Tamagawa Numbers for Motives with (Non-Commutative) Coefficients

D. Burns and M. Flach

Received: September 6, 2001
Revised: January 28, 2002

Communicated by Don Blasius

Abstract. Let M be a motive which is defined over a number field and admits an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. We formulate and study a conjecture for the leading coefficient of the Taylor expansion at 0 of the A-equivariant L-function of M. This conjecture simultaneously generalizes and refines the Tamagawa number conjecture of Bloch, Kato, Fontaine, Perrin-Riou et al. and also the central conjectures of classical Galois module theory as developed by Fröhlich, Chinburg, M. Taylor et al. The precise formulation of our conjecture depends upon the choice of an order $\mathfrak{A}$ in A for which there exists a 'projective $\mathfrak{A}$-structure' on M. The existence of such a structure is guaranteed if $\mathfrak{A}$ is a maximal order, and also occurs in many natural examples where $\mathfrak{A}$ is non-maximal. In each such case the conjecture with respect to a non-maximal order refines the conjecture with respect to a maximal order. We develop a theory of determinant functors for all orders in A by making use of the category of virtual objects introduced by Deligne.

2000 Mathematics Subject Classification: Primary 11G40; Secondary 11R65 19A31 19B28

1. Introduction

The study of values of L-functions attached to varieties over number fields occupies a prominent place in number theory and has led to some remarkably general conjectures. A seminal step was made by Bloch and Kato who conjecturally described up to sign the leading coefficient at zero of L-functions attached to motives of negative weight [4]. A little later, Fontaine and Perrin-Riou and (independently) Kato used the determinant functor to extend this conjecture to motives of any weight and with commutative coefficients, thereby taking into account the action of endomorphisms of the variety under consideration (cf. [19, 20, 27, 28]). In this article we shall formulate and study a yet more general conjecture which deals with motives with coefficients which need not be commutative and which in the commutative case recovers all of the above conjectures. We remark that the motivation for such a general conjecture is that in many natural cases, ranging from the central conjectures of classical Galois module theory to the recent attempts to develop an Iwasawa theory for elliptic curves which do not possess complex multiplication, it is necessary to consider motives with respect to coefficients which are not commutative.

We now fix a motive M which is defined over a number field K and carries an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. The precise formulation of our conjecture depends upon the choice of an order $\mathfrak{A}$ in A for which there exists a 'projective $\mathfrak{A}$-structure' on M (as defined in §3.3). We observe that if $\mathfrak{A}$ is any maximal order in A (as in the case considered by Fontaine and Perrin-Riou in [19]), then there always exists a projective $\mathfrak{A}$-structure on M, and in addition that if M arises by base change of a motive through a finite Galois extension L/K and $A := \mathbb{Q}[G]$ with $G := \mathrm{Gal}(L/K)$ (as in the case considered by Kato in [27]), then there exists a projective $\mathbb{Z}[G]$-structure on M. In general, we find that if there exists a projective $\mathfrak{A}$-structure on M, then there also exists a projective $\mathfrak{A}'$-structure on M for any order $\mathfrak{A} \subset \mathfrak{A}' \subset A$ but that the conjecture which we formulate for the pair $(M, \mathfrak{A}')$ is (in general strictly) weaker than that for the pair $(M, \mathfrak{A})$. This observation is important since we shall show that there are several natural examples (such as the case $\mathfrak{A} = \mathbb{Z}[G]$ described above) in which projective structures exist with respect to orders which are not maximal.

The key difficulty encountered when attempting to formulate Tamagawa number conjectures with respect to non-commutative coefficients is the fact that there is no determinant functor over non-commutative rings. In this article we circumvent this difficulty by making systematic use of the notion of 'categories of virtual objects' as described by Deligne in [17]. In our approach Tamagawa numbers are then elements of a relative algebraic K-group $K_0(\mathfrak{A}, \mathbb{R})$ and the Tamagawa number conjecture is an identity in this group. The group $K_0(\mathfrak{A}, \mathbb{R})$ is the relative K_0 which arises from the inclusion of rings $\mathfrak{A} \to A_{\mathbb{R}} := A \otimes_{\mathbb{Q}} \mathbb{R}$ and hence lies in a natural long exact sequence

$$K_1(\mathfrak{A}) \to K_1(A_{\mathbb{R}}) \to K_0(\mathfrak{A}, \mathbb{R}) \to K_0(\mathfrak{A}) \to K_0(A_{\mathbb{R}}).$$

We remark that in the non-equivariant setting originally considered by Bloch and Kato [4] one has $\mathfrak{A} = \mathbb{Z}$, $A = \mathbb{Q}$ and $K_0(\mathfrak{A}, \mathbb{R}) \cong \mathbb{R}^\times/\mathbb{Z}^\times$. This latter quotient identifies with the group of positive real numbers, and hence in this case Tamagawa numbers can be interpreted as volumes. For motives with non-commutative coefficients however, the only way we have at present been able to formulate a conjecture is by use of the group $K_0(\mathfrak{A}, \mathbb{R})$.

The basic content of this article is as follows. Various algebraic preliminaries relating to determinant functors, categories of virtual objects and relative algebraic K-theory, which may themselves be of some independent interest, are given in §2. In §3 we recall preliminaries on motives, define the notion of a 'projective $\mathfrak{A}$-structure' on M and give several natural examples of this notion. We henceforth assume that $\mathfrak{A}$ is an order in A for which there exists a projective $\mathfrak{A}$-structure on M. In the remainder of §3 we combine the results of §2 with certain standard assumptions on motives to define a canonical element $R\Omega(M,\mathfrak{A})$ of $K_0(\mathfrak{A},\mathbb{R})$. In §4 we review the A-equivariant L-function $L({}_AM,s)$ of M. This is a meromorphic function of the complex variable s which takes values in the center $\zeta(A_{\mathbb{C}})$ of $A_{\mathbb{C}} := A\otimes_{\mathbb{Q}}\mathbb{C}$, and the leading coefficient $L^*({}_AM,0)$ in its Taylor expansion at $s=0$ belongs to the group of units $\zeta(A_{\mathbb{R}})^\times$ of $\zeta(A_{\mathbb{R}})$. We also define a canonical 'extended boundary homomorphism' $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}} : \zeta(A_{\mathbb{R}})^\times \to K_0(\mathfrak{A},\mathbb{R})$ which has the property that the composite of $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ with the reduced norm map $K_1(A_{\mathbb{R}}) \to \zeta(A_{\mathbb{R}})^\times$ is equal to the boundary homomorphism $K_1(A_{\mathbb{R}}) \to K_0(\mathfrak{A},\mathbb{R})$ which occurs in the above long exact sequence. We then formulate the central conjecture of this article (Conjecture 4) which states that

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(L^*({}_AM,0)) = -R\Omega(M,\mathfrak{A}) \ \text{ in } \ K_0(\mathfrak{A},\mathbb{R}).$$

We remark that our use of the map $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ in this context is motivated by the central conjectures of classical Galois module theory. In the remainder of §4 we review some of the current evidence for our conjecture, establish its standard functorial properties and also derive several interesting consequences of these functorial properties. Finally, in §5 we use the Artin-Verdier Duality Theorem to investigate the compatibility of our conjecture with the functional equation of $L({}_AM,s)$.

In a sequel to this article [11] we shall give further evidence for our general conjectures by relating them to classical Galois module theory (in particular, to certain much studied conjectures of Chinburg [13, 14]) and by proving them in several nontrivial cases. In particular, we prove the validity of our central conjecture for pairs $(M,\mathfrak{A}) = (h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$ where $K=\mathbb{Q}$ and $L/\mathbb{Q}$ belongs to an infinite family of Galois extensions for which $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the Quaternion group of order 8.

This article together with its sequel [11] subsumes the contents of an earlier preprint of the same title, and also of the preprint [10]. In the preprint [5] the first named author described an earlier approach to formulating Tamagawa number conjectures with respect to non-commutative coefficients, by using the notions of 'trivialized perfect complex' and 'refined Euler characteristic' (cf. Remark 4 in §2.8 in this regard). However, by making systematic use of virtual objects the approach adopted here seems to be both more flexible and transparent, and in particular allows us to prove the basic properties of our construction in a very natural manner.

We would like to thank P. Deligne for a number of very valuable comments.

2. Determinant functors for orders in semisimple algebras

2.1. Picard categories. This section introduces the natural target categories for our generalized determinant functors. Recall that a groupoid is a nonempty category in which all morphisms are isomorphisms. A *Picard category* $\mathcal{P}$ is a groupoid equipped with a bifunctor $(L, M) \to L \boxtimes M$ with an associativity constraint [32] and so that all the functors $- \boxtimes M$, $M \boxtimes -$ for a fixed object M are autoequivalences of $\mathcal{P}$. In a Picard category there exists a unit object $\mathbf{1}_{\mathcal{P}}$, unique up to unique isomorphism, and for each object M an inverse M^{-1}, unique up to unique isomorphism, with an isomorphism $M \boxtimes M^{-1} \xrightarrow{\sim} \mathbf{1}_{\mathcal{P}}$. For a Picard category $\mathcal{P}$ define $\pi_0(\mathcal{P})$ to be the group of isomorphism classes of objects of $\mathcal{P}$ (with product induced by $\boxtimes$), and set $\pi_1(\mathcal{P}) := \mathrm{Aut}_{\mathcal{P}}(\mathbf{1}_{\mathcal{P}})$. We shall only have occasion to consider *commutative* Picard categories in which $\boxtimes$ also satisfies a commutativity constraint [32] and for which $\pi_0(\mathcal{P})$ is therefore abelian. The group $\pi_1(\mathcal{P})$ is always abelian. A monoidal functor $F : \mathcal{P}_1 \to \mathcal{P}_2$ between Picard categories induces homomorphisms $\pi_i(F) : \pi_i(\mathcal{P}_1) \to \pi_i(\mathcal{P}_2)$ for $i \in \{0, 1\}$, and F is an equivalence of categories if and only if $\pi_i(F)$ is an isomorphism for both $i \in \{0, 1\}$ (by a monoidal functor we mean a strong monoidal functor as defined in [31][Ch. XI.2]).

2.2. The fibre product of categories. Let $F_i : \mathcal{P}_i \to \mathcal{P}_3$, $i \in \{1, 2\}$ be functors between categories and consider the *fibre product category* $\mathcal{P}_4 := \mathcal{P}_1 \times_{\mathcal{P}_3} \mathcal{P}_2$ [2, Ch. VII, §3]

$$\begin{array}{ccc} \mathcal{P}_4 & \xrightarrow{G_2} & \mathcal{P}_2 \\ \big\downarrow{\scriptstyle G_1} & & \big\downarrow{\scriptstyle F_2} \\ \mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_3. \end{array} \tag{1}$$

Explicitly, $\mathcal{P}_4$ is the category with objects (L_1, L_2, λ) with $L_i \in \mathrm{Ob}(\mathcal{P}_i)$ for $i \in \{1, 2\}$ and $\lambda : F_1(L_1) \xrightarrow{\sim} F_2(L_2)$ an isomorphism in $\mathcal{P}_3$, and where morphisms $\alpha : (L_1, L_2, \lambda) \to (L_1', L_2', \lambda')$ are pairs $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \mathrm{Hom}_{\mathcal{P}_i}(L_i, L_i')$ so that the diagram

$$\begin{array}{ccc} F_1(L_1) & \xrightarrow{F_1(\alpha_1)} & F_1(L_1') \\ \big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle \lambda'} \\ F_2(L_2) & \xrightarrow{F_2(\alpha_2)} & F_2(L_2') \end{array}$$

in $\mathcal{P}_3$ commutes. If $\mathcal{P}_5$ is a category, $H_i : \mathcal{P}_5 \to \mathcal{P}_i$ for $i \in \{1, 2\}$ functors and $\beta : F_1 \circ H_1 \cong F_2 \circ H_2$ a natural isomorphism, then there exists a unique functor $H : \mathcal{P}_5 \to \mathcal{P}_4$ with $H_i = G_i \circ H$ for $i \in \{1, 2\}$ and such that β is induced by the natural isomorphism $F_1 \circ G_1 \cong F_2 \circ G_2$. If $\mathcal{P}_i$ for $i \in \{1, 2, 3\}$ are Picard categories and F_1 and F_2 are monoidal functors, then the fibre product category $\mathcal{P}_4$ is a Picard category with product $(L_1, L_2, \lambda) \boxtimes (L_1', L_2', \lambda') = (L_1 \boxtimes L_1', L_2 \boxtimes L_2', \lambda \boxtimes \lambda')$ and the functors $G_i : \mathcal{P}_4 \to \mathcal{P}_i$ for $i \in \{1, 2\}$ are both monoidal.

Lemma 1. *(Mayer-Vietoris sequence) For a fibre product diagram (1) of Picard categories one has an exact sequence*

$$0 \to \pi_1(\mathcal{P}_4) \xrightarrow{(\pi_1(G_1),\pi_1(G_2))} \pi_1(\mathcal{P}_1) \oplus \pi_1(\mathcal{P}_2) \xrightarrow{\pi_1(F_1)-\pi_1(F_2)} \pi_1(\mathcal{P}_3) \xrightarrow{\delta}$$
$$\to \pi_0(\mathcal{P}_4) \xrightarrow{(\pi_0(G_1),\pi_0(G_2))} \pi_0(\mathcal{P}_1) \oplus \pi_0(\mathcal{P}_2) \xrightarrow{\pi_0(F_1)-\pi_0(F_2)} \pi_0(\mathcal{P}_3).$$

Proof. The map δ is defined by $\delta(\beta) = (\mathbf{1}_{\mathcal{P}_1}, \mathbf{1}_{\mathcal{P}_2}, \beta)$. Given the explicit description of $\pi_0(-)$ and $\pi_1(-)$ it is an elementary computation to establish the exactness of this sequence. A general Mayer-Vietoris sequence for categories with product can be found in [2, Ch. VII, Th. (4.3)]. For Picard categories this sequence specialises to our Lemma (except for the injectivity of the first map). □

2.3. Determinant functors and virtual objects. Let $\mathcal{E}$ be an exact category [38, p. 91] and $(\mathcal{E}, \mathrm{is})$ the subcategory of all isomorphisms in $\mathcal{E}$. The main example we have in mind is the category $\mathrm{PMod}(R)$ of finitely generated projective modules over a (not necessarily commutative) ring R. By a *determinant functor* we mean a Picard category $\mathcal{P}$ together with the following data.

a) A functor $[\ ] : (\mathcal{E}, \mathrm{is}) \to \mathcal{P}$.
b) For each short exact sequence
$$\Sigma : 0 \to E' \to E \to E'' \to 0$$
a morphism $[\Sigma] : [E] \xrightarrow{\sim} [E'] \boxtimes [E'']$ in $\mathcal{P}$, functorial for isomorphisms of short exact sequences.
c) For each zero object 0 in $\mathcal{E}$ an isomorphism
$$\zeta(0) : [0] \xrightarrow{\sim} \mathbf{1}_{\mathcal{P}}.$$

This data is subject to the following axioms.

d) For an isomorphism $\phi : E \to E'$ and Σ the exact sequence $0 \to E \to E'$ (resp. $E \to E' \to 0$) $[\phi]$ (resp. $[\phi^{-1}]$) is the composite map
$$[E] \xrightarrow{[\Sigma]} [0] \boxtimes [E'] \xrightarrow{\zeta(0)\boxtimes\mathrm{id}} [E']$$
(resp.
$$[E'] \xrightarrow{[\Sigma]} [E] \boxtimes [0] \xrightarrow{\zeta(0)\boxtimes\mathrm{id}} [E]).$$
e) For admissible subobjects $0 \subseteq E'' \subseteq E' \subseteq E$ of an object E of $\mathcal{E}$ the diagram
$$\begin{tikzcd} {[E]} \arrow[r] \arrow[d] & {[E''] \boxtimes [E/E'']} \arrow[d] \\ {[E'] \boxtimes [E/E']} \arrow[r] & {[E''] \boxtimes [E'/E''] \boxtimes [E/E']} \end{tikzcd}$$
in $\mathcal{P}$ commutes.

The terminology here is borrowed from the key example in which $\mathcal{E}$ is the category of vector bundles on a scheme, $\mathcal{P}$ is the category of line bundles and the functor is taking the highest exterior power (see §2.5 below). However, as was shown by Deligne in [17, §4], there exists a universal determinant functor for any given exact category $\mathcal{E}$. More precisely, there exists a Picard category $V(\mathcal{E})$, called the 'category of virtual objects' of $\mathcal{E}$, together with data a)-c) which in addition to d) and e) also satisfies the following universal property.

f) For any Picard category $\mathcal{P}$ the category of monoidal functors $\mathrm{Hom}^{\boxtimes}(V(\mathcal{E}),\mathcal{P})$ is naturally equivalent to the category of determinant functors $(\mathcal{E},\mathrm{is})\to\mathcal{P}$.

Although comparatively inexplicit it is this construction which works best for the purposes of this paper.
We recall that the category $V(\mathcal{E})$ has a commutativity constraint defined as follows. Let

$$\tau_{E',E''}:[E']\boxtimes[E'']\xleftarrow{[\Sigma_1]}[E'\oplus E'']\xrightarrow{[\Sigma_2]}[E'']\boxtimes[E']$$

be the isomorphism induced by the short exact sequences

$$\Sigma_1: 0\to E'\to E'\oplus E''\to E''\to 0$$

$$\Sigma_2: 0\to E''\to E'\oplus E''\to E'\to 0.$$

Replacing $[\Sigma]$ by $\tau_{E',E''}\circ[\Sigma]$ yields a datum a), b), c) with values in $V(\mathcal{E})^{\boxtimes-op}$, the Picard category with product $(L,M)\mapsto M\boxtimes L$, and satisfying d),e). By the universal property f) of $V(\mathcal{E})$, this corresponds to a monoidal functor $F: V(\mathcal{E})\to V(\mathcal{E})^{\boxtimes-op}$. Since we have only changed the value of [ ] on short exact sequences, F is the identity on objects and morphisms and so the monoidality of F gives a commutativity constraint on $V(\mathcal{E})$.
The proof of the existence of $V(\mathcal{E})$ in [17, §4.2-5] also gives a topological model of $V(\mathcal{E})$ which in turn implies that there are isomorphisms

$$K_i(\mathcal{E})\xrightarrow{\sim}\pi_i(V(\mathcal{E})) \tag{2}$$

with the algebraic K-groups of the exact category $\mathcal{E}$ (see [38]) for $i\in\{0,1\}$. An exact functor $F:\mathcal{E}\to\mathcal{E}'$ induces a datum a),b),c) on $\mathcal{E}$ with values in $V(\mathcal{E}')$ and hence by f) a monoidal functor $V(F):V(\mathcal{E})\to V(\mathcal{E}')$. The isomorphism (2) then commutes with the maps induced by F on $K_i(\mathcal{E})$ and by $V(F)$ on $\pi_i(V(\mathcal{E}))$ for $i\in\{0,1\}$. Moreover, for $i=0$ the isomorphism (2) is the map induced by the functor [ ], and for $i=1$ the element in $K_1(\mathcal{E})$ represented by $\phi\in\mathrm{Aut}_{\mathcal{E}}(P)$ is sent to $[\phi]\boxtimes\mathrm{id}([P]^{-1})$ under (2).

2.4. Projective modules and extension to the derived category. For a ring R denote by $\mathrm{PMod}(R)$ the exact category of finitely generated projective left-R-modules and put $V(R):=V(\mathrm{PMod}(R))$. For a ring homomorphism $R\to R'$ we denote by $R'\otimes_R -$ both the scalar extension functor $\mathrm{PMod}(R)\to\mathrm{PMod}(R')$ and also the induced functor $V(R)\to V(R')$. It is known that the Whitehead group $K_1(R):=K_1(\mathrm{PMod}(R))$ of R is generated by automorphisms of objects of $\mathrm{PMod}(R)$.

We write $D(R)$ for the derived category of the homotopy category of complexes of R-modules, and $D^p(R)$ for the full triangulated subcategory of $D(R)$ which consists of perfect complexes. We say that an R-module X is *perfect* if the associated complex $X[0]$ belongs to $D^p(R)$, and we write $D^{p,p}(R)$ for the full subcategory of $D^p(R)$ consisting of those objects for which the cohomology modules are perfect in all degrees. The association $X \mapsto X[0]$ gives a full embedding of $\mathrm{PMod}(R)$ into $D^p(R)$.

In what follows we use the term 'true triangle' as synonymous for 'short exact sequence of complexes'. By a 'true nine term diagram' we shall mean a commutative diagram of complexes of the form

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle h'}\downarrow \\ X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' \end{array} \tag{3}$$

in which all of the rows and columns are true triangles.

Proposition 2.1. *The functor* $[\,] : (\mathrm{PMod}(R), \mathrm{is}) \to V(R)$ *extends to a functor* $[\,] : (D^p(R), \mathrm{is}) \to V(R)$. *Moreover, for each true triangle*

$$E = E(u,v) : X \xrightarrow{u} Y \xrightarrow{v} Z$$

in which X, Y, Z *are objects of* $D^p(R)$ *there exists an isomorphism* $[E] : [Y] \xrightarrow{\sim} [X] \boxtimes [Z]$ *in* $V(R)$ *which satisfies all of the following conditions:*

a) *If*

$$\begin{array}{ccccc} X & \xrightarrow{\quad u \quad} & Y & \xrightarrow{\quad v \quad} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ X' & \xrightarrow{\quad u' \quad} & Y' & \xrightarrow{\quad v' \quad} & Z' \end{array}$$

is a commutative diagram of true triangles and f, g, h *are all quasi-isomorphisms, then* $[f] \boxtimes [h] \circ [E(u,v)] \circ [g]^{-1} = [E(u', v')]$.

b) *If* u *(resp.* v*) is a quasi-isomorphism, then* $[E] = [u]^{-1}$ *(resp.* $[E] = [v]$*).*

c) $[\,]$ *commutes with the functors induced by any ring extension* $R \to R'$ *and for any true triangle* E *we have* $R' \otimes_R [E] = [R' \otimes_R E]$.

d) *For any true nine term diagram (3) in which all terms are objects of $D^p(R)$, the diagram*

$$\begin{array}{ccc} [Y'] & \xrightarrow{[E(u',v')]} & [X'] \boxtimes [Z'] \\ \big\downarrow{\scriptstyle [E(g,g')]} & & \big\downarrow{\scriptstyle [E(f,f')] \boxtimes [E(h,h')]} \\ [Y] \boxtimes [Y''] & \xrightarrow{[E(u,v)] \boxtimes [E(u'',v'')]} & [X] \boxtimes [Z] \boxtimes [X''] \boxtimes [Z''] \end{array}$$

in $V(R)$ commutes. (Note that we have suppressed any explicit reference to commutativity constraints in the above diagram).

e) *For any object X of $D^{p,p}(R)$ there exists a canonical isomorphism*

$$[X] \xrightarrow{\sim} \boxtimes_{i \in \mathbb{Z}} [H^i(X)]^{(-1)^i} \tag{4}$$

which is functorial with respect to quasi-isomorphisms.

Proof. This follows directly from [30, Prop. 4, Th. 2] where the same statement is proved for the determinant functor over a commutative ring R (see §2.5 below). Indeed, since the only properties of the determinant functor used in that proof are those listed in [loc. cit., Prop. 1] and all of these properties are satisfied by the functor [ ] (see [17, Lem. 4.8] for nine term diagrams) these arguments apply to give the desired extension of [ ] to $D^p(R)$ with properties a)-d). For e) see [30, Rem. b) after Th. 2]. □

Remark 1. As pointed out in [30, Rem. before Prop. 6], it is not possible to construct isomorphisms $[E]$ for all exact triangles E in $D^p(R)$ in such a way that the obvious generalisations of properties a)-d) hold. On the subcategory $D^{p,p}(R)$, however, one can at least construct isomorphisms so that a)-c) hold and so that d) holds under further assumptions (for example, that one of Y, X', Z' or Y'' is acyclic, or that X'' acyclic and $\mathrm{Hom}_{D^p(R)}(X,w) = 0$ where w is such that $Z[-1] \xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Z$ is an exact triangle).

2.5. Commutative rings. If R is a commutative ring, then one can consider the Picard category $\mathcal{P}(R)$ of *graded line bundles* on $\mathrm{Spec}(R)$ [30]. Recall that a graded line bundle is a pair (L,α) consisting of an invertible (that is, projective rank one) R-module L and a locally constant function $\alpha : \mathrm{Spec}(R) \to \mathbb{Z}$. A homomorphism $h : (L,\alpha) \to (M,\beta)$ is a module homomorphism $h : L \to M$ such that $\alpha(\mathfrak{p}) \neq \beta(\mathfrak{p})$ implies $h_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, and $\mathcal{P}(R)$ is the category of graded line bundles and isomorphisms of such. The category $\mathcal{P}(R)$ is a symmetric monoidal category with tensor product $(L,\alpha) \otimes (M,\beta) := (L \otimes_R M, \alpha+\beta)$, the usual associativity constraint, unit object $(R,0)$, and commutativity constraint

$$\psi(l \otimes m) := \psi_{(L,\alpha),(M,\beta)}(l \otimes m) = (-1)^{\alpha(\mathfrak{p})\beta(\mathfrak{p})} m \otimes l \tag{5}$$

for local sections $l \in L_{\mathfrak{p}}$ and $m \in M_{\mathfrak{p}}$. For a finitely generated projective R-module P one defines

$$\mathrm{Det}_R(P) := (\bigwedge_R^{\mathrm{rank}_R(P)} P, \mathrm{rank}_R(P)) \in \mathrm{Ob}(\mathcal{P}(R)).$$

The functor $\mathrm{Det}_R : (\mathrm{PMod}(R), \mathrm{is}) \to \mathcal{P}(R)$ is equipped with the data b) and c) of §2.3 and satisfies d) and e). Hence by f) there exists a unique monoidal functor

$$\mathrm{VDet}_R : V(R) \to \mathcal{P}(R),$$

which is also compatible with the commutativity constraints (this is the reason for the choice of signs in (5)). The functor VDet_R is an equivalence of categories if and only if the natural maps

$$K_0(R) \to \mathrm{Pic}(R) \times H^0(\mathrm{Spec}(R), \mathbb{Z}) \tag{6}$$

$$K_1(R) \to R^\times$$

are both bijective. For any commutative ring R, these maps are split surjections by [24, Exp. I, 6.11-6.14; Exp. X, Th. 5.3.2], [43, (1.8)]. They are known to be bijective if, for example, R is either a local ring, a semisimple ring or the ring of integers in a number field.

2.6. Semisimple rings. We recall here some facts about $K_0(R)$ and $K_1(R)$ for semisimple rings R. For the moment we let F be any field and assume that R is a central simple algebra over F. We fix a finite extension F'/F so that $R' := R \otimes_F F' \cong M_n(F')$ and an indecomposable idempotent e of R'.
The map $V \mapsto \dim_{F'} e(V \otimes_F F')$ is additive in $V \in \mathrm{Ob}(\mathrm{PMod}(R))$ and therefore induces a homomorphism

$$\mathrm{rr}_R : K_0(R) \to \mathbb{Z}.$$

This 'reduced rank' homomorphism is injective and has image $[\mathrm{End}_R(S) : F]^{\frac{1}{2}}\mathbb{Z}$ where here S is the unique simple R-module. Similarly, if $\phi \in \mathrm{End}_R(V)$, then we set $\mathrm{detred}(\phi) := \det_{F'}(\phi \otimes 1 | e(V \otimes_F F'))$. This is an element of F which is independent of the choices of both F' and e. Recalling that $K_1(R)$ is generated by pairs (V, ϕ) with $\phi \in \mathrm{Aut}_R(V)$ it is not hard to show that detred induces a homomorphism

$$\mathrm{nr}_R : K_1(R) \to F^\times$$

(cf. [15, §45A]). This 'reduced norm' homomorphism is in general neither injective nor surjective.

Proposition 2.2. *If F is either a local or a global field, then nr_R is injective. If F is a local field different from $\mathbb{R}$, then nr_R is bijective. If $F = \mathbb{R}$, then $\mathrm{im}(\mathrm{nr}_R) = (\mathbb{R}^\times)^2$ if R is a matrix algebra over the division ring of real quaternions, and $\mathrm{im}(\mathrm{nr}_R) = \mathbb{R}^\times$ otherwise. Finally, if F is a number field, then*

$$\mathrm{im}(\mathrm{nr}_R) = \{f \in F^\times : f_v > 0 \text{ for all } v \in S_A(F)\} \tag{7}$$

where $S_A(F)$ denotes the set of places v of F such that $F_v = \mathbb{R}$ and $A \otimes_{F,v} \mathbb{R}$ is a matrix algebra over the division ring of real quaternions.

Proof. See [15, (45.3)] $\square$

If now R is a general semisimple ring, then the above considerations apply to each of the Wedderburn factors of R. The center $\zeta(R)$ of R is a product of fields and we obtain maps

$$\mathrm{rr}_R : K_0(R) \to H^0(\mathrm{Spec}\,(\zeta(R)), \mathbb{Z}), \qquad \mathrm{nr}_R : K_1(R) \to \zeta(R)^\times.$$

If R is finite dimensional over either a local or a global field, then both of these maps are injective.

Lemma 2. *If R is any semisimple ring, then the maps* rr_R *and* nr_R *are both induced by a determinant functor* $(\mathrm{PMod}(R), \mathrm{is}) \to \mathcal{P}(\zeta(R))$.

Proof. We first observe that the target group of rr_R (resp. nr_R) does indeed coincide with $\pi_0(\mathcal{P}(\zeta(R)))$ (resp. $\pi_1(\mathcal{P}(\zeta(R)))$).
To construct a determinant functor it is clearly sufficient to restrict attention to each Wedderburn factor of R. Such a factor is isomorphic to $M_n(D)$, say, where D is a division ring with center F. By fixing an exact (Morita) equivalence $\mathrm{PMod}(M_n(D)) \to \mathrm{PMod}(D)$, it therefore suffices to construct a determinant functor for D. To this end we suppose that F'/F is a field extension such that $D \otimes_F F' \cong M_d(F')$, that e is an indecomposable idempotent of $M_d(F')$ and that $e_1, .., e_d$ is an ordered F'-basis of $eM_d(F')$. Any finitely generated projective D-module V is free, and for any D-basis $v_1, .., v_r$ of V the wedge product $b := \bigwedge e_i v_j$ (with the e_i in the fixed ordering) is an F'-basis of $\mathrm{Det}_{F'}(e(V \otimes_F F'))$. Since any change of basis v_i multiplies b by an element of $\mathrm{im}(\mathrm{nr}_D) \subseteq F^\times$, the F-space spanned by b yields a well defined graded F-line bundle. $\square$

This result shows that if the maps rr_R and nr_R are both injective, then one can dispense with virtual objects and instead use an explicit functor to graded line bundles over $\zeta(R)$. However, this approach no longer seems to be possible when one considers orders in non-commutative semisimple algebras, and it is in this setting that the existence of virtual objects will be most useful for us.

2.7. Orders in finite-dimensional $\mathbb{Q}$-algebras. Let A be a finite-dimensional $\mathbb{Q}$-algebra (associative and unital but not necessarily commutative) and put $A_F := A \otimes_{\mathbb{Q}} F$ for any field F of characteristic zero. For brevity we write A_p for $A_{\mathbb{Q}_p}$. Let R be a finitely generated subring of $\mathbb{Q}$. We call an R-subalgebra $\mathfrak{A}$ of A an *R-order* if $\mathfrak{A}$ is a finitely generated R-module and $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. We shall refer to a $\mathbb{Z}$-order more simply as an *order*. For any order $\mathfrak{A}$ we set $\mathfrak{A}_p := \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $\hat{\mathfrak{A}} := \mathfrak{A} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong \prod_p \mathfrak{A}_p$ and $\hat{A} := A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. The diagram of exact categories and exact (scalar extension) functors

$$\begin{array}{ccc} \mathrm{PMod}(\mathfrak{A}) & \longrightarrow & \mathrm{PMod}(A) \\ \downarrow & & \downarrow \\ \mathrm{PMod}(\hat{\mathfrak{A}}) & \longrightarrow & \mathrm{PMod}(\hat{A}) \end{array}$$

induces a corresponding diagram of Picard categories and monoidal functors. These diagrams commute up to a natural equivalence of functors. By the universal property of the fibre product category we therefore obtain a monoidal functor

$$V(\mathfrak{A}) \to V(\hat{\mathfrak{A}}) \times_{V(\hat{A})} V(A) =: \mathbb{V}(\mathfrak{A}). \tag{8}$$

We use the notation $\mathbb{V}(\mathfrak{A})$ in an attempt to stress the adelic nature of $\mathbb{V}(-)$.

Proposition 2.3. *The functor (8) induces an isomorphism on π_0 and a surjection on π_1.*

Proof. There is a map of long exact Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} & \longrightarrow & K_1(\mathfrak{A}) & \longrightarrow & K_1(\hat{\mathfrak{A}}) \oplus K_1(A) & \longrightarrow & K_1(\hat{A}) & \longrightarrow & ... \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(\mathbb{V}(\mathfrak{A})) & \longrightarrow & \pi_1(V(\hat{\mathfrak{A}})) \oplus \pi_1(V(A)) & \longrightarrow & \pi_1(V(\hat{A})) & \longrightarrow & ... \end{array}$$

$$\begin{array}{ccccccc} ... & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(\hat{\mathfrak{A}}) \oplus K_0(A) & \longrightarrow & K_0(\hat{A}) \\ & & \downarrow & & \downarrow & & \downarrow \\ ... & \longrightarrow & \pi_0(\mathbb{V}(\mathfrak{A})) & \longrightarrow & \pi_0(V(\hat{\mathfrak{A}})) \oplus \pi_0(V(A)) & \longrightarrow & \pi_0(V(\hat{A})) \end{array}$$

where the top sequence can be found in [15, (42.19)], the bottom sequence arises from Lemma 1 and the vertical maps are the isomorphisms (2) or, in the case of $K_i(\mathfrak{A})$, the isomorphisms (2) composed with the map induced by (8). The commutativity of the diagram follows from the naturality of (2) and, in the case of the boundary map, from an elementary computation using the explicit description of $K_1(-)$ (cf. [15, (38.28), (40.6)]). The statement of the proposition is then an easy consequence of the Five Lemma. □

Remark 2. The functor (8) may fail to be an equivalence of categories even if $\mathfrak{A}$ is commutative. Indeed, the map $K_1(\mathfrak{A}) \to \pi_1(\mathbb{V}(\mathfrak{A}))$ is an isomorphism if and only if the map $K_1(\mathfrak{A}) \to K_1(\hat{\mathfrak{A}}) \oplus K_1(A)$ is injective. This injectivity condition fails for $\mathfrak{A} = \mathbb{Z}[G]$ where G is any finite abelian group for which $SK_1(\mathbb{Z}[G])$ is nontrivial (see [15, Rem. after (48.8)] for examples of such groups G) because $\mathbb{Z}_p[G]$ is then a product of local rings and therefore $SK_1(\mathbb{Z}_p[G]) = 0$. However, if A is semisimple and $\mathfrak{A}$ is a maximal order, then the functor (8) is an equivalence of Picard categories. Indeed, under these hypotheses one can use the Wedderburn decompositions of A and $\mathfrak{A}$ and the Morita invariance of each functor $K_i(-)$ in order to reduce to the case in which A is a division ring. In this case the injectivity of $K_1(\mathfrak{A}) \to K_1(\hat{\mathfrak{A}}) \oplus K_1(A)$ is a consequence of [15, (45.15)].

Proposition 2.3 is crucial in what follows because it allows us to work in a Picard category $\mathbb{V}(\mathfrak{A})$ which has the 'correct' π_0 and in which objects localize in a similar manner to graded line bundles. Indeed, as the following result shows, it is quite reasonable to regard $\mathbb{V}(\mathfrak{A})$ as a generalisation of the category $\mathcal{P}(\mathfrak{A})$ to orders which need not be commutative.

Proposition 2.4. *If $\mathfrak{A}$ is a finite flat commutative $\mathbb{Z}$-algebra, then there is a natural equivalence of Picard categories $\mathcal{P}(\mathfrak{A}) \xrightarrow{\sim} \mathbb{V}(\mathfrak{A})$.*

Proof. We shall show that the natural monoidal functors in the diagram

$$\begin{array}{ccc} & & \mathbb{V}(\mathfrak{A}) = V(\hat{\mathfrak{A}}) \times_{V(\hat{A})} V(A) \\ (9) & & \Big\downarrow \hat{D}:=\mathrm{VDet}_{\hat{\mathfrak{A}}} \times \mathrm{VDet}_A \\ \mathcal{P}(\mathfrak{A}) & \longrightarrow & \mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A) \end{array}$$

are equivalences of Picard categories.

We recall that a ring R is said to be local (resp. semilocal) if $R/J(R)$ is a division ring (resp. is semisimple) where $J(R)$ denotes the Jacobson radical of R.

Lemma 3. *a) Suppose $R = \prod R_\iota$ is a (possibly infinite) product of semilocal rings R_ι. Then the natural map $K_i(R) \to \prod K_i(R_\iota)$ is injective for $i = 0$ and bijective for $i = 1$.*
b) If $R = \prod R_\iota$ with each R_ι local and commutative, then the functor VDet_R is an equivalence.

Proof. For a semilocal ring R_ι finitely generated projective modules P_ι and P'_ι are isomorphic if and only if their classes in $K_0(R_\iota)$ agree (see the discussion after [15, (40.35)]). In addition, for any finitely generated projective R-module P the natural map $P \to \prod_\iota P \otimes_R R_\iota$ is an isomorphism since both sides are additive and the map is an isomorphism for $P = R$. Hence the isomorphism class of P can be recovered from its image in $\prod K_0(R_\iota)$, and this implies a) for $i = 0$.

For any ring R we have $K_1(R) = \varinjlim_n GL_n(R)/E_n(R)$ where $E_n(R)$ is the subgroup generated by elementary matrices [15, (40.26)]. If R is semilocal, then the map $GL_n(R)/E_n(R) \to K_1(R)$ is an isomorphism for $n \geq 2$ [15, (40.31), (40.44)]. In addition, the minimal number of generators needed to express a matrix in $E_n(R_\iota)$ as a product of elementary matrices is bounded, depending on n but not on R_ι [15, (40.31)]. Hence $E_n(\prod R_\iota) = \prod E_n(R_\iota)$ and this implies that

$$K_1(R) = K_1(\prod R_\iota) \cong \varinjlim_n \prod GL_n(R_\iota)/E_n(R_\iota)$$
$$\cong \varinjlim_n \prod K_1(R_\iota) \cong \prod K_1(R_\iota),$$

i.e. statement a) for $i = 1$.
We remark now that b) follows from a) by using the decomposition $R^\times = \prod R_\iota^\times$, the fact that the functors VDet_{R_ι} are equivalences, and the fact that the image of the map

$$K_0(R) \to \prod K_0(R_\iota) = \prod H^0(\mathrm{Spec}(R_\iota), \mathbb{Z})$$

lies in (and is therefore isomorphic to) the subgroup $H^0(\mathrm{Spec}(R), \mathbb{Z})$. □

The functor $\mathrm{VDet}_{\hat{\mathfrak{A}}}$ in (9) is an equivalence by Lemma 3b) since $\hat{\mathfrak{A}}$ is a finite continuous commutative $\hat{\mathbb{Z}}$-algebra and hence a product of local rings. Similarly, VDet_A is an equivalence since A is Artinian and commutative and hence a product of local rings. The ring $\hat{A}$ is a filtered direct limit of rings $\mathfrak{A}_S := \prod_{p\in S} A_p \times \prod_{p\notin S} \mathfrak{A}_p$ for finite sets of primes S. As the ring $\mathfrak{A}_S$ is likewise a product of local rings the functor $\mathrm{VDet}_{\mathfrak{A}_S}$ is also an equivalence. It follows that in the commutative diagram

$$\begin{array}{ccc}
\varinjlim_S \pi_i(V(\mathfrak{A}_S)) & \longrightarrow & \pi_i(V(\hat{A})) \\
\downarrow & & \downarrow {\scriptstyle \pi_i(\mathrm{VDet}_{\hat{A}})} \\
\varinjlim_S \pi_i(\mathcal{P}(\mathfrak{A}_S)) & \longrightarrow & \pi_i(\mathcal{P}(\hat{A}))
\end{array}$$

the left hand vertical map is an isomorphism. Furthermore, the upper horizontal map is an isomorphism by (2) and [43, Lem. 5.9], and the lower horizontal map is an isomorphism for $i = 0$ (resp. $i = 1$) since $H^0(\mathrm{Spec}(\hat{A}), \mathbb{Z}) = \varinjlim_S H^0(\mathrm{Spec}(\mathfrak{A}_S), \mathbb{Z})$ by [23, 8.2.11] and the fact that an affine scheme is quasi-compact (resp. since $\hat{A}^\times = \varinjlim_S \mathfrak{A}_S^\times$).
We deduce that $\mathrm{VDet}_{\hat{A}}$, and as a consequence also $\hat{D}$, is an equivalence of Picard categories. It is known that the map (6) is an isomorphism for $R = \mathfrak{A}$ because $\mathfrak{A}$ is Noetherian of dimension 1 [24, Exp. VI 6.9]. Using Proposition 2.3 we find an isomorphism

$$\pi_0(\mathcal{P}(\mathfrak{A})) = \mathrm{Pic}(\mathfrak{A}) \times H^0(\mathrm{Spec}(\mathfrak{A}), \mathbb{Z}) \xleftarrow{\sim} K_0(\mathfrak{A})$$
$$\xrightarrow{\sim} \pi_0(\mathbb{V}(\mathfrak{A})) \xrightarrow{\pi_0(\hat{D})} \pi_0(\mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)).$$

This isomorphism coincides with π_0 of the lower horizontal functor in (9). Moreover, from the Mayer-Vietoris sequence for the fibre product $\mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)$ one easily deduces that

$$\pi_1(\mathcal{P}(\mathfrak{A})) = \mathfrak{A}^\times \cong \pi_1(\mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)).$$

Hence the lower horizontal functor in (9) is an equivalence, and this finishes the proof of Proposition 2.4. $\square$

This proposition makes it reasonable to think of objects of $\mathbb{V}(\mathfrak{A})$ as generalized graded line bundles. We invite the reader to do so when following the rest of this paper.

Lemma 4. *Assume that A is semisimple. Then the natural functor*

$$\mathbb{V}(\mathfrak{A}) = V(\hat{\mathfrak{A}}) \times_{V(\hat{A})} V(A) \to \prod_p V(\mathfrak{A}_p) \times_{\prod_p V(A_p)} V(A)$$

induces an injection on π_0 and an isomorphism on π_1.

Proof. By using Mayer-Vietoris sequences and the Five Lemma it suffices to show that the maps $K_1(\hat{A}) \to \prod_p K_1(A_p)$ and $K_0(\hat{\mathfrak{A}}) \to \prod_p K_0(\mathfrak{A}_p)$ are injective, and that the map $K_1(\hat{\mathfrak{A}}) \to \prod_p K_1(\mathfrak{A}_p)$ is bijective. For the latter two maps this is immediate from Lemma 3a). Since

$$K_1(\hat{A}) = \varinjlim{}_S K_1(\prod_{p\in S} A_p \times \prod_{p\notin S} \mathfrak{A}_p) \cong \prod_{p\in S} K_1(A_p) \times \prod_{p\notin S} K_1(\mathfrak{A}_p)$$

is a limit over finite sets S it therefore suffices to show that the map $K_1(\mathfrak{A}_p) \to K_1(A_p)$ is injective for any sufficiently large p. To prove this we may assume that A_p is a product of matrix algebras over finite field extensions F of $\mathbb{Q}_p$ and that $\mathfrak{A}_p$ is the corresponding product of matrix algebras over integer rings $\mathcal{O}_F$. By Morita equivalence the required result thus follows from the injectivity of the natural map $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \to K_1(F^\times) = F^\times$. $\square$

Remark 3. We believe that the assertion of Lemma 4 may well continue to hold without the assumption that A is semisimple, but we have no need for such additional generality in what follows.

2.8. The relative K_0. Let $\mathcal{P}_0$ be the Picard category with unique object $\mathbf{1}_{\mathcal{P}_0}$ and $\mathrm{Aut}_{\mathcal{P}_0}(\mathbf{1}_{\mathcal{P}_0}) = 0$. For A and $\mathfrak{A}$ as in §2.7 and an extension field F of $\mathbb{Q}$ we define $\mathbb{V}(\mathfrak{A}, F)$ to be the fibre product category in the diagram

$$\begin{tikzcd} \mathbb{V}(\mathfrak{A}, F) := \mathbb{V}(\mathfrak{A}) \times_{V(A_F)} \mathcal{P}_0 \arrow[r] \arrow[d] & \mathcal{P}_0 \arrow[d, "F_2"] \\ \mathbb{V}(\mathfrak{A}) \arrow[r, "F_1"] & V(A_F) \end{tikzcd}$$

where here F_2 is the unique monoidal functor and $F_1((L, M, \lambda)) = M \otimes_A A_F$ for each object (L, M, λ) of $\mathbb{V}(\mathfrak{A})$. We define the category $V(\mathfrak{A}_p, \mathbb{Q}_p) := V(\mathfrak{A}_p) \times_{V(A_p)} \mathcal{P}_0$ in a similar manner.

Proposition 2.5. *For any field extension F of $\mathbb{Q}$ one has an isomorphism*

$$\pi_0\mathbb{V}(\mathfrak{A}, F) \xrightarrow{\sim} K_0(\mathfrak{A}, F),$$

and for any prime p an isomorphism

$$\pi_0 V(\mathfrak{A}_p, \mathbb{Q}_p) \xrightarrow{\sim} K_0(\mathfrak{A}_p, \mathbb{Q}_p),$$

where the respective right hand sides are the relative algebraic K-groups as defined in [44, p. 215].

Proof. We recall that $K_0(\mathfrak{A}, F)$ is an abelian group with generators (X, g, Y), where X and Y are finitely generated projective $\mathfrak{A}$-modules and $g : X \otimes_{\mathbb{Z}} F \to Y \otimes_{\mathbb{Z}} F$ is an isomorphism of A_F-modules. For the defining relations we refer to [44, p.215]. By using these relations one checks that the map

$$(X, g, Y) \mapsto ([X] \boxtimes [Y]^{-1}, [g] \boxtimes \mathrm{id}([Y \otimes_{\mathbb{Z}} F]^{-1}) \in \pi_0\mathbb{V}(\mathfrak{A}, F)$$

induces a homomorphism $c : K_0(\mathfrak{A}, F) \to \pi_0\mathbb{V}(\mathfrak{A}, F)$. This homomorphism fits into a natural map of the relative K-theory exact sequence [44, Th. 15.5] to the Mayer-Vietoris sequence of the fibre product defining $\mathbb{V}(\mathfrak{A}, F)$

$$\begin{array}{ccccccccc}
K_1(\mathfrak{A}) & \longrightarrow & K_1(A_F) & \xrightarrow{\delta^1_{\mathfrak{A},F}} & K_0(\mathfrak{A}, F) & \xrightarrow{\delta^0_{\mathfrak{A},F}} & K_0(\mathfrak{A}) & \longrightarrow & K_0(A_F) \\
\downarrow & & \downarrow & & {\scriptstyle c}\downarrow & & \downarrow & & \downarrow \\
\pi_1\mathbb{V}(\mathfrak{A}) & \longrightarrow & \pi_1 V(A_F) & \longrightarrow & \pi_0\mathbb{V}(\mathfrak{A}, F) & \longrightarrow & \pi_0\mathbb{V}(\mathfrak{A}) & \longrightarrow & \pi_0 V(A_F).
\end{array}$$

The commutativity of this diagram is easy to check, given the explicit nature of all of the maps involved. For example, $\delta^0_{\mathfrak{A},F}((X, g, Y)) = [X] - [Y]$, $\delta^1_{\mathfrak{A},F}$ sends the element in $K_1(A_F)$ represented by an $n \times n$-matrix g to $(\mathfrak{A}^n, g, \mathfrak{A}^n)$ and the vertical maps are as described above. Given the commutativity of this diagram, the isomorphisms (2) combine with Proposition 2.3 and the Five Lemma to imply that c is bijective. The proof for $V(\mathfrak{A}_p, \mathbb{Q}_p)$ is entirely similar using the long exact relative K-theory sequence

$$K_1(\mathfrak{A}_p) \to K_1(A_p) \xrightarrow{\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}} K_0(\mathfrak{A}_p, \mathbb{Q}_p) \xrightarrow{\delta^0_{\mathfrak{A}_p,\mathbb{Q}_p}} K_0(\mathfrak{A}_p) \to K_0(A_p). \tag{10}$$

□

Remark 4. For any isomorphism $\lambda : X \to Y$ of A_F-modules we write λ_{Triv} for the isomorphism $[\lambda] \boxtimes \mathrm{id}([Y]^{-1}) : [X] \boxtimes [Y]^{-1} \xrightarrow{\sim} \mathbf{1}_{V(A_F)}$. For any $\mathbb{Z}$-graded module or morphism $X^\bullet$ we write X^+, resp. X^-, for the direct sum of X^i over all even, resp. odd, indices i.

If $P^\bullet$ is an object of the category $\mathrm{PMod}(\mathfrak{A})^\bullet$ of bounded complexes of objects of $\mathrm{PMod}(R)$ and ψ an A_F-equivariant isomorphism from $H^+(P^\bullet) \otimes F$ to $H^-(P^\bullet) \otimes F$, then we set

$$\begin{aligned}\langle P^\bullet, \psi\rangle :=& ([P^+] \boxtimes [P^-]^{-1}, [H^+(P^\bullet) \otimes F] \boxtimes [H^-(P^\bullet) \otimes F]^{-1}, h; \psi_{\mathrm{Triv}}) \\ \in& \mathbb{V}(\mathfrak{A}, F),\end{aligned}$$

where here h denotes the composite of the canonical isomorphism

$$[P^+ \otimes F] \boxtimes [P^- \otimes F]^{-1} \xrightarrow{\sim} [P^\bullet \otimes F],$$

the isomorphism (4) for $X = P^\bullet \otimes F$ and the canonical isomorphism

$$\boxtimes_{i\in\mathbb{Z}}[H^i(P^\bullet)\otimes F]^{(-1)^i} \xrightarrow{\sim} [H^+(P^\bullet)\otimes F] \boxtimes [H^-(P^\bullet)\otimes F]^{-1}.$$

If now Y is any object of $D^p(\mathfrak{A})$ and ψ is an A_F-equivariant isomorphism from $H^+(Y)\otimes F$ to $H^-(Y)\otimes F$, then the pair (Y, ψ^{-1}) constitutes, in the terminology of [5, §1.2], a 'trivialized perfect complex (of $\mathfrak{A}$-modules)'. Choose an $\mathfrak{A}$-equivariant quasi-isomorphism $\xi : P^\bullet \to Y$ with $P^\bullet$ an object of $\mathrm{PMod}(\mathfrak{A})^\bullet$, and write ψ_ξ for the composite isomorphism $H^-(\xi\otimes F)^{-1}\circ\psi\circ H^+(\xi\otimes F)$: $H^+(P^\bullet)\otimes F \xrightarrow{\sim} H^-(P^\bullet)\otimes F$. Then under the isomorphism $\pi_0\mathbb{V}(\mathfrak{A}, F) \xrightarrow{\sim} K_0(\mathfrak{A}, F)$ of Proposition 2.5, the image of the class of $\langle P^\bullet, \psi_\xi\rangle$ in $\pi_0(\mathbb{V}(\mathfrak{A}, F))$ is equal to the inverse of the 'refined Euler characteristic' class $\chi_{\mathfrak{A}}(Y, \psi^{-1})$ which is defined in [loc. cit., Th. 1.2.1].

In the remainder of this section we recall some useful facts concerning the groups $K_0(\mathfrak{A}, F)$.
If F is a field of characteristic 0, then one has a commutative diagram of long exact relative K-theory sequences (cf. [44, Th. 15.5])

$$\begin{array}{ccccccccc} K_1(\mathfrak{A}) & \longrightarrow & K_1(A_F) & \longrightarrow & K_0(\mathfrak{A},F) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(A_F) \\ \| & & \uparrow{\scriptstyle\beta} & & \uparrow & & \| & & \uparrow \\ K_1(\mathfrak{A}) & \longrightarrow & K_1(A) & \longrightarrow & K_0(\mathfrak{A},\mathbb{Q}) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(A). \end{array} \tag{11}$$

The scalar extension morphism β is injective and so, as a consequence of the Five Lemma, this diagram induces an inclusion

$$K_0(\mathfrak{A}, \mathbb{Q}) \subseteq K_0(\mathfrak{A}, F). \tag{12}$$

Furthermore, the map

$$(X, g, Y) \mapsto \prod_p (X_p, g_p, Y_p),$$

where $X_p := X \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $Y_p := Y \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $g_p := g \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for each prime p, induces an isomorphism

$$K_0(\mathfrak{A}, \mathbb{Q}) \xrightarrow{\sim} \bigoplus_p K_0(\mathfrak{A}_p, \mathbb{Q}_p) \tag{13}$$

where the sum is taken over all primes p (see the discussion following [15, (49.12)]).

2.9. The locally free class group. For any field F of characteristic 0 we define

$$\mathrm{Cl}(\mathfrak{A}, F) := \ker\big(K_0(\mathfrak{A}, F) \to K_0(\mathfrak{A}) \to \prod_p K_0(\mathfrak{A}_p)\big)$$

and

$$\mathrm{Cl}(\mathfrak{A}) := \ker\big(K_0(\mathfrak{A}) \to \prod_p K_0(\mathfrak{A}_p)\big).$$

The motivic invariants that we construct will belong to groups of the form $\mathrm{Cl}(\mathfrak{A}, \mathbb{R})$. The group $\mathrm{Cl}(\mathfrak{A})$ is the 'locally free class group' of $\mathfrak{A}$, as discussed in [15, §49].

We observe that the diagram (11) restricts to give a commutative diagram with exact rows

$$\begin{array}{ccccccc} K_1(A_F) & \xrightarrow{\delta^1_{\mathfrak{A},F}} & \mathrm{Cl}(\mathfrak{A}, F) & \xrightarrow{\delta^0_{\mathfrak{A},F}} & \mathrm{Cl}(\mathfrak{A}) & \longrightarrow & 0 \\ \uparrow{\scriptstyle\beta} & & \uparrow & & \| & & \\ K_1(A) & \xrightarrow{\delta^1_{\mathfrak{A},\mathbb{Q}}} & \mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) & \xrightarrow{\delta^0_{\mathfrak{A},\mathbb{Q}}} & \mathrm{Cl}(\mathfrak{A}) & \longrightarrow & 0, \end{array} \tag{14}$$

and hence that (12) restricts to give an inclusion $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) \subseteq \mathrm{Cl}(\mathfrak{A}, F)$. In addition, the restriction of the isomorphism (13) to $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$ combines with the exact sequence (10) to induce an isomorphism

$$\mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) \xrightarrow{\sim} \bigoplus_p K_1(A_p)/\mathrm{im}(K_1(\mathfrak{A}_p)). \tag{15}$$

In many cases of interest the maps

$$K_1(A_p)/\mathrm{im}(K_1(\mathfrak{A}_p)) \to K_0(\mathfrak{A}_p, \mathbb{Q}_p)$$

are bijective for all p, and hence one has

$$\mathrm{Cl}(\mathfrak{A}, F) = K_0(\mathfrak{A}, F).$$

For example, this is the case if $\mathfrak{A}$ is commutative, if $\mathfrak{A} = \mathbb{Z}[G]$ where G is any finite group [15, Rem. (49.11)(iv)] or if $\mathfrak{A}$ is a maximal order in A [loc. cit., Th. 49.32].

3. Motives

3.1. Motivic structures. We fix a number field K and denote by S_∞ the set of archimedean places of K. For each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$ we write $v(\sigma)$ for the corresponding element of S_∞. We also fix an algebraic closure $\bar{K}$ of K and let G_K denote the Galois group $\mathrm{Gal}(\bar{K}/K)$.

The category of (pure Chow) motives over K is a $\mathbb{Q}$-linear category with a functor to the category of realisations [26] and on which motivic cohomology functors are well defined. As is common in the literature on L-functions we shall treat motives in a formal sense: they are to be regarded as given by their realisations, motivic cohomology and the usual maps between these groups (that is, by a motivic structure in the sense of [20]). For example, if X is a smooth, projective variety over K, n a non-negative integer and r any integer, then $M := h^n(X)(r)$ is not in general known to exist as a Chow motive. However, the realisations of M are

$H_{dR}(M) := H^n_{dR}(X/K)$,

a filtered K-space, with its natural decreasing filtration $\left\{F^i H^n_{dR}(X/K)\right\}_{i\in\mathbb{Z}}$ shifted by r;

$H_l(M) := H^n_{et}(X \times_K \overline{K}, \mathbb{Q}_l(r))$,

a compatible system of l-adic representations of G_K;

$H_\sigma(M) := H^n(\sigma X(\mathbb{C}), (2\pi i)^r\mathbb{Q})$,

for each $\sigma \in \mathrm{Hom}(K,\mathbb{C})$ a $\mathbb{Q}$-Hodge structure over $\mathbb{R}$ or $\mathbb{C}$ according to whether $v(\sigma)$ is real or complex. If c denotes complex conjugation, then there is an obvious isomorphism of manifolds $\sigma X(\mathbb{C}) \xrightarrow{\sim} (c\circ\sigma)X(\mathbb{C})$ which we use to identify $H_\sigma(M)$ with $H_{c\circ\sigma}(M)$ if $v = v(\sigma)$ is complex. We then denote either of the two Hodge structures by $H_v(M)$, and we shall subsequently only make constructions which are independent of this choice.
One possible definition of the motivic cohomology of $M = h^n(X)(r)$ is

$$H^0(K,M) := \begin{cases} (CH^r(X)/CH^r(X)_{\mathrm{hom}\sim 0}) \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } n = 2r \\ 0, & \text{if } n \neq 2r \end{cases}$$

and

$$H^1(K,M) := \begin{cases} (K_{2r-n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)}, & \text{if } 2r-n-1 \neq 0 \\ CH^r(X)_{\mathrm{hom}\sim 0} \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } 2r-n-1 = 0. \end{cases}$$

Here $(K_{2r-n-1}(X)\otimes_{\mathbb{Z}}\mathbb{Q})^{(r)}$ is the eigenspace for the k-th Adams operator with eigenvalue k^r. One also defines a subspace

$$H^1_f(K,M) \subseteq H^1(K,M)$$

consisting of classes which are called 'finite' (or 'integral') at all non-archimedean places of K and puts $H^0_f(K,M) := H^0(K,M)$. In the K-theoretical version $H^1_f(K,M)$ is defined just as $H^1(K,M)$ but with $(K_{2r-n-1}(X)\otimes_{\mathbb{Z}}\mathbb{Q})^{(r)}$ replaced by

$$\mathrm{im}((K_{2r-n-1}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)} \to (K_{2r-n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)})$$

where $\mathcal{X}$ is a regular proper model of X over $\mathrm{Spec}\,(\mathcal{O}_K)$ (see [41] for the definition if such a model does not exist). The spaces $H^i_f(K,M)$ are expected to be finite dimensional, but this is not yet known to be true in general.
Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra. From now on we shall be interested in motives with coefficients in A, i.e. in pairs (M,ϕ) where $\phi : A \to \mathrm{End}(M)$ is a ring homomorphism. It suffices here to understand $\mathrm{End}(M)$ as endomorphisms of motivic structures. However, in all of the explicit examples considered in [11] A is in fact an algebra of correspondences, i.e. consists of endomorphisms in the category of Chow motives. If M has coefficients in A, then the dual motive M^* has coefficients in A^{op}.

3.2. Basic exact sequences. In this section we recall relevant material from [4] and [20].

We write K_v for the completion of K at a place v, and we fix an algebraic closure $\bar{K}_v$ of K_v and an embedding of $\bar{K}$ into $\bar{K}_v$. We denote by $G_v \subseteq G_K$ the corresponding decomposition group and, if v is non-archimedean, by $I_v \subset G_v$ and $f_v \in G_v/I_v$ the inertia subgroup and Frobenius automorphism respectively. For $v \in S_\infty$ and an $\mathbb{R}$-Hodge structure H over K_v (what we call) the Deligne cohomology of H is by definition the cohomology of the complex

$$R\Gamma_{\mathcal{D}}(K_v, H) := \left(H^{G_v} \xrightarrow{\alpha_v} (H \otimes_{\mathbb{R}} \bar{K}_v)^{G_v}/F^0 \right),$$

where here G_v acts diagonally on $H \otimes_{\mathbb{R}} \bar{K}_v$, and α_v is induced from the obvious inclusion $H \hookrightarrow H \otimes_{\mathbb{R}} \bar{K}_v$. Now if $H = H_v(M) \otimes_{\mathbb{Q}} \mathbb{R}$, then there is a canonical comparison isomorphism

$$H \otimes_{\mathbb{R}} \bar{K}_v \cong H_{dR}(M) \otimes_{K,v} \bar{K}_v$$

which is G_v-equivariant (the right hand side having the obvious G_v action). It follows that the complex

$$R\Gamma_{\mathcal{D}}(K, M) := \bigoplus_{v \in S_\infty} R\Gamma_{\mathcal{D}}(K_v, H_v(M) \otimes_{\mathbb{Q}} \mathbb{R})$$

can also be written as

$$\bigoplus_{v \in S_\infty} (H_v(M) \otimes_{\mathbb{Q}} \mathbb{R})^{G_v} \xrightarrow{\alpha_M} \left(\bigoplus_{v \in S_\infty} H_{dR}(M) \otimes_K K_v/F^0 \right) = \left(H_{dR}(M)/F^0\right) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{16}$$

For an $\mathbb{R}$-vector space W we write W^* for the linear dual $\mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$. If W is an A-module, then we always regard W^* as an A^{op}-module in the natural way.

Conjecture 1. *(cf.* [20]*[Prop. III.3.2.5]): There exists a long exact sequence of finite-dimensional $A_{\mathbb{R}}$-spaces*

$$\begin{aligned} 0 \longrightarrow H^0(K, M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\epsilon} \ker(\alpha_M) \xrightarrow{r_B^*} \left(H^1_f(K, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R}\right)^* \\ \xrightarrow{\delta} H^1_f(K, M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{r_B} \mathrm{coker}(\alpha_M) \xrightarrow{\epsilon^*} \left(H^0(K, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R}\right)^* \longrightarrow 0 \end{aligned} \tag{17}$$

where here ϵ is the cycle class map into singular cohomology; r_B is the Beilinson regulator map; and (if both $H^1_f(K, M)$ and $H^1_f(K, M^*(1))$ are non-zero so that M has weight -1, then) δ is a height pairing. Moreover, the $\mathbb{R}$-dual of (17) identifies with the corresponding sequence for $M^*(1)$ where the isomorphisms

$$\ker(\alpha_M)^* \cong \mathrm{coker}(\alpha_{M^*(1)}), \quad \mathrm{coker}(\alpha_M)^* \cong \ker(\alpha_{M^*(1)})$$

are constructed in Lemma 18 below.

For each prime number p we set $V_p := H_p(M)$. Following [8, (1.8)] we shall now construct for each place v a true triangle in $D^p(A_p)$

$$0 \to R\Gamma_f(K_v, V_p) \to R\Gamma(K_v, V_p) \to R\Gamma_{/f}(K_v, V_p) \to 0 \tag{18}$$

in which all of the terms will be defined as specific complexes, rather than only to within unique isomorphism in $D^p(A_p)$ as the notation would perhaps suggest (we have however chosen to keep the traditional notation for mnemonic purposes.)
For a profinite group Π and a continuous Π-module N we denote by $C^\bullet(\Pi, N)$ the standard complex of continuous cochains.
If $v \in S_\infty$, then we set

$$R\Gamma_f(K_v, V_p) := R\Gamma(K_v, V_p) := C^\bullet(G_v, V_p).$$

We also define $R\Gamma_{/f}(K_v, V_p) := 0$ and we take (18) to be the obvious true triangle (with second arrow equal to the identity map).
If $v \notin S_\infty$ and $v \nmid p$, then we set

$$\begin{aligned} R\Gamma(K_v, V_p) &:= C^\bullet(G_v, V_p), \\ R\Gamma_f(K_v, V_p) &:= C^\bullet(G_v/I_v, V_p^{I_v}) \subseteq C^\bullet(G_v, V_p). \end{aligned}$$

We define $R\Gamma_{/f}(K_v, V_p)$ to be the complex which in each degree i is equal to the quotient of $C^i(G_v, V_p)$ by $C^i(G_v/I_v, V_p^{I_v})$ (with the induced differential), and we take (18) to be the tautological true triangle. We observe that there is a canonical quasi-isomorphism

$$\begin{aligned} R\Gamma_f(K_v, V_p) &\xrightarrow{\pi} \big(V_p^{I_v} \xrightarrow{1-f_v^{-1}} V_p^{I_v}\big) \\ &=: \big(V_{p,v} \xrightarrow{\phi_v} V_{p,v}\big) \end{aligned} \tag{19}$$

where in the latter two complexes the spaces are placed in degrees 0 and 1, and π is equal to the identity map in degree 0 and is induced by evaluating a 1-cocycle at f_v^{-1} in degree 1.
If now $v \mid p$, then by [4, Prop. 1.17] one has an exact sequence of continuous G_v-modules

$$0 \to \mathbb{Q}_p \to B^0 \xrightarrow{\beta_1 - \beta_2} B^1 \to 0 \tag{20}$$

where here $B^0 := B_{cris} \times B_{dR}^+$ and $B^1 := B_{cris} \times B_{dR}$ are certain canonical algebras and $\beta_1(x, y) = (x, x)$ and $\beta_2(x, y) = (\phi(x), y)$ are algebra homomorphisms. We write $B^\bullet$ for the complex $B^0 \xrightarrow{\beta_1 - \beta_2} B^1$ where the modules are placed in degrees 0 and 1, and we set

$$\begin{aligned} R\Gamma(K_v, V_p) &:= \operatorname{Tot} C^\bullet(G_v, B^\bullet \otimes_{\mathbb{Q}_p} V_p), \\ R\Gamma_f(K_v, V_p) &:= H^0(K_v, B^\bullet \otimes_{\mathbb{Q}_p} V_p). \end{aligned}$$

We observe that, since $V_p \to B^\bullet \otimes_{\mathbb{Q}_p} V_p$ is a resolution of V_p, the natural map

$$C^\bullet(G_v, V_p) \to R\Gamma(K_v, V_p) \tag{21}$$

is a quasi-isomorphism. Also, since $R\Gamma_f(K_v, V_p)$ is a subcomplex of $R\Gamma(K_v, V_p)$ we can define $R\Gamma_{/f}(K_v, V_p)$ to be the complex which in each degree i is equal to the quotient of the i-th term of $R\Gamma(K_v, V_p)$ by the i-th term of $R\Gamma_f(K_v, V_p)$ (with the induced differential). With these definitions we take (18) to be the tautological true triangle. Further, using the notation

$$\begin{aligned} D_{cris}(V_p) &:= H^0(K_v, B_{cris} \otimes_{\mathbb{Q}_p} V_p), \\ D_{dR}(V_p) &:= H^0(K_v, B_{dR} \otimes_{\mathbb{Q}_p} V_p), \\ F^0 D_{dR}(V_p) &:= H^0(K_v, B_{dR}^+ \otimes_{\mathbb{Q}_p} V_p), \\ t_v(V_p) &:= D_{dR}(V_p)/F^0 D_{dR}(V_p) \end{aligned}$$

there is a commutative diagram of complexes

$$\begin{array}{ccc} 0 & \longrightarrow & t_v(V_p) \\ \uparrow & & \uparrow \\ F^0 D_{dR}(V_p) & \xrightarrow{\subseteq} & D_{dR}(V_p) \\ {\scriptstyle (0,-\mathrm{id})}\downarrow & & \downarrow{\scriptstyle (0,\mathrm{id})} \\ D_{cris}(V_p) \oplus F^0 D_{dR}(V_p) & \xrightarrow{d} & D_{cris}(V_p) \oplus D_{dR}(V_p) \end{array}$$

where d is induced by $(\beta_1 - \beta_2) \otimes \mathrm{id}_{V_p}$ so that the lower row is $R\Gamma_f(K_v, V_p)$, and the vertical maps are both quasi-isomorphisms. With $t_v^\bullet(V_p)$ denoting the central complex in the above diagram we obtain a canonical quasi-isomorphism

$$t_v^\bullet(V_p) \xrightarrow{\sim} t_v(V_p)[-1]$$

and a canonical true triangle

$$t_v^\bullet(V_p) \to R\Gamma_f(K_v, V_p) \to \left(V_{p,v} \xrightarrow{\phi_v} V_{p,v}\right) \tag{22}$$

where here $\left(V_{p,v} \xrightarrow{\phi_v} V_{p,v}\right) := \left(D_{cris}(V_p) \xrightarrow{1-\varphi_v} D_{cris}(V_p)\right)$ and the spaces are placed in degrees 0 and 1. In addition, from Faltings' fundamental comparison theorem between V_p and $H_{dR}(M)$ over K_v there exists a canonical A_p-equivariant isomorphism

$$(H_{dR}(M)/F^0) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \bigoplus_{v|p} t_v(V_p). \tag{23}$$

Let now V be any finitely generated (projective) A_p-module. If $\phi \in \mathrm{End}_{A_p}(V)$ and C denotes the (perfect) complex $V \xrightarrow{\phi} V$ (with the modules placed in degrees 0 and 1), then there is an isomorphism in $V(A_p)$

$$[C] = [V] \boxtimes [V]^{-1} \cong \mathbf{1}_{V(A_p)} \tag{24}$$

which corresponds to the canonical isomorphism $X \boxtimes X^{-1} \cong \mathbf{1}$ for any object X in a Picard category. Note however that if ϕ is an automorphism and C is therefore acyclic, then the isomorphism (24) differs from the isomorphism $[C] \cong \mathbf{1}_{V(A_p)}$ induced by the quasi-isomorphism $C \to 0$ (cf. also [8, Rem.

after (1.16)] in this regard). In the notation which was introduced in Remark 4 after Proposition 2.5 this latter isomorphism is denoted by ϕ_{Triv} whereas the isomorphism (24) is denoted by $\mathrm{id}_{V,\mathrm{Triv}}$.

We now fix a finite set S of places of K containing S_∞ and the places where M has bad reduction. We denote by S_p the union of S and the set of places of K above p, and we set $S_{p,f} := S_p \setminus S_\infty$. We denote by $\mathcal{O}_{K,S_p}$ the ring of S_p-integers in K and by G_{S_p} its étale fundamental group with respect to the previously chosen base point $\bar{K}$. For any continuous G_{S_p}-module N we set

$$\begin{aligned} R\Gamma(\mathcal{O}_{K,S_p}, N) &:= C^\bullet(G_{S_p}, N), \\ R\Gamma_c(\mathcal{O}_{K,S_p}, N) &:= \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{K,S_p}, N) \to \bigoplus_{v \in S_p} C^\bullet(G_v, N) \right)[-1] \end{aligned}$$

where the morphism here is induced by the natural maps $G_v \subseteq G_K \to G_{S_p}$. For $N = V_p$ we set

$$\begin{aligned} {}_1R\Gamma_c(\mathcal{O}_{K,S_p}, V_p) &:= \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{K,S_p}, V_p) \to \bigoplus_{v \in S_p} R\Gamma(K_v, V_p) \right)[-1], \\ R\Gamma_f(K, V_p) &:= \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{K,S_p}, V_p) \to \bigoplus_{v \in S_p} R\Gamma_{/f}(K_v, V_p) \right)[-1] \end{aligned}$$

where in both cases we have used the morphism (21) for each place $v \mid p$. Then there is a natural quasi-isomorphism

$$R\Gamma_c(\mathcal{O}_{K,S_p}, V_p) \xrightarrow{\sim} {}_1R\Gamma_c(\mathcal{O}_{K,S_p}, V_p)$$

and the maps

$$R\Gamma(\mathcal{O}_{K,S_p}, V_p) \to \bigoplus_{v \in S_p} R\Gamma(K_v, V_p) \leftarrow \bigoplus_{v \in S_p} R\Gamma_f(K_v, V_p) \tag{25}$$

induce a true nine term diagram

(26)

$$\begin{tikzcd} \bigoplus_{v \in S_p} R\Gamma_f(K_v, V_p)[-1] \arrow[r, equal] \arrow[d] & \bigoplus_{v \in S_p} R\Gamma_f(K_v, V_p)[-1] \arrow[d] & \\ \bigoplus_{v \in S_p} R\Gamma(K_v, V_p)[-1] \arrow[r] \arrow[d] & {}_1R\Gamma_c(\mathcal{O}_{K,S_p}, V_p) \arrow[r] \arrow[d] & R\Gamma(\mathcal{O}_{K,S_p}, V_p) \arrow[d, equal] \\ \bigoplus_{v \in S_p} R\Gamma_{/f}(K_v, V_p)[-1] \arrow[r] & R\Gamma_f(K, V_p) \arrow[r] & R\Gamma(\mathcal{O}_{K,S_p}, V_p). \end{tikzcd}$$

Note that in what follows we will systematically use the lower left numbering to distinguish between different but naturally quasi-isomorphic versions of a complex. When there is no danger of confusion we shall simply drop this numbering and leave implicit the resulting identifications.
The complex $R\Gamma_f(K, V_p)$ is acyclic outside degrees $0, 1, 2$ and 3 and in Lemma 19 below we will define a natural isomorphism in $D^p(A_p)$

$$\mathrm{AV}_f : R\Gamma_f(K, V_p) \cong R\Gamma_f(K, V_p^*(1))^*[-3].$$

Conjecturally therefore, the cohomology of $R\Gamma_f(K, V_p)$ is completely described by applying the following to both M and $M^*(1)$.

CONJECTURE 2. *For both $i \in \{0, 1\}$ there exists a canonical A_p-equivariant isomorphism*

$$c_p^i(M) : H_f^i(K, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} H^i R\Gamma_f(K, V_p). \tag{27}$$

In addition, we recall that for each $v \in S_\infty$ the comparison isomorphism between $H_v(M)$ and V_p induces an isomorphism in $D^p(A_p)$

$$R\Gamma_f(K_v, V_p) = R\Gamma(K_v, V_p) \cong V_p^{G_v}[0] \cong (H_v(M)^{G_v} \otimes_{\mathbb{Q}} \mathbb{Q}_p)[0]. \tag{28}$$

3.3. PROJECTIVE $\mathfrak{A}$-STRUCTURES. Let M be a motive which is defined over K and admits an action of the finite dimensional semisimple $\mathbb{Q}$-algebra A. If $\mathfrak{A}$ is an R-order in A (cf. §2.7) and V is an A-module, then an $\mathfrak{A}$-submodule T of V will be said to be an '$\mathfrak{A}$-lattice (in V)' if it is both finitely generated and full (i.e., satisfies $V = A \otimes_{\mathfrak{A}} T$).

DEFINITION 1. *Let $\mathfrak{A}$ be an R-order in A. An* $\mathfrak{A}$-structure *T on M is a set $\{T_v : v \in S_\infty\}$ where, for each $v \in S_\infty$, T_v is an $\mathfrak{A}$-lattice in $H_v(M)$ and for each prime $l \in \mathrm{Spec}(R)$ the image T_l of $T_v \otimes_{\mathbb{Z}} \mathbb{Z}_l$ under the comparison isomorphism $H_v(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H_l(M)$ is both independent of v and G_K-stable. An $\mathfrak{A}$-structure T on M is* projective*, resp.* free*, if each T_v is a projective, resp. free, $\mathfrak{A}$-module.*

If M is a motive with A-action, then there always exist $\mathfrak{A}$-structures on M. For example, if $M = h^n(X)$ for a smooth projective variety X defined over K, then there is an $\mathfrak{A}$-structure $h^n(X, \mathfrak{A})$ on M such that, for each $v \in S_\infty$, $h^n(X, \mathfrak{A})_v$ is the $\mathfrak{A}$-lattice in $H_v(M)$ which is generated by the image of $H^n(\sigma X(\mathbb{C}), \mathbb{Z})$ for an embedding $\sigma : K \to \mathbb{C}$ which corresponds to v. However, there need not exist projective $\mathfrak{A}$-structures on M. Indeed, even if there are full projective $\mathfrak{A}$-modules T_v in each space $H_v(M)$, it can occur that none of the corresponding modules T_l is G_K-stable (this is the case if, for example, $M = h^1(E)$ for an elliptic curve E defined over an imaginary quadratic field A for which $\mathrm{End}_A(E)$ is the maximal order in A and $\mathfrak{A}$ is any proper suborder of $\mathrm{End}_A(E)$). Nevertheless, the following examples show that a projective $\mathfrak{A}$-structure on M naturally exists in a variety of interesting cases.

Examples.

a) If $\mathfrak{A}$ is an hereditary R-order, and hence *a fortiori* if it is a maximal R-order, then there always exists a projective $\mathfrak{A}$-structure on M (cf. [15, Th. (26.12)]).
b) ('The Galois case') Let L/K be a finite Galois extension, and set $G := \mathrm{Gal}(L/K)$. If M_K is any motive which is defined over K, then the motive $M := h^0(\mathrm{Spec}(L)) \otimes M_K$ has a natural action of the semisimple algebra $\mathbb{Q}[G]$ via the first factor. Furthermore, if T_K is any $\mathbb{Z}$-structure on M_K, then $H^0(\mathrm{Spec}(L \otimes_{K,\sigma} \mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} T_K$ is a free $\mathbb{Z}[G]$-structure on M. Recall here that for any embedding $\sigma : K \to \mathbb{C}$ the scheme $\mathrm{Spec}(L \otimes_{K,\sigma} \mathbb{C})$ naturally identifies with the G-set $\Sigma := \{\tau \in \mathrm{Hom}(L, \mathbb{C}) : \tau|_K = \sigma\}$ and hence that $H^0(\mathrm{Spec}(L \otimes_{K,\sigma} \mathbb{C}), \mathbb{Z}) = \mathrm{Maps}(\Sigma, \mathbb{Z})$ is a free $\mathbb{Z}[G]$-module of rank one.
c) (Cf. [42, §4, Rem. following Cor. 2]). If X is a simple abelian variety defined over K which admits complex multiplication over K by a CM-field A, then the motive $h^1(X)$ has a natural A-action. In addition, if the order $\mathfrak{A} = \mathrm{End}_K(X) \subseteq A$, consisting of those elements which preserve each lattice $H^1(\sigma X(\mathbb{C}), \mathbb{Z})$, is Gorenstein then each $h^1(X, \mathfrak{A})_v$ is a projective $\mathfrak{A}$-lattice by [15, (37.13)] and hence there exists a projective $\mathfrak{A}$-structure on M. We note in particular that if X is an elliptic curve, then $\mathfrak{A}$ is automatically Gorenstein as a consequence of [1, 6.3].
d) Continuing the previous example, we assume now that X is an elliptic curve defined over K and so that $\mathfrak{A} := \mathrm{End}_K(X)$ is an order in an imaginary quadratic field A. Then A is contained in K and one can consider the $[K : A]$-dimensional abelian variety Y over A which is defined as the Weil restriction of X from K to A. If moreover K/A is an abelian Galois extension and X is isogenous to all of its Galois conjugates over A then $T := \mathrm{End}_A(Y) \otimes \mathbb{Q}$ is an algebra of dimension $[K : A]$ over A and $\mathfrak{T} := \mathrm{End}_A(Y)$ is an order in T, nonmaximal at primes dividing $[K : A][\mathcal{O}_A : \mathfrak{A}]$. This is shown in [21][15.1.6] for maximal $\mathfrak{A}$ but the arguments there extend to general $\mathfrak{A}$. The description of $\mathrm{End}_A(Y)$ in [21][15.1.5] also shows that $H^1(Y(\mathbb{C}), \mathbb{Z})$ is a projective $\mathfrak{T}$-module, and hence that the motive $M = h^1(Y)$ over the base field A admits a projective $\mathfrak{T}$-structure.
e) Let N be a prime number and consider the modular curve $X = X_0(N)$ defined over $K = \mathbb{Q}$. The *Hecke algebra* A is a finite dimensional commutative semisimple $\mathbb{Q}$-algebra consisting of correspondences which act on X, and $H^1(X(\mathbb{C}), \mathbb{Q})$ is known to be a free rank two A-space. If $\mathfrak{A}$ denotes the integral Hecke algebra (i.e., the subring of A which is generated by the Hecke correspondences over $\mathbb{Z}$), then $\mathfrak{A}$ is an order in A and Mazur shows in [34, II, (14.2), (16.3), (15.1)] that $H^1(X(\mathbb{C}), \mathbb{Z})_{\mathfrak{m}}$ is a free

module over $\mathfrak{A}_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of $\mathfrak{A}$ which do not contain 2. This implies that $H^1(X(\mathbb{C}),\mathbb{Z}[\frac{1}{2}])$ is a projective module over $\mathfrak{A}[\frac{1}{2}]$, and hence that there exists a projective $\mathfrak{A}[\frac{1}{2}]$-structure on M.

We shall henceforth assume that $\mathfrak{A}$ is an order (leaving to the reader the obvious modifications which are necessary for R-orders as discussed above) and that we are given a projective $\mathfrak{A}$-structure T on M. We assume that p is a prime number which satisfies neither of the following conditions:

(P1) The motive $\mathrm{Res}_{\mathbb{Q}}^{K} M$ has bad reduction at p.
(P2) $p - 2 < \min\{i \geq 0 | F^{i+1}H_{dR}(M) = 0\} - \max\{i \leq 0 | F^i H_{dR}(M) = H_{dR}(M)\}$.

Then, as explained in [8, 1.5.1], one can use the theory of Fontaine-Laffaille to define complexes $R\Gamma_f(K_v, T_p)$ for each place v in exactly the same way as for V_p. The same arguments which lead to diagram (26) can then be used to derive an analogous diagram in $D(\mathfrak{A}_p)$ in which V_p is replaced by T_p, and which naturally identifies with (26) after tensoring with $\mathbb{Q}_p$.
For any finite set of primes $\mathcal{S}$ we write $\mathfrak{A}_{\mathcal{S}}$ for the localisation of $\mathfrak{A}$ at the multiplicative set generated by the primes in $\mathcal{S}$. For $i \in \{2,3\}$ we set $H_f^i(K,M) := H_f^{3-i}(K, M^*(1))^*$ and $c_p^i(M) := H^i(\mathrm{AV}_f)^{-1} \circ (c_p^{3-i}(M^*(1))^*)^{-1}$. We now introduce an additional hypothesis on the pair (M, A).

Coherence hypothesis: For each $i \in \{0,1,2,3\}$ there exists a finitely generated $\mathfrak{A}$-module $H_f^i(K,M;T)$ and an $\mathfrak{A}$-equivariant map $\tau^i : H_f^i(K,M;T) \to H_f^i(K,M)$ such that $\tau^i \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. In addition, there exists a finite set $\mathcal{S}$ of primes containing all primes satisfying either (P1) or (P2) and such that for each $p \notin \mathcal{S}$ and $i \in \{0,1,2,3\}$ there is a commutative diagram of $\mathfrak{A}_p$-modules

$$\begin{array}{ccc} H_f^i(K,M;T) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \xrightarrow{\tau^i \otimes_{\mathbb{Z}} \mathbb{Q}_p} & H_f^i(K,M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \\ {\scriptstyle c_p^i(T)}\downarrow & & \downarrow{\scriptstyle c_p^i(M)} \\ H^i R\Gamma_f(K, T_p) & \xrightarrow{\otimes_{\mathbb{Z}_p} \mathbb{Q}_p} & H^i R\Gamma_f(K, V_p) \end{array}$$

in which $c_p^i(T)$ is an isomorphism.

Remark 5. This hypothesis is identical to an assumption on integral structures in motivic cohomology which is made in [8, §1.5], and is independent of the choices of both $\mathfrak{A}$ and T. As we shall see below, the hypothesis is not actually required in order to formulate conjectures on special values of motivic L-functions and is correspondingly not made in either of [4] or [20]. However, under the Coherence hypothesis one can define an invariant in $\mathrm{Cl}(\mathfrak{A})$ without reference to the L-function of M, and this ties in well with the approach of classical Galois module theory. Indeed, in concrete cases, the $\mathfrak{A}$-module structure of the groups $H_f^i(K,M;T)$ can be of considerable interest (see for example

[8, §1.6]), and this structure can be studied via the conjectures formulated in §4 below.

3.4. Virtual objects attached to motives. Let M be a motive which is defined over K and admits an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. In this section we fix an order $\mathfrak{A}$ in A and a projective $\mathfrak{A}$-structure T on M (always assuming that such a structure exists).
We shall henceforth use the following notational convention. When referring to the individual triangles in a true nine term diagram with equation number (n) we denote by $(n)_?$ with ? equal to 'top', 'bot', 'left', 'rght', 'hor' or 'vert' the top, bottom, left, right, central horizontal and central vertical triangle respectively.
We define an object $\Xi(M)$ of $V(A)$ by setting

$$\begin{aligned}\Xi(M) :=&[H^0_f(K,M)]\boxtimes [H^1_f(K,M)]^{-1}\boxtimes [H^1_f(K,M^*(1))^*]\\ &\boxtimes[H^0_f(K,M^*(1))^*]^{-1}\boxtimes \boxtimes_{v\in S_\infty}[H_v(M)^{G_v}]^{-1}\boxtimes [H_{dR}(M)/F^0].\end{aligned}\tag{29}$$

Note that this is the inverse of the space Ξ used in both [8] and [27] (because our choice of normalisation for the virtual object associated to a perfect complex is the inverse of that of [8, (0.2)]).
Applying the functor $[\,]$ to the isomorphisms (27), (28), (23), the isomorphisms (24), (19) or the triangle (22) for all $v\in S_{p,f}$ and finally to the triangle $(26)_{\mathrm{vert}}$, we obtain for each prime p an isomorphism in $V(A_p)$

$$\vartheta_p(M,S): A_p\otimes_A \Xi(M)\xrightarrow{\sim}[R\Gamma_c(\mathcal{O}_{K,S_p},V_p)]\cong A_p\otimes_{\mathfrak{A}_p}[R\Gamma_c(\mathcal{O}_{K,S_p},T_p)]$$

which we shall also abbreviate as $\vartheta_p(M)$ or even ϑ_p if there is no danger of confusion. We note here that $R\Gamma_c(\mathcal{O}_{K,S_p},T_p)$ is a perfect complex of $\mathfrak{A}_p$-modules by [18, Th. 5.1], and hence we obtain an object

$$\Xi(M,T_p,S) := ([R\Gamma_c(\mathcal{O}_{K,S_p},T_p)],\Xi(M),\vartheta_p)$$

of $V(\mathfrak{A}_p)\times_{V(A_p)}V(A)$.

Lemma 5. *For another choice T' of projective $\mathfrak{A}$-structure on M and another choice of the finite set of places S' the objects $\Xi(M,T_p,S)$ and $\Xi(M,T'_p,S')$ are isomorphic in $V(\mathfrak{A}_p)\times_{V(A_p)}V(A)$.*

Proof. By embedding S and S' into the union $S\cup S'$ we can assume that $S\subseteq S'$, and by induction we can then reduce to the case that $S'=S\cup\{w\}$ and $w\nmid p$. For any continuous G_{S_p}-module N one has a commutative diagram of complexes

$$\begin{array}{ccc} C^\bullet(G_{S_p},N) & \longrightarrow & \bigoplus_{v\in S_p} C^\bullet(G_v,N) \\ \downarrow & & \downarrow \\ C^\bullet(G_{S'_p},N) & \xrightarrow{r} & C^\bullet(G_w,N)/C^\bullet(G_w/I_w,N)\oplus \bigoplus_{v\in S_p} C^\bullet(G_v,N) \\ \| & & \uparrow \\ C^\bullet(G_{S'_p},N) & \longrightarrow & \bigoplus_{v\in S'_p} C^\bullet(G_v,N) \end{array}$$

which induces a quasi-isomorphism $R\Gamma_c(\mathcal{O}_{K,S_p},N)\xrightarrow{\sim}\mathrm{Cone}(r)[-1]$ and a true triangle

$$R\Gamma_f(K_w,N)[-1]\to R\Gamma_c(\mathcal{O}_{K,S'_p},N)\to \mathrm{Cone}(r)[-1] \tag{30}$$

where $R\Gamma_f(K_w,N)=C^\bullet(G_w/I_w,N)$ is naturally quasi-isomorphic to

$$N\xrightarrow{1-f_v^{-1}} N \tag{31}$$

(compare [36, Chap. II, Prop. 2.3d]). For $N=T_p$ the true triangle (30) lies in $D(\mathfrak{A}_p)$. In conjunction with isomorphisms of the form (24), it therefore induces an isomorphism $\iota:[R\Gamma_c(\mathcal{O}_{K,S_p},T_p)]\xrightarrow{\sim}[R\Gamma_c(\mathcal{O}_{K,S'_p},T_p)]$ in $V(\mathfrak{A}_p)$.

We have a natural map from diagram (25) to the diagram

(32)

$$R\Gamma(\mathcal{O}_{K,S'_p},V_p)\to \bigoplus_{v\in S_p} R\Gamma(K_v,V_p)\oplus R\Gamma_{/f}(K_w,V_p)\leftarrow \bigoplus_{v\in S_p} R\Gamma_f(K_v,V_p).$$

We now denote by $\mathrm{T9}(S)$, resp. T9, the diagram (26), resp. the true nine term diagram which is induced by (32). Then we obtain a map $\phi:\mathrm{T9}(S)\to\mathrm{T9}$ which restricts to give quasi-isomorphisms on all terms in the central column. In a similar way, there is a map $\psi:\mathrm{T9}(S')\to\mathrm{T9}$ which is moreover a termwise surjection. The kernel of ψ is naturally quasi-isomorphic to a sum of complexes (31) and hence naturally trivialized by isomorphisms of the form (24). Since the same trivializations are used in the construction of $\vartheta_p(M,S')$, we have $(A_p\otimes_{\mathfrak{A}_p}\iota)\circ\vartheta_p(M,S)=\vartheta_p(M,S')$. Hence the pair (ι,id) defines an isomorphism

$$(\iota,\mathrm{id}):([R\Gamma_c(\mathcal{O}_{K,S_p},T_p)],\Xi(M),\vartheta_p)\xrightarrow{\sim}([R\Gamma_c(\mathcal{O}_{K,S'_p},T_p)],\Xi(M),\vartheta'_p)$$

in the category $V(\mathfrak{A}_p)\times_{V(A_p)}V(A)$.

Replacing T_p by $p^nT_p\subseteq T_p\cap T'_p$ we can assume that $T_p\subseteq T'_p$. Then there is a true triangle of perfect complexes of $\mathfrak{A}_p$-modules

$$R\Gamma_c(\mathcal{O}_{K,S_p},T_p)\to R\Gamma_c(\mathcal{O}_{K,S_p},T'_p)\to R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p). \tag{33}$$

Since T'_p/T_p is finite $R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is acyclic and hence there is a canonical isomorphism $\tau_{\mathbb{Q}}:[R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p]\cong \mathbf{1}_{V(A_p)}$. By [18, Th. 5.1] the class of $(R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p),\mathbf{1}_{V(\mathfrak{A}_p)},\tau_{\mathbb{Q}})$ in $\pi_0(V(\mathfrak{A}_p,\mathbb{Q}_p))\cong$

$K_0(\mathfrak{A}_p, \mathbb{Q}_p)$ is 0. Upon unraveling the definition of $V(\mathfrak{A}_p, \mathbb{Q}_p)$ this means that $\tau_{\mathbb{Q}}$ is induced by an isomorphism $\tau : [R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p/T_p)] \cong \mathbf{1}_{V(\mathfrak{A}_p)}$. Hence the isomorphism induced by the triangle (33)

$$\begin{aligned}\iota : [R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p)] \cong &[R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)] \boxtimes [R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p/T_p)]\\ &\xrightarrow{\mathrm{id}\boxtimes\tau} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)]\end{aligned}$$

is part of an isomorphism

$$(\iota, \mathrm{id}) : ([R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p)], \Xi(M), \vartheta_p) \xrightarrow{\sim} ([R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \Xi(M), \vartheta_p)$$

in the category $V(\mathfrak{A}_p) \times_{V(A_p)} V(A)$. □

By taking the product over all primes p we now obtain an object

$$\Xi(M, T, S)_{\mathbb{Z}} := (\prod_p [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \Xi(M), \prod_p \vartheta_p)$$

of the fibre product category $\prod_p V(\mathfrak{A}_p) \times_{\prod_p V(A_p)} V(A)$.

Lemma 6. *If the Coherence hypothesis is satisfied, then* $\Xi(M, T, S)_{\mathbb{Z}}$ *is isomorphic to the image of an object of* $\mathbb{V}(\mathfrak{A})$ *under the functor of Lemma 4.*

Proof. Assume that $\mathcal{S}$ is a finite set of primes as in the Coherence hypothesis and also containing all primes p at which $\mathfrak{A}_p$ is not a maximal $\mathbb{Z}_p$-order in A_p. Then $\mathfrak{A}_{\mathcal{S}}$ is a (left) regular ring [15, Th. (26.12)], and so any finitely generated $\mathfrak{A}_{\mathcal{S}}$-module is of finite projective dimension. As in [8, (1.24)] there exists a full $\mathfrak{A}_{\mathcal{S}}$-sublattice D_{dR} of $H_{dR}(M)$ so that for $p \notin \mathcal{S}$ the isomorphism (23) is induced by an isomorphism

$$(D_{dR}/F^0) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \bigoplus_{v|p} D_{cr,v}(T_p)/F^0 \tag{34}$$

where here $D_{cr,v}(-)$ is an integral version of the functor $D_{cris}(-)$ for K_v [8, p. 82]. We define an object $\Xi_{\mathcal{S}}$ of $V(\mathfrak{A}_{\mathcal{S}})$ by setting

$$\begin{aligned}\Xi_{\mathcal{S}} := \Xi_{\mathcal{S}}(M, T, S) := &[H^0_f(K, M; T)_{\mathcal{S}}] \boxtimes [H^1_f(K, M; T)_{\mathcal{S}}]^{-1} \boxtimes [H^2_f(K, M; T)_{\mathcal{S}}]\\ &\boxtimes [H^3_f(K, M; T)_{\mathcal{S}}]^{-1} \boxtimes \boxtimes_{v\in S_\infty} [(T_v)^{G_v}_{\mathcal{S}}]^{-1} \boxtimes [D_{dR}/F^0].\end{aligned}$$

We set $\hat{\mathfrak{A}}' := \prod_{p\notin\mathcal{S}} \mathfrak{A}_p$. Then the finite product decomposition $\hat{\mathfrak{A}} \cong \prod_{p\in\mathcal{S}} \mathfrak{A}_p \times \hat{\mathfrak{A}}'$ induces a decomposition $V(\hat{\mathfrak{A}}) \cong \prod_{p\in\mathcal{S}} V(\mathfrak{A}_p) \times V(\hat{\mathfrak{A}}')$, and via this we define an object

$$\Xi' := ((\prod_{p\in\mathcal{S}} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \hat{\mathfrak{A}}' \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}), A \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}, \prod_{p\in\mathcal{S}} \vartheta_p \times \mathrm{id}_{\Xi_{\mathcal{S}}})$$

of $\mathbb{V}(\mathfrak{A})$. Under the Coherence hypothesis, there exists a natural isomorphism $A \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}(M, T, S) \xrightarrow{\sim} \Xi(M)$. The image of Ξ' under the functor of Lemma 4 is isomorphic to $\Xi(M, T, S)_{\mathbb{Z}}$ because for each $p \notin \mathcal{S}$ the isomorphism ϑ_p is induced by an isomorphism

$$\vartheta^T_p : \mathfrak{A}_p \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}(M, T, S) \xrightarrow{\sim} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)]$$

in $V(\mathfrak{A}_p)$ (see [8] for more details). This finishes the proof of Lemma 6. $\square$

Lemma 6, Lemma 5 and Lemma 4 now combine to imply that

$$\Xi(M)_{\mathbb{Z}} := \Xi(M,T,S)_{\mathbb{Z}}$$

is an object of $\mathbb{V}(\mathfrak{A})$ which is independent to within isomorphism in $\mathbb{V}(\mathfrak{A})$ of the choices of both S and T. The conjectural exact sequence (17) combines with (16) to induce an isomorphism in $V(A_{\mathbb{R}})$

$$\vartheta_\infty : A_{\mathbb{R}} \otimes_A \Xi(M) \cong \mathbf{1}_{V(A_{\mathbb{R}})}.$$

Under the Coherence hypothesis, we therefore obtain an object

$$(\Xi(M)_{\mathbb{Z}}, \vartheta_\infty) := (\prod_p [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \Xi(M), \prod_p \vartheta_p; \vartheta_\infty)$$

of $\mathbb{V}(\mathfrak{A}, \mathbb{R})$. We let $R\Omega(M,\mathfrak{A})$ denote the class of this element in $\pi_0(\mathbb{V}(\mathfrak{A},\mathbb{R})) \cong K_0(\mathfrak{A},\mathbb{R})$.

Lemma 7. $R\Omega(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{R})$.

Proof. We need to show that the class of $R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)$ in $K_0(\mathfrak{A}_p)$ vanishes, and this follows as an easy consequence of results in [18]. More precisely, if Γ denotes the image of G_S in $\mathrm{Aut}(T_p)$ and $\mathbb{Z}_p[[\Gamma]]$ the profinite group algebra of Γ, then [18, Prop. 5.1] shows that there exists a bounded complex $P_\bullet$ of finitely generated projective $\mathbb{Z}_p[[\Gamma]]$-modules and an isomorphism $R\Gamma_c(\mathcal{O}_{K,S_p}, N) \cong \mathrm{Hom}_{\mathbb{Z}_p[[\Gamma]]}(P_\bullet, N)$ in $D(\mathfrak{A}_p)$ for any continuous, profinite or discrete, $\mathfrak{A}_p[\Gamma]$-module N. If Σ is a set of representatives for the isomorphism classes of simple $\mathbb{Z}_p[[\Gamma]]$-modules and $P_I \to I$ a projective hull for each $I \in \Sigma$, then we have isomorphisms of $\mathbb{Z}_p[[\Gamma]]$-modules $P_i \cong \prod_{I\in\Sigma} P_I^{n_{I,i}}$ for some integers $n_{I,i}$ (note that Σ is finite since Γ contains a pro-p group of finite index). The $\mathfrak{A}_p$-module $N_I := \mathrm{Hom}_{\mathbb{Z}_p[[\Gamma]]}(P_I, N)$ is a direct summand of N, and hence is projective if N is projective.
We write $\mathrm{cl}_{\mathfrak{A}_p}(X)$, resp. $\mathrm{cl}_{\mathbb{Z}_p}(Y)$, for the class in $K_0(\mathfrak{A}_p)$ of any perfect complex of $\mathfrak{A}_p$-modules, resp. for the class in the Grothendieck group $K_0(\mathbb{Z}_p, \mathbb{Q}_p) \cong \mathbb{Z}$ of the category of finite $\mathbb{Z}_p$-modules of any bounded complex of finite $\mathbb{Z}_p$-modules Y. Then if either $\Lambda = \mathfrak{A}_p$, or if $\Lambda = \mathbb{Z}_p$ and N is finite, there is an identity

$$\mathrm{cl}_\Lambda(R\Gamma_c(\mathcal{O}_{K,S_p}, N)) = \sum_{I\in\Sigma}\left(\sum_{i\in\mathbb{Z}}(-1)^i n_{I,i}\right)\mathrm{cl}_\Lambda(N_I). \tag{35}$$

Assume now that $N \in \Sigma$. Then $N_I = 0$ for each $I \in \Sigma$ with $I \neq N$, and hence (35) with $\Lambda = \mathbb{Z}_p$ implies that

$$\sum_{i\in\mathbb{Z}}(-1)^i n_{N,i} \cdot \mathrm{cl}_{\mathbb{Z}_p}(N) = \mathrm{cl}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{K,S_p}, N))$$
$$= \sum_{i\in\mathbb{Z}}(-1)^i \,\mathrm{cl}_{\mathbb{Z}_p}(H_c^i(\mathcal{O}_{K,S_p}, N)) = 0,$$

where the last equality follows from Tate's formula for the Euler characteristic of a finite G_S-module. Since $\mathrm{cl}_{\mathbb{Z}_p}(N) \neq 0$, it follows that $\sum_{i\in\mathbb{Z}}(-1)^i n_{I,i} = 0$ for all $I \in \Sigma$.
From (35) with $\Lambda = \mathfrak{A}_p$ and $N = T_p$ we now deduce that $\mathrm{cl}_{\mathfrak{A}_p}(R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)) = 0$, as required. $\square$

3.5. Functorialities. Let $\rho : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism between orders $\mathfrak{A}$ and $\mathfrak{B}$ in finite-dimensional, semisimple $\mathbb{Q}$-algebras A and B respectively. We denote by $\rho_{\mathbb{Q}} : A \to B$ the induced homomorphism of algebras. For any field F of characteristic 0, the scalar extension functor $\mathfrak{B} \otimes_{\mathfrak{A}} -$ induces a natural homomorphism

$$\rho_* : K_0(\mathfrak{A}, F) \to K_0(\mathfrak{B}, F)$$

which sends the class of (X, g, Y) to that of $(\mathfrak{B} \otimes_{\mathfrak{A}} X, 1 \otimes g, \mathfrak{B} \otimes_{\mathfrak{A}} Y)$. If $\mathfrak{B}$ is a projective $\mathfrak{A}$-module via ρ, then there also exists a homomorphism in the reverse direction

$$\rho^* : K_0(\mathfrak{B}, F) \to K_0(\mathfrak{A}, F)$$

which is simply induced by restriction of scalars. If $\mathfrak{A}$ is commutative and $\mathfrak{B} = M_n(\mathfrak{A})$ is a matrix algebra over $\mathfrak{A}$, then we set

$$e := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{pmatrix} \in \mathfrak{B}. \tag{36}$$

In this case the exact functor $V \mapsto \mathrm{Im}(e) \subset V$ induces an equivalence of exact categories $\mu : \mathrm{PMod}(\mathfrak{B}) \to \mathrm{PMod}(\mathfrak{A})$ and hence also an isomorphism

$$\mu_* : K_0(\mathfrak{B}, F) \xrightarrow{\sim} K_0(\mathfrak{A}, F). \tag{37}$$

If M is a motive over K with A-action, then we define $B \otimes_A M$ to be the motive over K with B-action which occurs as the largest direct factor of $B \otimes_{\mathbb{Q}} M$ upon which the left action of A on M and the right action of A on B coincide. Here $B \otimes_{\mathbb{Q}} M$ is the direct sum of $[B : \mathbb{Q}]$ copies of M (see [16, 2.1]). With this definition one has

$$H(B \otimes_A M) \cong B \otimes_A H(M) \tag{38}$$

for $H(-)$ equal to any of the functors $H_v(-)$,$H_v(-)^{G_v}$, $H_{dR}(-)$, $F^n H_{dR}(-)$, $H_l(-)$, $H_l(-)^{I_v}$, $H^0_f(K, -)$ or $H^1_f(K, -)$. If now T is a projective $\mathfrak{A}$-structure in M (as defined in §3.3), then each $\mathfrak{B} \otimes_{\mathfrak{A}} T_v$ is a projective $\mathfrak{B}$-module and hence a lattice in $B \otimes_A H_v(M) \cong H_v(B \otimes_A M)$. It follows that if M admits a projective $\mathfrak{A}$-structure, then $B \otimes_A M$ admits a projective $\mathfrak{B}$-structure.
If M is a motive over K with B-action, then it can be regarded as a motive with A-action via $\rho_{\mathbb{Q}}$. Assuming that $\mathfrak{B}$ is a projective $\mathfrak{A}$-module via ρ, any projective $\mathfrak{B}$-lattice T_v in $H_v(M)$ is also a projective $\mathfrak{A}$-lattice (via ρ). Hence, if in this case M admits a projective $\mathfrak{B}$-structure, then it also admits a projective $\mathfrak{A}$-structure.

Suppose now that A is commutative, that $B = M_n(A)$ for a natural number n and that M is a motive over K with B-action. Since the category of motives is pseudo-abelian (i.e., contains images of idempotents) eM is a motive with A-action. Also, if T_v is a projective $M_n(\mathfrak{A})$-lattice in $H_v(M)$, then eT_v is a projective $\mathfrak{A}$-lattice in $eH_v(M) = H_v(eM)$. Hence, if M admits a projective $M_n(\mathfrak{A})$-structure, then eM admits a projective $\mathfrak{A}$-structure.

Theorem 3.1. *a) If M admits a projective $\mathfrak{A}$-structure, then $B \otimes_A M$ admits a projective $\mathfrak{B}$-structure and*

$$\rho_*(R\Omega(M, \mathfrak{A})) = R\Omega(B \otimes_A M, \mathfrak{B}).$$

b) If M admits a projective $\mathfrak{B}$-structure and $\mathfrak{B}$ is a projective $\mathfrak{A}$-module via ρ, then M admits a projective $\mathfrak{A}$-structure (via $\rho_{\mathbb{Q}}$) and

$$\rho^*(R\Omega(M, \mathfrak{B})) = R\Omega(M, \mathfrak{A}).$$

c) If $\mathfrak{A}$ is commutative and M admits a projective $M_n(\mathfrak{A})$-structure, then eM admits a projective $\mathfrak{A}$-structure and

$$\mu_*(R\Omega(M, M_n(\mathfrak{A}))) = R\Omega(eM, \mathfrak{A}).$$

Proof. In case a) the exact functor $B \otimes_A - : \mathrm{PMod}(A) \to \mathrm{PMod}(B)$ induces a monoidal functor $B \otimes_A - : V(A) \to V(B)$ and hence a natural isomorphism $[B \otimes_A -] \cong B \otimes_A [-]$. Together with (38) this yields an isomorphism of virtual B-modules

$$B \otimes_A \Xi(M) \cong \Xi(B \otimes_A M). \tag{39}$$

The map ϑ_p is induced by the A-equivariant isomorphisms and exact sequences (28), (27), (19), (22), (23), (24) for all $v \in S_{p,f}$ and $(26)_{\mathrm{vert}}$, all of which transform into the corresponding isomorphisms and exact sequences for $B \otimes_A M$ when tensored over A with B: this follows from the canonical isomorphisms (38) together with 'projection formula' isomorphisms of the type

$$B_p \otimes_{A_p} R\Gamma_?(X, H_p(M)) \cong R\Gamma_?(X, B_p \otimes_{A_p} H_p(M)) \tag{40}$$

for $(?, X)$ equal to any of the pairs $(c, \mathcal{O}_{K,S_p}), (f, K)$ or $(\acute{e}t, K_v)$. Hence, if $\vartheta_p^{B \otimes_A M}$ denotes the isomorphism ϑ_p for the B-equivariant motive $B \otimes_A M$, then one has a commutative diagram

$$\begin{tikzcd}
B \otimes_A \Xi(M) \arrow[r, "1 \otimes \vartheta_p"] \arrow[d, "(39)"'] & B_p \otimes_{A_p} R\Gamma_c(\mathcal{O}_{K,S_p}, H_p(M)) \arrow[d, "(40)"'] \\
\Xi(B \otimes_A M) \arrow[r, "\vartheta_p^{B \otimes_A M}"] & R\Gamma_c(\mathcal{O}_{K,S_p}, B_p \otimes_{A_p} H_p(M)).
\end{tikzcd}$$

Moreover, the isomorphism (40) for the pair $(?, X) = (c, \mathcal{O}_{K,S_p})$ is induced by an isomorphism

$$\mathfrak{B}_p \otimes_{\mathfrak{A}_p} R\Gamma_c(\mathcal{O}_{K,S_p}, T_p) \cong R\Gamma_c(\mathcal{O}_{K,S_p}, \mathfrak{B}_p \otimes_{\mathfrak{A}_p} T_p)$$

where T_p is a projective $\mathfrak{A}_p$-lattice in $H_p(M)$ [18, Prop. 4.2]. We deduce that there exists an isomorphism in $\mathbb{V}(\mathfrak{B})$

$$\Xi(B\otimes_A M)_{\mathbb{Z}} \cong \mathfrak{B}\otimes_{\mathfrak{A}} \Xi(M)_{\mathbb{Z}}.$$

The map ϑ_∞ is induced by the A-equivariant exact sequence (17) which, as a consequence of (38), is transformed into the corresponding B-equivariant exact sequence for $B\otimes_A M$ when one applies $B\otimes_A -$. So the map ϑ_∞ for $B\otimes_A M$, which we denote by $\vartheta_\infty^{B\otimes_A M}$, is equal to the composite

$$\Xi(B\otimes_A M)\otimes_{\mathbb{Q}}\mathbb{R}\cong B_{\mathbb{R}}\otimes_{A_{\mathbb{R}}}(\Xi(M)\otimes_{\mathbb{Q}}\mathbb{R})\xrightarrow{1\otimes\vartheta_\infty} B_{\mathbb{R}}\otimes_{A_{\mathbb{R}}}\mathbf{1}_{V(A_{\mathbb{R}})}\cong \mathbf{1}_{V(B_{\mathbb{R}})}.$$

Hence one has

$$\begin{aligned}\rho_*(R\Omega(M,\mathfrak{A})) = \rho_*((\Xi(M)_{\mathbb{Z}},\vartheta_\infty)) &= (\mathfrak{B}\otimes_{\mathfrak{A}}\Xi(M)_{\mathbb{Z}}, 1\otimes\vartheta_\infty)\\ &= (\Xi(B\otimes_A M)_{\mathbb{Z}},\vartheta_\infty^{B\otimes_A M}) = R\Omega(B\otimes_A M,\mathfrak{B}).\end{aligned}$$

This proves a).

We now simply observe that the proofs of b) and c) follow along exactly the same lines with the role of the functor $B\otimes_A -$ being played by the exact functor $\mathrm{Res}^B_A : \mathrm{PMod}(B)\to\mathrm{PMod}(A)$ which is restriction of scalars in b) and restriction of scalars and passage to the direct summand cut out by the idempotent e in c). The analogues of the isomorphisms (38) and (40) for the functor Res^B_A are in both of these cases obvious. □

4. L-functions

4.1. Equivariant L-factors and ϵ-factors. Let A be a finite-dimensional semisimple $\mathbb{Q}$-algebra and W a pseudo-abelian, $\mathbb{C}$-linear category. We define W_A to be the category of A-modules in W. Thus the objects of W_A are pairs $(V, A\to\mathrm{End}_W(V))$, and morphisms in W_A are morphisms in W which commute with the A-actions.

We fix a maximal set $i(A)$ of non-conjugate indecomposable idempotents in $A_{\mathbb{C}}$. More concretely, if

$$A_{\mathbb{C}}\cong\prod_{i=1}^r M_{n_i}(\mathbb{C}),$$

then we can take $i(A)=\{e_1,\cdots,e_r\}$ where e_i is the matrix (36) of size n_i in the i-th factor, and 0 in all others factors. The functors $V\mapsto(\mathrm{im}(e_i))_{1\le i\le r}$ and $(V_i)_{1\le i\le r}\mapsto\prod_{i=1}^r V_i\otimes_{\mathbb{C}}\mathbb{C}^{n_i}$ set up an equivalence of pseudo-abelian categories

$$W_A\cong\prod_{i=1}^r W. \tag{41}$$

If C is a set and $\epsilon:\mathrm{Ob}(W)\to C$ is any map which is constant on isomorphism classes, then we get a well defined induced map

$$\epsilon:\mathrm{Ob}(W_A)\to C^{i(A)}\qquad V\mapsto\epsilon(\mathrm{im}(e_i)) \tag{42}$$

which does not depend on the choice of $i(A)$ and is constant on isomorphism classes.
We now suppose given a motive M over K with an action of A. For any non-archimedean, resp. archimedean, place v of K we let W_v be the category of complex representations of the Weil-Deligne group, resp. of the Weil group, of K_v [45]. In order to apply the preceding considerations to W_v we need the following

Conjecture 3. *(Compatibility) For any finite place v of K, any rational prime $l \nmid v$ and any embedding $\tau : \mathbb{Q}_l \to \mathbb{C}$ consider the object $H_l(M) \otimes_{\mathbb{Q}_l,\tau} \mathbb{C}$ of $W_{v,A}$. Then the isomorphism class of the Frobenius semisimplification* [45, (4.1.3)] *of $H_l(M) \otimes_{\mathbb{Q}_l,\tau} \mathbb{C}$ in $W_{v,A}$ is independent of the choices of l and τ.*

Remark 6. If A is a number field, then this reduces to the compatibility conjecture formulated in [45, (4.2.4)].

Let $\mathcal{K}(\mathbb{C})$ be the multiplicative group of meromorphic functions on $\mathbb{C}$. As in [45] one attaches to any $V \in \mathrm{Ob}(W_v)$ an L-factor $L_v(V,s) \in \mathcal{K}(\mathbb{C})$ and an ϵ-factor $\epsilon_v(V,s,\psi_v,dx_v) \in \mathcal{K}(\mathbb{C})$ (also depending upon a choice of Haar measure dx_v on K_v and of an additive character $\psi_v : K_v \to \mathbb{C}^\times$). Assuming Conjecture 3 we use (42) to associate to the pair (M,A) equivariant L-factors $L_v({}_AM,s)$ and equivariant ϵ-factors $\epsilon_v({}_AM,s,\psi_v,dx_v)$ in $\mathcal{K}(\mathbb{C})^{i(A)}$, and we view these as meromorphic functions with values in

$$\mathbb{C}^{i(A)} \cong \zeta(A_{\mathbb{C}}) \cong \zeta(A) \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\sigma \in \mathrm{Hom}(\zeta(A),\mathbb{C})} \mathbb{C}. \tag{43}$$

We then define

$$\epsilon({}_AM,s) := \prod_v \epsilon_v({}_AM,s,\psi_v,dx_v)$$
$$\Lambda({}_AM,s) := \prod_v L_v({}_AM,s)$$

where the products are taken over all places v of K and ψ_v, dx_v are chosen as in [45, (3.5)]. We also set

$$L_\infty({}_AM,s) := \prod_{v \in S_\infty} L_v({}_AM,s)$$

and for any finite set S of places of K

$$L_S({}_AM,s) := \prod_{v \notin S} L_v({}_AM,s).$$

We usually abbreviate $L_{S_\infty}({}_AM,s)$ to $L({}_AM,s)$. We observe that the product for $L({}_AM,s)$ converges in a half plane $\mathrm{Re}(s) >> 0$ and that in the product for $\epsilon({}_AM,s)$ almost all of the terms are equal to 1. If there is no danger of confusion we shall often suppress the dependence on A and so write $L(M,s)$ etc.

Remark 7. Following [16, Rem. 2.12] one can define the L-factors $L_v({}_AM, s)$ in a more direct way than the above, and this allows one to assume a slightly weaker compatibility than that of Conjecture 3. To be more precise, for each finite place v of K of residue characteristic p and any prime $l \neq p$, one considers the A_l-module $V_l := H_l(M)^{I_v}$ together with its action of the Frobenius automorphism $f_v \in \mathrm{End}_{A_l}(V_l)$. Under the assumption that

$$P_v(H_l({}_AM), X) := \mathrm{detred}_{A_l}\big(1 - f_v^{-1} \cdot X|V_l\big) \in \zeta(A_l)[X].$$

belongs to $\zeta(A)[X]$ and is independent of the choice of l, one can define $L_v({}_AM, s)$ to be equal to $P_v(H_l({}_AM), \mathrm{N}\, v^{-s}) \in \zeta(A_{\mathbb{C}})$ for each $s \in \mathbb{C}$. We observe that the above assumption on $P_v(H_l({}_AM), X)$ is a consequence of Conjecture 3, and conversely that it implies Conjecture 3 if $H_l(M)$ is unramified at v.

Lemma 8. *If s is real, then $L_v({}_AM, s)$, $\epsilon({}_AM, s)$ and $L_S({}_AM, s)$ all belong to $\zeta(A) \otimes_{\mathbb{Q}} \mathbb{R} \cong \zeta(A_{\mathbb{R}})$.*

Proof. For any $\alpha \in \zeta(A_{\mathbb{C}})$ we denote by α_σ its σ-component under the isomorphism (43). If c denotes complex conjugation the isomorphism (43) identifies $\zeta(A) \otimes_{\mathbb{Q}} \mathbb{R} = \zeta(A_{\mathbb{R}}) \subset \zeta(A_{\mathbb{C}})$ with the set $\{(\alpha_\sigma) | \alpha_{c\circ\sigma} = c(\alpha_\sigma)\}$. If v is non-archimedean and s is real, then $L_v({}_AM, s)$ belongs to this set because $P_v(H_l({}_AM), X)$ has coefficients in $\zeta(A)$. Since the action of c is continuous, the same is therefore true for $L_S({}_AM, s)$. If v is archimedean and s is real, then $L_v({}_AM, s)_\sigma \in \mathbb{R}$ since the Γ-function is real valued for real arguments. On the other hand, if $V \in \mathrm{Ob}(W_v)$ arises from a $\mathbb{R}$-Hodge structure then there is an isomorphism $V^c \cong V$ in W_v given by complex conjugation of the coefficients. Finally for any v, $V \in \mathrm{Ob}(W_v)$ and $s \in \mathbb{R}$ one has $\epsilon(V^c, s, \psi_v^c, dx_v) = \epsilon(V, s, \psi_v, dx_v)^c$ by [45, 3.6]. If $V = V_\sigma$ arises from M and $\sigma \in \mathrm{Hom}(\zeta(A), \mathbb{C})$ we have $V_\sigma^c = V_{c\circ\sigma}$ and

$$\epsilon({}_AM, s)_\sigma^c := \prod_v \epsilon(V_\sigma, s, \psi_v, dx_v)^c = \prod_v \epsilon(V_\sigma^c, s, \psi_v^c, dx_v) = \epsilon({}_AM, s)_{c\circ\sigma}$$

where this last equality follows because ψ_v^c and dx_v also satisfy the conditions of [45, (3.5)]. □

4.2. The extended boundary homomorphism. Recall that the reduced norm homomorphism $\mathrm{nr}_{A_{\mathbb{R}}} : K_1(A_{\mathbb{R}}) \to \zeta(A_{\mathbb{R}})^\times$ is injective but not in general surjective (cf. Proposition 2.2). In this section we define a canonical homomorphism $\zeta(A_{\mathbb{R}})^\times \to \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$ which upon restriction to $\mathrm{im}(\mathrm{nr}_{A_{\mathbb{R}}})$ is equal to the composite $\delta^1_{\mathfrak{A},\mathbb{R}} \circ \mathrm{nr}_{A_{\mathbb{R}}}^{-1}$, where here $\delta^1_{\mathfrak{A},\mathbb{R}}$ is the homomorphism $K_1(A_{\mathbb{R}}) \to \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$ which occurs in diagram (14). This construction plays a key role in the formulation of conjectures in the next section.

Lemma 9. *There exists a canonical homomorphism*

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}} : \zeta(A_{\mathbb{R}})^\times \to \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$$

which satisfies $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}_{A_\mathbb{R}}(x)) = \delta^1_{\mathfrak{A},\mathbb{R}}(x)$ *for each* $x \in K_1(A_\mathbb{R})$.

Proof. In conjunction with the equality (7), the Weak Approximation Theorem guarantees that for each $y \in \zeta(A_\mathbb{R})^\times$ there exists an element λ of $\zeta(A)^\times$ such that $\lambda y \in \mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$. For each prime p we also view λ as an element of $\zeta(A_p)^\times = \mathrm{im}(\mathrm{nr}_{A_p})$, and we then set

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y) := \delta^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}^{-1}_{A_\mathbb{R}}(\lambda y)) - \sum_p \delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{A_p}(\lambda)) \in \mathrm{Cl}(\mathfrak{A},\mathbb{R}). \tag{44}$$

Here we view $K_1(A_p)/\mathrm{im}(K_1(\mathfrak{A}_p))$ as a subgroup of $\mathrm{Cl}(\mathfrak{A},\mathbb{R})$ via the isomorphism (15) and the inclusion (12). The sum is taken over all primes p but is finite since for almost all p both $\lambda \in \zeta(\mathfrak{A}_p)^\times$ and $\mathrm{nr}^{-1}_{A_p}(\zeta(\mathfrak{A}_p)^\times)$ is contained in the image of $K_1(\mathfrak{A}_p)$. If λ' is any other element of $\zeta(A)^\times$ such that $\lambda' y \in \mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$, then (7) implies that $\lambda/\lambda' \in \mathrm{im}(\mathrm{nr}_A)$. Hence, if $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)'$ is the element (44) formed with respect to λ' rather than λ, then

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y) - \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)' = \delta^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}^{-1}_{A_\mathbb{R}}(\lambda/\lambda')) - \sum_p \delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{A_p}(\lambda/\lambda'))$$

and this difference is zero since both terms on the right hand side are equal to $\delta^1_{\mathfrak{A},\mathbb{Q}}(\mathrm{nr}^{-1}_A(\lambda/\lambda'))$. It now only remains to check that the assignment $y \mapsto \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)$ is a homomorphism, and this is easy to verify directly. □

4.3. The main conjectures. We can now formulate the central conjecture of this paper. This conjecture is a generalisation to non-commutative coefficients of [8, Conj. 4] (which in turn generalized the central conjectures of [4, 20, 27]).

Conjecture 4. *Let M be a motive which carries an action of the finite-dimensional semisimple $\mathbb{Q}$-algebra A, and let $\mathfrak{A}$ be any order in A for which M admits a projective $\mathfrak{A}$-structure. Assume that (M, A) satisfies the Coherence hypothesis.*

(i) *$L({}_AM, s)$ can be analytically continued to $s = 0$.*

(ii) *Regarding $\mathrm{ord}_{s=0}L({}_AM, s)$ as a locally constant function on $\mathrm{Spec}(\zeta(A_\mathbb{C}))$ one has*

$$\mathrm{ord}_{s=0}L({}_AM, s) = \mathrm{rr}_A(H^1_f(K, M^*(1))^*) - \mathrm{rr}_A(H^0_f(K, M^*(1))^*)$$

where the map rr_A is as defined in §2.6.

(iii) *(Rationality) Set*

$$L^*({}_AM, 0) := \lim_{s\to 0} s^{-\mathrm{ord}_{s=0}L({}_AM,s)} L({}_AM, s) \in \zeta(A_\mathbb{R})^\times,$$
$$L(M, \mathfrak{A}) := \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(L^*({}_AM, 0)) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$$

and

$$T\Omega(M, \mathfrak{A}) := L(M, \mathfrak{A}) + R\Omega(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

Then $T\Omega(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$.

(iv) *(Integrality)* $T\Omega(M,\mathfrak{A}) = 0$.

Remark 8. It is possible to formulate an equivalent conjecture without assuming the Coherence hypothesis (which was required to define $R\Omega(M,\mathfrak{A})$). To do this, we note that the pair $(\Xi(M), \vartheta_\infty)$ represents an object of $V(A,\mathbb{R}) := V(A) \times_{V(A_\mathbb{R})} \mathcal{P}_0$, and we consider the Mayer-Vietoris sequence for this fibre product

$$\cdots \to K_1(A_\mathbb{R}) \xrightarrow{\delta} \pi_0(V(A,\mathbb{R})) \to K_0(A) \to K_0(A_\mathbb{R}) \to \cdots$$

Just as in the proof of Lemma 9, we let $\lambda \in \zeta(A)^\times$ be any element such that $\lambda L^*({}_AM, 0)$ belongs to $\mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$. Then Conjecture 4(iii) is equivalent to

Conjecture 5. *In* $\pi_0(V(A,\mathbb{R}))$ *one has* $[\Xi(M), \vartheta_\infty] + \delta(\mathrm{nr}_{A_\mathbb{R}}^{-1}(\lambda L^*({}_AM,0))) = 0$.

It is clear that this conjecture does not involve $\Xi(M)_\mathbb{Z}$. Further, as a consequence of the definition of δ and the definition of isomorphism in the category $V(A,\mathbb{R})$, Conjecture 5 implies the existence of an isomorphism in $V(A)$

$$\vartheta^{(\lambda)} : \Xi(M) \cong \mathbf{1}_{V(A)}$$

which maps to $-\,\mathrm{nr}_{A_\mathbb{R}}^{-1}(\lambda L^*({}_AM,0)) \circ \vartheta_\infty$ in $V(A_\mathbb{R})$. Since the map $K_1(A) \cong \pi_1(V(A)) \to \pi_1(V(A_\mathbb{R})) \cong K_1(A_\mathbb{R})$ is injective, the isomorphism $\vartheta^{(\lambda)}$ is unique. One can therefore define an object

$$\xi(M,\mathfrak{A}_p,\lambda) := ([R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], (\vartheta^{(\lambda)} \otimes \mathbb{Q}_p) \circ \vartheta_p^{-1})$$

of $V(\mathfrak{A}_p, \mathbb{Q}_p)$ and formulate the following

Conjecture 6. *Assuming Conjecture 5, the class*

$$T\Omega(M,\mathfrak{A}_p) := [\xi(M,\mathfrak{A}_p,\lambda)] - \delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}_{A_p}^{-1}(\lambda))$$

vanishes in $\pi_0(V(\mathfrak{A}_p,\mathbb{Q}_p)) \cong K_0(\mathfrak{A}_p,\mathbb{Q}_p)$.

Under the Coherence hypothesis (and Conjecture 5) one can show that $T\Omega(M,\mathfrak{A}_p)$ is equal to the p-component of the element $T\Omega(M,\mathfrak{A})$ of $K_0(\mathfrak{A},\mathbb{Q})$ under the decomposition (13). This implies that, under the Coherence hypothesis, Conjecture 6 is valid for all but finitely many primes p, and that its validity for all p is equivalent to the validity of Conjecture 4(iv).

Remark 9. In this remark we assume that $\mathfrak{A}$ is commutative. Then Proposition 2.4 implies that the Picard category $\mathbb{V}(\mathfrak{A})$ is equivalent to the category $\mathcal{P}(\mathfrak{A})$ of graded invertible $\mathfrak{A}$-modules. Hence one can work with the graded determinant functor and the category $\mathcal{P}(\mathfrak{A})$ to formulate conjectures which are equivalent to those of Conjecture 4. This is the approach taken in [8], and also in [20] and [28], except that in each of these references ordinary rather than graded determinants are used.

We recall that, for any commutative ring R, an isomorphism in $\mathcal{P}(R)$ of the form

$$\mathrm{Det}_R(\bigoplus_{i\in I} P_i) \cong \bigotimes_{i\in I} \mathrm{Det}_R(P_i) \tag{45}$$

is well defined because one can define an isomorphism for any given ordering of I and the isomorphisms so obtained are compatible with reordering I in the same way on both sides. (This is a consequence of standard coherence theorems for symmetric monoidal categories [32]). However, if one ignores rank data, then (45) depends upon an ordering of I. As a consequence, for example, unless an ordering of the set S_p is specified the definition of the isomorphism ϑ_p in [8, §1.4] is ambiguous to within multiplication by an element η of $A_p^\times$ which corresponds to a locally constant map $\mathrm{Spec}(A_p) \to \{\pm 1\}$ (see also the remarks in [20, 0.4] or [28, Rem. 3.2.3(3) and 3.2.6] to this effect). It is clear that such ambiguity cannot be permitted in the formulation of Conjecture 4(iv) because in general $\eta \notin \mathfrak{A}_p^\times$. By working in $\mathcal{P}(A)$ the definition of ϑ_p in [8] becomes unambiguous and the same is true for all of the other determinant computations in loc. cit. All computations involving the determinant functor in both loc. cit. and [9], and also in the work [20, 28] of other authors, should therefore be understood to take place in categories of the form $\mathcal{P}(R)$.

Remark 10. We quickly review some of the current evidence for Conjecture 4. At the outset, we remark that any proven case of the central conjectures of [4, 20] provides evidence for Conjecture 4 for pairs of the form $(M, \mathbb{Z})$ (in this regard see also Remark 11 in §4.5). Moreover, in [11] it is shown that Conjecture 4 implies the central conjecture of Kato's paper [27] (in all cases to which the latter applies), and that in the context of Tate motives Conjecture 4(iv) refines a number of previously formulated (and much studied) conjectures. For example, if L/K is a finite Galois extension of number fields, then it is shown in [11] that Conjecture 4(iv) for $M = h^0(\mathrm{Spec}(L))$ and with $\mathfrak{A}$ equal to $\mathbb{Z}[\mathrm{Gal}(L/K)]$, resp. equal to any maximal order in $\mathbb{Q}[\mathrm{Gal}(L/K)]$ which contains $\mathbb{Z}[\mathrm{Gal}(L/K)]$, is a refinement of the main conjecture formulated by Chinburg in [13], resp. is equivalent to the so called 'Strong Stark Conjecture' (that is, [loc. cit., Conj. 2.2]). In this direction, the reader can also consult [6].

We now fix a Galois extension L of $\mathbb{Q}$ and set $G := \mathrm{Gal}(L/\mathbb{Q})$. The main result of [25] is equivalent to the validity of Conjecture 4(iv) for pairs $(h^0(\mathrm{Spec}\,(L))(r), \mathfrak{M}_{(2)})$ where here G is abelian, $\mathfrak{M}_{(2)}$ denotes the maximal $\mathbb{Z}[\frac{1}{2}]$-order in $\mathbb{Q}[G]$ and r is any integer. In addition, the main result of [12] implies that Conjecture 4(iv) is valid for all pairs $(h^0(\mathrm{Spec}\,(L))(r), \mathbb{Z}[\frac{1}{2}][G])$ with G abelian and r any integer less than 1 (in this regard see also Remark 19 in §5.3). Relaxing the condition that G is abelian, it is also known that Conjecture 4(iv) is valid for the pairs $(h^0(\mathrm{Spec}\,(L)), \mathbb{Z}[G])$ where L ranges over a natural (infinite) family of fields for which G is isomorphic to the quaternion group of order 8 [11].

The above examples can all be regarded as providing evidence for Conjecture 4 in the setting of Example b) in §3.3 ('The Galois case'). The equivariant Birch-Swinnerton Dyer conjecture for elliptic curves with CM by the maximal order $\mathcal{O}$ of the CM-field, as formulated by Gross in [22], is perhaps the earliest integral equivariant special value conjecture in a setting other than the Galois case. Moreover, the relative algebraic K-group $K_0(\mathcal{O},\mathbb{R})$ is introduced in [22] in an ad-hoc manner in order to formulate the conjecture (which can indeed be shown to be equivalent to Conjecture 4(iv) in all relevant cases). Some instances of Gross' conjecture have been proved by Rubin [40]. However, at present we are unaware of any examples in which Conjecture 4(iv) has been verified in a non-Galois case and with $\mathfrak{A}$ non-maximal.

4.4. Functorialities. In this section we shall discuss the behaviour of the element $L(M,\mathfrak{A})$, and hence (given Theorem 3.1) also of $T\Omega(M,\mathfrak{A})$, under the functorialities discussed in §3.5.

We let $\rho : \mathfrak{A} \to \mathfrak{B}$ be as in §3.5, and we use the notation ρ_* for any of the maps induced by the exact functor $\mathfrak{B} \otimes_{\mathfrak{A}} - : \mathrm{PMod}(\mathfrak{A}) \to \mathrm{PMod}(\mathfrak{B})$ or its scalar extensions on algebraic K-groups. These maps ρ_* combine to give a map of the localization sequence (11) into the corresponding sequence with $\mathfrak{A}$ replaced by $\mathfrak{B}$. The same holds for the maps ρ^* (resp. μ_*) induced by the functor $\mathrm{res}^{\mathfrak{B}}_{\mathfrak{A}} : \mathrm{PMod}(\mathfrak{B}) \to \mathrm{PMod}(\mathfrak{A})$ if $\mathfrak{B}$ is a projective $\mathfrak{A}$-module (resp. by the functor $\mu : \mathrm{PMod}(M_n(\mathfrak{A})) \to \mathrm{PMod}(\mathfrak{A})$ if $\mathfrak{A}$ is commutative).

Our first result describes the functorial properties of the extended boundary homomorphism.

Lemma 10. *There exists a homomorphism $\rho_* : \zeta(A_{\mathbb{R}})^\times \to \zeta(B_{\mathbb{R}})^\times$ which fits into a commutative diagram*

$$
\begin{array}{ccccc}
K_1(A_{\mathbb{R}}) & \xrightarrow{\mathrm{nr}_{A_{\mathbb{R}}}} & \zeta(A_{\mathbb{R}})^\times & \xrightarrow{\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}} & \mathrm{Cl}(\mathfrak{A},\mathbb{R}) \\
{\scriptstyle\rho_*}\big\downarrow & & {\scriptstyle\rho_*}\big\downarrow & & {\scriptstyle\rho_*}\big\downarrow \\
K_1(B_{\mathbb{R}}) & \xrightarrow{\mathrm{nr}_{B_{\mathbb{R}}}} & \zeta(B_{\mathbb{R}})^\times & \xrightarrow{\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}} & \mathrm{Cl}(\mathfrak{B},\mathbb{R}).
\end{array}
\tag{46}
$$

The analogous statements also hold for both ρ^* and μ_*.

Proof. For any field F of characteristic 0 the ring homomorphism $\rho_F : A_F \to B_F$ induces an exact functor $B_F \otimes_{A_F} - : \mathrm{PMod}(A_F) \to \mathrm{PMod}(B_F)$ and hence also a group homomorphism $\rho_{F,*} : K_1(A_F) \to K_1(B_F)$. Although the reduced norm map nr_{A_F} is not in general bijective it identifies $\zeta(A_F)^\times$ with the sheafification of the presheaf $F \mapsto K_1(A_F)$ for the étale topology on $\mathrm{Spec}(F)$. If $\bar{F}$ is an algebraic closure of F and $\Gamma = \mathrm{Gal}(\bar{F}/F)$, then we have

$$H^0(\Gamma, K_1(A_{\bar{F}})) \cong H^0(\Gamma, \zeta(A_{\bar{F}})^\times) \cong H^0(\Gamma, (\zeta(A_F) \otimes_F \bar{F})^\times) \cong \zeta(A_F)^\times,$$

and the map $\rho_{F,*}$ can be defined on $\zeta(A_F)^\times$ via this formula. By construction then, the left hand square in (46) commutes (even with $\mathbb{R}$ replaced by any field of characteristic 0) and we also have a commutative diagram

$$\begin{array}{ccc} \zeta(A_F)^\times & \xrightarrow{\rho_{F,*}} & \zeta(B_F)^\times \\ \subseteq\downarrow & & \downarrow\subseteq \\ \zeta(A_E)^\times & \xrightarrow{\rho_{E,*}} & \zeta(B_E)^\times \end{array} \tag{47}$$

for any fields $E \supseteq F \supseteq \mathbb{Q}$. The localisation sequences (11) for A and B form a commutative diagram with the maps induced by ρ. With notation as in the proof of Lemma 9, the commutativity of the left hand square in (46) therefore implies that

$$\begin{aligned} \rho_*(\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)) &= \rho_*(\delta^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}^{-1}_{A_\mathbb{R}}(\lambda y))) - \sum_p \rho_*(\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{A_p}(\lambda))) \\ &= \delta^1_{\mathfrak{B},\mathbb{R}}(\mathrm{nr}^{-1}_{B_\mathbb{R}}(\rho_{\mathbb{R},*}(\lambda y))) - \sum_p \delta^1_{\mathfrak{B}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{B_p}(\rho_{\mathbb{Q}_p,*}(\lambda))). \end{aligned}$$

From the commutativity of (47) with $E/F = \mathbb{Q}_p/\mathbb{Q}$ and $E/F = \mathbb{R}/\mathbb{Q}$ it is clear that one can use the element $\rho_{\mathbb{Q},*}(\lambda)$ of $\zeta(B)^\times$ to compute $\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}(\rho_{\mathbb{R},*}(y))$. It follows that the above displayed formula is equal to $\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}(\rho_{\mathbb{R},*}(y))$, and hence that the right hand square in (46) commutes.
The arguments for ρ^* and μ_* are entirely similar. $\square$

Theorem 4.1. *All assertions of Theorem 3.1 remain valid with $R\Omega(-,-)$ replaced by either $L(-,-)$ or $T\Omega(-,-)$.*

Proof. We have a commutative diagram of exact functors

$$\begin{array}{ccc} \mathrm{PMod}(A_\mathbb{C}) & \xrightarrow{\kappa_A} & \prod_{i(A)} \mathrm{PMod}(\mathbb{C}) \\ {\scriptstyle B\otimes_A -}\downarrow & & {\scriptstyle\eta}\downarrow \\ \mathrm{PMod}(B_\mathbb{C}) & \xrightarrow{\kappa_B} & \prod_{i(B)} \mathrm{PMod}(\mathbb{C}) \end{array} \tag{48}$$

where κ_A and κ_B are given by (41), and $\eta := \kappa_B \circ (B \otimes_A -) \circ \kappa_A^{-1}$ is essentially given by an $|i(A)| \times |i(B)|$−matrix $N = (n_{i,j})$ with non-negative integer entries. Indeed, one checks easily that η sends $(V_i)_{i\in i(A)} \in \mathrm{Ob}(\prod_{i(A)} \mathrm{PMod}(\mathbb{C}))$ to $(\oplus_i V_i^{n_{i,j}})_{j\in i(B)}$. There exists a similar diagram involving the same matrix N for any pseudo-abelian $\mathbb{C}$-linear category W in place of $\mathrm{PMod}(\mathbb{C})$. For an abelian group C and any map ϵ defined as in (42) which is *additive* for direct sums in W, we therefore obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Ob}(W_A) & \xrightarrow{\epsilon} & C^{i(A)} \\ {\scriptstyle B\otimes_A -}\downarrow & & {\scriptstyle N}\downarrow \\ \mathrm{Ob}(W_B) & \xrightarrow{\epsilon} & C^{i(B)}. \end{array}$$

This observation applies to $L_v(M,s)$ and so by taking into account (38) it follows that $N(L_v({}_AM,s)) = L_v({}_B(B\otimes_A M),s) \in \mathbb{C}^{i(B)}$. Now since

$\mathbb{C}^{i(A)} \xrightarrow{N} \mathbb{C}^{i(B)}$ is an analytic map it commutes with the Euler product and hence $N(L({}_AM,s)) = L({}_B(B\otimes_A M),s) \in \mathbb{C}^{i(B)}$. By analytic continuation it follows that $N(L^*({}_AM,0)) = L^*({}_B(B\otimes_A M),0) \in \zeta(B_{\mathbb{R}})^\times$. We now recall that the map ρ_* defined on $\zeta(A_{\mathbb{C}})^\times \cong (\mathbb{C}^\times)^{i(A)}$ in Lemma 10 is compatible with the induced map on $K_1(A_{\mathbb{C}})$. The diagram of exact categories (48) then shows that ρ_* is given by the matrix N after making the canonical identification $K_1(\mathbb{C}) \cong \mathbb{C}^\times$. Lemma 10 then implies that $\rho_*(L(M,\mathfrak{A})) = L(B\otimes_A M,\mathfrak{B})$, i.e. the precise analogue of Theorem 3.1a).

The analogues of b) and c) in Theorem 3.1 follow by exactly the same argument using the maps ρ^* and μ_*. □

4.5. Consequences of functoriality. In terms of the notation of Theorem 3.1, Theorem 4.1 implies that if Conjecture 4(iv) is valid for the pair $(M,\mathfrak{A})$, then it is also valid for the pair $(B\otimes_A M,\mathfrak{B})$. In addition, if ρ_* is injective, then the converse is also true. Analogous statements also hold for ρ^*. It is therefore of some interest to know when the maps ρ_* and ρ^* are injective. The next result investigates $\ker(\rho_*)$ in the case that ρ is injective.

Lemma 11. *Let $\iota : \mathfrak{A} \to \mathfrak{B}$ denote the inclusion map between orders in finite dimensional semisimple $\mathbb{Q}$-algebras $A \subseteq B$. Assume that $\zeta(B)\cap A = \zeta(A)$.*

a) *The natural map $\iota_* : \mathrm{Cl}(\mathfrak{A},\mathbb{R}) \to \mathrm{Cl}(\mathfrak{B},\mathbb{R})$ has finite kernel contained in $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$. Moreover, $\iota_*^{-1}(\mathrm{Cl}(\mathfrak{B},\mathbb{Q})) = \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.*
b) *If either $\mathfrak{A}$ is a maximal order, or B is commutative and $\mathfrak{B}\cap A = \mathfrak{A}$, then ι_* is injective.*
c) *The group $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ is torsion-free if and only if for each prime p the image of the natural map $K_1(\mathfrak{A}_p) \to K_1(A_p) \cong \zeta(A_p)^\times$ is equal to the group of units of the maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$. This condition holds if $\mathfrak{A}$ is a maximal order in A.*
d) *If $\mathfrak{B}$ is a maximal order, then $\ker(\iota_*)$ is the torsion subgroup of $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.*

Proof. For any finite dimensional semisimple $\mathbb{Q}$-algebra C and field F of characteristic 0 we set $\zeta(C_F)^{\times+} := \mathrm{im}(\mathrm{nr}_{C_F}) \subset \zeta(C_F)^\times$. The map nr_{C_F} induces an isomorphism $K_1(C_F) \cong \zeta(C_F)^{\times+}$, and in what follows we regard this as an identification. We will often use the fact that since $\zeta(B)\cap A = \zeta(A)$ the natural map $K_1(A_F) \to K_1(B_F)$ corresponds under the above identifications to the inclusion $\zeta(A_F)^{\times+} \subseteq \zeta(B_F)^{\times+}$ [15, (45.3)]. We also use the fact that Proposition 2.2 implies an explicit description of $\zeta(C_F)^{\times+}$ in terms of positivity conditions at each quaternion component of C.

We first prove that $\iota_*^{-1}(\mathrm{Cl}(\mathfrak{B},\mathbb{Q})) = \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$. We thus suppose that x is any element of $\mathrm{Cl}(\mathfrak{A},\mathbb{R})$ for which $\iota_*(x) \in \mathrm{Cl}(\mathfrak{B},\mathbb{Q})$. The fact that diagram (14) is exact implies that, after possibly adding to x an element of $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$, we can assume that there exists an element $\tilde{x}$ of $K_1(A_{\mathbb{R}}) = \zeta(A_{\mathbb{R}})^{\times+}$ such that $x = \delta^1_{\mathfrak{A},\mathbb{R}}(\tilde{x})$. Since the image of $\tilde{x}$ in $K_0(\mathfrak{B},\mathbb{R})$ lies in $K_0(\mathfrak{B},\mathbb{Q})$ diagram (11) (with A replaced by B) implies that $\tilde{x} \in \zeta(B)^{\times+}$. Now $B\cap A_{\mathbb{R}} = A$ and so $\zeta(B)^{\times+}\cap\zeta(A_{\mathbb{R}})^{\times+} = \zeta(A)^{\times+}$ (as a consequence of Proposition 2.2). Hence

$x \in \mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) \subset \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$, as required. We note in particular that this implies that $\ker(\iota_*) \subseteq \mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$.
We write

$$\iota_{*,p} : \zeta(A_p)^\times / \operatorname{im}(K_1(\mathfrak{A}_p)) \to \zeta(B_p)^\times / \operatorname{im}(K_1(\mathfrak{B}_p)) \tag{49}$$

for the natural map which is induced by the inclusion $\zeta(A_p) \subseteq \zeta(B_p)$. We observe that the decomposition (15) induces a corresponding decomposition $\iota_* = \bigoplus_p \iota_{*,p}$, and hence that ι_* is injective if and only if each map $\iota_{*,p}$ is injective.
We now consider b). If firstly B and hence A are commutative, then the image of $K_1(\mathfrak{A}_p)$ in $K_1(A_p) = A_p^\times$ is isomorphic to $\mathfrak{A}_p^\times$ [15, (45.12)] and similarly for B_p. Hence if $\mathfrak{B}_p \cap A_p = \mathfrak{A}_p$, then the map $\iota_{*,p}$ is injective, as required. We assume now that $\mathfrak{A}_p$ is a maximal $\mathbb{Z}_p$-order in A_p. In this case $\zeta(\mathfrak{A}_p)$ is the (unique) maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$ and the map nr_{A_p} induces an identification $\operatorname{im}(K_1(\mathfrak{A}_p)) = \zeta(\mathfrak{A}_p)^\times \subset \zeta(A_p)^\times$ by [15, (45.8)]. To prove injectivity of $\iota_{*,p}$ we embed $\mathfrak{B}_p$ in a maximal $\mathbb{Z}_p$-order $\mathfrak{M}_p$ of B_p. Then $\operatorname{im}(K_1(\mathfrak{B}_p)) \subseteq \operatorname{im}(K_1(\mathfrak{M}_p)) = \zeta(\mathfrak{M}_p)^\times$. In addition, the intersection $C_p := \zeta(\mathfrak{M}_p) \cap \zeta(A_p)$ is a $\mathbb{Z}_p$-order in $\zeta(A_p)$ and is therefore contained in $\zeta(\mathfrak{A}_p)$. Hence one has

$$\operatorname{im}(K_1(\mathfrak{B}_p)) \cap \zeta(A_p)^\times \subseteq C_p^\times \subseteq \zeta(\mathfrak{A}_p)^\times = \operatorname{im}(K_1(\mathfrak{A}_p))$$

and so $\iota_{*,p}$ is indeed injective. This finishes the proof of b).
We next prove c) and also the first (and only remaining) assertion of a). We observe that

$$\ker(\iota_{*,p}) = (\operatorname{im}(K_1(\mathfrak{B}_p)) \cap \zeta(A_p)^\times) / \operatorname{im}(K_1(\mathfrak{A}_p)),$$

and that this quotient is finite since its numerator and denominator are both of finite index in the unit group of the maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$ (cf. [15, Exer. (45.4)]). In addition if $\mathfrak{A}_p$ is maximal, then $\zeta(A_p)$ is a product of local fields and $\zeta(\mathfrak{A}_p)$ is the corresponding product of valuation rings, and hence $K_0(\mathfrak{A}_p, \mathbb{Q}_p) \cong \zeta(A_p)^\times / \zeta(\mathfrak{A}_p)^\times$ is torsion free (in fact free abelian of finite rank). Hence in this case $\ker(\iota_{*,p})$ is trivial. This implies that $\ker(\iota_*)$ is finite (as claimed in a)) since $\mathfrak{A}_p$ is a maximal $\mathbb{Z}_p$-order for almost all p. This argument also implies that the torsion subgroup of $\zeta(A_p)^\times / \operatorname{im}(K_1(\mathfrak{A}_p))$ is equal to $\mathfrak{M}_p^\times / \operatorname{im}(K_1(\mathfrak{A}_p))$ where $\mathfrak{M}_p$ is the maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$. It follows that $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$ is indeed torsion free if and only if the condition in c) is satisfied. We remark that if $\mathfrak{A}$ is a maximal order, then this condition is satisfied as a consequence of [15, (45.8)].
We observe finally that d) follows immediately upon combining a) and c). □

Remark 11. The original conjecture of Bloch and Kato (as formulated in [4] and reworked in [20]) is equivalent to Conjecture 4(iv) for the pair $(M, \mathbb{Z})$. Now for any order $\mathfrak{A}$ the unique homomorphism $\mathbb{Z} \to \mathfrak{A}$ is flat. Hence, if $\mathfrak{A}$ is any order in A for which M admits a projective $\mathfrak{A}$-structure, then Conjecture 4(iv) for the pair $(M, \mathfrak{A})$ implies the conjecture of Bloch and Kato.

Remark 12. From Lemma 11a), it follows that Conjecture 4(iii) for any given pair $(M, \mathfrak{A})$ is equivalent to Conjecture 4(iii) for any pair $(B \otimes_A M, \mathfrak{B})$ where $\mathfrak{A} \subseteq \mathfrak{B}$. As a consequence, it suffices to verify Conjecture 4(iii) after an arbitrary extension $A \subseteq B$ of the operating algebra and for any choice of order in B.

Remark 13. In the Galois case (cf. Example b) in §3.3) there is a natural interplay between a change of coefficients and a change of field extension. This situation is described precisely by the following result.

Proposition 4.1. *Let M_K be a motive over K and L/K a Galois extension with group G so that $\mathbb{Q}[G]$ acts on $M := M_L = h^0(\mathrm{Spec}(L)) \otimes M_K$. Let H be a subgroup of G.*

a) Let $K' = L^H$ denote the fixed field of H and $T\Omega(M', \mathbb{Z}[H])$ the element constructed from the base change $M'_{K'}$ of M_K to K' and the extension L/K' with group H. Then

$$\rho_H^{G,*}(T\Omega(M, \mathbb{Z}[G])) = T\Omega(M, \mathbb{Z}[H]) = T\Omega(M', \mathbb{Z}[H]) \tag{50}$$

where $\rho_H^G : \mathbb{Z}[H] \to \mathbb{Z}[G]$ is the natural inclusion morphism (which is flat).

b) Set $Q := G/H$, $L' := L^H$ and $M_{L'} := h^0(\mathrm{Spec}(L')) \otimes M_K$. Then

$$q_{Q,*}^G(T\Omega(M, \mathbb{Z}[G])) = T\Omega(\mathbb{Q}[Q] \otimes_{\mathbb{Q}[G]} M, \mathbb{Z}[Q]) = T\Omega(M_{L'}, \mathbb{Z}[Q]) \tag{51}$$

where $q_Q^G : \mathbb{Z}[G] \to \mathbb{Z}[Q]$ is the natural projection.

Proof. After taking into account Theorem 4.1, we need only prove the second equalities of (50) and (51).
We observe first that the second equality of (51) is an immediate consequence of the isomorphism $\mathbb{Q}[Q] \otimes_{\mathbb{Q}[G]} h^0(\mathrm{Spec}(L)) \cong h^0(\mathrm{Spec}(L'))$ of motives over K with $\mathbb{Q}[Q]$-action.
On the other hand, the second equality of (50) is best understood by thinking of M_K as arising from a variety $X \to \mathrm{Spec}(K)$. Then both M and M' will arise from the same variety $X' = \mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} X$, respectively viewed over K and K' (and with H-action in both cases). It is well known that the L-functions taken over either K or K' are the same [16, Rem 2.9]. In addition, the groups $H^i_f(-,-)$ and $H_{dR}(-,-)$ are the same from both points of view since they only depend on the underlying scheme X'. Since also $H_v(M) = \bigoplus_{v'|v} H_{v'}(M')$ for each $v \in S_\infty$ it follows that $\Xi(M) = \Xi(M')$. The exact sequence (17) is the same for M and M'. Further, if $\pi : \mathrm{Spec}(\mathcal{O}_{K',S_p}) \to \mathrm{Spec}(\mathcal{O}_{K,S_p})$ denotes the natural finite morphism, then $\pi_*(H_p(M')) = H_p(M)$ and so $R\Gamma_c(\mathcal{O}_{K',S_p}, H_p(M')) \cong R\Gamma_c(\mathcal{O}_{K,S_p}, H_p(M))$. The map ϑ_p is therefore the same for both M and M' and hence $\Xi(M)_{\mathbb{Z}} = \Xi(M')_{\mathbb{Z}}$. This in turn implies that $T\Omega(M, \mathbb{Z}[H]) = T\Omega(M', \mathbb{Z}[H])$, as required. ☐

4.6. Reduction to the commutative case. In this section we use Theorem 4.1 to prove that Conjecture 4(iii), and also Conjecture 4(iv) for all pairs $(M,\mathfrak{A})$ with $\mathfrak{A}$ a maximal order, can be verified by restricting to motives with commutative coefficients.

Proposition 4.2. *a) Conjecture 4(iii) holds for all pairs* $(M,\mathfrak{A})$ *if it holds for all such pairs with* $\mathfrak{A}$ *commutative and maximal.*
b) Conjecture 4(iv) holds for all pairs $(M,\mathfrak{A})$ *where* $\mathfrak{A}$ *is a maximal order, if it holds for all such pairs with* $\mathfrak{A}$ *commutative and maximal.*

Proof. By remark 12 after Lemma 11 we may assume throughout that $\mathfrak{A}$ is maximal. Consider the Wedderburn decomposition $A\cong\prod_{i=1}^r M_{m_i}(D_i)$ of A and put $F_i:=\zeta(D_i)$. Pick a splitting field E_i for each i so that $M_{m_i}(D_i)\otimes_{F_i}E_i\cong M_{n_i}(E_i)$ with $n_i=m_i\sqrt{[D_i:F_i]}$. Then $B=\prod_{i=1}^r M_{n_i}(E_i)$ contains A and we have $\zeta(B)\cap A=\zeta(A)$. The image of $\mathfrak{A}$ can be embedded into a maximal order $\mathfrak{B}$ in B, and we write $\iota:\mathfrak{A}\to\mathfrak{B}$ for the corresponding morphism. One has $\iota_*(T\Omega(M,\mathfrak{A}))=T\Omega(B\otimes_A M,\mathfrak{B})$ by Theorem 4.1, and so Lemma 11 implies that Conjecture 4(iii), resp. (iv), is valid for $(M,\mathfrak{A})$ if and only if it is valid for $(B\otimes_A M,\mathfrak{B})$.
Now [15, Th. (26.25)] implies that, perhaps after enlarging each field E_i, we can assume that $\mathfrak{B}=b\cdot\mathfrak{B}'\cdot b^{-1}$ with $\mathfrak{B}'=\prod_{i=1}^r M_{n_i}(\mathcal{O}_{E_i})$ and $b\in B^\times$. In this case multiplication by b gives an isomorphism of pairs $(B\otimes_A M,B)\cong(B\otimes_A M,b\cdot B\cdot b^{-1})$ which in turn induces an equality

$$\begin{aligned}T\Omega(B\otimes_A M,\mathfrak{B})&=T\Omega(B\otimes_A M,\mathfrak{B}')\\&=\prod_{i=1}^r T\Omega(\epsilon_i(B\otimes_A M),M_{n_i}(\mathcal{O}_{E_i}))\\&\in\bigoplus_{i=1}^r K_0(M_{n_i}(\mathcal{O}_{E_i}),\mathbb{R})\cong K_0(\mathfrak{B}',\mathbb{R})\end{aligned}$$

where ϵ_i are the central idempotents of B. From Theorem 4.1 one has

$$\mu_{i,*}(T\Omega(\epsilon_i(B\otimes_A M),M_{n_i}(\mathcal{O}_{E_i})))=T\Omega(e_i\epsilon_i(B\otimes_A M),\mathcal{O}_{E_i})$$

where here e_i is the matrix (36) of size n_i, and $\mu_{i,*}:K_0(M_{n_i}(\mathcal{O}_{E_i}),\mathbb{R})\xrightarrow{\sim}K_0(\mathcal{O}_{E_i},\mathbb{R})$ is the associated isomorphism (37). Since also $K_0(M_{n_i}(\mathcal{O}_{E_i}),\mathbb{Q})=\mu_{i,*}^{-1}(K_0(\mathcal{O}_{E_i},\mathbb{Q}))$, it is clear that Conjecture 4(iii), resp. (iv), is true for $(B\otimes_A M,\mathfrak{B})$ if and only if it is true for each pair $(e_i\epsilon_i(B\otimes_A M),\mathcal{O}_{E_i})$. This finishes the proof of the proposition. □

Remark 14. Let A be a central simple algebra over a number field F with ring of integers $\mathcal{O}$. If $\mathfrak{A}$ is any maximal order in A, then the reduction to commutative coefficients effected by Proposition 4.2b) implies that Conjecture 4(iv) for the pair $(M,\mathfrak{A})$ can only determine $L^*({}_AM,0)$ to within multiplication by *all* elements of $\mathcal{O}^\times$ (inside $(F\otimes_{\mathbb{Q}}\mathbb{R})^\times=\zeta(A_{\mathbb{R}})^\times$). This reflects the general fact that if $\mathfrak{A}$ is any maximal order in a finite dimensional semisimple $\mathbb{Q}$-algebra

A, then $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ vanishes on all of $\zeta(\mathfrak{A})^\times$ rather than only on $\zeta(\mathfrak{A})^\times \cap \mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$. (The latter fact follows as an easy consequence of [15, (45.7), (45.8)]).

Remark 15. Non-maximal non-commutative orders $\mathfrak{A}$ arise as natural operating rings in many interesting examples. In general, when attempting to verify Conjecture 4(iv) for any such pair $(M, \mathfrak{A})$ no reduction to commutative coefficients is possible. In [11] we give a detailed discussion of Conjecture 4(iv) for a number of such examples.

5. Kummer duality

We recall that if M is any motive with an action of a semisimple $\mathbb{Q}$-algebra A, then the dual motive M^* is naturally endowed with an action of the opposite algebra A^{op}. After fixing an isomorphism $A^* \cong A^{op}$ of A^{op}-modules [15, (9.8)], we then have a functorial isomorphism of A^{op}-modules

$$W^* = \mathrm{Hom}_\mathbb{Q}(W, \mathbb{Q}) \cong \mathrm{Hom}_\mathbb{Q}(A \otimes_A W, \mathbb{Q}) \cong \mathrm{Hom}_A(W, \mathrm{Hom}_\mathbb{Q}(A^{op}, \mathbb{Q})) \cong \mathrm{Hom}_A(W, A) \tag{52}$$

for any A-module W. It follows that if M has a projective $\mathfrak{A}$-structure $\{T_v : v \in S_\infty\}$, then $M^*(1)$ has a projective $\mathfrak{A}^{op}$-structure $\{\mathrm{Hom}_\mathfrak{A}(T_v, \mathfrak{A})(1) : v \in S_\infty\}$. In this section we shall compare the elements $T\Omega(M, \mathfrak{A})$ and $T\Omega(M^*(1), \mathfrak{A}^{op})$. This comparison is naturally motivated by the problem of deciding whether Conjecture 4(iv) is compatible with the functional equation of $L({}_AM, s)$. The comparison result we prove in this section is most conveniently formulated in terms of an element $T\Omega^{loc}(M, \mathfrak{A})$ of $\mathrm{Cl}(\mathfrak{A}, \mathbb{R})$ the theory of which is strikingly parallel to that of $T\Omega(M, \mathfrak{A})$ but involves no assumptions on the motivic cohomology of M. Indeed, $T\Omega^{loc}(M, \mathfrak{A})$ takes the form $L^{loc}(M, \mathfrak{A}) + R\Omega^{loc}(M, \mathfrak{A})$ where the first term is defined in terms of the equivariant archimedean Euler factors and epsilon factors which are attached to M and $M^*(1)$ and the second term is of an algebraic nature, involving the realisations of M.

5.1. Definition of $R\Omega^{loc}(M, \mathfrak{A})$.

We first define a virtual A-module

$$\Xi^{loc}(M) := [H_{dR}(M)] \boxtimes [H_B(M)]^{-1}$$

where

$$H_B(M) := \bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} H_\sigma(M).$$

Recall that for each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$ we write $v(\sigma)$ for the corresponding element of S_∞. The action of $\mathrm{Gal}(\mathbb{C}/K_{v(\sigma)})$ on each space $H_\sigma(M)$ induces upon $H_B(M)$ an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In addition, by taking the direct sum over the $A \times \mathrm{Gal}(\mathbb{C}/K_{v(\sigma)})$-equivariant period isomorphisms

$$H_\sigma(M) \otimes_\mathbb{Q} \mathbb{C} \cong H_{dR}(M) \otimes_{K,\sigma} \mathbb{C}$$

one obtains an $A \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant isomorphism

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} H_{dR}(M) \otimes_{K,\sigma} \mathbb{C} = H_{dR}(M) \otimes_{\mathbb{Q}} \mathbb{C} \tag{53}$$

and after taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants this in turn induces an $A_{\mathbb{R}}$-equivariant isomorphism

$$(H_B(M) \otimes_{\mathbb{Q}} \mathbb{C})^+ \cong H_{dR}(M) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{54}$$

Here and in what follows, for any commutative ring R and $R[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module X we write X^+ and X^- for the R-submodules of X upon which complex conjugation acts as multiplication by 1 and -1 respectively. There is also an A-equivariant direct sum decomposition

$$(H_B(M) \otimes_{\mathbb{Q}} \mathbb{C})^+ = (H_B(M)^+ \otimes_{\mathbb{Q}} \mathbb{R}) \oplus (H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R}(2\pi i)^{-1}) \tag{55}$$

and an isomorphism

$$H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R}(2\pi i)^{-1} \cong H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R} \tag{56}$$

which is induced by identifying $\mathbb{R}(2\pi i)^{-1}$ with $\mathbb{R}$ by sending $(2\pi i)^{-1}$ to 1. Let ϵ_B (resp. ϵ_{dR}) be the automorphism $[-1]$ in $\pi_1 V(A_{\mathbb{R}}) \cong K_1(A_{\mathbb{R}})$ which is induced by multiplication by -1 on $H_B(M)^+ \otimes_{\mathbb{Q}} \mathbb{R}$ (resp. $F^0 H_{dR}(M) \otimes_{\mathbb{Q}} \mathbb{R}$). We write

$$\vartheta_\infty^{loc} : \Xi^{loc}(M) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbf{1}_{V(A_{\mathbb{R}})} \tag{57}$$

for the isomorphism of virtual $A_{\mathbb{R}}$-modules which is obtained by applying the functor $[\quad]$ to (54), (55) and (56) and then multiplying by $\epsilon_B \epsilon_{dR}$. The reason for the introduction of $\epsilon_B \epsilon_{dR}$ will become clear in the proof of Theorem 5.3 below.

As in previous sections, we now fix a finite set S of places of K which contains S_∞ and all places at which M has bad reduction, and for each rational prime p we set $V := V_p := H_p(M)$. For a finite group Π and a Π-module N we denote by $C^\bullet_{\mathrm{Tate}}(\Pi, N)$ the standard complex computing Tate cohomology. By a slight abuse of notation we also set $C^\bullet_{\mathrm{Tate}}(\Pi, N) := C^\bullet(\Pi, N)$ for any infinite profinite group Π.

For any continuous G_{S_p}-module N we set

$$\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, N) := \mathrm{Cone}\left(C^\bullet(G_{S_p}, N) \to \bigoplus_{v \in S_p} C^\bullet_{\mathrm{Tate}}(G_v, N) \right)[-1]$$

and if $N = V_p$ is a $\mathbb{Q}_p$-vector space we define

$${}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p) := \mathrm{Cone}\left(C^\bullet(G_{S_p}, N) \to \bigoplus_{v \in S_{p,f}} R\Gamma(K_v, V_p) \right)[-1]$$

so that there is a natural quasi-isomorphism

$$\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p) \to {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p). \tag{58}$$

We fix once and for all an injective resolution $\mathfrak{A}_p \to I^\bullet$ of $\mathfrak{A}_p$-$\mathfrak{A}_p$-bimodules, and for any complex N of $\mathfrak{A}_p$-modules (which is cohomologically bounded above) we define a complex of $\mathfrak{A}_p^{op}$-modules by $N^* := \mathrm{Hom}_{\mathfrak{A}_p}(N, I^\bullet)$. Note here that, since the natural map $\mathfrak{A}_p \to \mathfrak{A}_p \otimes_{\mathbb{Z}_p} \mathfrak{A}_p^{op}$ is flat, each I^n is an injective $\mathfrak{A}_p$-module and hence that $N^* = \mathrm{RHom}_{\mathfrak{A}_p}(N, \mathfrak{A}_p)$ in $D(\mathfrak{A}_p^{op})$. We shall moreover assume that $I^0 = A_p$ and that each I^n is torsion if $n \geq 1$. There is then an isomorphism of complexes of A_p-A_p-bimodules

$$I^\bullet_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I^\bullet \cong A_p[0]. \tag{59}$$

If $N = T = T_p \subset V_p$ is a projective $\mathfrak{A}_p$-lattice, and in that case only, we put $T^* = \mathrm{Hom}_{\mathfrak{A}_p}(T, \mathfrak{A}_p)$. The isomorphism (52) then induces an identification

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T^*(1) \cong \mathrm{Hom}_{A_p}(V, A_p(1)) \cong \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1)) = V^*(1) \tag{60}$$

of $T^*(1)$ with an $\mathfrak{A}_p^{op}$-lattice in $V^*(1)$.

Lemma 12. *a) There is a commutative diagram of maps of complexes*

$$\begin{array}{ccccc}
R\Gamma(\mathcal{O}_{K,S_p}, V) & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma(K_v, V) & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma_{/f}(K_v, V) \\
{\scriptstyle {}_1\mathrm{AV}}\big\downarrow & & {\scriptstyle \oplus\,\mathrm{AV}_v}\big\downarrow & & {\scriptstyle \oplus\,\mathrm{AV}_{f,v}}\big\downarrow \\
{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1))^*[-3] & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma(K_v, V^*(1))^*[-2] & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v, V^*(1))^*[-2]
\end{array}$$

in which all of the vertical maps are quasi-isomorphisms. Moreover $[{}_1\mathrm{AV}]$, $[\mathrm{AV}_v]$ *and* $[\mathrm{AV}_{f,v}]$ *are independent of any choices made in the construction of this diagram.*
b) There is a natural quasi-isomorphism

$$R\Gamma(\mathcal{O}_{K,S_p}, T) \xrightarrow{\mathrm{AV}} \tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, T^*(1))^*[-3] \tag{61}$$

so that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{AV} = \nu[-3] \circ {}_1\mathrm{AV}$ *where* ν *is the composite isomorphism*

$$\begin{aligned}
{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1))^* &= \mathrm{Hom}_{\mathbb{Q}_p}({}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), \mathbb{Q}_p) \\
&\xrightarrow{(52)} \mathrm{Hom}_{A_p}({}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), A_p) \\
&\xrightarrow{(59)} \mathrm{Hom}_{A_p}({}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), I^\bullet_{\mathbb{Q}_p}) \\
&\xrightarrow{(58)} \mathrm{Hom}_{A_p}(\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), I^\bullet_{\mathbb{Q}_p}) \\
&\xrightarrow{(60)} \mathrm{Hom}_{\mathfrak{A}_p}(\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, T^*(1)), I^\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\end{aligned}$$

Proof. We first define local pairings for places $v \mid p$. To do this we continue to use the notation introduced in §3.2.
Recall that B^i is an algebra for $i = 0, 1$ and that the differential of $B^\bullet$ is a difference of two algebra homomorphisms β_1 and β_2 (cf. (20)). There therefore exists a natural morphism of complexes $\mu : B^\bullet \otimes_{\mathbb{Q}_p} B^\bullet \to B^\bullet$ for which $\mu^0 : B^0 \otimes_{\mathbb{Q}_p} B^0 \to B^0$ is given by multiplication, $\mu^1 : (B^0 \otimes_{\mathbb{Q}_p} B^1) \oplus (B^1 \otimes_{\mathbb{Q}_p} B^0) \to$

B^1 is defined by $\mu_1(x\otimes y, y'\otimes x') = \beta_2(x)y + \beta_1(x')y'$, and $\mu^2 : B^1\otimes_{\mathbb{Q}_p} B^1 \to 0$ is the zero map. This morphism induces a commutative diagram of pairings

$$\begin{tikzcd} V\otimes_{\mathbb{Q}_p} V^*(1) \arrow[r] \arrow[d] & \mathbb{Q}_p(1) \arrow[d] \\ (B^\bullet\otimes_{\mathbb{Q}_p} V)\otimes_{\mathbb{Q}_p}(B^\bullet\otimes_{\mathbb{Q}_p} V^*(1)) \arrow[r] & B^\bullet\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(1) \end{tikzcd}$$

and also a commutative diagram of cup product pairings

$$\begin{tikzcd} C^\bullet(G_v,V)\times C^\bullet(G_v,V^*(1)) \arrow[r, "\cup"] \arrow[d] & C^\bullet(G_v,\mathbb{Q}_p(1)) \arrow[d] \\ \operatorname{Tot} C^\bullet(G_v, B^\bullet\otimes_{\mathbb{Q}_p} V)\times \operatorname{Tot} C^\bullet(G_v,(B^\bullet\otimes_{\mathbb{Q}_p} V^*(1))) \arrow[r] & \operatorname{Tot} C^\bullet(G_v, B^\bullet\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(1)). \end{tikzcd}$$

We thereby obtain a commutative diagram of local and global cup product pairings

$$\begin{tikzcd} C^\bullet(G_{S_p},V)\times C^\bullet(G_{S_p},V^*(1)) \arrow[r, "\cup"] \arrow[d, "\mathrm{res}_V\ \times\ \mathrm{res}_{V^*(1)}"] & C^\bullet(G_{S_p},\mathbb{Q}_p(1)) \arrow[d, "\mathrm{res}_{\mathbb{Q}_p(1)}"] \\ \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V)\times \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V^*(1)) \arrow[r, "\cup"] & \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,\mathbb{Q}_p(1)) \end{tikzcd} \tag{62}$$

and hence an induced pairing on the mapping cone

$$C^\bullet(G_{S_p},V)\times {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V^*(1)) \xrightarrow{\cup} {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1)) \xrightarrow{\tilde{\mathrm{Tr}}} \mathbb{Q}_p[-3] \tag{63}$$

so that

$$\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}\big(\mathrm{res}_V(x)\cup y\big) = x\cup \mathrm{res}^{ad}_{V^*(1)}(y) \tag{64}$$

where here

$$\mathrm{res}^{ad}_V : \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V)\to {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V)[1]$$

is the natural map. The morphism $\tilde{\mathrm{Tr}}$ in (63) is chosen to be a lift of the map

$$\begin{aligned} {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1)) \supseteq \tau^{\le 3}{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1)) \\ \to H^3_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1))[-3] \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p[-3] \end{aligned}$$

(such a lift exists because $\mathbb{Q}_p$ is an injective $\mathbb{Q}_p$-module).
In the diagram of claim a) the map ${}_1\,\mathrm{AV}$ is induced by (63) and the maps AV_v by the local cup product pairing composed with $\tilde{\mathrm{Tr}}\circ\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}$. These maps are quasi-isomorphisms by local and global duality and the compatibility of local and global trace maps [36, Chap. II, §3]. In addition, the commutativity of the left hand square of the diagram in a) is a consequence of (64).
The right hand square of the diagram in a) arises as a direct sum of commutative squares over the places in $S_{p,f}$, and for each such place v the existence of the appropriate square will follow directly if we can show that the complexes $R\Gamma_f(K_v,V)$ and $R\Gamma_f(K_v,V^*(1))$ (and not only their cohomology) annihilate

each other under the pairing $\tilde{\mathrm{Tr}}\circ\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}\circ\cup$ constructed above. To prove the required annihilation property, we consider separately the cases $v\nmid p$ and $v\mid p$. If firstly $v\nmid p$, then $R\Gamma_f(K_v,V)$ coincides with the subcomplex $C^\bullet(G_v/I_v,V^{I_v})$ of $C^\bullet(G_v,V)$. In addition, since $H^2(G_v/I_v,\mathbb{Q}_p(1))=0$, we can certainly choose the lifting $\tilde{\mathrm{Tr}}$ in such a way that it vanishes on the subcomplex

$$\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}\left(\bigoplus_{v\in S_{p,f}}C^\bullet(G_v/I_v,\mathbb{Q}_p(1))\right)\subseteq {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1))[1].$$

If now $v\mid p$, then the subcomplex $R\Gamma_f(K_v,V)=H^0(G_v,B^\bullet\otimes_{\mathbb{Q}_p}V)$ is concentrated in degrees 0 and 1 and the cup product of $x\in H^0(G_v,B^1\otimes_{\mathbb{Q}_p}V)$ and $x'\in H^0(G_v,B^1\otimes_{\mathbb{Q}_p}V^*(1))$ is given by $\mu_2(x\otimes x')=0$. Hence $R\Gamma_f(K_v,V)$ and $R\Gamma_f(K_v,V^*(1))$ do indeed annihilate each other.
We observe that, for each $v\in S_{p,f}$, the resulting morphism $\mathrm{AV}_{f,v}$: $R\Gamma_f(K_v,V)\to R\Gamma_f(K_v,V^*(1))^*[-2]$ is a quasi-isomorphism as a consequence of [4, Prop. 3.8].
Also, since all of the maps which are induced on cohomology by ${}_1\mathrm{AV}$, AV_v and $\mathrm{AV}_{f,v}$ are independent of the choice of the lift $\tilde{\mathrm{Tr}}$, and their sources and targets all belong to $D^{p,p}(A_p)=D^p(A_p)$, Proposition 2.1e) implies the final assertion of claim a).
To prove claim b), we argue in a similar way starting with the diagram

(65)
$$\begin{array}{ccc}
C^\bullet(G_{S_p},T)\times C^\bullet(G_{S_p},T^*(1)) & \xrightarrow{\cup} & C^\bullet(G_{S_p},\mathfrak{A}_p(1))\\
\Big\downarrow{\scriptstyle \mathrm{res}_T\times\mathrm{res}_{T^*(1)}} & & \Big\downarrow{\scriptstyle \mathrm{res}_{\mathfrak{A}_p(1)}}\\
\bigoplus_{v\in S_p}C^\bullet_{\mathrm{Tate}}(G_v,T)\times\bigoplus_{v\in S_p}C^\bullet_{\mathrm{Tate}}(G_v,T^*(1)) & \xrightarrow{\cup} & \bigoplus_{v\in S_p}C^\bullet_{\mathrm{Tate}}(G_v,\mathfrak{A}_p(1))
\end{array}$$

and using a lift $\tilde{\mathrm{Tr}}_{\mathfrak{A}}$ of the map

$$\begin{aligned}\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))\supseteq\tau^{\le 3}\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))\\
\to H^3_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))[-3]\xrightarrow{\mathrm{Tr}_{\mathfrak{A}}}\mathfrak{A}_p[-3]\to I^\bullet[-3].\end{aligned}$$

The resulting map AV (as in (61)) is a quasi-isomorphism by [9, Lem. 16]. In addition, there is a natural map from diagram (65) to diagram (62), inducing a commutative diagram

$$\begin{array}{ccccc}
C^\bullet(G_{S_p},T)\times\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},T^*(1)) & \to & \tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))) & \xrightarrow{\tilde{\mathrm{Tr}}_{\mathfrak{A}}} & I^\bullet[-3]\\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow\\
C^\bullet(G_{S_p},V)\times{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V^*(1)) & \to & {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1))) & \xrightarrow{\tilde{\mathrm{Tr}}} & \mathbb{Q}_p[-3]
\end{array}$$

where the left vertical arrow involves (60) and the middle and right vertical arrows involve the map $A \to \mathbb{Q}$ which is the image of $1 \in A^{op}$ under the isomorphism $A^{op} \cong A^* = \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q})$ chosen before (52). The second statement in b) then follows easily from this last commutative diagram. $\square$

We now define a virtual A_p-module $\Lambda_p(S, V_p)$ by setting

$$\Lambda_p(S,V_p) := \boxtimes_{v \in S_{p,f}} [R\Gamma(K_v, V_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} V_p]^{-1}. \tag{66}$$

We also define

$$\theta_p : A_p \otimes_A \Xi^{loc}(M) \cong \Lambda_p(S, V_p) \tag{67}$$

to be the isomorphism in $V(A_p)$ which results from composing the isomorphisms obtained by applying $[\quad]$ to the canonical $A \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant comparison isomorphism

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{Ind}_K^{\mathbb{Q}} V_p, \tag{68}$$

to the A-equivariant (Poincaré duality) exact sequence

$$0 \to (H_{dR}(M^*(1))/F^0)^* \to H_{dR}(M) \to H_{dR}(M)/F^0 \to 0, \tag{69}$$

to (23) for both M and $M^*(1)$, to (18) and the maps $\mathrm{AV}_{f,v}$ for each $v \in S_{p,f}$, to (19) for each $v \in S$ with $v \nmid p$ and (22) for each $v \mid p$, and by then using the isomorphisms (24) for $V = V_{p,v}$ and $V = V_p^*(1)_v$ and each $v \in S_{p,f}$.

Given a projective $\mathfrak{A}$-structure T on M we define $C(K, T_p)$ to be the mapping cone of the composite map

(70)

$$\begin{array}{ccc} R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)[-1] \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p}, T_p)[-1] & \\ & \Big\downarrow{\scriptstyle \mathrm{AV}} & \\ & \tilde{R\Gamma}_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*[-4] & \longrightarrow R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*[-4] \end{array}$$

and we set

$$\Lambda_p(S, T_p) := [C(K, T_p)]. \tag{71}$$

We next define a canonical isomorphism in $V(A_p)$

$$\theta_p' : \Lambda_p(S, V_p) \xrightarrow{\sim} A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p).$$

To do this we first define ${}_1C(K, V_p)$ just as $C(K, T_p)$ but using diagram $A_p \otimes_{\mathfrak{A}_p}$ (70) with $R\Gamma_c(\mathcal{O}_{K,S_p}, V_p)$ replaced by ${}_1R\Gamma_c(\mathcal{O}_{K,S_p}, V_p)$, and we define ${}_2C(K, V_p)$ by also replacing $\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p}, V_p^*(1))$, $R\Gamma_c(\mathcal{O}_{K,S_p}, V_p^*(1))$ and AV by their respective versions indexed by 1. Then there are natural quasi-isomorphisms

$$A_p \otimes_{\mathfrak{A}_p} C(K, T_p) \xrightarrow{\sim} {}_1C(K, V_p) \xleftarrow{\sim} {}_2C(K, V_p) \tag{72}$$

where we have used the last assertion in Lemma 12b) for the second quasi-isomorphism. Setting

$$
\begin{aligned}
&{}_2L(S_p,V_p) := \\
&\mathrm{Cone}\left({}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \to R\Gamma(\mathcal{O}_{K,S_p},V_p) \xrightarrow{{}_1\mathrm{AV}} {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-3]\right)
\end{aligned}
$$

we obtain a true nine term diagram

$$
\begin{tikzcd}
\bigoplus\limits_{v\in S_\infty} R\Gamma(K_v,V_p^*(1))^*[-4] \arrow[r, equal] \arrow[d] & \bigoplus\limits_{v\in S_\infty} R\Gamma(K_v,V_p^*(1))^*[-4] \arrow[d] & \\
{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-4] \arrow[r] \arrow[d] & {}_2L(S_p,V_p)[-1] \arrow[r] \arrow[d] & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \arrow[d, equal] \\
{}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-4] \arrow[r] & {}_2C(K,V_p) \arrow[r] & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p).
\end{tikzcd}
\tag{73}
$$

There is also a commutative diagram of true triangles

$$
\begin{tikzcd}
R\Gamma(\mathcal{O}_{K,S_p},V_p)[-1] \arrow[r] \arrow[d, "{}_1\mathrm{AV}"'] & {}_1L(S_p,V_p)[-1] \arrow[r] \arrow[d, "\lambda"'] & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \arrow[d, equal] \\
{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-4] \arrow[r] & {}_2L(S_p,V_p)[-1] \arrow[r] & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p)
\end{tikzcd}
\tag{74}
$$

where the bottom row coincides with the central row in (73) and

$$
{}_1L(S_p,V_p) := \mathrm{Cone}\left({}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \to R\Gamma(\mathcal{O}_{K,S_p},V_p)\right).
$$

Lemma 13. *Let*

$$
E : 0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0
$$

be any true triangle, and let E' denote the associated canonical true triangle

$$
0 \longrightarrow C \longrightarrow \mathrm{Cone}(\pi) \longrightarrow B[1] \longrightarrow 0.
$$

Then there is a natural quasi-isomorphism $A[1] \xrightarrow{q} \mathrm{Cone}(\pi)$ for which the following diagram commutes

$$
\begin{tikzcd}
{[A[1]]} \arrow[r] \arrow[d, "{[q]}"'] & {[A[1]] \boxtimes [C] \boxtimes [C]^{-1}} \arrow[d, "{[E]}"'] \\
{[\mathrm{Cone}(\pi)]} \arrow[r, "{[E']}"] & {[C] \boxtimes [B[1]].}
\end{tikzcd}
$$

Proof. This is an immediate consequence of the true nine term diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A[1] \arrow[r, equal] \arrow[d, "q"'] & A[1] \arrow[d] \\ C \arrow[r] \arrow[d, equal] & \mathrm{Cone}(\pi) \arrow[r] \arrow[d] & B[1] \arrow[d] \\ C \arrow[r] & Z \arrow[r] & C[1], \end{tikzcd}$$

where here $q(x) = (0, \iota(x)) \in C \oplus B[1] = \mathrm{Cone}(\pi)$ and Z is acyclic. □

By applying Lemma 13 to the short exact sequence given by the central row in (26) we obtain a canonical quasi-isomorphism

$$L(S_p, V_p) := \bigoplus_{v \in S_p} R\Gamma(K_v, V_p) \xrightarrow{q} {}_1L(S_p, V_p). \tag{75}$$

Upon composing the isomorphisms in $V(A_p)$ which are induced by (72), the central column in (73) and the isomorphisms λ^{-1} from (74), q^{-1} from (75),

$$\bigoplus_{v \in S_\infty} R\Gamma(K_v, V_p^*(1))^* \cong \mathrm{Ind}_K^{\mathbb{Q}} V_p^*(1)^{*+}[0],$$

$$(\mathrm{Ind}_K^{\mathbb{Q}} V_p^*(1))^{*+} \cong (\mathrm{Ind}_K^{\mathbb{Q}} V_p(-1))^+ \cong (\mathrm{Ind}_K^{\mathbb{Q}} V_p)^- \tag{76}$$

and

$$\mathrm{Ind}_K^{\mathbb{Q}} V_p \cong (\mathrm{Ind}_K^{\mathbb{Q}} V_p)^+ \oplus (\mathrm{Ind}_K^{\mathbb{Q}} V_p)^-, \tag{77}$$

we obtain the desired isomorphism

$$\theta_p' : \Lambda_p(S, V_p) = [L(S_{p,f}, V_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} V_p]^{-1} \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p). \tag{78}$$

LEMMA 14. *If p is odd, then the isomorphism θ_p' is induced by an isomorphism*

$$\boxtimes_{v \in S_{p,f}} [R\Gamma(K_v, T_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} T_p]^{-1} \cong \Lambda_p(S, T_p). \tag{79}$$

Proof. For each $v \in S_\infty$ and continuous G_{S_p}-module N we define $R\Gamma_\Delta(K_v, N)$ by the short exact sequence

$$0 \to C^\bullet(G_v, N) \to C^\bullet_{\mathrm{Tate}}(G_v, N) \to R\Gamma_\Delta(K_v, N)[1] \to 0$$

where the second map is the natural inclusion. We then have a true nine term diagram

$$
\begin{array}{ccccc}
\bigoplus_{v\in S_p} C^\bullet(G_v,N)[-1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p},N) & \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p},N) \\
\downarrow & & \downarrow & & \| \\
\bigoplus_{v\in S_p} C^\bullet_{\mathrm{Tate}}(G_v,N)[-1] & \longrightarrow & \tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},N) & \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p},N) \\
\downarrow & & \downarrow & & \\
\bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,N) & = & \bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,N). & &
\end{array}
\tag{80}
$$

Upon defining

$$
\begin{aligned}
&{}_3L(S_p,T_p) := \\
&\quad \mathrm{Cone}\left(R\Gamma_c(\mathcal{O}_{K,S_p},T_p) \to R\Gamma(\mathcal{O}_{K,S_p},T_p) \xrightarrow{\mathrm{AV}} \tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-3]\right)
\end{aligned}
$$

there is in addition a true nine term diagram

$$
\begin{array}{ccccc}
\bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,T_p^*(1))^*[-4] & = & \bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,T_p^*(1))^*[-4] & & \\
\downarrow & & \downarrow & & \\
\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-4] & \longrightarrow & {}_3L(S_p,T_p)[-1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p},T_p) \\
\downarrow & & \downarrow & & \| \\
R\Gamma_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-4] & \longrightarrow & C(K,T_p) & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p},T_p)
\end{array}
\tag{81}
$$

in which the left hand column is the dual of the central column in (80) with $N = T_p^*(1)$. If p is odd, then all terms in the central column of (81) belong to $D^p(\mathfrak{A}_p)$, and $R\Gamma_\Delta(K_v,T_p^*(1))$ is naturally quasi-isomorphic to $R\Gamma(K_v,T_p^*(1))$. In addition, setting

$$
{}_4L(S_p,T_p) := \mathrm{Cone}\left(R\Gamma_c(\mathcal{O}_{K,S_p},T_p) \to R\Gamma(\mathcal{O}_{K,S_p},T_p)\right)
$$

there exists a commutative diagram of true triangles similar to (74) and quasi-isomorphisms

$$
\bigoplus_{v\in S_p} R\Gamma(K_v,T_p) \xrightarrow{q} {}_4L(S_p,T_p) \leftarrow {}_3L(S_p,T_p)
$$

which together give (79). □

Remark 16. If p is odd, then the isomorphism (79) allows a more direct definition of $\Lambda_p(S,T_p)$ than that given by (71). However, we do not expect the statement of Lemma 14 to hold for $p=2$. More concretely, if for example $\mathfrak{A}=\mathbb{Z}$, $p=2$ and we interpret virtual objects as graded determinants, then the $\mathbb{Z}_2$-lattices in the $\mathbb{Q}_2$-line $\Lambda_p(S,V_p)$ given respectively by $\Lambda_p(S,T_p)$ and $\boxtimes_{v\in S_{p,f}} [R\Gamma(K_v,T_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} T_p]^{-1}$ may well differ.

We now define

$$\vartheta_p^{loc} := \epsilon(S,p) \circ \theta_p' \circ \theta_p : A_p \otimes_A \Xi^{loc}(M) \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S,T_p) \tag{82}$$

where θ_p was defined in (67), θ_p' in (78) and where $\epsilon(S,p) \in \pi_1(V(A_p))$ is the automorphism $[-1]$ which is induced by multiplication by -1 on $\bigoplus_{v\in S_{p,f}} R\Gamma_{/f}(K_v,V_p)$ (again, the reason for the introduction of $\epsilon(S,p)$ will become clear in the proof of Theorem 5.3 below). We then define an object of the category $V(\mathfrak{A}_p)\times_{V(A_p)} V(A)$ by setting

$$\Xi^{loc}(M,T_p,S) := (\Lambda_p(S,T_p), \Xi^{loc}(M), \vartheta_p^{loc}).$$

The following result is a natural analogue of Lemmas 5 and 6 for $\Xi^{loc}(M,T_p,S)$.

Lemma 15. *a) For a different choice of projective $\mathfrak{A}$-structure T' on M and a different set of places S' the objects $\Xi^{loc}(M,T_p,S)$ and $\Xi^{loc}(M,T_p',S')$ are isomorphic in $V(\mathfrak{A}_p)\times_{V(A_p)} V(A)$.*
b) Let M be a direct factor of $h^n(X)(r)$ for a smooth projective variety X over K. If (M,A) satisfies Conjecture 3, then the object

$$\Xi^{loc}(M)_{\mathbb{Z}} := (\prod_p \Lambda_p(S,T_p), \Xi^{loc}(M), \prod_p \vartheta_p^{loc})$$

of the category $\prod_p V(\mathfrak{A}_p)\times_{\prod_p V(A_p)} V(A)$ is isomorphic to the image of an object of $\mathbb{V}(\mathfrak{A})$ under the functor of Lemma 4.

Proof. Under further assumptions on M both of these claims follow from the proof of Theorem 5.3 given below. For brevity, we shall therefore just sketch a proof here.

For a second lattice $T'\subseteq T$ one has a commutative diagram

$$\begin{tikzcd}
R\Gamma_c(\mathcal{O}_{K,S_p},T_p')[-1] \arrow[r] \arrow[d, "\mathrm{AV}"'] & R\Gamma_c(\mathcal{O}_{K,S_p},T_p)[-1] \arrow[d, "\mathrm{AV}"'] \\
R\Gamma_c(\mathcal{O}_{K,S_p},(T_p')^*(1))^*[-4] \arrow[r] & R\Gamma_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-4].
\end{tikzcd}$$

One can then argue just as in the proof of Lemma 5 using [18, Th. 5.1] for both of the modules $\mathcal{F} := T_p/T_p'$ and $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{F},\mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathrm{Ext}^1_{\mathfrak{A}_p}(\mathcal{F},\mathfrak{A}_p(1))$. For the independence of S it is enough for us to consider the case $S' = S\cup\{w\}$ with $w\notin S_p$. In this case, the true triangle (18) is induced from a triangle in $D^p(\mathfrak{A}_p)$

$$\left(T_p \xrightarrow{1-f_w^{-1}} T_p\right) \to R\Gamma(K_w,T_p) \to \left(T_p^*(1) \xrightarrow{1-f_w^{-1}} T_p^*(1)\right)^*[-2] \tag{83}$$

and the isomorphism (24) is induced by an isomorphism $[T_p \xrightarrow{1-f_w^{-1}} T_p] \cong \mathbf{1}_{V(\mathfrak{A}_p)}$, and similarly for $T_p^*(1)$. Moreover, $\epsilon(S',p)\epsilon(S,p)^{-1}$ coincides with the automorphism which is induced by multiplication by -1 on $[T_p^*(1) \xrightarrow{1-f_w^{-1}}$

$T_p^*(1)]^*[-2]$ and hence lies in the image of $\pi_1(V(\mathfrak{A}_p))$. This suffices to construct an isomorphism between $\Xi^{loc}(M, T_p, S)$ and $\Xi^{loc}(M, T_p, S')$.
Claim b) is proved by choosing a smooth proper model of X over $\mathrm{Spec}(\mathcal{O}_{K,S})$ and then arguing just as in [8, pp. 81-83]. □

Following the last result, we define $R\Omega^{loc}(M, \mathfrak{A})$ to be the class of $(\Xi^{loc}(M)_{\mathbb{Z}}, \vartheta_\infty^{loc})$ in $\pi_0(\mathbb{V}(\mathfrak{A}, \mathbb{R})) \cong K_0(\mathfrak{A}, \mathbb{R})$.
We observe that, as a consequence of (71), one has

$$R\Omega^{loc}(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

5.2. Definition of $L^{loc}(M, \mathfrak{A})$. Recall that $L_\infty({}_AM, s) = \prod_{v \in S_\infty} L_v({}_AM, s)$ and $\Lambda({}_AM, s) = L_\infty({}_AM, s)L({}_AM, s)$. The following conjecture is standard.

Conjecture 7. *There is an identity of meromorphic $\zeta(A_{\mathbb{C}})$-valued functions of the complex variable s*

$$\Lambda({}_AM, s) = \epsilon({}_AM, s)\Lambda({}_{A^{op}}M^*(1), -s).$$

Letting $\rho \in \mathbb{Z}^{\pi_0(\mathrm{Spec}(\zeta(A_{\mathbb{R}})))}$ denote the algebraic order at $s = 0$ of the meromorphic function $\Lambda({}_{A^{op}}M^*(1), s)$, we set

$$\mathcal{E}({}_AM) := (-1)^\rho \epsilon({}_AM, 0)\frac{L_\infty^*({}_{A^{op}}M^*(1), 0)}{L_\infty^*({}_AM, 0)} \in \zeta(A_{\mathbb{R}})^\times$$

and

$$L^{loc}(M, \mathfrak{A}) := \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(\mathcal{E}({}_AM)) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

We then set

$$T\Omega^{loc}(M, \mathfrak{A}) := L^{loc}(M, \mathfrak{A}) + R\Omega^{loc}(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

The following result can be proved by mimicking the proofs of Theorems 3.1 and 4.1.

Theorem 5.1. *All assertions of Theorem 3.1 remain valid with $R\Omega(-,-)$ replaced by either $R\Omega^{loc}(-,-)$, $L^{loc}(-,-)$ or $T\Omega^{loc}(-,-)$.* □

We now describe conditions under which $T\Omega^{loc}(M, \mathfrak{A})$ can be shown to belong to $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$.

Theorem 5.2. *If Deligne's conjecture* [16, Conj. 6.6] *on the nature of rank one motives over* $\mathbb{Q}$ *is valid, then* $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$. *More precisely, if (in the notation of the proof of Proposition 4.2b) in* §4.6*) for each index* $i \in \{1,\dots,r\}$ *there exists an integer* p_i*, an* E_i*-valued Dirichlet character* χ_i *and an isomorphism of motives over* $\mathbb{Q}$ *with coefficients in* E_i

$$\bigwedge\nolimits_{E_i}^{max} e_i\epsilon_i(\mathrm{Res}^K_{\mathbb{Q}}(B\otimes_A M)) \cong E_i(p_i)(\chi_i), \tag{84}$$

then $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.

Proof. Upon combining the functorial behaviour of $T\Omega^{loc}(-,-)$ which is described in Theorem 5.1 together with the arguments used to prove Proposition 4.2 one finds that the containment $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ can be decided by considering the motives which occur on the left hand side of (84). Indeed, it follows that $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ if and only if $T\Omega^{loc}(e_i\epsilon_i(\mathrm{Res}^K_{\mathbb{Q}}(B\otimes_A M)),\mathcal{O}_{E_i}) \in K_0(\mathcal{O}_{E_i},\mathbb{Q})$ for each index $i \in \{1,\dots,r\}$.
We now fix such an index i and set $N := e_i\epsilon_i(\mathrm{Res}^K_{\mathbb{Q}}(B\otimes_A M))$ and $E := E_i$. Following [11, Lemma 1a)], one has $T\Omega^{loc}(N,\mathcal{O}_E) \in K_0(\mathcal{O}_E,\mathbb{Q})$ if and only if

$$\mathcal{E}({}_EN)^{-1}\vartheta^{loc}_\infty(\Xi^{loc}(N)) \subseteq E,$$

where here ϑ^{loc}_∞ is the isomorphism (57) for the pair (N,E). It therefore suffices to prove the displayed inclusion and to do this we adapt the proof of [16, Th. 5.6].
After fixing E-bases of $H_B(N)$ and $H_{dR}(N)$ we let $\delta(N) \in E_{\mathbb{C}}$ denote the corresponding determinant of the isomorphism (53) (with $M = N$ and $K = \mathbb{Q}$). We set $d^\pm := \mathrm{rank}_E(H_B(N)^\pm)$. After adjoining E-bases of $H_B(N)^+$ and $H_B(N)^-$ the isomorphism (56) (with $M = N$) implies that $\vartheta^{loc}_\infty(\Xi^{loc}(N))$ is the E-subspace of $E_{\mathbb{R}}$ which is generated by the element $(2\pi i)^{-d^-}\delta(N)$, and so we need to prove that

$$\mathcal{E}({}_EN)^{-1}(2\pi i)^{-d^-}\delta(N) \in E. \tag{85}$$

Now from [37, Lem. C.3.7] one has

$$\frac{L^*_\infty({}_EN^*(1),0)}{L^*_\infty({}_EN,0)}.(2\pi)^{d^-+t} \in E$$

where here $t := \frac{1}{2}w(d^+ + d^-) \in \mathbb{Z}$ with w equal to the weight of N. On the other hand, by assuming that there is an isomorphism of the form (84) Deligne has proved that

$$\epsilon({}_EN,0)i^{d^-}(2\pi)^{-t}\delta(N)^{-1} \in E$$

([16, second formula on p. 331]). Upon combining the last two displayed containments we obtain (85). □

5.3. The comparison of $T\Omega(M,\mathfrak{A})$ and $T\Omega(M^*(1),\mathfrak{A}^{op})$. The exact functor $P \mapsto P^* := \mathrm{Hom}_{\mathfrak{A}^{op}}(P,\mathfrak{A}^{op})$ induces an equivalence of exact categories $\mathrm{PMod}(\mathfrak{A}^{op}) \to \mathrm{PMod}(\mathfrak{A})^{op}$. We obtain induced equivalences $\mathrm{PMod}(A^{op}) \to \mathrm{PMod}(A)^{op}$ under scalar extension and also induced equivalences of Picard categories

$$\mathbb{V}(\mathfrak{A}^{op}) \xrightarrow{*} \mathbb{V}(\mathfrak{A})^{op} \xrightarrow{\iota} \mathbb{V}(\mathfrak{A}), \quad \mathbb{V}(\mathfrak{A}^{op},\mathbb{R}) \xrightarrow{*} \mathbb{V}(\mathfrak{A},\mathbb{R})^{op} \xrightarrow{\iota} \mathbb{V}(\mathfrak{A},\mathbb{R})$$

where the functor ι sends each morphism to its inverse. We denote each of these composite functors by $X \mapsto X^*$ and we use ψ^* to denote the induced isomorphisms on algebraic K-groups.
If F is any field of characteristic 0, then the maps ψ^* combine to give an isomorphism of localisation sequences

(86)
$$\begin{tikzcd}
\cdots \arrow[r] & K_1(A_F^{op}) \arrow[r, "\delta^1_{\mathfrak{A}^{op},F}"] \arrow[d, "\psi^*"'] & \mathrm{Cl}(\mathfrak{A}^{op},F) \arrow[r, "\delta^0_{\mathfrak{A}^{op},F}"] \arrow[d, "\psi^*"'] & \mathrm{Cl}(\mathfrak{A}^{op}) \arrow[r] \arrow[d, "\psi^*"'] & 0 \\
\cdots \arrow[r] & K_1(A_F) \arrow[r, "\delta^1_{\mathfrak{A},F}"] & \mathrm{Cl}(\mathfrak{A},F) \arrow[r, "\delta^0_{\mathfrak{A},F}"] & \mathrm{Cl}(\mathfrak{A}) \arrow[r] & 0.
\end{tikzcd}$$

Lemma 16. *One has*

$$\psi^* \circ \hat{\delta}^1_{\mathfrak{A}^{op},\mathbb{R}} = -\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$$

on $\zeta(A_{\mathbb{R}}^{op})^\times = \zeta(A_{\mathbb{R}})^\times$.

Proof. For any field F of characteristic 0 there is a commutative diagram

(87)
$$\begin{tikzcd}
K_1(A_F^{op}) \arrow[r, "\psi^*"] \arrow[d, "\mathrm{nr}_{A_F^{op}}"'] & K_1(A_F) \arrow[d, "\mathrm{nr}_{A_F}"'] \\
\zeta(A_F)^\times \arrow[r, "-1"] & \zeta(A_F)^\times.
\end{tikzcd}$$

This is a consequence of the fact that if $V \in \mathrm{Ob}(\mathrm{PMod}(A_F^{op}))$ and $\phi \in \mathrm{Aut}_{A_F^{op}}(V)$, then $\psi^*(\phi) = \mathrm{Hom}_F(\phi,F)^{-1}$.
The claimed equality thus follows from the definition of $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ in terms of $\delta^1_{\mathfrak{A},\mathbb{R}}$ and $\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}$ by using the commutativity of (86) and (87) (cf. the proof of Lemma 10). □

Theorem 5.3. *Assume that Conjectures 1 and 2 and the Coherence hypothesis are valid for both* (M,A) *and* $(M^*(1),A^{op})$*, and also that Conjecture 7 is valid for* (M,A)*. Let* $\mathfrak{A}$ *be an order in* A *for which* M *has a projective* $\mathfrak{A}$*-structure. Then* $M^*(1)$ *has a projective* $\mathfrak{A}^{op}$*-structure and there is an equality*

(88) $$T\Omega(M,\mathfrak{A}) + \psi^*(T\Omega(M^*(1),\mathfrak{A}^{op})) = T\Omega^{loc}(M,\mathfrak{A})$$

in $K_0(\mathfrak{A},\mathbb{R})$.

Corollary 1. *Assume that Conjecture 4 is valid for the pair* $(M, \mathfrak{A})$. *Then Conjecture 4(iii), resp. 4(iv), is valid for the pair* $(M^*(1), \mathfrak{A}^{op})$ *if and only if* $T\Omega^{loc}(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$, *resp.* $T\Omega^{loc}(M, \mathfrak{A}) = 0$.

Proof. This follows as an immediate consequence of Theorem 5.3 and the fact that ψ^* restricts to give an isomorphism $\mathrm{Cl}(\mathfrak{A}^{op}, \mathbb{Q}) \cong \mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$. □

Remark 17. If $\mathfrak{A}$ is commutative, then $\mathfrak{A} = \mathfrak{A}^{op}$ and ψ^* coincides with multiplication by -1 on $K_0(\mathfrak{A}, \mathbb{R})$ and so (88) simplifies to give an equality

$$T\Omega(M, \mathfrak{A}) - T\Omega(M^*(1), \mathfrak{A}) = T\Omega^{loc}(M, \mathfrak{A}).$$

To justify the claim that $\psi^* = -1$ whenever $\mathfrak{A}$ is commutative we first recall that, as a consequence of Propositions 2.5 and 2.4, all elements in $K_0(\mathfrak{A}, \mathbb{R})$ can be represented by pairs $((L, \alpha), g)$ where (L, α) is a graded invertible $\mathfrak{A}$-module and $g : A_{\mathbb{R}} \otimes_{\mathfrak{A}} (L, \alpha) \cong \mathbf{1}_{A_{\mathbb{R}}} = (A_{\mathbb{R}}, 0)$ is an isomorphism in $\mathcal{P}(A_{\mathbb{R}})$. Since the image of $\mathrm{Spec}(A_{\mathbb{R}})$ is dense in $\mathrm{Spec}(\mathfrak{A})$, this implies that $\alpha = 0$ and also that $g : L \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} A_{\mathbb{R}}$ is an isomorphism of ordinary line bundles. Then $\psi^*(L, 0) = (L^*, 0)$ and

$$\psi^*((L, 0)_{\mathbb{R}} \xrightarrow{g} (A_{\mathbb{R}}, 0)) = ((L^*, 0)_{\mathbb{R}} \xrightarrow{(g^*)^{-1}} (\mathfrak{A}^*_{\mathbb{R}}, 0) \cong (\mathfrak{A}_{\mathbb{R}}, 0))$$

where this last isomorphism sends the identity map in $\mathfrak{A}^* = \mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A})$ to the identity element of $\mathfrak{A}$. Now since $(L \otimes_{\mathfrak{A}} L^*)_{\mathbb{R}} \xrightarrow{g \otimes (g^*)^{-1}} \mathfrak{A}_{\mathbb{R}} \otimes \mathfrak{A}_{\mathbb{R}} = \mathfrak{A}_{\mathbb{R}}$ is isomorphic to $\mathfrak{A}_{\mathbb{R}} \xrightarrow{\mathrm{id}} \mathfrak{A}_{\mathbb{R}}$ via the evaluation map $L \otimes_{\mathfrak{A}} L^* \to \mathfrak{A}$, it follows that $\psi^*((L, 0), g)$ does indeed represent the inverse of $((L, 0), g)$ in $K_0(\mathfrak{A}, \mathbb{R})$.

Proof of Theorem 5.3. By applying the (monoidal) functor $(-)^*$ to the object $\Xi(M^*(1))$ of $V(A^{op})$ (as defined in (29)) one finds that there is an isomorphism in $V(A)$

$$\begin{aligned}\Xi(M^*(1))^* \cong\ & [H^0_f(K, M^*(1))^*] \boxtimes [H^1_f(K, M^*(1))^*]^{-1} \\ & \boxtimes [H^1_f(K, M)] \boxtimes [H^0_f(K, M)]^{-1} \\ & \boxtimes \left(\boxtimes_{v \in S_\infty} [H_v(M^*(1))^{G_v,*}]^{-1} \right) \boxtimes [(H_{dR}(M^*(1))/F^0)^*]\end{aligned}$$

and hence also an isomorphism

$$\begin{aligned}(89)\quad \Xi(M) \boxtimes \Xi(M^*(1))^* \cong\ & [\bigoplus_{v \in S_\infty} H_v(M)^{G_v}]^{-1} \boxtimes [(H_{dR}(M)/F^0)] \\ & \boxtimes [\bigoplus_{v \in S_\infty} H_v(M^*(1))^{G_v,*}]^{-1} \boxtimes [(H_{dR}(M^*(1))/F^0)^*].\end{aligned}$$

We now observe that

$$(90)\qquad \bigoplus_{v \in S_\infty} H_v(M)^{G_v} = H_B(M)^+$$

and that there are natural A-equivariant isomorphisms

$$H_B(M^*(1))^{+*} \cong H_B(M^*(1))^{*+} \cong H_B(M)^-(-1) \cong H_B(M)^- \tag{91}$$

where the last map is induced by sending each element $y \otimes (2\pi i)^{-1}$ to y. After applying $[\quad]$ to (90), to the linear dual of (90) for $M^*(1)$, to (91), to the natural isomorphism

$$H_B(M) \cong H_B(M)^+ \oplus H_B(M)^- \tag{92}$$

and to the Poincaré duality sequence (69), the right hand side of (89) identifies with $\Xi^{loc}(M)$, and hence one obtains an isomorphism of virtual A-modules

$$\vartheta^{PD} : \Xi(M) \boxtimes \Xi(M^*(1))^* \cong \Xi^{loc}(M).$$

Lemma 17. *a)* $\vartheta_\infty^{loc} \circ (A_{\mathbb{R}} \otimes_A \vartheta^{PD}) = \vartheta_\infty(M) \boxtimes \vartheta_\infty(M^*(1))^*$.
b) For each projective $\mathfrak{A}$-structure T on M and each prime p there is a commutative diagram in $V(A_p)$

$$\begin{array}{ccc}
A_p \underset{A}{\otimes} (\Xi(M) \boxtimes \Xi(M^*(1))^*) & \xrightarrow{A_p \underset{A}{\otimes} \vartheta^{PD}} & A_p \underset{A}{\otimes} \Xi^{loc}(M) \\
{\scriptstyle \vartheta_p(M) \boxtimes \vartheta_p(M^*(1))^*} \downarrow & & \downarrow {\scriptstyle \vartheta_p^{loc}} \\
A_p \underset{\mathfrak{A}_p}{\otimes} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)] \boxtimes [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*] & \xrightarrow{A_p \underset{\mathfrak{A}_p}{\otimes} \vartheta_p^{AV}} & A_p \underset{\mathfrak{A}_p}{\otimes} \Lambda_p(S, T_p)
\end{array}$$

where

$$\vartheta_p^{AV} : [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)] \boxtimes [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*] \xrightarrow{\sim} \Lambda_p(S, T_p)$$

is the isomorphism in $V(\mathfrak{A}_p)$ which is induced by the definition (71) of $\Lambda_p(S, T_p)$.

We assume for the moment that this lemma is true. Then from claim b) we deduce that there is an isomorphism in $\mathbb{V}(\mathfrak{A})$

$$\vartheta_{\mathbb{Z}}^{PD} : \Xi(M)_{\mathbb{Z}} \boxtimes \Xi(M^*(1))_{\mathbb{Z}}^* \cong \Xi^{loc}(M)_{\mathbb{Z}}.$$

Taken in conjunction with the equality of claim a), this isomorphism in turn implies that there is an equality

$$R\Omega(M, \mathfrak{A}) + \psi^*(R\Omega(M^*(1), \mathfrak{A}^{op})) = R\Omega^{loc}(M, \mathfrak{A}) \tag{93}$$

in $K_0(\mathfrak{A}, \mathbb{R})$.
On the other hand, by taking leading coefficients at $s = 0$ in Conjecture 7 we find that

$$L^*({}_AM, 0) = \mathcal{E}({}_AM) L^*({}_{A^{op}}M^*(1), 0)$$

in $\zeta(A_{\mathbb{R}})^{\times}$. By applying $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ to this equality and then using Lemma 16 we obtain an equality

$$\begin{aligned} L(M,\mathfrak{A}) =& L^{loc}(M,\mathfrak{A}) + \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(L^*(M^*(1),0)) \\ =& L^{loc}(M,\mathfrak{A}) - \psi^*(L(M^*(1),\mathfrak{A}^{op})) \end{aligned}$$

in $K_0(\mathfrak{A},\mathbb{R})$. Upon comparing this equality to (93) we finally obtain the formula of Theorem 5.3.

It therefore only remains to prove Lemma 17, and our proof of this result will occupy the rest of this section. Before starting the proof however we introduce another useful convention. For any integer n the symbol $(n)^*$ refers to the equality, isomorphism or exact triangle which is obtained by applying the functor * to the displayed formula (n) with $M^*(1)$ in place of M and $V_p^*(1)$ in place of V_p; $(n)^+$ refers to the equality, isomorphism or exact triangle obtained by taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants of (n); $(n)_v$ indicates that the formula (n) is to be used for all places v in $S_{p,f}$ to which it applies.

Proof of Lemma 17. We begin with the proof of part a).

Lemma 18. *(cf.* [20]*[Prop. III.1.1.6 iii)]) With notation as in section 3.2 there are natural isomorphisms of $A_{\mathbb{R}}$-modules*

$$\ker(\alpha_M) \cong \mathrm{coker}(\alpha_{M^*(1)})^*, \quad \mathrm{coker}(\alpha_M) \cong \ker(\alpha_{M^*(1)})^*.$$

Proof. For any $\mathbb{Q}$-space W and field of characteristic zero F we set $W_F := W \otimes_F \mathbb{Q}$. There is a commutative diagram of $A_{\mathbb{R}} \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-modules

$$\begin{array}{ccc} H_B(M)_{\mathbb{R}} & \xrightarrow{\alpha_{M,\mathbb{C}}} & H_{dR}(M)_{\mathbb{C}}/F^0 \\ \uparrow & & \uparrow \\ F^0H_{dR}(M)_{\mathbb{C}} \oplus H_B(M)_{\mathbb{R}} & \longrightarrow & H_{dR}(M)_{\mathbb{C}} \\ \| & & \downarrow{\scriptstyle (53)} \\ F^0H_{dR}(M)_{\mathbb{C}} \oplus H_B(M)_{\mathbb{R}} & \longrightarrow & H_B(M)_{\mathbb{C}} \\ \downarrow & & \downarrow \\ F^0H_{dR}(M)_{\mathbb{C}} & \longrightarrow & H_B(M)_{\mathbb{C}}/H_B(M)_{\mathbb{R}} \\ {\scriptstyle \beta_1}\downarrow & & {\scriptstyle \beta_2}\downarrow \\ (H_{dR}(M^*(1))_{\mathbb{C}}/F^0)^* & \xrightarrow{\alpha^*_{M^*(1),\mathbb{C}}} & H_B(M^*(1))^*_{\mathbb{R}} \end{array} \tag{94}$$

where all arrows other than β_1, β_2 are natural projections, inclusions or sum maps, possibly combined with the comparison isomorphism (53). The maps β_1

and β_2 arise as follows. There is a perfect duality of $\mathbb{R}$-vector spaces

$$\begin{array}{ccccc} H_B(M)_{\mathbb{C}} \times H_B(M^*(1))_{\mathbb{C}} & \longrightarrow & H_B(\mathbb{Q}(1))_{\mathbb{C}} & & \\ \downarrow \cong & & \downarrow \cong & & \\ H_{dR}(M)_{\mathbb{C}} \times H_{dR}(M^*(1))_{\mathbb{C}} & \longrightarrow & H_{dR}(\mathbb{Q}(1))_{\mathbb{C}} & \xrightarrow{\tau} & \mathbb{R} \end{array}$$

where the vertical isomorphisms are given by (53) for M,$M^*(1)$ and $\mathbb{Q}(1)$, and τ is the $\mathbb{R}$-linear splitting of the inclusion

$$\mathbb{R} = H_{dR}(\mathbb{Q}(1))_{\mathbb{R}} \subset H_{dR}(\mathbb{Q}(1))_{\mathbb{C}} = \mathbb{C}$$

with kernel $H_B(\mathbb{Q}(1))_{\mathbb{R}} = 2\pi i \cdot \mathbb{R}$. One verifies that $H_B(M)_{\mathbb{R}}$ is the orthogonal complement of $H_B(M^*(1))_{\mathbb{R}}$ under this pairing, and it is also well known that $F^0 H_{dR}(M)$ is the orthogonal complement of $F^0 H_{dR}(M^*(1))$. Hence we obtain the isomorphisms β_i. Viewing the rows of (94) as complexes concentrated in degrees zero and one, an easy inspection shows that all rows are quasi-isomorphic. The same is then true for the diagram $(94)^+$ whose top (resp. bottom) row coincides with $R\Gamma_{\mathcal{D}}(K, M)$ (resp. $R\Gamma_{\mathcal{D}}(K, M^*(1))^*[-1]$). This proves the Lemma. $\square$

We shall next establish existence of the following commutative diagram in $V(A_{\mathbb{R}})$

(95)

$$\begin{array}{ccc} A_{\mathbb{R}} \otimes_A \left(\Xi(M) \boxtimes \Xi(M^*(1))^*\right) & \xrightarrow{A_{\mathbb{R}} \otimes_A \vartheta^{PD}} & A_{\mathbb{R}} \otimes_A \Xi^{loc}(M) \\ A_{\mathbb{R}} \otimes_A (89) \Big\downarrow A_{\mathbb{R}} \otimes_A (90) & & \Vert \\ \begin{array}{c} [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [(H_{dR}(M)/F^0)_{\mathbb{R}}] \boxtimes \\ [H_B(M^*(1))^{*,+}_{\mathbb{R}}]^{-1} \boxtimes [(H_{dR}(M^*(1))/F^0)^*_{\mathbb{R}}] \end{array} & \xrightarrow{\gamma_1} & [H_B(M)_{\mathbb{R}}]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\ \Big\downarrow [\beta_1],[\beta_2] & & (55) \Big\downarrow (56) \\ \begin{array}{c} [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [(H_{dR}(M)/F^0)_{\mathbb{R}}] \boxtimes \\ [(H_B(M)_{\mathbb{C}})^+ / H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [F^0 H_{dR}(M)_{\mathbb{R}}] \end{array} & \longrightarrow & [(H_B(M)_{\mathbb{C}})^+]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\ \beta_3 \Big\downarrow & & \epsilon_B \Big\downarrow \epsilon_{dR} \\ \begin{array}{c} [F^0 H_{dR}(M)_{\mathbb{R}}]^{-1} \boxtimes [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\ \boxtimes [F^0 H_{dR}(M)_{\mathbb{R}}] \boxtimes [H_B(M)^+_{\mathbb{R}}] \boxtimes [(H_B(M)_{\mathbb{C}})^+]^{-1} \end{array} & \xrightarrow{\gamma_2} & [(H_B(M)_{\mathbb{C}})^+]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\ \beta_4 \Big\downarrow & & (54) \Big\downarrow \\ \mathbf{1}_{V(A_{\mathbb{R}})} & = & \mathbf{1}_{V(A_{\mathbb{R}})}. \end{array}$$

The first square in (95) is commutative by the definition of ϑ^{PD} if we define γ_1 to be induced by equations $A_{\mathbb{R}} \otimes_A (91)$, $A_{\mathbb{R}} \otimes_A (92)$ and $A_{\mathbb{R}} \otimes_A (69)$. The second square in (95) is the ⊠-product of the two commutative squares

$$\begin{array}{ccc} [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes & & \\ [H_B(M^*(1))^{*,+}_{\mathbb{R}}]^{-1} & \xrightarrow{A_{\mathbb{R}}\otimes\{(91),(92)\}} & [H_B(M)_{\mathbb{R}}]^{-1} \\ \Big\downarrow [\beta_2] & & (55) \Big\downarrow (56) \\ [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes & & \\ [(H_B(M)_{\mathbb{C}})^+/H_B(M)^+_{\mathbb{R}}]^{-1} & \longrightarrow & [(H_B(M)_{\mathbb{C}})^+]^{-1} \end{array} \tag{96}$$

and

$$\begin{array}{ccc} [(H_{dR}(M)/F^0)_{\mathbb{R}}] \boxtimes & & \\ [(H_{dR}(M^*(1))/F^0)^*_{\mathbb{R}}] & \xrightarrow{A_{\mathbb{R}}\otimes_A(69)} & [H_{dR}(M)_{\mathbb{R}}] \\ \Big\downarrow [\beta_1] & & \| \\ [(H_{dR}(M)/F^0)_{\mathbb{R}}] \boxtimes & & \\ [F^0 H_{dR}(M)_{\mathbb{R}}] & \longrightarrow & [H_{dR}(M)_{\mathbb{R}}]. \end{array}$$

In both of those squares the bottom horizontal maps are induced by the obvious short exact sequences. The square (96) commutes since the identification $H_B(M)^-_{\mathbb{R}} \cong H_B(M)^- \otimes (2\pi i)^{-1}\mathbb{R}$ used in $A_{\mathbb{R}}\otimes_A(91)$ is inverse to that used in (56).

Concerning the third square in (95), the map β_3 is the ⊠-product of the isomorphism

$$[R\Gamma_{\mathcal{D}}(K,M)]^{-1} = [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [(H_{dR}(M)/F^0)_{\mathbb{R}}] \xrightarrow{\sim} \\ [F^0 H_{dR}(M)_{\mathbb{R}}]^{-1} \boxtimes [H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}]$$

induced by the quasi-isomorphism between the first and second row in $(94)^+$ and a similar isomorphism induced by the quasi-isomorphism between the fourth and third row in $(94)^+$. The map γ_2 is induced by canceling mutually inverse terms in the first and second row of its source term. Effectively then, the third square in (95) is the ⊠-product of the squares

$$\begin{array}{ccc} \mathbf{1}_{V(A_{\mathbb{R}})} \boxtimes & & \\ [F^0 H_{dR}(M)_{\mathbb{R}}] & \longrightarrow & [F^0 H_{dR}(M)_{\mathbb{R}}] \\ \Big\downarrow & & \epsilon_{dR} \Big\downarrow \\ [F^0 H_{dR}(M)_{\mathbb{R}}]^{-1} \boxtimes [F^0 H_{dR}(M)_{\mathbb{R}}] & & \\ [F^0 H_{dR}(M)_{\mathbb{R}}] & \longrightarrow & \mathbf{1}_{V(A_{\mathbb{R}})} \boxtimes [F^0 H_{dR}(M)_{\mathbb{R}}] \end{array} \tag{97}$$

and

$$
(98)\qquad
\begin{array}{ccc}
\begin{array}{c}[H_B(M)^+_{\mathbb{R}}]^{-1}\\ \boxtimes\, \mathbf{1}_{V(A_{\mathbb{R}})}\end{array} & \longrightarrow & [(H_B(M)^+_{\mathbb{R}}]^{-1}\\
\Big\downarrow & & \Big\downarrow{\scriptstyle \epsilon_B}\\
\begin{array}{c}[H_B(M)^+_{\mathbb{R}}]^{-1}\\ \boxtimes\,[H_B(M)^+_{\mathbb{R}}]\boxtimes[H_B(M)^+_{\mathbb{R}}]^{-1}\end{array} & \longrightarrow & \mathbf{1}_{V(A_{\mathbb{R}})}\boxtimes[(H_B(M)^+_{\mathbb{R}}]^{-1}
\end{array}
$$

with (the identity maps on) $[(H_{dR}(M)/F^0)_{\mathbb{R}}]$ and $[(H_B(M)_{\mathbb{C}})^+/H_B(M)^+_{\mathbb{R}}]^{-1}$. In both diagrams (97) and (98) the left and bottom arrows are respectively induced by the two different ways to parenthesize the lower left term, and we have written this term so that the positions of its factors roughly match with their position in (95). We refer to [17][(4.1.1)] for the commutativity of (97) and (98), with the particular correcting factors ϵ_{dR} and ϵ_B given in [17][(4.9)]. Finally, the map β_4 in (95) is induced by the quasi-isomorphism between the second and third row in $(94)^+$, and the commutativity of the bottom square in (95) simply follows from the identity $(53)^+ = (54)$.

We now observe that the right hand vertical map in (95) coincides with ϑ_∞^{loc} by definition, and that the left hand vertical map

$$
\begin{aligned}
&A_{\mathbb{R}}\otimes_A\big(\Xi(M)\boxtimes\Xi(M^*(1))^*\big)\\
&\qquad\xrightarrow{\sim}[R\Gamma_{\mathcal{D}}(K,M)]^{-1}\boxtimes[R\Gamma_{\mathcal{D}}(K,M^*(1))^*[-1]]\xrightarrow{(94)^+}\mathbf{1}_{V(A_{\mathbb{R}})}
\end{aligned}
$$

does indeed coincide with $\vartheta_\infty(M)\boxtimes\vartheta_\infty(M^*(1))^*$ (using the second condition in Conjecture 1). Lemma 17a) then follows from the commutativity of diagram (95).

We now consider claim b) of Lemma 17. To further shorten notation we henceforth write $R\Gamma_?$ for $R\Gamma_?(\mathcal{O}_{K,S_p},V_p)$, $R\Gamma_f$ for $R\Gamma_f(K,V_p)$, $\tilde R\Gamma_c$ for $\tilde R\Gamma_c(\mathcal{O}_{K,S_p},V_p)$ and $L_?(S)$ for $\bigoplus_{v\in S}R\Gamma_?(K_v,V_p)$. In addition we use $R\Gamma_?^*$ as an abbreviation for $R\Gamma_?(\mathcal{O}_{K,S_p},V_p^*(1))^*$, and also introduce similar abbreviations $R\Gamma_f^*,\tilde R\Gamma_c^*$ and $L_?(S)^*$.

We shall first establish the existence of the following commutative diagram in $V(A_p)$

(99)

$$\begin{array}{ccc}
A_p \otimes_A (\Xi(M)\boxtimes \Xi(M^*(1))^*) & \xrightarrow{A_p\otimes_A \vartheta^{PD}} & A_p\otimes_A \Xi^{loc}(M)\\
{\scriptstyle \alpha_1}\big\downarrow & & {\scriptstyle \theta_p}\big\downarrow\\
[R\Gamma_f]\boxtimes[L_f(S_{p,f})]^{-1}\boxtimes[L(S_\infty)]^{-1} & & \\
 & \xrightarrow{\beta_1} & [L(S_{p,f})]^{-1}\boxtimes[\,\mathrm{Ind}\,V_p]^{-1}\\
\boxtimes[R\Gamma_f^*]\boxtimes[L_f(S_{p,f})^*]^{-1}\boxtimes[L(S_\infty)^*]^{-1} & & =:\Lambda(S,V_p)\\
{\scriptstyle (26)_{\mathrm{bot}}\boxtimes}\big\downarrow{\scriptstyle (100)_{\mathrm{bot}}\circ[\alpha_2]} & & \|\\
[L_{/f}(S_{p,f})]^{-1}\boxtimes[R\Gamma]\boxtimes[L_f(S_{p,f})]^{-1}\boxtimes[L(S_\infty)]^{-1} & & \\
 & \xrightarrow{\beta_2} & [L(S_{p,f})]^{-1}\boxtimes[\,\mathrm{Ind}\,V_p]^{-1}\\
\boxtimes[L_f(S_{p,f})^*]\boxtimes[{}_1\tilde{R}\Gamma_c^*]\boxtimes[L_f(S_{p,f})^*]^{-1}\boxtimes[L(S_\infty)^*]^{-1} & & \\
{\scriptstyle (18)_{v\in S_p}}\big\downarrow & & {\scriptstyle (76)\,(77)}\big\downarrow{\scriptstyle \epsilon(S,p)}\\
[L(S_p)]^{-1}\boxtimes[R\Gamma]\boxtimes[{}_1\tilde{R}\Gamma_c^*[-4]]\boxtimes[L(S_\infty)^*]^{-1} & \xrightarrow{1\ \mathrm{AV}_{\mathrm{Triv}}} & [L(S_p)]^{-1}\boxtimes[L(S_\infty)^*]^{-1}\\
{\scriptstyle (26)_{\mathrm{hor}}\boxtimes}\big\downarrow{\scriptstyle (73)_{\mathrm{left}}} & & {\scriptstyle (73)_{\mathrm{vert}}}\big\downarrow{\scriptstyle \circ[\lambda]\circ[q]}\\
[{}_1R\Gamma_c]\boxtimes[{}_1R\Gamma_c^*[-4]] & \xrightarrow{(73)_{\mathrm{bot}}} & [{}_2C(K,V_p)]\\
{\scriptstyle \alpha_3}\big\downarrow & & {\scriptstyle (72)}\big\downarrow\\
[R\Gamma_c]\boxtimes[R\Gamma_c^*[-4]] & \xrightarrow{\vartheta_p^{\mathrm{AV}}} & A_p\otimes_{\mathfrak{A}_p}\Lambda_p(S,T_p).
\end{array}$$

The maps $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2$ which occur in this diagram will all be defined in the course of our proof that the diagram is commutative.
Concerning the first square, we recall that the construction of ϑ^{PD} involves the set of equations

$$\vartheta^{PD}:\quad (29)\quad (29)^*\quad \mathrm{AV}^{\mathrm{mot}}\quad (69)\quad (91)\quad (92)$$

where here we denote by $\mathrm{AV}^{\mathrm{mot}}$ the collection of four isomorphisms

$$[H^i_f(K,M)]^{-1}\boxtimes[H^i_f(K,M)]\cong\mathbf{1},\ [H^i_f(K,M^*(1))^*]^{-1}\boxtimes[H^i_f(K,M^*(1))^*]\cong\mathbf{1}$$

with $i\in\{0,1\}$. The construction of θ_p involves the following equations (with v running through the set $S_{p,f}$):

$$\begin{array}{llllllll}
\theta_p: & (68) & (69) & (18)_v & \mathrm{AV}_{f,v} & (19)_{v\nmid p} & (22)_{v\mid p} & (23) \quad (24)_v\\
 & & & & & (19)^*_{v\nmid p} & (22)^*_{v\mid p} & (23)^* \quad (24)^*_v
\end{array}$$

Note now that when listing all of the equations that are involved in the composition $\kappa:=\theta_p\circ(A_p\otimes_A\vartheta^{PD})$, the equations (91) and (92) are transformed into (76) and (77) respectively as they are 'conjugated' by (68), and that (68) is in turn equivalent to a combination of $(28)^+$ and $(28)^{*,+}$ for $v\in S_\infty$. The isomorphism $\mathrm{AV}^{\mathrm{mot}}$ is a combination of AV_f, (27) and $(27)^*$. Finally note that the isomorphism induced by (69) is used in ϑ^{PD} whilst its inverse is used in

θ_p, so that (69) does not in fact occur in the composition κ. In summary, we find that the isomorphism κ involves the sets of equations

$$\begin{array}{llllllll} \alpha_1: & (27) & (28)_{v|\infty} & (29) & (19)_{v\nmid p} & (22)_{v|p} & (23) & (24)_v \\ & (27)^* & (28)^*_{v|\infty} & (29)^* & (19)^*_{v\nmid p} & (22)^*_{v|p} & (23)^* & (24)^*_v \end{array}$$

and

$$\beta_1: \quad (18)_v \quad \mathrm{AV}_f \quad \mathrm{AV}_{f,v} \quad (76) \quad (77).$$

If we then define α_1 (resp. β_1) as the isomorphism induced by the set of equations carrying the label α_1 (resp. β_1) we have $\kappa = \beta_1 \circ \alpha_1$, i.e. commutativity of the first square in (99).

In order to consider the second square in (99) we first define the map AV_f.

Lemma 19. *There exists a commutative diagram of true triangles*

(100)
$$\begin{array}{ccccc} \bigoplus_{v\in S_{p,f}} R\Gamma_{/f}(K_v,V_p)[-1] & \longrightarrow & R\Gamma_f(K,V_p) & \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p},V_p) \\ \oplus\,\mathrm{AV}_{f,v}[-1]\big\downarrow & & \mathrm{AV}_f\big\downarrow & & {}_1\mathrm{AV}\big\downarrow \\ \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,V_p^*(1))^*[-3] & \longrightarrow & {}_1R\Gamma_f(K,V_p^*(1))^*[-3] & \longrightarrow & {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-3], \end{array}$$

in which the upper row coincides with $(26)_{\mathrm{bot}}$*, all of the vertical maps are quasi-isomorphisms, and there exists a natural quasi-isomorphism*

$$R\Gamma_f(K,V_p^*(1))^*[-3] \xrightarrow{\alpha_2} {}_1R\Gamma_f(K,V_p^*(1))^*[-3].$$

Proof. In view of Lemma 12a) it suffices to show that the mapping cone of the lower composite map in Lemma 12a) is naturally quasi-isomorphic to $R\Gamma_f(K,V_p^*(1))^*[-2]$. Indeed, if this is true, then (100) is simply induced by taking the mapping cones of the composite horizontal maps in the diagram of Lemma 12a).

We observe that there is a map from diagram (25) into the diagram

$$R\Gamma(\mathcal{O}_{K,S_p},V_p) \to \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V_p) \hookleftarrow \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,V_p) \qquad (101)$$

and hence an induced map of the true nine term diagram (26) into the true nine term diagram that is induced by (101). In particular on the central columns we obtain a map which coincides with the map between the second and third column in the following true nine term diagram

$$
\begin{array}{ccccc}
\bigoplus_{v\in S_\infty} R\Gamma(K_v,V_p)[-1] & \longrightarrow & \bigoplus_{v\in S_p} R\Gamma_f(K_v,V_p)[-1] & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,V_p)[-1] \\
\| & & \downarrow & & \downarrow \\
\bigoplus_{v\in S_\infty} R\Gamma(K_v,V_p)[-1] & \longrightarrow & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) & \longrightarrow & {}_1\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},V_p) \\
 & & \downarrow & & \downarrow \\
 & & R\Gamma_f(K,V_p) & = & R\Gamma_f(K,V_p).
\end{array}
\tag{102}
$$

Upon replacing V_p by $V_p^*(1)$, dualizing the third column of this diagram and shifting by [-4] we obtain a true triangle

$$
R\Gamma_f^*[-4] \to {}_1\tilde{R\Gamma}_c^*[-4] \to L_f(S_{p,f})^*[-3]. \tag{103}
$$

The map α_2 then arises as the map q of Lemma 13 when the latter is applied to the triangle (103). □

The diagram (100) induces a commutative diagram

$$
\begin{array}{ccc}
[R\Gamma_f]\boxtimes[{}_1R\Gamma_f^*] & \xrightarrow{[\,\mathrm{AV}_f\,]_{\mathrm{Triv}}} & \mathbf{1}_{V(A_p)} \\
{\scriptstyle (26)_{\mathrm{bot}}\boxtimes}\downarrow{\scriptstyle (100)_{\mathrm{bot}}} & & \| \\
\begin{array}{c}[L_{/f}(S_{p,f})]^{-1}\boxtimes[R\Gamma] \\ \boxtimes[L_f(S_{p,f})^*]\boxtimes[{}_1\tilde{R\Gamma}_c^*]\end{array} & \xrightarrow{[\oplus \mathrm{AV}_{f,v}\,]_{\mathrm{Triv}}\boxtimes[{}_1\,\mathrm{AV}\,]_{\mathrm{Triv}}} & \mathbf{1}_{V(A_p)}.
\end{array}
\tag{104}
$$

The second square in (99) is obtained by taking the $\boxtimes$-product of the rows in (104) with the isomorphism

$$
\begin{array}{c}[L_f(S_{p,f})]^{-1}\boxtimes[L(S_\infty)]^{-1} \\ \boxtimes[L_f(S_{p,f})^*]^{-1}\boxtimes[L(S_\infty)^*]^{-1}\end{array} \xrightarrow{\beta_2'} [L(S_{p,f})]^{-1}\boxtimes[\,\mathrm{Ind}\,V_p]^{-1},
$$

where here β_2' involves the equations

$$
\beta_2': \quad (18)_v \quad \mathrm{AV}_{f,v} \quad (76) \quad (77).
$$

In order to establish the third square in (99) we begin with the commutative diagram

$$
\begin{array}{ccc}
[L(S_\infty)]^{-1}\boxtimes[L(S_\infty)^*]^{-1} & \xrightarrow{(76)\,(77)} & [\,\mathrm{Ind}\,V_p] \\
\downarrow & & {\scriptstyle (76)\,(77)}\downarrow \\
[L(S_\infty)]^{-1}\boxtimes[L(S_\infty)^*]^{-1} & = & [L(S_\infty)]^{-1}\boxtimes[L(S_\infty)^*]^{-1}.
\end{array}
\tag{105}
$$

Next we consider the diagram

$$(106)\qquad \begin{tikzcd} \begin{matrix}[L_{/f}(S_{p,f})]^{-1}\boxtimes[L_f(S_{p,f})]^{-1}\\ \boxtimes[L_f(S_{p,f})^*]\boxtimes[L_f(S_{p,f})^*]^{-1}\end{matrix} \arrow[r, "\beta_2''"] \arrow[d, "{[\alpha_2]}"'] & {[L(S_{p,f})]^{-1}} \arrow[d, "\epsilon(S,p)"] \\ {[L(S_{p,f})]^{-1}} \arrow[r, equal] & {[L(S_{p,f})]^{-1}.} \end{tikzcd}$$

Here β_2'' is induced by applying $\mathrm{AV}_{f,v}$, $(18)_v$ to the two right hand terms in the top left item and by trivializing the two left hand terms via another application of $\mathrm{AV}_{f,v}$. For the map α_2 on the other hand one applies $(18)_v$ to the upper two terms and trivializes the lower two terms. The commutativity of (106) then follows from [17, (4.1.1)] where the particular correcting factor $\epsilon(S,p)$ is as computed in [loc. cit., 4.9]. Indeed, upon writing diagram [loc. cit., (4.1.1)] with $X=[L_{/f}(S_{p,f})]$, taking $\boxtimes$-product with $[L_f(S_{p,f})]^{-1}$ and using $\mathrm{AV}_{f,v}$ twice we obtain (106). The third square in (99) is obtained by taking the $\boxtimes$-product of the diagrams (105) and (106) and then taking the $\boxtimes$-product of both rows of the resulting diagram with the isomorphism

$$[R\Gamma]\boxtimes[{}_1\tilde{R}\Gamma_c^*]\xrightarrow{{}_1\,\mathrm{AV}_{\mathrm{Triv}}}\mathbf{1}_{V(A_p)}.$$

We now consider the fourth square in (99). We observe first that there is a commutative diagram in $V(A_p)$

$$\begin{tikzcd} {[L(S_p)]^{-1}\boxtimes[R\Gamma]\boxtimes[R\Gamma]^{-1}\boxtimes[L(S_\infty)^*]^{-1}} \arrow[r, "\tau"] \arrow[d, "(26)_{\mathrm{hor}}"'] & {[L(S_p)]^{-1}\boxtimes[L(S_\infty)^*]^{-1}} \arrow[d, "{[q]}"'] \\ {[{}_1R\Gamma_c]\boxtimes[R\Gamma]^{-1}\boxtimes[L(S_\infty)^*]^{-1}} \arrow[r, "(74)_{\mathrm{top}}"] \arrow[d, "{[{}_1\,\mathrm{AV}]}"'] & {[{}_1L(S_p)]^{-1}\boxtimes[L(S_\infty)^*]^{-1}} \arrow[d, "{[\lambda]}"'] \\ {[{}_1R\Gamma_c]\boxtimes[{}_1\tilde{R}\Gamma_c^*[-4]]\boxtimes[L(S_\infty)^*]^{-1}} \arrow[r, "(73)_{\mathrm{hor}}"] \arrow[d, "(73)_{\mathrm{left}}"] & {[{}_2L(S_p)]^{-1}\boxtimes[L(S_\infty)^*]^{-1}} \arrow[d, "(73)_{\mathrm{vert}}"'] \\ {[{}_1R\Gamma_c]\boxtimes[{}_1R\Gamma_c^*[-4]]} \arrow[r, "(73)_{\mathrm{bot}}"] & {[{}_2C(K,V_p)]} \end{tikzcd}$$

in which the lower square is induced by the true nine term diagram (73), the central square by (74) and the upper square by Lemma 13. The map τ is the canonical isomorphism $[R\Gamma]\boxtimes[R\Gamma]^{-1}\cong\mathbf{1}_{V(A_p)}$. Upon replacing both occurrences of $[R\Gamma]^{-1}$ in this diagram by $[{}_1\tilde{R}\Gamma_c^*[-4]]$ and τ by ${}_1\,\mathrm{AV}_{\mathrm{Triv}}$, the upper square is still commutative and the resulting total square gives the fourth square in (99).

Finally, in the bottom square in (99) the map α_3 is defined by replacing complexes by their versions indexed by 1 in exactly the same order as in (72). The commutativity of this square is then clear.

It is clear that the right vertical map of (99) is equal to ϑ_p^{loc}. Hence, having now established the commutativity of (99), we shall prove Lemma 17b) if we can show that the left vertical map in (99) is equal to $\vartheta_p(M) \boxtimes \vartheta_p(M^*(1))^*$. In view of the definition of α_1 it therefore suffices for us to show that

$$(26)_{\mathrm{vert}} = (26)_{\mathrm{bot}} \circ (18)_{v \in S_p} \circ (26)_{\mathrm{hor}} \tag{107}$$

and also

$$(26)^*_{\mathrm{vert}} = [\alpha_2] \circ (100)_{\mathrm{bot}} \circ (73)_{\mathrm{left}}. \tag{108}$$

The identity (107) coincides with the identity in $V(A_p)$ induced by the true nine term diagram (26) since $(26)_{\mathrm{left}}$ coincides with the sum of (18) over $v \in S_p$. On the other hand, the identity (108) is a consequence of the following commutative diagram

$$\begin{tikzcd}
[{}_1R\Gamma^*_c] \arrow[r, "(26)^*_{\mathrm{vert}}"] \arrow[d, "(73)_{\mathrm{left}}"'] & {[R\Gamma^*_f] \boxtimes [L_f(S_p)^*]^{-1}} \arrow[d, equal] \\
{[{}_1\tilde{R\Gamma}^*_c] \boxtimes [L(S_\infty)^*]^{-1}} \arrow[r, "(103)"] \arrow[d, "(100)_{\mathrm{bot}}"'] & {[R\Gamma^*_f] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1}} \arrow[d, equal] \\
{[{}_1R\Gamma^*_f] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1}} \arrow[r, "{[\alpha_2]}"] & {[R\Gamma^*_f] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1}}
\end{tikzcd}$$

where here the first square is induced by the dual of diagram (102) with V_p replaced by $V_p^*(1)$ and the second square results from applying Lemma 13 to (103) and then taking the $\boxtimes$-product of each vertex of the resulting square with $[L_f(S_{p,f})^*]^{-1}$. $\square$

In view of Corollary 1 we are naturally led to make the following

CONJECTURE 8. $T\Omega^{loc}(M, \mathfrak{A}) = 0$.

This conjecture is itself of some independent interest, and will be considered in greater detail elsewhere. We therefore restrict ourselves here to a few brief remarks concerning the Galois case. We fix a finite Galois extension of number fields L/K and set $G := \mathrm{Gal}(L/K)$.

Remark 18. In this remark we assume that G is abelian. If M is any motive which is defined over K, then Conjecture 8 for the pair $(h^0(\mathrm{Spec}\,(L)) \otimes M, \mathbb{Z}[G])$ can be interpreted in terms of the 'local epsilon conjecture' formulated by Kato in [29]. In particular, [loc. cit., Th. 4.1] can be used to verify (at least modulo the 'sign ambiguities' discussed in Remark 9 of §4.3) that if $K = \mathbb{Q}$, then for all integers r Conjecture 8 is valid for the pair $(h^0(\mathrm{Spec}\,(L))(r), \mathbb{Z}[\frac{1}{2}][G])$. (Details of this deduction will be given elsewhere.) When combined with Corollary 1 and the main result of [12] (cf. Remark 10 in §4.3) this implies (again modulo the same sign ambiguities as above) that $T\Omega(h^0(\mathrm{Spec}\,(L))(r), \mathbb{Z}[\frac{1}{2}][G]) = 0$ for all integers r.

Remark 19. In [7] it is shown that (for any G)

$$\delta^0_{\mathbb{Z}[G],\mathbb{R}}(T\Omega^{loc}(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}[G])) = \Omega(L/K, 2) - w(L/K)$$

where here $\Omega(L/K, 2)$ and $w(L/K)$ are respectively equal to the 'second Chinburg invariant' and the 'Cassou-Noguès-Fröhlich root number class' as defined in [14]. This implies that Conjecture 8 is compatible with the conjectures formulated by Chinburg in loc. cit. For a further discussion of connections between Theorem 5.3 and the extensive existing theory concerning the conjectures of Chinburg, the reader can consult [11].

References

[1] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963) 8-28.

[2] H. Bass, Algebraic K-theory, Benjamin, New York, 1968.

[3] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, In: Analyse et topologie sur les éspaces singuliers, Asterisque 100, Soc. Math. France, 1982.

[4] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, In: 'The Grothendieck Festschrift' vol. 1, Progress in Math. 86, Birkhäuser, Boston, (1990) 333-400.

[5] D. Burns, Iwasawa theory and p-adic Hodge theory over non-commutative algebras I, preprint, 1997.

[6] D. Burns, Equivariant Tamagawa Numbers and Galois module theory I, Compositio Math. 129 (2001) 203-237.

[7] D. Burns, Equivariant Tamagawa Numbers and Galois module theory II, preprint, 1998.

[8] D. Burns, M. Flach, Motivic L-functions and Galois module structures, Math. Ann. 305 (1996) 65-102.

[9] D. Burns, M. Flach, On Galois structure invariants associated to Tate motives, Amer. J. Math. 120 (1998) 1343-1397.

[10] D. Burns, M. Flach, Equivariant Tamagawa numbers of motives, preprint 1999.

[11] D. Burns, M. Flach, Tamagawa numbers for motives with (non-commutative) coefficients II, preprint 2001.

[12] D. Burns, C. Greither, On the Equivariant Tamagawa Number Conjecture for Tate motives, preprint 2000.

[13] T. Chinburg, On the Galois structure of algebraic integers and S-units, Invent. math. 74 (1983) 321-349.

[14] T.Chinburg, Exact sequences and Galois module structure, Annals of Math. 121 (1985) 351-376.

[15] C. W. Curtis, I. Reiner, Methods of Representation Theory, Vol. I and II, John Wiley and Sons, New York, 1987.

[16] P. Deligne, Valeurs de fonctions L et périods d'intégrales, Proc. Sym. Pure Math. 33 (2), (1979) 313-346.

[17] P. Deligne, Le déterminant de la cohomologie, in: Current Trends in Arithmetical Algebraic Geometry, Cont. Math. 67, Amer. Math. Soc., 313-346, 1987.

[18] M. Flach, Euler characteristics in relative K-groups, Bull. London Math. Soc. 32 (2000) 272-284.

[19] J.-M. Fontaine, Valeurs spéciales des fonctions L des motifs, Séminaire Bourbaki, exposé 751, Feb. 1992.

[20] J.-M. Fontaine, B. Perrin-Riou, Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L, In: Motives (Seattle) Proc. Symp. Pure Math. 55, I, (1994) 599-706.

[21] B. H. Gross, Arithmetic on Elliptic Curves with Complex Multiplication, Lecture Notes in Math. 776, Springer 1980

[22] B. H. Gross, On the conjecture of Birch and Swinnerton-Dyer for elliptic curves with complex multiplication, In: Number Theory Related to Fermat's Last Theorem. Prog. Math. vol. 26, Birkhäuser, Boston (1982) 219-236.

[23] A. Grothendieck, J. Dieudonneé, Eléments de Géométrie Algébrique, Étude locale des schemas et des morphismes de schémas (EGA IV), Publ. Math. I.H.E.S. 28 (1966).

[24] A. Grothendieck, P. Berthelot, L. Illusie, Théorie des Intersections et Théorème de Riemann-Roch (SGA 6) Lecture Notes in Math. 225, Springer, Heidelberg, (1971).

[25] A. Huber, G. Kings, Bloch-Kato Conjecture and Main Conjecture of Iwasawa Theory for Dirichlet Characters, preprint 2000.

[26] U. Jannsen, Mixed motives and algebraic K-theory, Lecture Notes in Math. 1400, Springer 1990.

[27] K. Kato, Iwasawa theory and p-adic Hodge theory, Kodai Math. J. 16 no 1 (1993) 1-31.

[28] K. Kato, Lectures on the approach to Iwasawa theory of Hasse-Weil L-functions via B_{dR}, Part I, In: Arithmetical Algebraic Geometry (ed. E.Ballico), Lecture Notes in Math. 1553 (1993) 50-163, Springer, New York, 1993.

[29] K. Kato, Lectures on the approach to Iwasawa theory of Hasse-Weil L-functions via B_{dR}, Part II, preprint, 1993.

[30] F. Knudsen, D. Mumford, The projectivity of the moduli space of stable curves I: Preliminaries on 'det' and 'Div', Math. Scand. 39 (1976) 19-55.

[31] S. MacLane, Categories for the Working Mathematician, Graduate Texts in Mathematics 5, 2nd ed., Springer, New York, 1998.

[32] S. MacLane, Natural associativity and commutativity, Rice Univ. Studies 49 (1963) 28-46.

[33] B. Mazur, Notes on étale cohomology of number fields, Ann. Sci. Ec. Norm. Sup. 6 (1973) 521-556.
[34] B. Mazur, Modular curves and the Eisenstein ideal, Publ. Math. IHES 47 (1977) 33-186.
[35] J. S. Milne, Étale Cohomology, Princeton Math. Series 17, Princeton University Press, 1980.
[36] J. S. Milne, Arithmetic duality theorems, Perspectives in Math. 1, Academic Press, 1986.
[37] B. Perrin-Riou, Fonctions L p-adiques des répresentations p-adique, Astérisque 229, Soc. Math. France (1995).
[38] D. Quillen, Higher algebraic K-theory I, in Algebraic K-theory I, Batelle conference 1972, Springer Lect. Notes in Math. 341 (1973) 85-147.
[39] R. Rapoport, P. Schneider, N. Schappacher (eds.), Beilinson's Conjectures on Special Values of L-functions, Perspectives in Math. 4, Academic Press, 1988.
[40] K. Rubin, The main conjectures of Iwasawa Theory for imaginary quadratic fields, Invent. Math. 103 (1991) 25-68.
[41] A. J. Scholl, Integral elements in K-theory and products of modular curves, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 467–489, NATO Sci. Ser. C. Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000
[42] J.-P. Serre, J. Tate, Good reduction of abelian varieties, Ann. of Math. 88 (1968) 492-517.
[43] V. Srinivas, Algebraic K-Theory, Progress in Math. 90, Birkhäuser, Boston, 1996.
[44] R. G. Swan, Algebraic K-theory, Lecture Note in Math. 76, Springer, 1968.
[45] J. Tate, Number Theoretic Background, Proc. Symp. Pure Math. 33 (Vol. 2) 3-26, Amer. Math. Soc. 1979.
[46] J. Tate, Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$ (notes par D.Bernardi et N. Schappacher), Progress in Math., 47, Birkhäuser, Boston, 1984.
[47] J. L. Verdier, Des Catégories Dérivées des Catégories Abéliennes, Asterisque 239, Soc. Math. France 1996.

D. Burns
Dept. of Math.
King's College London
Strand
London WC2R 2LS
England
david.burns@kcl.ac.uk

M. Flach
Dept. of Math.
Caltech
Pasadena CA 91125, USA
flach@its.caltech.edu