Unimodular Covers of Multiples of Polytopes

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Abstract. Let P be a d-dimensional lattice polytope. We show that there exists a natural number c_d, only depending on d, such that the multiples cP have a unimodular cover for every natural number $c \geq c_d$. Actually, an explicit upper bound for c_d is provided, together with an analogous result for unimodular covers of rational cones.

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1. Statement of results

All polytopes and cones considered in this paper are assumed to be convex. A polytope $P \subset \mathbb{R}^d$ is called a *lattice polytope*, or *integral polytope*, if its vertices belong to the standard lattice $\mathbb{Z}^d$. For a (not necessarily integral) polytope $P \subset \mathbb{R}^d$ and a real number $c \geq 0$ we let cP denote the image of P under the dilatation with factor c and center at the origin $O \in \mathbb{R}^d$. A polytope of dimension e is called an *e-polytope*.

A *simplex* Δ is a polytope whose vertices $v_0, \dots, v_e$ are affinely independent (so that $e = \dim \Delta$). The *multiplicity* $\mu(\Delta)$ of a lattice simplex is the index of the subgroup U generated by the vectors $v_1 - v_0, \dots, v_e - v_0$ in the smallest direct summand of $\mathbb{Z}^d$ containing U, or, in other words, the order of the torsion subgroup of $\mathbb{Z}^d/U$. A simplex of multiplicity 1 is called *unimodular*. If $\Delta \subset \mathbb{R}^d$ has the full dimension d, then $\mu(\Delta) = d! \operatorname{vol}(\Delta)$, where vol is the Euclidean volume. The union of all unimodular d-simplices inside a d-polytope P is denoted by $\mathrm{UC}(P)$.

In this paper we investigate for which multiples cP of a lattice d-polytope one can guarantee that $cP = \mathrm{UC}(cP)$. To this end we let $\mathfrak{c}_d^{\mathrm{pol}}$ denote the infimum of the natural numbers c such that $c'P = \mathrm{UC}(c'P)$ for all lattice d-polytopes

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P and all natural numbers $c' \geq c$. A priori, it is not excluded that $\mathfrak{c}_d^{\mathrm{pol}} = \infty$ and, to the best of our knowledge, it has not been known up till now whether $\mathfrak{c}_d^{\mathrm{pol}}$ is finite except for the cases $d = 1, 2, 3$: $\mathfrak{c}_1^{\mathrm{pol}} = \mathfrak{c}_2^{\mathrm{pol}} = 1$ and $\mathfrak{c}_3^{\mathrm{pol}} = 2$, where the first equation is trivial, the second is a crucial step in the derivation of Pick's theorem, and a proof of the third can be found in Kantor and Sarkaria [KS]. Previous results in this direction were obtained by Lagarias and Ziegler (Berkeley 1997, unpublished).

The main result of this paper is the following upper bound, positively answering Problem 4 in [BGT2]:

Theorem 1.1. *For all natural numbers $d > 1$ one has*

$$\mathfrak{c}_d^{\mathrm{pol}} \leq O\big(d^5\big)\left(\frac{3}{2}\right)^{\lceil\sqrt{d-1}\rceil(d-1)}.$$

Theorem 1.1 is proved by passage to cones, for which we establish a similar result on covers by unimodular subcones (Theorem 1.3 below). This result, while interesting of its own, implies Theorem 1.1 and has the advantage of being amenable to a proof by induction on d.

We now explain some notation and terminology. The convex hull of a set $X \subset \mathbb{R}^d$ is denoted by $\operatorname{conv}(X)$, and $\operatorname{Aff}(X)$ is its affine hull. Moreover, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.

A lattice simplex is called *empty* if its vertices are the only lattice points in it. Every unimodular simplex is empty, but the opposite implication is false in dimensions ≥ 3. (In dimension 2 empty simplices are unimodular.)

A *cone* (without further predicates) is a subset of $\mathbb{R}^d$ that is closed under linear combinations with coefficients in $\mathbb{R}_+$. All cones considered in this paper are assumed to be *polyhedral*, *rational* and *pointed* (i. e. not to contain an affine line); in particular they are generated by finitely many rational vectors. For such a cone C the semigroup $C \cap \mathbb{Z}^d$ has a unique finite minimal set of generators, called the *Hilbert basis* and denoted by $\operatorname{Hilb}(C)$. The *extreme (integral) generators* of a rational cone $C \subset \mathbb{R}^d$ are, by definition, the generators of the semigroups $l \cap \mathbb{Z}^d \approx \mathbb{Z}_+$ where l runs through the edges of C. The extreme integral generators of C are members of $\operatorname{Hilb}(C)$. We define Δ_C to be the convex hull of O and the extreme integral generators of C.

A cone C is *simplicial* if it has a linearly independent system of generators. Thus C is simplicial if and only if Δ_C is a simplex. We say that C is *empty simplicial* if Δ_C is an empty simplex. The *multiplicity* of a simplicial cone is $\mu(\Delta_C)$. If Δ is a lattice simplex with vertex O, then the multiplicity of the cone $\mathbb{R}_+\Delta$ divides $\mu(\Delta)$. This follows easily from the fact that each non-zero vertex of Δ is an integral multiple of an extreme integral generator of $\mathbb{R}_+\Delta$.

A *unimodular* cone $C \subset \mathbb{R}^d$ is a rational simplicial cone for which Δ_C is a unimodular simplex. Equivalently we could require that C is simplicial and its extreme integral generators generate a direct summand of $\mathbb{Z}^d$. A *unimodular cover* of an arbitrary rational cone C is a finite system of unimodular cones

whose union is C. A *unimodular triangulation* of a cone is defined in the usual way – it is a unimodular cover whose member cones coincide along faces.
In addition to the cones C with apex in the origin O, as just introduced, we will sometimes have to deal with sets of the form $v + C$ where $v \in \mathbb{R}^d$. We call $v + C$ a *cone with apex* v.
We define $\mathfrak{c}_d^{\mathrm{cone}}$ to be the infimum of all natural numbers c such that every rational d-dimensional cone $C \subset \mathbb{R}^d$ admits a unimodular cover $C = \bigcup_{j=1}^k C_j$ for which

$$\mathrm{Hilb}(C_j) \subset c\Delta_C \quad j \in [1, k].$$

Remark 1.2. We will often use that a cone C can be triangulated into empty simplicial cones C' such that $\Delta_{C'} \subset \Delta_C$. In fact, one first triangulates C into simplicial cones generated by extreme generators of C. After this step one can assume that C is simplicial with extreme generators $v_1, \dots, v_d$. If Δ_C is not empty, then we use stellar subdivision along a ray through some $v \in \Delta_C \cap \mathbb{Z}^d$, $v \neq 0, v_1, \dots, v_d$, and for each of the resulting cones C' the simplex $\Delta_{C'}$ has a smaller number of integral vectors than Δ_C. In proving a bound on $\mathfrak{c}_d^{\mathrm{cone}}$ it is therefore enough to consider empty simplicial cones.
Similarly one triangulates every lattice polytope into empty simplices.

Results on $\mathfrak{c}_d^{\mathrm{cone}}$ seem to be known only in dimensions ≤ 3. Since the empty simplicial cones in dimension 2 are exactly the unimodular 2-cones (by a well known description of Hilbert bases in dimension 2, see Remark 4.2) we have $\mathfrak{c}_2^{\mathrm{cone}} = 1$. Moreover, it follows from a theorem of Sebő [S1] that $\mathfrak{c}_3^{\mathrm{cone}} = 2$. In fact Sebő has shown that a 3-dimensional cone C can be triangulated into unimodular cones generated by elements of $\mathrm{Hilb}(C)$ and that $\mathrm{Hilb}(C) \subset (d-1)\Delta_C$ in all dimensions d (see Remark 1.4(f)).
We can now formulate the main result for unimodular covers of rational cones:

Theorem 1.3. *For all $d \geq 2$ one has*

$$\mathfrak{c}_d^{\mathrm{cone}} \leq \left\lceil \sqrt{d-1}\, \right\rceil (d-1) \frac{d(d+1)}{2} \left(\frac{3}{2}\right)^{\left\lceil \sqrt{d-1} \right\rceil (d-1) - 2}.$$

Remark 1.4. (a) We have proved in [BGT1, Theorem 1.3.1] that there is a natural number c_P for a lattice polytope $P \subset \mathbb{R}^d$ such that $cP = \mathrm{UC}(cP)$ whenever $c \geq c_P$, $c \in \mathbb{N}$. However, neither did the proof in [BGT1] provide an explicit bound for c_P, nor was it clear that the numbers c_P can be uniformly bounded with respect to all d-dimensional polytopes. The proof we present below is an essential extension of that of [BGT1, Theorem 1.3.1].
(b) It has been proved in [KKMS, Theorem 4, Ch. III] that for every lattice polytope P there exists a natural number c such that cP admits even a regular triangulation into unimodular simplices. This implies that $c'cP$ also admits such a triangulation for $c' \in \mathbb{N}$. However, the question whether there exists a natural number c_P^{triang} such that the multiples $c'P$ admit unimodular triangulations for all $c' \geq c_P^{\mathrm{triang}}$ remains open. In particular, the existence of a uniform bound $\mathfrak{c}_d^{\mathrm{triang}}$ (independent of P) remains open.

(c) The main difficulty in deriving better estimates for $\mathfrak{c}_d^{\mathrm{pol}}$ lies in the fundamental open problem of an effective description of the empty lattice d-simplices; see Haase and Ziegler [HZ] and Sebő [S2] and the references therein.
(d) A chance for improving the upper bound in Theorem 1.1 to, say, a polynomial function in d would be provided by an algorithm for resolving toric singularities which is faster then the standard one used in the proof of Theorem 4.1 below. Only there exponential terms enter our arguments.
(e) A lattice polytope $P \subset \mathbb{R}^d$ which is covered by unimodular simplices is *normal*, i. e. the additive subsemigroup

$$S_P = \sum_{x \in P \cap \mathbb{Z}^d} \mathbb{Z}_+(x,1) \subset \mathbb{Z}^{d+1}$$

is normal and, moreover, $\mathrm{gp}(S_P) = \mathbb{Z}^{d+1}$. (The normality of S_P is equivalent to the normality of the K-algebra $K[S_P]$ for a field K.) However, there are normal lattice polytopes in dimension ≥ 5 which are not unimodularly covered [BG]. On the other hand, if $\dim P = d$ then cP is normal for arbitrary $c \geq d-1$ [BGT1, Theorem 1.3.3(a)] (and $\mathrm{gp}(S_{cP}) = \mathbb{Z}^{d+1}$, as is easily seen). The example found in [BG] is far from being of type cP with $c > 1$ and, correspondingly, we raise the following question: is $\mathfrak{c}_d^{\mathrm{pol}} = d-1$ for all natural numbers $d > 1$? As mentioned above, the answer is 'yes' for $d = 2, 3$, but we cannot provide further evidence for a positive answer.
(f) Suppose $C_1, \dots, C_k$ form a unimodular cover of C. Then $\mathrm{Hilb}(C_1) \cup \cdots \cup \mathrm{Hilb}(C_k)$ generates $C \cap \mathbb{Z}^d$. Therefore $\mathrm{Hilb}(C) \subset \mathrm{Hilb}(C_1) \cup \cdots \cup \mathrm{Hilb}(C_k)$, and so $\mathrm{Hilb}(C)$ sets a lower bound to the size of $\mathrm{Hilb}(C_1) \cup \cdots \cup \mathrm{Hilb}(C_k)$ relative to Δ_C. For $d \geq 3$ there exist cones C such that $\mathrm{Hilb}(C)$ is not contained in $(d-2)\Delta_C$ (see Ewald and Wessels [EW]), and so one must have $\mathfrak{c}_d^{\mathrm{cone}} \geq d-1$. On the other hand, $d-1$ is the best lower bound for $\mathfrak{c}_d^{\mathrm{cone}}$ that can be obtained by this argument since $\mathrm{Hilb}(C) \subset (d-1)\Delta_C$ for all cones C. We may assume that C is empty simplicial by Remark 1.2, and for an empty simplicial cone C we have

$$\mathrm{Hilb}(C) \subset \square_C \setminus (v_1 + \cdots + v_d - \Delta_C) \subset (d-1)\Delta_C$$

where

(i) $v_1, \dots, v_d$ are the extreme integral generators of C,
(ii) $\square_C$ is the semi-open parallelotope spanned by $v_1, \dots, v_d$, that is,

$$\square_C = \{\xi_1 v_1 + \cdots + \xi_d v_d : \xi_1, \dots, \xi_d \in [0,1)\}.$$

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2. Slope independence

By $[0,1]^d = \{(z_1, \dots, z_d) \mid 0 \leq z_1, \dots, z_d \leq 1\}$ we denote the standard unit d-cube. Consider the system of simplices

$$\Delta_\sigma \subset [0,1]^d, \quad \sigma \in S_d,$$

where S_d is the permutation group of $\{1,\dots,d\}$, and Δ_σ is defined as follows:

(i) $\Delta_\sigma = \operatorname{conv}(x_0, x_1, \dots, x_d)$,
(ii) $x_0 = O$ and $x_d = (1,\dots,1)$,
(iii) x_{i+1} differs from x_i only in the $\sigma(i+1)$st coordinate and $x_{i+1,\sigma(i+1)} = 1$ for $i \in [0, d-1]$.

Then $\{\Delta_\sigma\}_{\sigma\in S_d}$ is a unimodular triangulation of $[0,1]^d$ with additional good properties [BGT1, Section 2.3]. The simplices Δ_σ and their integral parallel translates triangulate the entire space $\mathbb{R}^d$ into affine Weyl chambers of type A_d. The induced triangulations of the integral multiples of the simplex

$$\operatorname{conv}(O, e_1, e_1+e_2, \dots, e_1+\dots+e_d\} \subset \mathbb{R}^d$$

are studied in great detail in [KKMS, Ch. III]. All we need here is the very existence of these triangulations. In particular, the integral parallel translates of the simplices Δ_σ cover (actually, triangulate) the cone

$$\mathbb{R}_+e_1 + \mathbb{R}_+(e_1+e_2) + \dots + \mathbb{R}_+(e_1+\dots+e_d) \approx \mathbb{R}^d_+$$

into unimodular simplices.
Suppose we are given a real linear form

$$\alpha(X_1,\dots,X_d) = a_1X_1 + \dots + a_dX_d \neq 0.$$

The *width* of a polytope $P \subset \mathbb{R}^d$ in direction $(a_1,\dots,a_d)$, denoted by $\operatorname{width}_\alpha(P)$, is defined to be the Euclidean distance between the two extreme hyperplanes that are parallel to the hyperplane $a_1X_1+\dots+a_dX_d = 0$ and intersect P. Since $[0,1]^d$ is inscribed in a sphere of radius $\sqrt{d}/2$, we have $\operatorname{width}_\alpha(\Delta_\sigma) \le \sqrt{d}$ whatever the linear form α and the permutation σ are. We arrive at

Proposition 2.1. *All integral parallel translates of Δ_σ, $\sigma \in S_d$, that intersect a hyperplane H are contained in the $\sqrt{d}$-neighborhood of H.*

In the following we will have to consider simplices that are unimodular with respect to an affine sublattice of $\mathbb{R}^d$ different from $\mathbb{Z}^d$. Such lattices are sets

$$\mathcal{L} = v_0 + \sum_{i=1}^{e} \mathbb{Z}(v_i - v_0)$$

where $v_0,\dots,v_e$, $e \le d$, are affinely independent vectors. (Note that $\mathcal{L}$ is independent of the enumeration of the vectors $v_0,\dots,v_d$.) An e-simplex $\Delta = \operatorname{conv}(w_0,\dots,w_e)$ defines the lattice

$$\mathcal{L}_\Delta = w_0 + \sum_{i=0}^{e} \mathbb{Z}(w_i - w_0).$$

Let $\mathcal{L}$ be an affine lattice. A simplex Δ is called $\mathcal{L}$-*unimodular* if $\mathcal{L} = \mathcal{L}_\Delta$, and the union of all $\mathcal{L}$-unimodular simplices inside a polytope $P \subset \mathbb{R}^d$ is denoted by $\mathrm{UC}_{\mathcal{L}}(P)$. For simplicity we set $\mathrm{UC}_\Delta(P) = \mathrm{UC}_{\mathcal{L}_\Delta}(P)$.
Let $\Delta \subset \Delta'$ be (not necessarily integral) d-simplices in $\mathbb{R}^d$ such that the origin $O \in \mathbb{R}^d$ is a common vertex and the two simplicial cones spanned by Δ and Δ'

at O are the same. The following lemma says that the $\mathcal{L}_\Delta$-unimodularly covered area in a multiple $c\Delta'$, $c \in \mathbb{N}$, approximates $c\Delta'$ with a precision independent of Δ'. The precision is therefore independent of the "slope" of the facets of Δ and Δ' opposite to O. The lemma will be critical both in the passage to cones (Section 3) and in the treatment of the cones themselves (Section 6).

Lemma 2.2. *For all d-simplices $\Delta \subset \Delta'$ having O as a common vertex at which they span the same cone, all real numbers ε, $0 < \varepsilon < 1$, and $c \geq \sqrt{d}/\varepsilon$ one has*

$$(c - \varepsilon c)\Delta' \subset \mathrm{UC}_\Delta(c\Delta').$$

Proof. Let $v_1, \dots, v_d$ be the vertices of Δ different from O, and let w_i, $i \in [1, d]$ be the vertex of Δ' on the ray $\mathbb{R}_+ v_i$. By a rearrangement of the indices we can achieve that

$$\frac{|w_1|}{|v_1|} \geq \frac{|w_2|}{|v_2|} \geq \dots \geq \frac{|w_d|}{|v_d|} \geq 1.$$

where $|\ |$ denotes Euclidean norm. Moreover, the assertion of the lemma is invariant under linear transformations of $\mathbb{R}^d$. Therefore we can assume that

$$\Delta = \operatorname{conv}(O, e_1, e_1 + e_2, \dots, e_1 + \dots + e_d).$$

Then $\mathcal{L}_\Delta = \mathbb{Z}^d$. The ratios above are also invariant under linear transformations. Thus

$$\frac{|w_1|}{|e_1|} \geq \frac{|w_2|}{|e_1 + e_2|} \geq \dots \geq \frac{|w_d|}{|e_1 + \dots + e_d|} \geq 1.$$

Now Lemma 2.4 below shows that the distance h from O to the affine hyperplane $\mathcal{H}$ through $w_1, \dots, w_d$ is at least 1.

By Proposition 2.1, the subset

$$(c\Delta') \setminus U_{\sqrt{d}}(c\mathcal{H}) \subset c\Delta'$$

is covered by integral parallel translates of the simplices Δ_σ, $\sigma \in S_d$ that are contained in $c\Delta$. ($U_\delta(M)$ is the δ-neighborhood of M.) In particular,

$$(c\Delta') \setminus U_{\sqrt{d}}(c\mathcal{H}) \subset \mathrm{UC}_\Delta(c\Delta'). \tag{1}$$

Therefore we have

$$\big(1-\varepsilon\big)c\Delta' \subset \left(1-\frac{\sqrt{d}}{c}\right)c\Delta' \subset \left(1-\frac{\sqrt{d}}{ch}\right)c\Delta' = \frac{ch - \sqrt{d}}{ch}c\Delta' = (c\Delta')\setminus U_{\sqrt{d}}(c\mathcal{H}),$$

and the lemma follows from (1). □

Remark 2.3. One can derive an analogous result using the trivial tiling of $\mathbb{R}^d_+$ by the integral parallel translates of $[0,1]^d$ and the fact that $[0,1]^d$ itself is unimodularly covered. The argument would then get simplified, but the estimate obtained is $c \geq d/\varepsilon$, and thus worse than $c \geq \sqrt{d}/\varepsilon$.

We have formulated the Lemma 2.2 only for full dimensional simplices, but it holds for simplices of smaller dimension as well: one simply chooses all data relative to the affine subspace generated by Δ'.

Above we have used the following

Lemma 2.4. *Let $e_1, \dots, e_d$ be the canonical basis of $\mathbb{R}^d$ and set $w_i = \lambda_i(e_1 + \dots + e_i)$ where $\lambda_1 \geq \dots \geq \lambda_d > 0$. Then the affine hyperplane $\mathcal{H}$ through $w_1, \dots, w_d$ intersects the set $Q = \lambda_d(e_1 + \dots + e_d) - \mathbb{R}^d_+$ only in the boundary ∂Q. In particular the Euclidean distance from O to $\mathcal{H}$ is $\geq \lambda_d$.*

Proof. The hyperplane $\mathcal{H}$ is given by the equation

$$\frac{1}{\lambda_1}X_1 + \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)X_2 + \dots + \left(\frac{1}{\lambda_d} - \frac{1}{\lambda_{d-1}}\right)X_d = 1.$$

The linear form α on the left hand side has non-negative coefficients and $w_d \in \mathcal{H}$. Thus a point whose coordinates are strictly smaller than λ_d cannot be contained in $\mathcal{H}$. □

3. Passage to cones

In this section we want to relate the bounds for $\mathfrak{c}_d^{\mathrm{pol}}$ and $\mathfrak{c}_d^{\mathrm{cone}}$. This allows us to derive Theorem 1.1 from Theorem 1.3.

Proposition 3.1. *Let d be a natural number. Then $\mathfrak{c}_d^{\mathrm{pol}}$ is finite if and only if $\mathfrak{c}_d^{\mathrm{cone}}$ is finite, and, moreover,*

$$\mathfrak{c}_d^{\mathrm{cone}} \leq \mathfrak{c}_d^{\mathrm{pol}} \leq \sqrt{d}(d+1)\mathfrak{c}_d^{\mathrm{cone}}. \tag{2}$$

Proof. Suppose that $\mathfrak{c}_d^{\mathrm{pol}}$ is finite. Then the left inequality is easily obtained by considering the multiples of the polytope Δ_C for a cone C: the cones spanned by those unimodular simplices in a multiple of Δ_C that contain O as a vertex constitute a unimodular cover of C.

Now suppose that $\mathfrak{c}_d^{\mathrm{cone}}$ is finite. For the right inequality we first triangulate a polytope P into lattice simplices. Then it is enough to consider a lattice d-simplex $\Delta \subset \mathbb{R}^d$ with vertices $v_0, \dots, v_d$.

Set $c' = \mathfrak{c}_d^{\mathrm{cone}}$. For each i there exists a unimodular cover (D_{ij}) of the corner cone C_i of Δ with respect to the vertex v_i such that $c'\Delta - c'v_i$ contains $\Delta_{D_{ij}}$ for all j. Thus the simplices $\Delta_{D_{ij}} + c'v_i$ cover the corner of $c'\Delta$ at $c'v_i$, that is, their union contains a neighborhood of $c'v_i$ in $c'\Delta$.

We replace Δ by $c'\Delta$ and can assume that each corner of Δ has a cover by unimodular simplices. It remains to show that the multiples $c''\Delta$ are unimodularly covered for every number $c'' \geq \sqrt{d}(d+1)$ for which $c''P$ is an integral polytope.

Let

$$\omega = \frac{1}{d+1}(v_0 + \dots + v_d)$$

be the barycenter of Δ. We define the subsimplex $\Delta_i \subset \Delta$ as follows: Δ_i is the homothetic image of Δ with respect to the center v_i so that ω lies on the facet of Δ_i opposite to v_i. In dimension 2 this is illustrated by Figure 1. The factor of the homothety that transforms Δ into Δ_i is $d/(d+1)$. In particular, the simplices Δ_i are pairwise congruent. It is also clear that

$$\bigcup_{i=0}^{d} \Delta_i = \Delta. \tag{3}$$

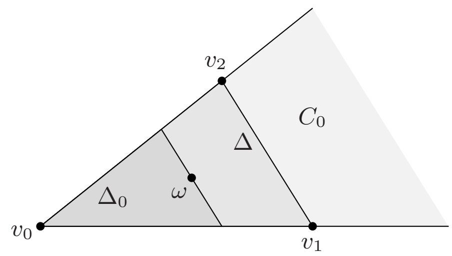

Figure 1.

The construction of ω and the subsimplices Δ_i commutes with taking multiples of Δ. It is therefore enough to show that $c''\Delta_i \subset \mathrm{UC}(c''\Delta)$ for all i. In order to simplify the use of dilatations we move v_i to O by a parallel translation.
In the case in which $v_i = O$ the simplices $c''\Delta$ and $c''\Delta_i$ are the unions of their intersections with the cones D_{ij}. This observation reduces the critical inclusion $c''\Delta_i \subset c''\Delta$ to

$$c''(\Delta_i \cap D_{ij}) \subset c''(\Delta \cap D_{ij})$$

for all j. But now we are in the situation of Lemma 2.2, with the unimodular simplex $\Delta_{D_{ij}}$ in the role of the Δ of 2.2 and $\Delta \cap D_{ij}$ in that of Δ'. For $\varepsilon = 1/(d+1)$ we have $c'' \geq \sqrt{d}/\varepsilon$ and so

$$c''(\Delta_i \cap D_{ij}) = c''\frac{d}{d+1}(\Delta \cap D_{ij}) = c''(1-\varepsilon)(\Delta \cap D_{ij}) \subset \mathrm{UC}(\Delta \cap D_{ij}),$$

as desired. $\square$

At this point we can deduce Theorem 1.1 from Theorem 1.3. In fact, using the bound for $\mathfrak{c}_d^{\mathrm{cone}}$ given in Theorem 1.3 we obtain

$$\begin{aligned}\mathfrak{c}_d^{\mathrm{pol}} &\leq \sqrt{d}(d+1)\mathfrak{c}_d^{\mathrm{cone}}\\ &\leq \sqrt{d}(d+1)\left\lceil\sqrt{d-1}\right\rceil(d-1)\frac{d(d+1)}{2}\left(\frac{3}{2}\right)^{\left\lceil\sqrt{d-1}\right\rceil(d-1)-2}\\ &\leq O\big(d^5\big)\left(\frac{3}{2}\right)^{\left\lceil\sqrt{d-1}\right\rceil(d-1)},\end{aligned}$$

as desired. (The left inequality in (2) has only been stated for completeness; it will not be used later on.)

4. Bounding toric resolutions

Let C be a simplicial rational d-cone. The following lemma gives an upper bound for the number of steps in the standard procedure to equivariantly resolve the toric singularity $\mathrm{Spec}(k[\mathbb{Z}^d \cap C])$ (see [F, Section 2.6] and [O, Section 1.5] for the background). It depends on d and the multiplicity of Δ_C. Exponential factors enter our estimates only at this place. Therefore any improvement of the toric resolution bound would critically affect the order of magnitude of the estimates of $\mathfrak{c}_d^{\mathrm{pol}}$ and $\mathfrak{c}_d^{\mathrm{cone}}$.

Theorem 4.1. *Every rational simplicial d-cone $C \subset \mathbb{R}^d$, $d \geq 3$, admits a unimodular triangulation $C = D_1 \cup \cdots \cup D_T$ such that*

$$\mathrm{Hilb}(D_t) \subset \left(\frac{d}{2}\left(\frac{3}{2}\right)^{\mu(\Delta_C)-2}\right)\Delta_C, \quad t \in [1, T].$$

Proof. We use the sequence h_k, $k \geq -(d-2)$, of real numbers defined recursively as follows:

$$h_k = 1, \quad k \leq 1, \qquad h_2 = \frac{d}{2}, \qquad h_k = \frac{1}{2}(h_{k-1} + \cdots + h_{k-d}), \quad k \geq 3.$$

One sees easily that this sequence is increasing, and that

$$\begin{aligned} h_k &= \frac{1}{2}h_{k-1} + \frac{1}{2}(h_{k-2} + \cdots + h_{k-d-1}) - \frac{1}{2}h_{k-d-1} = \frac{3}{2}h_{k-1} - \frac{1}{2}h_{k-d-1} \\ &\leq \frac{d}{2}\left(\frac{3}{2}\right)^{k-2} \end{aligned}$$

for all $k \geq 2$.

Let $v_1, \ldots, v_d$ be the extreme integral generators of C and denote by $\square_C$ the semi-open parallelotope

$$\big\{z \mid z = \xi_1 v_1 + \cdots + \xi_d v_d, \;\; 0 \leq \xi_1, \ldots, \xi_d < 1\big\} \subset \mathbb{R}^d.$$

The cone C is unimodular if and only if

$$\square_C \cap \mathbb{Z}^d = \{O\}.$$

If C is unimodular then the bound given in the theorem is satisfied (note that $d \geq 3$). Otherwise we choose a non-zero lattice point, say w, from $\square_C$,

$$w = \xi_{i_1} v_{i_1} + \cdots + \xi_{i_k} v_{i_k}, \quad 0 < \xi_{i_j} < 1.$$

We can assume that w is in $(d/2)\Delta_C$. If not, then we replace w by

$$v_{i_1} + \cdots + v_{i_k} - w. \tag{4}$$

The cone C is triangulated into the simplicial d-cones

$$C_j = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_{i_j-1} + \mathbb{R}_+ w + \mathbb{R}_+ v_{i_j+1} + \cdots + \mathbb{R}_+ v_d, \quad j = 1, \ldots, k.$$

Call these cones the *second generation cones*, C itself being of *first generation.* (The construction of the cones C_j is called *stellar subdivision* with respect to w.)

For the second generation cones we have $\mu(\Delta_{C_i}) < \mu(\Delta_C)$ because the volumes of the corresponding parallelotopes are in the same relation. Therefore we are done if $\mu(\Delta_C) = 2$.

If $\mu(\Delta_C) \geq 3$, we generate the $(k+1)$st generation cones by successively subdividing the kth generation *non-unimodular* cones. It is clear that we obtain a triangulation of C if we use each vector produced to subdivide all kth generation cones to which it belongs. Figure 2 shows a typical situation after 2 generations of subdivision in the cross-section of a 3-cone.

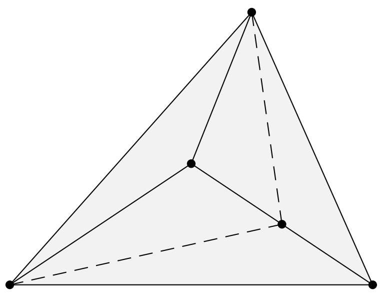

FIGURE 2.

If C'' is a next generation cone produced from a cone C', then $\mu(\Delta_{C''}) < \mu(\Delta_{C'})$, and it is clear that there exists $g \le \mu(\Delta_C)$ for which all cones of generation g are unimodular.
We claim that each vector $w^{(k)}$ subdividing a $(k-1)$st generation cone $C^{(k-1)}$ is in

$$h_k\ \Delta_C.$$

For $k = 2$ this has been shown already. So assume that $k \ge 3$. Note that all the extreme generators $u_1, \dots, u_d$ of $C^{(k-1)}$ either belong to the original vectors $v_1, \dots, v_d$ or were created in *different* generations. By induction we therefore have

$$u_i \in h_{t_i}\ \Delta_C, \qquad t_1, \dots, t_d \text{ pairwise different.}$$

Using the trick (4) if necessary, one can achieve that

$$w^{(k)} \in c\Delta_C, \qquad c \le \frac{1}{2}(h_{t_1} + \cdots + h_{t_d}).$$

Since the sequence (h_i) is increasing,

$$c \le \frac{1}{2}(h_{k-1} + \cdots + h_{k-d}) = h_k. \qquad \square$$

REMARK 4.2. (a) In dimension $d = 2$ the algorithm constructs a triangulation into unimodular cones D_t with $\operatorname{Hilb}(D_t) \subset \Delta_C$.
(b) For $d = 3$ one has Sebő's [S1] result $\operatorname{Hilb}(D_t) \subset 2\Delta_C$. It needs a rather tricky argument for the choice of w.

5. CORNER COVERS

Let C be a rational cone and v one of its extreme generators. We say that a system $\{C_j\}_{j=1}^k$ of subcones $C_j \subset C$ *covers the corner of* C *at* v if $v \in \operatorname{Hilb}(C_j)$ for all j and the union $\bigcup_{j=1}^k C_j$ contains a neighborhood of v in C.

LEMMA 5.1. *Suppose that $\mathfrak{c}_{d-1}^{\text{cone}} < \infty$, and let C be a simplicial rational d-cone with extreme generators $v_1, \dots, v_d$.*

(a) *Then there is a system of unimodular subcones $C_1, \dots, C_k \subset C$ covering the corner of C at v_1 such that* $\operatorname{Hilb}(C_1), \dots, \operatorname{Hilb}(C_k) \subset (\mathfrak{c}_{d-1}^{\text{cone}}+1)\Delta_C$.

(b) *Moreover, each element $w \neq v_1$ of a Hilbert basis of C_j, $j \in [1,k]$, has a representation $w = \xi_1 v_1 + \cdots + \xi_d v_d$ with $\xi_1 < 1$.*

Proof. For simplicity of notation we set $\mathfrak{c} = \mathfrak{c}_{d-1}^{\text{cone}}$. Let C' be the cone generated by $w_i = v_i - v_1$, $i \in [2,d]$, and let V be the vector subspace of $\mathbb{R}^d$ generated by the w_i. We consider the linear map $\pi : \mathbb{R}^d \to V$ given by $\pi(v_1) = 0$, $\pi(v_i) = w_i$ for $i > 0$, and endow V with a lattice structure by setting $\mathcal{L} = \pi(\mathbb{Z}^d)$. (One has $\mathcal{L} = \mathbb{Z}^d \cap V$ if and only if $\mathbb{Z}^d = \mathbb{Z}v_1 + (\mathbb{Z}^d \cap V)$.) Note that $v_1, z_2, \dots, z_d$ with $z_j \in \mathbb{Z}^d$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$ if and only if $\pi(z_2), \dots, \pi(z_d)$ are a $\mathbb{Z}$-basis of $\mathcal{L}$. This holds since $\mathbb{Z}v_1 = \mathbb{Z}^d \cap \mathbb{R}v_1$, and explains the unimodularity of the cones C_j constructed below.
Note that $w_i \in \mathcal{L}$ for all i. Therefore $\Delta_{C'} \subset \operatorname{conv}(O, w_2, \dots, w_d)$. The cone C' has a unimodular covering (with respect to $\mathcal{L}$) by cones C'_j, $j \in [1,k]$, with $\operatorname{Hilb}(C'_j) \subset \mathfrak{c}\Delta_{C'}$. We "lift" the vectors $x \in \operatorname{Hilb}(C'_j)$ to elements $\tilde{x} \in C$ as follows. Let $x = \alpha_2 w_2 + \cdots + \alpha_d w_d$ (with $\alpha_i \in \mathbb{Q}_+$). Then there exists a unique integer $n \geq 0$ such that

$$\begin{aligned} \tilde{x} := nv_1 + x &= nv_1 + \alpha_2(v_2 - v_1) + \cdots + \alpha_d(v_d - v_1) \\ &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_d v_d \end{aligned}$$

with $0 \leq \alpha_1 < 1$. (See Figure 3.) If $x \in \mathfrak{c}\Delta_{C'} \subset \mathfrak{c} \cdot \operatorname{conv}(O, w_2, \dots, w_d)$, then $\tilde{x} \in (\mathfrak{c}+1)\Delta_C$.

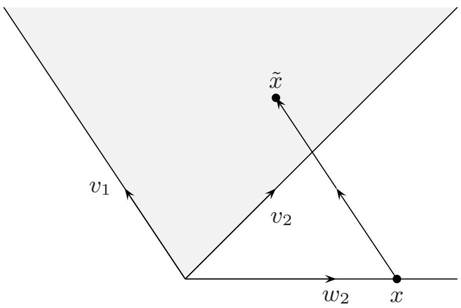

FIGURE 3.

We now define C_j as the cone generated by v_1 and the vectors $\tilde{x}$ where $x \in \operatorname{Hilb}(C'_j)$. It only remains to show that the C_j cover a neighborhood of v_1 in C. To this end we intersect C with the affine hyperplane $\mathcal{H}$ through $v_1, \dots, v_d$. It is enough that a neighborhood of v_1 in $C \cap \mathcal{H}$ is contained in $C_1 \cup \cdots \cup C_k$. For each $j \in [1,k]$ the coordinate transformation from the basis $w_2, \dots, w_d$ of V to the basis $x_2, \dots, x_d$ with $\{x_2, \dots, x_d\} = \operatorname{Hilb}(C'_j)$ defines a linear operator on $\mathbb{R}^{d-1}$. Let M_j be its $\| \ \|_\infty$ norm.
Moreover, let N_j be the maximum of the numbers n_i, $i \in [2,d]$ defined by the equation $\tilde{x}_i = n_i v_1 + x_i$ as above. Choose ε with

$$0 < \varepsilon \leq \frac{1}{(d-1)M_j N_j}, \qquad j \in [1,k].$$

and consider

$$y = v_1 + \beta_2 w_2 + \cdots + \beta_d w_d, \qquad 0 \le \beta_i < \varepsilon.$$

Since the C'_j cover C', one has $\beta_2 w_2 + \cdots + \beta_d w_d \in C'_j$ for some j, and therefore

$$y = v_1 + \gamma_2 x_2 + \cdots + \gamma_d x_d,$$

where $\{x_2, \dots, x_d\} = \operatorname{Hilb}(C'_j)$ and $0 \le \gamma_i \le M_j \varepsilon$ for $i \in [2, d]$. Then

$$y = \left(1 - \sum_{i=2}^{d} n_i \gamma_i\right) v_1 + \gamma_2 \tilde{x}_2 + \cdots + \gamma_d \tilde{x}_d$$

and

$$\sum_{i=2}^{d} n_i \gamma_i \le (d-1) N_j M_j \varepsilon \le 1,$$

whence $\left(1 - \sum_{i=2}^{d} n_i \gamma_i\right) \ge 0$ and $y \in C_j$, as desired. $\square$

6. The bound for cones

Before we embark on the proof of Theorem 1.3, we single out a technical step. Let $\{v_1, \dots, v_d\} \subset \mathbb{R}^d$ be a linearly independent subset. Consider the hyperplane

$$\mathcal{H} = \operatorname{Aff}(O, v_1 + (d-1)v_2, v_1 + (d-1)v_3, \dots, v_1 + (d-1)v_d) \subset \mathbb{R}^d$$

It cuts a simplex δ off the simplex $\operatorname{conv}(v_1, \dots, v_d)$ so that $v_1 \in \delta$. Let Φ denote the closure of

$$\mathbb{R}_+\delta \setminus \big(\big((1 + \mathbb{R}_+)v_1 + \mathbb{R}_+ e_2 + \cdots + \mathbb{R}_+ v_d\big) \cup \Delta\big) \subset \mathbb{R}^d.$$

where $\Delta = \operatorname{conv}(O, v_1, \dots, v_d)$. See Figure 4 for the case $d = 2$. The polytope

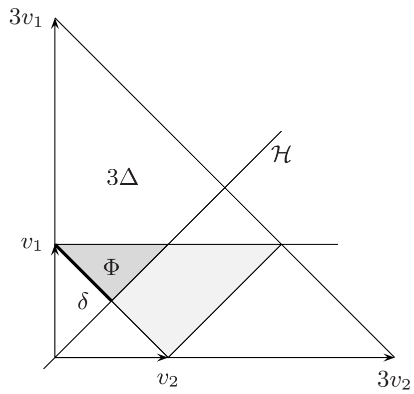

Figure 4.

$$\Phi' = -\frac{1}{d-1} v_1 + \frac{d}{d-1} \Phi$$

is the homothetic image of the polytope Φ under the dilatation with factor $d/(d-1)$ and center v_1. We will need that

$$\Phi' \subset (d+1)\Delta. \tag{5}$$

The easy proof is left to the reader.

Proof of Theorem 1.3. We want to prove the inequality

$$\mathfrak{c}_d^{\mathrm{cone}} \leq \left\lceil \sqrt{d-1} \right\rceil (d-1) \frac{d(d+1)}{2} \left(\frac{3}{2}\right)^{\left\lceil \sqrt{d-1} \right\rceil (d-1)-2} \tag{6}$$

for all $d \geq 2$ by induction on d.

The inequality holds for $d = 2$ since $\mathfrak{c}_2^{\mathrm{cone}} = 1$ (see the remarks preceding Theorem 1.3 in Section 1), and the right hand side above is 2 for $d = 2$. By induction we can assume that (6) has been shown for all dimensions $< d$. We set

$$\gamma = \left\lceil \sqrt{d-1}\, \right\rceil (d-1) \quad \text{and} \quad \kappa = \gamma \frac{d(d+1)}{2} \left(\frac{3}{2}\right)^{\gamma-2}.$$

As pointed out in Remark 1.2, we can right away assume that C is empty simplicial with extreme generators $v_1, \dots, v_d$.

Outline. The following arguments are subdivided into four major steps. The first three of them are very similar to their analogues in the proof of Proposition 3.1. In Step 1 we cover the d-cone C by $d+1$ smaller cones each of which is bounded by the hyperplane that passes through the barycenter of $\operatorname{conv}(v_1, \dots, v_d)$ and is parallel to the facet of $\operatorname{conv}(v_1, \dots, v_d)$ opposite of v_i, $i = 1, \dots, d$. We summarize this step in Claim A below.

In Step 2 Lemma 5.1 is applied for the construction of unimodular corner covers. Claim B states that it is enough to cover the subcones of C 'in direction' of the cones forming the corner cover.

In Step 3 we extend the corner cover far enough into C. Lemma 2.2 allows us to do this within a suitable multiple of Δ_C. The most difficult part of the proof is to control the size of all vectors involved.

However, Lemma 2.2 is applied to simplices $\Gamma = \operatorname{conv}(w_1, \dots, w_e)$ where $w_1, \dots, w_e$ span a unimodular cone of dimension $e \leq d$. The cones over the unimodular simplices covering $c\Gamma$ have multiplicity dividing c, and possibly equal to c. Nevertheless we obtain a cover of C by cones with *bounded* multiplicities. So we can apply Theorem 4.1 in Step 4 to obtain a unimodular cover.

Step 1. The facet $\operatorname{conv}(v_1, \dots, v_d)$ of Δ_C is denoted by Γ_0. (We use the letter Γ for $(d-1)$-dimensional simplices, and Δ for d-dimensional ones.) For $i \in [1, d]$ we put

$$\mathcal{H}_i = \operatorname{Aff}(O,\ v_i+(d-1)v_1, \dots, v_i+(d-1)v_{i-1},\ v_i+(d-1)v_{i+1}, \dots, v_i+(d-1)v_d)$$

and

$$\Gamma_i = \operatorname{conv}\left(v_i, \Gamma_0 \cap \mathcal{H}_i\right).$$

Observe that $v_1 + \dots + v_d \in \mathcal{H}_i$. In particular, the hyperplanes $\mathcal{H}_i$, $i \in [1,d]$ contain the barycenter of Γ_0, i. e. $(1/d)(v_1 + \dots + v_d)$. In fact, $\mathcal{H}_i$ is the vector subspace of dimension $d-1$ through the barycenter of Γ_0 that is parallel to the facet of Γ_0 opposite to v_i. Clearly, we have the representation $\bigcup_{i=1}^d \Gamma_i = \Gamma_0$, similar to (3) in Section 3. In particular, each of the Γ_i is homothetic to Γ_0 with factor $(d-1)/d$.
To prove (6) it is enough to show the following

Claim A. For each index $i \in [1,d]$ there exists a system of unimodular cones

$$C_{i1}, \dots, C_{ik_i} \subset C$$

such that $\mathrm{Hilb}(C_{ij}) \subset \kappa\Delta_C$, $j \in [1,k_i]$, and $\Gamma_i \subset \bigcup_{j=1}^{k_i} C_{ij}$.

The step from the original claim to the reduction expressed by Claim A seems rather small – we have only covered the cross-section Γ_0 by the Γ_i, and state that it is enough to cover each Γ_i by unimodular subcones. The essential point is that these subcones need not be contained in the cone spanned by Γ_i, but just in C. This gives us the freedom to start with a corner cover at v_i and to extend it far enough into C, namely beyond $\mathcal{H}_i$. This is made more precise in the next step.

Step 2. To prove Claim A it is enough to treat the case $i = 1$. The induction hypothesis implies $\mathfrak{c}_{d-1}^{\mathrm{cone}} \le \kappa - 1$ because the right hand side of the inequality (6) is a strictly increasing function of d. Thus Lemma 5.1 provides a system of unimodular cones $C_1, \dots, C_k \subset C$ covering the corner of C at v_1 such that

$$\mathrm{Hilb}(C_j) \setminus \{v_1, \dots, v_d\} \subset \bigl(\kappa\Delta_C\bigr) \setminus \Delta_C, \quad j \in [1,k]. \tag{7}$$

Here we use the emptiness of Δ_C – it guarantees that $\mathrm{Hilb}(C_j) \cap (\Delta_C \setminus \Gamma_0) = \emptyset$ which is crucial for the inclusion (9) in Step 3.
With a suitable enumeration $\{v_{j1}, \dots, v_{jd}\} = \mathrm{Hilb}(C_j)$, $j \in [1,k]$ we have $v_{11} = v_{21} = \dots = v_{k1} = v_1$ and

$$0 \le (v_{jl})_{v_1} < 1, \quad j \in [1,k], \quad l \in [2,d], \tag{8}$$

where $(-)_{v_1}$ is the first coordinate of an element of $\mathbb{R}^d$ with respect to the basis $v_1, \dots, v_d$ of $\mathbb{R}^d$ (see Lemma 5.1(b)).
Now we formulate precisely what it means to extend the corner cover beyond the hyperplane $\mathcal{H}_1$. Fix an index $j \in [1,k]$ and let $D \subset \mathbb{R}^d$ denote the simplicial d-cone determined by the following conditions:

(i) $C_j \subset D$,
(ii) the facets of D contain those facets of C_j that pass through O and v_1,
(iii) the remaining facet of D is in $\mathcal{H}_1$.

Figure 5 describes the situation in the cross-section Γ_0 of C.
By considering all possible values $j = 1, \dots, k$, it becomes clear that to prove Claim A it is enough to prove

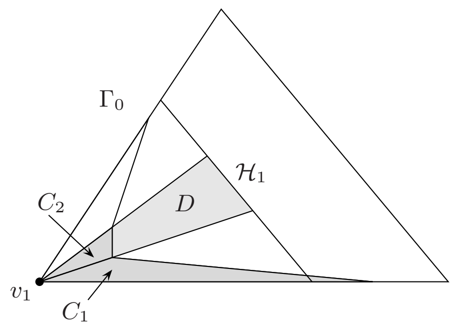

Figure 5.

Claim B. There exists a system of unimodular cones $D_1, \dots, D_T \subset C$ such that

$$\operatorname{Hilb}(D_t) \subset \kappa\Delta_C, \quad t \in [1, T] \qquad \text{and} \qquad D \subset \bigcup_{t=1}^{T} D_t.$$

Step 3. For simplicity of notation we put $\Delta = \Delta_{C_j}$, $\mathcal{H} = \mathcal{H}_1$. (Recall that Δ is of dimension d, spanned by O and the extreme integral generators of C_j.) The vertices of Δ, different from O and v_1 are denoted by $w_2, \dots, w_d$ in such a way that there exists i_0, $1 \le i_0 \le d$, for which

(i) $w_2, \dots, w_{i_0} \in D \setminus \mathcal{H}$ ('bad' vertices, on the same side of $\mathcal{H}$ as v_1),
(ii) $w_{i_0+1}, \dots w_d \in \overline{C_j \setminus D}$ ('good' vertices, beyond or on $\mathcal{H}$),

neither $i_0 = 1$ nor $i_0 = d$ being excluded. ($\overline{X}$ is the closure of $X \subset \mathbb{R}^d$ with respect to the Euclidean topology.) In the situation of Figure 5 the cone C_2 has two bad vertices, whereas C_1 has one good and one bad vertex. (Of course, we see only the intersection points of the cross-section Γ_0 with the rays from O through the vertices.)

If all vertices are good, there is nothing to prove since $D \subset C_j$ in this case. So assume that there are bad vertices, i. e. $i_0 \ge 2$. We now show that the bad vertices are caught in a compact set whose size with respect to Δ_C depends only on d, and this fact makes the whole proof work.

Consider the $(d-1)$-dimensional cone

$$E = v_1 + \mathbb{R}_+(w_2 - v_1) + \cdots + \mathbb{R}_+(w_d - v_1).$$

In other words, E is the $(d-1)$-dimensional cone with apex v_1 spanned by the facet $\operatorname{conv}(v_1, w_2, \dots, w_d)$ of Δ opposite to O. It is crucial in the following that the simplex $\operatorname{conv}(v_1, w_2, \dots, w_d)$ is unimodular (with respect to $\mathbb{Z}^d \cap \operatorname{Aff}(v_1, w_2, \dots, w_d)$), as follows from the unimodularity of C_j.

Due to the inequality (8) the hyperplane $\mathcal{H}$ cuts a $(d-1)$-dimensional (possibly non-lattice) simplex off the cone E. We denote this simplex by Γ. Figure 6 illustrates the situation by a vertical cross-section of the cone C.

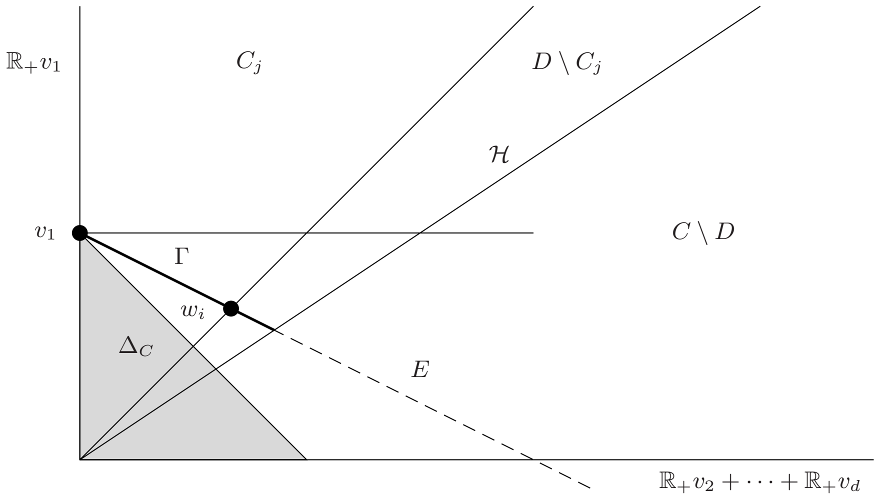

FIGURE 6.

By (7) and (8) we have

$$\Gamma \subset \Phi = \overline{\mathbb{R}_+\Gamma_1 \setminus \big((v_1 + C) \cup \Delta_C\big)}.$$

Let ϑ be the dilatation with center v_1 and factor $d/(d-1)$. Then by (5) we have the inclusion

$$\vartheta(\Gamma) \subset (d+1)\Delta_C. \tag{9}$$

One should note that this inclusion has two aspects: first it shows that Γ is not too big with respect to Δ_C. Second, it guarantees that there is some $\zeta > 0$ *only depending on* d, namely $\zeta = 1/(d-1)$, such that the dilatation with factor $1+\zeta$ and center v_1 keeps Γ inside C. If ζ depended on C, there would be no control on the factor c introduced below.

Let $\Sigma_1 = \operatorname{conv}(v_1, w_2, \dots, w_{i_0})$ and Σ_2 be the smallest face of Γ that contains Σ_1. These are d'-dimensional simplices, $d' = i_0 - 1$. Note that $\Sigma_2 \subset \vartheta(\Sigma_2)$.

We want to apply Lemma 2.2 to the pair

$$\gamma v_1 + (\Sigma_1 - v_1) \subset \gamma v_1 + (\Sigma_2 - v_1).$$

of simplices with the common vertex γv_1. The lattice of reference for the unimodular covering is

$$\mathcal{L} = \mathcal{L}_{\gamma v_1 + (\Sigma_1 - v_1)} = \gamma v_1 + \sum_{j=2}^{i_0} \mathbb{Z}(w_j - v_1).$$

Set

$$\varepsilon = \frac{1}{d} \quad \text{and} \quad c = \frac{d}{d-1}\gamma = \left\lceil \sqrt{d-1}\, \right\rceil d.$$

Since $d' \leq d-1$, Lemma 2.2 (after the parallel translation of the common vertex to O and then back to γv_1) and (9) imply

$$\gamma\Sigma_2 \subset \mathrm{UC}_{\mathcal{L}}\big(\gamma\vartheta(\Sigma_2)\big) \subset \gamma(d+1)\Delta_C. \tag{10}$$

Step 4. Consider the i_0-dimensional simplices spanned by O and the unimodular (i_0-1)-simplices appearing in (10). Their multiplicities with respect to the i_0-rank lattice $\mathbb{Z}\mathcal{L}_{\Sigma_1}$ are all equal to γ, since Σ_1, a face of $\mathrm{conv}(v_1, w_2, \dots, w_d)$ is unimodular and, thus, we have unimodular simplices σ on height γ. The cones $\mathbb{R}_+\sigma$ have multiplicity dividing γ. Therefore, by Lemma 4.1 we conclude that the i_0-cone $\mathbb{R}_+\Sigma_2$ is in the union $\delta_1 \cup \dots \cup \delta_T$ of unimodular (with respect to the lattice $\mathbb{Z}\mathcal{L}_{\Sigma_1}$) cones such that

$$\begin{aligned}\mathrm{Hilb}(\delta_1), \dots, \mathrm{Hilb}(\delta_T) &\subset \left(\frac{d}{2}\left(\frac{3}{2}\right)^{\gamma-2}\right)\Delta_{\mathbb{R}_+\Sigma_2}\\ &\subset \left(\frac{d}{2}\left(\frac{3}{2}\right)^{\gamma-2}\right)\gamma(d+1)\Delta_C = \kappa\Delta_C.\end{aligned}$$

In view of the unimodularity of $\mathrm{conv}(v_1, w_2, \dots, w_d)$, the subgroup $\mathbb{Z}\mathcal{L}_{\Sigma_1}$ is a direct summand of $\mathbb{Z}^d$. It follows that

$$D_t = \delta_t + \mathbb{R}_+w_{i_0+1} + \dots + \mathbb{R}_+w_d, \quad t \in [1, T],$$

is the desired system of unimodular cones. $\square$

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