Group C^*-Algebras as Compact Quantum Metric Spaces

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Abstract. Let ℓ be a length function on a group G, and let M_ℓ denote the operator of pointwise multiplication by ℓ on $\ell^2(G)$. Following Connes, M_ℓ can be used as a "Dirac" operator for $C_r^*(G)$. It defines a Lipschitz seminorm on $C_r^*(G)$, which defines a metric on the state space of $C_r^*(G)$. We investigate whether the topology from this metric coincides with the weak-$*$ topology (our definition of a "compact quantum metric space"). We give an affirmative answer for $G = \mathbb{Z}^d$ when ℓ is a word-length, or the restriction to $\mathbb{Z}^d$ of a norm on $\mathbb{R}^d$. This works for $C_r^*(G)$ twisted by a 2-cocycle, and thus for non-commutative tori. Our approach involves Connes' cosphere algebra, and an interesting compactification of metric spaces which is closely related to geodesic rays.

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0. Introduction

The group C^*-algebras of discrete groups provide a much-studied class of "compact non-commutative spaces" (that is, unital C^*-algebras). In [11] Connes showed that the "Dirac" operator of an unbounded Fredholm module over a unital C^*-algebra provides in a natural way a metric on the state space of the algebra. Unbounded Fredholm modules (i.e. spectral triples) also provide smooth structure, important homological data and much else. In the subsequent years Connes has been strongly advocating this use of Dirac operators as the way to deal with the Riemannian geometry of non-commutative spaces [12], [15], [14], [13]. The class of examples most discussed in [11] consists

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of the group C^*-algebras of discrete groups, with the Dirac operator coming in a simple way from a length function on the group. Connes obtained in [11] strong relationships between the growth of a group and the summability of Fredholm modules over its group C^*-algebra. However he did not explore much the metric on the state space.

In [39], [40] I pointed out that, motivated by what happens for ordinary compact metric spaces, it is natural to desire that the topology from the metric on the state space coincides with the weak-$*$ topology (for which the state space is compact). This property was verified in [39] for certain examples, notably the non-commutative tori, with "metric" structure coming from a different construction. (See [40], [41], [42] for further developments.) But in general I have found this property to be difficult to verify for many natural examples.

The main purpose of this paper is to examine this property for Connes' initial class of examples, the group C^*-algebras with the Dirac operator coming from a length function. To be more specific, let G be a countable (discrete) group, and let $C_c(G)$ denote the convolution $*$-algebra of complex-valued functions of finite support on G. Let π denote the usual $*$-representation of $C_c(G)$ on $\boldsymbol{\ell}^2(G)$ coming from the unitary representation of G by left translation on $\boldsymbol{\ell}^2(G)$. The norm-completion of $\pi(C_c(G))$ is by definition the reduced group C^*-algebra, $C_r^*(G)$, of G. We identify $C_c(G)$ with its image in $C_r^*(G)$, so that it is a dense $*$-subalgebra.

Let a length function ℓ be given on G. We let M_ℓ denote the (usually unbounded) operator on $\boldsymbol{\ell}^2(G)$ of pointwise multiplication by ℓ. Then M_ℓ will serve as our "Dirac" operator. One sees easily [11] that the commutators $[M_\ell, \pi_f]$ are bounded operators for each $f \in C_c(G)$. We can thus define a seminorm, L_ℓ, on $C_c(G)$ by $L_\ell(f) = \|[M_\ell, \pi_f]\|$.

In general, if L is a seminorm on a dense $*$-subalgebra A of a unital C^*-algebra $\bar{A}$ such that $L(1) = 0$, we can define a metric, ρ_L, on the state space $S(\bar{A})$ of $\bar{A}$, much as Connes did, by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A,\ L(a) \leq 1\}.$$

(Without further hypotheses ρ_L may take value $+\infty$.) In [40] we define L to be a *Lip-norm* if the topology on $S(\bar{A})$ from ρ_L coincides with the weak-$*$ topology. We consider a unital C^*-algebra equipped with a Lip-norm to be a compact quantum metric space [40].

The main question dealt with in this paper is whether the seminorms L_ℓ coming as above from length functions on a group are Lip-norms. In the end we only have success in answering this question for the groups $\mathbb{Z}^d$. The situation there is already somewhat complicated because of the large variety of possible length-functions. But we carry out our whole discussion in the slightly more general setting of group C^*-algebras twisted by a 2-cocycle (definitions given later), and so this permits us to treat successfully also the non-commutative tori [38]. The main theorem of this paper is:

Main Theorem 0.1. *Let ℓ be a length function on $\mathbb{Z}^d$ which is either the word-length function for some finite generating subset of $\mathbb{Z}^d$, or the restriction* *to* $\mathbb{Z}^d$ *of some norm on* $\mathbb{R}^d$. *Let* c *be a* 2*-cocycle on* $\mathbb{Z}^d$*, and let* π *be the regular representation of* $C^*(\mathbb{Z}^d, c)$ *on* $\ell^2(\mathbb{Z}^d)$. *Then the seminorm* L_ℓ *defined on* $C_c(\mathbb{Z}^d)$ *by* $L_\ell(f) = \|[M_\ell, \pi_f]\|$ *is a Lip-norm on* $C^*(\mathbb{Z}^d, c)$.

The path which I have found for the proof of this theorem is somewhat long, but it involves some objects which are of considerable independent interest, and which may well be useful in treating more general groups. Specifically, we need to examine Connes' non-commutative cosphere algebra [14] for the examples which we consider. This leads naturally to a certain compactification which one can construct for any locally compact metric space. We call this "the metric compactification". Actually, this compactification had been introduced much earlier by Gromov [24], but it is different from the famous Gromov compactification for a hyperbolic metric space, and it seems not to have received much study. Our approach gives a new way of defining this compactification. We also need to examine the strong relationship between geodesic rays and points in the boundary of this compactification, since this will provide us with enough points of the boundary which have finite orbits. For word-length functions on $\mathbb{Z}^d$ this is already fairly complicated.

The contents of the sections of this paper are as follows. In Section 1 we make more precise our notation, and we make some elementary observations showing that on any separable unital C^*-algebra there is an abundance of Lip-norms, and that certain constructions in the literature concerning groups of "rapid decay" yield natural Lip-norms on $C_r^*(G)$. In Section 2 we begin our investigation of the Dirac operators for $C_r^*(G)$ coming from length functions. In Section 3 we examine Connes' cosphere algebra for our situation. We show in particular that if the action of the group on the boundary of its metric compactification is amenable, then the cosphere algebra has an especially simple description. Then in Section 4 we study the metric compactification in general, with attention to the geodesic rays.

In Section 5 we begin our study of specific groups by considering the group $\mathbb{Z}$. This is already interesting. (Consider a generating set such as $\{\pm 3, \pm 8\}$.) The phenomena seen there for $\mathbb{Z}$ indicate some of the complications which we will encounter in trying to deal with $\mathbb{Z}^d$. In Section 6 we study the metric compactification of $\mathbb{R}^d$ for any given norm, and then in Section 7 we apply this to prove the part of our Main Theorem for length functions on $\mathbb{Z}^d$ which are the restrictions of norms on $\mathbb{R}^d$. In Section 8 we study the metric compactification of $\mathbb{Z}^d$ for word-length functions, and in Section 9 we apply this to prove the remaining part of our Main Theorem. We conclude in Section 10 with a brief examination of the free (non-Abelian) group on two generators, to see both how far our approach works, and where we become blocked from proving for it the corresponding version of our Main Theorem.

Last-minute note: I and colleagues believe we have a proof that the Main Theorem is also true for word-hyperbolic groups with word-length functions, using techniques which are entirely different from those used here, and which do not seem to apply to the case of $\mathbb{Z}^d$ treated here.

A substantial part of the research reported here was carried out while I visited the Institut de Mathématique de Luminy, Marseille, for three months. I would like to thank Gennady Kasparov, Etienne Blanchard, Antony Wasserman, and Patrick Delorme very much for their warm hospitality and their mathematical stimulation during my very enjoyable visit.

1. AN ABUNDANCE OF LIP-NORMS

In this section we establish some of our notation, and show that on any separable unital C^*-algebra there is an abundance of Lip-norms. In the absence of further structure these Lip-norms appear somewhat artificial. But we then show that some known constructions for group C^*-algebras yield somewhat related but more natural Lip-norms.

Our discussion in the next few paragraphs works in the greater generality of order-unit spaces which was used in [40]. But we will not use that generality in later sections, and so the reader can have in mind just the case of dense unital *-subalgebras of unital C^*-algebras, with the identity element being the order unit. We recall that a (possibly discontinuous) seminorm L on an order-unit space is said to be lower semicontinuous if $\{a \in A : L(a) \leq r\}$ is norm-closed for any $r > 0$.

PROPOSITION 1.1. *Let A be an order-unit space which is separable. For any countable subset E of A there are many lower semicontinuous Lip-norms on A which are defined and finite on E.*

Proof. The proof is a minor variation on the fact that the weak-$*$ topology on the unit ball of the dual of a separable Banach space is metrizable (theorem V.5.1 of [16]). We scale each non-zero element of E so that it is in the unit ball of A (and $\neq 0$), and we incorporate E into a sequence, $\{b_n\}$, of elements of A which is dense in the unit ball of A. Let $\{\omega_n\}$ be any sequence in $\mathbb{R}$ such that $\omega_n > 0$ for each n and $\Sigma\omega_n < \infty$. Define a norm, M, on the dual space A' of A by

$$M(\lambda) = \Sigma\omega_n|\lambda(b_n)|.$$

The metric from this norm, when restricted to the unit ball of A', gives the weak-$*$ topology, because it is easily checked that if a net in the unit ball of A' converges for the weak-$*$ topology then it converges for the metric from M, and then we can apply the fact that the unit ball is weak-$*$ compact.

We let $S(A)$ denote the state space of A. Since $S(A)$ is a subset of the unit ball of A', the restriction to $S(A)$ of the metric from the norm M gives $S(A)$ the weak-$*$ topology. Let L_M denote the corresponding Lipschitz seminorm on $C(S(A))$ from this metric, allowing value $+\infty$. View each element of A as a function on $S(A)$ in the usual way. Then $L_M(b_n) \leq \omega_n^{-1} < \infty$ for each n, because if $\mu, \nu \in S(A)$ then

$$|b_n(\mu) - b_n(\nu)| = |(\mu - \nu)(b_n)| \leq \omega_n^{-1}M(\mu - \nu) = \omega_n^{-1}\rho_M(\mu, \nu).$$

Let B denote the linear span of $\{b_n\}$ together with the order unit. Then B is a dense subspace of A containing the order-unit, and L_M restricted to B is a seminorm which can be verified to be lower semicontinuous. The inclusion of A into $C(S(A))$ is isometric (on self-adjoint elements if A is a C^*-algebra) and since L_M comes from an ordinary metric, it follows that L_M on A is a Lip-norm. (For example, use theorem 1.9 of [39].) $\square$

The considerations above are close to those of theorem 9.8 of [40]. Let me take advantage of this to mention here that Hanfeng Li showed me by clever counterexample that theorem 9.8 of [40] is not correct as presented, because A may not be big enough. However, if A is taken to be norm-complete, then there is no difficulty. Theorem 9.11 needs to be adjusted accordingly. But this change does not affect later sections of [40] nor the subsequent papers [41], [42].

We now turn to (twisted) group C^*-algebras, and we use a different approach, which takes advantage of the fact that the group elements provide a natural "basis" for the group C^*-algebras. Thus let G be a countable discrete group, and let c be a 2-cocycle [47] on G with values in the circle group $\mathbb{T}$. We assume that c is normalized so that $c(x,y) = 1$ if $x = e$ or $y = e$. We let $C^*(G,c)$ denote the full c-twisted group C^*-algebra of G, and we let $C^*_r(G,c)$ denote the reduced c-twisted group C^*-algebra [47], [35] coming from the left regular representation, π, on $\boldsymbol{\ell}^2(G)$. Both C^*-algebras are completions of $C_c(G)$, the space of finitely supported $\mathbb{C}$-valued functions on G, with convolution twisted by c. Our conventions, following [47], are that

$$\begin{aligned} (f*g)(x) &= \Sigma f(y)g(y^{-1}x)c(y,y^{-1}x), \\ f^*(x) &= \bar{f}(x^{-1})\bar{c}(x,x^{-1}). \end{aligned}$$

The left regular representation is given by the same formula as the above twisted convolution, but with g viewed as an element of $\boldsymbol{\ell}^2(G)$. Then $C^*_r(G,c)$ is the completion of $C_c(G)$ for the operator norm coming from the left regular representation. We will often set $A = C^*_r(G,c)$. We note that when π is restricted to G we have

$$(\pi_y\xi)(x) = \xi(y^{-1}x)c(y,y^{-1}x)$$

for $\xi \in \boldsymbol{\ell}^2(G)$ and $x, y \in G$. In particular $\pi_y\pi_z = c(y,z)\pi_{yz}$.

There is a variety of norms on $C_c(G)$ which have been found to be useful in addition to the C^*-norms. These other norms are not necessarily algebra norms. To begin with, there is the $\boldsymbol{\ell}^1$-norm, as well as the $\boldsymbol{\ell}^p$-norms for $1 < p \le \infty$. But let ℓ be a length function on G, so that $\ell(xy) \le \ell(x) + \ell(y)$, $\ell(x^{-1}) = \ell(x)$, $\ell(x) \ge 0$, and $\ell(x) = 0$ exactly if $x = e$, the identity element of G. Then in connection with groups of "rapid decay" (such as word-hyperbolic groups) one defines norms on $C_c(G)$ of the following form [30], [29], [27], [28]:

$$\|f\|_{p,k} = (\Sigma(|f(x)|(1+\ell(x))^k)^p)^{1/p}.$$

These norms clearly have the properties that

1) $\|f\|_{p,k} \le \||f|\|_{p,k}$ (actually $=$),
2) if $|f| \le |g|$ then $\||f|\|_{p,k} \le \||g|\|_{p,k}$.

Their interest lies in the fact that for a rapid-decay group and an appropriate choice of p and k depending on the group, one has (see the first line of the proof of theorem 1.3 of [27], combined, in the case of nontrivial cocycle, with proposition 3.10b of [28]):

3) There is a constant, K, such that $\|f\|_{C_r^*} \leq K\|f\|_{p,k}$.

Notice also that if the cocycle c is trivial and if G is amenable [35] then the C^*-norm itself satisfies the above three properties, because from the trivial representation we see that for $f \in C_c(G)$ we have

$$\|f\|_{C^*(G)} \leq \|f\|_1 = \||f|\|_{C^*(G)},$$

while if $|f| \leq |g|$ then

$$\||f|\|_{C^*(G)} = \|f\|_1 \leq \|g\|_1 = \||g|\|_{C^*(G)}.$$

Finally, for any group and any cocycle we always have at least the $\boldsymbol{\ell}^1$-norm which satisfies the above three properties.

With these examples in mind, we make

Definition 1.2. *Let $\|\cdot\|_A$ denote the C^*-norm on $A = C_r^*(G,c)$. We will say that a norm, $\|\cdot\|$, on $C_c(G)$ is* order-compatible *with $\|\cdot\|_A$ if for all $f, g \in C_c(G)$ we have:*

1) $\|f\| \leq \||f|\|$.
2) *If $|f| \leq |g|$ then $\||f|\| \leq \||g|\|$.*
3) *There is a constant, K, such that $\|f\|_A \leq K\|f\|$.*

We remark that these conditions are a bit weaker than those required for a "good norm" in [32].

Suppose now that ω is a real-valued function on G such that $\omega(e) = 0$ and $\omega(x) > 0$ for $x \neq e$. Fix an order-compatible norm $\|\cdot\|$ on $C_c(G)$, and set

$$L(f) = \|\omega|f|\|.$$

It is clear that L is a seminorm which is 0 only on the span of the identity element of the convolution algebra $C_c(G,c)$. (Thus L is a Lipschitz seminorm as defined in [40].) In the way discussed in the introduction, L defines a metric, ρ_L, on $S(C_r^*(G,c))$ by

$$\rho_L(\mu,\nu) = \sup\{|\mu(f) - \nu(f)| : L(f) \leq 1\},$$

which may take value $+\infty$. Denote $C_r^*(G,c)$ by A, and its C*-norm by $\|\cdot\|_A$, as above.

Lemma 1.3. *Suppose that there is a constant $s > 0$ such that $\omega(x) \geq s$ for all $x \neq e$. Then ρ_L gives $S(A)$ finite radius. (In particular, ρ_L does not take the value $+\infty$.)*

Proof. Let $f \in C_c(G)$, and assume that $f(e) = 0$. Let K be the constant in the definition of "order-compatible". Then

$$\|f\|_A \leq K\|f\| \leq K\||f|\| \leq Ks^{-1}\|\omega|f|\| = Ks^{-1}L(f).$$

The desired conclusion then follows from proposition 2.2 of [40]. $\square$

LEMMA 1.4. *Suppose that $\omega(x) = 0$ only if $x = e$ and that the function ω is "proper", in the sense that for any n the set $\{x \in G : \omega(x) \leq n\}$ is finite (so, in particular, there exists a constant s as in the above lemma). Then the topology from the metric ρ_L on $S(A)$ coincides with the weak-$*$ topology. Thus L is a Lip-norm.*

Proof. We apply theorem 1.9 of [39]. As in that theorem, we set

$$\mathcal{B}_1 = \{f \in C_c(G) : \|f\|_A \leq 1 \text{ and } L(f) \leq 1\}.$$

The theorem tells us that it suffices to show that $\mathcal{B}_1$ is totally bounded for $\|\cdot\|_A$. So let $\varepsilon > 0$ be given. Adjust K if necessary so that $K \geq 1$, and set

$$E = \{x \in G : \omega(x) \leq 3K/\varepsilon\}.$$

Then E is a finite set because ω is proper. Set $A^E = \{f \in C_c(G) : f(x) = 0 \text{ for } x \notin E\}$, so that A^E is a finite-dimensional subspace of $C_c(G)$. In particular, $A^E \cap \mathcal{B}_1$ is totally bounded.

Let $f \in \mathcal{B}_1$. Then $f = g + h$ where $g \in A^E$ and $h(x) = 0$ for $x \in E$. Now $|h| \leq |f|$, and $\omega(x) \geq 3K/\varepsilon$ on the support of h, and so

$$\begin{aligned} \|h\|_A &\leq K\|h\| \leq K\||h|\| \leq K(\varepsilon/3K)\|\omega|h|\| \\ &\leq (\varepsilon/3)\|\omega|f|\| = (\varepsilon/3)L(f) \leq \varepsilon/3. \end{aligned}$$

Thus $\|f - g\|_A = \|h\|_A \leq \varepsilon/3$. In particular, $\|g\|_A \leq 1 + (\varepsilon/3)$. Note also that $L(g) = \|\omega|g|\| \leq \|\omega|f|\| = L(f) \leq 1$. Thus upon scaling g by $(1 + \varepsilon/3)^{-1}$ if necessary to obtain an element of $\mathcal{B}_1$, we see that f is within distance $2\varepsilon/3$ of $\mathcal{B}_1 \cap A^E$. Thus a finite subset of $\mathcal{B}_1 \cap A^E$ which is $\varepsilon/3$ dense in $\mathcal{B}_1 \cap A^E$ will be ε-dense in $\mathcal{B}_1$. ☐

LEMMA 1.5. *Even without ω being proper, or satisfying the condition of Lemma 1.3, the seminorm L is lower semicontinuous (with respect to $\|\cdot\|_A$).*

Proof. Let $\{f_n\}$ be a sequence in $C_c(G)$ which converges to $g \in C_c(G)$ for $\|\cdot\|_A$, and suppose that there is an $r \in \mathbb{R}$ such that $L(f_n) \leq r$ for all n. Now $\pi_f\delta_0 = f$ where on the right f is viewed as an element of ℓ^2 and δ_0 is the "delta-function" at 0. Consequently $\|f\|_A \geq \|f\|_2 \geq \|f\|_\infty$. Thus f_n converges uniformly on G to g. Let S denote the support of g, and let χ_S be its characteristic function. Then the sequence $\omega\chi_S|f_n|$ converges uniformly to $\omega|g|$. But all norms on a finite-dimensional vector space are equivalent, and so $\omega\chi_S|f_n|$ converges to $\omega|g|$ for $\|\cdot\|$. This says that $L(\chi_S f_n)$ converges to $L(g)$. But $L(\chi_S f_n) = \|\omega\chi_S f_n\| \leq L(f) \leq r$. Thus $L(g) \leq r$. ☐

We combine the above lemmas to obtain:

PROPOSITION 1.6. *Let ω be a proper non-negative function on G such that $\omega(x) = 0$ exactly if $x = e$. Let $\|\cdot\|$ be an order-compatible norm on $C_c(G)$, and set*

$$L(f) = \|\omega|f|\|$$

for $f \in C_c(G)$. Then L is a lower semicontinuous Lip-norm on $C_r^*(G)$.

We remark that when ω is a length function on G and when $\|\cdot\| = \|\cdot\|_1$, it is well-known and easily seen that L satisfies the Leibniz rule with respect to $\|\cdot\|_1$, that is

$$L(f * g) \leq L(f)\|g\|_1 + \|f\|_1 L(g).$$

But there seems to be no reason why many of the above Lip-norms should satisfy the Leibniz rule with respect to $\|\cdot\|_A$. And it is not clear to me what significance the Leibniz rule has for the metric properties which we are examining.

2. Dirac operators from length functions

In this section we make various preliminary observations about the seminorms L which come from using length functions on a group as "Dirac" operators, as described in the introduction. We also reformulate our main question as concrete questions concerning $C_r^*(G)$ itself.

We use the notation of the previous section, and we let M_ℓ denote the (usually unbounded) operator on $\boldsymbol{\ell}^2(G)$ of pointwise multiplication by the length function ℓ. We recall from [11] why the commutators $[M_\ell, \pi_f]$ are bounded for $f \in C_c(G)$. Let $y \in G$ and $\xi \in \boldsymbol{\ell}^2(G)$. Then we quickly calculate that

$$([M_\ell, \pi_y]\xi)(x) = (\ell(x) - \ell(y^{-1}x))\xi(y^{-1}x)c(y, y^{-1}x).$$

From the triangle inequality for ℓ we know that $|\ell(x) - \ell(y^{-1}x)| \leq \ell(y)$, and so $\|[M_\ell, \pi_y]\| \leq \ell(y)$. In fact, this observation indicates the basic property of ℓ which we need for the elementary part of our discussion, namely that, although ℓ is usually unbounded, it differs from any of its left translates by a bounded function.

This suggests that we work in the more general context of functions having just this latter property, as this may clarify some aspects. Additional motivation for doing this comes from the importance which Connes has demonstrated for examining the effect of automorphisms of the C^*-algebra as gauge transformations, and the resulting effect on the metric. In Connes' approach the inner automorphisms play a distinguished role, giving "internal fluctuations" of the metric [9], [10] (called "internal perturbations" in [15]). However, in our setting we usually do not have available the "first order" condition which is crucial in Connes' setting. We discuss this briefly at the end of this section.

Anyway, in our setting the algebra $C_r^*(G, c)$ has some special inner automorphisms, namely those coming from the elements of G. The automorphism corresponding to $z \in G$ is implemented on $\boldsymbol{\ell}^2(G)$ by conjugating by π_z. When this automorphism is composed with the representation, the effect is to change $D = M_\ell$ to $M_{\alpha_z(\ell)}$, where $\alpha_z(\ell)$ denotes the left translate of ℓ by z. But $\alpha_z(\ell)$ need not again be a length function, although it is translation bounded. (In order to try to clarify contexts, we will from now on systematically use α to denote ordinary left translation of functions, especially when those functions are not to be viewed as being in $\boldsymbol{\ell}^2(G)$. Our convention is that $(\alpha_z\ell)(x) = \ell(z^{-1}x)$.)

We will make frequent use of the easily-verified commutation relation that

$$\pi_y M_h = M_{\alpha_y(h)} \pi_y$$

for any function h on G and any $y \in G$, as long as the domains of definitions of the product operators are respected. This commutation relation is what we used above to obtain the stated fact about the effect of inner automorphisms.

In what follows we will only use real-valued functions to define our Dirac operators, so that the latter are self-adjoint. But much of what follows generalizes easily to complex-valued functions, or to functions with values in C^*-algebras such as Clifford algebras. These generalizations deserve exploration.

To formalize our discussion above we make:

Definition 2.1. *We will say that a (possibly unbounded) real-valued function, ω, on G is (left)* translation-bounded *if $\omega - \alpha_y \omega$ is a bounded function for every $y \in G$. For $y \in G$ we set $\varphi_y = \omega - \alpha_y(\omega)$. So the context must make clear what ω is used to define φ. For each $y \in G$ we set $\ell^\omega(y) = \|\varphi_y\|_\infty$.*

Thus every length-function on G is translation-bounded. Any group homomorphism from G into $\mathbb{R}$ is translation bounded. (E.g., the homomorphism $\omega(n) = n$ from $\mathbb{Z}$ to $\mathbb{R}$ which is basically the Fourier transform of the usual Dirac operator on $\mathbb{T}$.) Linear combinations of translation-bounded functions are translation bounded. In particular, the sum of a translation-bounded function with any bounded function is translation bounded. (As a more general context one could consider any faithful unitary representation $(\pi, \mathcal{H})$ of G together with an unbounded self-adjoint operator D on $\mathcal{H}$ such that $D - \pi_z D \pi_z^*$ is densely defined and bounded for each $z \in G$, and D satisfies suitable non-triviality conditions. Our later discussion will indicate why one may also want to require that the $(\pi_z D \pi_z^*)$'s all commute with each other.)

It is simple to check that the φ_y's satisfy the 1-cocycle identity

$$\varphi_{yz} = \varphi_y + \alpha_y(\varphi_z). \tag{2.2}$$

We will make use of this relation a number of times. This type of relation occurs in various places in the literature in connection with dynamical systems.

Simple calculations show that ℓ^ω satisfies the axioms for a length function except that we may have $\ell^\omega(x) = 0$ for some $x \neq e$. Notice also that if ω is already a length function, then $\ell^\omega = \omega$. We also remark that in general we can always add a constant function to ω without changing the corresponding φ_y's, ℓ^ω, or the commutators $[M_\ell, \pi_y]$. In particular, we can always adjust ω in this way so that $\omega(e) = 0$ if desired.

We now fix a translation-bounded function, ω, on G, and we consider the operator, M_ω, of pointwise multiplication on $\ell^2(G)$. It is self-adjoint. We use it as a "Dirac operator". The calculation done earlier becomes

$$[M_\omega, \pi_y] = M_{\varphi_y} \pi_y.$$

From this we see that for each $y \in G$ we have

$$\|[M_\omega, \pi_y]\| = \ell^\omega(y).$$

For any $f \in C_c(G)$ we have

$$[M_\omega, \pi_f] = \Sigma f(y) M_{\varphi_y} \pi_y,$$

and consequently we have

$$\|[M_\omega, \pi_f]\| \leq \|\ell^\omega f\|_1,$$

where $\ell^\omega f$ denotes the pointwise product. We set

$$L^\omega(f) = \|[M_\omega, \pi_f]\|.$$

Then L^ω is a seminorm on $C_c(G) \subseteq C_r^*(G, c)$, and L^ω is lower semicontinuous by proposition 3.7 of [40]. A calculation above tells us that $L^\omega(\delta_x) = \ell^\omega(x)$ for all $x \in G$. In particular, $L^\omega(\delta_e) = 0$, with δ_e the identity element of the convolution algebra $C_c(G)$.

If we view δ_z as the usual basis element at z for $\ell^2(G)$, then for any $f \in C_c(G)$ we have

$$[M_\omega, \pi_f]\delta_z = \Sigma f(y) M_{\varphi_y} c(y, z) \delta_{yz}$$

for each z. From this we easily obtain:

Proposition 2.3. *Let $f \in C_c(G)$. Then $L^\omega(f) = 0$ exactly if $\varphi_y = 0$ for each y in the support of f, that is, exactly if $\ell^\omega f = 0$. Thus if $\ell^\omega(x) > 0$ for all $x \neq e$, then L^ω is a Lipschitz seminorm in the sense that its null space is spanned by δ_e.*

We would like to know when L^ω is a Lip-norm. Of course, L^ω defines, as earlier, a metric on the state space $S(C_r^*(G, c))$, which may take value $+\infty$. We denote this metric by ρ_ω. As a first step, we would like to know whether ρ_ω gives $S(C_r^*(G, c))$ finite radius. We recall from proposition 2.2 of [40] that this will be the case if there is an $r \in \mathbb{R}$ such that $\|f\|^\sim \leq rL(f)$ for all $f \in C_c(G)$, where $\|f\|^\sim = \inf\{\|f - \alpha\delta_e\| : \alpha \in \mathbb{C}\}$. Officially speaking we should work with self-adjoint f's, but by the comments before definition 2.1 of [41] we do not need to make this restriction because clearly $L^\omega(f^*) = L^\omega(f)$ for each f. However we find it convenient to use the following alternative criterion for finite radius, which is natural in our situation because we have a canonical tracial state:

Proposition 2.4. *Let L be a Lipschitz seminorm on an order-unit space A, and let μ be a state of A. If the metric ρ_L from L gives $S(A)$ finite radius r, then $\|a\| \leq 2rL(a)$ for all $a \in A$ such that $\mu(a) = 0$. Conversely, if there is a constant k such that*

$$\|a\| \leq kL(a)$$

for all $a \in A$ such that $\mu(a) = 0$, then ρ_L gives $S(A)$ radius no greater than k.

Proof. Suppose the latter condition holds. For any given $a \in A$ set $b = a - \mu(a)e$. (Here e is the order-unit.) Then $\mu(b) = 0$, and so $\|a - \mu(a)e\| \leq kL(a)$. It follows that $\|a\|^\sim \leq kL(a)$, so that the ρ_L-radius of $S(A)$ is no greater than k.

Suppose conversely that $\|a\|^\sim \leq rL(a)$ for all a. Let $a \in A$ with $\mu(a) = 0$. There is a $t \in \mathbb{R}$ such that $\|a - te\| \leq rL(a)$. Then

$$|t| = |\mu(a) - t| = |\mu(a - te)| \leq \|a - te\| \leq rL(a).$$

Thus

$$\|a\| \leq \|a - te\| + \|te\| \leq 2rL(a).$$

So for $k = 2r$ we have $\|a\| \leq kL(a)$ if $\mu(a) = 0$. □

We see that the constant k is not precisely related to the radius. But for our twisted group algebras there is a very natural state to use, namely the tracial state τ defined by $\tau(f) = f(e)$, which is the vector state for $\delta_e \in \boldsymbol{\ell}^2(G)$.

Suppose now that ρ_ω gives $S(C_r^*(G, c))$ finite radius, so that as above, if $\tau(f) = 0$ then $\|\pi(f)\| \leq 2rL(f)$. Let $x \in G$ with $x \neq e$. Then $\tau(\delta_x) = 0$, and so

$$1 = \|\pi(\delta_x)\| \leq 2rL^\omega(\delta_x) = 2r\ell^\omega(x).$$

We thus obtain:

Proposition 2.5. *If ρ_ω gives $S(C_r^*(G, c))$ finite radius r, then $\ell^\omega(x) \geq (2r)^{-1}$ for all $x \neq e$.*

Thus, for example, if θ is an irrational number, then neither the (unbounded) length function ℓ defined on $\mathbb{Z}^2$ by $\ell(m, n) = |m + n\theta|$, nor the homomorphism $\omega(m, n) = m + n\theta$, will give metrics for which $S(C^*(\mathbb{Z}^2))$ has finite radius.

But the condition of Proposition 2.5 is not at all sufficient for finite radius. For example, for any G we can define a length function ℓ by $\ell(x) = 1$ if $x \neq e$. Then it is easily checked that if $f = f^*$ then

$$L^\ell(f) = \|f - \tau(f)\delta_e\|_2.$$

If L^ℓ gives $S(C^*(G))$ finite radius, so that there is a constant k such that $\|\pi_f\| \leq kL^\ell(f)$ if $f(e) = 0$, then it follows that $\|\pi_f\| \leq 2k\|f\|_2$ when $f(e) = 0$. Since for any f we have $|f(e)| \leq \|f\|_2$, it follows that $\|\pi_f\| \leq (2k+1)\|f\|_2$, so that for any $g \in C_c(G)$ we have

$$\|f * g\|_2 \leq (2k+1)\|f\|_2\|g\|_2.$$

This quickly says that the norm on $\boldsymbol{\ell}^2(G)$ can be normalized so that $\boldsymbol{\ell}^2(G)$ forms an H^*-algebra, as defined in section 27 of [34]. But our algebra is unital, and the theory of H^*-algebra in [34] shows that G must then have finite-dimensional square-integrable unitary representations. But Weil pointed out on page 70 of [46] that this means that G is compact (so finite), because if $x \to U_x$ is the unitary matrix representation for a finite-dimensional square integrable representation, then the matrix coefficients of

$$x \mapsto I = U_x U_x^*$$

are integrable.

But beyond these elementary comments it is not clear to me what happens even for word-length functions. Thus we have the basic:

Question 2.6. *For which finitely generated groups G with cocycle c does the word-length function ℓ corresponding to a finite generating subset give a metric ρ_ℓ which gives $S(C_r^*(G,c))$ finite diameter? That is, when is there a constant, k, such that if $f \in C_c(G)$ and $f(e) = 0$ then*

$$\|\pi(f)\| \leq k\|[M_\ell, \pi(f)]\|?$$

(Is the answer independent of the choice of the generating set?)

I do not know the answer to this question when the cocycle c is trivial and, for example, G is the discrete Heisenberg group, or the free group on two generators. In later sections we will obtain some positive answers for $G = \mathbb{Z}^d$, but even that case does not seem easy.

Even less do I know answers to the basic:

Question 2.7. *For which finitely generated groups G with 2-cocycle c does the word-length function ℓ corresponding to a finite generating subset give a metric ρ_ℓ which gives $S(C_r^*(G,c))$ the weak-$*$ topology. That is* [39], *given that ρ_ℓ does give $S(C_r^*(G,c))$ finite diameter, when is*

$$\mathcal{B}_1 = \{f \in C_c(G) : \|\pi_f\| \leq 1 \text{ and } L_\ell(f) \leq 1\}$$

a totally-bounded subset of $C_r^*(G)$?

But we now make some elementary observations about this second question.

Proposition 2.8. *Let L be a Lip-norm on an order-unit space A. If L is continuous for the norm on A, then A is finite-dimensional.*

Proof. Much as just above we set

$$\mathcal{B}_1 = \{a \in A : \|\alpha\| \leq 1 \text{ and } L(a) \leq 1\}.$$

Since L is a Lip-norm, $\mathcal{B}_1$ is totally bounded by theorem 1.9 of [39]. But if L is also norm-continuous, then there is a constant $k \geq 1$ such that $L(a) \leq k\|a\|$ for all $a \in A$. Consequently $\{a : \|a\| \leq k^{-1}\} \subseteq \mathcal{B}_1$. It follows that the unit ball for the norm is totally bounded, and so the unit ball in the completion of A is compact. But it is well-known that the unit ball in a Banach space is not norm-compact unless the Banach space is finite-dimensional. ☐

Corollary 2.9. *Let A be an order-unit space which is represented faithfully as operators on a Hilbert space $\mathcal{H}$. Let D be a self-adjoint operator on $\mathcal{H}$, and set $L(a) = \|[D,a]\|$. Assume that L is (finite and) a Lip-norm on A. If D is a bounded operator, then A is finite-dimensional.*

From this we see that in our setting of $D = M_\omega$ for $C_r^*(G,c)$, if we want L^ω to be a Lip-norm, then we must use unbounded ω's unless G is finite. But it is not clear to me whether ω must always be a proper function, that is, whether $\{x : |\omega(x)| \leq k\}$ must be finite for every k. However, the referee has pointed out to me that if ω is actually a length function, then ω must be proper if L^ω is to be a Lip-norm. For if it is not proper, then there is a constant, r, with

$0 < r \leq 1$, such that $S = \{x : \omega(x) \leq r^{-1}\}$ is infinite. But if ω is a length function then $L(\delta_x) = \omega(x)$. Thus $\{r\delta_x : x \in S\}$ is a norm-discrete subset of

$$\mathcal{B}_1 = \{f \in C_c(G) : \|f\|_A \leq 1 \text{ and } L(f) \leq 1\},$$

so that $\mathcal{B}_1$ can not be totally bounded. (See the first three sentences of the proof of Lemma 1.4.)

Finally, we will examine briefly three of Connes' axioms for a non-commutative Riemannian geometry [15]. We begin first with the axiom of "reality" (axiom $7'$ on page 163 of [15] and condition 4 on page 483 of [23]). For any C^*-algebra A with trace τ there is a natural and well-known "charge-conjugation" operator, J, on the GNS Hilbert space for τ, determined by $Ja = a^*$. We are in that setting, and so our J is given by

$$(J\xi)(x) = \bar{\xi}(x^{-1})$$

for $\xi \in \ell^2(G)$. For any $f \in C_c(G)$ one checks that $J\pi_f J$ is the operator of *right*-convolution by f^*, where $f^*(x) = \bar{f}(x^{-1})$. In particular, $J\pi_f J$ will commute with any π_g for $g \in C_c(G)$. This means exactly that the axiom of reality is true if one considers our geometry to have dimension 0.

With the axiom of reality in place, Connes requires that D be a "first-order operator" (axiom $2'$ of [15], or condition 5 on page 484 of [23], where the terminology "first order" is used). This axiom requires that $[D, a]$ commutes with JbJ for all $a, b \in A$. For our situation, let ρ_z denote right c-twisted translation on $\ell^2(G)$ by $z \in G$, so that $J\pi_z^* J = \rho_z$. Then in terms of the notation we have established, the first-order condition requires that ρ_z commutes with M_{φ_y} for each z and y. This implies that for each $x \in G$ we have

$$\omega(x) - \omega(y^{-1}x) = \omega(xz) - \omega(y^{-1}xz).$$

If we choose $z = x^{-1}$ and rearrange, we obtain

$$\omega(x) + \omega(y^{-1}) = \omega(y^{-1}x) + \omega(e).$$

This says that if we subtract the constant function $\omega(e)$, then ω is a group homomorphism from G into $\mathbb{R}$. Thus the first-order condition is rarely satisfied in our context. In fact, if we want ω to give $S(C_r^*(G))$ finite radius then it follows from Proposition 2.5 that $G \cong \mathbb{Z}$ or is finite.

Lastly, we consider the axiom of smoothness (axiom 3 on page 159 of [15], or condition 2 on page 482 of [23], where it is called "regularity" rather than "smoothness"). This requires that a and $[D, a]$ are in the domains of all powers of the derivation $T \mapsto [|D|, T]$. In our context $|D| = M_{|\omega|}$. But

$$||\omega(x)| - |\omega(z^{-1}x)|| \leq |\omega(x) - \omega(z^{-1}x)|,$$

so that $|\omega|$ is translation-bounded when ω is. From this it is easily seen that the axiom of smoothness is always satisfied in our setting.

3. The cosphere algebra

We now begin to establish some constructions which will permit us to obtain positive answers to Questions 2.6 and 2.7 for the groups $\mathbb{Z}^d$, and which may eventually be helpful in dealing with other groups.

Connes has shown (section 6 of [13], [22]) how to construct for each spectral triple $(A, \mathcal{H}, D)$ a certain C^*-algebra, denoted S^*A. He shows that if $A = C^\infty(\mathcal{M})$ where $\mathcal{M}$ is a compact Riemannian spin manifold, and if $(\mathcal{H}, D)$ is the corresponding Dirac operator, then S^*A is canonically isomorphic to the algebra of continuous functions on the unit cosphere bundle of $\mathcal{M}$. Thus in the general case it seems reasonable to call S^*A the cosphere algebra of $(A, \mathcal{H}, D)$. (In [22] S^*A is called the "unitary cotangent bundle".) In this section we will explore what this cosphere algebra is for our (almost) spectral triples of form $(C_c(G), \ell^2(G), M_\omega)$. (I thank Pierre Julg for helpful comments about this at an early stage of this project.)

We now review the general construction. But for our purposes we do not need the usual further hypothesis of finite summability for D. Thus we just require that we have $(A, \mathcal{H}, D)$ such that $[D, a]$ is bounded for all $a \in A$. But, following Connes, we also make the smoothness requirement that $[|D|, a]$ be bounded for all $a \in A$. We saw in the previous section that this latter condition is always satisfied in our setting where $D = M_\omega$.

Connes' construction of the algebra S^*A is as follows. (See also the introduction of [22].) Form the strongly continuous one-parameter unitary group $U_t = \exp(it|D|)$. Let $\mathcal{C}_D$ be the C^*-algebra of operators on $\mathcal{H}$ generated by the algebra $\mathcal{K}$ of compact operators on $\mathcal{H}$ together with all of the algebras U_tAU_{-t} for $t \in \mathbb{R}$. (Note that usually $U_tAU_{-t} \not\subseteq A$.) Clearly the action of conjugating by U_t carries $\mathcal{C}_D$ into itself. We denote this action of $\mathbb{R}$ on $\mathcal{C}_D$ by η. Because of the requirement that $[|D|, a]$ be bounded, the action η is strongly continuous on $\mathcal{C}_D$. (See the first line of the proof of corollary 10.16 of [23].) Since $\mathcal{K}$ is an ideal (η-invariant) in $\mathcal{C}_D$, we can form $\mathcal{C}_D/\mathcal{K}$. Then by definition $S^*A = \mathcal{C}_D/\mathcal{K}$. The action η drops to an action of $\mathbb{R}$ on S^*A, which Connes calls the "geodesic flow".

We now work out what the above says for our case in which we have $(C_r^*(G, c), \ell^2(G), M_\omega)$. We will write $\mathcal{C}_\omega$ instead of $\mathcal{C}_D$. Since only $|\omega|$ is pertinent, we assume for a while that $\omega \geq 0$. Set $u_t(x) = \exp(it\omega(x))$ for $t \in \mathbb{R}$, so that the U_t of the above construction becomes M_{u_t}. Then for each $y \in G$ our algebra $\mathcal{C}_\omega$, defined as above, must contain

$$U_t\pi_yU_t^* = M_{u_t}M_{\alpha_y(u_t^*)}\pi_y = M_{u_t\alpha_y(u_t^*)}\pi_y.$$

But $\mathcal{C}_\omega$ must also contain $(\pi_y)^{-1}$, and thus it contains each $u_t\alpha_y(u_t^*)$, where for notational simplicity we omit M. But

$$(u_t\alpha_y(u_t^*))(x) = \exp(it(\omega(x) - \omega(y^{-1}x))) = \exp(it\varphi_y(x)).$$

Since φ_y is bounded, the derivative of $U_t\alpha_y(U_t^*)$ at $t = 0$ will be the norm-limit of the difference quotients. Thus we see that also $\varphi_y \in \mathcal{C}_\omega$ for each $y \in G$. But $\mathcal{C}_\omega \supseteq \mathcal{K}$, and so $\mathcal{C}_\omega \supseteq C_\infty(G)$, the space of continuous functions vanishing at infinity, where the elements of $C_\infty(G)$ are here viewed as multiplication operators. Note also that $\mathcal{C}_\omega$ contains the identity element.

All of this suggests that we consider, inside the algebra $C_b(G)$ of bounded functions on G, the unital norm-closed subalgebra generated by $C_\infty(G)$ together with $\{\varphi_y : y \in G\}$. We denote this subalgebra by E_ω. Let $\bar{G}^\omega$ denote the maximal ideal space of E_ω, with its compact topology, so that $E_\omega = C(\bar{G}^\omega)$. Note that G sits in $\bar{G}^\omega$ as a dense open subset because $E_\omega \supseteq C_\infty(G)$. That is, $\bar{G}^\omega$ is a compactification of the discrete set G. We will call it the ω-*compactification* of G. Note that $C(\bar{G}^\omega)$ is separable because G is countable and so there is only a countable number of φ_y's. Thus the compact topology of $\bar{G}^\omega$ has a countable base.

The action α of G on $C_b(G)$ by left translation clearly carries E_ω into itself. From this we obtain an induced action on $\bar{G}^\omega$ by homeomorphisms. We denote this action again by α.

Of course $C(\bar{G}^\omega)$ is faithfully represented as an algebra of pointwise multiplication operators on $\ell^2(G)$. This representation, M, together with the representation π of G on $\ell^2(G)$ form a covariant representation [35], [47] of $(C(\bar{G}^\omega), G, \alpha, c)$. We have already seen earlier several instances of the covariance relation $\pi_x M_f = M_{\alpha_x(f)}\pi_x$. The integrated form of this covariant representation, which we denote again by π, gives then a representation on $\ell^2(G)$ of the full twisted crossed product algebra $C^*(G, C(\bar{G}^\omega), \alpha, c)$. It is clear from the above discussion that our algebra $\mathcal{C}_\omega$ contains $\pi(C^*(G, C(\bar{G}^\omega), \alpha, c))$. But for any $y \in G$ and $t \in \mathbb{R}$ we have $\exp(it\varphi_y) \in C(\bar{G}^\omega)$. From our earlier calculation this means that $\pi(C^*(G, C(\bar{G}^\omega), \alpha, c))$ contains $U_t\pi_y U_t^*$. Thus it also contains $U_t\pi(C_c(G))U_t^*$. Consequently:

Lemma 3.1. *We have* $\mathcal{C}_\omega = \pi(C^*(G, C(\bar{G}^\omega), \alpha, c))$.

Now $C(\bar{G}^\omega)$ contains $C_\infty(G)$ as an α-invariant ideal. The following fact must be known, but I have not found a reference for it.

Lemma 3.2. *With notation as above,*

$$C^*(G, C_\infty(G), \alpha, c) \cong \mathcal{K}(\ell^2(G)),$$

the algebra of compact operator on $\ell^2(G)$*, with the isomorphism given by* π.

Proof. If we view elements of $C_c(G, C_\infty(G))$ as functions on $G \times G$, and if for $f \in C_c(G, C_\infty(G))$ we set $(\Phi f)(x, y) = f(x, y)c(x, x^{-1}y)$, then

$$\Phi(f *_c g) = (\Phi f) * (\Phi g),$$

where only here we let $*_c$ denote convolution (in the crossed product) twisted by c, while $*$ denotes ordinary convolution. The verification requires using the 2-cocycle identity to see that

$$c(y, y^{-1}z)c(y^{-1}x, x^{-1}z) = c(y, y^{-1}x)c(x, x^{-1}z).$$

The untwisted crossed product $C_\infty(G) \times_\alpha G$ is well-known to be carried onto $\mathcal{K}(\ell^2(G))$ by π. (See [37].) (For non-discrete groups one must be more careful, because cocycles are often only measurable, not continuous.) $\square$

Because $\mathcal{K}(\boldsymbol{\ell}^2(G))$ is simple, it follows that the reduced C^*-algebra $C_r^*(G, C_\infty(G), \alpha, c)$ coincides with the full twisted crossed product, even when G is not amenable. Anyway, the consequence of this discussion is:

Proposition 3.3. *With notation as above, the cosphere algebra is*

$$S^*A = \pi(C^*(G, C(\bar{G}^\omega), \alpha, c))/\mathcal{K}(\boldsymbol{\ell}^2(G)).$$

For an element of $\pi(C^*(G, C(\bar{G}^\omega), \alpha, c)$ it is probably appropriate to call its image in S^*A its "symbol", in analogy with the situation for pseudodifferential operators.

We can use recently-developed technology to obtain a simpler picture in those cases in which the action α of G on $\bar{G}^\omega$ is amenable [1], [3], [2], [26], [25]. This action will always be amenable if G itself is amenable, which will be the case when we consider $\mathbb{Z}^d$ in detail later. So the following comments will only be needed there for that case. But we will see in Section 10 that the action can be amenable also in some situations for which G is not amenable, namely for the free group on two generators and its standard word-length function.

Let $\partial_\omega G = \bar{G}^\omega \setminus G$. It is reasonable to call $\partial_\omega G$ the "ω-boundary" of G. Notice that α carries $\partial_\omega G$ into itself. Suppose that the action α of G on $\partial_\omega G$ is amenable [2], [3]. One of the equivalent conditions for amenability of α (for discrete G) is that the quotient map from $C^*(G, C(\partial_\omega G))$ onto $C_r^*(G, C(\partial_\omega G))$ is an isomorphism (theorem 4.8 of [1] or theorem 3.4 of [2]). (No cocycle c is involved here.) In proposition 2.4 of [31] it is shown that for situations like this the amenability of the action on $\partial_\omega G$ is equivalent to amenability of the action on $\bar{G}^\omega$. (I thank Claire Anantharaman–Delaroche for bringing this reference to my attention, and I thank both her and Jean Renault for helpful comments on related matters.) The proof in [31] uses the characterization of amenability of the action in terms of nuclearity of the crossed product. Here is another argument which does not use nuclearity. Following remark 4.10 of [36], we consider the exact sequence of full crossed products

$$0 \to C^*(G, C_\infty(G), \alpha) \to C^*(G, C(\bar{G}^\omega), \alpha) \to C^*(G, C(\partial_\omega G), \alpha) \to 0$$

and its surjective maps onto the corresponding sequence of reduced crossed products (which initially is not known to be exact). A simple diagram-chase shows that if the quotient map onto $C_r^*(G, C(\partial_\omega G), \alpha)$ is in fact an isomorphism, then the sequence of reduced crossed products is in fact exact. Also, as discussed above, $C^*(G, C_\infty(G), \alpha)$ is the algebra of compact operators, so simple, and so the quotient map from it must be an isomorphism. A second simple diagram-chase then shows that the quotient map from $C^*(G, C(\bar{G}^\omega), \alpha)$ must be an isomorphism, so that the action α of G on $\bar{G}^\omega$ is amenable. (The verification that if the action on $\bar{G}^\omega$ is amenable then so is that on $\partial_\omega G$ follows swiftly from the equivalent definition of amenability in terms of maps whose values are probability measures on G. This definition is given further below and in example 2.2.14(2) of [3].)

For our general functions ω it is probably not reasonable to hope to find a nice criterion for amenability of the action. But in the case in which ω is a length-function ℓ (in which case we write $\partial_\ell G$ instead of $\partial_\omega G$), we will obtain in the next sections considerable information about $\partial_\ell G$, and so it is reasonable to pose:

Question 3.4. *Let G be a finitely generated group, and let ℓ be the word-length function for some finite set of generators. Under what conditions will the action of G on $\partial_\ell G$ be amenable? For which class of groups will there exist a finite set of generators for which the action is amenable? For which class of groups will this amenability be independent of the choice of generators?*

It is known that if G is a word-hyperbolic group, then its action on its Gromov boundary is amenable. See the appendix of [3], written by E. Germain, and the references given there. We would have a positive answer to Question 3.4 for word-hyperbolic groups if we had a positive answer to:

Question 3.5. *Is it the case that for any word-hyperbolic group G and any word-length function on G for a finite generating set, there is an equivariant continuous surjection from $\partial_\ell G$ onto the Gromov boundary of G?*

This seems plausible in view of our discussion of geodesic rays in the next section, since the Gromov boundary considers geodesic rays which stay a finite distance from each other to be equivalent.

We now explore briefly the consequences of the action being amenable. The first consequence is that the full and reduced twisted crossed products coincide. We have discussed the case of a trivial cocycle c above. I have not seen the twisted case stated in the literature, but it follows easily from what is now known. We outline the proof. To every 2-cocycle there is associated an extension, E, of G by $\mathbb{T}$. As a topological space $E = \mathbb{T} \times G$, and the product is given by $(s,x)(t,y) = (stc(x,y), xy)$. (See III.5.12 of [20].) We can compose the evident map from E onto G with α to obtain an action, α, of E on $\bar{G}^\omega$. Let W be any compact space on which G acts, with the action denoted by α. If α is amenable, then by definition (example 2.2.14(2) of [3], [2], [26], [25]) there is a sequence $\{m_j\}$ of weak-$*$ continuous maps from W into the space of probability measures on G such that, for α denoting also the corresponding action on probability measures, we have for every $x \in G$

$$\lim_j \sup_{w \in W} \|\alpha_x(m_j(w)) - m_j(\alpha_x(w))\|_1 = 0.$$

Let h denote normalized Haar measure on $\mathbb{T}$, and for each j and each $w \in W$ let $n_j(w)$ be the product measure $h \otimes m_j(w)$ on E. Thus each $n_j(w)$ is a probability measure on E. It is easily verified that the function $w \mapsto n_j(w)$ is weak-$*$ continuous. Furthermore, a straight-forward calculation shows that

$$\alpha_{(s,x)}(n_j) = h \otimes \alpha_x(m_j)$$

for each $(s,x) \in E$ and each j. Now E is not discrete. But from this calculation it is easily seen that the action of E on W is amenable, where now we use definition 2.1 of [2]. Then from theorem 3.4 of [2] (which is a special case of proposition 6.1.8 of [3]), it follows that $C^*(E, C(W), \alpha)$ coincides with $C^*_r(E, C(W), \alpha)$.

Now let p be the function on $\mathbb{T}$ defined by $p(t) = \exp(2\pi i t)$, where here we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$. Since $\mathbb{T}$ is an open subgroup of E, we can view p as a function on E by giving it value 0 off of $\mathbb{T}$. Since $\mathbb{T}$ is central in E, and α is trivial on $\mathbb{T}$, and $C(W)$ is unital, it follows that p is a central projection in $C^*(E, C(W), \alpha)$. From this it follows that the cut-down algebras $pC^*(E, C(W), \alpha)$ and $pC^*_r(E, C(W), \alpha)$ coincide. But it is easily seen (see page 84 of [18] or page 144 of [19]) that $pC^*(E, C(W), \alpha) = C^*(G, C(W), \alpha, c)$, and similarly for C^*_r. In this way we obtain:

Proposition 3.6. *Let G be a discrete group, let α be an action of G on a compact space W, and let c be a 2-cocycle on G. If the action α is amenable, then $C^*(G, C(W), \alpha, c)$ coincides with $C^*_r(G, C(W), \alpha, c)$.*

With some additional care the above proposition can be extended to the case in which W is only locally compact. In that case the projection p is only in the multiplier algebras of the twisted crossed products.

We now return to the case in which G acts on $\bar{G}^\omega$ and $\partial_\omega G$. From the above proposition it follows that if G acts amenably on $\partial_\omega G$, and so on $\bar{G}^\omega$, then we can view π as a representation of the reduced crossed product $C^*_r(G, C(\bar{G}^\omega), \alpha, c)$. This has the benefit that we can apply corollary 4.19 of [47] to conclude that π is a faithful representation of $C^*_r(G, C(\bar{G}^\omega), \alpha, c)$. The hypotheses of this corollary 4.19 are that M be a faithful representation of $C(\bar{G}^\omega)$, which is clearly true, and that M be G-almost free (definition 1.12 of [47]). This latter means that for any non-zero subrepresentation N of M and any $x \in G$ with $x \neq e$ there is a non-zero subrepresentation P of N whose composition with the inner automorphism from x is disjoint from P. But subrepresentations of M correspond to non-empty subsets of G, and for P we can take any one-point subset of a given subset. Thus our algebra $\mathcal{C}^\omega$ coincides (under π) with $C^*(G, C(\bar{G}^\omega), \alpha, c)$.

Now from Lemma 3.2 we know that $C^*(G, C_\infty(G), \alpha, c)$ coincides with $\mathcal{K}(\ell^2(G))$, and the process of forming full twisted crossed products preserves short exact sequences. (See the top of page 149 of [47].) Thus from Proposition 3.3, and on removing our requirement that $\omega \geq 0$, we obtain:

Theorem 3.7. *Let ω be a translation bounded function on G such that the action of G on $\partial_{|\omega|} G$ is amenable. Then the cosphere algebra $S^*_\omega A$ for $(C^*_r(G, c), \ell^2(G), M_\omega)$ is (naturally identified with)*

$$S^*_\omega A = C^*(G, C(\partial_\omega G), \alpha, c) = C^*_r(G, C(\partial_\omega G), \alpha, c).$$

4. The metric compactification

The purpose of this section is to show that when ω is a length-function on G then geodesic rays in G for the metric on G from ω give points in the compactification $\bar{G}^\omega$. This will be a crucial tool for us in dealing with $\mathbb{Z}^d$, since it will supply us with a sufficient collection of points in the boundary which have finite orbits. We will also see that $\bar{G}^\omega$ is then a special case of a compactification of complete locally compact metric spaces introduced by Gromov [24] some time ago. (This is probably related to the comment which Connes makes about nilpotent groups in the second paragraph after the end of the proof of proposition 2 of section 6 of [14].) Gromov's definition appears fairly different from that which we gave in the previous section, and so our treatment here can also be viewed as showing how to define Gromov's compactification as the maximal ideal space of a unital commutative C^*-algebra. We will refrain from using here the terms "Gromov compactification" and "Gromov boundary", since these terms seem already reserved in the literature for use with hyperbolic spaces, where they have a different meaning and give objects which depend only on the coarse quasi-isometry class of the metric. (See IIIH3 of [6].) We will instead use the terms "metric compactification" and "metric boundary", and our notation will often show the dependence on the metric. We will see in Example 5.2 that for a hyperbolic metric space the metric boundary and the Gromov boundary can fail to be homeomorphic.

Let (X, ρ) be a metric space, and let $C_b(X)$ denote the algebra of continuous bounded functions on X, equipped with the supremum norm $\|\cdot\|_\infty$. Motivated by the observations in the previous section, we define $\varphi_{y,z}$ on X for $y, z \in X$ by

$$\varphi_{y,z}(x) = \rho(x, y) - \rho(x, z).$$

Then the triangle inequality tells us that $\|\varphi_{y,z}\|_\infty \leq \rho(y, z)$, so that $\varphi_{y,z} \in C_b(X)$. But on setting $x = z$ we see that, in fact, $\|\varphi_{y,z}\|_\infty = \rho(y, z)$. Let H_ρ denote the linear span in $C_b(X)$ of $\{\varphi_{y,z} : y, z \in X\}$. Suppose that we fix some base point $z_0 \in X$. Then it is easily checked that $\varphi_{y,z} = \varphi_{z_0,z} - \varphi_{z_0,y}$. Thus H_ρ is equally well the linear span of $\{\varphi_{z_0,y} : y \in X\}$, but is independent of the choice of z_0. (It will be useful to us that we can change base-points at will.) We often find it convenient to fix z_0, and to set $\varphi_y = \varphi_{z_0,y}$, so that H_ρ is the linear span of the φ_y's. When X is a group, it is natural to choose $z_0 = e$. We were implicitly doing this in the previous section. We note that $\|\varphi_y\|_\infty = \rho(y, z_0)$.

Much as above, we have $\varphi_y - \varphi_z = \varphi_{z,y}$, and so $\|\varphi_y - \varphi_z\|_\infty = \|\varphi_{z,y}\|_\infty = \rho(y, z)$. Thus the mapping $y \mapsto \varphi_y$ is an isometry from (X, ρ) into $C_b(X)$. The latter space is complete, and so this isometry extends to the completion of X.

We desire to obtain a compactification of X to which all of the functions φ_y extend as continuous functions. We want X to be an open subset of the compactification, and so we must require that X is locally compact. Then the various compactifications of X in which X is open are just the maximal-ideal spaces of the various unital closed *-subalgebras of $C_b(X)$ which contain $C_\infty(X)$. Thus we set:

Definition 4.1. *Let (X, ρ) be a metric space whose topology is locally compact. Let $\mathcal{G}(X, \rho)$ be the norm-closed subalgebra of $C_b(X)$ which is generated by $C_\infty(X)$, the constant functions, and H_ρ. Let $\bar{X}^\rho$ denote the maximal ideal space of $\mathcal{G}(X, \rho)$. We call $\bar{X}^\rho$ the* metric compactification *of X for ρ.*

Then, essentially by construction, $\bar{X}^\rho$ is a compactification of X (within which X is open). We remark that if, instead, we take the norm-closed subalgebra of $C_b(X)$ generated by all of the bounded Lipschitz functions, then we obtain the algebra of all bounded uniformly continuous (for ρ) functions on X. (See the bottom of page 23 of [45].)

It is natural to think of $X_\rho \backslash X$ as a boundary at infinity for X. But from a metric standpoint this is not always reasonable. Suppose that X is not complete. Each of the functions φ_y is a Lipschitz function, and so extends to the completion $\hat{X}^\rho$ of X. Each $f \in C_\infty(X)$ extends continuously to $\hat{X}^\rho$ by setting it equal to 0 off X. The constant functions obviously extend to $\hat{X}^\rho$. Thus the algebraic algebra generated by H_ρ, $C_\infty(X)$ and the constant functions extends to an algebra of functions on $\hat{X}^\rho$, and the supremum norm is preserved under this extension. Thus our completed algebra $\mathcal{G}(X,\rho)$ can be viewed as a unital subalgebra of $C_b(\hat{X}^\rho)$. It is easily seen that this algebra separates the points of $\hat{X}^\rho$. (E.g., use the fact that ρ extends to the completion.) Thus we obtain a (continuous) injection of $\hat{X}^\rho$ into $\bar{X}^\rho$. But there is no reason that $\hat{X}^\rho$ should be open in $\bar{X}^\rho$, notably if the completion is not locally compact. Even if $\hat{X}^\rho$ is locally compact, the points of $\hat{X}^\rho \backslash X$ will all be of finite distance from the points of X, and so are not "at infinity". For this reason it seems best to define the "boundary" only for *complete* locally compact metric spaces. Thus we make:

Definition 4.2. *Let (X,ρ) be a metric space which is complete and locally compact. Then its* metric boundary *is $\bar{X}^\rho \backslash X$. We will denote the metric boundary by $\partial_\rho X$.*

We now show that the metric compactification and the metric boundary which we have defined above coincide with those constructed by Gromov [24] in a somewhat different way. Gromov proceeds as follows. (See also 3.1 of [5], II.1 of [4] and II.8.12 of [6].) Let (X,ρ) be a complete locally compact metric space, let $C(X)$ denote the vector space of all continuous (possibly unbounded) functions on X, and equip X with the topology of uniform convergence on compact subsets of X. Let $C_*(X)$ denote the quotient of $C(X)$ by the subspace of constant functions, with the quotient topology. For $f \in C(X)$ denote its image in $C_*(X)$ by $\bar{f}$. For $y \in X$ set $\psi_y(x) = \rho(x,y)$. Then $x \mapsto \psi_x$ is an embedding of X into $C(X)$. Let ι denote the corresponding embedding of X into $C_*(X)$, and let $\mathcal{C}\ell(X)$ be the closure of $\iota(X)$ in $C_*(X)$. Then $\mathcal{C}\ell(X)$ can be shown to be compact, and $\iota(X)$ can be shown to be open in $\mathcal{C}\ell(X)$, so that $\mathcal{C}\ell(X)\backslash X$ is a boundary at infinity for X.

We now explain the relationship between this construction of Gromov and our construction given earlier in this section. Fix a base point z_0. For any given $u \in \bar{X}^\rho$ define the function g_u by $g_u(x) = -\varphi_x(u)$, where φ_x is now viewed as a function on $\bar{X}^\rho$. If $u \in X$ then $g_u(x) = \rho(u,x) - \rho(u,z_0)$. Since $\rho(u,z_0)$ is constant in x, the image of g_u in $C_*(X)$ is exactly Gromov's $\iota(u)$. On the other hand, suppose that $u \in \partial_\rho X$. Because X is dense in $\bar{X}^\rho$, there is a net $\{y_\alpha\}$ of elements of X which converges to u. Then for each $x \in X$ we have

$$g_u(x) = -\varphi_x(u) = -\lim \varphi_x(y_\alpha) = \lim g_{y_\alpha}(x).$$

That is, g_{y_α} converges to g_u pointwise on X. But each g_y for $y \in X$ is clearly a Lipschitz function of Lipschitz constant 1, and pointwise convergence of a net of functions of bounded Lipschitz constant implies uniform convergence on compact sets. Thus g_{y_α} converges uniformly to g_u on compact subsets of X, so that $\bar{g}_u \in \mathcal{C}\ell(X)$. (In the literature cited above, g_u would be called a *horofunction* if $u \in \partial_\rho X$.) In this way we obtain a mapping, $u \mapsto \bar{g}_u$, from $\bar{X}^\rho$ to $\mathcal{C}\ell(X)$. If $\bar{g}_u = \bar{g}_v$ for some $u, v \in \bar{X}^\rho$, then there is a constant, k, such that $\varphi_x(u) = \varphi_x(v) + k$ for all $x \in X$. From this it is easily seen that $u = v$. Thus the mapping $u \mapsto \bar{g}_u$ is injective on $\bar{X}^\rho$. Finally, if $\{u_\alpha\}$ is a net in $\bar{X}^\rho$ which converges to $u \in \bar{X}^\rho$, then, much as above, g_{u_α} converges to g_u pointwise, and so uniformly on compact sets. Thus the mapping $u \mapsto \bar{g}_u$ is continuous from $\bar{X}^\rho$ into $\mathcal{C}\ell(X)$. Since $\bar{X}^\rho$ is compact, it follows that this mapping is a homeomorphism onto its image. But the image of X in $\mathcal{C}\ell(X)$ is dense, and so the mapping is a homeomorphism from $\bar{X}^\rho$ onto $\mathcal{C}\ell(X)$, and so from $\partial_\rho X$ to $\mathcal{C}\ell(X)\backslash X$, as desired.

For our later purposes it is important for us to examine the relationship between geodesics and points of $\partial_\rho X$. Much of the content of the next paragraphs appears in some form in various places in the literature [5], [4], [6], though usually not in the generality we consider here. And here we reformulate it in terms of our approach to the construction of $\partial_\rho X$.

We will not assume that our metric spaces are connected. For example, we will later consider $\mathbb{Z}^d$ with its Euclidean metric from $\mathbb{R}^d$. Every ray (half-line) in $\mathbb{R}^d$ should give a direction toward infinity for $\mathbb{Z}^d$. But if the direction involves irrational angles, the ray may not meet $\mathbb{Z}^d$ at an infinite number of points. So we need a slight generalization of geodesic rays. For perspective we also include a yet weaker definition.

Definition 4.3. *Let (X, ρ) be a metric space, let T be an unbounded subset of $\mathbb{R}^+$ which contains 0, and let γ be a function from T into X. We will say that:*

a) *γ is a* geodesic ray *if $\rho(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in T$.*

b) *γ is an* almost-geodesic ray *if it satisfies the condition:*

For every $\varepsilon > 0$ there is an integer N such that if $t, s \in T$ and $t \geq s \geq N$, then

$$|\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t| < \varepsilon.$$

c) *γ is a* weakly-geodesic ray *if for every $y \in X$ and every $\varepsilon > 0$ there is an integer N such that if $s, t \geq N$ then*

$$|\rho(\gamma(t), \gamma(0)) - t| < \varepsilon$$

and

$$|\rho(\gamma(t), y) - \rho(\gamma(s), y) - (t - s)| < \varepsilon.$$

It is evident that any geodesic ray is an almost-geodesic ray. (I thank Simon Wadsley for pointing out to me that my definition of weakly-geodesic rays in the first version of this paper was defective.)

Lemma 4.4. *Let γ be an almost-geodesic ray, and let (ε, N) be as in Definition 4.3b. Then for $t \geq s \geq N$ we have:*

a) $|\rho(\gamma(t), \gamma(0)) - t| < \varepsilon$.
b) $|\rho(\gamma(t), \gamma(s)) - (t-s)| < 2\varepsilon$.
c) $\rho(\gamma(t), \gamma(s)) < \rho(\gamma(t), \gamma(0)) - \rho(\gamma(s), \gamma(0)) + 2\varepsilon$.

Proof. For a) set $s = t$ in the condition of Definition 4.3b. For b) we have

$$\begin{aligned}&|\rho(\gamma(t)), \gamma(s)) - (t-s)| \\ &\quad = |(\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t) - (\rho(\gamma(s), \gamma(0)) - s)| < 2\varepsilon.\end{aligned}$$

Finally, for c) we have

$$\begin{aligned}&\rho(\gamma(t), \gamma(s)) \\ &\quad = (\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t) - \rho(\gamma(s), \gamma(0)) \\ &\quad +\rho(\gamma(t), \gamma(0)) - (\rho(\gamma(t), \gamma(0)) - t) \\ &\quad < \rho(\gamma(t), \gamma(0)) - \rho(\gamma(s), \gamma(0)) + 2\varepsilon.\end{aligned}$$

□

Lemma 4.5. *Any almost-geodesic ray is weakly geodesic. Let γ be a weakly-geodesic ray. Take $\gamma(0)$ as the base-point for defining φ_y for any $y \in X$. Then $\lim_{t\to\infty} \varphi_y(\gamma(t))$ exists for every $y \in X$. If γ is actually a geodesic ray, then $t \mapsto \varphi_y(\gamma(t))$ is a non-decreasing (bounded) function.*

Proof. To motivate the rest of the proof, suppose first that γ is a geodesic ray. We show that $t \mapsto \varphi_y(\gamma(t))$ is a non-decreasing function (so has a limit). For $t \geq s$ we have

$$\begin{aligned}\varphi_y(\gamma(t)) \quad &- \quad \varphi_y(\gamma(s)) \\ &= \quad \rho(\gamma(t), \gamma(0)) - \rho(\gamma(t), y) - \rho(\gamma(s), \gamma(0)) + \rho(\gamma(s), y) \\ &= \quad t - s + \rho(\gamma(s), y) - \rho(\gamma(t), y) \\ &= \quad \rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), y) - \rho(\gamma(t), y) \geq 0\end{aligned}$$

by the triangle inequality.

Next, let γ be an almost-geodesic ray. It is useful and instructive to first see why $\lim_{t\to\infty} \varphi_y(\gamma(t))$ exists. Given $\varepsilon > 0$, take N as in Definition 4.3b. We will show first that if $t \geq s \geq N$ then $\varphi_y(\gamma(t)) > \varphi_y(\gamma(s)) - 3\varepsilon$. In fact,

$$\begin{aligned}\varphi_y(\gamma(t)) \quad &- \quad \varphi_y(\gamma(s)) \\ &= \quad \rho(\gamma(t), \gamma(0)) - \rho(\gamma(t), y) - \rho(\gamma(s), \gamma(0)) + \rho(\gamma(s), y) \\ &\geq \quad -\rho(\gamma(t), \gamma(s)) + \rho(\gamma(t), \gamma(0)) - \rho(\gamma(s), \gamma(0)) > -3\varepsilon,\end{aligned}$$

by part c) of Lemma 4.4.

Now let $m = \overline{\lim}\varphi_y(\gamma(t))$. Since $\varphi_y(x) \leq \rho(y, \gamma(0))$ for all $x \in X$, we must have $m \leq \rho(y, \gamma(0))$. Now there is an $s_0 \geq N$ such that $\varphi_y(\gamma(s_0)) \geq m - \varepsilon$.

Set $M = s_0$. Then for $t \geq M$ we must have $m \geq \varphi_y(\gamma(t)) \geq m - 4\varepsilon$ according to the previous paragraph. It follows that $\lim \varphi_y(\gamma(t)) = m$.

We can now show that γ is weakly-geodesic. Given $\varepsilon > 0$, choose N and $M \geq N$ as above. Then for $t \geq s \geq M$ the first condition of Definition 4.3c is satisfied by Lemma 4.4a, while for the second condition we have from above

$$\begin{aligned}
|\rho(\gamma(t), y) \quad - \quad & \rho(\gamma(s), y) - (t - s)| \\
\leq \quad & |\rho(\gamma(t), y) - \rho(\gamma(t), \gamma(0)) - \rho(\gamma(s), y) + \rho(\gamma(s), \gamma(0))| \\
+ \quad & |\rho(\gamma(t), \gamma(0)) - t| + |\rho(\gamma(s), \gamma(0)) - s| \\
\leq \quad & |\varphi_y(\gamma(t)) - \varphi_y(\gamma(s))| + 2\varepsilon < 6\varepsilon.
\end{aligned}$$

Finally, suppose that γ is a weakly-geodesic ray. For any $y \in X$ we show that $\{\varphi_y(\gamma(t))\}$ is a Cauchy net. Let ε and N be as in Definition 4.3c. Then for $t, s \geq N$ we have

$$\begin{aligned}
|\varphi_y(\gamma(t)) \quad - \quad & \varphi_y(\gamma(s))| \\
= \quad & |\rho(\gamma(t), \gamma(0)) - \rho(\gamma(t), y) - \rho(\gamma(s), \gamma(0)) + \rho(\gamma(s), y)| \\
\leq \quad & |\rho(\gamma(s), y) - \rho(\gamma(t), y) - (s - t)| \\
+ \quad & |\rho(\gamma(t), \gamma(0)) - t| + |s - \rho(\gamma(s), \gamma(0))| < 3\varepsilon.
\end{aligned}$$

□

For the next theorem we will need:

PROPOSITION 4.6. *Let (X, ρ) be a locally compact metric space. If the topology of X has a countable base, then so do the topologies of $\bar{X}^\rho$ and $\partial_\rho X$.*

Proof. If (X, ρ) is a locally compact metric space whose topology has a countable base, then $C_\infty(X)$ has a countable dense set. Also, X has a countable dense set, and the corresponding φ_y's can be used to construct a countable dense subset of H_ρ. Thus the C^*-algebra $\mathcal{G}(X, \rho)$ will have a countable dense set, and so the underlying spaces will have countable bases for their topologies. □

We recall that a metric is said to be *proper* if every closed ball of finite radius is compact.

THEOREM 4.7. *Let (X, ρ) be a complete locally compact metric space, and let γ be a weakly-geodesic ray in X. Then $\lim_{t\to\infty} f(\gamma(t))$ exists for every $f \in \mathcal{G}(X, \rho)$, and defines an element of $\partial_\rho X$. Conversely, if ρ is proper and if the topology of (X, ρ) has a countable base, then every point of $\partial_\rho X$ is determined as above by a weakly-geodesic ray.*

Proof. It is clear that the limit exists for the constant functions. From the definition of a weakly geodesic ray we see that γ must leave any compact set. Thus the limit exists and is 0 for all $f \in C_\infty(X)$. Choose $\gamma(0)$ as the base-point in defining φ_y for any $y \in X$. Then from Lemma 4.5 we know that $\lim \varphi_y(\gamma(t))$ exists for all $y \in X$.

Let $\tilde{\mathcal{G}}(X,\rho)$ denote the subalgebra of $C_b(X)$ generated by $C_\infty(X)$, the constant functions, and the φ_y's, before taking the norm-closure. It is clear from the above that $\lim f(\gamma(t))$ exists for every $f \in \tilde{\mathcal{G}}(X,\rho)$, and that $|\lim f(\gamma(t))| \le \|f\|_\infty$. Thus the limit defines a homomorphism from $\tilde{\mathcal{G}}(X,\rho)$ to $\mathbb{C}$ which is norm-continuous, and so extends to all of $\mathcal{G}(X,\rho)$ by continuity. It thus defines a point, say u, of $\bar{X}^\rho$. But because γ leaves any compact subset of X, the point defined by the limit must be in $\partial_\rho X$. It is easy to check now that $\lim f(\gamma(t))$ exists and equals $f(u)$ for all $f \in \mathcal{G}(X,\rho)$.

Suppose now that the topology of (X,ρ) has a countable base, and that ρ is proper. Let $u \in \partial_\rho X$. Then we can apply Proposition 4.6 to conclude that there is a sequence, $\{w_n\}$, in X which converges in $\bar{X}^\rho$ to u. Since $u \notin X$ and ρ is proper, the sequence $\{w_n\}$ must be unbounded. Thus we can find a subsequence, which we denote again by $\{w_n\}$, such that if $n > m$ then $\rho(w_n, w_0) > \rho(w_m, w_0)$. Let T denote the set of $\rho(w_n, w_0)$'s, and for any $t \in T$ with $t = \rho(w_n, w_0)$ set $\gamma(t) = w_n$. Then $\lim \gamma(t) = u$. We show that γ is weakly-geodesic. Notice that by construction $\rho(\gamma(t), \gamma(0)) = t$ for each $t \in T$, so that the first condition of Definition 4.3c is satisfied. Let $y \in X$. Use $\gamma(0)$ as the base-point for defining φ_y. Now $\varphi_y(\gamma(t))$ converges to $\varphi_y(u)$, and so, given $\varepsilon > 0$, we can find an N such that whenever $s, t \in T$ with $s, t \ge N$ then $|\varphi_y(t) - \varphi_y(s)| \le \varepsilon$. Then for such s, t we have

$$|\rho(\gamma(t), y) - \rho(\gamma(s), y) - (t - s)| = |\varphi_y(\gamma(t)) - \varphi_y(\gamma(s))| \le \varepsilon.$$

□

In view of the history of these ideas (see 1.2 of [24]), we make:

Definition 4.8. *A point of $\partial_\rho X$ which is defined as above by an almost-geodesic ray γ will be called a* Busemann point *of $\partial_\rho X$, and we will denote the point by b_γ.*

For any (X,ρ) it is an interesting question as to whether every point of $\partial_\rho X$ is a Busemann point. This is known to be the case for CAT(0) spaces (corollary II.8.20 of [6]). But in the next section we will need to deal with metric spaces which are not CAT(0). We will also see there by example that two metrics ρ_1 and ρ_2 on X which are Lipschitz equivalent, in the sense that there are positive constants k, K such that

$$k\rho_1 \le \rho_2 \le K\rho_1,$$

can give metric boundaries for X which are not homeomorphic.

Here is an example of a complete locally compact non-compact metric space X which has no geodesic rays, but for which every point of $\partial_\rho X$ is a Busemann point. Let X be the subset $X = \{(n, 1/n) : n \ge 1\}$ of $\mathbb{R}^2$, with the restriction to it of the Euclidean metric on $\mathbb{R}^2$. This suggests the usefulness of almost-geodesic rays. Just before Proposition 5.4 we will give an example of a proper metric on $\mathbb{Z}$ for which there are no almost-geodesic rays, so no Busemann points (but there are sufficiently many weakly-geodesic rays).

We will later need:

PROPOSITION 4.9. *Let $z_0 \in X$ and let γ and γ' be almost-geodesic rays from z_0 (i.e., $\gamma(0) = z_0 = \gamma'(0)$). If for any positive integer N and any $\varepsilon > 0$ we can find s and t in the domains of γ and γ' respectively such that $s, t \geq N$ and $\rho(\gamma(s), \gamma'(t)) < \varepsilon$, then $b_\gamma = b_{\gamma'}$.*

Proof. Each φ_y has Lipschitz constant ≤ 2, so
$$|\varphi_y(\gamma(s)) - \varphi_y(\gamma'(t))| \leq 2\rho(\gamma(s), \gamma'(t)).$$
The desired result follows quickly from this. □

We now briefly consider isometries. Suppose that α is an isometry of (X, ρ) onto itself. Then for $y, z \in X$ we have $\varphi_{y,z} \circ \alpha^{-1} = \varphi_{\alpha(y),\alpha(z)}$. Thus H_ρ is carried onto itself by α. Clearly so are $C_\infty(X)$ and the constant functions, and so α gives an automorphism of the algebra $\mathcal{G}(X, \rho)$. It follows that α gives a homeomorphism of $\bar{X}^\rho$ onto itself which extends α on X. This homeomorphism carries $\partial_\rho X$ onto itself. Thus:

PROPOSITION 4.10. *Every isometry of a complete locally compact metric space (X, ρ) extends uniquely to a homeomorphism of $\bar{X}^\rho$ onto itself which carries $\partial_\rho X$ onto itself.*

Later we will need to consider (cartesian) products of metric spaces. There are many ways to define a metric on a product. One of these ways meshes especially simply with the construction of the metric compactification. If (X, ρ_X) and (Y, ρ_Y) are metric spaces, we define ρ on $X \times Y$ by
$$\rho((x_1, y_1), (x_2, y_2)) = \rho_X(x_1, x_2) + \rho_Y(y_1, y_2).$$
We will call ρ the "sum of metrics".

PROPOSITION 4.11. *Let (X, ρ_X) and (Y, ρ_Y) be locally compact metric spaces, and let ρ be the sum of metrics on $X \times Y$. Then*
$$(X \times Y)^{-\rho} = (\bar{X}^{\rho_X}) \times (\bar{Y}^{\rho_Y}).$$

Proof. We need to show that the evident map from $X \times Y$ to $(\bar{X}^{\rho_X}) \times (\bar{Y}^{\rho_Y})$ extends to a homeomorphism from $(X \times Y)^{-\rho}$. For this it suffices to show that the restriction map from $C((\bar{X}^{\rho_X}) \times (\bar{Y}^{\rho_Y}))$ to $C_b(X \times Y)$ maps into $C((X \times Y)^{-\rho})$ and is onto. Let x_0, y_0 be base-points in X and Y respectively, and use (x_0, y_0) as a base-point for $X \times Y$. Then for $(u, v) \in X \times Y$ we have
$$\begin{aligned} \varphi_{(u,v)}(x, y) &= \rho((x, y), (x_0, y_0)) - \rho((x, y), (u, v)) \\ &= \rho_X(x, x_0) - \rho_X(x, u) + \rho_Y(y, y_0) - \rho_Y(y, v) \\ &= \varphi_u(x) + \varphi_v(y). \end{aligned}$$
In particular, $\varphi_{(u,y_0)} = \varphi_u \otimes 1_Y$ and $\varphi_{(x_0,v)} = 1_X \otimes \varphi_v$. Thus the restrictions of $\varphi_u \otimes 1_Y$ and $1_X \otimes \varphi_v$ are in $C((X \times Y)^{-\rho})$. The same is true for any $f \otimes 1_Y$ and $1_X \otimes g$ where $f \in C_c(X)$ and $g \in C_c(Y)$, or for constant functions. Thus the range of the restriction map is in $C((X \times Y)^{-\rho})$. But from the calculation above we also see that any $\varphi_{(u,v)}$ is in the range of the restriction map, and from this it is easily seen that the restriction map is onto $C((X \times Y)^{-\rho})$. □

5. The case of $G = \mathbb{Z}$

In this section we will see how the constructions of the previous sections can be used to deal with Questions 2.6 and 2.7 when $G = \mathbb{Z}$. This case already reveals some phenomena which we will have to deal with later for the case $G = \mathbb{Z}^d$.

Example 5.1. We examine first the case in which ℓ is the standard length function on $G = \mathbb{Z}$ defined by $\ell(n) = |n|$, so that $\rho(m,n) = |m-n|$. Note that ℓ is the word-length function for the generating set $S = \{\pm 1\}$. We determine $\partial_\ell G$. For any $k \in \mathbb{Z}$ we have

$$\varphi_k(n) = |n| - |n-k|.$$

In particular,

$$\varphi_k(n) = \begin{cases} k & \text{for } n \geq 0 \text{ and } n \geq k \\ -k & \text{for } n \leq 0 \text{ and } n \leq k. \end{cases}$$

From this it is clear that $\bar{\mathbb{Z}}^\ell$ is just $\mathbb{Z}$ with the points $\{\pm\infty\}$ adjoined in the traditional way. The action α of $\mathbb{Z}$ on $\bar{\mathbb{Z}}^\ell$ is by translation leaving the points at infinity fixed. Thus $\partial_\ell Z = \{\pm\infty\}$ with the trivial action α of $\mathbb{Z}$.

Now let $f \in C_c(\mathbb{Z})$ be given. Since $\mathbb{Z}$ is amenable, we know that $[M_\ell, \pi(f)]$ is in $C(\bar{\mathbb{Z}}^\ell) \times_\alpha \mathbb{Z}$, and that this crossed product is faithfully represented on $\boldsymbol{\ell}^2(\mathbb{Z})$, as discussed in Section 3. We can factor by $\mathcal{K} = C_\infty(\mathbb{Z}) \times_\alpha Z$, and so look at the image of $[M_\ell, \pi_f]$ in the cosphere algebra S^*A, which by the discussion of Section 3 is exactly $C(\partial_\ell\mathbb{Z}) \times_\alpha \mathbb{Z}$. This latter is isomorphic to two copies of $C^*(\mathbb{Z})$. The image of $\Sigma f(y) M_{\varphi_y}\pi_y$ in the copy at $+\infty$ will be $\{k \mapsto kf(k)\}$, while the image in the copy at $-\infty$ will be $\{k \to -kf(k)\}$. Let us take here the convention that the Fourier series for any $g \in C_c(\mathbb{Z})$ is given by $\hat{g}(t) = \Sigma g(k)e^{ikt}$, so that $\hat{g}'(t) = i\Sigma kg(k)e^{ikt}$. Then we see from just above that

$$L(f) = \|\Sigma f(y) M_{\varphi_y}\pi_y\| \geq \|\hat{f}'\|_\infty.$$

But $\|\hat{f}'\|_\infty$ agrees with the standard Lip-norm on $C^*(\mathbb{Z}) = C(\mathbb{T})$ which gives the circle a circumference of 2π. From the comparison lemma 1.10 of [39] it follows that L is a Lip-norm, and that it gives $\mathbb{T}$ (and so the state space $S(C^*(Z))$) radius no larger than π.

Example 5.2. Again we take $G = \mathbb{Z}$, but now we take the word-length function ℓ corresponding to the generating set $\{\pm 1, \pm 2\}$. Then ℓ is given by

$$\ell(n) = [|n|/2],$$

where $[\cdot]$ denotes "least integer not less than". Thus for any $k \in \mathbb{Z}$

$$\varphi_k(n) = [|n|/2] - [|n-k|/2].$$

From this one finds that if k is even then

$$\varphi_k(n) = \begin{cases} k/2 & \text{for } n \geq 0 \text{ and } n \geq k \\ -k/2 & \text{for } n \leq 0 \text{ and } n \leq k, \end{cases}$$

whereas if k is odd then

$$\varphi_k(n) = \begin{cases} \left.\begin{array}{l} (k-1)/2 \text{ for } n \text{ even} \\ (k+1)/2 \text{ for } n \text{ odd} \end{array}\right\} & \text{for } n \geq 0 \text{ and } n \geq k \\ \left.\begin{array}{l} -(k+1)/2 \text{ for } n \text{ even} \\ -(k-1)/2 \text{ for } n \text{ odd} \end{array}\right\} & \text{for } n \leq 0 \text{ and } n \leq k. \end{cases}$$

From this it is easily seen that $\partial_\ell \mathbb{Z}$ will consist of 4 points, two at $+\infty$ and two at $-\infty$, which we can label "even" and "odd". The action of $\mathbb{Z}$ on $\partial_\ell \mathbb{Z}$ will at each end be that of $\mathbb{Z}$ on $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. In particular, the boundary contains no fixed-points for this action.

We learn several things from comparing this example with the one just before. First, two word-length metrics on a given group can give metric boundaries which are not homeomorphic. But it is well-known (e.g., proposition 8.3.18 of [8]) and easily seen that if G is a finitely-generated group and if ℓ_1 and ℓ_2 are the word-length functions for two finite generating sets, then the corresponding left-invariant metrics are (Lipschitz) equivalent in the sense defined in the previous section. Thus we see that equivalent metrics which give (the same) locally compact topologies (even discrete) and for which the set is complete, can give metric boundaries which are not homeomorphic.

Next, $\mathbb{Z}$ is an example of a hyperbolic group [21], and so for the metric from either of these generating sets it is a hyperbolic metric space. But the Gromov boundary of a hyperbolic space is independent of the metrics as long as the metrics are equivalent, or at least coarsely equivalent. The Gromov boundary for $\mathbb{Z}$ is just $\{\pm\infty\}$. One way of viewing what is happening is that for the metric of the Example 5.2 the maps $m \mapsto 2m$ and $m \mapsto 2m+1$ are geodesic rays which determine Busemann points in the boundary which are our two points at $+\infty$. But for the Gromov boundary any two geodesic rays which stay a bounded distance from each other define the same point at infinity. In particular, our present example shows that for a given hyperbolic metric space the metric boundary and the Gromov boundary can fail to be homeomorphic.

For our next observation, let (X, ρ) be a proper metric space with basepoint z_0, and let $T \subset \mathbb{R}^+$ be a fixed domain for geodesic rays, so that $0 \in T$ and T is unbounded. On the set of geodesic rays from z_0 whose domain is T we put the topology of pointwise convergence (which, because geodesic rays are Lipschitz maps of Lipschitz constant 1, is equivalent to the topology of uniform convergence on bounded subsets of T). This is done in various places in the literature. Because ρ is proper, it is easy to see that the set of all such geodesic rays is compact. For groups G with a word-length ℓ (or for graphs in general) it is natural to take $T = \mathbb{Z}^+$. It is reasonable to wonder then whether $\partial_\ell G$ is the quotient of this compact set of geodesics, with the quotient topology. If it were, then for each $y \in G$ the function which assigns to each such geodesic ray γ from e the number $\lim \varphi_y(\gamma(t))$ should be a continuous function on this compact set. But this already fails for Example 5.2. For each $k \geq 1$ let γ^k be the geodesic ray from 0 defined by

$$\gamma^k(n) = \begin{cases} 2n & \text{if } n \le k \\ 2n-1 & \text{if } n \ge k+1. \end{cases}$$

Then γ^k converges pointwise to the geodesic ray defined by $\gamma^\infty(n) = 2n$ for all n. But it is easy to see that b_{γ^∞} is the even point at $+\infty$ while b_{γ^k} is the odd point at $+\infty$ for all k. We also remark that in our present example there is no geodesic line which joins the two points at $+\infty$ (so this example fails to have the property of "visibility" [21]).

Our Example 5.2 also shows that the metric compactification is not in general well-related to the Higson compactification, as defined in 5.4 of [43]. For that definition let (X, ρ) be a proper metric space. For any $r > 0$ we define the variational function, $V_r f$, of any function f by

$$(V_r f)(x) = \sup\{|f(x) - f(y)| : \rho(y, x) \le r\}.$$

The Higson compactification is the maximal ideal space of the unital commutative C^*-algebra of all bounded continuous functions on X such that for each $r > 0$ the function $V_r f$ vanishes at infinity. For Example 5.2 let us consider $V_2\varphi_1$. Easy calculation shows that for any $n \ge 1$ we have $\varphi_1(2k) = 0$ while $\varphi_1(2k+1) = 1$. But $\rho(2k, 2k+1) = \ell(1) = 1$ for all k. Thus $(V_2\varphi_1)(k) \ge 1$ for all k. Consequently φ_1 does not extend to the Higson compactification. More generally, if a complete locally compact metric space (X, ρ) has geodesic rays which determine distinct Busemann points of $\partial_\rho X$ and yet stay a finite distance from each other, then $\bar{X}^\rho$ is not a quotient of the Higson compactification. Indeed, since the φ_y's separate the points of $\partial_\rho X$, there will be some y such that its φ_y separates the two Busemann points, and $V_r\varphi_y$ will not vanish at infinity if r is larger than the distance between the two rays.

The situation becomes yet more interesting when we consider generating sets such as $\{\pm 3, \pm 8\}$. But the proof given above that we obtain a Lip-norm when we use the generating set $\{\pm 1\}$ extends without too much difficulty to the case of arbitrary finite generating sets for $\mathbb{Z}$. We do not include this proof here since in Section 9 we will treat the general case of $\mathbb{Z}^d$ by similar techniques, though the details are certainly more complicated.

However we will discuss here another approach for the case of $G = \mathbb{Z}$ which uses a classical argument which was pointed out to me by Michael Christ. (I thank him for his guidence in this matter). This second approach seems less likely to generalize to more complicated groups, but it gives a stronger result for $\mathbb{Z}$. For any β with $0 < \beta \le 1$ and any metric ρ on a set, ρ^β will again be a metric, because $t \to |t|^\beta$ is a length function on $\mathbb{R}$. In particular, if we set $\ell_\beta(n) = |n|^\beta$ then ℓ_β is a length function on $\mathbb{Z}$.

Theorem 5.3. *Let ω be a translation-bounded function on $\mathbb{Z}$ such that $\omega(0) = 0$. If ℓ_β/ω is a bounded function (ignoring $n = 0$) for some β with $1/2 < \beta \le 1$, then L_ω is a Lip-norm on $C^*(\mathbb{Z}) = C(\mathbb{T})$.*

Proof. For any group G and any ω we have

$$[M_\omega, \pi_f]\delta_e = \Sigma f(y)\varphi_y(y)\delta_y = \Sigma\omega(y)f(y)\delta_y,$$

where $\{\delta_y\}$ here denotes the standard basis for $\boldsymbol{\ell}^2(G)$. Thus

$$\|\omega f\|_2 \leq \|[M_\omega, \pi_f]\| = L_\omega(f).$$

What is special about $\mathbb{Z}$ is that $\|\omega f\|_2$ can control the norms we need. (This is related to our discussion of "rapid decay" in Section 1.) For this we need that $\ell_\beta^{-1} \in \boldsymbol{\ell}^2(\mathbb{Z})$, which happens exactly for $\beta > 1/2$. (Here and below we ignore $n = 0$ or set $\ell_\beta^{-1}(0) = 0$.) Let $f \in C_c(\mathbb{Z})$, with $\hat{f}$ its Fourier transform on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, viewed as a periodic function on $\mathbb{R}$. For $s, t \in \mathbb{R}$ with $s < t$ and $|t - s| < 1$ let $\chi_{[s,t]}$ denote the characteristic function of the interval $[s, t]$, extended by periodicity. Then

$$|\hat{f}(s) - \hat{f}(t)| = \left|\int_s^t \hat{f}'(r)dr\right| = |\langle \hat{f}', \chi_{[s,t]}\rangle| = |\langle (\hat{f}')^\vee, (\chi_{[s,t]})^\vee\rangle|.$$

But $(\chi_{[s,t]})^\vee(n) = (1/i2\pi n)(e(nt) - e(ns))$ if we set $e(r) = e^{2\pi i n r}$, while $(\hat{f}')^\vee(n) = -2\pi i n f(n)$. Thus if we set $g_{s,t}(n) = (e(nt) - e(ns))$, the above becomes $|\langle f, g_{s,t}\rangle|$ as a pairing between functions in $\boldsymbol{\ell}^1(\mathbb{Z})$ and $\boldsymbol{\ell}^\infty(\mathbb{Z})$. But (with $\omega^{-1}(0) = 0$) we can rewrite this as

$$|\langle \omega f, \omega^{-1} g_{s,t}\rangle| \leq \|\omega f\|_2 \|\omega^{-1} g_{s,t}\|_2,$$

and notice that

$$\begin{aligned} \|\omega^{-1} g_{s,t}\|_2 &= \|(\ell_\beta/\omega)\ell_\beta^{-1} g_{s,t}\|_2 \\ &\leq \|\ell_\beta/\omega\|_\infty \|\ell_\beta^{-1} g_{s,t}\|_2 < \infty, \end{aligned}$$

since $\ell_\beta^{-1} \in \boldsymbol{\ell}^2(\mathbb{Z})$. Set $m(s,t) = \|\ell_\beta^{-1} g_{s,t}\|_2$. Then putting the above together, we obtain

$$|\hat{f}(t) - \hat{f}(s)| \leq m(s,t)\|\ell_\beta/\omega\|_\infty \|L_\omega(f)\|.$$

A simple estimate using the fact that $\ell_\beta^{-1} \in \boldsymbol{\ell}^2(\mathbb{Z})$ shows that for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $|t - s| < \delta$ then $m(s,t) < \varepsilon$. From this we see that the set of $\hat{f}$'s for which $L_\omega(f) \leq 1$ and $f(0) = 0$ forms a bounded subset of $C(\mathbb{T})$ which is equicontinuous, so totally bounded by the Arzela–Ascoli theorem. From this it is clear that L_ω gives finite radius and, by theorem 1.9 of [39], that it is a Lip-norm. □

I suspect that when $\beta < 1/2$ then L_{ℓ_β} fails to be a Lip-norm, but I have not found a proof of this.

Notice that Theorem 5.3 applies if $|\omega(n)| \geq 1$ for $n \neq 0$ and if there are positive constants c and K such that $|\omega - c\ell^\beta| \leq K$, for then $|\ell/\omega| \leq (K+1)c$. This is the situation which occurs for the various word-length functions on $\mathbb{Z}$ (for $\beta = 1$).

It is interesting to see what the metric compactification of $\mathbb{Z}$ is when $\beta < 1$. For any $p \in \mathbb{Z}$ we have

$$\varphi_p(n) = |n|^\beta - |n-p|^\beta = \int_{|n-p|}^{|n|} \beta t^{\beta-1}\, dt.$$

Since $t^{\beta-1} \to 0$ at $+\infty$ because $\beta < 1$, it follows that $\varphi_p(n) \to 0$ as $n \to \pm\infty$. Thus $\varphi_p \in C_\infty(\mathbb{Z})$, and so the metric compactification is just the one-point compactification of $\mathbb{Z}$. Note also that $[M_{\ell_\beta}, \pi_f]$ is a compact operator for each $f \in C_c(\mathbb{Z})$. Thus the cosphere algebra for $(C^*(\mathbb{Z}), \boldsymbol{\ell}^2(\mathbb{Z}), M_{\ell_\beta})$ is $C^*(\mathbb{Z})$, and the image of $[M_{\ell_\beta}, \pi_f]$ in it is 0. We also remark that it is easily verified that if we set $\gamma(n^\beta) = n$, then γ is a weakly-geodesic ray, but that there are no almost-geodesic rays in $\mathbb{Z}$ for this metric, since by parts a) and b) of Lemma 4.4 if γ were such a ray we would have, for any fixed big r, that $|\gamma(t)|^\beta - |\gamma(t-r)|^b$ would be approximately r as $t \to \infty$, contradicting our observation above that it must go to 0.

We conclude this section with the following observation, which applies to our more general case of $\mathbb{Z}^d$.

Proposition 5.4. *Let ω be a translation-bounded function on a countable discrete Abelian group G, let L_ω on $C_c(G)$ be defined as earlier by $L_\omega(f) = \|[M_\omega, \pi_f]\|$, and let ρ_ω be the corresponding metric on $\hat{G}$ (which may not give the usual topology of $\hat{G}$). Then ρ_ω is invariant under translation on $\hat{G}$.*

Proof. Let us denote the pairing between G and $\hat{G}$ by $\langle m, t\rangle$. Then translation on $\hat{G}$ corresponds to the dual action, β, of $\hat{G}$ on $C^*(G)$ given on $C_c(G)$ by $(\beta_t(f))(m) = \langle m, t\rangle f(m)$. This is unitarily implemented in $\boldsymbol{\ell}^2(G)$ by M_t, where $(M_t\xi)(m) = \langle m, t\rangle \xi(m)$. Then

$$\begin{aligned} [M_\omega, \beta_t(\pi_f)] &= [M_\omega, M_t \pi_f M_t^*] \\ &= M_t [M_\omega, \pi_f] M_t^*, \end{aligned}$$

so that $L_\omega(\beta_t(f)) = L_\omega(f)$. (In other words, β is an action by isometries as defined in [41].) $\square$

From Theorem 5.3 one begins to see that $\mathbb{T}^d$ has a bewildering variety of translation invariant metrics which give its topology. For example, if ρ is such a metric then so is ρ^r for any r with $0 < r < 1$, as is any convex function of ρ. The sum of two metrics and the supremum of two metrics are again metrics. More generally, the "$\boldsymbol{\ell}^p$-sum" of two metrics is a metric. These operations all preserve translation invariance. For the case of $\mathbb{T}$, any strictly increasing continuous function ℓ on $[0, 1/2]$ such that $\ell(0) = 0$ and $\ell(s+t) \le \ell(s) + \ell(t)$ if $s + t \le 1/2$ gives in an evident way a continuous length function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and all continuous length functions on $\mathbb{T}$ arise in this way. It would be interesting to determine which generating sets for $\mathbb{Z}$ determine which length functions on $\mathbb{T}$, but I have not investigated this question.

6. The metric compactification for norms on $\mathbb{R}^d$

One of our eventual aims is to show that when ℓ is a length function on $\mathbb{Z}^d$ which is the restriction to $\mathbb{Z}^d$ of a norm on $\mathbb{R}^d$, then L_ℓ is a Lip-norm. In preparation for this we examine here the metric compactification of $\mathbb{R}^d$ for any given norm. We begin by considering the usual $\boldsymbol{\ell}^1$-norm, both because it is simple to treat and displays some interesting phenomena, and also because its restriction to $\mathbb{Z}^d$ gives the word-length function for the standard generating set. Following up on Example 5.1, we set $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ in the usual way, with the action of $\mathbb{R}$ fixing the points $\pm\infty$.

Proposition 6.1. *The metric compactification of* $(\mathbb{R}^d, \|\cdot\|_1)$ *is just* $(\bar{\mathbb{R}})^d$ *with its product action of* $\mathbb{R}^d$*. Thus the metric boundary is the set of* $(\tilde{x}_j) \in (\bar{\mathbb{R}})^d$ *such that at least one entry is* $+\infty$ *or* $-\infty$*.*

Proof. The metric from $\|\cdot\|_1$ on $\mathbb{R}^d$ is easily seen to be the sum of the metrics on $\mathbb{R}$ in the sense used in Proposition 4.11. Thus we just need to apply that proposition a number of times. □

We note that now there are orbits in the boundary which are not finite, but there are also fixed points (only a finite number of them).

We now investigate what happens for other norms on $\mathbb{R}^d$. It is notationally convenient for us just to consider a finite-dimensional vector space V with some given norm $\|\cdot\|$. We will denote the corresponding metric boundary simply by $\partial_\ell V$, where $\ell(x) = \|x\|$ for all $x \in V$.

For any $v \in V$ with $\|v\| = 1$ it is evident that the function $\gamma(t) = tv$ for $t \in T = [0, \infty)$ is a geodesic ray, and so from our earlier discussion it will determine a Busemann point, b_v, in $\partial_\ell V$. We now convert to this picture some of the known elementary facts about tangent functionals of convex sets, as explained for example in section V.9 of [17]. There is at least one linear functional, say σ, on V such that $\|\sigma\| = 1 = \sigma(v)$. We call such a σ a "support functional" at v. Then for any $y \in V$ we have

$$\varphi_y(\gamma(t)) = \|tv\| - \|tv - y\| \le t - \sigma(tv - y) = \sigma(y).$$

In particular, $\varphi_{-y}(\gamma(t)) \le -\sigma(y)$. On letting t go to $+\infty$ we find that

$$-\varphi_{-y}(b_v) \ge \sigma(y) \ge \varphi_y(b_v).$$

But theorem 5 of section V.9 of [17] (which uses the Hahn–Banach theorem) tells us that for any real number r such that $-\varphi_{-y}(b_v) \ge r \ge \varphi_y(b_v)$ there is a support functional σ at v such that $\sigma(y) = r$. To see that theorem V.9.5 really applies here, we note that if we set $s = t^{-1}$ then

$$\|tv\| - \|tv - y\| = (\|v\| - \|v - sy\|)/s,$$

and that $s \to +0$ as $t \to +\infty$. From this viewpoint we are thus looking at the negative of the tangent functional to the unit ball at v in the direction of $-y$, which fits the setting of theorem V.9.5.

The point v is called a *smooth* point of the unit sphere if there is only one support functional σ at v. We denote this unique σ by σ_v. Then the above considerations tell us that if v is smooth then $\varphi_y(b_v) = -\varphi_{-y}(b_v)$. On combining this with the inequalities found above, we obtain:

PROPOSITION 6.2. *Let v be a smooth point of the unit sphere of V. Then*

$$\varphi_y(b_v) = \sigma_v(y)$$

for all $y \in V$.

For us the following proposition will be of considerable importance. We consider the action of V on itself by translation, and the corresponding action on $\partial_\ell V$.

PROPOSITION 6.3. *Let v be a smooth point of the unit sphere of V. Then b_v is a fixed point under the action of V on $\partial_\ell V$.*

Proof. We use the 1-cocycle relation 2.2 and Proposition 6.2 to calculate that for any $x, y \in V$ we have

$$\begin{aligned}(\alpha_x\varphi_y)(b_v) &= \varphi_{x+y}(b_v) - \varphi_x(b_v) \\ &= \sigma_v(x+y) - \sigma_v(x) = \sigma_v(y) = \varphi_y(b_v).\end{aligned}$$

□

Finally, we note that theorem 8 of section V.9 of [17] says that, for any norm, the set of smooth points of the unit sphere is dense in the unit sphere. This does not imply that there are infinitely many fixed points in $\partial_\ell V$, as the next example shows. But we will see later that it does show that there are enough for our purposes.

EXAMPLE 6.4. We examine the case of $\mathbb{R}^2$ with $\|\cdot\|_1$, whose metric compactification is described by Proposition 6.1. Let us see how our considerations concerning geodesics fit this example. We identify the dual space V' in the usual way with $\mathbb{R}^2$ with the norm

$$\|(r, s)\|_\infty = \max\{|r|, |s|\}.$$

All but 4 points of the unit sphere of V are smooth. However, for any $v = (a, b)$ with $0 < a$, $0 < b$ and $a + b = 1$ we see that $\sigma_v = (1, 1) \in V'$. Thus all these different v's determine the same Busemann point of $\partial_\ell V$. This accords with Proposition 6.2 and the fact that $\partial_\ell V$ has only 4 fixed-points for the action of $\mathbb{R}^2$.

If instead we let v be the non-smooth point $(1, 0)$ and let γ be the corresponding geodesic ray, then for any $y = (p, q) \in \mathbb{R}^2$ we have

$$\begin{aligned}\varphi_y(\gamma(t)) &= \|\gamma(t)\| - \|\gamma(t) - (p, q)\| \\ &= |t| - |t - p| - |q|.\end{aligned}$$

The limit as $t \to +\infty$ is clearly $p - |q|$, so that

$$\varphi_{(p,q)}(b_v) = p - |q| = \varphi_p(+\infty) + \varphi_q(0),$$

where φ_p and φ_q are for $\mathbb{R}$. Thus $b_v = (+\infty, 0)$ in the description of $\partial_\ell V$ given by Proposition 6.1. Clearly b_v is not given by an element of V'. It is easily seen that this b_v is not invariant under translation.

We see in this way that the *linear* geodesic rays from 0, corresponding to the points of the unit sphere, determine only 8 Busemann points of $\partial_\ell V$. But we can show that every point of $\partial_\ell V$ is determined by at least one (possibly non-linear) geodesic ray from 0. For example, if we consider $(+\infty, s) \in \partial_\ell V$ for some fixed $s \in \mathbb{R}$, we can pick any $t_0 \geq 0$ and let γ consist of the unit-speed straight-line path from $(0,0)$ to $(t_0, 0)$, followed by that from $(t_0, 0)$ to (t_0, s), followed by the linear ray from (t_0, s) in the direction $(1, 0)$. (We deal here with the "Manhattan metric".) It is easy to check that γ is a geodesic ray whose Busemann point corresponds to $(+\infty, s)$. We see in this way that every point of $\partial_\ell V$ is a Busemann point. It is also easy to see that for each of the 4 points $(\pm\infty, 0)$ and $(0, \pm\infty)$ of $\partial_\ell V$ there is only one geodesic ray to them from 0, but that for every other point of $\partial_\ell V$ there are uncountably many geodesic rays to it from 0.

Question 6.5. *Is it true that, for every finite-dimensional vector space and every norm on it, every point of $\partial_\ell V$ is a Busemann point?*

One says that $(V, \|\cdot\|)$ is *smooth* if every point of the unit sphere, S, of V is a smooth point. Let S' denote the unit sphere of V'. Then our earlier mapping $v \mapsto \sigma_v$ is defined on all of S. Furthermore it is onto S', because V, being finite dimensional, is reflexive. This mapping σ can also be seen to be continuous. This is essentially the fact that, as remarked at the bottom of page 60 of [33], a compactness argument shows that smoothness implies uniform smoothness. However, if S has "flat spots" then σ will not be injective. It is not difficult to show that for $(V, \|\cdot\|)$ smooth, $\partial_\ell V$ can be naturally identified with S', glued at ∞ using σ. In this case each point of $\partial_\ell V$ will be fixed by the action of V.

Question 6.6. *For a general $(V, \|\cdot\|)$ is there an attractive description of $\partial_\ell V$ and of the action of V on it?*

We have seen in Example 6.4 that the number of support functionals σ_v coming from smooth points v of the unit sphere can be finite. The reason that they nevertheless are adequate for our later purposes is given by the following proposition (which must be already known):

Proposition 6.7. *Let $\|\cdot\|$ be a norm on a finite-dimensional vector space V. Let $w \in V$, and suppose that $|\sigma_v(w)| \leq r$ for all smooth points v of the unit sphere. Then $\|w\| \leq r$. Furthermore, the closed convex hull of $\{\sigma_v : v \text{ is a smooth point}\}$ is the unit ball in the dual space V' for the dual norm $\|\cdot\|'$.*

Proof. Let $\|w\| = s$. Because the smooth points are dense in the unit sphere by theorem 8 of section V.9 of [17], for any $\varepsilon > 0$ we can find a smooth point v such that $\|w - sv\| < \varepsilon$. Then $|\sigma_v(w) - s| = |\sigma_v(w - sv)| < \varepsilon$. Since $|\sigma_v(w)| \leq r$ and ε is arbitrary, it follows that $\|w\| = s \leq r$.

Suppose now that $\tau \in V'$ and that $\tau \notin \bar{co}\{\sigma_v : v \text{ smooth}\}$. Then by the Hahn–Banach theorem there is a $w \in V$ and an $r \in \mathbb{R}$ such that $|\sigma_v(w)| \leq r < \tau(w)$ for all smooth v. But we have just seen that then $\|w\| \leq r$. Thus $\|\tau\|' > 1$. □

7. Restrictions of norms to $\mathbb{Z}^d$

In this section we will examine what happens when norms on $V = \mathbb{R}^d$ are restricted to $\mathbb{Z}^d$. We begin with the case of the norm $\|\cdot\|_1$. Following up on Example 6.4 we set $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$ in the usual way, with its action of $\mathbb{Z}$ leaving fixed the points at infinity. The proof of the following proposition is basically the same as that of Proposition 6.1.

Proposition 7.1. *For $\ell = \|\cdot\|_1$, the metric compactification of $(\mathbb{Z}^d, \ell)$ is $(\bar{\mathbb{Z}})^d$ with its product action of $\mathbb{Z}^d$. The metric boundary is the set of $(\tilde{n}_j) \in (\bar{\mathbb{Z}})^d$ such that at least one entry is $+\infty$ or $-\infty$.*

Suppose now that $\ell = \|\cdot\|$ is any norm on $V = \mathbb{R}^d$, and that we restrict it to $\mathbb{Z}^d$. For any $y \in \mathbb{Z}^d$ the function φ_y clearly extends to $\bar{V}^\ell$, and then restricts to the closure of $\mathbb{Z}^d$ in $\bar{V}^\ell$. It is not evident to me whether the φ_y's for $y \in \mathbb{Z}^d$ separate the points of this closure. But even if they did, it is not clear to me that we could then use this to apply the results of the previous section to show that there are sufficient fixed-points in $\partial_\ell \mathbb{Z}^d$ for the action of $\mathbb{Z}^d$. It is this supply of fixed-points which we need later. So we take a more direct tack. We show that every *linear* geodesic ray in V can be approximated by an almost-geodesic ray in $\mathbb{Z}^d$. The following lemma is closely related to Kronecker's theorem [7], so we just sketch the proof.

Lemma 7.2. *Let $v \in V$ with $\|v\| = 1$. Then there is an unbounded strictly increasing sequence $\{s_n\}$ of positive real numbers such that for every $\varepsilon > 0$ there is an N such that if $s_n > N$ then there is an $x \in \mathbb{Z}^d$ for which $\|x - s_n v\| < \varepsilon$.*

Proof. If there is an $r \in \mathbb{R}^+$ with $rv \in \mathbb{Z}^d$ then we simply take $s_n = nr$. Suppose now that no such r exists. Consider the image of $\mathbb{R}v$ in $V/\mathbb{Z}^d$. Its closure is a connected subgroup, and so is a torus. The dimension of this torus must be ≥ 2 for otherwise there would be an r as above. But for any finite closed interval I of $\mathbb{R}$ the image of Iv is compact, and so must stay away from 0 except at 0. Thus for any neighborhood of 0 there must be a t outside of I such that the image of tv is in that neighborhood. □

Let $\{s_n\}$ be as in the lemma. Then we can find a subsequence, $\{t_k\}$, of the sequence $\{s_n\}$, and for each k we can choose a $x_k \in \mathbb{Z}^d$, such that $\|x_k - t_k v\| < 1/k$ for all k.

Lemma 7.3. *For v, $\{t_k\}$ and $\{x_k\}$ as above, define γ by $\gamma(0) = 0$ and $\gamma(t_k) = x_k$. Then γ is an almost-geodesic ray in V which determines the same Busemann point in $\partial_\ell V$ as does the ray $t \mapsto tv$.*

Proof. Given $\varepsilon > 0$, choose N such that $1/N < \varepsilon/3$. Then for $t_n \geq t_m \geq N$ we have from the triangle inequality

$$\begin{aligned} |\|x_n - x_m\| &+ \|x_m\| - t_n| = |\|x_n - x_m\| - \|(t_n - t_m)v\| + \|x_m\| - t_m| \\ &\leq \|(x_n - t_n v) - (x_m - t_m v)\| + \|x_m - t_m v\| < \varepsilon. \end{aligned}$$

From this it follows that γ is an almost-geodesic ray. The fact that it determines the same Busemann point as does v now follows from Proposition 4.9. □

Proposition 7.4. *Let v be a smooth point of the unit sphere of V, with support functional σ_v. Then there is a Busemann point $b_v \in \partial_\ell \mathbb{Z}^d$ such that for any $y \in \mathbb{Z}^d$ we have*

$$\varphi_y(b_v) = \sigma_v(y).$$

Furthermore, b_v is a fixed-point for the action of $\mathbb{Z}^d$ on $\partial_\ell \mathbb{Z}^d$.

Proof. Let γ be an almost-geodesic ray associated with v as in the above lemmas. By Proposition 6.2 we know that

$$\lim \varphi_y(x_k) = \sigma_v(y)$$

for all $y \in V$. But γ is equally well an almost-geodesic ray in $\mathbb{Z}^d$, and so defines a Busemann point $b_\gamma \in \partial \mathbb{Z}^d$. But for $y \in \mathbb{Z}^d$ its φ_y for $\mathbb{Z}^d$ is just the restriction to $\mathbb{Z}^d$ of its φ_y for V. Thus $\varphi_y(b_\gamma) = \sigma_v(y)$ for $y \in \mathbb{Z}^d$. The proof that b_γ is a fixed-point for the action is the same as that for Proposition 6.3. □

We remark that, just as for V, different smooth points v may have the same σ_v, and so determine the same Busemann point of $\partial_\ell \mathbb{Z}^d$, and so it can happen that only a finite number of points of $\partial_\ell \mathbb{Z}^d$ arise from smooth points v.

We are now ready to prove one part of our Main Theorem 0.1, namely:

Theorem 7.5. *Let ℓ on $\mathbb{Z}^d$ be defined by $\ell(x) = \|x\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^d$. Let L_ℓ be defined on $C_c(\mathbb{Z}^d, c)$ as before by*

$$L_\ell(f) = \|[M_\ell, \pi_f]\|.$$

Then L_ℓ is a Lip-norm on $C^*(\mathbb{Z}^d, c)$.

Proof. Let v be a smooth point of the unit sphere of V for $\|\cdot\|$. Let σ_v denote its support functional, and b_v its corresponding Busemann point as above in $\partial_\ell \mathbb{Z}^d$. Since b_v is a fixed-point, it determines a homomorphism from the cosphere algebra $C^*(G, C(\partial_\ell G), \alpha, c)$ onto $C^*(G, c)$ which takes M_{φ_y} to the constant $\sigma_v(y)$. (We use here the amenability of $\mathbb{Z}^d$.) Then under this homomorphism $[M_\ell, \pi_f]$ is sent to the operator

$$\Sigma f(y)\varphi_y(b_v)\pi_y = \Sigma f(y)\sigma_v(y)\pi_y$$

in $C^*(\mathbb{Z}^d, c)$. Let us denote this operator, and the corresponding function, by $X_v f$. Of course $\|X_v f\| \leq L_\ell(f)$.

We let β denote the usual dual action [35] of the dual group $\hat{G}$ on $C^*(G, c)$ determined by

$$(\beta_p(f))(x) = \langle x, p\rangle f(x)$$

for $f \in C_c(\mathbb{Z}^d)$ and $p \in \hat{G}$, where $\langle \cdot, \cdot \rangle$ denotes the pairing of G and $\hat{G}$. Each $\tau \in V'$ determines an element of $\hat{G}$ by $\langle x, \tau \rangle = \exp(i\tau(x))$ for $x \in \mathbb{Z}^d$. Let Γ denote the lattice in V' consisting of elements which on $\mathbb{Z}^d$ take values in $2\pi\mathbb{Z}$. Then we can identify $\hat{G}$ with the torus V'/Γ, and then V' is identified with the Lie algebra of $\hat{G}$, so that the exponential mapping is just the quotient map from V' to V'/Γ. The action β has an infinitesimal version which is a Lie algebra homomorphism from the (Abelian) Lie algebra V' into the Lie algebra of derivations on $C^*(G, c)$. We denote it by $d\beta$, and it is determined by

$$(d\beta_\tau(f))(x) = i\tau(x)f(x).$$

Each $f \in C_c(G)$ then determines a linear mapping, $\tau \mapsto d\beta_\tau(f)$, from V' into $C^*(G, c)$, which we denote by df, much as done for theorem 3.1 of [39].

In terms of the notation just introduced, we see that for any smooth point v we have

$$iX_v f = d\beta_{\sigma_v} f = df(\sigma_v).$$

With this notation our earlier inequality becomes

$$\|df(\sigma_v)\| \leq L_\ell(f).$$

Now V' has the dual norm $\|\cdot\|'$, and $C^*(G, c)$ has its C^*-norm. So the norm of the linear map df between them is well-defined. We denote it by $\|df\|$. But by Proposition 6.7 the closed convex hull of the set of σ_v's is the unit ball in V'. It follows that

$$\|df\| \leq L_\ell(f).$$

But in theorem 3.1 of [39] it is shown that $f \mapsto \|df\|$ is a Lip-norm. Thus we can apply comparison lemma 1.10 of [39] to conclude that L_ℓ is a Lip-norm as well. □

8. The boundary of $(\mathbb{Z}^d, S)$

Let S be a finite generating subset of $G = \mathbb{Z}^d$ such that $S = -S$ and $0 \notin S$. Let ℓ denote the corresponding word-length function on G. I do not know how to give a concrete description of $\partial_\ell G$. (But note that $\partial_\ell G$ is totally disconnected since each φ_y takes only integer values, in contrast to what happens if ℓ comes, for example, from the Euclidean norm on $\mathbb{R}^d$.) We will show here how to construct a substantial supply of geodesic rays. (Somewhat related considerations appear in [44], but geodesic rays and compactifications are not considered there.) In the next section we will show that our supply is sufficient to prove that when M_ℓ is used as the Dirac operator for $C^*(G, c)$, then the corresponding metric on the state space of $C^*(G, c)$ gives the weak-$*$ topology.

Our construction is motivated by several features which we found in Sections 6 and 7. For convenience we view $G = \mathbb{Z}^d$ as embedded in $\mathbb{R}^d$. We let $K = K_S$ denote the (closed) convex hull in $\mathbb{R}^d$ of S. Because K is balanced (since $S = -S$), it determines a norm, $\|\cdot\|_S$, on $\mathbb{R}^d$, for which it is the unit ball. (In fact, $(\mathbb{R}^d, \|\cdot\|_S)$ is the "asymptotic cone" of $(\mathbb{Z}^d, \ell)$—see exercise 8.2.12 of [8].) We will see later that this norm is relevant. The set of extreme points of K_S is a subset of S, which we will denote by S^e. The faces of K_S (of all dimensions) will have certain subsets of S^e as their extreme points, and will intersect S in certain subsets F. Such an F is characterized by the fact that there is a linear functional σ on $\mathbb{R}^n$ (not necessarily unique) such that $\sigma(s) \leq 1$ for all $s \in S$ and $F = \{s \in S : \sigma(s) = 1\}$. We call any such σ a *support functional* for F. Note that $|\sigma(s)| \leq 1$ for all $s \in S$. By abuse of terminology we will refer to F itself as a face of K_S, and we will not distinguish between F and the usual face which F determines.

LEMMA 8.1. *Let σ be a support functional for a face F of K_S. Then*

$$|\sigma(x)| \leq \ell(x)$$

for all $x \in G$.

Proof. Suppose that $x = \Sigma q(s)s$ for some function q from S to $\mathbb{Z}$. Then

$$\sigma(x) = \Sigma q(s)\sigma(s) \leq \Sigma|q(s)|.$$

On considering the minimum for all such q, we see that $\sigma(x) \leq \ell(x)$. But this holds for $-x$ as well, which gives the desired result. $\square$

Let F be a face of K_S. Any function γ from $\mathbb{Z}^+$ to G which consists of successively adding elements of F (i.e., $\gamma(n+1) - \gamma(n) \in F$ for $n \geq 0$) is a geodesic ray. In fact, for any support functional σ for F the above lemma tells us that we have $n \geq \ell(\gamma(n)) \geq \sigma(\gamma(n)) = n$. Since F is finite, some (perhaps all) elements of F will have to be added in an infinite number of times. One can see that if the order in which the elements of F are added-in is changed, but the number of times they ultimately appear is the same, then one obtains an equivalent geodesic ray. A class of such geodesic rays can be specified by a function on F which has values either in $\mathbb{Z}^+$ or $+\infty$. But it seems to be tricky to decide when two such functions (possibly for different faces) determine the same Busemann point. For our present purposes we do not need to concern ourselves with this issue. It is sufficient for us to associate a canonical geodesic ray to each face. This will be a special case of forming geodesic rays by successively adding elements of the semigroup generated by F (so that the domain of the ray may be a proper subset of $\mathbb{Z}$).

NOTATION 8.2. *For a face F of K_S set $z_f = \Sigma\{s : s \in F\}$, and let γ_F denote the geodesic ray whose domain is $|F|\mathbb{Z}^+$ (where $|F|$ denotes the number of elements of F) and which is defined by $\gamma(|F|n) = nz_F$. We denote by b_F the corresponding Busemann point. We denote by G_F the subgroup of G generated by F.*

Again Lemma 8.1 quickly shows that the above ray is geodesic. The following proposition is analogous to Proposition 6.2.

PROPOSITION 8.3. *Let σ be a support functional for a face F of K_S. For every $u \in G_F$ we have*

$$\varphi_u(b_F) = \sigma(u).$$

Proof. Since $u \in G_F$, there is a positive integer N such that whenever $n \geq N$ then $nz_F - u$ can be expressed as a sum of elements of F, so that $\ell(nz_F - u) = \sigma(nz_F - u)$. Of course $\ell(nz_F) = \sigma(nz_F)$. Thus for $n \geq N$

$$\varphi_u(nz_F) = \sigma(nz_F) - \sigma(nz_F - u) = \sigma(u).$$

□

Proposition 8.4. *Let F and σ be as above. For any $y \in G$ and $u \in G_F$ we have*

$$\varphi_{y+u}(b_F) = \varphi_y(b_F) + \sigma(u).$$

Proof. Consider the set of u's such that this equation holds for all $y \in G$. It is easy to verify that this set is a subsemigroup of G. But for u in this set we have

$$\varphi_{y-u}(\beta_F) = \varphi_{(y-u)+u}(\beta_F) - \sigma(u) = \varphi_y(\beta_F) + \sigma(-u),$$

so that this set is a group. It thus suffices to verify the above equation for each $u = s \in F$.

So let $s \in S$. Since $n \mapsto \varphi_y(nz_F)$ is integer-valued, non-decreasing by Lemma 4.5, and bounded, we can find a positive integer N such that

$$\varphi_y(b_F) = \ell((N+m)z_F) - \ell((N+m)z_F - y)$$

for all $m \geq 0$. We can find a larger N such that also

$$\varphi_{y+s}(b_F) = \ell((N+m)z_F) - \ell((N+m)z_F - (y+s))$$

for all $m \geq 0$. Since $\sigma(s) = 1$ it is then clear that we need to show that

$$\ell((N+m)z_F - (y+s)) = \ell((N+m)z_F - y) - 1$$

for some $m \geq 0$. Let $\bar{y} = y - Nz_F$. Then what we need becomes

$$\ell(mz_F - (\bar{y}+s)) = \ell(mz_F - \bar{y}) - 1$$

for some $m \geq 0$. Note that $\ell(mz_F) - \ell(mz_F - \bar{y})$ is independent of $m \geq 0$ because of our choice of N, and similarly for $\bar{y}+s$ instead of $\bar{y}$.

Since $S = -S$ and $0 \notin S$, we can find a subset, S^+, such that $S^+ \cup (-S^+) = S$ and $S^+ \cap (-S^+) = \emptyset$. Since $F \cap (-F) = \emptyset$, we can require that $F \subseteq S^+$. Index the elements of S^+ in such a way that $s_1 = s$, and $F = \{s_1, \ldots, s_{|F|}\}$, where $|F|$ denote the number of elements in F. Since S generates G, we can express $\bar{y}$ as $\bar{y} = \Sigma n_j s_j$ where $n_j \in \mathbb{Z}$ for each j. Then $\ell(\bar{y})$ will be the minimum of the sums $\Sigma|n_j|$ over all such expressions for $\bar{y}$. We make a specific choice of such a minimizing set $\{n_j\}$. (It need not be unique.)

Since $\ell(mz_F) = m|F|$ by Lemma 8.1, the stability described earlier says that $m|F| - \ell(mz_F - \bar{y})$ is independent of $m \geq 0$. We combine this for $m = 0$ and $m = 1$ to obtain $-\ell(-\bar{y}) = |F| - \ell(z_F - \bar{y})$. We use this to calculate

$$\begin{aligned} |F| \quad + \quad \Sigma|n_j| &= |F| + \ell(-\bar{y}) = \ell(z_F - \bar{y}) \\ &= \ell\left(\sum_{j \leq |F|}(1-n_j)s_j + \sum_{j > |F|} n_j s_j\right) \leq \sum_{j \leq |F|}|1-n_j| + \sum_{j>|F|}|n_j|. \end{aligned}$$

On comparing the two ends, we see that we must have $n_j \le 0$ for $j \le |F|$, and that the two ends must be equal. Thus

$$\ell(z_F - \bar{y}) = \sum_{j \le |F|} (1 - n_j) + \sum_{j > |F|} |n_j|.$$

Now

$$\begin{aligned} z_F - \bar{y} - s &= \sum_{j \le |F|} (1 - n_j) s_j + \sum_{j > |F|} n_j s_j - s_1 \\ &= -n_1 s_1 + \sum_2^{|F|} (1 - n_j) s_j + \sum_{j > |F|} n_j s_j. \end{aligned}$$

From the fact that $n_j \le 0$ for $j \le |F|$ it follows that

$$\begin{aligned} \ell(z_F - \bar{y} - s) &\le -n_1 + \sum_2^{|F|} (1 - n_j) + \sum_{j > |F|} |n_j| \\ &= -1 + \ell(z_F - \bar{y}). \end{aligned}$$

From the triangle inequality and the fact that $\ell(s) = 1$ it follows that

$$\ell(z_F - \bar{y} - s) = -1 + \ell(z_F - \bar{y}),$$

as needed. □

COROLLARY 8.5. *For any $y, z \in G$ and any $u \in G_F$, and for any support functional σ for F, we have*

$$\varphi_{y+u}(\alpha_z(b_F)) = \varphi_y(\alpha_z(b_F)) + \sigma(u).$$

Proof. Using the 1-cocycle identity 2.2 and Proposition 8.4 we obtain

$$\begin{aligned} \varphi_{y+u}(\alpha_z(b_F)) &= (\alpha_{-z}\varphi_{y+u})(b_F) = \varphi_{y-z+u}(b_F) - \varphi_{-z}(b_F) \\ &= \varphi_{y-z}(b_F) + \sigma(u) - \varphi_{-z}(b_F) \\ &= (\alpha_{-z}\varphi_y)(b_F) + \sigma(u) = \varphi_y(\alpha_z(b_F)) + \sigma(u). \end{aligned}$$

□

PROPOSITION 8.6. *Let F be a face of K. For each $u \in G_F$ the homeomorphism α_u of $\bar{G}^\ell$ leaves fixed each point of the α-orbit of b_F. That is, for each $z \in G$ we have*

$$\alpha_u(\alpha_z(b_F)) = \alpha_z(b_F).$$

Proof. Because G is Abelian, it suffices to show that $\alpha_u(b_F) = b_F$. For this we must verify that $f(\alpha_u(b_F)) = f(b_F)$ for all $f \in C(\bar{G}^\ell)$. It suffices to verify this for $f = \varphi_y$ for each $y \in G$. But from the 1-cocycle identity 2.2 and Proposition 8.4 we have

$$\begin{aligned} \varphi_y(\alpha_u(b_F)) &= (\alpha_{-u}\varphi_y)(b_F) = \varphi_{y-u}(b_F) - \varphi_{-u}(b_F) \\ &= \varphi_y(b_F) + \sigma(-u) + \sigma(u) = \varphi_y(b_F). \end{aligned}$$

□

We will also need the following fact:

Proposition 8.7. *If* $y \notin G_F$ *then* φ_y *is not constant on the* G*-orbit of* b_F*, and in fact there is an* $s \in S$ *such that* $s \notin F$ *and*

$$\varphi_y(\alpha_s(b_F)) = \varphi_y(b_F) + (1 - \varphi_{-s}(b_F)),$$

with $\varphi_{-s}(b_F) = 0$ *or* -1.

Proof. Let S^+ and the indexing $\{s_j\}$ be as in the proof of Proposition 8.4. Much as in that proof, we can find a large enough N that $\varphi_{y \pm s_j}((N+m)z_F)$ is constant for $m \geq 0$ for all $\pm s_j$ simultaneously, as is $\varphi_y((N+m)z_F)$. Set $\bar{y} = y - Nz_F$. For this $\bar{y}$ choose $\{n_j\}$ as before so that $\bar{y} = \Sigma n_j s_j$ and $\ell(\bar{y}) = \Sigma |n_j|$. Since $y \notin G_F$, also $\bar{y} \notin G_F$, and so there is a $k > |F|$ such that $n_k \neq 0$. Suppose that $n_k \geq 1$. Then

$$\bar{y} - s_k = \sum_{j \neq k} n_j s_j + (n_k - 1)s_k,$$

so that

$$\ell(\bar{y} - s_k) \leq \sum_{j \neq k} |n_j| + n_k - 1 = \ell(\bar{y}) - 1.$$

From the triangle inequality we then obtain $\ell(\bar{y} - s_k) = \ell(\bar{y}) - 1$, that is,

$$\ell(Nz_F - y + s_k) = \ell(Nz_F - y) - 1.$$

From our choice of N (and with $m = 0$) we then get

$$\begin{aligned} \varphi_{y-s_k}(b_F) &= \varphi_{y-s_k}(Nz_F) \\ &= N|F| - \ell(Nz_F - y + s_k) = N|F| - \ell(N|F| - y) + 1 \\ &= \varphi_y(b_F) + 1. \end{aligned}$$

We combine this with the 1-cocycle identity 2.2 to obtain

$$\begin{aligned} \varphi_y(\alpha_{s_k}(b_F)) &= (\alpha_{-s_k}\varphi_y)(b_F) \\ &= \varphi_{y-s_k}(b_F) - \varphi_{-s_k}(b_F) \\ &= \varphi_y(b_F) + (1 - \varphi_{-s_k}(b_F)). \end{aligned}$$

Since φ_{-s_k} takes only the values 0, ± 1, the desired conclusion is then obtained from:

Lemma 8.8. *If* $s \in S$ *and* $\varphi_s(b_F) = 1$ *then* $s \in F$.

Proof. If $\varphi_s(b_F) = 1$, then for large n, and for a support functional σ for F, we have

$$\begin{aligned} n|F| - 1 &= \ell(nz_F) - 1 = \ell(nz_F - s) \\ &\geq \sigma(nz_F - s) = n|F| - \sigma(s), \end{aligned}$$

so that $1 \leq \sigma(s)$, and so $s \in F$. $\square$

The above argument for the proof of Proposition 8.7 was under the assumption that $n_k \geq 1$. If instead we have $n_k \leq -1$, then we carry out a similar argument using $-s_k$ instead of s_k. This concludes the proof of Proposition 8.7. $\square$

9. Word-length functions give Lip-norms on $C^*(\mathbb{Z}^d, c)$

We will now see how to use the results of the previous section to prove the part of our Main Theorem 0.1 concerning word-length functions. We use the notation of the previous section, and in particular, the norm $\|\cdot\|_S$ determined by $K = K_S$. Here we will consider the (proper) faces of K of maximal dimension, namely of dimension $d-1$. We will call them "facets" of K, as is not infrequently done. The interior points of the facets are the smooth points of the unit sphere for $\|\cdot\|_S$. Again our terminology and notation will not distinguish between facets as intersections of K with hyperplanes, and as the corresponding subsets of S. Because K has only a finite number of extreme points, every point of the boundary of K is contained in at least one facet, and there are only a finite number of facets. Each facet F has a unique support functional, which we denote by σ_F. Furthermore, F contains a basis for $\mathbb{R}^d$, and consequently G_F is of finite index in G. This has the crucial consequence for us that the orbit, $\mathcal{O}_F$, of b_F in $\partial_\ell G$ under the action α, is finite. (Apply Proposition 8.6.) We consider the restriction map from $C(\partial_\ell G)$ onto $C(\mathcal{O}_F)$. Since it is α-equivariant, it gives an algebra homomorphism, Π_F, from $C^*(G, C(\partial_\ell G), \alpha, c)$ onto $C^*(G, C(\mathcal{O}_F), \alpha, c)$. If we let π and M denote also the corresponding homomorphisms of G and $C(\mathcal{O}_F)$ into this latter algebra, and if for each $y \in G$ we let ψ_y denote the restriction of φ_y to $\mathcal{O}_F$, then

$$\Pi_F([M_\ell, \pi_f]) = \Sigma f(y) M_{\psi_y} \pi_y.$$

Let Q be a set of coset representatives for G_F in G containing 0. Then we can express the above as

$$\Sigma_{q \in Q}(\Sigma_{u \in G_F} f(u+q) M_{\psi_{u+q}} \bar{c}(u,q) \pi_u) \pi_q.$$

From Corollary 8.5 we see that $\psi_{u+q} = \psi_q + \sigma_F(u)$. For each q let g^q be the function on G_F defined by $g^q(u) = f(u+q)\bar{c}(u,q)$. We can also view g^q as a function on G by giving it value 0 off G_F. Then we can rewrite our previous expression for $\Pi_F([M_\ell, \pi_f])$ as

$$\Sigma_q(\Sigma_u g^q(u)(\sigma_F(u) + M_{\psi_q})\pi_u)\pi_q.$$

As in Section 7 let $\hat{G} = \mathbb{T}^d$ be the dual group of G, and denote the pairing between G and $\hat{G}$ by $\langle x, s\rangle$. Let β now denote the usual dual action of $\hat{G}$ on $C^*(G, C(\mathcal{O}_F), \alpha, c)$, so that

$$\beta_s(M_\psi \pi_x) = \langle x, s\rangle M_\psi \pi_x.$$

Then the finite group $(G/G_F)^\wedge$ can be identified with the set of characters on G which take value 1 on G_F. We can thus restrict β to $(G/G_F)^\wedge$ and average over $(G/G_F)^\wedge$. This gives a projection of norm 1 onto the subalgebra of elements supported on G_F, and this projection on functions on G is just restriction of functions to G_F. If for each fixed q we apply this projection to the product with π_q^* of the above expression for $\Pi_F([M_\ell, \pi_f])$, we find that

$$\|[M_\ell, \pi_f]\| \geq \|\Sigma_u g^q(u)(\sigma_F(u) + M_{\psi_q})\pi_u\|$$

for each q. The norm on the right is that of $C^*(G, C(\mathcal{O}_F), \alpha, c)$. But section 2.27 of [47] tells us that $C^*(G_F, C(\mathcal{O}_F), \alpha, c)$ is a C^*-subalgebra of $C^*(G, C(\mathcal{O}_F), \alpha, c)$ under the evident identification of functions. Thus we can view the operator on the right as being in $C^*(G_F, C(\mathcal{O}_F), \alpha, c)$, where we are here restricting α and c to G_F. But from Proposition 8.6 we know that the action α of G_F on $\mathcal{O}_F$ is trivial. Thus we have the decomposition

$$C^*(G_F, C(\mathcal{O}_F), \alpha, c) \cong C(\mathcal{O}_F) \otimes C^*(G_F, c).$$

Let $a_q = \Sigma g^q(u)\pi_u$ and $b_q = \Sigma g^q(u)\sigma_F(u)\pi_u$. Then in terms of the above decomposition we are looking at $I \otimes b_q + \psi_q \otimes a_q$. From Proposition 8.7 we know that ψ_q is not constant on $\mathcal{O}_F$ for $q \neq 0$. Note that $\psi_0 \equiv 0$. For given $q \neq 0$ let m_j for $j = 1, 2$ be two distinct values of ψ_q. Upon evaluating at the points where ψ_q takes these values, and using our earlier inequality, we see that

$$\|b_q + m_j a_q\| \leq \|[M_\ell, \pi_f]\| = L_\ell(f)$$

for $j = 1, 2$. Upon writing the inequalities as

$$\|m_j^{-1} b_q + a_q\| \leq |m_j|^{-1} L_\ell(f)$$

and using the triangle inequality to eliminate a_q, and simplifying, we find that

$$\|b_q\| \leq (|m_1| + |m_2|)/|m_1 - m_2| L_\ell(f).$$

(If either m_j is 0 the path is simpler.) Of course m_1 and m_2 depend on q. Thus we see that we have found a constant, k_q, such that $\|b_q\| \leq k_q L_\ell(f)$. For $q = 0$ we have the same inequality with $k_0 = 1$ since $\psi_0 = 0$. Much as in Section 7 set $X_F f = \Sigma \sigma_F(x) f(x) \pi_x$. Then

$$\begin{aligned} X_F f &= \Sigma \sigma_F(x) f(x) \pi_x = \Sigma_q (\Sigma_{u \in G_F} \sigma_F(u+q) f(u+q) \bar{c}(u,q) \pi_u) \pi_q \\ &= \Sigma_q \sigma_F(q) (\Sigma_u \sigma_F(u) g^q(u) \pi_u) \pi_q = \Sigma \sigma_F(q) b_q \pi_q. \end{aligned}$$

When we combine this with the inequality obtained earlier for $\|b_q\|$, we obtain

$$\|X_F f\| \leq (\Sigma |\sigma_F(q)| k_q) L_\ell(f).$$

Observe that the $\sigma_F(q)$'s and k_q's do not depend on f, but only on F and the choice Q of coset representatives. Thus for each facet F we have obtained a constant, k_F, such that

$$\|X_F f\| \leq k_F L_\ell(f)$$

for all $f \in C_c(G)$. Note that knowing that k_F is finite is the crucial place where we use that the number of coset representatives in Q is finite.

Just as toward the end of Section 7, we have the dual action β of $\mathbb{T}^d$ on $C^*(G, c)$, and the corresponding differential df of any $f \in C_c(G)$, such that $df(\sigma_F) = iX_F f$. Then our inequality above gives, much as in Section 7,

$$\|df(\sigma_F)\| \leq k_F L_\ell(f).$$

Recall now the norm $\|\cdot\|_S$ determined by $K = K_S$. The σ_F's are exactly the support functionals corresponding to the smooth points of the unit sphere for $\|\cdot\|_S$. Let $\|df\|_S$ denote the norm of the linear map df using the dual norm

$\|\cdot\|'_S$. Also let $k = \max\{k_F : F \text{ is a facet}\}$. Then from Proposition 6.7 we conclude, much as in Section 7, that

$$\|df\|_S \leq kL_\ell(f).$$

Then just as in Section 7 we conclude that L_ℓ is a Lip-norm. This concludes the proof of Main Theorem 0.1.

□

Since the norm $\|\cdot\|'_S$ on V' does not come from an inner product, and V' can be thought of as the analogue of the tangent space at the non-existent points of the quantum space $C^*(G,c)$, we can consider that we have here a non-commutative Finsler geometry (as also in section 3 of [39]). The metric geometry from L_ℓ also, in a vague way, seems Finsler-like.

I imagine that the above considerations can be generalized so that the Main Theorem can be extended to weighted-word-length functions, where each generator has been assigned a weight. I imagine that they can also be generalized to deal with extensions of $\mathbb{Z}^d$ by finite groups. But I have not explored these possibilities.

Since our estimates for the proof of the Main Theorem depend just on the behavior of the φ_y's on the boundary, the conclusions of the Main Theorem will also be valid if ℓ is replaced by the translation-bounded function $\ell + h$ where h is any function in $C_\infty(\mathbb{Z}^d)$.

10. The free group

We briefly discuss here how the ideas developed earlier apply to the free (non-Abelian) group on two generators, $G = F_2$. Denote the two generators by a and b, and take them and their inverses as our generating set S. Let ℓ denote the corresponding length function. It is well-known [21] that F_2 is a hyperbolic group, and that its Gromov boundary, $\partial_h G$, is described as the set of all infinite (to the right) reduced words in the elements of S. (The "h" in $\partial_h G$ is for "hyperbolic"—it does not denote a length function.) The action of G on $\partial_h G$ is the evident one by "left concatenation" (and then reduction). We can obtain the topology of $\partial_h G$ and of the compactification of G as follows. (See comment ii) on page 104 of [21].) To include the elements of G we need a "stop" symbol. We denote it by p. We let S' denote S with p added, and we let $\prod^\infty S'$ denote the set of sequences with values in S', with its compact topology of "index-wise" convergence.

Notation 10.1. *Let $\bar{G}^h$ be the subset of $\prod^\infty S'$ consisting of all sequences such that*

1) *If p occurs in the sequence then all subsequent letters in that sequence are p.*
2) *The sequence is reduced, in the sense that a and a^{-1} are never adjacent entries, and similarly for b and b^{-1}.*

It is easily seen that $\bar{G}^h$ is a closed subset of $\prod^\infty S'$, so compact. We identify the elements of G with the words containing p (and in particular, we identify the identity element of G with the constant sequence with value p). With this understanding, it is easily seen that G is an open dense subset of $\bar{G}^h$. We identify $\partial_h G$ with the infinite words which do not contain p.

The group G again acts on $\bar{G}^h$ by left concatenation. It is easily seen that this action is by homeomorphisms. Consider the function φ_a on G. For any word w we have $\ell(a^{-1}w) = \ell(w)+1$ if w begins with the letters a^{-1}, b or b^{-1}, or is the identity element, while $\ell(a^{-1}w) = \ell(w)-1$ if w begins with the letter a. Thus $\varphi_a(w) = \ell(w) - \ell(a^{-1}w)$ has value 1 if w begins with the letter a, and value -1 otherwise. But we can extend φ_a to $\bar{G}^h$ by exactly this same prescription, and it is easily seen that this extended φ_a is continuous on $\bar{G}^h$. We do the same with φ_b, $\varphi_{a^{-1}}$ and $\varphi_{b^{-1}}$. By using the 1-cocycle identity 2.2 inductively, we see that each φ_x for $x \in G$ extends to a continuous function on $\bar{G}^h$ (in a unique way since G is dense). Of course the functions in $C_\infty(G)$ extend by giving them value 0 on $\partial_h G$, and the constant functions also extend. In this way we identify $C(\bar{G}^\ell)$ with a unital subalgebra of $C(\bar{G}^h)$.

Let us see now that the subalgebra $C(\bar{G}^\ell)$ separates the points of $\bar{G}^h$. Because the subalgebra contains $C_\infty(G)$, it is clear that we only need to treat the points of $\partial_h G$. Let $v, w \in \partial_h G$ with $v \neq w$. Then there must be a first entry where they differ. That is, we can write them as $v = x\tilde{v}$, $w = x\tilde{w}$ where x is a finite word while $\tilde{v}$ and $\tilde{w}$ differ in their first entry. Suppose the first entry of $\tilde{v}$ is a while the first entry of $\tilde{w}$ is not a. Then from what we saw above

$$(\alpha_x\varphi_a)(v) = \varphi_a(x^{-1}v) = \varphi_a(\tilde{v}) = 1,$$

while in the same way $(\alpha_x\varphi_a)(w) = -1$. Thus the subalgebra $C(\bar{G}^\ell)$ separates the points of $\bar{G}^h$, and so by the Stone–Weierstrass theorem $C(\bar{G}^\ell) = C(\bar{G}^h)$, so that $\bar{G}^\ell = \bar{G}^h$. Thus in this case the metric and hyperbolic boundaries coincide. (The referee has pointed out that if instead we take as generating set $\{a^{\pm 1}, a^{\pm 2}, b^{\pm 1}\}$, then the resulting metric compactification will be different from that described above, because just as in Example 5.2 we will obtain two "parallel" geodesic rays, namely $(e, a^2, a^4, \dots)$ and $(a, a^3, a^5, \dots)$, which will give different Busemann points.)

Each $w \in \partial_h G$ specifies a unique geodesic ray to it from e, namely $e, w_1, w_1w_2, w_1w_2w_3, \dots$. Thus every point of $\partial_\ell G$ is a Busemann point. It is well-known [2] that the action of G on $\partial_h G$ is amenable. If one uses the definition of amenability in terms of maps from $\partial_h G$ to probability measures on G which was stated in Section 3, then this is seen by letting the n-th map, m_n, be the map which assigns to $w \in \partial_h G$ the probability measure which gives mass $1/n$ to the first n points of the geodesic ray from e to w [2]. In view of Theorem 3.7 this implies that the cosphere algebra $S^*_\ell A$ for the spectral triple $(A = C^*_r(F_2), \boldsymbol{\ell}^2(F_2), M_\ell)$ is $C^*(G, C(\partial_h F_2), \alpha)$.

However, the action α on $\partial_h F_2$ does not have any finite orbits, and so I do not see how to continue along the lines of the previous section to determine whether the metric on the state space $S(C^*_r(F_2))$ coming from the above spectral triple gives the state space the weak-$*$ topology, or even just finite diameter. The difficulty remains: What information can one obtain about $\|\pi_f\|$ if one knows that $\|[M_\ell, \pi_f]\| \leq 1$?

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