L^2-Invariants of Locally Symmetric Spaces

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Abstract. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type, $\Gamma \subset G$ a discrete, torsion-free, cocompact subgroup, and let $Y = \Gamma\backslash X$ be the corresponding locally symmetric space. In this paper we explain how the Harish-Chandra Plancherel Theorem for $L^2(G)$ and results on $(\mathfrak{g}, K)$-cohomology can be used in order to compute the L^2-Betti numbers, the Novikov-Shubin invariants, and the L^2-torsion of Y in a uniform way thus completing results previously obtained by Borel, Lott, Mathai, Hess and Schick, Lohoue and Mehdi. It turns out that the behaviour of these invariants is essentially determined by the fundamental rank $m = \mathrm{rk}_{\mathbb{C}} G - \mathrm{rk}_{\mathbb{C}} K$ of G. In particular, we show the nonvanishing of the L^2-torsion of Y whenever $m = 1$.

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1 Introduction

During the last two decades L^2-invariants have proved to be a powerful tool in the topology of compact manifolds (see [15] for an overview). Although they can be defined in purely combinatorial terms we are interested here in their equivalent analytic versions: They are spectral invariants of the p-form Laplacians of the universal cover of the manifold. For particular nice manifolds these might be computable. Indeed, the aim of the present paper is to extract from the representation theoretic work of Harish-Chandra [9] and Borel-Wallach [2] information on the spectral decomposition of the form Laplacians on Riemannian symmetric spaces of the non-compact type which is sufficiently explicit in order to compute the spectral invariants of interest. We try to do this in a rather detailed way which, we hope, keeps the paper readable for nonspecialists in harmonic analysis. [1]

Let $X \to Y$ be the universal cover of a compact Riemannian manifold. Set $\Gamma := \pi_1(Y)$. The form Laplacian $\Delta_p = d^*d + dd^*$ defines a non-negative, elliptic, self-adjoint operator acting on $L^2(X, \Lambda^p T^* X)$, the square integrable p-forms on X. By Δ'_p and $\Delta^c_p = (d^*d)'$ we denote the restriction of Δ_p to the orthogonal complement of its kernel and the coclosed forms in this orthogonal complement, respectively. We consider the corresponding heat kernels $e^{-t\Delta^*_p}(x, x') := (P_* e^{-t\Delta_p})(x, x')$, $x, x' \in X$, for $* = \emptyset, \prime$ or c. Here P_* denotes the orthogonal projection to the corresponding subspace. The local traces $\operatorname{tr} e^{-t\Delta^*_p}(x, x)$ are Γ-invariant functions on X. For a thorough discussion of the following definitions we refer to [14], [17], [16], and [15].

We set

$$\mathrm{Tr}_\Gamma e^{-t\Delta^*_p} := \int_F \operatorname{tr} e^{-t\Delta^*_p}(x, x)\, dx \;,$$

where $F \subset X$ is a fundamental domain of the action of Γ on X and dx is the Riemannian volume element of X. Then the L^2-Betti numbers are given by

$$b^{(2)}_p(Y) := \lim_{t\to\infty} \mathrm{Tr}_\Gamma e^{-t\Delta_p} \in [0, \infty) \;.$$

They are equal to the von Neumann dimension of $\ker \Delta_p$ viewed as Hilbert $\mathcal{N}(\Gamma)$-module, where $\mathcal{N}(\Gamma)$ is the group von Neumann algebra of Γ. If the spectrum of Δ_p has no gap around 0 the Novikov-Shubin invariants of Y are defined by

$$\tilde{\alpha}_p(Y) := \sup\{\beta \mid \mathrm{Tr}_\Gamma e^{-t\Delta'_p} = O(t^{-\frac{\beta}{2}}) \text{ as } t \to \infty\} \in [0, \infty] \;.$$

It measures the asymptotic behaviour of the spectral density function of Δ_p at 0. In case of a gap around 0 we set $\tilde{\alpha}_p(Y) := \infty^+$. Replacing Δ'_p by Δ^c_p we obtain the analogously defined Novikov-Shubin invariants $\alpha_p(Y)$ of d^*d. Using the action of the exterior differential d on the Hodge decomposition of L^2-forms we obtain

$$\tilde{\alpha}_p(Y) = \min\{\alpha_p(Y), \alpha_{p-1}(Y)\} \;. \tag{1}$$

Finally, if $\alpha_p(Y) > 0$ for all p (or, more generally, if X is of determinant class (see [15])), then the L^2-torsion of Y is defined by

$$\rho^{(2)}(Y) := \frac{1}{2}\sum(-1)^{p+1} p \,\log \det{}_\Gamma(\Delta'_p) = \frac{1}{2}\sum(-1)^p \log \det{}_\Gamma(\Delta^c_p) \;,$$

where for $* = \prime, c$

$$\begin{aligned} -\log \det{}_\Gamma(\Delta^*_p) \quad &:= \quad \frac{d}{ds}_{|s=0} \left(\frac{1}{\Gamma(s)} \int_0^\varepsilon \mathrm{Tr}_\Gamma e^{-t\Delta^*_p} t^{s-1} dt \right) \\ &\qquad + \int_\varepsilon^\infty \mathrm{Tr}_\Gamma e^{-t\Delta^*_p} t^{-1} dt \end{aligned}$$

[1] *Note added in proof*: In the meanwhile the interested reader can find a discussion without proofs of the results of the present paper in the recent monograph [16] which gives a comprehensive treatment of the theory of L^2-invariants.

for any $\varepsilon > 0$, where the first integral is considered as a meromorphic function in s. By Poincaré duality $\rho^{(2)}(Y) = 0$ for even dimensional manifolds Y.
From now on let $X = G/K$ be a Riemannian symmetric space of the noncompact type. Here G is a real, connected, linear, semisimple Lie group without compact factors, and $K \subset G$ is a maximal compact subgroup. It is the universal cover of compact locally symmetric spaces of the form $Y = \Gamma\backslash X$, where $\Gamma \cong \pi_1(Y)$ can be identified with a discrete, torsion-free, cocompact subgroup of G. It will be convenient to consider also the compact dual X^d of X. X^d is defined as follows: Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{g}^d := \mathfrak{k} \oplus i\mathfrak{p}$ is another subalgebra of the complexification of $\mathfrak{g}$. Let G^d be the corresponding analytic subgroup of the complexification $G_{\mathbb{C}}$ of G. Then G^d is a compact group, and $X^d = G^d/K$. We normalize the Riemannian metric on X^d such that multiplication by i becomes an isometry $T_{eK}X \cong \mathfrak{p} \to i\mathfrak{p} \cong T_{eK}X^d$. In this paper we are going to prove the following theorem:

Theorem 1.1 *Let $n = \dim Y$ and $m = m(X) := \mathrm{rk}_{\mathbb{C}}G - \mathrm{rk}_{\mathbb{C}}K$ be the fundamental rank of G. Let $\chi(Y)$ be the Euler characteristic of Y. Then*

(a) $b_p^{(2)}(Y) \neq 0 \Leftrightarrow m = 0$ *and* $p = \frac{n}{2}$.
In particular, $b_{\frac{n}{2}}^{(2)}(Y) = (-1)^{\frac{n}{2}}\chi(Y) = \dfrac{\mathrm{vol}(Y)}{\mathrm{vol}(X^d)}\chi(X^d)$.

(b) $\alpha_p(Y) \neq \infty^+ \Leftrightarrow m > 0$ *and* $p \in [\frac{n-m}{2}, \frac{n+m}{2} - 1]$.
In this range $\alpha_p(Y) = m$.

(c) $\rho^{(2)}(Y) \neq 0 \Leftrightarrow m = 1$.

Note that $n - m$ is always even and positive. Part (a) of the theorem was known for a long time, at least since Borel's paper [1]. For special cases see also [4], [5]. For the convenience of the reader we include a proof here. In fact, we prove a stronger statement which should have been known to the experts although we were not able to find it in the literature:

Proposition 1.2 *The discrete spectrum of Δ_p on $L^2(X, \Lambda^p T^*X)$ is empty unless $m(X) = 0$ and $p = \frac{n}{2}$. In this case 0 is the only eigenvalue of Δ_p.*

The strategy of the proof of (b) can already be found in Lott's paper [14], Section VII. To be more precise, Equation (1) implies the slightly weaker result

$$\tilde{\alpha}_p(Y) = \begin{cases} \infty^+ & p \notin [\frac{n-m}{2}, \frac{n+m}{2}] \text{ or } m = 0 \\ m & p \in [\frac{n-m}{2}, \frac{n+m}{2}] \text{ and } m \neq 0 \end{cases} . \tag{2}$$

Lott proved the first line of (2) and that $\tilde{\alpha}_p(Y)$ is finite and independent of p for the remaining values of p. He indicated how one should be able to compute the precise value of $\tilde{\alpha}_p(Y)$. But he finished the computation in the real rank one case, only. In addition, already Borel [1] showed that the range of the

differential d_p of the L^2-de-Rham-complex is not closed for $p \in [\frac{n-m}{2}, \frac{n+m}{2}-1]$. Theorem 1.1 (b) can be considered as a quantitative refinement of this result. After the present paper was written I was informed by S. Mehdi that there is a joint paper of him with N. Lohoue [13] which has recently appeared in print and which contains a proof of (2). The interested reader will also find there more information concerning the material presented here in Sections 2 and 3. But he should be aware that in that paper the range of finiteness of $\tilde{\alpha}_p(Y)$ is constantly misprinted and that the proof of the second line of (2) as written down there is not quite complete (it is not mentioned that it is important to know that $p_\xi(0) > 0$, see Equation (14) below).
The main motivation to do the present work was to obtain part (c) of the theorem. In our locally homogeneous situation we have for any $x \in X$

$$\mathrm{Tr}_\Gamma e^{-t\Delta_p^*} = \mathrm{vol}(Y) \cdot \mathrm{tr}\, e^{-t\Delta_p^*}(x,x)$$

and thus

$$\rho^{(2)}(Y) = \mathrm{vol}(Y) \cdot T^{(2)}(X)$$

for a certain real number $T^{(2)}(X)$. Note that in contrast to $\rho^{(2)}(Y)$ the number $T^{(2)}(X)$ depends on the normalization of the invariant metric on X (of course only via the volume form). A well-known symmetry argument ([18], Proposition 2.1) yields that $T^{(2)}(X) = 0$ whenever $m \neq 1$. Lott [14] (see also [17]) showed that $T^{(2)}(H^n) \neq 0$ for $n = 3, 5, 7$, where H^n is the real hyperbolic space (his values for $T^{(2)}(H^n)$ for $n = 5, 7$ were not correct). This led to the conjecture that $T^{(2)}(H^n) \neq 0$ for all odd n which was open until the work of Hess-Schick [11] who found a trick in order to control the sign of $\log \det_\Gamma(\Delta_p^c)$ in terms of p and n. So they were able to show that there is a *positive* rational number q_n such that

$$T^{(2)}(H^n) = \left(-\frac{1}{\pi}\right)^{\frac{n-1}{2}} q_n \ . \tag{3}$$

Here the metric on H^n is normalized to have sectional curvature -1. Along the same lines Hess [10] obtained

$$T^{(2)}(SL(3,\mathbb{R})/SO(3)) \neq 0 \ . \tag{4}$$

Let us introduce

$$Q_n = \frac{2q_n}{(\frac{n-1}{2})!}$$

and rewrite (3) as

$$T^{(2)}(H^n) = (-1)^{\frac{n-1}{2}} \frac{\pi Q_n}{\mathrm{vol}(S^n)} \ . \tag{5}$$

Recall that S^n is the compact dual of H^n. The rational number Q_n which does not depend on the normalization of the metric has a nice interpretation in terms of Weyl's dimension polynomial for finite-dimensional representations, see Proposition 5.3. But its significance remains to be clarified further, and it

seems to be difficult to write down a practical formula valid for all odd n. One has $Q_3 = \frac{1}{3}$, $Q_5 = \frac{31}{45}$, $Q_7 = \frac{221}{210}$. For further information see [11].
We will reduce Theorem 1.1 (c) to (4) and the positivity of Q_n. Let X be an arbitrary symmetric space satisfying $m(X) = 1$. By the classification of simple Lie groups $X = X_1 \times X_0$, where $m(X_0) = 0$ and $X_1 = SL(3,\mathbb{R})/SO(3)$ or $X_1 = X_{p,q} := SO(p,q)^0/SO(p) \times SO(q)$ for p, q odd. (Here as throughout the paper a superscript 0 denotes the connected component of the identity.) Note that a corresponding decomposition of Y does not necessarily exist. We show

Proposition 1.3

(a) $T^{(2)}(X_{p,q}) = (-1)^{\frac{pq-1}{2}} \chi(X^d_{p-1,q-1}) \dfrac{\pi Q_{p+q-1}}{\mathrm{vol}(X^d_{p,q})}$.

(b) If $m(X) = 1$, *then* $T^{(2)}(X) = \dfrac{(-1)^{\frac{n_0}{2}} \chi(X_0^d)}{\mathrm{vol}(X_0^d)} T^{(2)}(X_1)$.
Here $n_0 = \dim X_0$.

Note that $X^d_{p-1,q-1} = SO(p+q-2)/SO(p-1) \times SO(q-1)$ and for $p, q > 1$

$$\chi(X^d_{p-1,q-1}) = 2 \binom{\frac{p+q-2}{2}}{\frac{p-1}{2}} .$$

In fact, it is a classical result that $\chi(X^d) > 0$ whenever $m(X) = 0$ (compare Theorem 1.1 (a)). It is equal to the quotient of the orders of certain Weyl groups (see Section 5). Now Theorem 1.1 (c) follows from Proposition 1.3, (4) and the positivity of Q_n.
We are also able to identify the missing constant in (4).

Proposition 1.4 *If* $X = SL(3,\mathbb{R})/SO(3)$, *then*

$$T^{(2)}(X) = \frac{2\pi}{3\mathrm{vol}(X^d)} .$$

If the invariant metric on X *is induced from twice the trace form of the standard representation of* $\mathfrak{sl}(3,\mathbb{R})$, *then* $\mathrm{vol}(X^d) = 4\pi^3$, *and we have*

$$T^{(2)}(X) = \frac{1}{6\pi^2} .$$

In particular, we see that $(-1)^{\frac{n-1}{2}} T^{(2)}(X)$ is positive for all X with $m(X) = 1$. Proposition 5.3 provides a uniform formula for the L^2-torsion of all these spaces.
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2 The Harish-Chandra Plancherel Theorem

We want to understand the action of the Laplacian and of the corresponding heat kernels on $L^2(X, \Lambda^p T^*X)$. Since the Laplacian coincides (up to the sign) with the action of the Casimir operator Ω of G (Kuga's Lemma [2], Thm. 2.5.) it is certainly enough to understand the "decomposition" of $L^2(X, \Lambda^p T^*X)$ into irreducible unitary representations of G. There is an isomorphism of homogeneous vector bundles $\Lambda^p T^*X \cong G \times_K \Lambda^p \mathfrak{p}^*$, and, hence, of G-representations

$$L^2(X, \Lambda^p T^*X) \cong [L^2(G) \otimes \Lambda^p \mathfrak{p}^*]^K .$$

Thus our task consists of two steps: First to understand $L^2(G)$ as a representation of $G \times G$ which is accomplished by the Harish-Chandra Plancherel Theorem recalled in the present section and, second, to understand spaces of the form $[V_\pi \otimes \Lambda^p \mathfrak{p}^*]^K$, where (π, V_π) is an irreducible unitary representation of G occurring in the Plancherel decomposition. For general G, the second step will resist a naive approach. However, if $\pi(\Omega) = 0$, then the space $[V_\pi \otimes \Lambda^p \mathfrak{p}^*]^K$ has cohomological meaning, and the theory of relative $(\mathfrak{g}, K)$-cohomology as recalled in the next section will provide a sufficient amount of information.
An irreducible unitary representation (π, V_π) of G is called a representation of the discrete series if there is a G-invariant embedding $V_\pi \hookrightarrow L^2(G)$. Let $\hat{G}_d$ denote the set of equivalence classes of discrete series representations of G. Then we have

Theorem 2.1 (Harish-Chandra [7]) *$\hat{G}_d$ is non-empty if and only if $m(X) = 0$.*

Note that $m(X) = 0$ means that G has a compact Cartan subgroup. The Plancherel Theorem provides a decomposition of $L^2(G)$ which is indexed by discrete series representations of certain subgroups $M \subset G$ which we are going to define now.
Let $\mathfrak{a}_0 \subset \mathfrak{p}$ be a maximal abelian subspace. It induces a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}_0)} \mathfrak{g}_\alpha .$$

Choose a decomposition $\Delta(\mathfrak{g}, \mathfrak{a}_0) = \Delta^+ \mathbin{\dot\cup} -\Delta^+$ into positive and negative roots, and let $\Pi \subset \Delta^+$ be the subset of simple roots. For any subset $F \subset \Pi$ we define

$$\mathfrak{a}_F = \{H \in \mathfrak{a}_0 \mid \alpha(H) = 0 \text{ for all } \alpha \in \Pi\} , \qquad A_F = \exp(\mathfrak{a}_F) ,$$
$$\mathfrak{n}_F = \bigoplus_{\{\alpha \in \Delta^+ \mid \alpha_{|\mathfrak{a}_F} \neq 0\}} \mathfrak{g}_\alpha , \qquad N_F = \exp(\mathfrak{n}_F) .$$

Furthermore, there is a unique (possibly disconnected) subgroup $M_F \subset G$ with Lie algebra $\mathfrak{m}_F$ such that $M_F A_F$ is the centralizer of $\mathfrak{a}_F$ in G and $\mathfrak{m}_F$

is orthogonal to $\mathfrak{a}_F$ with respect to any invariant bilinear form on $\mathfrak{g}$. M_F is a reductive subgroup with compact center. The corresponding parabolic subgroup $P_F := M_F A_F N_F$ is called a standard parabolic. P_F is called cuspidal if M_F has a compact Cartan subgroup. If for two subsets $F, I \subset \Pi$ the spaces $\mathfrak{a}_F$ and $\mathfrak{a}_I$ are conjugated by an element of K (thus by an element of the Weyl group $W(\mathfrak{g}, \mathfrak{a}_0)$) we call P_F and P_I associate. (In many cases this already implies $F = I$.) The assignment $P_F \mapsto A_F T$, where T is a compact Cartan subgroup of M_F, gives a one-to-one correspondence between association classes of cuspidal parabolic subgroups and conjugacy classes of Cartan subgroups of G.

For two subsets $F \subset I \subset \Pi$ we have $P_F \subset P_I$, $M_F \subset M_I$, $A_F \supset A_I$, $N_F \supset N_I$. If F is understood we will often suppress the subscript F.

For illustration let us consider the two extreme cases. The minimal parabolic arises for $F = \emptyset$. Since $M_\emptyset$ is compact $P_\emptyset$ is always cuspidal. For $F = \Pi$ we have $P = M = G$, and G is cuspidal iff $m(X) = 0$. For any cuspidal parabolic subgroup we have $\dim A \geq m(X)$, and there is exactly one association class of cuspidal parabolic subgroups, called fundamental, with $\dim A = m(X)$.

Theorem 2.1 also holds in the context of such reductive groups like M. Thus a parabolic $P = MAN$ is cuspidal iff M has a non-empty discrete series $\hat{M}_d$. Let $\mathfrak{a}^*_{\mathbb{C}}$ be the complexified dual of the Lie algebra $\mathfrak{a}$ of A. For a discrete series representation (ξ, W_ξ) of M and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ we form the induced representation $(\pi_{\xi,\nu}, H^{\xi,\nu})$ by

$$\begin{aligned} H^{\xi,\nu} &= \left\{ f : G \to W_\xi \mid \begin{array}{l} f(gman) = a^{-(\nu+\rho_{\mathfrak{a}})}\xi(m)^{-1} f(g) \text{ for all} \\ g \in G, man \in MAN,\ f_{|K} \in L^2(K, W_\xi) \end{array} \right\}, \\ & (\pi_{\xi,\nu}(g) f)(x) = f(g^{-1}x) \ . \end{aligned}$$

Here $\rho_{\mathfrak{a}} = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha_{|\mathfrak{a}}$. If $\nu \in i\mathfrak{a}^*$, then $\pi_{\xi,\nu}$ is unitary. An invariant bilinear form on $\mathfrak{g}$ induces corresponding forms on $\mathfrak{m}$ and $\mathfrak{a}^*_{\mathbb{C}}$ and determines Casimir operators Ω and Ω_M of G and M, respectively. Then we have

$$\pi_{\xi,\nu}(\Omega) = \langle \nu, \nu \rangle - \langle \rho_{\mathfrak{a}}, \rho_{\mathfrak{a}} \rangle + \xi(\Omega_M) \ . \tag{6}$$

Note that $\xi(\Omega_M)$ is a non-negative real scalar.

Let $\mathcal{C}(G) \subset L^2(G)$ be the Harish-Chandra Schwartz space (for a definition see e.g. [19], 7.1.2). It is stable under the left and right regular representations l and r of G. Let $\mathcal{C}(G)_{K\times K} = \{f \in \mathcal{C}(G) \mid \dim \operatorname{span}\{l_{k_1} r_{k_2} f \mid k_1, k_2 \in K\} < \infty\}$ be the subspace of Schwartz functions which are K-finite from the left and the right. Note that $\mathcal{C}(G)_{K\times K}$ is dense in $L^2(G)$. Suppose that $\nu \in i\mathfrak{a}^*$. Then for $f \in \mathcal{C}(G)$

$$\pi_{\xi,\nu}(f) := \int_G f(g) \pi_{\xi,\nu}(g)\, dg$$

is a well-defined trace class operator on $H^{\xi,\nu}$ which has finite rank if $f \in \mathcal{C}(G)_{K\times K}$. Note that the map $f \mapsto \pi_{\xi,\nu}(f)$ intertwines the G-actions in the following way: $\pi_{\xi,\nu}(l_x r_y f) = \pi_{\xi,\nu}(x) \pi_{\xi,\nu}(f) \pi_{\xi,\nu}(y^{-1})$ for $x, y \in G$.

The Harish-Chandra Plancherel Theorem can now be formulated as follows:

THEOREM 2.2 (HARISH-CHANDRA [9]) *For each cuspidal parabolic subgroup as constructed above and any discrete series representation ξ of the corresponding group M there exists an explicitly computable analytic function $p_\xi : i\mathfrak{a}^* \to [0,\infty)$ of polynomial growth (the Plancherel density) such that for any $f \in \mathcal{C}(G)_{K\times K}$ and $g \in G$*

$$f(g) = \sum_P \sum_{\xi\in\hat{M}_d} \int_{\mathfrak{a}^*} \mathrm{Tr}(\pi_{\xi,i\nu}(f)\pi_{\xi,i\nu}(g^{-1}))\, p_\xi(i\nu)d\nu \ .$$

Here the first sum runs over a set of representatives $P = P_F$ of association classes of cuspidal parabolic subgroups of G.

For more details on the Plancherel Theorem and the structure theory behind it the interested reader may consult the textbooks [12], [19], [20].
Note that the Plancherel measures $p_\xi(i\nu)d\nu$ depend on the normalization of the Haar measure dg. In the remainder of the paper we use the following one. Let dx be the Riemannian volume form of $X = G/K$ and dk be the Haar measure of K with total mass one. Then $\int_G f(g)\, dg = \int_X \tilde{f}(x)\, dx$, where $\tilde{f}(gK) = \int_K f(gk)\, dk$. We normalize the invariant bilinear form on $\mathfrak{g}$ such that its restriction to $\mathfrak{p} \cong T_{eK}X$ coincides with the Riemannian metric of X. Let $d\nu$ be the Lebesgue measure corresponding to the induced form on $\mathfrak{a}^*$. By these choices p_ξ is uniquely determined.
We are now able to give a kind of spectral expansion of $\mathrm{Tr}_\Gamma e^{-t\Delta_p^*}$.

COROLLARY 2.3

$$\begin{aligned}\mathrm{Tr}_\Gamma e^{-t\Delta_p} &= \mathrm{vol}(Y)\sum_P \sum_{\xi\in\hat{M}_d} \int_{\mathfrak{a}^*} e^{-t(\|\nu\|^2+\|\rho_\mathfrak{a}\|^2-\xi(\Omega_M))} \\ &\qquad \dim[H^{\xi,i\nu}\otimes\Lambda^p\mathfrak{p}^*]^K p_\xi(i\nu)d\nu \qquad (7)\\ &= \mathrm{vol}(Y)\sum_P \sum_{\xi\in\hat{M}_d} \int_{\mathfrak{a}^*} e^{-t(\|\nu\|^2+\|\rho_\mathfrak{a}\|^2-\xi(\Omega_M))} \\ &\qquad \dim[W_\xi\otimes\Lambda^p\mathfrak{p}^*]^{K_M} p_\xi(i\nu)d\nu\ . \qquad (8)\end{aligned}$$

Here $K_M := K\cap M$ denotes the maximal compact subgroup of M. There are only finitely many pairs (P,ξ) with $[H^{\xi,i\nu}\otimes\Lambda^p\mathfrak{p}^*]^K \cong [W_\xi\otimes\Lambda^p\mathfrak{p}^*]^{K_M} \neq \{0\}$.

Proof. We define $k_t \in [\mathcal{C}(G)\otimes\mathrm{End}(\Lambda^p\mathfrak{p}^*)]^{K\times K}$ by $k_t(g) := e^{-t\Delta_p}(eK, gK)\circ g$. For $f \in L^2(X,\Lambda^pT^*X) \cong [L^2(G)\otimes\Lambda^p\mathfrak{p}^*]^K$ we have $e^{-t\Delta_p}f(g_0) = \int_G k_t(g)f(g_0g)\, dg$. In addition, $\mathrm{tr}\, e^{-t\Delta_p}(x,x) = \mathrm{tr}\, k_t(e)$ for any $x \in X$. We consider $\pi_{\xi,i\nu}(k_t)$ as an operator acting on $H^{\xi,i\nu}\otimes\Lambda^p\mathfrak{p}^*$. Using Kuga's

Lemma, Equation (6), and the $K \times K$-invariance of k_t one derives that $\pi_{\xi,i\nu}(k_t) = e^{-t(\|v\|^2+\|\rho_{\mathfrak{a}}\|^2-\xi(\Omega_M))}P$, where P is the orthogonal projection onto the subspace of K-invariants in $H^{\xi,i\nu} \otimes \Lambda^p\mathfrak{p}^*$. The Plancherel formula now yields

$$\operatorname{tr} k_t(e) = \sum_P \sum_{\xi \in \hat{M}_d} \int_{\mathfrak{a}^*} e^{-t(\|v\|^2+\|\rho_{\mathfrak{a}}\|^2-\xi(\Omega_M))} \dim[H^{\xi,i\nu} \otimes \Lambda^p\mathfrak{p}^*]^K \, p_\xi(i\nu)d\nu \ .$$

This proves (7). Since $H^{\xi,i\nu} \cong L^2(K \times_{K_M} W_\xi)$ as a representation of K Equation (8) follows by Frobenius reciprocity. The last assertion is a consequence of the Blattner formula (see e.g. [19], 6.5.4) for the K_M-types of discrete series representations of M. □

3 $(\mathfrak{g}, K)$-cohomology

If (π, V_π) is a representation of G on a complete locally convex Hausdorff topological vector space, then we can form its subspace $V_{\pi,K}$ consisting of all K-finite smooth vectors of V_π. $V_{\pi,K}$ becomes a simultaneous module under $\mathfrak{g}$ and K, where both actions satisfy the obvious compatibility conditions. Such a module is called a $(\mathfrak{g}, K)$-module (see [2], 0.2).
We are interested in the functor of $(\mathfrak{g}, K)$-cohomology $V \mapsto H^*(\mathfrak{g}, K, V)$ which goes from the category of $(\mathfrak{g}, K)$-modules to the category of vector spaces. It is the right derived functor of the left exact functor taking $(\mathfrak{g}, K)$-invariants. $H^*(\mathfrak{g}, K, V)$ can be computed using the standard relative Lie algebra cohomology complex $([V \otimes \Lambda^*\mathfrak{p}^*]^K, d)$, where

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i \pi(X_i)\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p) \ , \\ & \qquad \omega \in [V \otimes \Lambda^p\mathfrak{p}^*]^K, \ X_i \in \mathfrak{p} \ . \end{aligned}$$

Note that for $V = C^\infty(G)_K$ this complex is isomorphic to the de Rham complex of the symmetric space X.
Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra of $\mathfrak{g}$. If (π, V) is an irreducible $(\mathfrak{g}, K)$-module, then any $z \in \mathcal{Z}(\mathfrak{g})$ acts by a scalar $\chi_\pi(z)$ on V. The homomorphism $\chi_\pi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is called the infinitesimal character of V. The following basic result can be considered as an algebraic version of Hodge theory.

Proposition 3.1 ([2], II.3.1. and I.5.3.) *Let (π, V_π) be an irreducible unitary representation of G, and let (τ, F) be an irreducible finite-dimensional representation of G. Then*

$$H^p(\mathfrak{g}, K, V_{\pi,K} \otimes F) = \begin{cases} [V_\pi \otimes F \otimes \Lambda^p\mathfrak{p}^*]^K & \pi(\Omega) = \tau(\Omega) \\ \{0\} & \pi(\Omega) \neq \tau(\Omega) \end{cases} .$$

If $H^p(\mathfrak{g},K,V_{\pi,K}\otimes F)\neq\{0\}$, then $\chi_\pi=\chi_{\tilde\tau}$, where $\tilde\tau$ is the dual representation of τ.

The cohomology groups $H^p(\mathfrak{g},K,V_{\pi,K}\otimes F)$ for the representations $\pi=\pi_{\xi,i\nu}$ occurring in the Plancherel Theorem have been computed in [2]. We shall need the following information.

PROPOSITION 3.2 ([2],II.5.3., III.5.1., AND III.3.3.)

(a) Let (τ,F) be an irreducible finite-dimensional representation of G and $\pi\in\hat G_d$ with $\chi_\pi=\chi_{\tilde\tau}$. Then

$$\dim H^p(\mathfrak{g},K,V_{\pi,K}\otimes F)=\begin{cases}1 & p=\frac{n}{2}\\ 0 & \textit{otherwise}\end{cases}\quad .$$

(b) Let $(\pi_{\xi,\nu},H^{\xi,\nu})$ be a representation occurring in the Plancherel Theorem. Then

$$H^*(\mathfrak{g},K,H_K^{\xi,i\nu})=\{0\}$$

unless P is fundamental, $\nu=0$ and ξ belongs to a certain non-empty finite subset $\Xi\subset\hat M_d$. If P is fundamental and $\xi\in\Xi$, then

$$\dim H^p(\mathfrak{g},K,H_K^{\xi,0})=\begin{cases}\binom{m}{p-\frac{n-m}{2}} & p\in[\frac{n-m}{2},\frac{n+m}{2}]\\ 0 & \textit{otherwise}\end{cases}\quad .$$

Choose a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and a system of positive roots. Via the Harish-Chandra isomorphism any infinitesimal character $\chi_\pi:\mathcal{Z}(\mathfrak{g})\to\mathbb{C}$ is given by an element $\Lambda_\pi\in\mathfrak{h}^*_{\mathbb{C}}$, which is uniquely determined up to the action of the Weyl group $W(\mathfrak{g},\mathfrak{h})$ ([19], 3.2.4.). The set of infinitesimal characters of discrete series representations coincides (in case $m=0$) with the set of infinitesimal characters of finite-dimensional representations, which are of the form $\mu_\tau+\rho_{\mathfrak{g}}$, μ_τ and $\rho_{\mathfrak{g}}$ being the highest weight of τ and the half-sum of positive roots, respectively. In the following we will represent infinitesimal characters of discrete series representations by elements of this form. We introduce a partial order on $\mathfrak{h}^*_{\mathbb{C}}$ by saying that $\mu>\nu$, if $\mu-\nu$ is a sum of (not necessarily distinct) positive roots. Then a careful examination of the proof of Proposition II.5.3. in [2], which only rests on some basic knowledge of the possible K-types occurring in discrete series representations, shows that slightly more than Proposition 3.2 (a) is true.

PROPOSITION 3.3 *Let (τ,F) be an irreducible finite-dimensional representation of G and $\pi\in\hat G_d$ with $\Lambda_\pi\not<\Lambda_{\tilde\tau}$. Then*

$$\dim[V_{\pi,K}\otimes F\otimes\Lambda^p\mathfrak{p}^*]^K=\begin{cases}1 & p=\frac{n}{2},\chi_\pi=\chi_{\tilde\tau}\\ 0 & \textit{otherwise}\end{cases}\quad .$$

4 L^2-Betti numbers and Novikov-Shubin invariants

In this section we shall prove parts (a) and (b) of Theorem 1.1 as well as Proposition 1.2.
Let $L^2(X, \Lambda^p T^*X)_d$ be the discrete subspace of $L^2(X, \Lambda^p T^*X)$, i.e., the direct sum of the L^2-eigenspaces of the Laplacian. The Plancherel Theorem in particular says that as a representation of G

$$L^2(X, \Lambda^p T^*X)_d \cong \bigoplus_{\pi \in \hat{G}_d} V_{\tilde{\pi}} \otimes [V_\pi \otimes \Lambda^p \mathfrak{p}^*]^K \ .$$

Let $(\tau_0, \mathbb{C})$ be the trivial representation of G. Let $\pi \in \hat{G}_d$. Then $\Lambda_\pi \not< \Lambda_{\tau_0} = \rho_{\mathfrak{g}}$. Proposition 3.3 now yields

$$\dim[V_\pi \otimes \Lambda^p \mathfrak{p}^*]^K = \begin{cases} 1 & p = \frac{n}{2}, \chi_\pi = \chi_{\tau_0} \\ 0 & \text{otherwise} \end{cases} \ . \tag{9}$$

Since in case $m = 0$ discrete series representations with infinitesimal character χ_{τ_0} always exist (see e.g. [12], Thm. 9.20 or [19], Thm. 6.8.2) and $\tau_0(\Omega) = 0$ this implies Proposition 1.2. In particular, $b_p^{(2)}(Y)$ is non-zero exactly when $m = 0$ and $p = \frac{n}{2}$. Using that the L^2-Euler characteristic coincides with the usual Euler characteristic and applying Hirzebruch proportionality we obtain

$$b^{(2)}_{\frac{n}{2}}(Y) = (-1)^{\frac{n}{2}} \chi(Y) = \frac{\mathrm{vol}(Y)}{\mathrm{vol}(X^d)} \chi(X^d) \ . \tag{10}$$

We will give an alternative, purely analytic proof of that formula in the sequel of Corollary 5.2. This finishes the proof of part (a) of Theorem 1.1.
We now turn to part (b). In order to compute $\alpha_p(Y)$ we need an expression for $\mathrm{Tr}_\Gamma e^{-t\Delta_p^c}$.

Proposition 4.1 *For any triple (P, ξ, ν) appearing in (7) let $B^p(\xi, \nu) = d([H^{\xi, i\nu} \otimes \Lambda^{p-1}\mathfrak{p}^*]^K)$ be the space of coboundaries in the relative Lie algebra cohomology complex and $b^p(\xi, \nu)$ be its dimension. Then*

$$\mathrm{Tr}_\Gamma e^{-t\Delta_p^c} = \mathrm{vol}(Y) \sum_{P \neq G} \sum_{\xi \in \hat{M}_d} \int_{\mathfrak{a}^*} e^{-t(\|\nu\|^2 + \|\rho_{\mathfrak{a}}\|^2 - \xi(\Omega_M))} b^{p+1}(\xi, \nu)\, p_\xi(i\nu) d\nu \ . \tag{11}$$

Here the first sum runs over a set of representatives $P = P_F$ of association classes of proper cuspidal parabolic subgroups of G. If $m > 0$, P is fundamental, $\xi \in \Xi$ (see Proposition 3.2 (b)), and $\nu \neq 0$, then

$$b^{p+1}(\xi, \nu) = \begin{cases} \binom{m-1}{p - \frac{n-m}{2}} & p \in [\frac{n-m}{2}, \frac{n+m}{2} - 1] \\ 0 & \textit{otherwise} \end{cases} \ . \tag{12}$$

Proof. We proceed exactly as in the proof of Corollary 2.3. The kernel $e^{-t\Delta_p^c}$ determines a function $k_t^c \in [\mathcal{C}(G)\otimes \mathrm{End}(\Lambda^p\mathfrak{p}^*)]^{K\times K}$. Then one computes that $\pi_{\xi,i\nu}(k_t^c) = e^{-t(\|\nu\|^2+\|\rho_{\mathfrak{a}}\|^2-\xi(\Omega_M))}P^c$, where P^c is the projection to the orthogonal complement in $[H^{\xi,i\nu}\otimes\Lambda^p\mathfrak{p}^*]^K$ of the space of cocycles in the relative Lie algebra cohomology complex. The dimension of that complement is equal to $b^{p+1}(\xi,\nu)$. If $m=0$, $P=G$, and $\xi\in\hat{G}_d$, then $b^{p+1}(\xi)=0$ for all p by (9). This proves (11).
Let now $m>0$, P be fundamental, $\xi\in\Xi$, and $\nu\in\mathfrak{a}^*$. In this case $H^p(\mathfrak{g},K,H_K^{\xi,0})\neq\{0\}$ for some p. Hence Proposition 3.1 implies that $\pi_{\xi,0}(\Omega)=0$ and

$$h^p(\xi) := \dim[H^{\xi,i\nu}\otimes\Lambda^p\mathfrak{p}^*]^K = \dim[H^{\xi,0}\otimes\Lambda^p\mathfrak{p}^*]^K = \dim H^p(\mathfrak{g},K,H_K^{\xi,0})\ .$$

By Proposition 3.2 (b) we have $h^p(\xi) = \binom{m}{p-\frac{n-m}{2}}$. On the other hand, $\dim H^p(\mathfrak{g},K,H_K^{\xi,i\nu})=0$ for $\nu\neq 0$ implies that $h^p(\xi)=b^p(\xi,\nu)+b^{p+1}(\xi,\nu)$. (12) now follows inductively. □

If $H^p(\mathfrak{g},K,H_K^{\xi,i\nu})=\{0\}$ for all $\nu\in\mathfrak{a}^*$, then by Proposition 3.1 $\dim[H^{\xi,i\nu}\otimes\Lambda^p\mathfrak{p}^*]^K=0$ (which is independent of ν) or $\inf\limits_{\nu\in\mathfrak{a}^*}(\|\nu\|^2+\|\rho_{\mathfrak{a}}\|^2-\xi(\Omega_M))>0$. Now Proposition 4.1 implies that the spectrum of Δ_p^c has a gap around zero, which means $\alpha_p(Y)=\infty^+$, unless $m>0$ and $p\in[\frac{n-m}{2},\frac{n+m}{2}-1]$. In the latter case we obtain for some $c>0$

$$\mathrm{Tr}_\Gamma e^{-t\Delta_p^c} = \mathrm{vol}(Y)\binom{m-1}{p-\frac{n-m}{2}}\sum_{\xi\in\Xi}\int_{\mathfrak{a}^*} e^{-t\|\nu\|^2}\,p_\xi(i\nu)d\nu + O(e^{-ct})$$

as $t\to\infty$, where $\mathfrak{a}$ corresponds to a fundamental parabolic subgroup and thus has dimension m.
Let $P=MAN$ be fundamental and $\xi\in\hat{M}_d$. Then $\dim\mathfrak{n}=: 2u$ is even. Choose a compact Cartan subgroup $T\subset M$ with Lie algebra $\mathfrak{t}$ and a system $\Delta^+(\mathfrak{m},\mathfrak{t})$ of positive roots. Considering the pair $(\mathfrak{m},\mathfrak{t})$ instead of $(\mathfrak{g},\mathfrak{h})$ we can define $\Lambda_\xi\in\mathfrak{t}_{\mathbb{C}}^*$ in the same way as at the end of Section 3. Then $\mathfrak{h}:=\mathfrak{t}\oplus\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ^+ be a system of positive roots for $(\mathfrak{g},\mathfrak{h})$ containing $\Delta^+(\mathfrak{m},\mathfrak{t})$. Then there exists a positive constant c_X depending only on the normalization of the volume form dx such that

$$p_\xi(\nu) = c_X(-1)^u\prod_{\alpha\in\Phi^+}\frac{\langle\alpha,\Lambda_\xi+\nu\rangle}{\langle\alpha,\rho_{\mathfrak{g}}\rangle} \tag{13}$$

(see [9], Thm. 24.1, [20], Thm. 13.5.1 or [12], Thm. 13.11). In particular, p_ξ is an even polynomial of degree $\dim\mathfrak{n}$. The factor $(-1)^u$ makes it nonnegative on $i\mathfrak{a}^*$.
The element $\Lambda_\xi+\nu$ gives the infinitesimal character of $\pi_{\xi,\nu}$: $\Lambda_{\pi_{\xi,\nu}}=\Lambda_\xi+\nu$ ([12], Prop. 8.22). Let now $\xi\in\Xi$. Then Propositions 3.2 (b) and 3.1 imply that $\pi_{\xi,0}$ has the same infinitesimal character as the trivial representation. It

follows that Λ_ξ is conjugated in $\mathfrak{h}_{\mathbb{C}}^*$ by an element of the Weyl group $W(\mathfrak{g},\mathfrak{h})$ to $\rho_{\mathfrak{g}}$. By (13) we obtain $p_\xi(0) = \pm c_X \neq 0$. On the other hand $p_\xi(0) \geq 0$, hence

$$p_\xi(0) > 0 \ . \tag{14}$$

We decompose $p_\xi(\nu) = \sum_{k=0}^{u} p_{\xi,2k}(\nu)$ into homogeneous polynomials. Set $q_{\xi,k} = \int_{\|\nu\|=1} p_{\xi,2k}(\nu)\, d\nu$. Then $q_{\xi,0} > 0$ and

$$\begin{aligned} \int_{\mathfrak{a}^*} e^{-t\|\nu\|^2}\, p_\xi(i\nu) d\nu &= \sum q_{\xi,k} \int_0^\infty e^{-tr^2} r^{m-1+2k} dr \\ &= \sum t^{-(\frac{m}{2}+k)} q_{\xi,k} \int_0^\infty e^{-y^2} y^{m-1+2k} dy \ . \end{aligned}$$

Thus for $p \in [\frac{n-m}{2}, \frac{n+m}{2} - 1]$ the leading term of $\mathrm{Tr}_\Gamma e^{-t\Delta_p^c}$ as $t \to \infty$ is a non-zero multiple of $t^{-\frac{m}{2}}$. This completes the proof of Theorem 1.1 (b).

5 L^2-torsion

For even dimensional manifolds the L^2-torsion vanishes. Thus we may assume that m is odd, in particular $m \geq 1$. Then $\Delta_p' = \Delta_p$. We first want to compute

$$k_X(t) := \frac{1}{2\mathrm{vol}(Y)} \sum_{p=0}^{n} (-1)^p p \, \mathrm{Tr}_\Gamma e^{-t\Delta_p} \ .$$

Then

$$T^{(2)}(X) = \frac{d}{ds}_{|s=0} \left(\frac{1}{\Gamma(s)} \int_0^\varepsilon k_X(t) t^{s-1} dt \right) + \int_\varepsilon^\infty k_X(t) t^{-1} dt \ .$$

Let $P = MAN$ be a parabolic subgroup appearing in (8). Set $K_M = K \cap M$ and $\mathfrak{p}_{\mathfrak{m}} = \mathfrak{p} \cap \mathfrak{m}$. Then an elementary calculation in the representation ring $R(K_M)$ of K_M yields

$$\sum_{p=0}^{n} (-1)^p p \, \Lambda^p \mathfrak{p}^* = 0 \ , \qquad \text{if } \dim \mathfrak{a} \geq 2 \ ,$$

while for $\dim \mathfrak{a} = 1$ we have

$$\begin{aligned} \sum_{p=0}^{n} (-1)^p p \, \Lambda^p \mathfrak{p}^* &= \sum_{p=0}^{n-1} (-1)^{p+1} \Lambda^p (\mathfrak{p}_{\mathfrak{m}}^* \oplus \mathfrak{n}^*) \\ &= \sum_{l=0}^{\dim \mathfrak{n}} (-1)^{l+1} (\Lambda^{ev} \mathfrak{p}_{\mathfrak{m}}^* - \Lambda^{odd} \mathfrak{p}_{\mathfrak{m}}^*) \otimes \Lambda^l \mathfrak{n} \end{aligned}$$

(see [18], Prop. 2.1 and Lemma 2.3). It follows from (8) that $k_X(t) \equiv 0$ for $m > 1$, hence $\rho^{(2)}(Y) = 0$.
From now on let $m = 1$. Let $P = MAN$ be a fundamental parabolic subgroup of G. Then (8) gives

$$\begin{aligned} k_X(t) &= \frac{1}{2}\sum_{l=0}^{\dim\mathfrak{n}}(-1)^{l+1}\sum_{\xi\in\hat{M}_d}\dim\,[W_\xi\otimes(\Lambda^{ev}\mathfrak{p}_\mathfrak{m}^*-\Lambda^{odd}\mathfrak{p}_\mathfrak{m}^*)\otimes\Lambda^l\mathfrak{n}^*]^{K_M} \\ &\qquad\int_{\mathfrak{a}^*} e^{-t(\|\nu\|^2+\|\rho_\mathfrak{a}\|^2-\xi(\Omega_M))}\,p_\xi(i\nu)d\nu\ . \end{aligned} \tag{15}$$

Now $X = X_1 \times X_0$, $X_1 = G_1/K_1$, $X_0 = G_0/K_0$, $m(X_0) = 0$ as explained in the introduction. Although (15) can be evaluated directly for general X with $m(X) = 1$ we prefer to reduce the computation to the irreducible case $X = X_1$. In order to compute $T^{(2)}(X)$ it is sufficient to compare $\rho^{(2)}(Y)$ with $\mathrm{vol}(Y)$ for one particular $Y = \Gamma\backslash X$. If we choose Γ of the form $\Gamma_1 \times \Gamma_0$, where $\Gamma_0 \subset G_0$ and $\Gamma_1 \subset G_1$, then

$$\rho^{(2)}(Y) = \chi(Y_0)\rho^{(2)}(Y_1)\ ,$$

where $Y_1 = \Gamma_1\backslash X_1$, $Y_0 = \Gamma_0\backslash X_0$. Applying Hirzebruch proportionality we obtain the assertion of Proposition 1.3 (b)

$$\begin{aligned} T^{(2)}(X) &= \frac{\rho^{(2)}(Y)}{\mathrm{vol}(Y_0)\mathrm{vol}(Y_1)} = \frac{\chi(Y_0)}{\mathrm{vol}(Y_0)}T^{(2)}(X_1) \\ &= \frac{(-1)^{\frac{n_0}{2}}\chi(X_0^d)}{\mathrm{vol}(X_0^d)}T^{(2)}(X_1)\ . \end{aligned} \tag{16}$$

It remains to deal with the case $X = X_1$. We can assume that $G = SO(p,q)^0$, $p \le q$ odd, or $G = SL(3,\mathbb{R})$. Then $M \cong SO(p-1,q-1)$, $\mathfrak{n} \cong \mathbb{R}^{p+q-2}$ or $M \cong {}^0GL(2,\mathbb{R}) := \{A \in GL(2,\mathbb{R}) \mid |\det A| = 1\}$, $\mathfrak{n} \cong \mathbb{R}^2$, respectively, and M acts on $\mathfrak{n}$ via the standard representation. Note that M is not connected unless $G = SO(1,q)^0$. The M^0-representations $\Lambda^l\mathfrak{n}^* \otimes \mathbb{C}$ are irreducible unless $G = SO(p,q)$ and $l = u = \frac{1}{2}\dim\mathfrak{n}$. In the latter case $\Lambda^u\mathfrak{n}^* \otimes \mathbb{C}$ decomposes into two irreducible components $\Lambda^+\mathfrak{n} \oplus \Lambda^-\mathfrak{n}$. Since compact Cartan subgroups of M are connected the discrete series representations of M are induced from discrete series representations of M^0: $W_\xi = \mathrm{Ind}_{M^0}^M(W_{\xi^0})$, $\xi_0 \in (\hat{M}^0)_d$ (see [19], 6.9 and 8.7.1). As representations of K_M we have $W_\xi \cong \mathrm{Ind}_{K_M^0}^{K_M}(W_{\xi^0})$. By Frobenius reciprocity we obtain

$$\begin{aligned} &\dim[W_\xi\otimes(\Lambda^{ev}\mathfrak{p}_\mathfrak{m}^*-\Lambda^{odd}\mathfrak{p}_\mathfrak{m}^*)\otimes\Lambda^l\mathfrak{n}^*]^{K_M} \\ &\quad = \dim[W_{\xi_0}\otimes(\Lambda^{ev}\mathfrak{p}_\mathfrak{m}^*-\Lambda^{odd}\mathfrak{p}_\mathfrak{m}^*)\otimes\Lambda^l\mathfrak{n}^*]^{K_M^0}\ . \end{aligned}$$

Note that the infinitesimal characters χ_ξ and χ_{ξ_0} coincide. By $\chi(\mathfrak{m}, K_M^0, .)$ we denote the Euler characteristic of relative Lie algebra cohomology. Set

$v = \frac{1}{2}\dim\mathfrak{p}_{\mathfrak{m}}$. Applying Propositions 3.1 and 3.2 (a) to M^0 instead of G we obtain

$$\begin{aligned}
&\dim[W_{\xi_0}\otimes(\Lambda^{ev}\mathfrak{p}_{\mathfrak{m}}^* - \Lambda^{odd}\mathfrak{p}_{\mathfrak{m}}^*)\otimes\Lambda^*\mathfrak{n}^*]^{K_M^0}\\
&= \chi(\mathfrak{m}, K_M^0, W_{\xi_0,K_M^0}\otimes\Lambda^*\mathfrak{n}^*)\\
&= \begin{cases} (-1)^v & \chi_\xi = \chi_{\Lambda^*\mathfrak{n}} \\ 0 & \text{otherwise} \end{cases} .
\end{aligned}$$

Here $\Lambda^*\mathfrak{n}^*$, $* = l, +, -$, denotes an irreducible component of $\Lambda^l\mathfrak{n}^*\otimes\mathbb{C}$.
In all cases under consideration the set $\{\alpha_{|\mathfrak{a}} \mid \alpha\in\Delta^+, \alpha_{|\mathfrak{a}}\neq 0\}$ consists of a single element $\alpha_0\in\mathfrak{a}^*$. It follows that $\rho_{\mathfrak{a}} = u\alpha_0$. Moreover, Ω_M acts on $\Lambda^l\mathfrak{n}$ as $l(2u-l)\|\alpha_0\|^2\mathrm{id}$ (compare [18], Lemma 2.5).
In order to evaluate (15) further we have to determine the constant c_X in formula (13). This can be done in complete generality. So for a moment we drop the assumptions $m = 1$, $X = X_1$.

Lemma 5.1 *Let $P = MAN \subset G$ be fundamental. We retain the notation introduced before (13). Set $W_A = \{k\in K|\mathrm{Ad}(k)\mathfrak{a}\subset\mathfrak{a}\}/K_M$, $S_A^d = \exp(i\mathfrak{a})K \subset X^d$, and let $\Phi_{\mathfrak{k}}^+$ be a positive root system for $(\mathfrak{k},\mathfrak{t})$ with corresponding half sum $\rho_{\mathfrak{k}}$. Then*

$$\begin{aligned}
c_X &= \frac{1}{|W_A|(2\pi)^{\frac{n+m}{2}}}\frac{\prod_{\alpha\in\Phi^+}\langle\alpha,\rho_{\mathfrak{g}}\rangle}{\prod_{\alpha\in\Phi_{\mathfrak{k}}^+}\langle\alpha,\rho_{\mathfrak{k}}\rangle} &(17)\\
&= \frac{1}{|W_A|}\frac{\mathrm{vol}(S_A^d)}{(2\pi)^m}\frac{1}{\mathrm{vol}(X^d)} . &(18)
\end{aligned}$$

Proof. Formula (17) is a combination of [8], Thm. 37.1, with [9], Cor. 23.1, Thm. 24.1 and Thm. 27.3. In order to apply these results correctly one has to take into account that Harish-Chandra's and our normalizations of the measures dg and $d\nu$ all of them starting from a fixed invariant bilinear form on $\mathfrak{g}$ differ by the factors $2^{\frac{n-\dim\mathfrak{a}_0}{2}}$ ([8], Section 7 and Lemma 37.2) and $(2\pi)^m$, respectively. On the other hand we have

$$\prod_{\alpha\in\Phi_{\mathfrak{k}}^+}\langle\alpha,\rho_{\mathfrak{k}}\rangle = (2\pi)^{\frac{\dim K/T}{2}}\frac{\mathrm{vol}(T)}{\mathrm{vol}(K)} \tag{19}$$

(see e.g. [8], Lemma 37.4). Here the volumes are the Riemannian ones corresponding to the invariant bilinear form $\langle .,.\rangle$. Formula (19) holds for any pair (K,T) of a connected compact Lie group and a maximal torus. Applying it also to the pair (G^d, H^d), where H^d is the maximal torus of G^d with Lie algebra $\mathfrak{t}\oplus i\mathfrak{a}$, we obtain

$$\frac{\prod_{\alpha\in\Phi^+}\langle\alpha,\rho_{\mathfrak{g}}\rangle}{\prod_{\alpha\in\Phi_{\mathfrak{k}}^+}\langle\alpha,\rho_{\mathfrak{k}}\rangle} = (2\pi)^{\frac{n-m}{2}}\frac{\mathrm{vol}(K)\mathrm{vol}(H^d)}{\mathrm{vol}(G^d)\mathrm{vol}(T)} = (2\pi)^{\frac{n-m}{2}}\frac{\mathrm{vol}(S_A^d)}{\mathrm{vol}(X^d)} . \tag{20}$$

The second equality follows from the fact that the map $H^d/T \to X^d$, $hT \mapsto hK$, is an isometric embedding with image S_A^d. This proves (18). $\square$

In particular, specializing (13) and (18) to the case $m = 0$ we obtain as a consequence of Weyl's dimension formula

Corollary 5.2 *Let π be a discrete series representation of G having the same infinitesimal character as the finite-dimensional representation τ, then*

$$p_\pi = \frac{\dim \tau}{\mathrm{vol}(X^d)} \ .$$

Let us now give the promised analytic proof of (10). For a fixed finite-dimensional representation τ of G (which still is assumed to be connected) there are exactly $|W(\mathfrak{g},\mathfrak{t})|/|W(\mathfrak{k},\mathfrak{t})|$ equivalence classes of discrete series representations with infinitesimal character χ_τ (see [19], Thm. 8.7.1, or [12], Thm. 12.21). But this quotient of orders of Weyl groups is equal to $\chi(X^d)$ (see e.g. [3]). By (7), (9) and Corollary 5.2 we obtain

$$b^{(2)}_{\frac{n}{2}} = \mathrm{vol}(Y) \sum_{\pi \in \hat{G}_d, \chi_\pi = \chi_{\tau_0}} p_\pi = \mathrm{vol}(Y)\chi(X^d)\frac{1}{\mathrm{vol}(X^d)} \ .$$

We return to the evaluation of $k_X(t)$ for $m = 1$, $X = X_1$. The polynomial p_ξ only depends on the infinitesimal character of ξ. By $\Lambda_* \in i\mathfrak{t}^*$, $* = l, +, -$, we denote the infinitesimal character of the irreducible M-representation $\Lambda^*\mathfrak{n}$. We set

$$p_l(\nu) := \begin{cases} \prod_{\alpha\in\Phi^+} \frac{\langle\alpha,\Lambda_l+\nu\rangle}{\langle\alpha,\rho_\mathfrak{g}\rangle} & G = SL(3,\mathbb{R}) \text{ or } l \neq u \\ \prod_{\alpha\in\Phi^+} \frac{\langle\alpha,\Lambda_++\nu\rangle}{\langle\alpha,\rho_\mathfrak{g}\rangle} + \prod_{\alpha\in\Phi^+} \frac{\langle\alpha,\Lambda_-+\nu\rangle}{\langle\alpha,\rho_\mathfrak{g}\rangle} \\ \quad = 2\prod_{\alpha\in\Phi^+} \frac{\langle\alpha,\Lambda_++\nu\rangle}{\langle\alpha,\rho_\mathfrak{g}\rangle} & \text{otherwise} \end{cases} .$$

For fixed infinitesimal character there are $|W(\mathfrak{m},\mathfrak{t})|/|W_{K_M}|$ equivalence classes of discrete series representations, where $W_{K_M} = \{k \in K_M \mid \mathrm{Ad}(k)\mathfrak{t} \subset \mathfrak{t}\}/T$. Note that there is an embedding $W(\mathfrak{k}_\mathfrak{m},\mathfrak{t}) \hookrightarrow W_{K_M}$ which becomes an isomorphism if K_M is connected. Furthermore, $|W_A| = 1$ except for $G = SO(1,q)^0$, where $|W_A| = 2$. In any case $|W_{K_M}||W_A| = 2|W(\mathfrak{k}_\mathfrak{m},\mathfrak{t})|$. Let X_M^d be the compact dual of $X_M = M/K_M = M^0/K_M^0$. Then $|W(\mathfrak{m},\mathfrak{t})|/|W_{K_M}||W_A| = \frac{1}{2}\chi(X_M^d)$. In addition, $u + v = \frac{n-1}{2}$ and

$$\frac{\mathrm{vol}(S_A^d)}{2\pi} = \frac{1}{\|\alpha_0\|} \ .$$

Summarizing the above discussion we obtain

$$k_X(t) = (-1)^{\frac{n-1}{2}} \frac{\chi(X_M^d)}{4\|\alpha_0\|\mathrm{vol}(X^d)} \sum_{l=0}^{2u} (-1)^{l+1} k_l(t) \ ,$$

where

$$k_l(t) = \int_{\mathfrak{a}^*} e^{-t(\|\nu\|^2+(u-l)^2\|\alpha_0\|^2)} p_l(i\nu) d\nu \ .$$

For an even polynomial P and $c \geq 0$ set

$$k_{P,c}(t) = \int_{-\infty}^{\infty} e^{-t(y^2+c^2)} P(iy) dy \ .$$

Then (compare [6], Lemma 2 and Lemma 3, [17], Lemma 6.4, [11], p.332)

$$\begin{aligned}
&\frac{d}{ds}_{|s=0} \left(\frac{1}{\Gamma(s)} \int_0^{\varepsilon} k_{P,c}(t) t^{s-1} dt \right) + \int_{\varepsilon}^{\infty} k_{P,c}(t) t^{-1} dt \\
&= \left(\int_0^{\varepsilon} k_{P,c}(t) t^{s-1} dt \right)_{|s=0} + \int_{\varepsilon}^{\infty} k_{P,c}(t) t^{-1} dt \\
&= -2\pi \int_0^c P(y)\, dy \ .
\end{aligned}$$

Since $p_l = p_{2u-l}$ we obtain

Proposition 5.3 *Let $X = G/K$ with $m(X) = 1$, $P = MAN \subset G$ a fundamental parabolic subgroup, $u = \frac{1}{2}\dim \mathfrak{n}$, and $X_M = M/K_M$. Then*

$$T^{(2)}(X) = (-1)^{\frac{n-1}{2}} \chi(X_M^d) \frac{\pi Q_X}{\mathrm{vol}(X^d)} \ ,$$

where

$$Q_X = \sum_{l=0}^{u-1} (-1)^l \int_0^{u-l} p_l(y \cdot \alpha_0)\, dy \ .$$

In fact, we have proved the proposition for irreducible X, but (16) shows that it holds in the general case, too.
Let us first discuss the case $G = SO(p,q)^0$, $X = X_{p,q}$, $p \leq q$ odd. Then $n = pq$ and $X_M = X_{p-1,q-1}$. Furthermore, the polynomials p_l depend only on the complexification of G, i.e., on $p+q$. Thus $Q_{X_{p,q}} = Q_{H^{p+q-1}}$. This proves Proposition 1.3 (a). We emphasize again that $Q_{H^{p+q-1}}$ is a positive rational number [11]. In fact, Hess-Schick showed that

$$(-1)^l \int_0^{u-l} p_l(y \cdot \alpha_0)\, dy > 0 \ , \qquad l = 0, \ldots, u-1 \ .$$

Let $X = SL(3,\mathbb{R})/SO(3)$. Then $n = 5$ and $u = 1$. We find that

$$Q_X = \int_0^1 y^2 dy = \frac{1}{3} \ , \quad X_M^d = S^2, \ \chi(X_M^d) = 2 \ .$$

Using for instance (20) also $\mathrm{vol}(X^d)$ can be easily computed. This proves Proposition 1.4 and finishes the proof of Theorem 1.1.

References

[1] A. Borel. The L^2-cohomology of negatively curved Riemannian symmetric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 10 (1985), 95–105.

[2] A. Borel and N. Wallach. *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*. Princeton University Press, 1980.

[3] R. Bott. The index theorem for homogeneous differential operators. In *Differential and Combinatorial Topology. A Symposion in Honour of Marston Morse*, pages 167–186. Princeton University Press, 1965.

[4] J. Dodziuk. L^2-harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc.* 77 (1979), 395–400.

[5] H. Donnelly. The differential form spectrum of hyperbolic space. *manuscripta math.* 33 (1981), 365–385.

[6] D. Fried. Analytic torsion and closed geodesics on hyperbolic manifolds. *Invent. Math.* 84 (1986), 523–540.

[7] Harish-Chandra. Discrete series for semisimple Lie groups II. *Acta Math.* 116 (1966), 1–111.

[8] Harish-Chandra. Harmonic analysis on real reductive groups I. The theory of the constant term. *J. Funct. Anal.* 19 (1975), 104–204.

[9] Harish-Chandra. Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula. *Ann. of Math.* 104 (1976), 117–201.

[10] E. Hess. *Simpliziales Volumen und L^2-Invarianten bei asphärischen Mannigfaltigkeiten*. Dissertation Universität Mainz, 1998. Available at wwwmath.uni-muenster.de/u/lueck/publ.

[11] E. Hess and T. Schick. L^2-torsion of hyperbolic manifolds. *manuscripta math.* 97 (1998), 329–334.

[12] A. W. Knapp. *Representation Theory of Semisimple Lie Groups. An Overview Based on Examples*. Princeton University Press, 1986.

[13] N. Lohoue and S. Mehdi. The Novikov-Shubin invariants for locally symmetric spaces. *J. Math. Pures Appl.* 79, 2 (2000), 111–140.

[14] J. Lott. Heat kernels on covering spaces and topological invariants. *J. Differential Geom.* 35 (1992), 471–510.

[15] W. Lück. L^2-invariants of regular coverings of compact manifolds and CW-complexes. In *Handbook of geometric topology*, pages 735–817. Elsevier, Amsterdam, 2002.

[16] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*. Springer Verlag, 2002.

[17] V. Mathai. L^2-analytic torsion. *J. Funct. Anal.* 107 (1992), 369–386.

[18] H. Moscovici and R. Stanton. R-torsion and zeta functions for locally symmetric manifolds. *Invent. Math.* 105 (1991), 185–216.

[19] N. R. Wallach. *Real Reductive Groups*. Academic Press, 1988.

[20] N. R. Wallach. *Real Reductive Groups II*. Academic Press, 1992.

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