Divisible Subgroups of Brauer Groups and Trace Forms of Central Simple Algebras

Grégory Berhuy, David B. Leep

Received: October 31, 2001

Communicated by Alexander S. Merkurjev

Abstract.

Let F be a field of characteristic different from 2 and assume that F satisfies the strong approximation theorem on orderings (F is a SAP field) and that $I^3(F)$ is torsion-free. We prove that the 2-primary component of the torsion subgroup of the Brauer group of F is a divisible group and we prove a structure theorem on the 2-primary component of the Brauer group of F. This result generalizes well-known results for algebraic number fields. We apply these results to characterize the trace form of a central simple algebra over such a field in terms of its determinant and signatures.

2000 Mathematics Subject Classification: 16K50, 11E81, 11E04
Keywords and Phrases: Central Simple Algebras, Trace Forms, Brauer Groups

1 Introduction and Preliminaries

Let A be a central simple algebra over a field F of characteristic different from 2. The quadratic form $q : A \to F$ given by $x \mapsto \mathrm{Trd}_A(x^2) \in F$ is called *the trace form of* A, and is denoted by $\mathcal{T}_A$. This trace form has been studied by many authors (cf.[Le], [LM], [Ti] and [Se], Annexe §5 for example). In particular, its classical invariants are well-known (*loc.cit.*).
In this article, we prove some divisibility results for the Brauer group of fields F under the assumption that F satisfies the strong approximation theorem on orderings (F is a SAP field) and $I^3(F)$ is torsion-free. Then we apply these results to characterize the trace form of a central simple algebra over such a field in terms of its determinant and signatures.

First we review the necessary background for this article. For a field F, $\mathrm{Br}(F)$ denotes the Brauer group of F. If p is a prime number, ${}_p\mathrm{Br}(F)$ denotes the p-primary component of $\mathrm{Br}(F)$. If $n \geq 1$, $\mathrm{Br}_n(F)$ denotes the kernel of multiplication by n in the Brauer group. If A is a central simple algebra over F, *the exponent of* A, denoted by $\exp A$, is the order of $[A]$ in $\mathrm{Br}(F)$ and *the index of* A, denoted by $\mathrm{ind}\, A$, is the degree of the division algebra which corresponds to A. We know that $\exp A$ divides $\mathrm{ind}\, A$. If $a, b \in F^\times$, we denote by $(a,b)_F$ the corresponding quaternion algebra, or simply (a,b) if no confusion is possible. We also use the same symbol to denote its class in the Brauer group.

We refer to [D], [J], or [Sc] for more information on central simple algebras over general fields.

In the following, all quadratic forms are nonsingular. If q is a quadratic form over F, $\dim q$ is the dimension of q and $\det q \in F^\times/F^{\times 2}$ is the determinant of q. We denote by $\mathbb{H}$ the hyperbolic plane.

If $q \simeq \langle a_1, \cdots, a_n\rangle$, the *Hasse-Witt invariant of* q is given by $w_2(q) = \sum\limits_{i<j}(a_i, a_j) \in \mathrm{Br}_2(F)$.

If $a_1, \cdots, a_n \in F^\times$, the quadratic form $\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ is called an *n-fold Pfister form.*

If F is a formally real field, the space of orderings of F is denoted by Ω_F. We let $\mathrm{sign}_v(q) \in \mathbb{Z}$ denote the signature of q relative to an ordering $v \in \Omega_F$. Thus $\mathrm{sign}_v(q)$ is the difference between the number of positive elements and the number of negative elements in any diagonalization of q.

If $n \geq 1$, $I^n(F)$ is the n^{th} power of the fundamental ideal of the Witt ring $W(F)$ of F. We denote by $I^n(F)_t$ the kernel of the map $I^n(F) \to \prod\limits_{v \in \Omega_F} I^n(F_v)$. We will say that $I^n(F)$ is torsion-free if $I^n(F)_t = 0$. A field F satisfies property A_n if every torsion n-fold Pfister form defined over F is hyperbolic over F. See [EL2], section 4, for more details on property A_n. The absolute stability index of F, denoted $st_a(F)$ is the smallest nonnegative integer n such that $I^{n+1}(F) = 2I^n(F)$ (or ∞, if no such integer exists). See [EP], p. 1248 for more details. The reduced stability index of F, denoted $st_r(F)$ is the smallest nonnegative integer n such that $I^{n+1}(F) \equiv 2I^n(F) \mod W(F)_t$. See [La2], Chapter 13, for more details.

A field F satisfies the strong approximation property (SAP) if for every clopen set X of Ω_F there exists $a \in F^\times$ such that $a >_v 0$ if $v \in X$ and $a <_v 0$ otherwise. See [La2] for various equivalent definitions and basic properties of SAP fields. If q is a quadratic form defined over F, then $\hat{q} \in C(\Omega_F, \mathbb{Z})$ is the continuous function $\hat{q} : \Omega_F \longrightarrow \mathbb{Z}$ defined by $\hat{q}(v) = \mathrm{sign}_v(q)$ for every $v \in \Omega_F$.

If M is a discrete torsion Galois-module of exponent m, prime to the characteristic of F , $H^n(F, M)$ denotes the n-th cohomology group $H^n(Gal(F^{sep}/F), M)$. The group $H^n(F, M)_t$ denotes the kernel of the map $H^n(F, M) \to \prod\limits_{v \in \Omega_F} H^n(F_v, M)$. If L/F is any field extension, $\mathrm{Res}_{L/F}$ denotes the restriction map. We then have $\mathrm{Res}_{L/F}(w_2(q)) = w_2(q_L)$ for any quadratic form q over F. If L/F is a finite Galois extension, $\mathrm{Cor}_{L/F}$ denotes the core-

striction map.
In this paper, we deal only with the case when n is even, because we know that $\mathcal{T}_A \simeq n < 1 > \perp \frac{n(n-1)}{2}\mathbb{H}$ when n is odd (cf.[Se], Annexe §5 for example).
An abelian group G is *divisible* if for all $n \geq 1$, we have $G = nG$. If J is any set, $G^{(J)}$ is the group of families of elements of G indexed by J, with finite supports.
In the following, F always denotes a field of characteristic different from 2, and $K = F(\sqrt{-1})$.
We now recall some results about the classical invariants of trace forms of central simple algebras:

Theorem 1.1. *Let A be a central simple algebra over F of degree n. Then we have:*

1. $\dim \mathcal{T}_A = n^2$
2. $\det \mathcal{T}_A = (-1)^{\frac{n(n-1)}{2}}$
3. *We have* $\mathrm{sign}_v\, \mathcal{T}_A = \pm n$ *for each* $v \in \Omega_F$, *and* $\mathrm{sign}_v\, \mathcal{T}_A = n$ *if and only if* $\mathrm{Res}_{F_v/F}([A]) = 0$, *where F_v is the real closure of (F, v).*
4. *If $n = 2m \geq 2$, then* $w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1,-1) + m[A]$

The three first statements can be found in [Le], and the last one is proved in [LM] or [Ti] for example.

2 Divisibility results in the Brauer group

Proposition 2.1. *Let* $\theta : I^3(F) \longrightarrow \prod_{v\in\Omega_F} I^3(F_v)/I^4(F_v)$. *If* $st_r(F) \leq 4$, *then* $\ker(\theta) = I^3(F)_t + I^4(F)$.

Proof. It is clear that $\ker(\theta) \supseteq I^3(F)_t + I^4(F)$. Now let $q \in I^3(F)$ and assume $q \in \ker(\theta)$. Then $q_v \in I^4(F_v)$ and this implies $16|\mathrm{sign}_v(q)$ for each $v \in \Omega_F$. Thus $\hat{q} \in C(\Omega_F, 16\mathbb{Z})$. Since $st_r(F) \leq 4$, Theorem 13.1 of [La2] applied to the preorder $T = \sum F^2$ implies there exists $q_0 \in I^4(F)$ such that $\hat{q} = \hat{q_0}$. Then $q - q_0 \in I^3(F) \cap W(F)_t = I^3(F)_t$ and hence $q \in I^3(F)_t + I^4(F)$. □

Corollary 2.2. *Let* $\bar{\theta} : I^3(F)/I^4(F) \longrightarrow \prod_{v\in\Omega_F} I^3(F_v)/I^4(F_v)$. *If* $I^3(F)_t = 0$ *and* $st_r(F) \leq 4$, *then $\bar{\theta}$ is injective and* $H^3(F, \mu_2)_t = 0$.

Proof. The hypothesis and Proposition 2.1 imply $\ker(\theta) = I^4(F)$. Therefore $\ker(\bar{\theta}) = (0)$ and $\bar{\theta}$ is injective. Since $I^3(F)/I^4(F) \simeq H^3(F, \mu_2)$, and $I^3(F_v)/I^4(F_v) \simeq H^3(F_v, \mu_2)$ by [MS1] and [MS2], it follows $H^3(F, \mu_2)_t = 0$. □

Proposition 2.3. *Assume that* $I^3(F)_t = 0$ *and* $st_r(F) \leq 4$. *Let* $\alpha \in H^2(F, \mu_{2^r})_t$ $(r \geq 1)$. *Then there exists* $\beta \in H^2(F, \mu_{2^{r+1}})$ *such that* $\alpha = 2\beta$.

Proof. The exact sequence

$$1 \to \mu_2 \to \mu_{2^{r+1}} \to \mu_{2^r} \to 1$$

(where the last map is squaring) induces the following commutative diagram with exact rows

$$\begin{array}{ccccc}
H^2(F,\mu_{2^{r+1}}) & \longrightarrow & H^2(F,\mu_{2^r}) & \longrightarrow & H^3(F,\mu_2) \\
\downarrow & & \downarrow & & \downarrow \\
\prod\limits_{v\in\Omega_F} H^2(F_v,\mu_{2^{r+1}}) & \longrightarrow & \prod\limits_{v\in\Omega_F} H^2(F_v,\mu_{2^r}) & \longrightarrow & \prod\limits_{v\in\Omega_F} H^3(F_v,\mu_2).
\end{array}$$

Since the third vertical map is injective by Corollary 2.2, a diagram chase gives the conclusion. □

In Theorem 2.7 below, we need a hypothesis that is slightly stronger than the one occurring in Proposition 2.3. The following result gives a characterization of this hypothesis.

Proposition 2.4. *Let* $K = F(\sqrt{-1})$. *The following statements are equivalent.*

1. *F satisfies property* A_3 *and* $st_a(F) \leq 2$.
2. $I^3(F)_t = 0$ *and* $st_r(F) \leq 2$.
3. $st_a(K) \leq 2$.
4. $I^3(K) = 0$.
5. $H^3(K, \mu_2) = 0$.

Proof. (4) $\iff$ (5): $I^3(K) = 0$ if and only if $I^3(K)/I^4(K) = 0$ by the Arason-Pfister Hauptsatz, and $I^3(K)/I^4(K) \simeq H^3(K, \mu_2)$ by [MS1] and [MS2].
(3) $\iff$ (4): $st_a(K) \leq 2$ means $I^3(K) = 2I^2(K)$ and this holds if and only if $I^3(K) = 0$, since $\langle 1, 1\rangle = 0$ implies $2I^2(K) = 0$.
(1) $\iff$ (3): This is [EP], Theorem 3.3.
(1) $\Longrightarrow$ (2): Property A_3 implies $I^3(F)_t = 0$, by [EL1], Theorem 3 and Corollary 3. It is clear that $st_a(F) \leq 2$ implies $st_r(F) \leq 2$, by [La2], Theorem 13.1(3).
(2) $\Longrightarrow$ (1): Clearly $I^3(F)_t = 0$ implies F satisfies property A_3. Let q be a 3-fold Pfister form defined over F. Then there exists $q' \in I^2(F)$ such that $q - 2q' \in I^3(F)_t = 0$. Thus $q = 2q'$ with $q' \in I^2(F)$ and it follows $I^3(F) = 2I^2(F)$. □

Proposition 2.5. *If* $st_r(F) \leq 2$, *then for every* $\beta \in H^2(F, \mu_{2^{r+1}})$, *there exists* $\beta' \in H^2(F, \mu_{2^{r+1}})_t$ *such that* $2\beta' = 2\beta$.

Proof. Since the characteristic of F is not 2, we have $H^2(F, \mu_{2^{r+1}}) \simeq \mathrm{Br}_{2^{r+1}}(F)$. Let A be a central simple algebra over F such that $\beta = [A]$, and set $X = \{v \in \Omega_F, \mathrm{sign}_v \mathcal{T}_A = n\}$, where $n = \deg A$. Then $X^c = \{v \in \Omega_F, \mathrm{sign}_v \mathcal{T}_A = -n\}$ by Theorem 1.1. Since the total signature map is continuous with respect to the topology on Ω_F, the set X is clopen. Since $st_r(F) \leq 2$ and X is clopen, there exists $q \in I^2(F)$ such that $\mathrm{sign}_v(q) = \begin{cases} 4, & \text{if } v \notin X \\ 0, & \text{if } v \in X. \end{cases}$ In the Witt ring WF we have $q = \sum_{i=1}^n \langle\langle a_i, b_i \rangle\rangle$, with $a_i, b_i \in F^\times$. Let B be a central simple algebra over F such that $[B] = \sum_{i=1}^n (a_i, b_i)_F$. Let $\gamma \in H^2(F, \mu_{2^{r+1}})$ be such that $\gamma = [B]$ under the isomorphism $H^2(F, \mu_{2^{r+1}}) \simeq \mathrm{Br}_{2^{r+1}}(F)$.
Now set $\beta' = \beta + \gamma$. We clearly have $2\beta' = 2\beta$. Moreover, if $v \in X$, then $\mathrm{Res}_{F_v/F}(\beta) = 0$ by Theorem 1.1 and $\mathrm{Res}_{F_v/F}(\gamma) = 0$ by the choice of B. Similar arguments show that $\mathrm{Res}_{F_v/F}(\beta') = 0$ for all $v \notin X$. It follows that $\beta' \in H^2(F, \mu_{2^{r+1}})_t$. $\square$

Remark 2.6. In Proposition 2.5, a stronger conclusion is possible if we also assume that F is a SAP field. This is equivalent to assuming $st_r(F) \leq 1$. (See [La2].) In this case there exists an element $a \in F^\times$ such that $a >_v 0$ if $v \in X$ and $a <_v 0$ if $v \notin X$. Let $\gamma \in H^2(F, \mu_{2^{r+1}})$ be such that $\gamma = (-1, a)_F$ under the isomorphism $H^2(F, \mu_{2^{r+1}}) \simeq \mathrm{Br}_{2^{r+1}}(F)$. Now set $\beta' = \beta + \gamma$. We clearly have $2\beta' = 2\beta$. We finish as before. This observation will be used in the proof of Theorem 2.8.

Theorem 2.7. *Assume $I^3(F)_t = 0$ and $st_r(F) \leq 2$. Then ${}_2\mathrm{Br}(F)_t$ is a divisible group.*

Proof. It suffices to check that for all $[B] \in {}_2\mathrm{Br}(F)_t$ and all primes p, there exists $[A] \in {}_2\mathrm{Br}(F)_t$ such that $p[A] = [B]$. Let $[B] \in {}_2\mathrm{Br}(F)_t$. Then, there exists $r \geq 1$ such that $2^r[B] = 0$. Assume first that p is odd. Then $\gcd(p, 2^r) = 1$, so there exist $n, m \in \mathbb{Z}$ such that $np + m2^r = 1$. Then $[B] = (np + m2^r)[B] = p(n[B])$. If $p = 2$, apply Proposition 2.3 and Proposition 2.5. $\square$

We now give a structure theorem on the 2-primary component of the Brauer group. We denote by $\sum F^2$ the multiplicative subgroup of $F^\times$ of nonzero sums of squares. We use the notation of [K].

Theorem 2.8. *Assume that $I^3(F)_t = 0$ and F is SAP. Let T (resp. Λ) be an index set of a $\mathbb{Z}/2\mathbb{Z}$-basis of $\mathrm{Br}_2(F)_t$ (resp. of $F^\times/\sum F^{\times 2}$). Then we have the following group isomorphism*

$$ {}_2\mathrm{Br}(F) \simeq \mathbb{Z}(2^\infty)^{(T)} \times (\mathbb{Z}/2\mathbb{Z})^{(\Lambda)}. $$

Proof. Theorem 2.7 implies that ${}_2\mathrm{Br}(F)_t$ is a divisible group. Since every element of ${}_2\mathrm{Br}(F)_t$ has 2-power order, the structure theorems on divisible groups (see [K] for example) imply that this group is isomorphic to $\mathbb{Z}(2^\infty)^{(T)}$, where T is an index set of a basis of the 2-torsion part of ${}_2\mathrm{Br}(F)_t$, namely $\mathrm{Br}_2(F)_t$.

Let $[A] \in {}_2\operatorname{Br}(F)$. Remark 2.6 shows that there exists $a \in F^\times$ such that $[A'] := [A] + (-1, a)$ is a torsion element. Choose elements $a_\lambda \in F^\times$ such that $(a_\lambda \sum F^{\times 2})_{\lambda\in\Lambda}$ is a $\mathbb{Z}/2\mathbb{Z}$-basis of $F^\times / \sum F^{\times 2}$. Then $a = b \prod\limits_{\lambda\in\Lambda} a_\lambda^{r_\lambda}$, where $b \in \sum F^2$ and $r_\lambda = 0$ or 1. Since b is a sum of squares, $(-1, b)$ is a torsion element, so we have a decomposition $[A] = [B] + \sum r_\lambda(-1, a_\lambda)$, where $[B] = [A'] + (-1, b)$ is a torsion element. Now we show that $[B]$ and the r_λ's are uniquely determined. Assume that $[B] + \sum r_\lambda(-1, a_\lambda) = 0$. Then $(-1, \prod a_\lambda^{r_\lambda}) = -[B]$ is a torsion element. This implies that $\prod a_\lambda^{r_\lambda}$ is positive at all orderings of F, so $\prod a_\lambda^{r_\lambda}$ is a sum of squares. By choice of the a_λ's, this implies that $r_\lambda = 0$ for all $\lambda \in \Lambda$ and hence that $[B] = 0$. □

3 Trace forms of central simple algebras

In this section, we give realization theorems for trace forms of central simple algebras.

Theorem 3.1. *Let $n = 2m \geq 2$ be an even integer. Assume that F is SAP and $I^2(F)$ is torsion-free. Then a quadratic form q is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions are satisfied :*

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$
3. $\operatorname{sign}_v q = \pm n$, *for all* $v \in \Omega_F$.

Proof. The necessity follows from Theorem 1.1. Conversely, let q be a quadratic form satisfying the conditions above. Since $I^2(F)$ is torsion-free, it is well-known that quadratic forms are classified by dimension, determinant and signatures (see [EL1]). Let $X = \{v \in \Omega_F, \operatorname{sign}_v q = n\}$. This is a clopen set, so the *SAP* property of F implies there exists $a \in F^\times$ such that $a >_v 0$ if $v \in X$ and $a <_v 0$ otherwise. Set $A = M_m((-1, a))$. Then $\operatorname{Res}_{F_v/F}([A]) = 0$ if and only if $\operatorname{sign}_v q = n$, so $\mathcal{T}_A$ and q have the same signatures. Since they also have equal dimension and determinant, they are isomorphic. □
The following proposition gives a characterization of fields that satisfy the hypotheses of Theorem 3.1. Note the similarity to Proposition 2.4.

Proposition 3.2. *Let $K = F(\sqrt{-1})$. The following statements are equivalent.*

1. *F satisfies property A_2 and F is a SAP field ($st_a(F) \leq 1$).*
2. *$I^2(F)_t = 0$ and F is a SAP field ($st_r(F) \leq 1$).*
3. *$st_a(K) \leq 1$.*
4. *$I^2(K) = 0$.*

5. $u(K) \leq 2$.

6. $\tilde{u}(F) \leq 2$.

7. $I^2(F)_t = 0$ *and* F *is linked.*

Proof. The proof of the equivalence of (1)-(4) is very similar to the proof of the equivalence of the corresponding statements in Proposition 2.4. The equivalence of (4) and (5) is well-known. The equivalence of (6) and (7) appears in [E]. The equivalence of (2) and (6) appears in [ELP]. □

We now give a characterization of fields F such that $I^2(F)$ is torsion-free in terms of Brauer groups.

Proposition 3.3. $I^2(F)$ *is torsion-free if and only if* $\mathrm{Br}(F)$ *has no element of order* 4.

Proof. Assume that $[A] \in \mathrm{Br}(F)$ has order 4, so $[A] \in H^2(F, \mu_4)$. Then $2[A] \in H^2(F, \mu_2)$ has order 2. Moreover, it is well-known that the image of $[A] \in H^2(F, \mu_4) \mapsto 2[A] \in H^2(F, \mu_2)$ is the kernel of
$[B] \in H^2(F, \mu_2) \mapsto (-1) \cup [B] \in H^3(F, \mu_2)$ (see for example [LLT], Proposition A2 and Remark A3). So $(-1) \cup 2[A] = 0$, that is $2[A] = \mathrm{Cor}_{K/F}([B])$ for some $[B] \in H^2(K, \mu_2)$. Since $H^2(K, \mu_2)$ is generated by elements of the form $(a, b), a \in F^\times, b \in K^\times$, the transfer formula shows that $2[A] = \sum (a_i, \mathrm{N}_{K/F}(b_i))$ for some $a_i \in F^\times$ and $b_i \in K^\times$. Since $2[A]$ has order 2, it is not split, so there exists i such that $(a_i, \mathrm{N}_{K/F}(b_i))$ is not split. Then the norm form of this quaternion algebra is not hyperbolic, and it is a torsion 2-fold Pfister form, since $\mathrm{N}_{K/F}(b_i)$ is the sum of 2 squares.
Conversely, assume that $I^2(F)$ is not torsion-free. Then property A_2 fails (see [EL2], section 4). Theorem 4.3(3) in [EL2] (with $x = 1$) implies that there exists a binary form $\langle 1, -a \rangle$ and an element $b = u^2 + v^2$, with $u, v \in F$, such that $\langle 1, -a \rangle$ does not represent b. This means $\langle\langle a, b \rangle\rangle$ is an anisotropic 2-fold Pfister form and b is not a square. Let $L := F(\sqrt{b + v\sqrt{b}})$. Then L/F is a cyclic quartic extension which contains $F(\sqrt{b})$. Denote by σ a generator of $Gal(L/F)$ and let A be the cyclic algebra $(a, L/F, \sigma)$ (see [Sc] for the definition and basic properties of cyclic algebras). It is not difficult to show that $2[A] = (a, b)$ (for example use [J], Corollary 2.13.20). By construction, the norm form of this quaternion algebra is not hyperbolic, so $2[A]$ is not split, and $[A]$ has order 4. □

We now apply the results of section 2 to prove the following theorem:

Theorem 3.4. *Let* $n = 2m \geq 2$ *be an even integer.*
Write $n = 2^{r+1}s, r \geq 0, s \geq 1$ *odd. Assume that* F *satisfies the following conditions:*

(a) $I^3(F)$ *is torsion-free*

(b) *For every* $[A] \in \mathrm{Br}(F)$ *such that* $2^{r+1}[A] = 0$, *there exists* A', $\deg A' = 2^{r+1}$ *such that* $[A'] = [A]$

(c) If $r \geq 1$, assume $st_r(F) \leq 2$.

Then a quadratic form q is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions are satisfied :

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$
3. $\mathrm{sign}_v\, q = \pm n$, *for all* $v \in \Omega_F$.

Before we begin the proof of this theorem, we need the following calculation.

LEMMA 3.5. *Let $n = 2m$, $m \geq 1$, and assume F is a real closed field. Let $q_+ = n\langle 1\rangle \perp \dfrac{n(n-1)}{2}\mathbb{H}$ and let $q_- = n\langle -1\rangle \perp \dfrac{n(n-1)}{2}\mathbb{H}$. Then $w_2(q_+) = \dfrac{m(m-1)}{2}(-1,-1)_F$ and $w_2(q_-) = \left(\dfrac{m(m-1)}{2} + m\right)(-1,-1)_F$. In particular, if m is odd, then $w_2(q_+) \neq w_2(q_-)$.*

PROOF. Let $A = M_n(F)$ and let $B = M_m((-1,-1))$. Then $\deg A = \deg B = n$ and hence Theorem 1.1 implies $\mathrm{sign}(\mathcal{T}_A) = n$ and $\mathrm{sign}(\mathcal{T}_B) = -n$. This implies $\mathcal{T}_A \simeq q_+$ and $\mathcal{T}_B \simeq q_-$. In addition, Theorem 1.1 implies

$$w_2(q_+) = w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1,-1) + m[A] = \frac{m(m-1)}{2}(-1,-1)$$

and

$$\begin{aligned} w_2(q_-) = w_2(\mathcal{T}_B) &= \frac{m(m-1)}{2}(-1,-1) + m(-1,-1) \\ &= \left(\frac{m(m-1)}{2} + m\right)(-1,-1). \end{aligned}$$

The last statement of this Lemma is clear since $(-1,-1)_F \neq 0$ if F is real closed. □

PROOF OF THEOREM 3.4 Notice that property (a) implies that quadratic forms are classified by dimension, determinant, Hasse-Witt invariant and signatures (see [EL1]).

The necessity follows from Theorem 1.1. Now suppose q satisfies (1)-(3). Assume first that $r = 0$, so m is odd. By hypothesis, there exists a quaternion algebra Q such that $[Q] = w_2(q) + \dfrac{m(m-1)}{2}(-1,-1)_F$. Let $A = M_m(Q)$. Then

$$w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1,-1)_F + m[Q] = \frac{m(m-1)}{2}(-1,-1)_F + [Q] = w_2(q).$$

We have $\mathrm{sign}_v(\mathcal{T}_A) = n$ if and only if $\mathrm{Res}_{F_v/F}([Q]) = 0$, by Theorem 1.1, which is equivalent to $w_2(q_{F_v}) = \dfrac{m(m-1)}{2}(-1,-1)_{F_v}$. This occurs if and only if

$q_{F_v} \simeq q_+$, by Lemma 3.5, since m is odd and $\mathrm{sign}_v(q) = \pm n$. Thus q and $\mathcal{T}_A$ have the same signatures. Since q and $\mathcal{T}_A$ also have the same dimension, determinant and Hasse-Witt invariant, it follows that they are isomorphic.
Assume now that $r \geq 1$. Let B be a central simple algebra over F such that $[B] = w_2(q) + \dfrac{m(m-1)}{2}(-1,-1)_F$. Since m is even and $\mathrm{sign}_v(q) = \pm n$, it follows from Lemma 3.5 that

$$\mathrm{Res}_{F_v/F}([B]) = \mathrm{Res}_{F_v/F}(w_2(q) + \frac{m(m-1)}{2}(-1,-1)_{F_v}) = 0$$

for all $v \in \Omega_F$. By Theorem 2.7, there exists $[A_1] \in {}_2\,\mathrm{Br}(F)_t$ such that $2^r[A_1] = [B]$. Let $X = \{v \in \Omega_F, \mathrm{sign}_v\, q = n\}$. Since X is clopen and $st_r(F) \leq 2$, we can use the ideas in the proof of Proposition 2.5 to find a central simple algebra D over F such that $2[D] = 0$ and such that $[A_2] = [A_1] + [D]$ satisfies $\mathrm{Res}_{F_v/F}[A_2] = 0$ if and only if $\mathrm{sign}_v(q) = n$. Then $2^r[A_2] = [B]$ since $r \geq 1$. Since $2[B] = 0$, we have $2^{r+1}[A_2] = 0$, and so by assumption there exists a central simple algebra A_3, $\deg A_3 = 2^{r+1}$, such that $[A_3] = [A_2]$. Now set $A = M_s(A_3)$, and note that A has degree n. Since A and A_2 are Brauer equivalent, q and $\mathcal{T}_A$ have equal signatures by construction of A_2. Since

$$m[A] = 2^r s[A_2] = s[B] = [B] = \frac{m(m-1)}{2}(-1,-1)_F + w_2(q),$$

it follows that $w_2(\mathcal{T}_A) = \dfrac{m(m-1)}{2}(-1,-1)_F + m[A] = w_2(q)$. Thus q and $\mathcal{T}_A$ are isomorphic, since they have the same dimension, determinant, Hasse-Witt invariant, and signature. □

Corollary 3.6. *Assume F satisfies the following conditions.*

(a) *$I^3(F)$ is torsion-free*

(b′) *For every $r \geq 0$ and for every $[A] \in \mathrm{Br}(F)$ such that $2^{r+1}[A] = 0$, there exists A', $\deg A' = 2^{r+1}$ such that $[A'] = [A]$.*

Then a quadratic form q is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions are satisfied :

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$
3. $\mathrm{sign}_v\, q = \pm n$, *for all $v \in \Omega_F$.*

Proof. This follows immediately from the Theorem 3.4 and the following observation. Condition (b′) with $r = 0$ implies that F is a linked field. That is, a sum of quaternion algebras defined over F is similar to another quaternion algebra defined over F. A theorem of Elman ([E]) states that a field F is linked and has $I^3(F)_t = 0$ if and only if $\tilde{u}(F) \leq 4$. It is known that if $\tilde{u}(F) < \infty$, then F is a SAP field (see [ELP]). Thus condition (c) in Theorem 3.4 holds automatically in the situation of Corollary 3.6. □

Remark 3.7. Condition (b) is realized for example when $\exp A = \operatorname{ind} A$ for every central simple algebra. In particular, it is the case when every central simple algebra is cyclic. For example, condition (b) holds for local fields, global fields or quotient fields of excellent two-dimensional local domains with algebraically closed residue fields of characteristic zero, e.g. finite extensions of $\mathbb{C}((X,Y))$ (see [CTOP], Theorem 2.1 for the last example and [CF] for the others). Such fields also satisfy condition (a). This is well-known for local fields and global fields (see [CF]). If F is a field of the last type, then $I^3(F) = 0$ (see [CTOP], Corollary 3.3).

We finish this paper giving a local-global principle for trace forms over global fields.

Corollary 3.8. *Let F be a global field of characteristic different from 2, and let $n = 2m \geq 2$ be an even integer. Then a quadratic form q over F is isomorphic to the trace form of a central simple algebra of degree n defined over F if and only if q is isomorphic to the trace form of a central simple algebra of degree n defined over all completions of F.*

Proof. Assume that q is a trace form over all completions of F. Then $\dim q = n^2$. By assumption, $(-1)^{\frac{n(n-1)}{2}} \det q$ is a nonzero square over all completions of F, so it is a nonzero square in F, and hence $\det q = (-1)^{\frac{n(n-1)}{2}} \in F^\times/F^{\times 2}$. Since q is a trace form over all real completions of F, we have $\operatorname{sign}_v q = \pm n$ for all real places v of F, according to whether q_{F_v} is isomorphic to the trace form of the split algebra or that of $M_m((-1,-1)_{F_v})$. Now apply Theorem 3.4. The other implication is clear, since $(\mathcal{T}_A)_L \simeq \mathcal{T}_{A\otimes L}$ for every central simple algebra over F, and every field extension L/F. □

The fact that $q_{F_\mathfrak{p}} \simeq \mathcal{T}_{A_\mathfrak{p}}$ for all places $\mathfrak{p}$ implies that $q \simeq \mathcal{T}_A$ does not mean that $A \otimes F_\mathfrak{p} \simeq A_\mathfrak{p}$ for all places. We sketch below the construction of a counterexample.

Example 3.9. We refer to [CF] for the definition of $\operatorname{inv}_\mathfrak{p}$ and the theorems concerning central simple algebras over global fields.

Assume $n \equiv 0$ [8]. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two places of F. For $i = 1,2$, let A_i be a central simple of degree n over $F_{\mathfrak{p}_i}$ such that $\operatorname{inv}_{\mathfrak{p}_i}[A_i] = \dfrac{1}{n}$, and let $A_\mathfrak{p}$ be $M_n(F_\mathfrak{p})$ for the other places over F. Now let $q_\mathfrak{p}$ be the trace form of $A_\mathfrak{p}$. We have $w_2(q_\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2$. Moreover $\det q_\mathfrak{p} = (-1)^{\frac{n(n-1)}{2}}$ for all $\mathfrak{p}$, so by [Sc], 6.6.10, there exists a quadratic form q over F such that $q_{F_\mathfrak{p}} \simeq q_\mathfrak{p}$. So q is locally a trace form, then q is the trace form of some central simple algebra A over F, but we can never have $A \otimes F_\mathfrak{p} \simeq A_\mathfrak{p}$ for all $\mathfrak{p}$. Otherwise, we will have $\sum \operatorname{inv}_\mathfrak{p}([A]) = 0 \in \mathbb{Q}/\mathbb{Z}$, which is not the case by choice of the $A_\mathfrak{p}$'s.

References

[CF] Cassels J.W.S., Frölich A. *Algebraic number theory.* Acad. Press, New York (1967)

[CTOP] Colliot-Thélène , J.-L., Ojanguren M., Parimala R. *Quadratic forms over two-dimensional henselian rings and Brauer groups of related schemes.* To appear in "Algebra, Arithmetic and Geometry", Proceedings of the Bombay Colloquium 2000.

[D] Draxl P.K. *Skew fields.* LMS Lecture Notes 83, Cambridge University Press (1983)

[E] Elman R. *Quadratic forms and the u-invariant. III.* Conference on Quadratic Forms—1976 (Proc. Conf., Queen's Univ., Kingston, Ont., 1976), Queen's Papers in Pure and Appl. Math., No. 46, Queen's Univ., Kingston, Ont., 422-444 (1977)

[EL1] Elman R., Lam T.Y. *Classification theorems for quadratic forms over fields.* Comm. Math. Helv. 49, 373-381 (1974)

[EL2] Elman R., Lam T.Y. *Quadratic forms under algebraic extensions.* Math. Ann. 219, 21-42 (1976)

[ELP] Elman R., Lam T.Y. Prestel A. *On some Hasse principles over formally real fields.*Math. Z. 134, 291–301 (1973)

[EP] Elman R., Prestel A. *Reduced stability of the Witt ring of a field and its Pythagorean closure.* Amer. J. Math. 106, 1237-1260 (1984)

[J] Jacobson A. *Finite-Dimensional Algebras over Fields.* Springer (1996)

[K] Kaplansky I. *Infinite abelian groups, revised edition.* Ann Arbor, University of Michigan Press (1969)

[La1] Lam T.Y. *The algebraic theory of quadratic forms.* W.A. Benjamin, Inc., Reading, Mass., Mathematics Lecture Notes Series (1973)

[La2] Lam T.Y. *Orderings, valuations and quadratic forms.* Conference Board of the Mathematical Sciences, Amer. Math. Soc. (1983)

[Le] Lewis D.W. *Trace forms of central simple algebras.* Math.Z. 215, 367-375 (1994)

[LLT] Lam T.-Y., Leep D.B., Tignol J.-P. *Biquaternion algebras and quartic extensions.* Pub. Math. I.H.E.S. 77, 63-102 (1993)

[LM] Lewis D.W., Morales J. *The Hasse invariant of the trace form of a central simple algebra.* Pub. Math. de Besançon, Théorie des nombres, 1-6 (1993/94)

[MS1] Merkurjev A. S., Suslin A. A. *Norm residue homomorphism of degree three* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 2, 339–356; translation in Math. USSR-Izv. 36, no. 2, 349–367 (1991)

[MS2] Merkurjev A. S., Suslin A. A. *The group K_3 for a field* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 3, 522–545; translation in Math. USSR-Izv. 36, no. 3, 541–565 (1991)

[Sc] Scharlau W. *Quadratic and Hermitian forms* Grundlehren Math. Wiss. 270, Springer-Verlag, New York (1985)

[Se] Serre J.-P. *Cohomologie galoisienne.* Cinquième édition, Lecture Notes in Mathematics 5, Springer-Verlag (1994)

[Ti] Tignol J.-P. *La norme des espaces quadratiques et la forme trace des algèbres simples centrales.* Pub.Math.Besançon, Théorie des nombres (92/93-93/94)

Grégory Berhuy
Ecole Polytechnique
Fédérale de
Lausanne, D.M.A.
CH-1015 Lausanne Switzerland
gregory.berhuy@epfl.ch

David B. Leep
Department of Mathematics
University of Kentucky
Lexington, KY
40506-0027, USA
leep@ms.uky.edu