  $$H^i_{\text{ét}}(K, \mathbf{A}; M) \times \operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$$

  is a nondegenerate pairing of abelian groups inducing an isomorphism $H^i_{\text{ét}}(K, \mathbf{A}; M) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(n)), \mathbf{Q}/\mathbf{Z})$.

As a purely formal consequence we get duality for any bounded complex $\mathscr{C}$ of constructible étale sheaves defined over an open subset $U \subset \operatorname{Spec} \mathscr{O}_K$: we have for any $i \in \mathbf{Z}$ that

$$H^i_{\text{ét}}(K, \mathbf{A}; \mathscr{C}) \times \operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(\mathscr{C}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$$

is a nondegenerate pairing of abelian groups inducing an isomorphism $H^i_{\text{ét}}(K, \mathbf{A}; \mathscr{C}) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(\mathscr{C}, \mathbf{Q}/\mathbf{Z}(n)), \mathbf{Q}/\mathbf{Z})$.
Examples of fields that satisfy $(n+2)$-dimensional global duality in étale cohomology are

- Totally imaginary number fields (with $n = 1$).
- Function fields of curves over generalised $(n-1)$-local fields with k_1 of characteristic zero.

Remark 1.2. To get 3-dimensional global duality for number fields that admit real embeddings, one needs to take care of the real places separately (as in [Mi, §II.2]). Having done that, the methods of this paper still apply.

1.3 The reciprocity pairing

Let X be a nonsingular complete variety over a field K having $(n+2)$-dimensional global duality in étale cohomology.
For any i, $j \in \mathbf{Z}$ the restriction map gives pairings of sets

$$X(K) \times H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(K; \mathbf{Q}/\mathbf{Z}(j))$$
$$X(\mathbf{A}) \times H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$$

where we use the notation

$$H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j)) := H^i_{\text{ét}}(\mathbf{A}, R\Gamma(X/K, \mathbf{Q}/\mathbf{Z}(j))).$$

When we compare these two pairings, we see that composition with the restriction map $H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j))$ and the boundary map $H^i_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)) \to H^{i+1}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$ transforms the second pairing into a pairing

$$X(\mathbf{A}) \times H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^{i+1}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$$

with the property that the image of the map $X(K) \to X(\mathbf{A})$ lands into the subset

$$X(\mathbf{A})^{\perp H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j))} := \Big\{\{x_v\} \in X(\mathbf{A}) \colon \langle \{x_v\}, \omega\rangle = 0 \text{ for any } \omega \in H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j))\Big\}$$

Taking $i = n+1$, $j = n$, we get the *reciprocity pairing*

$$X(\mathbf{A}) \times H^{n+1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

mentioned in the introduction, and we have that

$$X(K) \hookrightarrow X(\mathbf{A})^{\text{rcpr}} = X(\mathbf{A})^{\perp H^{n+1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(n))}.$$

1.4 Generalised global duality beyond finite coefficients

Later in this paper we will use $(n+2)$-dimensional global duality to detect elements in $H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M))$ for a finitely generated group scheme over K. Here a *finitely generated group scheme* over a perfect field k is a group scheme G such that $G(\bar{k})$ is a finitely generated group. By

$$\mathbf{X}(M) := \mathscr{H}om(M, \mathbf{G}_{\mathrm{m}})$$

we denote the Cartier dual of M.
Morally speaking, one would expect a nondegenerate pairing

$$H^n_{\text{ét}}(K, M \overset{L}{\otimes} \mathbf{Z}(n-1))) \times H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M)) \to H^{n+3}_{\text{ét}}(K, \mathbf{A}; \mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}$$

for suitable 'motivic' complexes of sheaves $\mathbf{Z}(n-1)$ and $\mathbf{Z}(n)$ in the sense of Beilinson and Lichtenbaum (see [BMS], [L]; recall that we have $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{G}_{\mathrm{m}} = \mathbf{Z}(1)[1]$). Such a duality for 'integral' coefficients is known when K is a number field (cf. [Mi, §I.4]), but I do not know of such a full duality in any other case — even for K a p-adic function field, the results of [SvH] do give the required duality between $H^2_{\text{ét}}(K, M \otimes^L \mathbf{Z}(1))$ and $H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M))$, but this duality is obtained without introducing a complex $\mathbf{Z}(2)$.
To avoid these complications, we consider the torsion version

$$\begin{aligned} H^{n-1}_{\text{ét}}(K, M \overset{L}{\otimes} \mathbf{Q}/\mathbf{Z}(n-1))) &\times H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M)) \\ &\to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z} \quad (1) \end{aligned}$$

which can be defined as the Yoneda pairing associated to the isomorphisms

$$M \overset{L}{\otimes} \mu_m^{\otimes n-1} \simeq R\mathscr{H}om(\mathbf{X}(M), \mathbf{G}_{\mathrm{m}} \overset{L}{\otimes} \mu_m^{\otimes n-1}) = R\mathscr{H}om(\mathbf{X}(M), \mu_m^{\otimes n}[1])$$

for all $m \in \mathbf{N}$.

Proposition 1.3. *Let K be a field that has $(n+2)$-dimensional global duality, and let M be a finitely generated group scheme over K. If $Ш^2(K, \mathbf{X}(M))$ is of finite exponent, then the pairing* (1) *is nondegenerate on the right.*

Proof. By hypothesis, we have an $N \in \mathbf{N}$ such that $Ш^2(K, \mathbf{X}(M))$ is N-torsion. By Hilbert's Theorem 90 and a restriction–corestriction argument we also have an $N' \in \mathbf{N}$ such that $H^1_{\text{ét}}(\mathbf{A}, \mathbf{X}(M))$ is N'-torsion.
The long exact sequence of relative cohomology now implies that $H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M))$ is NN'-torsion, so this group embeds into $H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M) \otimes^L \mathbf{Z}/NN')$ by the Kummer sequence.
Global duality then implies that $H^2_{\text{ét}}(K, \mathbf{A}; \mathbf{X}(M))$ embeds into the dual of $H^{n-1}_{\text{ét}}(K, M \otimes^L \mathbf{Z}/NN'(n-1)))$, hence into the dual of $H^{n-1}_{\text{ét}}(K, M \otimes^L \mathbf{Q}/\mathbf{Z}(n-1)))$. ☐

We will also use the following easy lemma.

Lemma 1.4. *Let K be a field that has $(n+2)$-dimensional global duality. The pairing*

$$H^{n+1}_{ét}(K, \mathbf{Q}/\mathbf{Z}(n)) \times H^1_{ét}(K, \mathbf{A}; \mathbf{Z}) \to H^{n+2}_{ét}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z} \qquad (2)$$

is nondegenerate on the right.

Proof. This follows easily from the fact that $H^1_{\text{ét}}(K, \mathbf{A}; \mathbf{Z}) = \left(\prod_{v \in \Omega} \mathbf{Z}\right)/\mathbf{Z}$, whereas $H^{n+1}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(n))$ surjects onto the kernel of the map $\bigoplus_{v\in\Omega} \mathbf{Q}/\mathbf{Z} = H^{n+1}_{\text{ét}}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}$. $\square$

2 Pseudo-motivic homology

Let k be a field of characteristic zero. Let X be a nonsingular variety over k. We write X_{sm} for the smooth site over X (i.e., underlying category the smooth schemes of finite type over X and coverings the surjective smooth morphisms). For any sheaf $\mathscr{F}$ on X_{sm} we denote by $R\Gamma(X/K_{\text{sm}}, \mathscr{F})$ the total direct image in the derived category of sheaves on $(\operatorname{Spec} k)_{\text{sm}}$ of $\mathscr{F}$ under the structure morphism $X \to \operatorname{Spec} k$. With this notation we define

$$\begin{aligned}
\mathscr{C}^*(X, \mathbf{G}_\text{m}) &:= R\Gamma(X/k_{\text{sm}}, \mathbf{G}_\text{m}) \\
\mathscr{C}^\text{c}_*(X, \mathbf{Z}) &:= R\,\mathscr{H}om_{k_{\text{sm}}}(\mathscr{C}^*(X, \mathbf{G}_\text{m}), \mathbf{G}_\text{m}) \\
{}^1H^\text{c}_i(X, \mathbf{Z}) &:= H^{-i}(k_{\text{sm}}, \mathscr{C}^\text{c}_*(X, \mathbf{Z}))
\end{aligned}$$

When X is complete, we have that ${}^1H^\text{c}_i(X, \mathbf{Z}) = {}^1H_i(X, \mathbf{Z})$, *pseudo-motivic homology*, which was introduced and studied in [vH1] for nonsingular complete varieties. For a noncomplete X, we have that ${}^1H^\text{c}_i(X, \mathbf{Z})$ is *pseudo-motivic homology with compact supports*, which was studied in [vH2].
As in [vH2] we will work with a truncated version for technical reasons. We define

$$\begin{aligned}
\mathscr{C}^*(X, \mathbf{G}_\text{m})_\tau &:= \tau_{\leq 1} R\Gamma(X/k_{\text{sm}}, \mathbf{G}_\text{m}) \\
H^i(X, \mathbf{G}_\text{m})_\tau &:= H^i(k_{\text{sm}}, \mathscr{C}^*(X, \mathbf{G}_\text{m})_\tau) \\
\mathscr{C}^\text{c}_*(X, \mathbf{Z})_\tau &:= R\,\mathscr{H}om_{k_{\text{sm}}}(\mathscr{C}^*(X, \mathbf{G}_\text{m})_\tau, \mathbf{G}_\text{m}) \\
{}^1H^\text{c}_i(X, \mathbf{Z})_\tau &:= H^{-i}(k_{\text{sm}}, \mathscr{C}^\text{c}_*(X, \mathbf{Z})_\tau)
\end{aligned}$$

Finally, we will also need the associated 'truncated' cohomology theory with torsion coefficients, so we define

$$\begin{aligned}
\mathscr{C}^*(X, \mu_m^{\otimes j})_\tau &:= \mathscr{C}^*(X, \mathbf{G}_\text{m})_\tau \overset{L}{\otimes} \mu_m^{\otimes j} \\
H^i(X, \mu_m^{\otimes j})_\tau &:= H^i(k_{\text{sm}}, \mathscr{C}^*(X, \mu_m^{\otimes j})_\tau) \\
H^i(X, \mathbf{Q}/\mathbf{Z}(j))_\tau &:= \varinjlim_m H^i(X, \mu_m^{\otimes j})_\tau
\end{aligned}$$

Remark 2.1. We only need the smooth topology in the definition of the complexes $\mathscr{C}^{\mathrm{c}}_{*}(X,\mathbf{Z})_{(\tau)}$. After that, the comparison between smooth cohomology and étale cohomology of the complexes of sheaves that we are using assures that we might as well compute everything on the étale site. In particular, there is no need to distinguish in notation between $H^*(k_{\mathrm{sm}},-)$ and $H^*(k_{\text{ét}},-)$, and we will normally just write $H^*(k,-)$

2.1 Some calculations

In the present paper we are interested in varieties with a finitely generated geometric Picard group. For these varieties the truncated pseudo-motivic homology has a very simple structure.

LEMMA 2.2. *Assume X is a nonsingular complete geometrically irreducible variety over k such that the Picard scheme $\mathrm{Pic}_{X/k}$ is a finitely generated group scheme. Then we have a triangle*

$$\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}})[1]\to\mathscr{C}_{*}(X,\mathbf{Z})_{\tau}\to\mathbf{Z}.$$

Proof. By Cartier duality this follows from the fact that we have a triangle

$$\mathbf{G}_{\mathrm{m}}\to\mathscr{C}^{*}(X,\mathbf{G}_{\mathrm{m}})_{\tau}\to\mathrm{Pic}_{X/K}[-1]. \tag{3}$$

□

COROLLARY 2.3. *With X as above, we have a long exact sequence*

$$\cdots\to H^1(k,\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}}))\to {}^1H_0(X,\mathbf{Z})_{\tau}\to H^0(k,\mathbf{Z})$$
$$\to H^2(k,\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}}))\to\cdots$$

LEMMA 2.4. *Assume V is a nonsingular geometrically irreducible variety over k such that $\mathrm{Pic}_{V/k}=0$. Then we have a triangle*

$$\mathscr{H}om(\bar{k}[V]^*/\bar{k}^*,\mathbf{G}_{\mathrm{m}})\to\mathscr{C}_{*}(V,\mathbf{Z})_{\tau}\to\mathbf{Z}$$

Proof. Let $V\hookrightarrow X$ be a smooth compactification, let $Z\subset X$ be the closed complement, and let $\mathscr{Z}^1_Z(X/k)$ be the locally free abelian sheaf on k_{sm} associated to the Galois permutation module generated by the set of irreducible components of $\overline{Z}$ that are of codimension 1 in $\overline{X}$. Let M be the kernel of the surjective map of sheaves

$$\mathscr{Z}^1_Z(X/k)\to\mathrm{Pic}_{X/k}.$$

We have that M is a locally free abelian sheaf on k_{sm}, hence it is the sheaf associated to a torsion-free Galois module. It follows from the triangle (3) and the triangle [vH2, eq. (3)] that we have a triangle

$$\mathbf{G}_{\mathrm{m}}\to\mathscr{C}^{*}(V,\mathbf{Z})_{\tau}\to M.$$

Checking the global sections over $\bar{k}$ then gives that M is the sheaf corresponding to the finitely generated Galois module $\bar{k}[V]^*/\bar{k}^*$. □

Lemma 2.5. *Let X be a nonsingular projective variety over k and let $V \subset X$ be an open subvariety, then the natural map*

$$ {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})_\tau \to {}^1H_0(X, \mathbf{Z})_\tau $$

is surjective

Proof. This is part of [vH2, Cor. 1.5]. □

2.2 Homology classes of points

For any variety V over k we have that the covariantly functorial properties of pseudo-motivic homology give a natural map

$$ V(k) \to {}^1H_0^{\mathrm{c}}(V, \mathbf{Z}). $$

We denote the homology class of a k-valued point $x \in V(k)$ by $[x]$, If x corresponds to a map $i\colon \operatorname{Spec} k \to V$ then $[x]$ corresponds to the morphism

$$ R\Gamma(X/k, \mathbf{G}_{\mathrm{m}}) \to \mathbf{G}_{\mathrm{m}} $$

induced by the natural morphism

$$ \mathbf{G}_{\mathrm{m}} \to i_*\mathbf{G}_{\mathrm{m}} $$

of sheaves on X. We will not make a distinction in notation between the class $[x] \in {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})$ and its image under the truncation map ${}^1H_0^{\mathrm{c}}(V, \mathbf{Z}) \to {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})_\tau$
The sheafified version of this map gives a morphism of sheaves (of sets) over k_{sm}

$$ V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) $$

with the image of V landing in the inverse image of 1 under the degree map

$$ {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) \to \mathbf{Z}. $$

See [vH1] and [vH2] for more information.

Lemma 2.6. *Assume V is a nonsingular geometrically connected variety over k such that $\operatorname{Pic}_{V/k} = 0$. Then the morphism*

$$ V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) = R\,\mathscr{H}om_{k_{sm}}(\Gamma(V/k_{sm}, \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}}) $$

is given by locally sending a section $x \in V$ to the map that sends a local section f of $\Gamma(V/k_{sm}, \mathbf{G}_{\mathrm{m}})$ to $f(x)$.

Proof. This follows immediately from the definitions. □

3 The cap-product and partial generalised global duality for pseudo-motivic homology

3.1 Definition and basic properties of the cap product

Let X be a nonsingular variety over a field k of characteristic zero. Since $\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z}) = R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}}),\mathbf{G}_{\mathrm{m}})$, we have well-defined Yoneda-products

$$
\begin{aligned}
{}^1H^{\mathrm{c}}_i(X,\mathbf{Z}) \times H^j(X,\mathbf{G}_{\mathrm{m}}) &\to H^{j-i}(k,\mathbf{G}_{\mathrm{m}}) \\
{}^1H^{\mathrm{c}}_i(X,\mathbf{Z}) \times H^j(X,\mathbf{Q}/\mathbf{Z}(1)) &\to H^{j-i}(k,\mathbf{Q}/\mathbf{Z}(1)).
\end{aligned}
$$

Applying Tate twist to the torsion coefficients in the second pairing gives us

$$
{}^1H^{\mathrm{c}}_i(X,\mathbf{Z}) \times H^j(X,\mathbf{Q}/\mathbf{Z}(m)) \to H^{j-i}(k,\mathbf{Q}/\mathbf{Z}(m)).
$$

for any $m \in \mathbf{Z}$. Similarly, we have the truncated versions

$$
\begin{aligned}
{}^1H^{\mathrm{c}}_i(X,\mathbf{Z})_\tau \times H^j(X,\mathbf{G}_{\mathrm{m}})_\tau &\to H^{j-i}(k,\mathbf{G}_{\mathrm{m}}) \\
{}^1H^{\mathrm{c}}_i(X,\mathbf{Z})_\tau \times H^j(X,\mathbf{Q}/\mathbf{Z}(m))_\tau &\to H^{j-i}(k,\mathbf{Q}/\mathbf{Z}(m)).
\end{aligned}
$$

All these pairings can be called *cap-product pairings* and will be denoted by $-\cap-$.
For a k-valued point $x\colon \operatorname{Spec} k \hookrightarrow X$, and an $\omega \in H^j(X,\mathbf{Q}/\mathbf{Z}(m))_{(\tau)}$ we have that

$$
[x] \cap \omega = i^*\omega \in H^j(k,\mathbf{Q}/\mathbf{Z}(m)). \tag{4}
$$

This follows easily from the definitions, in particular from the fact that the homology class $[x]$ is defined using the the natural maps $\mathbf{G}_{\mathrm{m}} \to i_*\mathbf{G}_{\mathrm{m}}$ and the pull-back homomorphism i^* is defined using the natural map $\mathbf{Q}/\mathbf{Z}(m) \to i_*\mathbf{Q}/\mathbf{Z}(m)$.

3.2 Partial generalised global duality for pseudo-motivic homology

Let K be a field of characteristic zero with global $(n+2)$-dimensional duality in étale cohomology, and let X be a nonsingular variety over K. We define:

$$
\begin{aligned}
{}^1H^{\mathrm{c}}_i(X_{\mathbf{A}},\mathbf{Z})_{(\tau)} &:= H^{-i}(\mathbf{A},\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_{(\tau)}) \\
{}^1H^{\mathrm{c}}_i(X,X_{\mathbf{A}};\mathbf{Z})_{(\tau)} &:= H^{-i}(K,\mathbf{A};\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_{(\tau)}).
\end{aligned}
$$

The cap product and the maps in cohomology for the pair $(K,\mathbf{A})$ give us pairings

$$
\begin{aligned}
{}^1H^{\mathrm{c}}_i(X_{\mathbf{A}},\mathbf{Z})_{(\tau)} \times H^j(X,\mathbf{Q}/\mathbf{Z}(m))_{(\tau)} &\to H^{j-i+1}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(m)) \\
{}^1H^{\mathrm{c}}_i(X,X_{\mathbf{A}};\mathbf{Z})_{(\tau)} \times H^j(X,\mathbf{Q}/\mathbf{Z}(m))_{(\tau)} &\to H^{j-i+1}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(m)).
\end{aligned}
$$

We will be interested in the case $i = 0$, $j = n+1$, $m = n$. By equation (4) we get a commutative diagram of pairings

$$\begin{array}{ccccc} X(\mathbf{A}) & \times & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_{(\tau)} & \stackrel{\text{rcpr}}{\to} & \mathbf{Q}/\mathbf{Z} \\ \downarrow & & \| & & \| \\ {}^1H_0^{\mathrm{c}}(X_{\mathbf{A}}, \mathbf{Z})_{(\tau)} & \times & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_{(\tau)} & \stackrel{\cap}{\to} & \mathbf{Q}/\mathbf{Z} \end{array} \tag{5}$$

The aim of this section is to identify the left kernel of the bottom pairing in the truncated version of the above diagram with the image of ${}^1H_0^{\mathrm{c}}(X, \mathbf{Z})_\tau$. Since the long exact sequences in the cohomology of the pair $(K, \mathbf{A})$ gives a long exact sequences in pseudo-motivic homology:

$$\cdots \to {}^1H_i^{\mathrm{c}}(X, X_{\mathbf{A}}, \mathbf{Z})_\tau \to {}^1H_i^{\mathrm{c}}(X, \mathbf{Z})_\tau \to {}^1H_i^{\mathrm{c}}(X_{\mathbf{A}}, \mathbf{Z})_\tau \to \\ {}^1H_{i-1}^{\mathrm{c}}(X, X_{\mathbf{A}}, \mathbf{Z})_\tau \to \cdots \tag{6}$$

it will be sufficient to prove that $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau$ detects all elements in ${}^1H_{-1}^{\mathrm{c}}(X, X_{\mathbf{A}}, \mathbf{Z})_\tau$

Theorem 3.1. *Let X be a nonsingular complete variety over a field K having $(n+2)$-dimensional global duality in étale cohomology. Assume that $\mathrm{Pic}_{X/K}$ is a finitely generated group scheme and that $Ш^2(K, \mathbf{X}(\mathrm{Pic}_{X/K}))$ is of finite exponent. Then the cap-product pairing*

$${}^1H_{-1}(X, X_{\mathbf{A}}; \mathbf{Z})_\tau \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau \to H^{n+2}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the left.

Proof. The triangle of Lemma 2.2 gives the following diagram of compatible pairings with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & H^2(K, \mathbf{A}; \mathbf{X}(\mathrm{Pic}_{X/K})) & \longrightarrow & {}^1H_{-1}(X, X_{\mathbf{A}}; \mathbf{Z})_\tau & \longrightarrow & H^1(K, \mathbf{A}; \mathbf{Z}) \\ & \times & & \times & & \times \\ 0 \leftarrow & H^{n-1}(K, \mathrm{Pic}_{X/K} \otimes^L \mathbf{Q}/\mathbf{Z}(n-1)) & \leftarrow & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau & \leftarrow & H^{n+1}(K, \mathbf{Q}/\mathbf{Z}(n)) \\ & \downarrow & & \downarrow & & \downarrow \\ & \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} \end{array}$$

where the leftmost pairing is the pairing (1) and the rightmost pairing is the pairing of Lemma 1.4. Since those two pairings are nondegenerate on the $(K, \mathbf{A})$-side, it follows that the middle pairing is nondegenerate on the $(K, \mathbf{A})$-side as well. $\square$

Corollary 3.2. *Let X be as in Theorem 3.1. Then the left kernel of the cap-product pairing*

$${}^1H_0(X_{\mathbf{A}}; \mathbf{Z})_\tau \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}$$

is precisely the image of the map

$${}^1H_0(X, \mathbf{Z})_\tau \to H_0(X_{\mathbf{A}}, \mathbf{Z})_\tau$$

Proof. This follows from Theorem 3.1, the exact sequence (6) for the pseudo-motivic homology of the pair $(X, X_{\mathbf{A}})$, and the fact that we have a compatible diagram of pairings

$$\begin{array}{ccccc} {}^1H_0(X_{\mathbf{A}}; \mathbf{Z})_\tau & \times & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau & \to & \mathbf{Q}/\mathbf{Z} \\ \downarrow & & \| & & \| \\ {}^1H_{-1}(X, X_{\mathbf{A}}; \mathbf{Z})_\tau & \times & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau & \to & \mathbf{Q}/\mathbf{Z} \end{array}$$

□

COROLLARY 3.3. *Let X be as in Theorem 3.1. If $X(\mathbf{A})^{\mathrm{rcpr}} \neq \emptyset$, then the degree map*

$$H_0(X, \mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective.

Proof. Take an adèlic point $\{x_v\} \in X(\mathbf{A})^{\mathrm{rcpr}}$. The compatibility between cap-product and the map $X(\mathbf{A}) \to {}^1H_0(X_{\mathbf{A}}, \mathbf{Z})_\tau$ implies that its homology class

$$[\{x_v\}] \in {}^1H_0(X_{\mathbf{A}}, \mathbf{Z})_\tau$$

is orthogonal to any $\omega \in H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$, so certainly to any $\omega \in H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau$. Therefore, the homology class $[\{x_v\}]$ is the restriction of some $\gamma \in {}^1H_0(X, \mathbf{Z})$. Since each $[x_v] \in {}^1H_0(X_{K_v}, \mathbf{Z})$ is of degree 1, the degree of γ is 1. □

4 PSEUDO-MOTIVIC HOMOLOGY OF COMPACTIFICATIONS OF TORSORS UNDER TORI

Throughout this section V will be a torsor under a torus T over a field k of characteristic zero.
Since $\mathrm{Pic}_{V/k} = 0$, Lemma 2.4 gives us that the complex $\mathscr{C}_*(V, \mathbf{Z})_\tau$ is in fact quasi-isomorphic to a sheaf which is represented by a group scheme locally of finite type (the extension of $\mathbf{Z}$ by a torus). We denote this group scheme by ${}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})$.

PROPOSITION 4.1. (i) *The triangle of 2.4 is naturally isomorphic to the triangle of sheaves associated to the short exact sequence of group schemes*

$$0 \to T \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) \to \mathbf{Z} \to 0$$

(ii) *The natural map $V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})$ induces a T-equivariant isomorphism of V with the connected component of ${}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})$ mapping to $1 \in \mathbf{Z}$.*

Proof. The first part of the proposition follows by Cartier duality from the sheafified version of Rosenlicht's result that we have a short exact sequence

$$0 \to \mathbf{G}_{\mathrm{m}} \to \Gamma(V/k_{\mathrm{sm}}, \mathbf{G}_{\mathrm{m}}) \to \mathbf{X}(T) \to 0.$$

See [Ro], and also [Ray, Cor. VII.1.2.], [CTS, Prop. 1.4.2].
To get the second part of the proposition, we need the extra information that the map $\Gamma(V/k_{\mathrm{sm}}, \mathbf{G}_{\mathrm{m}}) \to \mathbf{X}(T)$ considered above is defined locally by sending a local section f of $\Gamma(V/k_{\mathrm{sm}}, \mathbf{G}_{\mathrm{m}})$ to the map that sends a local section t of T to $f(t \cdot x)/f(x)$ for any local section x of V. Comparing this with the description of the map $V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})$ in Lemma 2.6 gives the desired result. □

Corollary 4.2. *For any field extension k'/k we have that the natural map*

$$V(k') \to {}^1H_0^{\mathrm{c}}(V_{k'}, \mathbf{Z})_\tau$$

gives a $T(k')$-equivariant isomorphism of $V(k')$ onto the subset of elements of ${}^1H_0^{\mathrm{c}}(V_{k'}, \mathbf{Z})_\tau$ of degree 1.

Proof. Immediate from the above, since ${}^1H_0^{\mathrm{c}}(V_{k'}, \mathbf{Z})_\tau = {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})(k')$ by Lemma 2.4. □

Corollary 4.3. *Let X be a nonsingular complete variety over k containing V as a Zariski-dense subvariety.*

(i) *The degree map*

$${}^1H_0^{\mathrm{c}}(V, \mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective if and only if $V(k') \neq \emptyset$.

(ii) *The degree map*

$${}^1H_0(X, \mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective if and only if $X(k) \neq \emptyset$.

Proof. The first statement follows immediately from Corollary 4.2, whereas the second statement follows from the first combined with Lemma 2.5. □

Together with the results of Section 3 this implies the two theorems in the introduction.

Remark 4.4. It is clear from the above, that we can sharpen Theorem 1 by replacing the full group $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$ in the reciprocity pairing by the truncated group $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau$, or its image under the map $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau \to H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$.
In the case of a number field, this makes no difference, since for smooth compactifications of torsors under tori we have that our truncated cohomological Brauer group $H^2(X, \mathbf{G}_{\mathrm{m}})_\tau$ is equal to the full cohomological Brauer group $H^2(X, \mathbf{G}_{\mathrm{m}})$. (For an arbitrary variety X our truncated group $H^2(X, \mathbf{G}_{\mathrm{m}})_\tau$ is equal to the the so-called 'algebraic' cohomological Brauer group, i.e., the kernel of the map $H^2(X, \mathbf{G}_{\mathrm{m}}) \to H^2(\overline{X}, \mathbf{G}_{\mathrm{m}})$).

5 Comparison with the literature

Torsors under a torus T over a (generalised) global field K which are trivial everywhere locally are classified by $\text{Ш}^1(K,T)$. It follows from Rosenlicht's result and Hilbert Theorem 90 that $H^1(K,T)$ embeds into $H^2(K,\mathbf{X}(\mathrm{Pic}_{X/K}))$ for any smooth compactification X of a principal homogeneous space V under T. For a field K as in Theorem 1, duality gives that $\text{Ш}^1(K,T)$ embeds into the dual of $H^{n-1}(K,\mathrm{Pic}_{X/K}\otimes^L \mathbf{Q}/\mathbf{Z}(n-1))$, hence into the dual of $H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau$. Therefore, it is not very surprising that the reciprocity pairing detects a failure of the local–global principle.

The only problem is to relate the abstract 'arithmetic' pairing

$$\text{Ш}^1(K,T)\times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}$$

to the 'geometric' reciprocity pairing

$$X(\mathbf{A})\times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}.$$

We have seen that pseudo-motivic homology provides a nice conceptual intermediate to compare the two pairings, but there have been other approaches as well. The existing literature deals with number fields, so here we consider the Brauer group, rather than $H^2(X,\mathbf{Q}/\mathbf{Z}(1))_\tau$.

In [San] the comparison between the 'geometric' and the 'arithmetic' pairing is essentially done in Lemma 8.4, using explicit ways of representing classes in the Brauer group and explicit cochain calculations. If one would want to apply this approach to global duality fields of higher cohomological dimension, both the higher degree of the cochains and the fact that the coefficients would be in $\mathbf{Q}/\mathbf{Z}(n)$ should complicate things considerably.

A more conceptual approach, the *descent method*, due to Colliot-Thélène and Sansuc and described in [CTS] uses the concept of a *universal X-torsor* under groups of multiplicative type. The most streamlined version of this approach is probably presented in [Sk]. As in the present paper, the proof proceeds in two major steps. The first result is that for any nonsingular complete variety X over a number field K with $X(\mathbf{A}_K)^{\mathrm{Br}(X)_\tau}\neq\emptyset$ we have that the universal X-torsor exists. The second result is that for a smooth projective compactification of a torsor under a torus over K the universal X-torsor exists if and only if $X(K)\neq\emptyset$.

There is a very clear relation with the present paper: Colliot-Thélène and Sansuc show that the universal X-torsor exists if and only if the 2-fold extension of Galois modules

$$0\to\bar{k}^*\to\bar{k}(X)^*\to\mathrm{Div}(\overline{X})\to\mathrm{Pic}(\overline{X})\to 0$$

is trivial. This can be seen to be equivalent to the surjectivity of the degree map

$${}^1H_0(X,\mathbf{Z})_\tau\to\mathbf{Z}.$$

Therefore, the two steps of the proof in the present paper are equivalent to the two steps in the descent method, but in both steps the methods of proof are different. In particular in the first step the homology approach of the present paper seems much more efficient than the approach of Colliot-Thélène and Sansuc (or Skorobogatov's streamlined version in [Sk, Sec. 6.1]), where again the core of the proof is a comparison of the 'geometric' and the 'arithmetic' pairing using cocycle computations. Recently, Salberger has published a different proof of the first step in the descent method, which no longer needs explicit cocyle computations ([Sal, Prop. 1.26]). However, Salberger's proof does require a subtle cohomological construction that might not be easy to generalise to global fields of higher cohomological dimension.

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