Variations on the Bloch-Ogus Theorem

Ivan Panin, Kirill Zainoulline

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Abstract. Let R be a semi-local regular ring of geometric type over a field k. Let $\mathcal{U} = \operatorname{Spec} R$ be the semi-local scheme. Consider a smooth proper morphism $p : Y \to \mathcal{U}$. Let $Y_{k(u)}$ be the fiber over the generic point of a subvariety u of $\mathcal{U}$. We prove that the Gersten-type complex for étale cohomology

$$0 \to H^q_{\text{ét}}(Y, C) \to H^q_{\text{ét}}(Y_{k(\mathcal{U})}, C) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q-1}_{\text{ét}}(Y_{k(u)}, C(-1)) \to \dots$$

is exact, where C is a locally constant sheaf with finite stalks of $\mathbb{Z}/n\mathbb{Z}$-modules on Y_{et} and n is an integer prime to $\operatorname{char}(k)$.

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1 Introduction

The history of the subject of the present paper starts with the famous paper of D. Quillen [14] where he proves the geometric case of the Gersten's conjecture for K-functor. One may ask whether the similar result holds for étale cohomology.

The first answer on this question was given by S. Bloch and A. Ogus in [2]. They proved the analog of Gersten's conjecture for étale cohomology with coefficients in the twisted sheaf $\mu_n^{\otimes i}$ of n-th roots of unity. More precisely, let X be a smooth quasi-projective variety over a field k and let $x = \{x_1, \dots, x_m\} \subset X$ be a finite subset of points. We denote by $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ the semi-local scheme at x. Consider the sheaf μ_n of n-th roots of unity on the small étale site X_{et}, with n prime to $\operatorname{char}(k)$. Then the main result of [2] (see Theorem 4.2 and

Example 2.1) implies that the Gersten-type complex for étale cohomology with supports

$$0 \to H^q(\mathcal{U}, \mu_n^{\otimes i}) \to \coprod_{u \in \mathcal{U}^{(0)}} H^q_u(\mathcal{U}, \mu_n^{\otimes i}) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q+1}_u(\mathcal{U}, \mu_n^{\otimes i}) \to \cdots \qquad (\dagger)$$

is exact for all $i \in \mathbb{Z}$ and $q \geq 0$, where $\mathcal{U}^{(p)}$ denotes the set of all points of codimension p in $\mathcal{U}$.

The next step was done by O. Gabber in [8]. He proved that the complex ($\dagger$) is exact for cohomology with coefficients in any torsion sheaf C on X_{et} that comes from the base field k, i.e., $C = p^*C'$ for some sheaf C' on $(\operatorname{Spec} k)_{et}$ and the structural morphism $p : X \to \operatorname{Spec} k$.

It turned out that the proof of Gabber can be applied to any cohomology theory with supports that satisfies the same formalism as étale cohomology do. This idea was realized in the paper [4] by J.-L. Colliot-Thélène, R. Hoobler and B. Kahn. Namely, they proved that a cohomology theory with support h^* which satisfies some set of axioms [4, Section 5.1] is effaceable [4, Definition 2.1.1]. Then the exactness of ($\dagger$) follows immediately by trivial reasons [4, Proposition 2.1.2]. In particular, one gets the exactness of ($\dagger$) for the case when $\mathcal{U}$ is replaced by the product $\mathcal{U} \times_k T$, where T is a smooth variety over k [4, Theorem 8.1.1]. It was also proven [4, Remark 8.1.2.(3), Corollary B.3.3] the complex ($\dagger$) is exact for the case when $\dim \mathcal{U} = 1$ and the sheaf of coefficients $\mu_n^{\otimes i}$ is replaced by a bounded below complex of sheaves, whose cohomology sheaves are locally constant constructible, torsion prime to $\operatorname{char}(k)$. The goal of the present paper is to prove the latter case for any dimension of the scheme $\mathcal{U}$. Namely, we want to prove the following

1.1 Theorem. *Let X be a smooth quasi-projective variety over a field k. Let $x = \{x_1, \ldots, x_m\} \subset X$ be a finite subset of points and $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ be the semi-local scheme at x. Let $\mathcal{C}$ be a bounded below complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} with n prime to* $\operatorname{char}(k)$*, Then the E^1-terms of the coniveau spectral sequence yield an exact complex*

$$0 \to H^q(\mathcal{U}, \mathcal{C}) \to \coprod_{u \in \mathcal{U}^{(0)}} H^q_u(\mathcal{U}, \mathcal{C}) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q+1}_u(\mathcal{U}, \mathcal{C}) \to \cdots$$

of étale hypercohomology with supports.

1.2 Remark. For the definition of a constructible sheaf we refer to [1, IX] or [9, V.1.8]. Observe that a locally constant sheaf with finite stalks provides an example of a locally constant constructible sheaf (see [1, IX.2.13]).

1.3 Corollary. *Let R be a semi-local regular ring of geometric type over a field k. We denote by $\mathcal{U} = \operatorname{Spec} R$ the respective semi-local affine scheme. Let $\mathcal{C}$ be a bounded below complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on U_{et} with n prime to* $\operatorname{char}(k)$*, Then the complex*

$$0 \to H^q_{\acute{e}t}(\mathcal{U}, \mathcal{C}) \to H^q_{\acute{e}t}(\operatorname{Spec} k(\mathcal{U}), \mathcal{C}) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q-1}_{\acute{e}t}(\operatorname{Spec} k(u), \mathcal{C}(-1)) \to \cdots$$

is exact, where $k(\mathcal{U})$ *is the function field of* $\mathcal{U}$*,* $k(u)$ *is the residue field of* u *and* $\mathcal{C}(i) = \mathcal{C} \otimes \mu_n^{\otimes i}$.

Proof. Follows by purity for étale cohomology (see the proof of [4, 1.4]). □

1.4 Corollary. *Let* R *be a semi-local regular ring of geometric type over a field* k*. Let* $p : Y \to \mathcal{U}$ *be a smooth proper morphism, where* $\mathcal{U} = \operatorname{Spec} R$*. Let* C *be a locally constant constructible sheaf of* $\mathbb{Z}/n\mathbb{Z}$*-modules on* Y_{et} *with* n *prime to* $\operatorname{char}(k)$*. Then the complex*

$$0 \to H^q_{\acute{e}t}(Y, C) \to H^q_{\acute{e}t}(Y_{k(\mathcal{U})}, C) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q-1}_{\acute{e}t}(Y_{k(u)}, C(-1)) \to \cdots$$

is exact, where $Y_{k(u)} = \operatorname{Spec} k(u) \times_{\mathcal{U}} Y$.

Proof. The cohomology of Y with coefficients in C coincide with the hypercohomology of $\mathcal{U}$ with coefficients in the total direct image Rp_*C [9, VI.4.2]. Observe that the bounded complex Rp_*C has locally constant constructible cohomology sheaves. Now by the main result of paper [12], a bounded complex of sheaves on $\mathcal{U}_{et}$ with locally constant constructible cohomology sheaves is in the derived category isomorphic to a bounded complex of locally constant constructible sheaves. Hence, there exists a bounded complex $\mathcal{C}$ of locally constant constructible sheaves that is quasi-isomorphic to the complex Rp_*C. Replace Rp_*C by $\mathcal{C}$ and apply the previous corollary. □

1.5 Remark. The assumptions on the sheaf C are essential. As it was shown in [7] for the case $k = \mathbb{C}$ and $C = \mathbb{Z}$ there are examples of extensions $Y/\mathcal{U}$ for which the map $H^q_{\mathrm{DR}}(Y) \to H^q_{\mathrm{DR}}(Y_{k(\mathcal{U})})$ is not injective.

1.6 Remark. The injectivity part of Theorem 1.1 (i.e., the exactness at the first term) has been proven recently in [15] by extending the arguments of Voevodsky [16].

The structure of the proof of Theorem 1.1 is the following. First, we give some general formalism (sections 2, 3 and 4). Namely, we prove that any functor F that satisfies some set of axioms (homotopy invariance, transfers, finite monodromy) is effaceable (Theorem 4.7). Then we apply this formalism to étale cohomology (section 5). More precisely, we check that the étale cohomology functor $F(X, Z) = H^*_Z(X, \mathcal{C})$ satisfies all the axioms and, hence, is effaceable. It implies Theorem 1.1 immediately.
We would like to stress that our axioms for the functor F are different from those in [4]. The key point of the proof is that we use Geometric Presentation Lemma of Ojanguren and Panin (see Lemma 3.5) instead of Gabber's. This fact together with the notion of a functor with finite monodromy allows us to apply the techniques developed in [10], [11] and [17].

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2 Definitions and Notations

2.1 Notation. In the present paper all schemes are assumed to be Noetherian and separated. By k we denote a fixed ground field. A variety over k is an integral scheme of finite type over k. To simplify the notation sometimes we will write k instead of the scheme $\operatorname{Spec} k$. We will write $X_1 \times X_2$ for the fibered product $X_1 \times_k X_2$ of two k-schemes. By $\mathcal{U}$ we denote a regular semi-local scheme of geometric type over k, i.e., $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ for a smooth affine variety X over k and a finite set of points $x = \{x_1, \ldots, x_n\}$ of X. By $\mathcal{X}$ we denote a relative curve over $\mathcal{U}$ (see 3.1.(i)). By $\mathcal{Z}$ and $\mathcal{Y}$ we denote closed subsets of $\mathcal{X}$. By Z and Z' we denote closed subsets of $\mathcal{U}$. Observe that $\mathcal{X}$ and $\mathcal{U}$ are essentially smooth over k and all schemes $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{Y}$, Z, Z' are of finite type over $\mathcal{U}$.

2.2 Notation. Let U be a k-scheme. Denote by $Cp(U)$ a category whose objects are couples (X, Z) consisting of an U-scheme X of finite type over U and a closed subset Z of the scheme X (we assume the empty set is a closed subset of X). Morphisms from (X, Z) to (X', Z') are those morphisms $f : X \to X'$ of U-schemes that satisfy the property $f^{-1}(Z') \subset Z$. The composite of f and g is $g \circ f$.

2.3 Notation. Denote by $F : Cp(U) \to Ab$ a contravariant additive functor from the category of couples $Cp(U)$ to the category of (graded) abelian groups. Recall that F is additive if one has an isomorphism

$$F(X_1 \amalg X_2, Z_1 \amalg Z_2) \cong F(X_1, Z_1) \oplus F(X_2, Z_2).$$

Sometimes we shall write $F_Z(X)$ for $F(X, Z)$ having in mind the notation used for cohomology with supports.

Now notions of a homotopy invariant functor, a functor with transfers and a functor that satisfies vanishing property will be given.

2.4 Definition. A contravariant functor $F : Cp(U) \to Ab$ is said to be homotopy invariant if for each U-scheme X smooth or essentially smooth over k and for each closed subset Z of X the map $F_Z(X) \to F_{Z \times \mathbb{A}^1}(X \times \mathbb{A}^1)$ induced by the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism.

2.5 Definition. One says a contravariant functor $F : Cp(U) \to Ab$ satisfies vanishing property if for each U-scheme X one has $F(X, \emptyset) = 0$.

2.6 Definition. A contravariant functor $F : Cp(U) \to Ab$ is said to be endowed with transfers if for each finite flat morphism $\pi : X' \to X$ of U-schemes and for each closed subset $Z \subset X$ it is given a homomorphism of abelian groups $\operatorname{Tr}_X^{X'} : F_{\pi^{-1}(Z)}(X') \to F_Z(X)$ and the family $\{\operatorname{Tr}_X^{X'}\}$ satisfies the following properties:

(i) for each fibered product diagram of U-schemes with a finite flat morphism π

$$\begin{tikzcd} X' & X_1' \arrow[l, "f'"'] \arrow[d, "\pi_1"] \\ X \arrow[u, leftarrow, "\pi"'] & X_1 \arrow[l, "f"'] \end{tikzcd}$$

and for each closed subset $Z \subset X$ the diagram

$$\begin{tikzcd} F_{Z'}(X') \arrow[r, "F(f_1)"] \arrow[d, "\mathrm{Tr}_X^{X'}"'] & F_{Z_1'}(X_1') \arrow[d, "\mathrm{Tr}_{X_1}^{X_1'}"] \\ F_Z(X) \arrow[r, "F(f)"] & F_{Z_1}(X_1) \end{tikzcd}$$

is commutative, where $Z' = \pi^{-1}(Z)$, $Z_1 = f^{-1}(Z)$ and $Z_1' = \pi_1^{-1}(Z_1)$;

(ii) if $\pi : X_1' \amalg X_2' \to X$ is a finite flat morphism of U-schemes, then for each closed subset $Z \subset X$ the diagram

$$\begin{tikzcd} F_{Z'}(X_1' \amalg X_2') \arrow[r, "\text{``+''}"] \arrow[dr, "\mathrm{Tr}_X^{X_1' \amalg X_2'}"'] & F_{Z_1'}(X_1') \oplus F_{Z_2'}(X_2') \arrow[d, "\mathrm{Tr}_{X_1}^{X_1'} + \mathrm{Tr}_{X_2}^{X_2'}"] \\ & F_Z(X) \end{tikzcd}$$

is commutative, where $Z' = \pi^{-1}(Z)$, $Z_1' = X_1' \cap Z'$ and $Z_2' = X_2' \cap Z'$;

(iii) if $\pi : (X', Z') \to (X, Z)$ is an isomorphism in $Cp(U)$, then two maps $\mathrm{Tr}_X^{X'}$ and $F(\pi)$ are inverses of each other, i.e.

$$F(\pi) \circ \mathrm{Tr}_X^{X'} = \mathrm{Tr}_X^{X'} \circ F(\pi) = \mathrm{id}.$$

3 The Specialization Lemma

The following definition is inspired by the notion of a good triple used by Voevodsky in [16].

3.1 Definition. Let $\mathcal{U}$ be a regular semi-local scheme of geometric type over the field k. A triple $(\mathcal{X}, \delta, \mathfrak{f})$ consisting of an $\mathcal{U}$-scheme $p : \mathcal{X} \to \mathcal{U}$, a section $\delta : \mathcal{U} \to \mathcal{X}$ of the morphism p and a regular function $\mathfrak{f} \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is called a perfect triple over $\mathcal{U}$ if $\mathcal{X}$, δ and $\mathfrak{f}$ satisfy the following conditions:

(i) the morphism p can be factorized as $p : \mathcal{X} \xrightarrow{\pi} \mathbb{A}^1 \times \mathcal{U} \xrightarrow{pr} \mathcal{U}$, where π is a finite surjective morphism and pr is the canonical projection on the second factor;

(ii) the vanishing locus of the function $\mathfrak{f}$ is finite over $\mathcal{U}$;

(iii) the scheme $\mathcal{X}$ is essentially smooth over k and the morphism p is smooth along $\delta(\mathcal{U})$;

(iv) the scheme $\mathcal{X}$ is irreducible.

3.2 Remark. The property (i) says that $\mathcal{X}$ is an affine curve over $\mathcal{U}$. The property (iii) implies that $\mathcal{X}$ is a regular scheme. Since $\mathcal{X}$ and $\mathbb{A}^1 \times \mathcal{U}$ are regular schemes by [6, 18.17] the morphism $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$ from (i) is a finite flat morphism.

The following lemma will be used in the proof of Theorems 4.2 and 4.7.

3.3 Lemma. *Let $\mathcal{U}$ be a regular semi-local scheme of geometric type over an infinite field k. Let $(p : \mathcal{X} \to \mathcal{U}, \delta : \mathcal{U} \to \mathcal{X}, \mathfrak{f} \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$ be a perfect triple over $\mathcal{U}$. Let $F : Cp(\mathcal{U}) \to Ab$ be a homotopy invariant functor endowed with transfers which satisfies vanishing property (see 2.4, 2.6 and 2.5). Then for each closed subset $\mathcal{Z}$ of the vanishing locus of $\mathfrak{f}$ the following composite vanishes*

$$F_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{F(\delta)} F_{\delta^{-1}(\mathcal{Z})}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{p(\mathcal{Z})}(\mathcal{U})$$

3.4 Remark. The mentioned composite is the map induced by the morphism $\delta : (\mathcal{U}, p(\mathcal{Z})) \to (\mathcal{X}, \mathcal{Z})$ in the category $Cp(\mathcal{U})$. Observe that we have $\delta^{-1}(\mathcal{Z}) \subset p(\mathcal{Z})$, where $p(\mathcal{Z})$ is closed by (ii) of 3.1.

Proof. Consider the commutative diagram in the category $Cp(\mathcal{U})$

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{Y}) & \xrightarrow{\mathrm{id}_{\mathcal{X}}} & (\mathcal{X}, \mathcal{Z}) \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\delta} \\ (\mathcal{U}, Z') & \xrightarrow{\mathrm{id}_{\mathcal{U}}} & (\mathcal{U}, Z) \end{array}$$

where $Z = \delta^{-1}(\mathcal{Z})$, $Z' = p(\mathcal{Z})$ and $\mathcal{Y} = p^{-1}(Z')$. It gives the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(\delta) = F(\delta) \circ F(\mathrm{id}_{\mathcal{X}})$. Thus to prove the theorem it suffices to check that the following composite vanishes

$$F_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{F(\mathrm{id}_{\mathcal{X}})} F_{\mathcal{Y}}(\mathcal{X}) \xrightarrow{F(\delta)} F_{Z'}(\mathcal{U}) \qquad (\dagger)$$

By Lemma 3.5 below applied to the perfect triple $(\mathcal{X}, \delta, \mathfrak{f})$ we can choose the finite surjective morphism $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$ from (i) of 3.1 in such a way that it's fibers at the points 0 and 1 of $\mathbb{A}^1$ look as follows:

(a) $\pi^{-1}(\{0\} \times \mathcal{U}) = \delta(\mathcal{U}) \amalg \mathcal{D}_0$ (scheme-theoretically) and $\mathcal{D}_0 \subset \mathcal{X}_{\mathfrak{f}}$;

(b) $\pi^{-1}(\{1\} \times \mathcal{U}) = \mathcal{D}_1$ and $\mathcal{D}_1 \subset \mathcal{X}_{\mathfrak{f}}$.

Observe that $\mathcal{Y} = \pi^{-1}(\mathbb{A}^1 \times Z')$. Let $\mathcal{Z}'_0 = \pi^{-1}(\{0\} \times Z') \cap \mathcal{D}_0$ and $\mathcal{Z}'_1 = \pi^{-1}(\{1\} \times Z')$ be the closed subsets of $\mathcal{Y}$. By definition $\mathcal{Z}'_0$, $\mathcal{Z}'_1$ are the closed subsets of $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively. Since $\mathcal{Z}$ is contained in the vanishing locus of $\mathfrak{f}$ and $\mathcal{D}_0$, $\mathcal{D}_1 \subset \mathcal{X}_{\mathfrak{f}}$ we have $\mathcal{Z} \cap \mathcal{D}_0 = \mathcal{Z} \cap \mathcal{D}_1 = \emptyset$. The latter means that there are two commutative diagrams in the category $Cp(\mathcal{U})$

$$\begin{tikzcd}
(\mathcal{X},\mathcal{Y}) \arrow[r, "\mathrm{id}_{\mathcal{X}}"] & (\mathcal{X},\mathcal{Z}) \\
(\mathcal{D}_0,\mathcal{Z}'_0) \arrow[u, "I_0"] \arrow[r, "\mathrm{id}_{\mathcal{D}_0}"] & (\mathcal{D}_0,\emptyset) \arrow[u, "I_0"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(\mathcal{X},\mathcal{Y}) \arrow[r, "\mathrm{id}_{\mathcal{X}}"] & (\mathcal{X},\mathcal{Z}) \\
(\mathcal{D}_1,\mathcal{Z}'_1) \arrow[u, "I_1"] \arrow[r, "\mathrm{id}_{\mathcal{D}_1}"] & (\mathcal{D}_1,\emptyset) \arrow[u, "I_1"']
\end{tikzcd}$$

where I_0, I_1 are the closed embeddings $\mathcal{D}_0 \hookrightarrow \mathcal{X}$ and $\mathcal{D}_1 \hookrightarrow \mathcal{X}$ respectively. By vanishing property 2.5 we have $F(\mathcal{D}_0,\emptyset) = F(\mathcal{D}_1,\emptyset) = 0$. Then applying F to the diagrams we immediately get

$$F(I_0) \circ F(\mathrm{id}_{\mathcal{X}}) = 0 \text{ and } F(I_1) \circ F(\mathrm{id}_{\mathcal{X}}) = 0. \tag{1}$$

Let $i_0,\, i_1 : \mathcal{U} \hookrightarrow \mathbb{A}^1 \times \mathcal{U}$ be the closed embeddings which correspond to the points 0 and 1 of $\mathbb{A}^1$ respectively. The homotopy invariance property 2.4 implies that

$$F(i_0) = F(i_1) : F_{\mathbb{A}^1 \times Z'}(\mathbb{A}^1 \times \mathcal{U}) \to F_{Z'}(\mathcal{U}). \tag{2}$$

The base change property 2.6.(i) applied to the fibered product diagram

$$\begin{tikzcd}
(\mathcal{X},\mathcal{Y}) \arrow[d, "\pi"'] & (\mathcal{D}_1,\mathcal{Z}'_1) \arrow[l, "I_1"'] \arrow[d, "\pi"] \\
(\mathbb{A}^1 \times \mathcal{U}, \mathbb{A}^1 \times Z') & (\mathcal{U}, Z') \arrow[l, "i_1"']
\end{tikzcd}$$

gives the relation

$$F(i_1) \circ \mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} = \mathrm{Tr}^{\mathcal{D}_1}_{\mathcal{U}} \circ F(I_1), \tag{3}$$

where $\mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} : F_{\mathcal{Y}}(\mathcal{X}) \to F_{\mathbb{A}^1 \times Z'}(\mathbb{A}^1 \times \mathcal{U})$ and $\mathrm{Tr}^{\mathcal{D}_1}_{\mathcal{U}} : F_{\mathcal{Z}'_1}(\mathcal{D}_1) \to F_{Z'}(\mathcal{U})$ are the transfer maps for the finite flat morphism π and $\pi|_{\mathcal{D}_1}$ respectively.
Consider the commutative diagram

$$\begin{tikzcd}
F_{\mathcal{Z}}(\mathcal{X}) \arrow[r, "F(\mathrm{id}_{\mathcal{X}})"] & F_{\mathcal{Y}}(\mathcal{X}) \arrow[r, "{(F(\delta),F(I_0))}"] \arrow[d, "\mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}}"'] & F_{Z' \amalg \mathcal{Z}'_0}(\mathcal{U} \amalg \mathcal{D}_0) \arrow[r, "\text{``+''}"] \arrow[d, "\mathrm{Tr}"'] & F_{Z'}(\mathcal{U}) \oplus F_{\mathcal{Z}'_0}(\mathcal{D}_0) \arrow[dl, "\mathrm{id}+\mathrm{Tr}^{\mathcal{D}_0}_{\mathcal{U}}"] \\
& F_{\mathbb{A}^1 \times Z'}(\mathbb{A}^1 \times \mathcal{U}) \arrow[r, "F(i_0)"] & F_{Z'}(\mathcal{U}) &
\end{tikzcd} \tag{4}$$

where the central square commutes by 2.6.(i) and the right triangle commutes by 2.6.(ii). In the diagram we identify $\mathcal{U}$ with $\delta(\mathcal{U})$ by means of the isomorphism $\delta : \mathcal{U} \to \delta(U)$ and use the property 2.6.(iii) to identify $\mathrm{Tr}^{\delta(\mathcal{U})}_{\mathcal{U}}$ with $F(\delta)$.
The following chain of relations shows that the composite (†) vanishes and we finish the proof of the lemma.

$$F(\delta) \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(1)}{=} (\mathrm{id} + \mathrm{Tr}^{\mathcal{D}_0}_{\mathcal{U}}) \circ (F(\delta), F(I_0)) \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(4)}{=} F(i_0) \circ \mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(2)}{=}$$

$$F(i_1) \circ \mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(3)}{=} \mathrm{Tr}^{\mathcal{D}_1}_{\mathcal{U}} \circ F(I_1) \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(1)}{=} 0$$

□

The following lemma is the semi-local version of Geometric Presentation Lemma [10, 10.1]

3.5 Lemma. *Let R be a semi-local essentially smooth algebra over an infinite field k and A an essentially smooth k-algebra, which is finite over the polynomial algebra $R[t]$. Suppose that $e : A \to R$ is an R-augmentation and let $I = \ker e$. Assume that A is smooth over R at every prime containing I. Given $f \in A$ such that A/Af is finite over R we can find an $s \in A$ such that*

1. *A is finite over $R[s]$.*
2. *$A/As = A/I \times A/J$ for some ideal J of A.*
3. *$J + Af = A$.*
4. *A(s-1)+Af=A.*

Proof. In the proof of [10, 10.1] replace the reduction modulo maximal ideal by the reduction modulo radical of the semi-local ring. □

4 The Effacement Theorem

We start with the following definition which is a slightly modified version of [4, 2.1.1].

4.1 Definition. Let X be a smooth affine variety over a field k. Let $x = \{x_1, \ldots, x_n\}$ be a finite set of points of X and let $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ be the semi-local scheme at x. A contravariant functor $F : Cp(X) \to Ab$ is effaceable at x if the following condition satisfied:
Given $m \geq 1$, for any closed subset $Z \subset X$ of codimension m, there exist a closed subset $Z' \subset \mathcal{U}$ such that

(1) $Z' \supset Z \cap \mathcal{U}$ and $\mathrm{codim}_{\mathcal{U}}(Z') \geq m - 1$;

(2) the composite $F_Z(X) \xrightarrow{F(j)} F_{Z\cap\mathcal{U}}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{Z'}(\mathcal{U})$ vanishes, where $j : \mathcal{U} \to X$ is the canonical embedding and $Z \cap \mathcal{U} = j^{-1}(Z)$.

4.2 Theorem. *Let X be a smooth affine variety over an infinite field k and $x \subset X$ be a finite set of points. Let $G : Cp(k) \to Ab$ be a homotopy invariant functor endowed with transfers which satisfies vanishing property (see 2.4, 2.6 and 2.5). Let $F = p^*G$ denote the restriction of G to $Cp(X)$ by means of the structural morphisms $p : X \to \operatorname{Spec} k$. Then F is effaceable at x.*

Proof. We may assume $x \cap Z$ in non-empty. Indeed, if $x \cap Z = \emptyset$ then the theorem follows by the vanishing property of F (see 2.5).
Let $f \neq 0$ be a regular function on X such that Z is a closed subset of the vanishing locus of f. By Quillen's trick [14, 5.12], [13, 1.2] we can find a morphism $q : X \to \mathbb{A}^{n-1}$, where $n = \dim X$, such that

(a) $q|_{f=0} : \{f = 0\} \to \mathbb{A}^{n-1}$ is a finite morphism;

(b) q is smooth at the points x;

(c) q can be factorized as $q = pr \circ \Pi$, where $\Pi : X \to \mathbb{A}^n$ is a finite surjective morphism and $pr : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is a linear projection.

Consider the base change diagram for the morphism q by means of the composite $r : \mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{j} X \xrightarrow{q} \mathbb{A}^{n-1}$.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "r_X"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ \mathcal{U} \arrow[r, "r"] & \mathbb{A}^{n-1} \end{tikzcd}$$

So we have $\mathcal{X} = \mathcal{U} \times_{\mathbb{A}^{n-1}} X$ and p, r_X denote the canonical projections on $\mathcal{U}$, X respectively. Let $\delta : \mathcal{U} \to \mathcal{X} = \mathcal{U} \times_{\mathbb{A}} X$ be the diagonal embedding. Clearly δ is a section of p. Set $\mathfrak{f} = r_X^*(f)$. Take instead of $\mathcal{X}$ it's irreducible component containing $\delta(\mathcal{U})$ and instead of $\mathfrak{f}$ it's restriction to this irreducible component (since $x \cap Z$ is non-empty the vanishing locus of $\mathfrak{f}$ on the component containing $\delta(\mathcal{U})$ is non-empty as well).
Now assuming the triple $(p : \mathcal{X} \to \mathcal{U}, \delta, \mathfrak{f})$ is a perfect triple over $\mathcal{U}$ (see 3.1 for the definition) we complete the proof as follows:
Let $\mathcal{Z} = r_X^{-1}(Z)$ be the closed subset of the vanishing locus of $\mathfrak{f}$. Let $Z' = p(\mathcal{Z})$ be a closed subset of $\mathcal{U}$. Since $r_X \circ \delta = j$ we have $\delta^{-1}(\mathcal{Z}) = j^{-1}(Z) = Z \cap \mathcal{U}$. By Specialization Lemma 3.3 applied to the perfect triple $(\mathcal{X}, \delta, \mathfrak{f})$ and the functor $j^*F : Cp(\mathcal{U}) \to Ab$ the composite

$$F_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{F(\delta)} F_{Z\cap\mathcal{U}}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{Z'}(\mathcal{U})$$

vanishes. In particular, the composite

$$F_Z(X) \xrightarrow{F(j)} F_{Z\cap\mathcal{U}}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{Z'}(\mathcal{U}) \qquad (*)$$

vanishes as well. Clearly $Z' \supset Z \cap \mathcal{U}$. By 3.1.(i) we have $\dim \mathcal{X} = \dim \mathcal{U} + 1$. On the other hand the morphism $r_{\mathcal{X}} : \mathcal{X} \to X$ is flat (even essentially smooth) and, thus, $\operatorname{codim}_{\mathcal{X}}(\mathcal{Z}) = \operatorname{codim}_X Z$. Therefore, we have $\operatorname{codim}_{\mathcal{U}}(Z') = \operatorname{codim}_{\mathcal{X}}(\mathcal{Z}) - 1 = m - 1$. □

Hence, it remains to prove the following:

4.3 Lemma. *The triple* $(p : \mathcal{X} \to \mathcal{U}, \delta, \mathfrak{f})$ *is perfect over* $\mathcal{U}$.

Proof. By the property (c) one has $q = pr \circ \Pi$ with a finite surjective morphism $\Pi : X \to \mathbb{A}^n$ and a linear projection $pr : \mathbb{A}^n \to \mathbb{A}^{n-1}$. Taking the base change of Π by means of $r : \mathcal{U} \to \mathbb{A}^{n-1}$ one gets a finite surjective $\mathcal{U}$-morphism $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$. This checks (i) of 3.1. Since the closed subset $\{f = 0\}$ of X is finite over $\mathbb{A}^{n-1}$ the closed subset $\{\mathfrak{f} = 0\}$ of $\mathcal{X}$ is finite over $\mathcal{U}$ and we get 3.1.(ii). Since q is smooth at x the morphism $r : \mathcal{U} \to \mathbb{A}^{n-1}$ is essentially smooth. Thus the morphism $r_X : \mathcal{X} \to X$ is essentially smooth as the base change of the morphism r. The variety X is smooth over k implies that $\mathcal{X}$ is essentially smooth over k as well. Since q is smooth at x the morphism p is smooth at each point $y \in \mathcal{X}$ with $r_X(y) \in x$. In particular p is smooth at the points $\delta(x_i)$ $(x_i \in \mathcal{U})$. Since $\mathcal{U}$ is semi-local $\delta(\mathcal{U})$ is semi-local and p is smooth along $\delta(\mathcal{U})$. This checks (iii) of 3.1. Since $\mathcal{X}$ is irreducible 3.1.(iv) holds. And we have proved the lemma and the theorem. □

To prove Theorem 4.2 in the case when F is defined over some smooth affine variety we have to put an additional condition on F. In order to formulate this condition we introduce some notations.

4.4 Notation. Let $\rho : Y \to X \times X$ be a finite étale morphism together with a section $s : X \to Y$ over the diagonal embedding $\Delta : X \to X \times X$, i.e., $\rho \circ s = \Delta$. Let pr_1, $pr_2 : X \times X \to X$ be the canonical projections. We denote p_1, $p_2 : Y \to X$ to be the composite $pr_1 \circ \rho$, $pr_2 \circ \rho$ respectively.
For a contravariant functor $F : Cp(X) \to Ab$ consider it's pull-backs $p_1^* F$ and $p_2^* F : Cp(Y) \to Ab$ by means of p_1 and p_2 respectively. From this point on we denote $F_1 = p_1^* F$ and $F_2 = p_2^* F$. By definition we have

$$F_i(Y' \to Y, Z) = F(Y' \to Y \xrightarrow{p_i} X, Z).$$

4.5 Remark. In general case the functors F_1 and F_2 are not equivalent. Moreover, the functors $pr_1^* F$ and $pr_2^* F$ are different. But in the case when F comes from the base field k, i.e., $F = p^* G$ where $p : X \to \operatorname{Spec} k$ is the structural morphism and $G : Cp(k) \to Ab$ is a contravariant functor, these functors coincide with each other.

4.6 Definition. We say a contravariant functor $F : Cp(X) \to Ab$ has a finite monodromy of the type $(\rho : Y \to X \times X, s : X \to Y)$, where ρ is a finite étale morphism and s is a section of ρ over the diagonal, if there exists an isomorphism $\Phi : F_1 \to F_2$ of functors on $Cp(Y)$. A functor $F : Cp(X) \to Ab$ is said to be a functor with finite monodromy if F has a finite monodromy of some type.

4.7 Theorem. *Let X be a smooth affine variety over an infinite field k and $x \subset X$ be a finite set of points. Let $F : Cp(X) \to Ab$ be a homotopy invariant functor endowed with transfers which satisfies vanishing property (see 2.4, 2.6 and 2.5). If F is a functor with finite monodromy then F is effaceable at x.*

Proof. Similar to the proof of Theorem 4.2 let $f \neq 0$ be a regular function on X such that Z is a closed subset of the vanishing locus of f. We may assume $x \cap Z$ is non-empty. Consider the fibered product diagram from the proof of Theorem 4.2

$$\begin{tikzcd} \mathcal{X} \arrow[r, "r_X"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ \mathcal{U} \arrow[r, "r"] & \mathbb{A}^{n-1} \end{tikzcd}$$

We have the projection $p : \mathcal{X} = \mathcal{U} \times_{\mathbb{A}^{n-1}} X \to \mathcal{U}$, the section $\delta : \mathcal{U} \to \mathcal{X}$ of p and the regular function $\mathfrak{f} = r_X^*(f)$.
Since F is the functor with finite monodromy there is a finite étale morphism $\rho : Y \to X \times X$, a section $s : X \to Y$ of ρ over the diagonal embedding and a functor isomorphism $\Phi : F_1 \to F_2$ as in 4.4 and 4.6. Consider the base change diagram for the morphism $\rho : Y \to X \times X$ by means of the composite $g : \mathcal{X} \xrightarrow{(p,r_X)} \mathcal{U} \times X \xrightarrow{(j,\mathrm{id})} X \times X$

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\rho}"'] & Y \arrow[d, "\rho"] \\ \mathcal{X} \arrow[r, "g"] & X \times X \end{tikzcd}$$

Then $\tilde{\rho}$ is a finite étale morphism and there is the section $\tilde{\delta} : \mathcal{U} \to \tilde{\mathcal{X}}$ of the composite $\tilde{p} = p \circ \tilde{\rho} : \tilde{\mathcal{X}} \to \mathcal{U}$ such that $\tilde{\rho} \circ \tilde{\delta} = \delta$ ($\tilde{\delta}$ is the base change of the morphism $s : X \to Y$ by means of $\tilde{g} : \tilde{\mathcal{X}} \to Y$). Set $\tilde{\mathfrak{f}} = \tilde{\rho}^{-1}(\mathfrak{f})$. As in the proof of 4.2 we replace $\tilde{\mathcal{X}}$ by it's irreducible component containing $\tilde{\delta}(\mathcal{U})$ and $\tilde{\mathfrak{f}}$ by it's restriction to this component. By Lemma 4.8 below the triple $(\tilde{p} : \tilde{\mathcal{X}} \to \mathcal{U}, \tilde{\delta}, \tilde{\mathfrak{f}})$ is perfect.
Let $\mathcal{Z} = r_X^{-1}(Z)$ and $\tilde{\mathcal{Z}} = \tilde{\rho}^{-1}(\mathcal{Z})$. Set $Z' = p(\mathcal{Z})$. Observe that $\tilde{p}(\tilde{\mathcal{Z}}) = Z'$. The commutative diagram

$$\begin{tikzcd} F_Z(X) \arrow[r, "F(j)"] \arrow[d, "F(r_X)"'] & F_{Z\cap\mathcal{U}}(\mathcal{U}) \arrow[r, "F(\mathrm{id}_{\mathcal{U}})"] & F_{Z'}(\mathcal{U}) \\ F_{\mathcal{Z}}(\mathcal{X}) \arrow[r, "F(\tilde{\rho})"] \arrow[ur, "F(\delta)"] & F_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{X}}) \arrow[u, "F(\tilde{\delta})"'] & \end{tikzcd}$$

shows that to prove the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(j) = 0$ (compare with $(*)$ of the proof of 4.2) it suffices to check the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(\tilde{\delta}) = 0$.
Consider the pull-backs of the functors F_1, F_2 and the functor isomorphism Φ by means of the morphism $\tilde{g} : \tilde{\mathcal{X}} \to Y$. We shall use the same notation F_1, F_2 and Φ for these pull-backs till the end of this proof. So we have $F_1 = \tilde{g}^*(p_1^* F)$ and $F_2 = \tilde{g}^*(p_2^* F)$. The isomorphism $\Phi : F_1 \xrightarrow{\cong} F_2$ of functors over $Cp(\tilde{\mathcal{X}})$

provides us with the following commutative diagram

$$\begin{tikzcd}
(F_1)_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{X}}) \arrow[r, "F_1(\tilde{\delta})"] \arrow[d, "\Phi"', "\cong"] & (F_1)_{Z\cap\mathcal{U}}(\mathcal{U}) \arrow[r, "F_1(\mathrm{id}_{\mathcal{U}})"] \arrow[d, "\Phi"', "\cong"] & (F_1)_{Z'}(\mathcal{U}) \arrow[d, "\cong"', "\Phi"] \\
(F_2)_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{X}}) \arrow[r, "F_2(\tilde{\delta})"] & (F_2)_{Z\cap\mathcal{U}}(\mathcal{U}) \arrow[r, "F_2(\mathrm{id}_{\mathcal{U}})"] & (F_2)_{Z'}(\mathcal{U})
\end{tikzcd}$$

where the structure of an $\tilde{\mathcal{X}}$-scheme on $\mathcal{U}$ is given by $\tilde{\delta}$.
Since $r_X \circ \tilde{\rho} = p_2 \circ \tilde{g}$ we have $\tilde{\rho}^*(r_X^* F) = F_2$. Thus to check the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(\tilde{\delta}) = 0$ for the functor F we have to verify the same relation $F_2(\mathrm{id}_{\mathcal{U}}) \circ F_2(\tilde{\delta}) = 0$ for the functor F_2. Then by commutativity of the diagram it suffices to prove the relation $F_1(\mathrm{id}_{\mathcal{U}}) \circ F_1(\tilde{\delta}) = 0$ for the functor F_1.
Since $j \circ \tilde{p} = p_1 \circ \tilde{g}$ we have $\tilde{p}^*(j^*F) = F_1 : Cp(\tilde{\mathcal{X}}) \to Ab$. Thereby it suffices to prove the relation $G(\mathrm{id}_{\mathcal{U}}) \circ G(\tilde{\delta}) = 0$ for the functor $G = j^*F : Cp(\mathcal{U}) \to Ab$. This relation follows immediately from Theorem 3.3 applied to the functor G, the triple $(\tilde{p} : \tilde{\mathcal{X}} \to \mathcal{U}, \tilde{\delta}, \tilde{\mathfrak{f}})$ and the closed subset $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{X}}$. □

4.8 Lemma. *The triple $(\tilde{\mathcal{X}}, \tilde{\delta}, \tilde{\mathfrak{f}})$ is perfect over $\mathcal{U}$.*

Proof. Observe that the triple $(p : \mathcal{X} \to \mathcal{U}, \delta, \mathfrak{f})$ is perfect by Lemma 4.3, the morphism $\tilde{\rho} : \tilde{\mathcal{X}} \to \mathcal{X}$ is finite étale and $\tilde{\rho} \circ \tilde{\delta} = \delta$ for the section $\tilde{\delta} : \mathcal{U} \to \tilde{\mathcal{X}}$ of the morphism $\tilde{p} : \tilde{\mathcal{X}} \to \mathcal{U}$. For the finite surjective morphism of $\mathcal{U}$-schemes $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$ the composite $\tilde{\mathcal{X}} \xrightarrow{\tilde{\rho}} \mathcal{X} \xrightarrow{\pi} \mathbb{A}^1 \times \mathcal{U}$ is a finite surjective morphism of $\mathcal{U}$-schemes as well. This proves 3.1.(i). Since $\tilde{\rho}$ is finite and the vanishing locus of $\mathfrak{f}$ is finite over $\mathcal{U}$ the vanishing locus of the function $\tilde{\mathfrak{f}}$ is finite over $\mathcal{U}$ as well. This proves 3.1.(ii). Since $\tilde{\rho} \circ \tilde{\delta} = \delta$, $\tilde{\rho}$ is étale and p is smooth along $\delta(\mathcal{U})$ the morphism $\tilde{p}$ is smooth along $\tilde{\delta}(\mathcal{U})$. Since the scheme $\mathcal{X}$ is essentially smooth over k and $\tilde{\rho}$ is étale the scheme $\tilde{\mathcal{X}}$ is essentially smooth over k. This proves 3.1.(iii). Since $\tilde{\mathcal{X}}$ is irreducible we have 3.1.(iv). And the lemma is proven. □

5 Applications to Étale Cohomology

5.1 Definition. Let X be a smooth affine variety over a field k, n an integer prime to $char(k)$ and $\mathcal{C}$ a bounded below complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et}. Since étale cohomology commute with inductive limits, we may suppose the complex $\mathcal{C}$ is also bounded from above. We consider the functor $F : Cp(X) \to Ab$ which is given by

$$F(Y \xrightarrow{f} X, Z) = H^*_Z(Y, f^*\mathcal{C}),$$

where the object on the right hand side are étale hypercohomology. Since a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} is representable by a finite étale scheme over X, we may assume that the hypercohomology are taken on the big étale site of X [9, V.1]. Hence, we have a well-defined functor.

5.2 Lemma. *The étale cohomology functor* $F(X,Z) = H^*_Z(X,\mathcal{C})$ *is a functor endowed with transfers (2.6).*

Proof. Let $\pi : Y \to X$ be a finite flat morphism of schemes. For an X-scheme X' set $Y' = X' \times_X Y$ and denote the projection $Y' \to X'$ by π'. If $X'' \xrightarrow{g} X'$ is an X-scheme morphism then set $Y'' = X'' \times_X Y$ and denote by $\pi'' : Y'' \to X''$ the projection on X'' and by $g_Y : Y'' \to Y'$ the morphism $g \times \mathrm{id}_Y$. If $Z \subset X$ is a closed subset then we set $S = \pi^{-1}(Z)$, $Z' = X' \times_X Z$, $Z'' = X'' \times_X Z$, $S' = (\pi')^{-1}(Z')$, $S'' = (\pi'')^{-1}(Z'')$.
If $Y' = Y_1' \amalg Y_2'$ (disjoint union) then set $\pi_i' = \pi'|_{Y_i'}$, $Y_i'' = g_Y^{-1}(Y_i')$, $S_i' = Y_i' \cap S'$, $S_i'' = Y_i'' \cap S''$ and define $g_{Y,i} : Y_i'' \to Y_i'$ to be the restriction of g_Y.
Let C be a sheaf on the big étale site Et/X. If $Z \subset X$ is a closed subset then for an X-scheme X' we denote $\Gamma_{Z'}(X',C) = \ker(\Gamma(X',C) \to \Gamma(X'-Z',C))$ and if $Y' = Y_1' \amalg Y_2'$ we denote $\Gamma_{S_i'}(Y_i',C) = \ker(\Gamma(Y_i',C) \to \Gamma(Y_i' - S_i',C))$.
Deligne in [5] constructed trace maps for finite flat morphisms. In particular, for a X-scheme X' and for every presentation of the scheme Y' in the form $Y' = Y_1' \amalg Y_2'$ there are certain trace maps $\mathrm{Tr}_{\pi_i'} : \Gamma_{S_i'}(Y_i',C) \to \Gamma_{Z'}(X',C)$. These maps satisfy the following properties

(i) (base change) the diagram

$$\begin{tikzcd} \Gamma_{S_i''}(Y_i'',C) \arrow[d, "\mathrm{Tr}_{\pi_i''}"'] & \Gamma_{S_i'}(Y_i',C) \arrow[l, "g_{Y,i}^*"'] \arrow[d, "\mathrm{Tr}_{\pi_i'}"] \\ \Gamma_{Z''}(X'',C) & \Gamma_{Z'}(X',C) \arrow[l, "g_Y^*"'] \end{tikzcd}$$

commutes;

(ii) (additivity) the diagram

$$\begin{tikzcd} \Gamma_{S'}(Y',C) \arrow[r, "``+''"] \arrow[d, "\mathrm{Tr}_{\pi'}"'] & \Gamma_{S_1'}(Y_1',C) \oplus \Gamma_{S_2'}(Y_2',C) \arrow[d, "\mathrm{Tr}_{\pi_1'} + \mathrm{Tr}_{\pi_2'}"] \\ \Gamma_{Z'}(X',C) \arrow[r, "\mathrm{id}"] & \Gamma_{Z'}(X',C) \end{tikzcd}$$

commutes;

(iii) (normalization) if $\pi_1' : Y_1' \to X'$ is an isomorphism then the composite map

$$\Gamma_{Z'}(X',C) \xrightarrow{(\pi_1')^*} \Gamma_{S_1'}(Y_1',C) \xrightarrow{\mathrm{Tr}_{\pi_1'}} \Gamma_{Z'}(X',C)$$

is the identity;

(iv) maps $\mathrm{Tr}_{\pi_i'}$ are functorial with respect to sheaves C on Et/X.

Now let $0 \to C \to \mathfrak{I}^\bullet$ be an injective resolution of the sheaf C on Et/X. Then for a closed subset $Z \subset X$ and for a presentation $Y' = Y'_1 \amalg Y'_2$ one has

$$H^p_{S'_i}(Y'_i, C) := H^p(\Gamma_{S'_i}(Y'_i, \mathfrak{I}^\bullet)) \text{ , } H^p_{Z'}(X', C) := H^p(\Gamma_{Z'}(X', \mathfrak{I}^\bullet)).$$

Thereby the property (iv) shows that the trace maps $\mathrm{Tr}_{\pi'_i} : \Gamma_{S'_i}(Y'_i, \mathfrak{I}^r) \to \Gamma_{Z'}(X', \mathfrak{I}^r)$ determine a morphism of complexes $\Gamma_{S'_i}(Y'_i, \mathfrak{I}^\bullet) \to \Gamma_{Z'}(X', \mathfrak{I}^\bullet)$. Thus one gets the induced map which we will denote by $H^p(\mathrm{Tr}_{\pi'_i}) : H^p_{S'_i}(Y'_i, C) \to H^p_{Z'}(X', C)$. And these trace maps satisfy the following properties (the same as in Definition 2.6):

(i) the base changing property;

(ii) the additivity property;

(iii) the normalization property;

(iv) the functorality with respect to sheaves on Et/X.

□

5.3 Lemma. *The étale cohomology functor $F(X, Z) = H^*_Z(X, \mathcal{C})$ is a functor with finite monodromy (4.6).*

Proof. According to Definition 4.6 we have to show that there exist a finite étale morphism $\rho : Y \to X \times X$ together with a section $s : X \to Y$ of ρ over the diagonal and a functor isomorphism $\Phi : F_1 \to F_2$ on $Cp(Y)$, where $F_1 = \rho^* \circ pr_1^* F$ and $F_2 = \rho^* \circ pr_2^* F$.
To produce Y we use the following explicit construction suggested by H. Esnault (we follow [15]):
Let $\tilde{X}$ be a finite Galois covering with Galois group G such that the pull-back of $\mathcal{C}$ to $\tilde{X}$ is a complex of constant sheaves. Consider the étale covering $\tilde{X} \times_k \tilde{X} \to X \times_k X$ with Galois group $G \times G$. Let $\widetilde{X \times_k X} = (\tilde{X} \times_k \tilde{X})_G$ be the unique intermediate covering associated with the diagonal subgroup $G = (g, g) \in G \times G$. The diagonal map $\tilde{X} \to \tilde{X} \times_k \tilde{X}$ induces a map $s : X \to \widetilde{X \times_k X}$ which is a section to the projection $\widetilde{X \times_k X} \to X \times_k X$ over the diagonal $X \cong \Delta_X \subset X \times_k X$. Let Y be the connected component of $X = im(s)$ in $\widetilde{X \times_k X}$. Then $Y \xrightarrow{\rho} X \times_k X$ is a connected Galois covering having a section s over Δ_X.
To check that there is the functor isomorphism Φ we refer to the end of section 4 of [15]. □

We also need the following technical lemma that is a slightly modified version of Proposition 2.1.2, [4]

5.4 Lemma. *Let $\mathcal{U}$ be a semi-local regular scheme of geometric type over a field k, i.e., $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ for some smooth affine variety X and a finite set of points $x = \{x_1, \ldots, x_n\}$ of X. Suppose the étale cohomology functor*

$F(Y,Z) = H^*_Z(Y,\mathcal{C})$ from Theorem 1.1 is effaceable at x. Then, in the exact couple [4, 1.1] defining the coniveau spectral sequence for $(\mathcal{U},\mathcal{C})$, the map $i^{p,q}$ is identically 0 for all $p > 0$. In particular, we have $E_2^{p,q} = H^q(\mathcal{U},\mathcal{C})$ if $p = 0$ and $E_2^{p,q} = 0$ if $p > 0$. And the Cousin complex [4, 1.3] yields the exact complex from Theorem 1.1.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
H^n_Z(X,\mathcal{C}) & \longrightarrow & H^n_{Z\cap\mathcal{U}}(\mathcal{U},\mathcal{C}) & \longrightarrow & H^n_{Z'}(\mathcal{U},\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
H^n_{X^{(m)}}(X,\mathcal{C}) & \longrightarrow & H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) & \longrightarrow & H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})
\end{array}$$

The composition of arrows in the first row is identically 0 for any n. Therefore the compositions $H^n_Z(X,\mathcal{C}) \to H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \to H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})$ are 0. Passing to the limit over Z, this gives that the compositions $H^n_{X^{(m)}}(X,\mathcal{C}) \to H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \to H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})$ are 0. Passing to the limit over open neighborhoods of x, we get that the map $i^{m,n-m} : H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \to H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})$ is itself 0 for any $m \geq 1$. □

Now we are ready to prove the main result of this paper stated in the Introduction.

Proof of Theorem 1.1. Assume the ground field k is infinite. In this case, observe that the étale cohomology functor $F(X,Z) = H^*_Z(X,\mathcal{C})$ satisfies all the hypotheses of Theorem 4.7. Indeed, it is homotopy invariant according to [4, 7.3.(1)]. It satisfies vanishing property by the very definition. It has transfer maps by Lemma 5.2 and it is a functor with finite monodromy by Lemma 5.3. So that by Theorem 4.7 the functor F is effaceable. Now Theorem 1.1 follows immediately from Lemma 5.4.
To finish the proof, i.e., to treat the case of a finite ground field, we apply the standard arguments with transfers for finite field extensions (see the proof of [4, 6.2.5]). □

5.5 Remark. If the complex $\mathcal{C}$ comes from the base field k, i.e., each sheaf in $\mathcal{C}$ can be represented as p^*C' for some sheaf C' on $(\operatorname{Spec} k)_{et}$, where $p : X \to \operatorname{Spec} k$ is the structural morphism. Then the étale cohomology functor F satisfies all the hypotheses of Theorem 4.2. Hence, Theorem 1.1 holds without assuming that F is a functor with finite monodromy.

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Ivan Panin,
Laboratory of Algebra and Number Theory,
Steklov Mathematical Institute at St.Petersburg RAN,
27, Fontanka
191011 St.Petersburg
Russia

Kirill Zainoulline
Laboratory of Algebra and Number Theory,
Steklov Mathematical Institute at St.Petersburg RAN,
27, Fontanka
191011 St.Petersburg
Russia