The Connection Between May's Axioms for a Triangulated Tensor Product and Happel's Description of the Derived Category of the Quiver D_4

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Abstract. In an important recent paper [12], May gave an axiomatic description of the properties of triangulated categories with a symmetric tensor product. The main point of the current article is that there are two other results in the literature which can be used to shed considerable light on May's work. The first is a construction of Verdier's, which appeared in Beilinson, Bernstein and Deligne's [4, Prop. 1.1.11, pp. 24-25]. The second and more important is the beautiful work of Happel, in [9], which can be used to better organise May's axioms.

Keywords and Phrases: derived category, tensor product, quiver

1. Introduction

We should begin with a disclaimer. This article definitely *does not* attempt to give the definitive axiomatic description of tensor products in triangulated categories. In the opinion of the authors, the subject is not ripe for such a treatment. It is only very recently that there has been any real interest in the field. The subject is still at a very formative stage. Time will tell which properties of the tensor product really matter.

Let $\mathcal{T}$ be a triangulated category, and assume it has a (symmetric) tensor product. For example, $\mathcal{T}$ might be the derived category of a commutative ring R, or the homotopy category of spectra. It becomes interesting to know what are the "natural" properties that this tensor product has. In a lovely recent article [12], May made giant steps towards answering this question.

Some properties are obvious, and we do not repeat them here. The interest lies in the following. Given two distinguished triangles

$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$$

$$x' \longrightarrow y' \longrightarrow z' \longrightarrow \Sigma x'$$

one can form the tensor product

$$\begin{tikzcd}
x \otimes x' \arrow[r] \arrow[d] & y \otimes x' \arrow[r] \arrow[d] & z \otimes x' \arrow[r] \arrow[d] & \Sigma x \otimes x' \arrow[d] \\
x \otimes y' \arrow[r] \arrow[d] & y \otimes y' \arrow[r] \arrow[d] & z \otimes y' \arrow[r] \arrow[d] & \Sigma x \otimes y' \arrow[d] \\
x \otimes z' \arrow[r] \arrow[d] & y \otimes z' \arrow[r] \arrow[d] & z \otimes z' \arrow[r] \arrow[d] \arrow[dr, phantom, "(-)"] & \Sigma x \otimes z' \arrow[d] \\
\Sigma x \otimes x' \arrow[r] & \Sigma y \otimes x' \arrow[r] & \Sigma z \otimes x' \arrow[r] & \Sigma^2 x \otimes x'
\end{tikzcd}$$

It is natural to assume that the rows and columns are distinguished triangles. The question is what, if any, are the other reasonable properties one could postulate. It turns out that, at least for reasonable examples of triangulated categories $\mathcal{T}$ with tensor products, in this diagram the diagonal arrows

$$\begin{tikzcd}
x \otimes x' \arrow[r] \arrow[d] & y \otimes x' \arrow[r] \arrow[d] & z \otimes x' \arrow[r] \arrow[d] \arrow[dr] & \Sigma x \otimes x' \arrow[d] \\
x \otimes y' \arrow[r] \arrow[d] & y \otimes y' \arrow[r] \arrow[d] \arrow[dr] & z \otimes y' \arrow[r] \arrow[d] & \Sigma x \otimes y' \arrow[d] \\
x \otimes z' \arrow[r] \arrow[d] \arrow[dr] & y \otimes z' \arrow[r] \arrow[d] & z \otimes z' \arrow[r] \arrow[d] \arrow[dr, phantom, "(-)"] & \Sigma x \otimes z' \arrow[d] \\
\Sigma x \otimes x' \arrow[r] & \Sigma y \otimes x' \arrow[r] & \Sigma z \otimes x' \arrow[r] & \Sigma^2 x \otimes x'
\end{tikzcd}$$

all have a common mapping cone. May's axiom (TC3) describes very well the various diagrams involving this common mapping cone.

Of course, we could look at other diagonal arrows. In the diagram below

$$\begin{tikzcd}
x \otimes x' \arrow[r] \arrow[d] & y \otimes x' \arrow[r] \arrow[d] \arrow[dr, squiggly] & z \otimes x' \arrow[r] \arrow[d] \arrow[dr] & \Sigma x \otimes x' \arrow[d] \\
x \otimes y' \arrow[r] \arrow[d] \arrow[dr, squiggly] & y \otimes y' \arrow[r] \arrow[d] \arrow[dr] & z \otimes y' \arrow[r] \arrow[d] \arrow[dr, dashed] & \Sigma x \otimes y' \arrow[d] \\
x \otimes z' \arrow[r] \arrow[d] \arrow[dr] & y \otimes z' \arrow[r] \arrow[d] \arrow[dr, dashed] & z \otimes z' \arrow[r] \arrow[d] \arrow[dr, phantom, "(-)"] & \Sigma x \otimes z' \arrow[d] \\
\Sigma x \otimes x' \arrow[r] & \Sigma y \otimes x' \arrow[r] & \Sigma z \otimes x' \arrow[r] & \Sigma^2 x \otimes x'
\end{tikzcd}$$

the squiggly arrows have a common mapping cone Σu, the straight arrows a common mapping cone Σv and the broken arrows a common mapping cone Σw. It becomes interesting to describe what relations there should be among u, v and w. It turns out that there are many. May's axioms include one such relation.

In this paper, we will see that May's results are related to earlier work by Verdier and by Happel. We will show that the older approaches lead to new insights; in terms of the above, in general, they lead to infinitely many relations among u, v and w which May missed. We will see that the work of Happel is particularly illuminating.

As we have already said, we do not see this as an attempt to give the definitive foundational treatment. The subject is very young and active. Aside from May's paper there is the totally unrelated work by Balmer [3], and recent talks by Gaitsgory (no manuscript yet) show that his work is also related. At this point, all we want is to advertise widely the fact that the results of Verdier, Happel and May (in chronological order) are related.

Of course, we must also persuade the reader that this relation, among three existing articles in the literature, is interesting. Of most interest is how Happel's work in [9] leads to a better organisation of the theory. To illustrate this, we give examples of new results that can be obtained. We make no attempt to prove the best possible versions of these new results. That is not the point. We settle for weaker-than-optimal statements of our new results, to make transparent how they can be viewed as consequences of Happel's work.

Since we want the article to be accessible to a wide audience, we try not to assume much background knowledge. The experts in representations of quivers will undoubtedly find Section 5 painfully slow and detailed. The experts in topology will undoubtedly wonder why we assume the reader may never have heard of closed model categories. The guiding policy in writing this article was that the presentation should be as free of prerequisites as possible. The unfortunate side effect is that it adds to the length of the article. We ask the experts to be patient with us.

The structure of the article is as follows. Section 2 sets some notation. Section 3 establishes the relation among the three approaches. Sections 4 and 5 apply these to obtain new identities. The main results are

(i) Axiom (TC3) of May produces exactly the same diagram as Verdier found in [4]. (See Theorem 3.5). Haynes Miller noticed this independently.
(ii) The special case of $D^b(\square)$ is universal (Theorem 3.10).
(iii) May's axiom (TC4) follows from (TC3) and the octahedral axiom (Theorem 4.1).
(iv) There is an equivalence of categories $D^b(\square) = D^b(\mathsf{Y})$, where $D^b(\mathsf{Y})$ is the bounded derived category of the category of representations of the quiver D_4. Happel studied this in the special case where the categories are all linear over a field k. In the case of k-linear categories we can therefore glean a great deal of information from Happel's work (Section 5).

2. Notation for the octahedral axiom

An octahedron can be thought of as two pyramids glued together along their square bases. There are three planes along which we can split the octahedron into two pyramids. This gives three squares. For octahedra as in the octahedral axiom, each edge is a morphism in a triangulated category and has a direction. The four squares in an octahedron have arrows as follows

$$\begin{tikzcd} {} \arrow[r] \arrow[dr, phantom, "(1)"] & {} \arrow[d] \\ {} \arrow[u] & {} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

The convention we adopt is to always write the octahedron as a union of two pyramids, split along the unique square where the arrows cycle around as in (1) above. The octahedron splits into a "top pyramid" and a "bottom pyramid" (of course, it is somewhat arbitrary which pyramid is declared to be "top" and which "bottom").

If we project the top and bottom pyramids to their common base plane, we get diagrams

$$\begin{tikzcd} & a \arrow[d] \arrow[dr] & \\ d \arrow[ur] & x \arrow[l] \arrow[r] & b \arrow[dl] \\ & c \arrow[u] \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dr] & \\ d \arrow[ur] \arrow[r] & y \arrow[u] \arrow[d] & b \arrow[l] \arrow[dl] \\ & c \arrow[ul] & \end{tikzcd}$$

It turns out to be very convenient to twist and torture the octahedron. We wish to switch the positions of c and d. The pyramids become

$$
\begin{tikzcd}
& a \arrow[d] \arrow[dr] & \\
c \arrow[r] \arrow[dr] & x \arrow[r] \arrow[d] & b \\
& d &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& a \arrow[dr] & \\
c \arrow[dr] & y \arrow[l] \arrow[u] & b \arrow[l] \\
& d \arrow[u] &
\end{tikzcd}
$$

Of course, it now takes some imagination to see that these are pyramids. There are still four triangles to each pyramid; but two of them project to straight lines (the horizontal and vertical lines). We will frequently write our octahedra in this contorted form.

The octahedral axiom tells us that, in a triangulated category, certain diagrams can be completed to octahedra. The refined octahedral axiom tells us that the two commutative squares, which arise from the two "other" planes splitting the octahedron into pyramids, are homotopy pushouts. In the notation above, there are canonical distinguished triangles

$$x \longrightarrow b \oplus d \longrightarrow y \longrightarrow \Sigma x$$

$$y \longrightarrow a \oplus c \longrightarrow x \longrightarrow \Sigma y.$$

There is a choice of sign here which we do not wish to make explicit, and some of the morphisms are of degree 1. The important thing is that the maps $y \longrightarrow \Sigma x$ and $x \longrightarrow \Sigma y$ are very explicitly given by the octahedron. We will refer to them as the *differentials of the squares.*

3. The relation among the approaches

Suppose $\mathcal{T}$ is a triangulated category with a tensor product. We wish to study when this tensor product is well-behaved. To this end, we make a definition.

Definition 3.1. *We say that the tensor product on $\mathcal{T}$ is* decent *if the following holds.*

- (i) *There exists an abelian category $\mathcal{A}$ and a triangulated functor $F : D^b(\mathcal{A}) \longrightarrow \mathcal{T}$. [Here $D^b(\mathcal{A})$ means the bounded derived category of $\mathcal{A}$.]*
- (ii) *The category $\mathcal{A}$ has a natural tensor product.*
- (iii) *The category $\mathcal{A}$ comes with a collection of* special short exact sequences. *These form a subclass of all the short exact sequences.*
- (iv) *For any special short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and any $X \in \mathcal{A}$, the two sequences

$$0 \longrightarrow X \otimes A \longrightarrow X \otimes B \longrightarrow X \otimes C \longrightarrow 0$$

$$0 \longrightarrow A \otimes X \longrightarrow B \otimes X \longrightarrow C \otimes X \longrightarrow 0$$

are both exact.

(v) *Any distinguished triangle in $\mathcal{T}$ is isomorphic to*

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow \Sigma F(A)$$

for some special short exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

(vi) *With the notation as in (iv) and (v) above, the two triangles*

$$F(X \otimes A) \longrightarrow F(X \otimes B) \longrightarrow F(X \otimes C) \longrightarrow \Sigma F(X \otimes A)$$

$$F(A \otimes X) \longrightarrow F(B \otimes X) \longrightarrow F(C \otimes X) \longrightarrow \Sigma F(A \otimes X)$$

are canonically independent of the choice of the special short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

lifting the triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow \Sigma F(A).$$

(vii) *Suppose X and Y are objects of $\mathcal{A}$, which occur in some special short exact sequences. Then there is a canonical isomorphism*

$$F(X) \otimes F(Y) \quad = \quad F(X \otimes Y).$$

Example 3.2. Let R be a commutative ring and let $\mathcal{T} = D^-(R)$, the derived category of bounded-above chain complexes of R–modules. The tensor product on $\mathcal{T}$ is the derived tensor product. The category $\mathcal{A}$ is defined to be the abelian category of bounded-above chain complexes of R–modules, with the obvious tensor product. The special short exact sequences are the short exact sequences of bounded-above chain complexes of projectives.

Roughly speaking, the idea in May's article [12] is to study decent tensor products in triangulated categories. The existence of the abelian category $\mathcal{A}$ and $F : D^b(\mathcal{A}) \longrightarrow \mathcal{T}$ has many consequences, allowing us to create complicated diagrams in $\mathcal{T}$. What May does is postulate the existence of the diagrams in $\mathcal{T}$ as axioms for the tensor product, even in the absence of any explicit $F : D^b(\mathcal{A}) \longrightarrow \mathcal{T}$.

Remark 3.3. What we said above is slightly inaccurate. May handles a more general framework. Instead of a functor $F : D^b(\mathcal{A}) \longrightarrow \mathcal{T}$, he assumes only that $\mathcal{T}$ has a closed model structure with a compatible tensor product. Since we do not want to assume the reader knows what a closed model is, we have allowed ourselves to restrict to the simplified situation.

Suppose $\mathcal{T}$ is a triangulated category with a decent tensor product. Suppose we are given two distinguished triangles in $\mathcal{T}$. By Definition 3.1(v) these two triangles are the images under F of two special short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

The tensor product in $\mathcal{A}$ gives a 3×3 diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \otimes A' \arrow[r] \arrow[d] & B \otimes A' \arrow[r] \arrow[d] & C \otimes A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes B' \arrow[r] \arrow[d] & B \otimes B' \arrow[r] \arrow[d] & C \otimes B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes C' \arrow[r] \arrow[d] & B \otimes C' \arrow[r] \arrow[d] & C \otimes C' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The central idea in May's [12] is to write down all the distinguished triangles one can deduce from this diagram. It might be simplest to focus on one of May's results. From now until the end of the section, we consider (TC3).
Let X be the quotient of the injective map $A \otimes A' \longrightarrow B \otimes B'$. Then we have two diagrams with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \otimes A' \arrow[r] \arrow[d, "1"'] & B \otimes A' \arrow[r] \arrow[d] & C \otimes A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes A' \arrow[r] \arrow[d] & B \otimes B' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & B \otimes C' \arrow[r, "1"] \arrow[d] & B \otimes C' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A\otimes A' \arrow[r, "1"] \arrow[d] & A\otimes A' \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A\otimes B' \arrow[r] \arrow[d] & B\otimes B' \arrow[r] \arrow[d] & C\otimes B' \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A\otimes C' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C\otimes B' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The right column of the first of these diagrams and the bottom row of the second exhibit $F(X)$ as the mapping cone on two more maps, namely

$$\Sigma^{-1}F(B\otimes C') \longrightarrow F(C\otimes A')$$

and

$$\Sigma^{-1}F(C\otimes B') \longrightarrow F(A\otimes C').$$

In other words, the diagonal arrows in the following diagram

$$\begin{tikzcd}
 & & \Sigma^{-1}F(B\otimes C') \arrow[r] \arrow[d] \arrow[dr] & \Sigma^{-1}F(C\otimes C') \arrow[d] \\
 & F(A\otimes A') \arrow[r] \arrow[d] \arrow[dr] & F(B\otimes A') \arrow[r] \arrow[d] & F(C\otimes A') \arrow[d] \\
\Sigma^{-1}F(C\otimes B') \arrow[r] \arrow[d] \arrow[dr] & F(A\otimes B') \arrow[r] \arrow[d] & F(B\otimes B') \arrow[r] \arrow[d] & F(C\otimes B') \arrow[d] \\
\Sigma^{-1}F(C\otimes C') \arrow[r] & F(A\otimes C') \arrow[r] & F(B\otimes C') \arrow[r] & F(C\otimes C')
\end{tikzcd}$$

all have the common mapping cone $F(X)$. Playing around a little gives yet more distinguished triangles. Instead of following May's approach (which the reader can find in [12]), let us see how Verdier found them.

As we have explained, May's approach is based on beginning with a 3×3 diagram in an abelian category $\mathcal{A}$, with exact rows and columns, and reading off induced triangles in the derived category of $\mathcal{A}$. To simplify the notation, let us forget that the diagram arose from a tensor product of two short exact sequences in $\mathcal{A}$. We have a diagram in $\mathcal{A}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x' \arrow[r] \arrow[d] & y' \arrow[r] \arrow[d] & z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x'' \arrow[r] \arrow[d] & y'' \arrow[r] \arrow[d] & z'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. It is well known that, for a diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x' \arrow[r] \arrow[d] & y' \arrow[r] \arrow[d] & z' \arrow[r] & 0 \\
 & x'' \arrow[r] \arrow[d] & y'' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

one has:

Lemma 3.4. *the following assertions are equivalent:*

(i) *The map $z \longrightarrow z'$ is a monomorphism.*
(ii) *The map $x'' \longrightarrow y''$ is a monomorphism.*
(iii) *The map from the pushout of*

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \\
x' &
\end{tikzcd}$$

to y' is a monomorphism.

The entire 3×3 diagram is, up to canonical isomorphism, entirely determined by the commutative square of monomorphisms

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd}$$

satisfying the condition in Lemma 3.4(iii). Verdier's idea was to build the entire diagram using repeated applications of the octahedral axiom. We remind the reader.

Our commutative square may be viewed as two commutative triangles

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd}$$

The two commutative triangles may be completed to two octahedra. In the twisted notation of Section 2, the top pyramids of the octahedra may be written as

$$\begin{tikzcd} & x \arrow[d] \arrow[dr] & \\ \Sigma^{-1}z' \arrow[r] \arrow[dr] & x' \arrow[r] \arrow[d] & y' \\ & x'' & \end{tikzcd} \qquad \begin{tikzcd} & \Sigma^{-1}y'' \arrow[d] \arrow[dr] & \\ x \arrow[r] \arrow[dr] & y \arrow[r] \arrow[d] & z \\ & y' & \end{tikzcd}$$

We remind the reader how this should be read. The horizontal and vertical lines are projections of distinguished triangles. We have, so far, four distinguished triangles. They are the rows and columns below

$$
\begin{tikzcd}
& & \Sigma^{-1}y'' \arrow[d] & & \\
& x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] & \Sigma x \\
\Sigma^{-1}z' \arrow[r] & x' \arrow[r] \arrow[d] & y' \arrow[r] \arrow[d] & z' & \\
& x'' \arrow[d] & y'' & & \\
& \Sigma x & & &
\end{tikzcd}
$$

We also have four commutative triangles, as below

$$
\begin{tikzcd}
& & \Sigma^{-1}y'' \arrow[d] \arrow[dr] & \\
& x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[r] \arrow[d] & z \\
\Sigma^{-1}z' \arrow[r] \arrow[dr] & x' \arrow[r] \arrow[d] & y' & \\
& x'' & &
\end{tikzcd}
$$

Our two octahedra also have bottom pyramids. We deduce diagrams

$$
\begin{tikzcd}
& x \arrow[dr, "d"'] & \\
\Sigma^{-1}z' \arrow[dr, "d"] & w \arrow[l, dotted, "(1)"'] \arrow[u, "(1)"] & y' \arrow[l] \\
& x'' \arrow[u, dotted] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \Sigma^{-1}y'' \arrow[dr, "d"'] & \\
x \arrow[dr, "d"] & w \arrow[l, "(1)"'] \arrow[u, dotted, "(1)"] & z \arrow[l, dotted] \\
& y' \arrow[u] &
\end{tikzcd}
$$

What we mean by this is that we may complete $x \longrightarrow y'$ to a distinguished triangle

$$x \longrightarrow y' \longrightarrow w \longrightarrow \Sigma x.$$

This is our solid triangle. The octahedral lemma allows us to choose the dotted arrows to complete each of the two octahedra. The diagrams depicting the projections of the bottom pyramids exhibit the commutative triangles as straight lines, and the distinguished triangles as triangles. But now we have the bottom pyramid of an octahedron

$$\begin{tikzcd}
 & \Sigma^{-1}y'' \arrow[dr] & \\
\Sigma^{-1}z' \arrow[dr] & w \arrow[u, "(1)"] \arrow[l, "(1)"'] \arrow[ur, phantom, "d"] \arrow[dl, phantom, "d"] & z \arrow[l] \\
 & x'' \arrow[u] &
\end{tikzcd}$$

which we may complete to a top pyramid. We have three octahedra, with top pyramids

$$\begin{tikzcd}
 & & & \Sigma^{-1}y'' \arrow[d] \arrow[dr] & \\
 & & x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[r] \arrow[d] & z \\
 & \Sigma^{-1}z' \arrow[r] \arrow[d] \arrow[dr] & x' \arrow[r] \arrow[d] & y' & \\
\Sigma^{-1}y'' \arrow[r] \arrow[dr] & \Sigma^{-1}z'' \arrow[r] \arrow[d] & x'' & & \\
 & z & & &
\end{tikzcd}$$

The remaining maps, which define the bottom pyramids of the octahedra, can be written as

$$\Sigma^{-1}w \xrightarrow{\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}} \Sigma^{-1}y'' \oplus x \oplus \Sigma^{-1}z' \qquad\qquad z \oplus y' \oplus x'' \xrightarrow{\begin{pmatrix} a & b & c\end{pmatrix}} w$$

The three octahedra tell us, among other things, that the three diagonals have the same mapping cone, namely the object w. But we know more. Each octahedron gives two commutative squares. The "refined" octahedral lemma chooses these commutative squares to be homotopy pushout squares; see Section 2 for details. Using these six homotopy pushout squares, we can extend the above to a commutative diagram

$$(**)\qquad \begin{tikzcd}
 & & \Sigma^{-1}w \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(1)"] & \Sigma^{-1}y'' \arrow[d] \arrow[dr] & \\
 & \Sigma^{-1}w \arrow[r, "\beta"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "(2)"] & x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & z \arrow[d, "a"] \\
\Sigma^{-1}w \arrow[r, "\gamma"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(3)"] & \Sigma^{-1}z' \arrow[r] \arrow[d] \arrow[dr] & x' \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & y' \arrow[r, "b"'] \arrow[d, "b"] & w \\
\Sigma^{-1}y'' \arrow[r] \arrow[dr] & \Sigma^{-1}z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "(6)"] & x'' \arrow[r, "c"'] \arrow[d, "c"] & w & \\
 & z \arrow[r, "a"'] & w & &
\end{tikzcd}$$

In this diagram, the six squares labeled (1) − (6) are homotopy pushout squares, and the differential of the square labeled (n) is given by the diagonal of the square labeled $(n + 3)$, where we read the labels modulo 6. For example, the differential of (1) is given by the diagonal of $(1 + 3) = (4)$.

We also have, in the upper part of the three octahedra, six distinguished triangles. These assemble to a diagram

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & \Sigma x \arrow[d] \\
x' \arrow[r] \arrow[d] & y' \arrow[r] \arrow[d] & z' \arrow[r] \arrow[d] & \Sigma x' \arrow[d] \\
x'' \arrow[r] \arrow[d] & y'' \arrow[r] \arrow[d] & z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "(-)"] & \Sigma x'' \arrow[d] \\
\Sigma x \arrow[r] & \Sigma y \arrow[r] & \Sigma z \arrow[r] & \Sigma^2 x
\end{tikzcd}$$

The standard sign convention has all the squares commuting except the one at the bottom right, which anticommutes. We put the symbol $(-)$ in the bottom right square to remind ourselves that it anticommutes.

If we begin with a square of monomorphisms in an abelian category, satisfying the condition in Lemma 3.4(iii), then all the choices in the octahedra we constructed are canonically unique. Using only the octahedral axiom the above argument (of Verdier) shows how to extend a commutative square to an elaborate diagram with many distinguished triangles. In the special case where the square is the top left corner of a 3×3 diagram of short exact sequences in $\mathcal{A}$, the extension in $D^b(\mathcal{A})$ is unique. We recover the 3×3 diagram, the mapping cone w on the map $x \longrightarrow y'$, and many distinguished triangles. The first theorem is

Theorem 3.5. *Axiom* (TC3) *of May's is just the assertion that the tensor product of two distinguished triangles comes with* Verdier structure. *By this we mean that there exists an object w, and a diagram* $(**)$ *as on page 547.*

Proof: It needs to be checked that May's list of the properties of the object w precisely coincides with what we obtained above, from the octahedral axiom. We leave this to the interested reader; one needs to compare $(**)$ with May's beautifully drawn diagram on page 49 of [12]. □

Remark 3.6. In the discussion preceding Theorem 3.5 we indicated how, following either May or Verdier, one can prove that a triangulated category $\mathcal{T}$ with a decent tensor product satisfies (TC3).

Note that, both in May's and in Verdier's argument, the tensor product plays a very minor role. What matters is that in the abelian category $\mathcal{A}$ we have a 3×3 diagram with exact rows and columns. The fact that it happens to come from the tensor product of two short exact sequences is largely irrelevant.

Remark 3.7. Haynes Miller independently observed that Verdier's construction yields May's diagram.

So far we have explained the relation between May's work and Verdier's. Now we move to the more interesting observation. We will explain the relation between two approaches we have already seen and Happel's work.

What we have seen so far is the following. We started with a commutative square of monomorphisms in $\mathcal{A}$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd}$$

satisfying the condition in Lemma 3.4(iii). Then, either by pushing out in the abelian category $\mathcal{A}$ or by repeatedly applying the octahedral lemma, we extended to an elaborate diagram of triangles giving May's (TC3). Let k be a noetherian commutative ring. Suppose the category $\mathcal{A}$ is k-linear. (For any abelian category $\mathcal{A}$ we may take $k = \mathbb{Z}$.) A commutative square in $\mathcal{A}$ may be viewed as a k-linear functor $\Box \longrightarrow \mathcal{A}$, where $\Box$ is the k-category presented by the quiver (=oriented graph)

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd} \quad \text{with the relation} \quad \gamma\,\alpha = \delta\,\beta \ .$$

In other words, the category $\Box$ has four objects corresponding to the four vertices of the quiver, and its morphisms between two objects are obtained by taking all k-linear combinations of paths between the corresponding vertices of the quiver and dividing out all consequences of the relations. Let $\operatorname{mod} k$ be the category of finitely generated k-modules. Let $\operatorname{mod}\Box = \mathcal{C}at(\Box^{op}, \operatorname{mod} k)$ be the category of all k-linear functors $\Box^{op} \longrightarrow \operatorname{mod} k$. We remind the reader of the well-known

LEMMA 3.8. *Up to canonical isomorphism, any k-linear functor $\Box \longrightarrow \mathcal{A}$ may be factored uniquely as*

$$\Box \longrightarrow \operatorname{mod}\Box \xrightarrow{F} \mathcal{A},$$

with F a right exact k-linear functor of k-linear categories.

PROOF: Any right exact functor $F : \operatorname{mod}\Box \longrightarrow \mathcal{A}$ is uniquely determined by what it does on projective objects. And each projective object in the functor category $\operatorname{mod}\Box$ is a direct factor of a finite sum of the representable functors $P_i = \operatorname{Hom}_\Box(-, i)$. Since the Yoneda functor is covariant, the representable functors appear in a commutative square

$$(*) \qquad \begin{tikzcd} & P_2 \arrow[dr, "\gamma_*"] & \\ P_1 \arrow[ur, "\alpha_*"] \arrow[dr, "\beta_*"'] & & P_4 \\ & P_3 \arrow[ur, "\delta_*"'] & \end{tikzcd} \quad .$$

Given a functor $F : \operatorname{mod}\Box \longrightarrow \mathcal{A}$, F must take the commutative square $(*)$ in $\operatorname{mod}\Box$ to a commutative square in $\mathcal{A}$. Conversely, given a commutative square in $\mathcal{A}$, we want a functor F. It is clear how to define F on P_1, P_2, P_3 and P_4. This definition extends by additivity to direct summands of direct sums of the P_i's, that is to all projectives. Finally, to define F on an arbitrary object X, choose a projective presentation for X

$$P \longrightarrow Q \longrightarrow X \longrightarrow 0,$$

and $F(X)$ is defined to be the cokernel of $F(P) \longrightarrow F(Q)$. $\Box$

Taking the left derived functor of the F in Lemma 3.8, we have that any commutative square in $\mathcal{A}$ yields a functor $D^b(\square) \longrightarrow D^b(\mathcal{A})$, where we abbreviate $D^b(\operatorname{mod}\square)$ to $D^b(\square)$.

Remark 3.9. For much more detail see [10], [8], [11].

Theorem 3.10. *The relations which hold in $D^b(\square)$ are universal. The same diagram of triangles will exist in any triangulated category $\mathcal{T}$ with a decent tensor product.*

Proof: The commutative square, which we saw in Verdier's construction of the diagram, will give rise to a triangulated functor $D^b(\square) \longrightarrow D^b(\mathcal{A})$. The decency of the tensor product gives a triangulated functor $D^b(\mathcal{A}) \longrightarrow \mathcal{T}$. The composite takes the diagram of triangles in $D^b(\square)$ to $\mathcal{T}$. $\square$

Remark 3.11. The word "universal" is appearing here in an extended, somewhat unusual way. We are not asserting that the category $D^b(\square)$ has a decent tensor product. As far as we know it has no tensor product at all; there does not seem to be a Hopf algebra structure on the quiver algebra.
All we say is the following. Let $\mathcal{T}$ be a triangulated category with a decent tensor product. Then triangles appearing in $D^b(\square)$ will always be reflected in the tensor product of two distinguished triangles in $\mathcal{T}$. As we have already said, the tensor product in $\mathcal{T}$ plays a minor role in the proof, and the category $D^b(\square)$ does not seem to have a tensor product at all.

The real use of Theorem 3.10 is that Happel studied the category $D^b(\square)$ in great detail, in the case where the ground ring k is a field. By appealing to his results we can obtain a great deal of information, at least in the case of k-linear triangulated categories over fields k. In principle, it should not be particularly difficult to generalise Happel's work to the case where $k = \mathbb{Z}$. In this article we chose not to do so. We chose to highlight the idea, not to pursue it to obtain the sharpest results. The main reason is that we wanted to keep the article reasonably brief.
In the next two sections, we will show how the different approaches can yield new results.

4. Consequences of the octahedral axiom

First we establish some notation. Consider the diagram

$$\begin{tikzcd}
 & & x \arrow[r] \arrow[d] \arrow[dr, dashed] & y \arrow[r] \arrow[d] \arrow[dr, squiggly] & z \arrow[r] \arrow[d] \arrow[dr] & \Sigma x \arrow[d] \\
 & \Sigma^{-1}z' \arrow[r] \arrow[d] \arrow[dr, dashed] & x' \arrow[r] \arrow[d] \arrow[dr, squiggly] & y' \arrow[r] \arrow[d] \arrow[dr] & z' \arrow[r] \arrow[d] & \Sigma x' \arrow[d] \\
\Sigma^{-1}y'' \arrow[r] \arrow[d] \arrow[dr, dashed] & \Sigma^{-1}z'' \arrow[r] \arrow[d] \arrow[dr, squiggly] & x'' \arrow[r] \arrow[d] \arrow[dr] & y'' \arrow[r] \arrow[d] & z'' \arrow[r] \arrow[d] & \Sigma x'' \arrow[d] \\
y \arrow[r] & z \arrow[r] & \Sigma x \arrow[r] & \Sigma y \arrow[r] & \Sigma z \arrow[r] & \Sigma^2 x
\end{tikzcd}$$

The axiom (TC3) assigns a common mapping cone w to the three broken arrows. Applying (TC3) to rotations of the triangles, we expect a common mapping cone Σu to the curly arrows, and a common mapping cone Σv to the plain arrows. Needless to say, u, v and w should be related. May found one relation. We will use the different approaches to obtain more.

In this section we will, following Verdier's approach, see what the octahedral lemma buys us. We have:

Theorem 4.1. *May's axiom* (TC4) *is a formal consequence of* (TC3) *and the octahedral axiom. The proof will give us yet another distinguished triangle. It is a triangle May missed, whose existence also follows formally from* (TC3) *and the octahedral axiom.*

Proof: Recall that the octahedra defining w give a homotopy pushout square

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & z \arrow[d] \\
y' \arrow[r] & w
\end{tikzcd}$$

The mapping cone on the diagonal $y \longrightarrow w$ is just the sum of the mapping cones on the horizontal and vertical maps, that is $y'' \oplus \Sigma x$. The triangle

$$\Sigma^{-1}z' \longrightarrow x'' \longrightarrow w \longrightarrow z'$$

gives us a map $w \longrightarrow z'$, and hence a commutative square

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & z \arrow[d] \\
y' \arrow[r] & z'
\end{tikzcd}$$

The object Σu is the mapping cone of the diagonal map $y \longrightarrow z'$ in this square. In other words, Σu is the mapping cone on a composite

$$y \longrightarrow w \longrightarrow z'.$$

We now complete to an octahedron. We know all the objects of the octahedron. In the standard notation, where d stands for a distinguished triangle, $+$ for a commutative one and (1) for an arrow of degree one, we draw the octahedron. The top pyramid is

$$\begin{tikzcd}
y \arrow[rrrr] \arrow[ddrr] & & & & z' \arrow[dddd] \\
& & + & & \\
& d & w \arrow[uurr] \arrow[ddll] & d & \\
& & + & & \\
y'' \oplus \Sigma x \arrow[uuuu, "(1)"] & & & & \Sigma x'' \arrow[llll, "(1)"] \arrow[uull, "(1)"']
\end{tikzcd}$$

The bottom pyramid is

$$\begin{tikzcd}
y \arrow[rrrr] & & & & z' \arrow[dddd] \arrow[ddll] \\
& & d & & \\
& + & \Sigma u \arrow[uull, "(1)"'] \arrow[ddrr] & + & \\
& & d & & \\
y'' \oplus \Sigma x \arrow[uuuu, "(1)"] \arrow[uurr] & & & & \Sigma x'' \arrow[llll, "(1)"]
\end{tikzcd}$$

From this octahedron we deduce two homotopy pushout squares. There are therefore distinguished triangles

$$w \longrightarrow z' \oplus y'' \oplus \Sigma x \longrightarrow \Sigma u \longrightarrow \Sigma w$$

and

$$u \longrightarrow y \oplus x'' \longrightarrow w \longrightarrow \Sigma u$$

The first of these triangles is axiom (TC4) of May's [12]. The second is new. $\square$

In the next section we will see how to better organise all the triangles above, and more.

5. The relation with Happel's work

As we saw in Theorem 3.10, the problem reduces to understanding the category $D^b(\square)$. It helps to introduce an equivalent derived category. We define

Definition 5.1. *Let D_4 be the quiver(=oriented graph)*

$$(1)\qquad \begin{tikzcd} & & 2 \\ 1 \arrow[r, "\lambda"] & 5 \arrow[ur, "\mu"] \arrow[dr, "\nu"'] & & . \\ & & 3 \end{tikzcd}$$

Let Y be the k-category presented by the quiver D_4. Let $\operatorname{mod}\mathsf{Y}$ be the category of k-linear functors $\mathsf{Y}^{op} \longrightarrow \operatorname{mod} k$. We denote the bounded derived category $D^b(\operatorname{mod}\mathsf{Y})$ by $D^b(\mathsf{Y})$.

The interest in this definition comes from the well-known

Lemma 5.2. *The derived categories $D^b(\Box)$ and $D^b(\mathsf{Y})$ are equivalent, as k-linear triangulated categories.*

Proof: Recall that $\Box$ is the category presented by the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd} \qquad \text{with the relation} \quad \gamma\,\alpha = \delta\,\beta$$

and $D^b(\Box)$ the bounded derived category of the category $\operatorname{mod}\Box$ of k-linear functors $\Box^{op} \longrightarrow \operatorname{mod} k$. The categories $\operatorname{mod}\mathsf{Y}$ and $\operatorname{mod}\Box$ are related by a natural pair L, R of adjoint functors: If M is in $\operatorname{mod}\Box$, we complete the corresponding diagram of k-modules into

$$\begin{tikzcd} & & M2 \arrow[dll, bend right, "M\alpha"'] \arrow[dl, dotted, "M\mu"] & \\ M1 & M5 \arrow[l, dotted, "M\lambda"'] & & M4 \arrow[ul, "M\gamma"'] \arrow[dl, "M\delta"] \\ & & M3 \arrow[ull, bend left, "M\beta"] \arrow[ul, dotted, "M\nu"'] & \end{tikzcd} ,$$

where $M5$ is the pushout of $M\gamma$ and $M\delta$. We define the object $LM \in \operatorname{mod}\mathsf{Y}$ as the full subdiagram on $M1, M2, M3, M5$. Similarly, if N is in $\operatorname{mod}\mathsf{Y}$, we complete the corresponding diagram by defining $N4$ as the pullback of $N\mu$ and $N\nu$, and we define RN to be the full subdiagram on $N1, N2, N3, N4$. Note that the functors L and R are not equivalences (they take some non zero objects to zero). But the left derived functor of L is easily computed to be quasi-inverse to the right derived functor of R, giving an equivalence between $D^b(\Box)$ and $D^b(\mathsf{Y})$. $\Box$

Remark 5.3. The experts will note that the fact that L and R induce equivalences of derived categories is a special case [1] of tilting theory (cf. e.g. [9], [7], [2]).

Theorem 3.10 tells us that we are reduced to understanding the distinguished triangles in the category $D^b(\square) = D^b(\mathsf{Y})$. The proof of Theorem 3.10, more specifically Lemma 3.8, tells us that in the category $\operatorname{mod}\square \subset D^b(\square)$ we have a commutative square

$$\begin{tikzcd} & P_2 \arrow[dr, "\gamma_*"] & \\ P_1 \arrow[ur, "\alpha_*"] \arrow[dr, "\beta_*"'] & & P_4 \\ & P_3 \arrow[ur, "\delta_*"'] & \end{tikzcd}$$

and everything reduces to understanding the distinguished triangles in which it lies. The equivalence $D^b(\square) = D^b(\mathsf{Y})$ is explicit enough to be able to work out the image of this commutative square in $D^b(\mathsf{Y})$.

5.1. Happel's description of $D^b(\mathsf{Y})$. From now on, we assume k is a field. We will describe $D^b(\mathsf{Y})$ as a k-linear category, following Happel [9]. This description will also yield a great deal of information on the distinguished triangles of $D^b(\mathsf{Y})$. Note that Happel built on previous work by many researchers, notably Ringel [14], Riedtmann [13], Gabriel [6]. For more information, we refer to the books [15], [2], [7].

Each object of the category $D^b(\mathsf{Y})$ decomposes into a finite sum of indecomposable objects with local endomorphism rings and this decomposition is unique up to permutation and isomorphism. To describe $D^b(\mathsf{Y})$ as a k-linear category, it suffices therefore to describe the full subcategory formed by the indecomposable objects.

We will give a presentation of the category of indecomposables in $D^b(\mathsf{Y})$. Let us first describe its objects: The category $\operatorname{mod}\mathsf{Y}$ is the k-category of representations of a quiver without relations. Therefore, it is an abelian category of global dimension ≤ 1. This entails that in its derived category $D^b(\mathsf{Y})$, each object is (non canonically) isomorphic to the direct sum of its homologies placed in their respective degrees. Each indecomposable of $D^b(\mathsf{Y})$ is therefore concentrated in one degree, i.e. it is a shift of some indecomposable module. Now D_4 is a quiver whose underlying graph is a *Dynkin diagram.* So by Gabriel's theorem [5], there are only finitely many (isomorphism classes of) indecomposable modules; moreover, the indecomposables are in bijection with the twelve positive roots of the corresponding root system (the orientation of the quiver determines the positive cone). The bijection is given by sending each indecomposable M to its *dimension vector* $\underline{\dim}\, M$, i.e. to the function $i \mapsto \dim Mi$. For example, the dimension vector of the module $P_2 : i \mapsto \operatorname{Hom}_{\mathsf{Y}}(i, 2)$ is given by

$$\begin{smallmatrix} & 1 \\ 1 & 1 \\ & 0 \end{smallmatrix} \;.$$

Note that, by definition of P_i, we have

$$Mi = \operatorname{Hom}_{\mathsf{Y}}(P_i, M) \;\;,\;\; M \in \operatorname{mod}\mathsf{Y}\,.$$

Thus the i-component of the dimension vector of M is the dimension of the space of morphisms from P_i to M. The map $M \mapsto \underline{\dim} M$ induces an isomorphism

$$K_0(\operatorname{mod} \mathsf{Y}) \longrightarrow \mathbf{Z}^4.$$

The simple modules correspond to the vectors of the standard basis.

Caution 5.4. In what follows, we make frequent reference to figure 1 and figure 2. For the reader's convenience, both figures have been placed *on the last page,* after the bibliography.

Let us summarize the above: The indecomposable objects of $D^b(\mathsf{Y})$ are the shifts of the indecomposable modules; the indecomposable modules are determined by their dimension vectors. The positive dimension vectors in figure 1 are precisely the dimension vectors of indecomposable objects of Y. The negative vectors in the figure correspond to shifted indecomposable modules.

We now describe the morphisms between indecomposable objects of $D^b(\mathsf{Y})$. Let U and V be indecomposable. A *radical morphism* from U to V is a non invertible morphism $f : U \longrightarrow V$. Denote by $\operatorname{rad}(U, V)$ the space of radical morphisms from U to V. Clearly, rad is an ideal of the category of indecomposables. Denote its square by rad^2. Thus a morphism $f : U \longrightarrow V$ belongs to $\operatorname{rad}^2(U, V)$ iff it is *reducible*, i.e. we have

$$f = \sum_{i=1}^{n} g_i h_i$$

for some n and for radical morphisms $h_i : U \longrightarrow W_i$ and $g_i : W_i \longrightarrow V$. A morphism is *irreducible* if it is not reducible. The *Auslander-Reiten quiver* of $D^b(\mathsf{Y})$ is the quiver whose vertices are the isomorphism classes $[U]$ of indecomposable objects and which has $\dim \operatorname{rad}(U, V)/\operatorname{rad}^2(U, V)$ arrows from the vertex $[U]$ to the vertex $[V]$.

Happel's theorem [9, Cor. 4.5] yields as a special case that the Auslander-Reiten quiver of $D^b(\mathsf{Y})$ is the quiver R of figure 1.

We will obtain the required presentation of the category of indecomposables of $D^b(\mathsf{Y})$ by dividing the free k-category on the Auslander-Reiten quiver by suitable relations, which we now describe. To do so, we introduce the automorphism $\tau : R \longrightarrow R$ which is the shift by two units to the left. It is called the *Auslander-Reiten translation.* For example, we have

$$\tau\left(\begin{smallmatrix} & 0 & \\ 0 & 1 & \\ & 1 & \end{smallmatrix} \right) = \begin{smallmatrix} & 1 & \\ 1 & 1 & \\ & 0 & \end{smallmatrix} \;,\; \tau\left(\begin{smallmatrix} & 1 & \\ 1 & 2 & \\ & 1 & \end{smallmatrix} \right) = \begin{smallmatrix} & 0 & \\ 1 & 1 & \\ & 0 & \end{smallmatrix} \;.$$

The *mesh relation* associated to a vertex x of R is the relation

$$\sum \alpha\beta = 0 \;,$$

where the sum ranges over all subquivers

$$\tau(x) \xrightarrow{\beta} y \xrightarrow{\alpha} x.$$

Theorem 5.5 (Happel [9]). *There is an equivalence Φ from the k-category presented by the Auslander-Reiten quiver R of figure 1 subject to all mesh relations to the category of indecomposables of $D^b(\mathsf{Y})$. It can be chosen so that for each vertex x of R, $\underline{\dim}\,\Phi(x)$ is the dimension vector associated with x and that for each arrow $\alpha : x \longrightarrow y$ of R, $\Phi(\alpha)$ is an irreducible morphism from $\Phi(x)$ to $\Phi(y)$. Moreover, for each vertex x of R, there is a canonical triangle (called the* Auslander-Reiten triangle*)*

$$\Phi(\tau(x)) \longrightarrow \bigoplus \Phi(y) \longrightarrow \Phi(x) \longrightarrow \Sigma\Phi(\tau(x)) \ ,$$

where the sum ranges over all subquivers

$$\tau(x) \xrightarrow{\beta} y \xrightarrow{\alpha} x.$$

Remark 5.6. Under the equivalence of the theorem, the suspension $U \mapsto \Sigma U$ corresponds to τ^{-3}.

Remark 5.7. The group $S_3 \times \mathbf{Z}$ acts on the quiver R: The factor $\mathbf{Z}$ acts via τ; the factor S_3 fixes the τ-orbit of $\begin{smallmatrix} & 0 \\ 1 & 1 \\ & 0 \end{smallmatrix}$ and simultaneously permutes the vertices

$$\tau^i \left(\begin{smallmatrix} & 1 \\ 1 & 1 \\ & 0 \end{smallmatrix} \right) , \ \tau^i \left(\begin{smallmatrix} & 0 \\ 0 & 1 \\ & 0 \end{smallmatrix} \right) , \ \tau^i \left(\begin{smallmatrix} & 0 \\ 1 & 1 \\ & 1 \end{smallmatrix} \right)$$

for each $i \in \mathbf{Z}$. By Happel's theorem, we obtain an action on $D^b(\mathsf{Y})$. The autoequivalences of $D^b(\mathsf{Y})$ which occur are triangulated functors.

Remark 5.8. Lemma 5.2 gives an equivalence $D^b(\square) \longrightarrow D^b(\mathsf{Y})$. The natural commutative square in $\operatorname{mod}\square$ maps via the composite

$$\operatorname{mod}\square \longrightarrow D^b(\square) \longrightarrow D^b(\mathsf{Y})$$

to the square formed by the vertices labeled

$$\begin{smallmatrix} & 0 \\ 1 & 0 \\ & 0 \end{smallmatrix} \ , \ \begin{smallmatrix} & 1 \\ 1 & 1 \\ & 0 \end{smallmatrix} \ , \ \begin{smallmatrix} & 0 \\ 1 & 1 \\ & 1 \end{smallmatrix} \ , \ \begin{smallmatrix} & 1 \\ 1 & 1 \\ & 1 \end{smallmatrix} \ .$$

Remark 5.9. Suppose that x is a vertex of R corresponding to a representable functor P_i and y an arbitrary vertex. Then we have

$$\dim \operatorname{Hom}(\Phi(x), \Phi(y)) = \dim \operatorname{Hom}(P_i, \Phi(y)) = (\underline{\dim}\,\Phi(y))_i \ .$$

For an arbitrary vertex x, there is always an $i \in \mathbf{Z}$ so that $\tau^i x$ corresponds to a representable. This allows us to compute

$$\dim \operatorname{Hom}(\Phi(x), \Phi(y)) = \dim \operatorname{Hom}(\Phi(\tau^i x), \Phi(\tau^i y))$$

very easily by inspecting figure 1.

Remark 5.10. Happel's theorem allows us to exhibit many triangles produced by short exact sequences of the module category $\operatorname{mod}\mathsf{Y}$. It is clear that a sequence of modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is exact iff it is a complex, the left arrow is a monomorphism, the right arrow an epimorphism and we have $\underline{\dim}\, M = \underline{\dim}\, L + \underline{\dim}\, N$. These conditions are easy to check with the help of the Auslander-Reiten quiver. Using the action of $S_3 \times \mathbf{Z}$ of Remark 5.7 we obtain further triangles.

5.2. Application: Organising the triangles. Suppose that we have a commutative square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd}$$

in a k-linear triangulated category $\mathcal{T}$. Suppose that, as in Theorem 3.10, we also have a triangulated functor

$$F : D^b(\square) = D^b(\mathsf{Y}) \longrightarrow \mathcal{T}$$

extending the square. If we compose F with the isomorphism Φ of Happel's theorem, we obtain the mapping suggested by superposing figures 1 and 2. Here we use the notations of Sections 3 and 4, as well as some of the triangles of $D^b(\square) = D^b(\mathsf{Y})$ obtained from Remark 5.10.

Note that, miraculously, the twelve objects of Section 3 and 4 correspond to the twelve orbits of *indecomposable* Y-modules under the action of the group $\Sigma^{\mathbf{Z}}$ generated by Σ and that the 'interesting' objects u, v, w are in the same orbit under the action of the group $\tau^{\mathbf{Z}}$ generated by τ.

Remark 5.11. Perhaps the miracle deserves a small comment. The objects x, x', y and y' in the commutative square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd}$$

correspond, in $D^b(\square)$, to P_1, P_2, P_3 and P_4, all of which are (projective) indecomposables in $\square \subset D^b(\square)$. Being indecomposable in $D^b(\square)$, they must remain indecomposable under the equivalence $D^b(\square) \simeq D^b(\mathsf{Y})$ of Lemma 5.2. This much is not surprising.

The miracle, which the authors do not understand, is why the other naturally arising eight objects correspond precisely, up to suspension, to the other classes of indecomposables.

Let us call two triangles *equivalent* if they are obtained from one another by rotations and the action of $S_3 \times \mathbf{Z}$. Then the triangles constructed in Sections 3 and 4 belong to the (distinct) equivalence classes of the following seven triangles (those marked with $(*)$ are rotations of Auslander-Reiten triangles)

$$x \longrightarrow u \longrightarrow \Sigma^{-1}z'' \longrightarrow \Sigma x \qquad (*)$$

$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$$

$$x \longrightarrow v \longrightarrow z \oplus x'' \longrightarrow \Sigma x$$

$$x \longrightarrow y' \longrightarrow w \longrightarrow \Sigma x \qquad (*)$$

$$u \longrightarrow x'' \longrightarrow \Sigma x \oplus y'' \longrightarrow \Sigma u$$

$$u \longrightarrow w \longrightarrow \Sigma x \oplus y'' \oplus z' \longrightarrow \Sigma u \qquad (*)$$

$$u \longrightarrow v \longrightarrow x'' \oplus z' \longrightarrow \Sigma u \; ,$$

where the last triangle is equivalent to the new triangle constructed at the end of Section 4. Note that the morphism space $\mathrm{Hom}(u, v)$ is 2-dimensional so that the morphism $u \longrightarrow v$ is not even unique up to a scalar multiple. The morphism $u \longrightarrow v$ occurring in the last triangle is defined to be the composition $u \longrightarrow y \longrightarrow v$. Note that up to the action of $S_3 \times \mathbf{Z}$ this is the only 2-dimensional morphism space between indecomposables.
Let us construct some more triangles: The plane $\mathrm{Hom}(u, v)$ contains three distinguished lines given by the morphisms factoring respectively through x', y and $\Sigma^{-1}z''$. The mapping cone triangle over a morphism lying in one of the lines is equivalent to the last triangle of the list above. However, if we choose a morphism f outside of the three lines, we obtain a new triangle

$$u \xrightarrow{f} v \xrightarrow{g} w \xrightarrow{\varepsilon(f,g)} u$$

by looking at the corresponding short exact sequence of Y-modules. Thus, we obtain a whole new family of isomorphism classes of triangles, parametrized by the projective line over k punctured at 3 points.
We claim that this is the list of all equivalence classes of non-split triangles with two indecomposable vertices. To check this, one proceeds in two steps: (1) classify morphisms between indecomposables of $D^b(\mathsf{Y})$ up to conjugacy under the the group $S_3 \times \mathbf{Z}$; (2) inspect all mapping cone triangles over the morphisms obtained in (1) and eliminate duplicates. We leave the details to the interested reader.

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$$
\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}&-1\\0&0\\&0\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}&1\\1&1\\&0\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}&0\\0&1\\&1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}&1\\0&0\\&0\end{smallmatrix} & \\
\cdots & \begin{smallmatrix}&0\\1&0\\&0\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&0\\1&1\\&0\end{smallmatrix} \arrow[r] \arrow[ur] \arrow[dr] & \begin{smallmatrix}&0\\0&1\\&0\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&1\\1&2\\&1\end{smallmatrix} \arrow[r] \arrow[ur] \arrow[dr] & \begin{smallmatrix}&1\\1&1\\&1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&1\\0&1\\&1\end{smallmatrix} \arrow[r] \arrow[ur] \arrow[dr] & \begin{smallmatrix}&0\\-1&0\\&0\end{smallmatrix} & \cdots \\
 & \begin{smallmatrix}&0\\0&0\\&-1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&0\\1&1\\&1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&1\\0&1\\&0\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&0\\0&0\\&1\end{smallmatrix} &
\end{tikzcd}
$$

FIGURE 1.

$$
\begin{tikzcd}[column sep=small]
 & \Sigma^{-1}z' \arrow[dr] & & y \arrow[dr] & & x'' \arrow[dr] & & z' & \\
\cdots & x \arrow[r] & u \arrow[r] \arrow[ur] \arrow[dr] & \Sigma^{-1}z'' \arrow[r] & v \arrow[r] \arrow[ur] \arrow[dr] & y' \arrow[r] & w \arrow[r] \arrow[ur] \arrow[dr] & \Sigma x & \cdots \\
 & \Sigma^{-1}y'' \arrow[ur] & & x' \arrow[ur] & & z \arrow[ur] & & y'' &
\end{tikzcd}
$$

FIGURE 2.