Rost Projectors and Steenrod Operations

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Abstract. Let X be an anisotropic projective quadric possessing a Rost projector ρ. We compute the 0-dimensional component of the total Steenrod operation on the modulo 2 Chow group of the Rost motive given by the projector ρ. The computation allows to determine the whole Chow group of the Rost motive and the Chow group of every excellent quadric (the results announced by Rost). On the other hand, the computation is being applied to give a simpler proof of Vishik's theorem stating that the integer $\dim X + 1$ is a power of 2.

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M. Rost noticed that certain smooth projective anisotropic quadric hypersurfaces are decomposable in the category of Chow motives into a direct sum of some motives. The (in some sense) smallest direct summands are called the *Rost motives*. For example, the motive of a Pfister quadric is a direct sum of Rost motives and their Tate twists. The *Rost projectors* split off the Rost motives as direct summands of quadrics. In the present paper we study Rost projectors by means of modulo 2 Steenrod operations on the Chow groups of quadrics. The Steenrod operations in motivic cohomology were defined by V. Voevodsky. We use results of P. Brosnan who found in [1] an elementary construction of the Steenrod operations on the Chow groups.

As a consequence of our computations we give a description of the Chow groups of a Rost motive (Corollary 8.2). This result (which has been announced by M. Rost in [11]) allows to compute all the Chow groups of every excellent quadric (see Remark 8.4).

We also give a simpler proof of a theorem of A. Vishik [3, th. 6.1] stating that if an anisotropic quadric X possesses a Rost projector, then $\dim X + 1$ is a power of 2 (Theorem 5.1).

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Contents

1. Parity of binomial coefficients

Lemma 1.1. *Let i, n be any non-negative integers. The binomial coefficient $\binom{n+i}{i}$ is odd if and only if we don't carry over units while adding n and i in base 2.*

Proof. For any integer $a \geq 0$, let $s_2(a)$ be the sum of the digits in the base 2 expansion of a. By [9, Lemma 5.4(a)], $\binom{n+i}{i}$ is odd if and only if $s_2(n+i) = s_2(n) + s_2(i)$. □

The following statement is obvious:

Lemma 1.2. *For any non-negative integer m, we don't carry over units while adding m and $m+1$ in base 2 if and only if $m+1$ is a power of 2.* □

The following statement will be applied in the proof of Theorem 4.8:

Corollary 1.3. *For any non-negative integer m, the binomial coefficient $\binom{-m-2}{m}$ is odd if and only if $m+1$ is a power of 2.*

Proof. By Lemma 1.1, the binomial coefficient

$$\binom{-m-2}{m} = (-1)^m \binom{2m+1}{m}$$

is odd if and only if we don't carry over units while adding m and $m+1$ in base 2. It remains to apply Lemma 1.2. □

2. Integral and modulo 2 Rost projectors

Let F be a field, X a quasi-projective smooth equidimensional variety over F. We write $\mathrm{CH}(X)$ for the *modulo* 2 Chow group of X. The usual (integral) Chow group is denoted by $\mathbb{CH}(X)$. We are working mostly with $\mathrm{CH}(X)$, but several times we have to use $\mathbb{CH}(X)$ (for example, already the definition of a modulo 2 Rost correspondence cannot be given on the level of the modulo 2 Chow group).

Both groups are graded. We use the upper indices for the gradation by codimension of cycles and we use the lower indices for the gradation by the dimension of cycles.

For projective X_1 and X_2, an element $\rho \in \mathbb{CH}(X_1 \times X_2)$ (we do not consider the gradation on $\mathbb{CH}$ for the moment) can be viewed as a correspondence from X_1 to X_2 ([2, §16.1]). In particular, it gives a homomorphism [2, def. 16.1.2]

$$\rho_* \colon \mathbb{CH}(X_1) \to \mathbb{CH}(X_2), \quad \rho_*(\alpha) = pr_{2*}\left(pr_1^*(\alpha) \cdot \rho\right),$$

where pr_1 and pr_2 are the two projections of $X_1 \times X_2$ onto X_1 and X_2, and can be composed with another correspondence $\rho' \in \mathbb{CH}(X_2 \times X_3)$ [2, def. 16.1.1]. The same can be said and defined with $\mathbb{CH}$ replaced by CH.

Starting from here, we always assume that $\operatorname{char} F \neq 2$. Let φ be a non-degenerate quadratic form over F, and let X be the projective quadric $\varphi = 0$. We set $n = \dim X = \dim \varphi - 2$ and we assume that $n \geq 1$.

An element $\varrho \in \mathbb{CH}^n(X \times X)$ is called an (integral) *Rost correspondence*, if over an algebraic closure $\bar{F}$ of F one has:

$$\varrho_{\bar{F}} = [\bar{X} \times x] + [x \times \bar{X}] \in \mathbb{CH}^n(\bar{X} \times \bar{X})$$

with $\bar{X} = X_{\bar{F}}$ and a rational point $x \in \bar{X}$. A *Rost projector* is a Rost correspondence which is an idempotent with respect to the composition of correspondences.

Remark 2.1. Assume that the quadric X is isotropic, i.e., contains a rational closed point $x \in X$. Then $[X \times x] + [x \times X]$ is a Rost projector. Moreover, this is the unique Rost projector on X ([7, lemma 4.1]).

Remark 2.2. Let ϱ be a Rost correspondence on X. It follows from the Rost nilpotence theorem ([12, prop. 1]) that a certain power of ϱ is a Rost projector (see [7, cor. 3.2]). In particular, a quadric X possesses a Rost projector if and only if it possesses a Rost correspondence.

A *modulo* 2 *Rost correspondence* $\rho \in \mathrm{CH}^n(X \times X)$ is a correspondence which can be represented by an integral Rost correspondence. A *modulo* 2 *Rost projector* is an idempotent modulo 2 Rost correspondence. Clearly, a modulo 2 Rost correspondence represented by an integral Rost projector is a modulo 2 Rost projector. Conversely,

Lemma 2.3. *A modulo* 2 *Rost projector is represented by an integral Rost projector.*

Proof. Let ρ be a modulo 2 Rost projector and let ϱ be an integral Rost correspondence representing ρ. The correspondence $\varrho_{\bar{F}}$ is idempotent, therefore, by the Rost nilpotence theorem (see [7, th. 3.1]), ϱ^r is idempotent for some r; so, ϱ^r is an integral Rost projector. Since ρ is idempotent as well, ϱ^r still represents ρ. $\square$

Lemma 2.4. *Let ϱ be an integral Rost correspondence and let ρ be a modulo* 2 *Rost correspondence. Then ϱ_* is the identity on $\mathbb{CH}^0(X)$ and on $\mathbb{CH}_0(X)$;*

also ρ_* *is the identity on* $\mathrm{CH}^0(X)$ *and on* $\mathrm{CH}_0(X)$*. Moreover, for every* i *with* $0 < i < n$*, the group* $\varrho_* \,\mathbb{CH}^i(X)$ *vanishes over* $\bar{F}$*.*

Proof. It suffices to prove the statements on ϱ. Since $\mathbb{CH}^0(X)$ and $\mathbb{CH}_0(X)$ inject into $\mathbb{CH}^0(X_{\bar{F}})$ and $\mathbb{CH}_0(X_{\bar{F}})$ (see [5, prop. 2.6] or [13] for the statement on $\mathbb{CH}_0(X)$), it suffices to consider the case where the quadric X has a rational closed point x and $\varrho = [X \times x] + [x \times X]$. Since $[X \times x]_*([X]) = 0$, $[X \times x]_*([x]) = [x]$, $[x \times X]_*([X]) = [X]$, $[x \times X]_*([x]) = 0$ and since $[X]$ generates $\mathbb{CH}^0(X)$ while $[x]$ generates $\mathbb{CH}_0(X)$, we are done with the statements on $\mathbb{CH}^0(X)$ and on $\mathbb{CH}_0(X)$. Since $[X \times x]_*([Z]) = 0 = [x \times X]_*([Z])$ for any closed subvariety $Z \subset X$ of codimension $\neq 0, n$, we are done with the rest. □

3. Steenrod operations

In this section we briefly recall the basic properties of the Steenrod operations on the modulo 2 Chow groups constructed in [1].

Let X be a smooth quasi-projective equidimensional variety over a field F. For every $i \geq 0$ there are certain homomorphisms $S^i : \mathrm{CH}^*(X) \to \mathrm{CH}^{*+i}(X)$ called *Steenrod operations*; their sum (which is in fact finite because $S^i = 0$ for $i > \dim X$)

$$S = S_X = S^0 + S^1 + \cdots : \mathrm{CH}(X) \to \mathrm{CH}(X)$$

is the *total Steenrod operation* (we omit the $*$ in the notation of the Chow group to indicate that S is not homogeneous). They have the following basic properties (see [1] for the proofs): for any smooth quasi-projective F-scheme X, the total operation $S\colon \mathrm{CH}(X) \to \mathrm{CH}(X)$ is a ring homomorphism such that for every morphism $f\colon Y \to X$ of smooth quasi-projective F-schemes and for every field extension E/F, the squares

$$\begin{array}{ccc} \mathrm{CH}(Y) & \xrightarrow{S_Y} & \mathrm{CH}(Y) \\ \uparrow{\scriptstyle f^*} & & {\scriptstyle f^*}\uparrow \\ \mathrm{CH}(X) & \xrightarrow{S_X} & \mathrm{CH}(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{CH}(X_E) & \xrightarrow{S_{X_E}} & \mathrm{CH}(X_E) \\ \uparrow{\scriptstyle \mathrm{res}_{E/F}} & & {\scriptstyle \mathrm{res}_{E/F}}\uparrow \\ \mathrm{CH}(X) & \xrightarrow{S_X} & \mathrm{CH}(X) \end{array}$$

are commutative. Moreover, the restriction $S^i|_{\mathrm{CH}^n(X)}$ is 0 for $n < i$ and the map $\alpha \mapsto \alpha^2$ for $n = i$; finally S^0 is the identity.

Also, the total Steenrod operation satisfies the following Riemann-Roch type formula:

$$f_*\big(S_Y(\alpha) \cdot c(-T_Y)\big) = S_X\big(f_*(\alpha)\big) \cdot c(-T_X)$$

(in other words, S modified by $c(-T)$ this way, commutes with the push-forwards) for any proper $f\colon Y \to X$ and any $\alpha \in \mathrm{CH}(Y)$, where $f_*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ is the push-forward, c is the total Chern class, T_X is the tangent bundle of X, and $c(-T_X) = c^{-1}(T_X)$ (the expression $-T_X$ makes sense if one considers T_X as an element of $K_0(X)$). This formula is proved in [1]. It also follows from the previously formulated properties of S by the general Riemann-Roch theorem of Panin [10].

Lemma 3.1. *Assume that X is projective. For any $\alpha \in \mathrm{CH}(X)$ and for any $\rho \in \mathrm{CH}(X \times X)$, one has*

$$S_X(\rho_*(\alpha)) = S_{X\times X}(\rho)_*\big(S_X(\alpha)\cdot c(-T_X)\big)\ ,$$

where T_X is the class in $K_0(X)$ of the tangent bundle of X.

Proof. Let $pr_1, pr_2 \colon X \times X \to X$ be the first and the second projections. By the Riemann-Roch formula applied to the morphism pr_2, one has

$$S_X(\rho_*(\alpha)) = pr_{2*}\Big(S_{X\times X}\big(\,pr_1^*(\alpha)\cdot\rho\big)\cdot c(-T_{X\times X})\Big)\cdot c(T_X)\ .$$

By the projection formula for pr_2, this gives

$$S_X(\rho_*(\alpha)) = pr_{2*}\Big(S_{X\times X}(pr_1^*(\alpha)\cdot\rho)\cdot c\big(-T_{X\times X} + pr_2^*(T_X)\big)\Big)\ .$$

Since

$$T_{X\times X} = pr_1^*(T_X) + pr_2^*(T_X) \in K_0(X\times X)$$

and since S (as well as c) commutes with the products and the pull-backs, we get

$$pr_{2*}\Big(\,pr_1^*\left(S_X(\alpha)\cdot c(-T_X)\right)\cdot S_{X\times X}(\rho)\Big) = S_{X\times X}(\rho)_*\Big(S_X(\alpha)\cdot c(-T_X)\Big)\ .$$

□

4. Main theorem

In this section, let φ be an anisotropic quadratic form over F, and let X be the projective quadric $\varphi = 0$ with $n = \dim X = \dim\varphi - 2 \geq 1$. We are assuming that an integral Rost projector (see §2 for the definition) $\varrho \in \mathbb{CH}^n(X\times X)$ exists for our X and we write $\rho \in \mathrm{CH}^n(X\times X)$ for the modulo 2 Rost projector. We write h for the class in $\mathrm{CH}^1(X)$ (as well as in $\mathbb{CH}^1(X)$) of a hyperplane section of X.

Proposition 4.1. *One has for every $i \geq 0$:*

$$S\big(\rho_*(h^i)\big) = S(\rho)_*\big(h^i\cdot(1+h)^{i-n-2}\big)\ .$$

Proof. Since $h \in \mathrm{CH}^1(X)$, we have $S(h) = S^0(h) + S^1(h)$. Since $S^0 = \mathrm{id}$ while S^1 on $\mathrm{CH}^1(X)$ is the squaring, $S(h) = h{+}h^2 = h(1{+}h)$, hence $S(h^i) = S(h)^i = h^i(1+h)^i$. To finish the proof it suffices to check that $c(T_X) = (1+h)^{n+2}$ for the tangent bundle T_X of the quadric X: then the formula of Lemma 3.1 will give the formula of Proposition 4.1.

Let $i\colon X \hookrightarrow P$ be the embedding of X into the $(n+1)$-dimensional projective space P. Let us write H for the class in $\mathrm{CH}^1(P)$ of a hyperplane. Note that $h = i^*(H)$.

The exact sequence of vector X-bundles

$$0 \to T_X \to i^*(T_P) \to i^*(\mathcal{O}_P(2)) \to 0$$

gives the equality $c(T_X)\cdot i^*\big(c(\mathcal{O}_P(2))\big) = i^*\big(c(T_P)\big)$. Since $c(\mathcal{O}_P(2)) = 1{+}2H = 1$ (we are working with the *modulo* 2 Chow groups) and $c(T_P) = (1+H)^{n+2}$, we get $c(T_X) = (1+h)^{n+2}$. □

Lemma 4.2. *Let L/F be a field extension such that the quadric X_L is isotropic. Then $S^i(\rho_L) = 0$ for every $i > 0$.*

Proof. By the uniqueness of a modulo 2 Rost projector on an isotropic quadric (Remark 2.1) $\rho_L = [X] \times [x] + [x] \times [X]$, where $x \in X_L$ is a rational point. Since $S = S^0 = \mathrm{id}$ on $\mathrm{CH}^0(X_L) \ni [X]$ as well as on $\mathrm{CH}^n(X_L) \ni [x]$, we have

$$S(\rho_L) = S([X] \times [x] + [x] \times [X]) = S([X]) \times S([x]) + \\ S([x]) \times S([X]) = [X] \times [x] + [x] \times [X] = \rho_L = S^0(\rho_L)\ .$$

□

Lemma 4.3. *The Witt index of the quadratic form $\varphi_{F(X)}$ is 1.*

Proof. Let $Y \subset X$ be a subquadric of codimension 1. If $i_W(\varphi_{F(X)}) > 1$, Y has a rational point over $F(X)$. Therefore there exists a rational morphism $X \to Y$. Let $\alpha \in \mathbb{CH}^n(X \times X)$ be the correspondence given by the closure of the graph of this morphism. Let us show that $\Delta^*(\varrho \circ \alpha) \in \mathbb{CH}_0(X)$, where $\Delta\colon X \to X \times X$ is the diagonal morphism, is an element of $\mathbb{CH}_0(X)$ of degree 1 (giving a contradiction with the fact that the quadric X is anisotropic, this will finish the proof).

Clearly, verifying the assertion on the degree, we may replace F by a field extension of F. Therefore, we may assume that X has a rational point x. In this case $\varrho = [X \times x] + [x \times X]$. Since $[x \times X] \circ \alpha = 0$ (because $\dim Y < \dim X$) while $[X \times x] \circ \alpha = [X \times x]$, we get $\Delta^*(\varrho \circ \alpha) = [x]$. □

Lemma 4.4. *Let X be an anisotropic F-quadric such that the Witt index of the quadratic form $\varphi_{F(X)}$ is 1. Then for every $\alpha \in \mathbb{CH}_i(X_{F(X)})$, $i > 0$, the degree of the 0-cycle class $h^i \cdot \alpha$ is even.*

Proof. It is sufficient to consider the case $i = 1$. We have $\varphi_{F(X)} \simeq \psi \perp \mathbb{H}$ for an anisotropic quadratic form ψ over $F(X)$ (where $\mathbb{H}$ is a hyperbolic plane). Let X' be the quadric $\psi = 0$ over $F(X)$. There is an isomorphism [5, §2.2]

$$f : \mathbb{CH}_1(X_{F(X)}) \to \mathbb{CH}_0(X')$$

taking h^{n-1} to the class of a closed point of degree 2. Since the quadric X' is anisotropic, the group $\mathbb{CH}_0(X')$ is generated by the class of a degree 2 closed point (see [5, prop. 2.6] or [13]); therefore the group $\mathbb{CH}_1(X_{F(X)})$ is generated by h^{n-1}. Since $\deg(h \cdot h^{n-1}) = 2$, it follows that the integer $\deg(h \cdot \alpha)$ is even for every $\alpha \in \mathbb{CH}_1(X_{F(X)})$. □

For the modulo 2 Chow groups we get

Corollary 4.5. *Let $\mu \in \mathrm{CH}_i(X \times X)$ for some $0 < i \le n$ be a correspondence such that $\mu_{F(X)} = 0$. Then $\mu_*(h^i) = 0$.*

Proof. We replace μ by its representative in $\mathbb{CH}_i(X \times X)$ and we mean by h the integral class of a hyperplane section of X in the proof (while in the statement h is the class of a hyperplane section in the modulo 2 Chow group). Since the degree homomorphism $\deg : \mathbb{CH}^n(X) \to \mathbb{Z}$ is injective ([5, prop. 2.6] or [13]) with the image $2\mathbb{Z}$, it suffices to show that $\deg(\mu_*(h^i))$ is divisible by 4. Let us compute this degree. By definition of μ_*, we have $\mu_*(h^i) = pr_{2*}\big(\mu \cdot pr_1^*(h^i)\big)$. Note that the product $\mu \cdot pr_1^*(h^i)$ is in $\mathbb{CH}_0(X \times X)$ and the square

$$\begin{array}{ccc} \mathbb{CH}_0(X \times X) & \xrightarrow{pr_{2*}} & \mathbb{CH}_0(X) \\ {\scriptstyle pr_{1*}}\big\downarrow & & {\scriptstyle \deg}\big\downarrow \\ \mathbb{CH}_0(X) & \xrightarrow{\deg} & \mathbb{Z} \end{array}$$

commutes (the two compositions being the degree homomorphism of the group $\mathbb{CH}_0(X \times X)$). Therefore the degree of $\mu_*(h^i)$ coincides with the degree of $pr_{1*}\big(\mu \cdot pr_1^*(h^i)\big)$. By the projection formula for pr_{1*} the latter element coincides with the product $h^i \cdot pr_{1*}(\mu)$.
We are going to check that the degree of this element is divisible by 4. Since the degree does not change under extensions of the base field, it suffices to verify the divisibility relation over $F(X)$. The class $pr_{1*}(\mu)_{F(X)}$ is divisible by 2 by assumption, therefore the statement follows from Lemmas 4.3 and 4.4. □

Corollary 4.6. *$S^{n-i}(\rho)_*(h^i) = 0$ for every i with $0 < i < n$.*

Proof. We take $\mu = S^{n-i}(\rho)$. Since $i < n$, we have $\mu_{F(X)} = 0$ by Lemma 4.2. Since $i > 0$, we may apply Corollary 4.5 obtaining $\mu_*(h^i) = 0$. □

Putting together Corollary 4.6 and Proposition 4.1, we get

Corollary 4.7. *For every $i > 0$, one has:*

$$S^{n-i}\big(\rho_*(h^i)\big) = \binom{i-n-2}{n-i} \cdot \rho_*(h^n)\ .$$

Proof. By Proposition 4.1, $S^{n-i}\big(\rho_*(h^i)\big)$ is the n-codimensional component of $S(\rho)_*\big(h^i \cdot (1+h)^{i-n-2}\big)$; moreover, according to Corollary 4.6, $S(\rho)$ can be replaced by $S^0(\rho) = \rho$. □

Finally, by Corollary 1.3, computing the binomial coefficient modulo 2, together with Lemma 2.4, computing $\rho_*(h^n)$, we get

Theorem 4.8. *Suppose that the anisotropic quadric X of dimension n possesses a Rost projector. Let ρ be a modulo 2 Rost projector on X and let i be an integer with $0 < i < n$. Then*

$$S^{n-i}\big(\rho_*(h^i)\big) = h^n$$

in $\mathrm{CH}_0(X)$ if (and only if) the integer $n-i+1$ is a power of 2. □

As the quadric X is anisotropic, $\mathbb{CH}_0(X)$ is an infinite cyclic group generated by h^n (see [5, prop. 2.6] or [13]); in particular, h^n in $\mathrm{CH}_0(X)$ is not 0. Therefore we get

Corollary 4.9. *For every i such that $0 < i < n$ and $n-i+1$ is a power of 2, the element $S^{n-i}(\rho_*(h^i))$ (and consequently $\rho_*(h^i)$) is non-zero.* □

5. Dimensions of quadrics with Rost projectors

The following Theorem is proved in [3]. The proof given there makes use of the Steenrod operations in the motivic cohomology constructed by Voevodsky (since Voevodsky has announced that the operations were constructed in any characteristic $\neq 2$ only quite recently, the assumption $\operatorname{char} F = 0$ was made in [3]). Here we give an elementary proof.

Theorem 5.1 ([3, th. 6.1]). *If X is an anisotropic smooth projective quadric possessing a Rost projector, then $\dim X + 1$ is a power of 2.*

Proof. Let us assume that this is not the case. Let r be the largest integer such that $n > 2^r - 1$ where $n = \dim X$. Then Theorem 4.8 applies to $i = n - (2^r - 1)$, stating that $S^{n-i}\big(\rho_*(h^i)\big) \neq 0$. Note that $n - i \geq i$. Since the Steenrod operation S^i is trivial on $\mathrm{CH}^j(X)$ with $i > j$, it follows that $n - i = i$ and therefore $S^{n-i}\big(\rho_*(h^i)\big) = \rho_*(h^i)^2$. Since the element $\varrho_*(h^i)$ (where ϱ is the integral Rost projector) vanishes over $\bar{F}$ (Lemma 2.4), its square vanishes over $\bar{F}$ as well. The group $\mathbb{CH}_0(X)$ injects however into $\mathbb{CH}_0(X_{\bar{F}})$, hence $\varrho_*(h^i)^2 = 0$ and therefore $S^{n-i}\big(\rho_*(h^i)\big) = 0$, giving a contradiction with Corollary 4.9. □

Remark 5.2. It turns out that Theorem 5.1 is extremely useful in the theory of quadratic forms. For example, it is the main ingredient of Vishik's proof of the theorem that there is no anisotropic quadratic forms satisfying $2^r < \dim \varphi < 2^r + 2^{r-1}$ and $[\varphi] \in I^r(F)$ (see [14], [15]).

6. Rost motives

Let Λ be an associative commutative ring with 1. We set $\Lambda\mathbb{CH} = \Lambda \otimes_{\mathbb{Z}} \mathbb{CH}$ (we will only need $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{Z}/2$).

We briefly recall the construction of the category of Grothendieck $\Lambda\mathbb{CH}$-motives similar to that of [4]. A motive is a triple (X, p, n), where X is a smooth projective equidimensional F-variety, $p \in \Lambda\mathbb{CH}^{\dim X}(X \times X)$ an idempotent correspondence, and n an integer. Sometimes the reduced notations are used: (X, n) for (X, p, n) with p the diagonal class; (X, p) for (X, p, n) with $n = 0$; and (X) for $(X, 0)$, the motive of the variety X.

For a motive $M = (X, p, n)$ and an integer m, the m-th twist $M(m)$ of M is defined as $(X, p, n + m)$.

The set of morphisms is defined as

$$\operatorname{Hom}\big((X, p, n), (X', p', n')\big) = p' \circ \Lambda\mathbb{CH}^{\dim X' - n + n'}(X \times X') \circ p \, .$$

In particular, *every* homogeneous correspondence $\alpha \in \Lambda\mathbb{CH}(X \times X')$ determines a morphism of *every* twist of (X, p) to a certain twist of (X', p').

The Chow group $\Lambda\mathbb{CH}_*(X, p, n)$ of a motive (X, p, n) is defined as

$$\Lambda\mathbb{CH}_*(X, p, n) = p_* \, \Lambda\mathbb{CH}_{*-n}(X) \, .$$

It gives an additive functor of the category of $\Lambda\mathbb{CH}$-motives to the category of graded abelian groups (namely, the functor $\operatorname{Hom}(M(*), -)$, where M is the motive of a point).

For any Λ, there is an evident additive functor of the category of $\mathbb{CH}$-motives to the category of $\Lambda\mathbb{CH}$-motives (identical on the motives of varieties). In particular, every isomorphism of $\mathbb{CH}$-motives automatically produces an isomorphism of the corresponding $\Lambda\mathbb{CH}$-motives. This is why bellow we mostly formulate the results only on the integral motives.

We are coming back to the quadratic forms.

Definition 6.1. Let ϱ be an integral Rost projector on a projective quadric X. We refer to the motive (X, ϱ) as to an (integral) *Rost motive*. (While the CH-motive given by a modulo 2 Rost projector can be called a *modulo* 2 *Rost motive*.) A Rost motive is *anisotropic*, if the quadric X is so.

Let now π be a Pfister form and let φ be a neighbor of π which is *minimal*, that is, has dimension $\dim \pi/2+1$. As noticed by M. Rost (see [7, 5.2] for a proof), the projective quadric X given by φ possesses an integral Rost projector ϱ.

Proposition 6.2. *Let φ be as above. Let ϱ' be the Rost projector on the quadric X' given by a minimal neighbor φ' of another Pfister form π'. The Rost motives (X, ϱ) and (X', ϱ') are isomorphic if and only if the Pfister forms π and π' are isomorphic.*

Proof. First we assume that $(X, \varrho) \simeq (X', \varrho')$. Looking at the degrees of 0-cycles on X and on X', we see that φ is isotropic if and only if φ' is isotropic, thus π is isotropic if and only if π' is isotropic. Therefore, the forms $\pi_{F(\pi')}$ and $\pi'_{F(\pi)}$ are isotropic. Since π and π' are Pfister forms, it follows that $\pi \simeq \pi'$.
Conversely, assume that $\pi \simeq \pi'$. By [12, §3], in order to show that $(X, \varrho) \simeq (X', \varrho')$, it suffices to construct a morphism of motives $(X, \varrho) \to (X', \varrho')$ which becomes mutually inverse isomorphism over an algebraic closure $\bar{F}$ of F. We will do a little bit more: we construct two morphisms $(X, \varrho) \rightleftarrows (X', \varrho')$ which become mutually inverse isomorphisms over an algebraic closure $\bar{F}$ of F (in this case, the initial F-morphisms are isomorphisms, although possibly not mutually inverse ones, [7, cor. 3.3]).
Since $\pi \simeq \pi'$, the quadratic forms $\varphi'_{F(\varphi)}$ and $\varphi_{F(\varphi')}$ are isotropic. Therefore there exist rational morphisms $X \to X'$ and $X' \to X$. The closures of their graphs give two correspondences $\alpha \in \mathbb{CH}(X \times X')$ and $\beta \in \mathbb{CH}(X' \times X)$.
Over $\bar{F}$ we have: $\varrho' \circ \alpha \circ \varrho = [X \times x'] + a[x \times X']$, where $x \in X_{\bar{F}}$ and $x' \in X'_{\bar{F}}$ are closed rational points, while a is an integer (which coincides, in fact, with the degree of the rational morphism $X \to X'$). Similarly, $\varrho \circ \beta \circ \varrho' = [X' \times x] + b[x' \times X]$ with some $b \in \mathbb{Z}$ over $\bar{F}$.
We are going to check that the integers a and b are odd. For this we consider the composition

$$(\varrho \circ \beta \circ \varrho') \circ (\varrho' \circ \alpha \circ \varrho) \in \mathbb{CH}(X \times X) .$$

Over $\bar{F}$ this composition gives $[X \times x] + ab[x \times X]$. Consequently, by [6, th. 6.4] and Lemma 4.3, the integer ab is odd.

Let us take now

$$\alpha' = \alpha - \frac{a-1}{2} \cdot [y \times X'] \quad \text{and} \quad \beta' = \beta - \frac{b-1}{2} \cdot [y' \times X]$$

with some degree 2 closed points $y \in X$ and $y' \in X'$. Then over $\bar{F}$

$$\varrho' \circ \alpha' \circ \varrho = [X \times x'] + [x \times X'] \quad \text{while} \quad \varrho \circ \beta' \circ \varrho' = [X' \times x] + [x' \times X] \ ,$$

therefore the two F-morphisms $(X, \varrho) \rightleftarrows (X', \varrho')$ given by these α' and β' become mutually inverse isomorphisms over $\bar{F}$. □

Definition 6.3. The motive (X, ϱ) for X and ϱ as in Proposition 6.2 (more precisely, the isomorphism class of motives) is called the *Rost motive of the Pfister form* π and denoted $R(\pi)$.

Remark 6.4. It is conjectured in [7, conj. 1.6] (with a proof given for 3 and 7-dimensional quadrics) that every anisotropic Rost motive is the Rost motive of some Pfister form.

7. Motivic decompositions of excellent quadrics

Theorem 7.1 (announced in [11]). *Let φ be a neighbor of a Pfister form π and let φ' be the complementary form (that is, φ' is such that the form $\varphi \perp \varphi'$ is similar to π). Then*

$$(X) \simeq \Big(\bigoplus_{i=0}^{m-1} R(\pi)(i) \Big) \bigoplus (X')(m) \ ,$$

where $m = (\dim \varphi - \dim \varphi')/2$, X is the quadric defined by φ, and X' is the quadric defined by φ'.

Proof. Similar to [12, th. 17] (see also [7, prop. 5.3]). □

We recall that a quadratic form φ over F is called *excellent*, if for every field extension E/F the anisotropic part of the form φ_E is defined over F. An anisotropic quadratic form is excellent if and only if it is a Pfister neighbor whose complementary form is excellent as well [8, §7].

Let $\pi_0 \supset \pi_1 \supset \cdots \supset \pi_r$ be a strictly decreasing sequence of embedded Pfister forms. Let φ be the quadratic form such that the class $[\varphi]$ of φ in the Witt ring of F is the alternating sum $[\pi_0] - [\pi_1] + \cdots + (-1)^r [\pi_r]$, while the dimension of φ is the alternating sum of the dimensions of the Pfister forms. Clearly, φ is excellent. Moreover, every anisotropic excellent quadratic form is similar to a form obtained this way. Let us require additionally that $2 \dim \pi_r < \dim \pi_{r-1}$. Then every anisotropic excellent quadratic form is still similar to a form obtained this way and, moreover, the Pfister forms $\pi_0, \ldots, \pi_r$ are uniquely determined by the initial excellent quadratic form.

Let X be an *excellent* quadric, that is, the quadratic form φ giving X is excellent. As Theorem 7.1 shows, the motive of X is a direct sum of twisted Rost motives. More precisely, Corollary 7.2 (announced in [11]). *Let X be the excellent quadric determined by Pfister forms $\pi_0 \supset \cdots \supset \pi_r$. Then*

$$(X) \simeq \Big(\bigoplus_{i=0}^{m_0-1} R(\pi_0)(i) \Big) \bigoplus \Big(\bigoplus_{i=m_0}^{m_0+m_1-1} R(\pi_1)(i) \Big) \bigoplus \dots$$
$$\dots \bigoplus \Big(\bigoplus_{i=m_0+\cdots+m_{r-1}}^{m_0+\cdots+m_r} R(\pi_r)(i) \Big)$$

with $m_j = \dim \pi_j/2 - \dim \pi_{j+1} + \dim \pi_{j+2} - \dots$. □

Here are three examples of excellent forms which are most important for us:

Example 7.3 (Pfister forms, [12, prop. 19]). Let $\varphi = \pi$ be a Pfister form. Then

$$(X) \simeq \bigoplus_{i=0}^{\dim \pi/2-1} R(\pi)(i) \; .$$

Example 7.4 (Maximal neighbors, [12, th. 17]). Let φ be a *maximal* neighbor of a Pfister form π (that is, $\dim \varphi = \dim \pi - 1$). Then

$$(X) \simeq \bigoplus_{i=0}^{\dim \pi/2-2} R(\pi)(i)$$

Example 7.5 (Norm forms, [12, th. 17]). Let φ be a norm quadratic form, that is, φ is a minimal neighbor of a Pfister form π containing a 1-codimensional subform which is similar to a Pfister form π'. Then

$$(X) \simeq R(\pi) \bigoplus \Big(\bigoplus_{i=1}^{\dim \pi'/2-1} R(\pi')(i) \Big) \; .$$

8. Chow groups of Rost motives

The following theorem computes the Chow groups of the modulo 2 Rost motive of a Pfister form.

Theorem 8.1 (announced in [11]). *Let ρ be the modulo 2 Rost projector on the projective n-dimensional quadric X given by an anisotropic minimal Pfister neighbor. Let i be an integer with $0 \le i \le n$. If $i+1$ is a power of 2, then the Chow group $\mathrm{CH}_i(X,\rho) = \rho_* \mathrm{CH}_i(X)$ is cyclic of order 2 generated by $\rho_*(h^{n-i})$. Otherwise this group is 0.*

Proof. According to Proposition 6.2, we may assume that X is a *norm quadric*, that is, X contains a 1-codimensional subquadric Y being a Pfister quadric. Let r be the integer such that $n = \dim X = 2^r - 1$.

We proceed by induction on r. Let $Y' \subset Y$ be a subquadric of dimension $2^{r-1}-2$ which is a Pfister quadric. Let X' be a norm quadric of dimension $2^{r-1}-1$ such that $Y' \subset X' \subset Y$. Let ρ' be a modulo 2 Rost projector on X'. By Example 7.5, passing from $\mathbb{CH}$-motives to the category of CH-motives, we see that the motive of X is the direct sum of the motive (X,ρ) and the motives

(X', ρ', i) with $i = 1, \dots, 2^{r-1} - 1$. Therefore

$$\mathrm{CH}(X) \simeq \rho_* \mathrm{CH}(X) \bigoplus \Big(\bigoplus_{i=1}^{2^{r-1}-1} \rho'_* \mathrm{CH}(X') \Big)$$

(we do not care about the gradations on the Chow groups).
Also the motive of Y decomposes in the direct sum of the motives (X', ρ', i) with $i = 0, \dots, 2^{r-1} - 1$ (Example 7.3). Therefore

$$\mathrm{CH}(Y) \simeq \bigoplus_{i=0}^{2^{r-1}-1} \rho'_* \mathrm{CH}(X') \ .$$

It follows that the order of the group $\mathrm{CH}(Y)$ is $|\rho'_* \mathrm{CH}(X')|^{2^{r-1}}$, while the order of $\mathrm{CH}(X)$ is $|\rho'_* \mathrm{CH}(X')|^{2^{r-1}-1} \cdot |\rho_* \mathrm{CH}(X)|$.
In the exact sequence

$$\mathrm{CH}(Y) \to \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0$$

with $U = X \setminus Y$, the Chow group $\mathrm{CH}(U)$ of the affine norm quadric U is computed by M. Rost ([7, th. A.4]): $\mathrm{CH}(U) = \mathrm{CH}^0(U) \simeq \mathbb{Z}/2$. Therefore, the orders of these groups satisfy

$$|\,\mathrm{CH}(X)| \le |\,\mathrm{CH}(Y)| \cdot |\,\mathrm{CH}(U)| = 2|\,\mathrm{CH}(Y)| \ ,$$

thus $|\rho_* \mathrm{CH}(X)| \le 2|\rho'_* \mathrm{CH}(X')|$.
The group $\rho'_* \mathrm{CH}(X')$ is known by induction. In particular, the order of this group is 2^r. It follows that the order of $\rho_* \mathrm{CH}(X)$ is at most 2^{r+1}. Corollary 4.9 gives already $r+1$ non-zero elements of $\rho_* \mathrm{CH}_*(X)$ living in different dimensions (more precisely, $\rho_*(h^{n-2^s+1}) \neq 0$ for $s = 1, \dots, r-1$ by Corollary 4.9 and for $s = 0, r$ by Lemma 2.4) and therefore generating a subgroup of order 2^{r+1}. It follows that the order of $\rho_* \mathrm{CH}(X)$ is precisely 2^{r+1} and the non-zero elements we have found generate the group $\rho_* \mathrm{CH}(X)$. □

The integral version of Theorem 8.1 is given by

Corollary 8.2 (announced in [11]). *For X as in Theorem 8.1, let ϱ be the integral Rost projector on X. Then for every i with $0 \le i \le n$, the Chow group $\mathbb{CH}_i(X, \varrho) = \varrho_* \mathbb{CH}_i(X)$ is a cyclic group generated by $\varrho_*(h^{n-i})$. Moreover, the element $\varrho_*(h^{n-i})$ is*

- *0, if $i+1$ is not a power of 2;*
- *of order 2, if $i+1$ is a power of 2 and $i \notin \{0, n\}$;*
- *of infinite order, if $i \in \{0, n\}$.*

Proof. The statements on $\mathbb{CH}_n(X)$ and on $\mathbb{CH}_0(X)$ are clear. The rest follows from Theorem 8.1, if we show that $2 \cdot \varrho_* \mathbb{CH}_i(X) = 0$ for every i with $0 < i < n$. Let L/F be a quadratic extension such that X_L is isotropic. Then $(\varrho_L)_* \mathbb{CH}_i(X_L) = 0$ for such i by [7, cor. 4.2] (cf. Lemma 2.4). Since the composition of the restriction $\mathbb{CH}_i(X) \to \mathbb{CH}_i(X_L)$ with the transfer $\mathbb{CH}_i(X_L) \to \mathbb{CH}_i(X)$ coincides with the multiplication by 2, it follows that $2 \cdot \varrho_* \mathbb{CH}_i(X) = 0$. □

REMARK 8.3. The result of Corollary 8.2 was announced in [11]. A proof has never appeared.

REMARK 8.4. Clearly, Corollary 8.2 describes the Chow group of the Rost motive of an anisotropic Pfister form. Since the motive of any anisotropic excellent quadric is a direct sum of twists of such Rost motives (Corollary 7.2), we have computed the Chow group of an arbitrary anisotropic excellent projective quadric. Note that the answer depends only on the dimension of the quadric.

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