Complex Structure on the Smooth Dual of $GL(n)$

Jacek Brodzki[1] and Roger Plymen

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Abstract. Let G denote the p-adic group $GL(n)$, let $\Pi(G)$ denote the smooth dual of G, let $\Pi(\Omega)$ denote a Bernstein component of $\Pi(G)$ and let $\mathcal{H}(\Omega)$ denote a Bernstein ideal in the Hecke algebra $\mathcal{H}(G)$. With the aid of Langlands parameters, we equip $\Pi(\Omega)$ with the structure of complex algebraic variety, and prove that the periodic cyclic homology of $\mathcal{H}(\Omega)$ is isomorphic to the de Rham cohomology of $\Pi(\Omega)$. We show how the structure of the variety $\Pi(\Omega)$ is related to Xi's affirmation of a conjecture of Lusztig for $GL(n, \mathbb{C})$. The smooth dual $\Pi(G)$ admits a deformation retraction onto the tempered dual $\Pi^t(G)$.

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Introduction

The use of unramified quasicharacters to create a complex structure is well established in number theory. The group of unramified quasicharacters of the idele class group of a global field admits a complex structure: this complex structure provides the background for the functional equation of the zeta integral $Z(\omega, \Phi)$, see [39, Theorem 2, p. 121].

Let now G be a reductive p-adic group and let M be a Levi subgroup of G. Let $\Pi^{sc}(M)$ denote the set of equivalence classes of irreducible supercuspidal representations of M. Harish-Chandra creates a complex structure on the set $\Pi^{sc}(M)$ by using unramified quasicharacters of M [16, p.84]. This complex structure provides the background for the Harish-Chandra functional equations [16, p. 91].

Bernstein considered the set $\Omega(G)$ of all conjugacy classes of pairs (M, σ) where M is a Levi subgroup of G and σ is an irreducible supercuspidal representation of M. Making use of unramified quasicharacters of M, Bernstein gave the set

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$\Omega(G)$ the structure of a complex algebraic variety. Each irreducible component Ω of $\Omega(G)$ has the structure of a complex affine algebraic variety [5].
Let $\Pi(G)$ denote the set of equivalence classes of irreducible smooth representations of G. We will call $\Pi(G)$ the smooth dual of G. Bernstein defines the *infinitesimal character* from $\Pi(G)$ to $\Omega(G)$:

$$inf.ch. : \Pi(G) \to \Omega(G).$$

The infinitesimal character is a finite-to-one map from the set $\Pi(G)$ to the variety $\Omega(G)$.
Let F be a nonarchimedean local field and from now on let $G = GL(n) = GL(n, F)$. Let now W_F be the Weil group of the local field F, then W_F admits unramified quasicharacters, namely those which are trivial on the inertia subgroup I_F. Making use of the unramified quasicharacters of W_F, we introduced in [8] a complex structure on the set of Langlands parameters for $GL(n)$. In view of the local Langlands correspondence for $GL(n)$ this creates, by transport of structure, a complex structure on the smooth dual of $GL(n)$.
In Section 1 of this article, we describe in detail the complex structure on the set of L-parameters for $GL(n)$. We prove that the smooth dual $\Pi(GL(n))$ has the structure of complex manifold. The local L-factors $L(s, \pi)$ then appear as complex valued functions of several complex variables. We illustrate this with the local L-factors attached to the unramified principal series of $GL(n)$.
The complex structure on $\Pi(GL(n))$ is well adapted to the periodic cyclic homology of the Hecke algebra $\mathcal{H}(GL(n))$. The identical structure arises in the work of Xi on Lusztig's conjecture [40]. Let W be the extended affine Weyl group associated to $GL(n, \mathbb{C})$, and let J be the associated based ring (asymptotic algebra) [27, 40]. Xi confirms Lusztig's conjecture and proves that $J \otimes_{\mathbb{Z}} \mathbb{C}$ is Morita equivalent to the coordinate ring of the complex algebraic variety $\widetilde{(\mathbb{C}^\times)^n/S_n}$, the *extended* quotient by the symmetric group S_n of the complex n-dimensional torus $(\mathbb{C}^\times)^n$. In Section 2 we describe the theorem of Xi on the structure of the based ring J.
So the structure of extended quotient, which runs through our work, occurs in the work of Xi *at the level of algebras.* The link with our work is now provided by the theorem of Baum and Nistor [3, 4]

$$\mathrm{HP}_*(\mathcal{H}(n, q)) \simeq \mathrm{HP}_*(J)$$

where $\mathcal{H}(n, q)$ is the associated extended affine Hecke algebra.
Let Ω be a component in the Bernstein variety $\Omega(GL(n))$, and let $\mathcal{H}(G) = \bigoplus \mathcal{H}(\Omega)$ be the Bernstein decomposition of the Hecke algebra.
Let

$$\Pi(\Omega) = (inf.ch.)^{-1}\Omega.$$

Then $\Pi(\Omega)$ is a smooth complex algebraic variety with finitely many irreducible components. We have the following Bernstein decomposition of $\Pi(G)$:

$$\Pi(G) = \bigsqcup \Pi(\Omega).$$

Let M be a compact C^∞ manifold. Then $C^\infty(M)$ is a Fréchet algebra, and we have Connes' fundamental theorem [14, Theorem 2, p. 208]:

$$\mathrm{HP}_*(C^\infty(M)) \cong \mathrm{H}^*(M;\mathbb{C}).$$

Now the ideal $\mathcal{H}(\Omega)$ is a purely algebraic object, and, in computing its periodic cyclic homology, we would hope to find an algebraic variety to play the role of the manifold M. This algebraic variety is $\Pi(\Omega)$.

Theorem 0.1. *Let Ω be a component in the Bernstein variety $\Omega(G)$. Then the periodic cyclic homology of $\mathcal{H}(\Omega)$ is isomorphic to the periodised de Rham cohomology of $\Pi(\Omega)$:*

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) \cong \mathrm{H}^*(\Pi(\Omega);\mathbb{C}).$$

This theorem constitutes the main result of Section 3, which is then used to show that the periodic cyclic homology of the Hecke algebra of $GL(n)$ is isomorphic to the periodic cyclic homology of the Schwartz algebra of $GL(n)$. We also provide an explicit numerical formula for the dimension of the periodic cyclic homology of $\mathcal{H}(\Omega)$ in terms of certain natural number invariants attached to Ω.

The smooth dual $\Pi(GL(n))$ has a natural stratification-by-dimension. We compare this stratification with the Schneider-Zink stratification [34]. Stratification-by-dimension is finer than the Schneider-Zink stratification, see Section 3.

A *scheme* X is a topological space, called the *support* of X and denoted $|X|$, together with a sheaf $\mathcal{O}_X$ of rings on X, such that the pair $(|X|, \mathcal{O}_X)$ is locally affine, see [15, p. 21]. The smooth dual $\Pi(G)$ determines a reduced scheme, see [18, Prop. 2.6]. If S is the reduced scheme determined by the Bernstein variety $\Omega(G)$, then $\Pi(G)$ is a *scheme over* S, i.e. a scheme together with a morphism $\Pi(G) \to S$. This morphism is the q-projection introduced in [8]:

$$\pi_q : \Pi(G) \to S.$$

In Section 4 we give a detailed description of the q-projection and prove that the q-projection is a finite morphism.

From the point of view of noncommutative geometry it is natural to seek the spaces which underlie the noncommutative algebras $\mathcal{H}(G)$ and $\mathcal{S}(G)$. The space which underlies the Hecke algebra $\mathcal{H}(G)$ is the complex manifold $\Pi(G)$. The space which underlies the Schwartz algebra is the Harish-Chandra parameter space, which is a disjoint union of compact orbifolds. In Section 5 we construct a deformation retraction of the smooth dual onto the tempered dual. We view this deformation retraction as a geometric counterpart of the Baum-Connes assembly map for $GL(n)$.

In Section 6 we track the fate of supercuspidal representations of G through the diagram which appears in Section 5. In particular, the index map μ manifests itself as an example of Ahn reciprocity.

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1. The complex structure on the smooth dual of $GL(n)$

The field F is a nonarchimedean local field, so that F is a finite extension of $\mathbb{Q}_p$, for some prime p or F is a finite extension of the function field $\mathbb{F}_p((x))$. The residue field k_F of F is the quotient $\mathbf{o}_F/\mathbf{m}_F$ of the ring of integers $\mathbf{o}_F$ by its unique maximal ideal $\mathbf{m}_F$. Let q be the cardinality of k_F.

The essence of local class field theory, see [29, p.300], is a pair of maps

$$(d : G \longrightarrow \widehat{\mathbb{Z}}, v : F^\times \longrightarrow \mathbb{Z})$$

where G is a profinite group, $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and v is the valuation.

Let $\overline{F}$ be a separable algebraic closure of F. Then the absolute Galois group $G(\overline{F}|F)$ is the projective limit of the finite Galois groups $G(E|F)$ taken over the finite extensions E of F in $\overline{F}$. Let $\widetilde{F}$ be the maximal unramified extension of F. The map d is in this case the projection map

$$d : G(\overline{F}|F) \longrightarrow G(\tilde{F}|F) \cong \hat{\mathbb{Z}}$$

The group $G(\tilde{F}|F)$ is procyclic. It has a single topological generator: the Frobenius automorphism ϕ_F of $\tilde{F}|F$. The Weil group W_F is by definition the pre-image of $< \phi_F >$ in $G(\overline{F} \mid F)$. We thus have the surjective map

$$d : W_F \longrightarrow \mathbb{Z}$$

The pre-image of 0 is the inertia group I_F. In other words we have the following short exact sequence

$$1 \to I_F \to W_F \to \mathbb{Z} \to 0$$

The group I_F is given the profinite topology induced by $G(\overline{F}|F)$. The topology on the Weil group W_F is dictated by the above short exact sequence. The Weil group W_F is a locally compact group with maximal compact subgroup I_F. The map

$$W_F \longrightarrow G(\tilde{F}|F)$$

is a continuous homomorphism with dense image.

A detailed account of the Weil group for local fields may be found in [37]. For a topological group G we denote by G^{ab} the quotient $G^{\mathrm{ab}} = G/G^c$ of G by the closure G^c of the commutator subgroup of G. Thus G^{ab} is the maximal abelian Hausdorff quotient of G. The local reciprocity laws [29, p.320]

$$r_{E|F} : G(E|F)^{ab} \cong F^\times/N_{E|F}E^\times$$

now create an isomorphism [30, p.69]:

$$r_F : W_F^{ab} \cong F^\times$$

We have $W_F = \sqcup \Phi^n I_F, n \in \mathbb{Z}$. The Weil group is a locally compact, totally disconnected group, whose maximal compact subgroup is I_F. This subgroup is also open. There are three models for the Weil-Deligne group.

One model is the crossed product $W_F \ltimes \mathbb{C}$, where the Weil group acts on $\mathbb{C}$ by $w \cdot x = \|w\|x$, for all $w \in W_F$ and $x \in \mathbb{C}$.
The action of W_F on $\mathbb{C}$ extends to an action of W_F on $SL(2,\mathbb{C})$. The semidirect product $W_F \ltimes SL(2,\mathbb{C})$ is then isomorphic to the direct product $W_F \times SL(2,\mathbb{C})$, see [22, p.278]. Then a complex representation of $W_F \times SL(2,\mathbb{C})$ is determined by its restriction to $W_F \times SU(2)$, where $SU(2)$ is the standard compact Lie group.
From now on, we shall use this model for the Weil-Deligne group:

$$\mathcal{L}_F = W_F \times SU(2).$$

Definition 1.1. An L-parameter is a continuous homomorphism

$$\phi : \mathcal{L}_F \to GL(n,\mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$. Two L-parameters are equivalent if they are conjugate under $GL(n,\mathbb{C})$. The set of equivalence classes of L-parameters is denoted $\Phi(G)$.

Definition 1.2. A representation of G on a complex vector space V is *smooth* if the stabilizer of each vector in V is an open subgroup of G. The set of equivalence classes of irreducible smooth representations of G is the *smooth dual* $\Pi(G)$ of G.

Theorem 1.3. *Local Langlands Correspondence for $GL(n)$. There is a natural bijection between $\Phi(GL(n))$ and $\Pi(GL(n))$.*

The naturality of the bijection involves compatibility of the L-factors and ϵ-factors attached to the two types of objects.
The local Langlands conjecture for $GL(n)$ was proved by Laumon, Rapoport and Stuhler [25] when F has positive characteristic and by Harris-Taylor [17] and Henniart [19] when F has characteristic zero.
We recall that a *matrix coefficient* of a representation ρ of a group G on a vector space V is a function on G of the form $f(g) = \langle \rho(g)v, w\rangle$, where $v \in V$, $w \in V^*$, and V^* denotes the dual space of V. The inner product is given by the duality between V and V^*. A representation ρ of G is called *supercuspidal* if and only if the support of every matrix coefficient is compact modulo the centre of G.
Let $\tau_j = \mathrm{spin}(j)$ denote the $(2j+1)$-dimensional complex irreducible representation of the compact Lie group $SU(2)$, $j = 0, 1/2, 1, 3/2, 2, \ldots$.
For $GL(n)$ the local Langlands correspondence works in the following way.

- Let ρ be an irreducible representation of the Weil group W_F. Then $\pi_F(\rho \otimes 1)$ is an irreducible supercuspidal representation of $GL(n)$, and every irreducible supercuspidal representation of $GL(n)$ arises in this way. If $\det(\rho)$ is a unitary character, then $\pi_F(\rho\otimes 1)$ has unitary central character, and so is pre-unitary.
- We have $\pi_F(\rho\otimes \mathrm{spin}(j)) = Q(\Delta)$, the Langlands quotient associated to the segment $\{|\ |^{-(j-1)/2}\pi_F(\rho), \ldots, |\ |^{(j-1)/2}\pi_F(\rho)\}$. If $\det(\rho)$ is unitary,

then $Q(\Delta)$ is in the discrete series. In particular, if $\rho = 1$ then $\pi_F(1 \otimes \mathrm{spin}(j))$ is the Steinberg representation $St(2j+1)$ of $GL(2j+1)$.

- If ϕ is an L-parameter for $GL(n)$ then $\phi = \phi_1 \oplus \ldots \oplus \phi_m$ where $\phi_j = \rho_j \otimes \mathrm{spin}(j)$. Then $\pi_F(\rho)$ is the Langlands quotient $Q(\Delta_1, \ldots, \Delta_m)$. If $\det(\rho_j)$ is a unitary character for each j, then $\pi_F(\phi)$ is a tempered representation of $GL(n)$.

This correspondence creates, as in [23, p. 381], a natural bijection

$$\pi_F : \Phi(GL(n)) \to \Pi(GL(n)).$$

A quasi-character $\psi : W_F \to \mathbb{C}^\times$ is *unramified* if ψ is trivial on the inertia group I_F. Recall the short exact sequence

$$0 \to I_F \to W_F \xrightarrow{d} \mathbb{Z} \to 0$$

Then $\psi(w) = z^{d(w)}$ for some $z \in \mathbb{C}^\times$. Note that ψ is not a *Galois* representation unless z has finite order in the complex torus $\mathbb{C}^\times$, see [37]. Let $\Psi(W_F)$ denote the group of all unramified quasi-characters of W_F. Then

$$\begin{array}{ccc} \Psi(W_F) & \simeq & \mathbb{C}^\times \\ \psi & \mapsto & z \end{array}$$

Each L-parameter $\phi : \mathcal{L}_F \to GL(n, \mathbb{C})$ is of the form $\phi_1 \oplus \cdots \oplus \phi_m$ with each ϕ_j irreducible. Each irreducible L-parameter is of the form $\rho \otimes \mathrm{spin}(j)$ with ρ an irreducible representation of the Weil group W_F.

Definition 1.4. The orbit $\mathcal{O}(\phi) \subset \Phi_F(G)$ is defined as follows

$$\mathcal{O}(\phi) = \{\bigoplus_{r=1}^{m} \psi_r \phi_r \mid \psi_r \in \Psi(W_F), 1 \le r \le m\}$$

where each ψ_r is an unramified quasi-character of W_F.

Definition 1.5. Let $\det \phi_r$ be a unitary character, $1 \le r \le m$ and let $\phi = \phi_1 \oplus \ldots \oplus \phi_m$. The compact orbit $\mathcal{O}^t(\phi) \subset \Phi^t(G)$ is defined as follows:

$$\mathcal{O}^t(\phi) = \{\bigoplus_{r=1}^{m} \psi_r \phi_r \mid \psi_r \in \Psi^t(W_F), 1 \le r \le m\}$$

where each ψ_r is an unramified unitary character of W_F.

We note that $I_F \times SU(2) \subset W_F \times SU(2)$ and in fact $I_F \times SU(2)$ is the maximal compact subgroup of $\mathcal{L}_F$. Now let ϕ be an L-parameter. Moving (if necessary) to another point in the orbit $\mathcal{O}(\phi)$ we can write ϕ in the canonical form

$$\phi = \phi_1 \oplus \ldots \oplus \phi_1 \oplus \ldots \oplus \phi_k \oplus \ldots \oplus \phi_k$$

where ϕ_1 is repeated l_1 times, …, ϕ_k is repeated l_k times, and the representations

$$\phi_j|_{I_F \times SU(2)}$$

are irreducible and pairwise inequivalent, $1 \le j \le k$. We will now write $k = k(\phi)$. This natural number is an invariant of the orbit $\mathcal{O}(\phi)$. We have

$$\mathcal{O}(\phi) = \mathrm{Sym}^{l_1}\mathbb{C}^\times \times \ldots \times \mathrm{Sym}^{l_k}\mathbb{C}^\times$$

the product of symmetric products of $\mathbb{C}^\times$.

Theorem 1.6. *The set $\Phi(GL(n))$ has the structure of complex algebraic variety. Each irreducible component $\mathcal{O}(\phi)$ is isomorphic to the product of a complex affine space and a complex torus*

$$\mathcal{O}(\phi) = \mathbb{A}^l \times (\mathbb{C}^\times)^k$$

where $k = k(\phi)$.

Proof. Let $Y = \mathbb{V}(x_1y_1 - 1, \ldots, x_ny_n - 1) \subset \mathbb{C}^{2n}$. Then Y is a Zariski-closed set in $\mathbb{C}^{2n}$, and so is an affine complex algebraic variety. Let $X = (\mathbb{C}^\times)^n$. Set $\alpha : Y \to X, \alpha(x_1, y_1, \ldots, x_n, y_n) = (x_1, \ldots, x_n)$ and $\beta : X \to Y, \beta(x_1, \ldots, x_n) = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$. So X can be embedded in affine space $\mathbb{C}^{2n}$ as a Zariski-closed subset. Therefore X is an affine algebraic variety, as in [36, p.50].
Let $A = \mathbb{C}[X]$ be the coordinate ring of X. This is the restriction to X of polynomials on $\mathbb{C}^{2n}$, and so $A = \mathbb{C}[X] = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, the ring of Laurent polynomials in n variables $x_1, \ldots, x_n$. Let S_n be the symmetric group, and let Z denote the quotient variety X/S_n. The variety Z is an affine complex algebraic variety.
The coordinate ring of Z is

$$\mathbb{C}[Z] \simeq \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]^{S_n}.$$

Let σ_i, $i = 1, \ldots, n$ be the elementary symmetric polynomials in n variables. Then from the last isomorphism we have

$$\begin{aligned}
\mathbb{C}[Z] &\simeq \mathbb{C}[x_1, \ldots, x_n]^{S_n} \otimes \mathbb{C}[\sigma_n^{-1}] \\
&\simeq \mathbb{C}[\sigma_1, \ldots, \sigma_n] \otimes \mathbb{C}[\sigma_n^{-1}] \\
&\simeq \mathbb{C}[\sigma_1, \ldots, \sigma_{n-1}] \otimes \mathbb{C}[\sigma_n, \sigma_n^{-1}] \\
&\simeq \mathbb{C}[\mathbb{A}^{n-1}] \otimes \mathbb{C}[\mathbb{A} - \{0\}] \\
&\simeq \mathbb{C}[\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})]
\end{aligned}$$

where $\mathbb{A}^n$ denotes complex affine n-space. The coordinate ring of the quotient variety $\mathbb{C}^{\times n}/S_n$ is isomorphic to the coordinate ring of $\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})$. Now the categories of affine algebraic varieties and of finitely generated reduced $\mathbb{C}$-algebras are equivalent, see [36, p.26]. Therefore the variety $\mathbb{C}^{\times n}/S_n$ is isomorphic to the variety $\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})$.
Consider $\mathbb{A} - \{0\} = \mathbb{V}(f)$ where $f(x) = x_1x_2 - 1$. Then $\partial f/\partial x_1 = x_2 \neq 0$ and $\partial f/\partial x_2 = x_1 \neq 0$ on the variety $\mathbb{V}(f)$. So $\mathbb{A} - \{0\}$ is smooth. Then $\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})$ is smooth. Therefore the quotient variety $\mathbb{C}^{\times n}/S_n$ is a smooth complex affine algebraic variety of dimension n. Now each orbit $\mathcal{O}(\phi)$ is a product of symmetric products of $\mathbb{C}^\times$. Therefore each orbit $\mathcal{O}(\phi)$ is a smooth complex affine algebraic variety. We have

$$\mathcal{O}(\phi) = \mathrm{Sym}^{l_1}\mathbb{C}^\times \times \ldots \times \mathrm{Sym}^{l_k}\mathbb{C}^\times = \mathbb{A}^l \times (\mathbb{C}^\times)^k$$

where $l = l_1 + \ldots + l_k - k$ and $k = k(\phi)$. $\square$

We now transport the complex structure from $\Phi(GL(n))$ to $\Pi(GL(n))$ via the local Langlands correspondence. This leads to the next result.

Theorem 1.7. *The smooth dual* $\Pi(GL(n))$ *has a natural complex structure. Each irreducible component is a smooth complex affine algebraic variety.*

The smooth dual $\Pi(GL(n))$ has countably many irreducible components of each dimension d with $1 \leq d \leq n$. The irreducible supercuspidal representations of $GL(n)$ arrange themselves into the 1-dimensional tori.
It follows from Theorems 1.6 and 1.7 that the smooth dual $\Pi(GL(n))$ is a complex manifold. Then $\mathbb{C} \times \Pi(GL(n))$ is a complex manifold. So the local L-factor $L(s, \pi_v)$ and the local ϵ-factor $\epsilon(s, \pi_v)$ are functions of *several complex variables*:

$$L : \mathbb{C} \times \Pi(GL(n)) \longrightarrow \mathbb{C}$$

$$\epsilon : \mathbb{C} \times \Pi(GL(n)) \longrightarrow \mathbb{C}.$$

Example 1.8. Unramified representations. Let $\psi_1, \ldots, \psi_n$ be unramified quasicharacters of the Weil group W_F. Then we have

$$\psi_j(w) = z_j^{d(w)}$$

with $z_j \in \mathbb{C}^\times$ for all $1 \leq j \leq n$. Let ϕ be the L-parameter given by $\psi_1 \oplus \ldots \oplus \psi_n$. Then the image $\pi_F(\phi)$ of ϕ under the local Langlands correspondence π_F is an unramified principal series representation.
For the local L-factors $L(s,\pi)$ see [23, p. 377]. The local L-factor attached to such an unramified representation of $GL(n)$ is given by

$$L(s, \pi_F(\phi)) = \prod_{j=1}^{n} (1 - z_j q^{-s})^{-1}.$$

This exhibits the local L-factor as a function on the complex manifold $\mathbb{C} \times \mathrm{Sym}^n \mathbb{C}^\times$.

2. The structure of the based ring J

Let W be the extended affine Weyl group associated to $GL(n, \mathbb{C})$. For each two-sided cell $\mathbf{c}$ of W we have a corresponding partition λ of n. Let μ be the dual partition of λ. Let u be a unipotent element in $GL(n, \mathbb{C})$ whose Jordan blocks are determined by the partition μ. Let the distinct parts of the dual partition μ be $\mu_1, \ldots, \mu_p$ with μ_r repeated n_r times, $1 \leq r \leq p$.
Let $C_G(u)$ be the centralizer of u in $G = GL(n, \mathbb{C})$. Then the maximal reductive subgroup $F_\mathbf{c}$ of $C_G(u)$ is isomorphic to $GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \times \cdots \times GL(n_p, \mathbb{C})$. Following Lusztig [27] and Xi [40, 1.5] let J be the free $\mathbb{Z}$-module with basis $\{t_w \mid w \in W\}$. The multiplication $t_w t_u = \sum_{v \in W} \gamma_{w,u,v} t_v$ defines an associative ring structure on J. The ring J is the based ring of W. For each two-sided cell $\mathbf{c}$ of W the $\mathbb{Z}$-submodule $J_\mathbf{c}$ of J, spanned by all t_w, $w \in \mathbf{c}$, is a two-sided ideal of J. The ring $J_\mathbf{c}$ is the based ring of the two-sided cell $\mathbf{c}$. Let $|Y|$ be the

number of left cells contained in $\mathbf{c}$. The Lusztig conjecture says that there is a ring isomorphism

$$J_{\mathbf{c}} \simeq M_{|Y|}(R_{F_{\mathbf{c}}}), \quad t_w \mapsto \pi(w)$$

where $R_{F_{\mathbf{c}}}$ is the rational representation ring of $F_{\mathbf{c}}$. This conjecture for $GL(n,\mathbb{C})$ has been proved by Xi [40, 1.5, 4.1, 8.2].
Since $F_{\mathbf{c}}$ is isomorphic to a direct product of the general linear groups $GL(n_i,\mathbb{C})$ $(1 \leq i \leq p)$ we see that $R_{F_{\mathbf{c}}}$ is isomorphic to the tensor product over $\mathbb{Z}$ of the representation rings $R_{GL(n_i,\mathbb{C})}$, $1 \leq i \leq p$. For the ring $R_{GL(n,\mathbb{C})}$ we have

$$R_{GL(n,\mathbb{C})} \simeq \mathbb{Z}[X_1, X_2, \ldots, X_n][X_n^{-1}]$$

where the elements $X_1, X_2, \ldots, X_n, X_n^{-1}$ are described in [40, 4.2][6, IX.125]. Then

$$\begin{aligned} R_{GL(n,\mathbb{C})} &\simeq \mathbb{Z}[\sigma_1, \ldots, \sigma_n, \sigma_n^{-1}] \\ &\simeq \mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]^{S_n} \end{aligned}$$

We have

$$R_{GL(n,\mathbb{C})} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[\mathrm{Sym}^n \mathbb{C}^\times]$$

and

$$R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[\mathrm{Sym}^{n_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{n_p}\mathbb{C}^\times]$$

We recall the *extended quotient*. Let the finite group Γ act on the space X. Let $\widetilde{X} = \{(x,\gamma) : \gamma x = x\}$, let Γ act on $\widehat{X}$ by $\gamma_1(x,\gamma) = (\gamma_1 x, \gamma_1\gamma\gamma_1^{-1})$. Then $\widetilde{X}/\Gamma$ is the extended quotient of X by Γ, and we have

$$\widetilde{X}/\Gamma = \bigsqcup X^\gamma / Z(\gamma)$$

where one γ is chosen in each Γ-conjugacy class.
There is a canonical projection $\widetilde{X}/\Gamma \to X/\Gamma$.
Let $\gamma \in S_n$ have cycle type μ, let $X = (\mathbb{C}^\times)^n$. Then

$$\begin{aligned} X^\gamma &\simeq (C^\times)^{n_1} \times \cdots \times (C^\times)^{n_p} \\ Z(\gamma) &\simeq (\mathbb{Z}/\mu_1\mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/\mu_p\mathbb{Z}) \wr S_{n_p} \\ X^\gamma/Z(\gamma) &\simeq \mathrm{Sym}^{n_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{n_p}\mathbb{C}^\times \end{aligned}$$

and so

$$R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[X^\gamma/Z(\gamma)]$$

Then

$$J \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{\mathbf{c}}(J_{\mathbf{c}} \otimes_{\mathbb{Z}} \mathbb{C}) \sim \oplus_{\mathbf{c}}(R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C}) \simeq \mathbb{C}[\widetilde{X}/S_n]$$

The algebra $J \otimes_{\mathbb{Z}} \mathbb{C}$ is Morita equivalent to a reduced, finitely generated, commutative unital $\mathbb{C}$-algebra, namely the coordinate ring of the extended quotient $\widetilde{X}/S_n$.

3. Periodic cyclic homology of the Hecke algebra

The Bernstein variety $\Omega(G)$ of G is the set of G-conjugacy classes of pairs (M, σ), where M is a Levi (i.e. block-diagonal) subgroup of G, and σ is an irreducible supercuspidal representation of M. Each irreducible smooth representation of G is a subquotient of an induced representation $i_{GM}\sigma$. The pair (M, σ) is unique up to conjugacy. This creates a finite-to-one map, the infinitesimal character, from $\Pi(G)$ onto $\Omega(G)$.
Let $\Omega(G)$ be the Bernstein variety of G. Each point in $\Omega(G)$ is a conjugacy class of cuspidal pairs (M, σ). A quasicharacter $\psi : M \longrightarrow \mathbb{C}^\times$ is *unramified* if ψ is trivial on M°. The group of unramified quasicharacters of M is denoted $\Psi(M)$. We have $\Psi(M) \cong (\mathbb{C}^\times)^\ell$ where ℓ is the parabolic rank of M. The group $\Psi(M)$ now creates orbits: the orbit of (M, σ) is $\{(M, \psi \otimes \sigma) : \psi \in \Psi(M)\}$. Denote this orbit by D, and set $\Omega = D/W(M, D)$, where $W(M)$ is the Weyl group of M and $W(M, D)$ is the subgroup of $W(M)$ which leaves D globally invariant. The orbit D has the structure of a complex torus, and so Ω is a complex algebraic variety. We view Ω as a component in the algebraic variety $\Omega(G)$.
The Bernstein variety $\Omega(G)$ is the disjoint union of ordinary quotients. We now replace the ordinary quotient by the extended quotient to create a new variety $\Omega^+(G)$. So we have

$$\Omega(G) = \bigsqcup D/W(M, D) \quad \text{and} \quad \Omega^+(G) = \bigsqcup \widetilde{D}/W(M, D)$$

Let Ω be a component in the Bernstein variety $\Omega(GL(n))$, and let $\mathcal{H}(G) = \bigoplus \mathcal{H}(\Omega)$ be the Bernstein decomposition of the Hecke algebra.
Let

$$\Pi(\Omega) = (inf.ch.)^{-1}\Omega.$$

Then $\Pi(\Omega)$ is a smooth complex algebraic variety with finitely many irreducible components. We have the following Bernstein decomposition of $\Pi(G)$:

$$\Pi(G) = \bigsqcup \Pi(\Omega).$$

Let M be a compact C^∞ manifold. Then $C^\infty(M)$ is a Fréchet algebra, and we have Connes' fundamental theorem [14, Theorem 2, p. 208]:

$$\mathrm{HP}_*(C^\infty(M)) \cong \mathrm{H}^*(M; \mathbb{C}).$$

Now the ideal $\mathcal{H}(\Omega)$ is a purely algebraic object, and, in computing its periodic cyclic homology, we would hope to find an algebraic variety to play the role of the manifold M. This algebraic variety is $\Pi(\Omega)$.

Theorem 3.1. *Let Ω be a component in the Bernstein variety $\Omega(G)$. Then the periodic cyclic homology of $\mathcal{H}(\Omega)$ is isomorphic to the periodised de Rham cohomology of $\Pi(\Omega)$:*

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) \cong \mathrm{H}^*(\Pi(\Omega); \mathbb{C}).$$

Proof. We can think of Ω as a vector $(\tau_1, \ldots, \tau_r)$ of irreducible supercuspidal representations of smaller general linear groups, the entries of this vector being

only determined up to tensoring with unramified quasicharacters and permutation. If the vector is equivalent to $(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_r, \ldots, \sigma_r)$ with σ_j repeated e_j times, $1 \le j \le r$, and $\sigma_1, \ldots, \sigma_r$ are pairwise distinct, then we say that Ω has *exponents* $e_1, \ldots, e_r$.
Then there is a Morita equivalence

$$\mathcal{H}(\Omega) \sim \mathcal{H}(e_1, q_1) \otimes \ldots \otimes \mathcal{H}(e_r, q_r)$$

where $q_1, \ldots, q_r$ are natural number invariants attached to Ω.
This result is due to Bushnell-Kutzko [11, 12, 13]. We describe the steps in the proof. Let (ρ, W) be an irreducible smooth representation of the compact open subgroup K of G. As in [12, 4.2], the pair (K, ρ) is an Ω-type in G if and only if, for $(\pi, V) \in \Pi(G)$, we have $\text{inf.ch.}(\pi) \in \Omega$ if and only if π contains ρ. The existence of an Ω-type in $GL(n)$, for each component Ω in $\Omega(GL(n))$, is established in [13, 1.1]. So let (K, ρ) be an Ω-type in $GL(n)$. As in [12, 2.9], let

$$e_\rho(x) = (\text{vol}K)^{-1}(\dim\, \rho)\ \text{Trace}_W(\rho(x^{-1}))$$

for $x \in K$ and 0 otherwise.
Then e_ρ is an idempotent in the Hecke algebra $\mathcal{H}(G)$. Then we have

$$\mathcal{H}(\Omega) \cong \mathcal{H}(G) * e_\rho * \mathcal{H}(G)$$

as in [12, 4.3] and the two-sided ideal $\mathcal{H}(G) * e_\rho * \mathcal{H}(G)$ is Morita equivalent to $e_\rho * \mathcal{H}(G) * e_\rho$. Now let $\mathcal{H}(K, \rho)$ be the endomorphism-valued Hecke algebra attached to the semisimple type (K, ρ). By [12, 2.12] we have a canonical isomorphism of unital $\mathbb{C}$-algebras :

$$\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}\, W \cong e_\rho * \mathcal{H}(G) * e_\rho$$

so that $e_\rho * \mathcal{H}(G) * e_\rho$ is Morita equivalent to $\mathcal{H}(G, \rho)$. Now we quote the main theorem for semisimple types in $GL(n)$ [13, 1.5]: there is an isomorphism of unital $\mathbb{C}$-algebras

$$\mathcal{H}(G, \rho) \cong \mathcal{H}(G_1, \rho_1) \otimes \ldots \otimes \mathcal{H}(G_r, \rho_r)$$

The factors $\mathcal{H}(G_i, \rho_i)$ are (extended) affine Hecke algebras whose structure is given explicitly in [11, 5.6.6]. This structure is in terms of generators and relations [11, 5.4.6]. So let $\mathcal{H}(e, q)$ denote the affine Hecke algebra associated to the affine Weyl group $\mathbb{Z}^e \rtimes S_e$. Putting all this together we obtain a Morita equivalence

$$\mathcal{H}(\Omega) \sim \mathcal{H}(e_1, q_1) \otimes \ldots \otimes \mathcal{H}(e_r, q_r)$$

The natural numbers $q_1, \ldots, q_r$ are specified in [11, 5.6.6]. They are the cardinalities of the residue fields of certain extension fields $E_1/F, \ldots, E_r/F$.
Using the Künneth formula the calculation of $\text{HP}_*(\mathcal{H}(\Omega))$ is reduced to that of the affine Hecke algebra $\mathcal{H}(e, q)$. Baum and Nistor demonstrate the spectral invariance of periodic cyclic homology in the class of finite type algebras [3, 4]. Now $\mathcal{H}(e, q)$ is the Iwahori-Hecke algebra associated to the extended affine

Weyl group $\mathbb{Z}^e \rtimes S_e$, and let J be the asymptotic Hecke algebra (based ring) associated to $\mathbb{Z}^e \rtimes S_e$. According to [3, 4], Lusztig's morphisms $\phi_q : \mathcal{H}(e,q) \to J$ induce isomorphisms

$$(\phi_q)_* : \mathrm{HP}_*(\mathcal{H}(e,q)) \to \mathrm{HP}_*(J)$$

for all $q \in \mathbb{C}^\times$ that are not proper roots of unity. At this point we can back track and deduce that

$$\mathrm{HP}_*(\mathcal{H}(e,q)) \simeq \mathrm{HP}_*(J) \simeq \mathrm{HP}_*(\mathcal{H}_1)$$

and use the fact that $\mathcal{H}(e,1) \simeq \mathbb{C}[\mathbb{Z}^e \rtimes S_e]$. It is much more illuminating to quote Xi's proof of the Lusztig conjecture for the based ring J, see Section 2. Then we have

$$\mathrm{HP}_*(\mathcal{H}(e,q)) \simeq \mathrm{HP}_*(J) \simeq \mathrm{HP}_*(\mathbb{C}[\widetilde{(\mathbb{C}^\times)^e/S_e}]) \simeq \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^e/S_e};\mathbb{C}).$$

If Ω has exponents $e_1, \ldots, e_r$ then $e_1 + \ldots + e_r = d(\Omega) = \dim_{\mathbb{C}} \Omega$, and $W(\Omega)$ is a product of symmetric groups:

$$W(\Omega) = S_{e_1} \times \ldots \times S_{e_r}$$

We have

$$\begin{array}{rcl}
\mathrm{HP}_*(\mathcal{H}(\Omega)) & \simeq & \mathrm{HP}_*(\mathcal{H}(e_1,q_1) \otimes \cdots \otimes \mathcal{H}(e_r,q_r)) \\
& \simeq & \mathrm{HP}_*(\mathcal{H}(e_1,q_1)) \otimes \cdots \otimes \mathrm{HP}_*(\mathcal{H}(e_r,q_r)) \\
& \simeq & \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^{e_1}/S_{e_1}};\mathbb{C}) \otimes \cdots \otimes \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^{e_r}/S_{e_r}};\mathbb{C})
\end{array}$$

Now the extended quotient is multiplicative, i.e.

$$\widetilde{(\mathbb{C}^\times)^{d(\Omega)}/W(\Omega)} = \widetilde{(\mathbb{C}^\times)^{e_1}/S_{e_1}} \times \cdots \times \widetilde{(\mathbb{C}^\times)^{e_r}/S_{e_r}}$$

which implies that

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) = \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^{d(\Omega)}/W(\Omega)};\mathbb{C})$$

Recall that

$$\begin{array}{rcl}
\Omega & = & (\mathbb{C}^\times)^{d(\Omega)}/W(\Omega) \\
\Omega^+ & = & \widetilde{(\mathbb{C}^\times)^{d(\Omega)}/W(\Omega)}
\end{array}$$

and by [8, p. 217] we have $\Pi(\Omega) \simeq \Omega^+$. It now follows that

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) \simeq \mathrm{H}^*(\Pi(\Omega);\mathbb{C})$$

□

Lemma 3.2. *Let Ω be a component in the variety $\Omega(G)$ and let Ω have exponents $\{e_1, \ldots, e_r\}$. Then for $j = 0, 1$ we have*

$$\dim_{\mathbb{C}} \mathrm{HP}_j\, \mathcal{H}(\Omega) = 2^{r-1}\beta(e_1)\cdots\beta(e_r)$$

where

$$\beta(e) = \sum_{|\lambda|=e} 2^{\alpha(\lambda)-1}$$

and where $\alpha(\lambda)$ is the number of unequal parts of λ. Here $|\lambda|$ is the weight of λ, i.e. the sum of the parts of λ so that λ is a partition of e.

Proof. Suppose first that Ω has the single exponent e. By Theorem 3.1 the periodic cyclic homology of $\mathcal{H}(\Omega)$ is isomorphic to the periodised de Rham cohomology of the extended quotient of $(\mathbb{C}^\times)^e$ by the symmetric group S_e. The components in this extended quotient correspond to the partitions of e. In fact, if $\alpha(\lambda)$ is the number of unequal parts in the partition λ then the corresponding component is homotopy equivalent to the compact torus of dimension $\alpha(\lambda)$. We now proceed by induction, using the fact that the extended quotient is multiplicative and the Künneth formula. □

Theorem 3.1, combined with the calculation in [7], now leads to the next result.

Theorem 3.3. *The inclusion $\mathcal{H}(G) \longrightarrow \mathcal{S}(G)$ induces an isomorphism at the level of periodic cyclic homology:*

$$\mathrm{HP}_*(\mathcal{H}(G)) \simeq \mathrm{HP}_*(\mathcal{S}(G)).$$

Remark 3.4. We now consider further the disjoint union

$$\Phi(\Omega) = \mathcal{O}(\phi_1) \sqcup \cdots \sqcup \mathcal{O}(\phi_r) \simeq \Omega^+$$

If we apply the local Langlands correspondence π_F then we obtain

$$\Pi(\Omega) = \pi_F(\mathcal{O}(\phi_1)) \sqcup \cdots \sqcup \pi_F(\mathcal{O}(\phi_r)) \simeq \Omega^+$$

This partition of $\Pi(\Omega)$ is *identical* to that in Schneider-Zink [34, p. 198], modulo notational differences. In their notation, for each $\mathcal{P} \in \mathcal{B}$ there is a natural map

$$Q_{\mathcal{P}} : X_{nr}(N_{\mathcal{P}}) \to \mathrm{Irr}(\Omega)$$

such that

$$\mathrm{Irr}(\Omega) = \bigsqcup_{\mathcal{P} \in \mathcal{B}} \mathrm{im}(Q_{\mathcal{P}}).$$

In fact this is a special stratification of $\mathrm{Irr}(\Omega)$ in the precise sense of their article [34, p.198].

Let

$$Z_{\mathcal{P}} = \bigcup_{\mathcal{P}' \leq \mathcal{P}} \mathrm{im}(Q_{\mathcal{P}})$$

Then $Z_{\mathcal{P}}$ is a Jacobson closed set, in fact $Z_{\mathcal{P}} = V(J_{\mathcal{P}})$, where $J_{\mathcal{P}}$ is a certain 2-sided ideal [34, p.198]. We note that the set $Z_{\mathcal{P}}$ is also closed in the topology of the present article: each component in Ω^+ is equipped with the classical (analytic) topology.

Issues of stratification play a dominant role in [34]. The stratification of the tempered dual $\Pi^t(GL(n))$ arises from their construction of *tempered* K-types, see [34, p. 162, p. 189]. In the context of the present article, there is a natural stratification-by-dimension as follows. Let $1 \leq k \leq n$ and define

$$k\text{-stratum} = \{\mathcal{O}(\phi) \mid \dim_{\mathbb{C}} \mathcal{O}(\phi) \leq k\}$$

If $\pi_F(\mathcal{O}(\phi))$ is the complexification of the component $\Theta \subset \Pi^t(G)$ then we have

$$\dim_{\mathbb{R}} \Theta = \dim_{\mathbb{C}} \mathcal{O}(\phi).$$

The partial order in [34] on the components Θ transfers to a partial order on complex orbits $\mathcal{O}(\phi)$. This partial order originates in the opposite of the natural partial order on partitions, and the partitions manifest themselves in terms of Langlands parameters. For example, let

$$\begin{array}{rcl} \phi & = & \rho\otimes\mathrm{spin}(j_1)\oplus\cdots\oplus\rho\otimes\mathrm{spin}(j_r) \\ \phi' & = & \rho\otimes\mathrm{spin}(j_1')\oplus\cdots\oplus\rho\otimes\mathrm{spin}(j_r') \end{array}$$

Let $\lambda_1 = 2j_1+1,\ldots,\lambda_r = 2j_r+1$, $\mu_1 = 2j_1'+1,\ldots,\mu_r = 2j_r'+1$ and define partitions as follows

$$\begin{array}{rcl} \lambda & = & (\lambda_1,\ldots,\lambda_r), \quad \lambda_1\geq\lambda_2\geq\ldots \\ \mu & = & (\mu_1,\ldots,\mu_r), \quad \mu_1\geq\mu_2\geq\ldots \end{array}$$

The natural partial order on partitions is: $\lambda\leq\mu$ if and only if

$$\lambda_1+\cdots+\lambda_i\leq\mu_1+\cdots+\mu_i$$

for all $i\geq 1$, see [28, p.6]. Let $l(\lambda)$ be the length of λ, that is the number of parts in λ. Then $\dim_{\mathbb{C}}\mathcal{O}(\phi) = l(\lambda)$. Let λ', μ' be the dual partitions as in [28]. Then we have [28, 1.11] $\lambda\geq\mu$ if and only if $\mu'\geq\lambda'$. Note that $l(\lambda) = \lambda_1'$, $l(\mu) = \mu_1'$. Then

$$\Theta_\lambda\leq\Theta_\mu \Leftrightarrow \lambda\geq\mu \Leftrightarrow \mu'\geq\lambda' \Rightarrow \lambda_1'\leq\mu_1'$$

So if $\Theta_\lambda \leq \Theta_\mu$ then $\dim_{\mathbb{R}}\Theta_\lambda\leq\dim_{\mathbb{R}}\Theta_\mu$, similarly $\mathcal{O}(\phi)\leq\mathcal{O}(\phi')$ implies $\dim_{\mathbb{C}}\mathcal{O}(\phi)\leq\dim_{\mathbb{C}}\mathcal{O}(\phi')$. Stratification-by-dimension is finer than the Schneider-Zink stratification [34].

Let now R denote the ring of all regular functions on $\Pi(G)$. The ring R is a commutative, reduced, unital ring over $\mathbb{C}$ which is not finitely generated. We will call R the *extended centre* of G. It is natural to believe that the extended centre R of G is the centre of an 'extended category' made from smooth G-modules. The work of Schneider-Zink [34, p. 201] contains various results in this direction.

4. The q-projection

Let Ω be a component in the Bernstein variety. This component is an ordinary quotient D/Γ. We now consider the extended quotient $\widetilde{D}/\Gamma = \bigsqcup D^\gamma/Z_\gamma$, where D is the complex torus $\mathbb{C}^{\times m}$. Let γ be a permutation of n letters with cycle type

$$\gamma = (1\ldots\alpha_1)\cdots(1\ldots\alpha_r)$$

where $\alpha_1+\cdots+\alpha_r = m$. On the fixed set D^γ the map π_q, by definition, sends the element $(z_1,\ldots,z_1,\ldots,z_r,\ldots,z_r)$ where z_j is repeated α_j times, $1\leq j\leq r$, to the element

$$(q^{(\alpha_1-1)/2}z_1,\ldots,q^{(1-\alpha_1)/2}z_1,\ldots,q^{(\alpha_r-1)/2}z_r,\ldots,q^{(1-\alpha_r)/2}z_r)$$

The map π_q induces a map from D^γ/Z_γ to D/Γ, and so a map, still denoted π_q, from the extended quotient $\widetilde{D}/\Gamma$ to the ordinary quotient D/Γ. This creates a map π_q from the extended Bernstein variety to the Bernstein variety:

$$\pi_q : \Omega^+(G) \longrightarrow \Omega(G).$$

Definition 4.1. The map π_q is called the *q-projection.*

The q-projection π_q occurs in the following commutative diagram [8]:

$$\begin{array}{ccc} \Phi(G) & \longrightarrow & \Pi(G) \\ {\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{inf. ch.}} \\ \Omega^+(G) & \xrightarrow{\pi_q} & \Omega(G) \end{array}$$

Let A, B be commutative rings with $A \subset B, 1 \in A$. Then the element $x \in B$ is *integral* over A if there exist $a_1, \ldots, a_n \in A$ such that

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$

Then B is *integral* over A if each $x \in B$ is integral over A. Let X, Y be affine varieties, $f : X \longrightarrow Y$ a regular map such that $f(X)$ is dense in Y. Then the pull-back $f^{\#}$ defines an isomorphic inclusion $\mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$. We view $\mathbb{C}[Y]$ as a subring of $\mathbb{C}[X]$ by means of $f^{\#}$. Then f is a *finite* map if $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$, see [35]. This implies that the pre-image $F^{-1}(y)$ of each point $y \in Y$ is a finite set, and that, as y moves in Y, the points in $F^{-1}(y)$ may merge together but not disappear. The map $\mathbb{A}^1 - \{0\} \longrightarrow \mathbb{A}^1$ is the classic example of a map which is *not* finite.

Lemma 4.2. *Let X be a component in the extended variety $\Omega^+(G)$. Then the q-projection π_q is a finite map from X onto its image $\pi_q(X)$.*

Proof. Note that the fixed-point set D^γ is a complex torus of dimension r, that $\pi_q(D^\gamma)$ is a torus of dimension r and that we have an isomorphism of affine varieties $D^\gamma \cong \pi_q(D^\gamma)$. Let $X = D^\gamma/Z_\gamma, Y = \pi_q(D^\gamma)/\Gamma$ where Z_γ is the Γ-centralizer of γ. Now each of X and Y is a quotient of the variety D^γ by a finite group, hence X, Y are affine varieties [35, p.31]. We have $D^\gamma \longrightarrow X \longrightarrow Y$ and $\mathbb{C}[Y] \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C}[D^\gamma]$. According to [35, p.61], $\mathbb{C}[D^\gamma]$ is integral over $\mathbb{C}[Y]$ since $Y = D^\gamma/\Gamma$. Therefore the subring $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$. So the map $\pi_q : X \longrightarrow Y$ is finite. □

Example 4.3. $GL(2)$. Let T be the diagonal subgroup of $G = GL(2)$ and let Ω be the component in $\Omega(G)$ containing the cuspidal pair $(T, 1)$. Then $\sigma \in \Pi(GL(2))$ is *arithmetically unramified* if $inf.ch.\sigma \in \Omega$. If $\pi_F(\phi) = \sigma$ then ϕ is a 2-dimensional representation of $\mathcal{L}_F$ and there are two possibilities:
ϕ is reducible, $\phi = \psi_1 \oplus \psi_2$ with ψ_1, ψ_2 unramified quasicharacters of W_F. So $\psi_j(w) = z_j^{d(w)}, z_j \in \mathbb{C}^\times, j = 1, 2$. We have $\pi_F(\phi) = Q(\psi_1, \psi_2)$ where ψ_1 does not precede ψ_2. In particular we obtain the 1-dimensional representations of G as follows:

$$\pi_F(|\ |^{1/2}\psi \oplus |\ |^{-1/2}\psi) = Q(|\ |^{1/2}\psi, |\ |^{-1/2}\psi) = \psi \circ det.$$

ϕ *is irreducible*, $\phi = \psi \otimes \mathrm{spin}(1/2)$. Then $\pi_F(\phi) = Q(\Delta)$ with $\Delta = \{|\ |^{-1/2}\psi, |\ |^{1/2}\psi\}$ so $\pi_F(\phi) = \psi \otimes St(2)$ where $St(2)$ is the Steinberg representation of $GL(2)$.
The orbit of $(T,1)$ is $D = (\mathbb{C}^\times)^2$, and $W(T,D) = \mathbb{Z}/2\mathbb{Z}$. Then $\Omega \cong (\mathbb{C}^\times)^2/\mathbb{Z}/2\mathbb{Z} \cong \mathrm{Sym}^2\,\mathbb{C}^\times$. The extended quotient is $\Omega^+ = \mathrm{Sym}^2\ \mathbb{C}^\times \sqcup \mathbb{C}^\times$. The $q-$projection works as follows:

$$\pi_q : \{z_1, z_2\} \mapsto \{z_1, z_2\}$$

$$\pi_q : z \mapsto \{q^{1/2}z, q^{-1/2}z\}$$

where q is the cardinality of the residue field of F.

Let $A = \mathcal{H}(GL(2)//I)$ be the Iwahori-Hecke algebra of $GL(2)$. This is a finite type algebra. Following [21, p. 327], denote by $\mathrm{Prim}_n(A) \subset \mathrm{Prim}(A)$ the set of primitive ideals $B \subset A$ which are kernels of irreducible representations of A of dimension n. Set $X_1 = \mathrm{Prim}_1(A)$, $X_2 = \mathrm{Prim}_1(A) \sqcup \mathrm{Prim}_2(A) = \mathrm{Prim}(A)$. Then X_1 and X_2 are closed sets in $\mathrm{Prim}(A)$ defining an increasing filtration of $\mathrm{Prim}(A)$. Now A is Morita equivalent to the Bernstein ideal $\mathcal{H}(\Omega)$, and $\Pi(\Omega) \simeq \mathrm{Prim}(A)$.
Let $\phi_1 = 1 \otimes \mathrm{spin}(1/2)$, $\phi_2 = 1 \otimes 1 \oplus 1 \otimes 1$. The 1-dimensional representations of $GL(2)$ determine 1-dimensional representations of $\mathcal{H}(G//I)$ and so lie in X_1. The L-parameters of the 1-dimensional representations of $GL(2)$ do *not* lie in the 1-dimensional orbit $\mathcal{O}(\phi_1)$: they lie in the 2-dimensional orbit $\mathcal{O}(\phi_2)$. The Kazhdan-Nistor-Schneider stratification [21] does *not* coincide with stratification-by-dimension.

Example 4.4. $GL(3)$. In the above example, the q-projection is stratified-injective, i.e. injective on each orbit type. This is not so in general, as shown by the next example. Let T be the diagonal subgroup of $GL(3)$ and let Ω be the component containing the cuspidal pair $(T,1)$. Then $\Omega = \mathrm{Sym}^3\,\mathbb{C}^\times$ and

$$\Omega^+ = \mathrm{Sym}^3\,\mathbb{C}^\times \sqcup (\mathbb{C}^\times)^2 \sqcup \mathbb{C}^\times$$

.
The map π_q works as follows:

$$\{z_1, z_2, z_3\} \mapsto \{z_1, z_2, z_3\}$$
$$(z, w, w) \mapsto \{z, q^{1/2}w, q^{-1/2}w\}$$
$$(z, z, z) \mapsto \{qz, z, q^{-1}z\}.$$

Consider the L-parameter

$$\phi = \psi_1 \otimes 1 \oplus \psi_2 \otimes \mathrm{spin}(1/2) \in \Phi(GL(3)).$$

If $\psi(w) = z^{d(w)}$ then we will write $\psi = z$. With this understood, let

$$\phi_1 = q \otimes 1 \oplus q^{-1/2} \otimes \mathrm{spin}(1/2)$$

$$\phi_2 = q^{-1} \otimes 1 \oplus q^{1/2} \otimes \mathrm{spin}(1/2).$$

Then $\alpha(\phi_1), \alpha(\phi_2)$ are distinct points in the same stratum of the extended quotient, but their image under the q-projection π_q is the single point $\{q^{-1}, 1, q\} \in \mathrm{Sym}^3\, \mathbb{C}^\times$.
Let

$$\phi_3 = 1 \otimes \mathrm{spin}(3/2)$$
$$\phi_4 = q^{-1} \otimes 1 \oplus 1 \otimes 1 \oplus q \otimes 1.$$

Then the distinct L-parameters $\phi_1, \phi_2, \phi_3, \phi_4$ all have the same image under the q-projection π_q.

5. The diagram

In this section we create a diagram which incorporates several major results. The following diagram serves as a framework for the whole article:

$$\begin{tikzcd}
K_*^{\mathrm{top}}(G) \arrow[rr, "\mu"] \arrow[d, "\mathrm{ch}"'] & & K_*(C_r^*(G)) \arrow[d, "\mathrm{ch}"] \\
\mathrm{H}_*(G;\beta G) \arrow[r] \arrow[d, dotted] & \mathrm{HP}_*(\mathcal{H}(G)) \arrow[r, "\imath_*"] \arrow[d] & \mathrm{HP}_*(\mathcal{S}(G)) \arrow[d] \\
\mathrm{H}_c^*(\Phi(G);\mathbb{C}) \arrow[r] & \mathrm{H}_c^*(\Pi(G);\mathbb{C}) \arrow[r] & \mathrm{H}_c^*(\Pi^t(G);\mathbb{C})
\end{tikzcd}$$

The Baum-Connes assembly map μ is an isomorphism [1, 24]. The map

$$\mathrm{H}_*(G;\beta G) \to \mathrm{HP}_*(\mathcal{H}(G))$$

is an isomorphism [20, 33]. The map $\imath_*$ is an isomorphism by Theorem 3.3. The right hand Chern character is constructed in [9] and is an isomorphism after tensoring over $\mathbb{Z}$ with $\mathbb{C}$ [9, Theorem 3]. The Chern character on the left hand side of the diagram is the unique map for which the top half of the diagram is commutative.
In the diagram, $\mathrm{H}_c^*(\Pi^t(G);\mathbb{C})$ denotes the (periodised) compactly supported de Rham cohomology of the tempered dual $\Pi^t(G)$, and $\mathrm{H}_c^*(\Pi(G);\mathbb{C})$ denotes the (periodised) de Rham cohomology supported on finitely many components of the smooth dual $\Pi(G)$. The map

$$\mathrm{HP}_*(\mathcal{S}(G)) \to \mathrm{H}_c^*(\Pi^t(G);\mathbb{C})$$

is constructed in [7] and is an isomorphism [7, Theorem 7].
The map

$$\mathrm{H}_c^*(\Pi(G);\mathbb{C}) \to \mathrm{H}_c^*(\Pi^t(G);\mathbb{C})$$

is constructed in the following way. Given an L-parameter $\phi : \mathcal{L}_F \to GL(n, \mathbb{C})$ we have

$$\phi = \phi_1 \oplus \ldots \oplus \phi_m$$

with each ϕ_j an irreducible representation. We have $\phi_j = \rho_j \otimes \mathrm{spin}(j)$ where each ρ_j is an irreducible representation of the Weil group W_F. We shall assume that $\det \rho_j$ is a unitary character. Let $\mathcal{O}(\phi)$ be the orbit of ϕ as in Definition 1.4. The map $\mathcal{O}(\phi) \to \mathcal{O}^t(\phi)$ is now defined as follows

$$\psi_1\phi_1 \oplus \ldots \oplus \psi_m\phi_m \mapsto |\psi_1|^{-1} \cdot \psi_1\phi_1 \oplus \ldots \oplus |\psi_m|^{-1} \cdot \psi_m\phi_m.$$

This map is a deformation retraction of the complex orbit $\mathcal{O}(\phi)$ onto the compact orbit $\mathcal{O}^t(\phi)$. Since $\Phi(G)$ is a disjoint union of such complex orbits this formula determines, via the local Langlands correspondence for $GL(n)$, a deformation retraction of $\Pi(G)$ onto the tempered dual $\Pi^t(GL(n))$, which implies that the induced map on cohomology is an isomorphism.
The map

$$\mathrm{H}^*_c(\Phi(G); \mathbb{C}) \to \mathrm{H}^*_c(\Pi(G); \mathbb{C})$$

is an isomorphism, induced by the local Langlands correspondence π_F.
The map

$$\mathrm{HP}_*(\mathcal{H}(G)) \to \mathrm{H}^*_c(\Pi(G); \mathbb{C})$$

is an isomorphism by Theorem 3.1.
There is at present no direct definition of the map

$$\mathrm{H}_*(G; \beta G) \to \mathrm{H}^*_c(\Phi(G); \mathbb{C}).$$

Suppose for the moment that F has characteristic 0 and has residue field of characteristic p. An irreducible representation ρ of the Weil group W_F is called wildly ramified if $\dim \rho$ is a power of p and $\rho \not\simeq \rho \otimes \psi$ for any unramified quasicharacter $\psi \neq 1$ of W_F. We write $\Phi^{wr}_m(F)$ for the set of equivalence classes of such representations of dimension p^m. An irreducible supercuspidal representation π of $GL(n)$ is wildly ramified if n is a power of p and $\pi \not\simeq \pi \otimes (\psi \circ \det)$ for any unramified quasicharacter $\psi \neq 1$ of $F^\times$. We write $\Pi^{wr}_m(F)$ for the set of equivalence classes of such representations of $GL(p^m, F)$. In this case Bushnell-Henniart [10] construct, for each m, a canonical bijection

$$\pi_{F,m} : \Phi^{wr}_m(F) \to \Pi^{wr}_m(F).$$

Now the maximal simple type (J, λ) of an irreducible supercuspidal representation determines an element in the chamber homology of the affine building [2, 6.7]. The construction of Bushnell-Henniart therefore determines a map from a *subspace* of $\mathrm{H}^{\mathrm{even}}_c(\Phi(G); \mathbb{C})$ to a *subspace* of $\mathrm{H}_0(G; \beta G)$.
In the context of the above diagram the Baum-Connes map has a geometric counterpart: it is induced by the deformation retraction of $\Pi(GL(n))$ onto the tempered dual $\Pi^t(GL(n))$.

6. Supercuspidal representations of $GL(n)$

In this section we track the fate of supercuspidal representations of $GL(n)$ through the diagram constructed in the previous Section. Let ρ be an irreducible n-dimensional complex representation of the Weil group W_F such that $\det \rho$ is a unitary character and let $\phi = \rho \otimes 1$. Then ϕ is the L-parameter for a pre-unitary supercuspidal representation ω of $GL(n)$. Let $\mathcal{O}(\phi)$ be the orbit of ϕ and $\mathcal{O}^t(\phi)$ be the compact orbit of ϕ. Then $\mathcal{O}(\phi)$ is a component in the

Bernstein variety isomorphic to $\mathbb{C}^\times$ and $\mathcal{O}^t(\phi)$ is a component in the tempered dual, isomorphic to $\mathbb{T}$. The L-parameter ϕ now determines the following data.

6.1. Let (J, λ) be a maximal simple type for ω in the sense of Bushnell and Kutzko [11, chapter 6]. Then J is a compact open subgroup of G and λ is a smooth irreducible complex representation of J.
We will write

$$\mathbb{T} = \{\psi \otimes \omega : \psi \in \Psi^t(G)\}$$

where $\Psi^t(G)$ denotes the group of unramified unitary characters of G.

Theorem 6.1. *Let K be a maximal compact subgroup of G containing J and form the induced representation $W = Ind_J^K(\lambda)$. We then have*

$$\ell^2(G \times_K W) \simeq Ind_K^G(W) \simeq Ind_J^G(\lambda) \simeq \int_{\mathbb{T}} \pi d\pi.$$

Proof. The supercuspidal representation ω contains λ and, modulo unramified unitary twist, is the only irreducible unitary representation with this property [11, 6.2.3]. Now the Ahn reciprocity theorem expresses Ind_J^G as a direct integral [26, p.58]:

$$\mathrm{Ind}_J^G(\lambda) = \int n(\pi, \lambda)\pi d\pi$$

where $d\pi$ is Plancherel measure and $n(\pi, \lambda)$ is the multiplicity of λ in $\pi|_J$. But the Hecke algebra of a maximal simple type is commutative (a Laurent polynomial ring). Therefore $\omega|_J$ contains λ with multiplicity 1 (thanks to C. Bushnell for this remark). We then have $n(\psi \otimes \omega, \lambda) = 1$ for all $\psi \in \Psi^t(G)$. We note that Plancherel measure induces Haar measure on $\mathbb{T}$, see [31].
The affine building of G is defined as follows [38, p. 49]:

$$\beta G = \mathbb{R} \times \beta SL(n)$$

where $g \in G$ acts on the affine line $\mathbb{R}$ via $t \mapsto t + val(det(g))$. Let $G^\circ = \{g \in G : val(det(g)) = 0\}$. We use the standard model for $\beta SL(n)$ in terms of equivalence classes of $\mathbf{o}_F$ - lattices in the n-dimensional F-vector space V. Then the vertices of $\beta SL(n)$ are in bijection with the maximal compact subgroups of G°, see [32, 9.3]. Let $P \in \beta G$ be the vertex for which the isotropy subgroup is $K = GL(n, \mathbf{o}_F)$. Then the G-orbit of P is the set of all vertices in βG and the discrete space G/K can be identified with the set of vertices in the affine building βG. Now the base space of the associated vector bundle $G \times_K W$ is the discrete coset space G/K, and the Hilbert space of ℓ^2-sections of this homogeneous vector bundle is a realization of the induced representation $\mathrm{Ind}_K^G(W)$. □

The $C_0(\beta G)$-module structure is defined as follows. Let $f \in C_0(\beta G)$, $s \in \ell^2(G \times_K W)$ and define

$$(fs)(v) = f(v)s(v)$$

for each vertex $v \in \beta G$. We proceed to construct a K-cycle in degree 0. This K-cycle is

$$(C_0(\beta G), \ell^2(G \times_K W) \oplus 0, 0)$$

interpreted as a $\mathbb{Z}/2\mathbb{Z}$-graded module. This triple satisfies the properties of a (pre)-Fredholm module [14, IV] and so creates an element in $K_0^{\text{top}}(G)$. By Theorem 5.1 this generator creates a free $C(\mathbb{T})$-module of rank 1, and so provides a generator in $K_0(C_r^*(G))$.

6.2. The Hecke algebra of the maximal simple type (J, λ) is commutative (the Laurent polynomials in one complex variable). The periodic cyclic homology of this algebra is generated by 1 in degree zero and dz/z in degree 1.
The corresponding summand of the Schwartz algebra $\mathcal{S}(G)$ is Morita equivalent to the Fréchet algebra $C^\infty(\mathbb{T})$. By an elementary application of Connes' theorem [14, Theorem 2, p. 208], the periodic cyclic homology of this Fréchet algebra is generated by 1 in degree 0 and $d\theta$ in degree 1.

6.3. The corresponding component in the Bernstein variety is a copy of $\mathbb{C}^\times$. The cohomology of $\mathbb{C}^\times$ is generated by 1 in degree 0 and $d\theta$ in degree 1.
The corresponding component in the tempered dual is the circle $\mathbb{T}$. The cohomology of $\mathbb{T}$ is generated by 1 in degree 0 and $d\theta$ in degree 1.

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Jacek Brodzki
Faculty of Mathematical Studies
University of Southampton
Southampton SO17 1BJ, U.K.
j.brodzki@maths.soton.ac.uk

Roger Plymen
Department of Mathematics
University of Manchester
Manchester M13 9PL, U.K.
roger@maths.man.ac.uk