Realizing Countable Groups As Automorphism Groups of Riemann Surfaces

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Abstract. Every countable group can be realized as the full automorphism group of a Riemann surface as well as the full group of isometries of a Riemannian manifold.

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Saerens and Zame [4], and independently Bedford and Dadok [2] proved that every compact real Lie group K can be realized as the group of holomorphic automorphisms of a complex manifold as well as the group of isometries of a Riemannian manifold.
Here we deduce a similar result for countable discrete groups.
Thus the purpose of this paper is to prove the following theorem:

Theorem. *Let G be a (finite or infinite) countable group.*
Then there exists a (connected) Riemann surface M such that G is isomorphic to the group $\mathrm{Aut}_{\mathcal{O}}(M)$ of all holomorphic automorphisms of M.
Moreover, there exists a Riemannian metric h on M such that $\mathrm{Aut}_{\mathcal{O}}(M)$ equals the group of all isometries of (M, h).

Our strategy is as follows: Using Galois theory of coverings, we first construct a Riemann surface M_1 on which G acts. Then we remove a discrete subset $S \subset M_1$ to kill excess automorphisms. However, we have to show that passing from M_1 to $M_1 \setminus S$ we do not risk enlarging the automorphism group, i.e.,

we will show that every automorphism from $M_1 \setminus S$ extends to M_1. For this purpose we employ the Freudenthal's theory of topological ends.
Finally, hyperbolicity of the Riemann surface is exploited to ensure that there is a hermitian metric of constant negative curvature such that the group of all holomorphic automorphisms coincides with the group of all isometries.
Let us remark that by uniformization theory it is well-known that the following is the list of all Riemann surface with positive-dimensional automorphism group and that their automorphism groups are well-known:

- $\mathbb{P}_1(\mathbb{C})$,
- $\mathbb{C}$,
- $\mathbb{C}^*$,
- $H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$,
- $E_\tau = \mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$ with $\tau \in H^+$,
- $A(r,1) = \{z \in \mathbb{C} : r < |z| < 1\}$ with $0 \le r < +\infty$.

(See [1] for this and other basic facts on Riemann surfaces.)
Therefore our result yields a complete characterization which groups may occur as automorphism group of a Riemann surface.
The above list furthermore has the following consequence which we will use later on:

Fact. *Let M be a Riemann surface with non-commutative fundamental group $\pi_1(M)$. Then $\mathrm{Aut}_{\mathcal{O}}(M)$ is discrete and acting properly discontinuously on M. In particular, every orbit is closed.*

1 Hyperbolic Riemannian Surfaces

A Riemannian surface M is "hyperbolic" (in the sense of Kobayashi) if and only if its universal covering is isomorphic to the unit disk. In this case the Poincaré metric on the unit disk induces a unique hermitian metric of constant Gaussian curvature -1 on M. Every holomorphic automorphism of M is an isometry and conversely every isometry is either holomorphic or antiholomorphic. Thus the group of holomorphic automorphisms of M is a subgroup of index 1 or 2 in the group of isometries and the group of isometries coincides with the group of all holomorphic or antiholomorphic diffeomorphisms of M.

2 Galois theory of coverings

Proposition 1. *Let G be a countable group. Let M_0 be a Riemann surface whose fundamental group is not finitely generated.*
Then there exists an unramified covering $M_1 \to M_0$ such that there is an effective G-action on M_1, $\mathrm{Aut}_{\mathcal{O}}(M_1)$ is discrete and acting properly discontinuously on M_1.

Proof. Let F_∞ be a free group with countably infinitely many generators $\alpha_1, \alpha_2, \ldots$. By standard results on Riemann surfaces (see e.g. [1]) we have $\pi_1(M_0) \simeq F_\infty$. Since G is countable, there is a surjective group homomorphism $\zeta : F_\infty \to G$. Furthermore, we may require that $\alpha_1, \alpha_2 \in \ker \zeta$. Then we obtain a short exact sequence of groups

$$\{e\} \to N \to F_\infty \xrightarrow{\zeta} G \to \{e\}$$

where N is non-commutative, because it contains a free group with two generators (viz. α_1 and α_2).
By Galois theory of coverings, this implies that there exists an unramified covering $M_1 \to M_0$ with $\pi_1(M_1) \simeq N$ and an effective G-action on M_1.
Finally, discreteness of $\mathrm{Aut}_{\mathcal{O}}(M_1)$ as well as the action of $\mathrm{Aut}_{\mathcal{O}}(M_1)$ being properly discontinuous is implied by the "Fact" established above. □

3 Topological ends

Let us recall the basic facts from the theory of ends as developed by Freudenthal [3].
Let X be a locally compact topological space. Then the set of "ends" $e(X)$ is defined by

$$e(X) = \lim_{K} \pi_0(X \setminus K).$$

Thus, if K_n is an exhaustion of X by an increasing sequence of compact subsets, then every end $\epsilon \in e(X)$ can be represented by a sequence U_n of connected components of $X \setminus K_n$ with $U_n \supset U_{n+1}$. For every connected component $W_{n,i}$ of $X \setminus K_n$ we now define $E_{n,i}$ as the set of ends ϵ which can be represented with $U_n = W_{n,i}$. Now $\bar{X} = X \cup e(X)$ becomes a compact topological space as follows: As a basis of the topology we take the family of all open subsets of X together with $V_{n,i} = W_{n,i} \cup E_{n,i}$ for all n, i.
Then every proper continuous map between locally compact topological spaces X and Y extends to a continuous maps between $\bar{X}$ and $\bar{Y}$. In particular, every homeomorphism of X extends to a homeomorphism of $\bar{X}$.

Definition. *An end ϵ of a Riemann surface X is called a "*puncture*" if there is an open neighbourhood W of ϵ in $\bar{X}$ such that*

- *$W \setminus \{\epsilon\} \subset X$ and*
- *there is a homeomorphism $\xi : W \to D = \{z \in \mathbb{C} : |z| < 1\}$ such that $\xi|_{W \setminus \{\epsilon\}}$ is holomorphic.*

We now prove that the ends of a certain special class of Riemann surfaces cannot be punctures.

Proposition 2. *Let a_n be a diverging sequence in $D = \{z \in \mathbb{C} : |z| < 1\}$ and $r_n \in \mathbb{R}^{>0}$ such that all the closed balls $B_n = \{z \in \mathbb{C} : |z - a_n| \leq r_n\}$ are disjoint subsets of D. Then $A = \cup_n B_n$ is a closed subset of D.*

Let $X_0 = D \setminus A$ and let $X_1 \to X_0$ be an unramified covering. Then no end of X_1 is a puncture.

Proof. First we show that A is indeed closed. Let $vol(B_n) = \pi r_n^2$ denote the euclidean volume. Since the balls B_n are disjoint, we have $\sum vol(B_n) \leq \pi$ and therefore $\lim r_n = 0$. It follows that for every $s < 1$ there is a natural number N such that $|a_n| - r_n > s$ for all $n \geq N$. Therefore $A = \cup_n B_n$ is actually a locally finite union, and closedness of the B_n implies that A is closed.
Now let us assume that there exists an end which is a puncture. The natural embeddings $X_0 \hookrightarrow D \hookrightarrow \mathbb{C}$ composed with the projection $\pi : X_1 \to X_0$ yield a bounded holomorphic function f on X_1. Let $\epsilon \in e(X_1)$ be a puncture with a connected open neighbourhood W as in the above definition of the notion "puncture". Then the Riemann extension theorem implies that f extends through ϵ. In other words, $\lim_{x\to\epsilon} f(x) = a$ exists. Evidently a is contained in the closure of X_0 in $\mathbb{C}$. However, the boundary ∂X_0 is given by $\partial X_0 = \partial D \cup (\cup_n \partial B_n)$, and the openness of holomorphic maps implies that a cannot lie on either ∂D or on one of the sets ∂B_n. Therefore $a \in X_0$. Now choose contractible open neighbourhoods U of a in X_0 and V of ϵ in W such that $V \setminus \{\epsilon\} \subset \pi^{-1}(U)$. Since $\pi : X_1 \to X_0$ is an unramified covering and U is simply-connected, we obtain $\pi^{-1}(U) \simeq G \times U$ where G is equipped with the discrete topology. Being connected, $V^* = V \setminus \{\epsilon\}$ must be contained in one connected component of $\pi^{-1}(U)$. This implies the following: If a_n is a sequence in V^* such that $\lim \pi(a_n) = a$, then there is a point $p \in X_1$ with $\lim a_n = p$. But this contradicts the fact that by construction there is sequence a_n in V^* with $\lim a_n = \epsilon \notin X_1$ and $\lim \pi(a_n) = a$. Thus this case can be ruled out as well, i.e., there cannot exist an end which is a puncture. □

4 Proof of the theorem

Proof. Let a_n be a diverging sequence in $D = \{z \in \mathbb{C} : |z| < 1\}$ and $r_n \in \mathbb{R}^{>0}$ such that all the closed balls $B_n = \{z \in \mathbb{C} : |z - a_n| \leq r_n\}$ are disjoint subsets of D. Let $A = \cup_n B_n$. Then A is a closed subset of D and the fundamental group of $X_0 = D \setminus A$ is not finitely generated. Hence, by prop. 1 there is an unramified covering $\pi : X_1 \to X_0$ with an effective G-action on X_1. Let $\mathrm{Aut}_{\mathcal{O}}(X_1)$ denote the group of all holomorphic automorphisms of X_1 and A the group of all diffeomorphisms of X_1 which are either holomorphic or antiholomorphic. Again by prop. 1 we may assume that $\mathrm{Aut}_{\mathcal{O}}(X_1)$ is discrete and acting properly discontinuously. Since $\mathrm{Aut}_{\mathcal{O}}(X_1)$ is of finite index in A, the group A is likewise discrete and acting properly discontinuously on X_1.
Now, for every $g \in A \setminus \{e\}$ the fixed point set $X_1^g = \{x \in X_1 : g(x) = x\}$ is a nowhere dense real analytic subset of X_1. Hence $\Sigma = \cup_{g \in A\setminus\{e\}} X_1^g$ is a set of measure zero. In particular, $\Sigma \neq X_1$. Let $p \in X_1 \setminus \Sigma$, $S = G(p)$ and $X = X_1 \setminus S$.
Note that the conditions $p \in X_1 \setminus \Sigma$, $S = G(p)$ imply that $g(p) \notin S$ if $g \in A \setminus G$.

Therefore

$$G = \{g \in A : g(S) = S\}.$$

Let h be the unique hermitian metric of constant Gaussian curvature -1 on X and I its isomorphism group. We claim that $I = \mathrm{Aut}_{\mathcal{O}}(X) \simeq G$. To show this, it suffices to show that every holomorphic or antiholomorphic automorphism of X extends to a holomorphic or antiholomorphic automorphism of X_1. If ϕ is a holomorphic or antiholomorphic automorphism of X, it is in particular a self-homeomorphism and therefore extends to a homeomorphism $\bar{\phi}$ of the compact topological space $\bar{X} = X \cup e(X)$ (where $e(X)$ is the set of ends as explained in §3. above). Now $e(X) = e(X_1) \cup S$. Evidently every end of X given by a point of S is a puncture as defined in §3. On the other hand, due to prop. 2 none of the ends of X_1 is a puncture. Now $\bar{\phi}|_{e(X)}$ is a permutation of the elements of $e(X)$ which stabilizes the set of those ends which are punctures. Hence $\bar{\phi}(S) = S$. Thus ϕ extends to a continuous self-map of $X_1 = X \cup S$. However, a continuous map which is holomorphic or antiholomorphic everywhere except for some isolated points, is necessarily holomorphic resp. antiholomorphic everywhere (by Riemann extension theorem). Hence every $\phi \in I$ of X extends to a holomorphic or antiholomorphic automorphism of X_1. Consequently

$$\mathrm{Aut}_{\mathcal{O}}(X) = I = \{g \in A : g(S) = S\} = G.$$

□

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