Partition-Dependent Stochastic Measures and q-Deformed Cumulants

Michael Anshelevich

Received: June 26, 2001

Communicated by Joachim Cuntz

Abstract. On a q-deformed Fock space, we define multiple q-Lévy processes. Using the partition-dependent stochastic measures derived from such processes, we define partition-dependent cumulants for their joint distributions, and express these in terms of the cumulant functional using the number of restricted crossings of P. Biane. In the single variable case, this allows us to define a q-convolution for a large class of probability measures. We make some comments on the Itô table in this context, and investigate the q-Brownian motion and the q-Poisson process in more detail.

2000 Mathematics Subject Classification: Primary 46L53; Secondary 05A18, 47D, 60E, 81S05

1. Introduction

In [RW97], Rota and Wallstrom introduced, in the context of usual probability theory, the notion of partition-dependent stochastic measures. These objects give precise meaning to the following heuristic expressions. Start with a Lévy process $X(t)$. For a set partition $\pi = (B_1, B_2, \ldots, B_k)$, temporarily denote by $c(i)$ the number of the class $B_{c(i)}$ to which i belongs. Then, heuristically,

$$\mathrm{St}_\pi(t) = \int_{\substack{[0,t)^k \\ \text{all } s_i\text{'s distinct}}} dX(s_{c(1)})dX(s_{c(2)}) \cdots dX(s_{c(n)}).$$

In particular, denote by Δ_n the higher diagonal measures of the process defined by

$$\Delta_n(t) = \int_{[0,t)} (dX(s))^n.$$

These objects were used to define the Itô multi-dimensional stochastic integrals through the usual product measures, by employing the Möbius inversion on the lattice of all partitions. In particular this approach unifies a number of combinatorial results in probability theory.

The formulation of the algebraic (noncommutative, quantum) probability goes back to the beginnings of quantum mechanics and operator algebras. While a number of results have been obtained in a general context, in many cases the lack of tight hypotheses guaranteed that the conclusions of the theory would be somewhat loose. In the last twenty years of the twentieth century a particular noncommutative probability theory, the free probability theory [VDN92, Voi00], appeared, whose wealth of results approaches that of the classical one. This theory is based on a new notion of independence, the so-called *free independence*. In particular, one defines the (additive) free convolution, a new binary operation on probability measures: $\mu \boxplus \nu$ is the distribution of the sum of freely independent operators with distributions μ, ν. Note that this is precisely the relation between independence and the usual convolution. Many limit theorems for independent random variables carry over to free probability [BP99] by adapting the method of characteristic functions, using the R-transform of Voiculescu in place of the Fourier transform. Applications of the theory range from von Neumann algebras to random matrix theory and asymptotic representations of the symmetric group.

In [Ans00, Ans01a] (see also [Ans01b]) we investigated the analogs of the multiple stochastic measures of Rota and Wallstrom in the context of free probability theory. In this analysis, the starting object $X(t)$ is a stationary process with *freely* independent *bounded* increments. One important fact observed was that in the classical case the expectation of $\mathrm{St}_\pi(t)$ is the combinatorial cumulant of the distribution of $X(t)$. This means that the expectation of $\mathrm{St}_\pi(t)$ is equal to $\prod_{j=1}^k r_{|B_j|}$, where r_i is the i-th coefficient in the Taylor series expansion of the logarithm of the Fourier transform of the distribution of $X(t)$. See [Shi96, Nic95] or Section 6.1. The importance of cumulants lies in their relation to independence: since independence corresponds to a factorization property of the joint Fourier transforms, it can also be expressed as a certain additivity property of cumulants, the "mixed cumulants of independent quantities equal 0" condition. See Section 4.1.

It was observed by Speicher that in free probability, the condition of free independence can also be expressed in terms of a certain different family of cumulants, the so-called free cumulants; see [Spe97a] for a review. One also naturally obtains partition-dependent free cumulants, but only for partitions that are *noncrossing*. In [Ans00], we showed that in the free case $\mathrm{St}_\pi(t) = 0$ if the partition π is crossing, and for a noncrossing partition π the expectation of $\mathrm{St}_\pi(t)$ is the corresponding free cumulant of the distribution of $X(t)$. Thus both in the classical and in the free case $\mathrm{St}_\pi(t)$ can be considered as an operator-valued version of combinatorial cumulants (no relation to [Spe98]).

We try out this idea on the q-deformed probability theory. This is a noncommutative probability theory in an algebra on the q-deformed Fock space, developed by a number of authors, see the References. Free probability corresponds to $q = 0$, while classical (bosonic) and anti-symmetric (fermionic) theories correspond to $q = 1, -1$ (in these cases q-Fock spaces degenerate to the symmetric and anti-symmetric Fock spaces, respectively). For the intermediate values of q, it is known that the theory cannot be as good as the classical and the free ones, since one cannot define a notion of q-independence satisfying all the desired properties [vLM96, Spe97b].

In this paper we try a different approach. As mentioned above, whenever we have a family of operators which corresponds to a family of measures that is in some sense infinitely divisible, we should be able to define the partition-dependent stochastic measures, and then *define* the combinatorial cumulants as their expectations. One definition of cumulants appropriate for the q-deformed probability theory has already been given in [Nic95], based on an analog of the canonical form introduced by Voiculescu in the context of free probability. The advantage of the approach of that paper is that Nica's cumulants are defined for any probability distribution all of whose moments are finite. However, the canonical form of that paper is not self-adjoint, and it is also not appropriate for our approach since it does not provide us with a natural additive process. Instead, as a canonical form we choose the q-analog of the families of [HP84, AH84, Sch91, GSS92], which in the classical and the free case provide representations for all (classically, resp. freely) infinitely divisible distributions all of whose moments are finite. We provide an explicit formula for the resulting combinatorial cumulants, involving as expected a notion of the number of crossings of a partition. The appropriate one for our context happens to be the number of "restricted crossings" of [Bia97]; in particular the resulting cumulants are different from those of [Nic95]. Our approach makes sense only for distributions corresponding to q-infinitely divisible families (although strictly speaking, one can use our definition in general). However, our canonical form of an operator is self-adjoint, and this in turn leads to a notion of q-convolution on a large class of probability measures. The fact that this convolution is not defined on all probability measures is actually to be expected, see Section 6.1.

After finishing this article, we learned about a physics paper [NS94] which seems to have been overlooked by the authors of both [Bia97] and [SY00b]. The goals of that paper are different from ours, but in particular it defines and investigates the same q-Poisson process as we do (Section 6.3) and, in that case, points out the relation between the moments of the process and the number of restricted crossings of corresponding partitions. It would be interesting to see if the results of that paper can be extended to our context, and how our results fit together with the "partial cumulants" approach.

The paper is organized as follows. Following the Introduction, in Section 2 we provide some background on the combinatorics of partitions and on q-Fock spaces, and define the q-Lévy processes. In Section 3, we define the joint moments and q-cumulants, and express partition-dependent q-cumulants in terms of the q-cumulant functional. In Section 4 we show that the cumulant functional is the generator, in the sense defined in that section, of the family of time-dependent moment functionals, and characterize all such generators. We also discuss the notion of a product state that arises from this construction. In Section 5, we provide some information about the Itô product formula in this context, by calculating the quadratic co-variation of two q-Lévy processes. In a long Section 6 we define the q-convolution, describe how our construction relates to the Bercovici-Pata bijection, and investigate the q-Brownian motion and the q-Poisson process in more detail. Finally, in the last section we make a few preliminary comments on the von Neumann algebras generated by the processes.

Acknowledgments: I would like to thank Prof. Bożejko for encouraging me to look at the q-analogs of [Ans00], and Prof. Voiculescu for many talks I had to give in his seminar, during which I learned the background for this paper. I am also grateful to Prof. Speicher for a number of suggestions about Section 7, and to Daniel Markiewicz for numerous comments. This paper was written while I was participating in a special Operator Algebras year at MSRI, and is supported in part by an MSRI postdoctoral fellowship.

2. Preliminaries

2.1. Notation. Fix a parameter $q \in (-1, 1)$; we will usually omit the dependence on q in the notation. The analogs of the results of this paper for $q = \pm 1$ are in most cases well-known; we will comment on them throughout the paper. For n a non-negative integer, denote by $[n]_q$ the corresponding q-integer, $[0]_q = 0$, $[n]_q = \sum_{i=0}^{n-1} q^i$.

For a collection $\left\{ y_j^{(i)} \right\}$ of numbers and two multi-indices $\vec{v} = (v(1), \ldots, v(k))$ and $\vec{u} = (u(1), \ldots, u(k))$, we will throughout the paper use the notation $\mathbf{y}_{\vec{v}}^{(\vec{u})}$ to denote $\prod_{j=1}^{k} y_{v(j)}^{(u(j))}$.

Denote by $[k \ldots n]$ the ordered set of integers in the interval $[k, n]$.

For a family of functions $\{F_j\}$, where F_j is a function of j arguments, $\vec{v}$ a vector with k components, and $B \subset [1 \ldots k]$, denote $F(\vec{v}) = F_k(\vec{v})$ and

$$F(B : \vec{v}) = F_{|B|}(v(i(1)), v(i(2)), \ldots, v(i(|B|))),$$

where $B = (i(1), i(2), \ldots, i(|B|))$. In particular, we use this notation for joint moments and cumulants (see below).

2.2. Partitions. For an ordered set S, denote by $\mathcal{P}(S)$ the lattice of set partitions of that set. Denote by $\mathcal{P}(n)$ the lattice of set partitions of the set $[1 \ldots n]$, and by $\mathcal{P}_2(n)$ the collection of its pair partitions, i.e. of partitions into 2-element classes. Denote by $\leq$ the lattice order, and by $\hat{1}_n = ((1, 2, \ldots, n))$ the largest and by $\hat{0}_n = ((1)(2) \ldots (n))$ the smallest partition in this order.

Fix a partition $\pi \in \mathcal{P}(n)$, with classes $\{B_1, B_2, \ldots, B_l\}$. We write $B \in \pi$ if B is a class of π. Call a class of π a *singleton* if it consists of one element. For a class B, denote by $a(B)$ its first element, and by $b(B)$ its last element. Order the classes according to the order of their last elements, i.e. $b(B_1) < b(B_2) < \ldots < b(B_l)$. Call a class $B \in \pi$ an interval if $B = [a(B) \ldots b(B)]$. Call π an *interval partition* if all the classes of π are intervals.

Following [Bia97], we define the number of *restricted crossings* of a partition π as follows. For B a class of π and $i \in B, i \neq a(B)$, denote

$p(i) = \max\{j \in B, j < i\}$. For two classes $B, C \in \pi$, a restricted crossing is a quadruple $(p(i) < p(j) < i < j)$ with $i \in B, j \in C$. The number of restricted crossings of B, C is

$$\begin{aligned}\operatorname{rc}(B,C) = &|\{i \in B, j \in C : p(i) < p(j) < i < j\}| \\ &+ |\{i \in B, j \in C : p(j) < p(i) < j < i\}|,\end{aligned}$$

and the number of restricted crossings of π is $\operatorname{rc}(\pi) = \sum_{i<j} \operatorname{rc}(B_i, B_j)$. It has the following graphical representation. Draw the points $[1 \ldots n]$ in a sequence on the x-axis, and to represent the partition π connect each i with $p(i)$ (if it is well-defined) by a semicircle above the x axis. Then the number of intersections of the resulting semicircles is precisely $\operatorname{rc}(\pi)$. See Figure 1 for an example. We say that a partition π is *noncrossing* if $\operatorname{rc}(\pi) = 0$. Denote by $NC(n) \subset \mathcal{P}(n)$ the collection of all noncrossing partitions, which in fact form a sub-lattice of $\mathcal{P}(n)$.

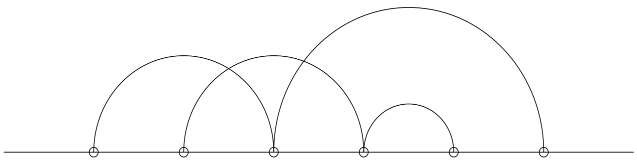

Figure 1. A partition of 6 elements with 2 restricted crossings.

We need some auxiliary notation. For $\sigma, \pi \in \mathcal{P}(n)$, we define $\pi \wedge \sigma \in \mathcal{P}(n)$ to be the meet of π and σ in the lattice, i.e.

$$i \overset{\pi\wedge\sigma}{\sim} j \;\Leftrightarrow\; i \overset{\pi}{\sim} j \text{ and } i \overset{\sigma}{\sim} j.$$

For $\pi \in \mathcal{P}(n)$, we define $\pi^{op} \in \mathcal{P}(n)$ to be π taken in the opposite order, i.e.

$$i \overset{\pi^{op}}{\sim} j \;\Leftrightarrow\; (n-i+1) \overset{\pi}{\sim} (n-j+1).$$

For $\pi \in \mathcal{P}(n), \sigma \in \mathcal{P}(k)$, we define $\pi + \sigma \in \mathcal{P}(n+k)$ by

$$i \overset{\pi+\sigma}{\sim} j \Leftrightarrow ((i,j \leq n, i \overset{\pi}{\sim} j) \text{ or } (i,j > n, (i-n) \overset{\sigma}{\sim} (j-n))).$$

We'll denote $m\pi = \pi + \pi + \ldots + \pi \quad m$ times.
Finally, using the above notation, for a subset $B \subset [1 \ldots n]$ and $\pi \in \mathcal{P}(n)$, $(B : \pi)$ is the restriction of π to B.

2.3. The q-Fock space. Let H be a (complex) Hilbert space. Let $\mathcal{F}_{\mathrm{alg}}(H)$ be its algebraic full Fock space, $\mathcal{F}_{\mathrm{alg}}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}$, where $H^{\otimes 0} = \mathbb{C}\Omega$ and Ω is the vacuum vector. For each $n \geq 0$, define the operator P_n on $H^{\otimes n}$ by

$$\begin{aligned}&P_0(\Omega) = \Omega,\\ &P_n(\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n) = \sum_{\alpha \in \mathrm{Sym}(n)} q^{i(\alpha)} \eta_{\alpha(1)} \otimes \eta_{\alpha(2)} \otimes \ldots \otimes \eta_{\alpha(n)},\end{aligned}$$

where $\mathrm{Sym}(n)$ is the group of permutations of n elements, and $i(\alpha)$ is the number of inversions of the permutation α. For $q = 0$, each $P_n = \mathrm{Id}$. For $q = 1$, $P_n = n! \times$ the projection onto the subspace of symmetric tensors. For $q = -1$, $P_n = n! \times$ the projection onto the subspace of anti-symmetric tensors. Define the q-deformed inner product on $\mathcal{F}_{\mathrm{alg}}(H)$ by the rule that for $\zeta \in H^{\otimes k}$, $\eta \in H^{\otimes n}$,

$$\langle \zeta, \eta \rangle_q = \delta_{nk} \langle \zeta, P_n \eta \rangle ,$$

where the inner product on the right-hand-side is the usual inner product induced on $H^{\otimes n}$ from H. All inner products are linear in the second variable. It is a result of [BS91] that the inner product $\langle \cdot, \cdot \rangle_q$ is positive definite for $q \in (-1, 1)$, while for $q = -1, 1$ it is positive semi-definite. Let $\mathcal{F}_q(H)$ be the completion of $\mathcal{F}_{\mathrm{alg}}(H)$ with respect to the norm corresponding to $\langle \cdot, \cdot \rangle_q$. For $q = -1, 1$ one first needs to quotient out by the vectors of norm 0 and then complete; the result is the anti-symmetric, respectively, symmetric Fock space, with the inner product multiplied by $n!$ on the n-particle space.

For ξ in H, define the (left) creation and annihilation operators on $\mathcal{F}_{\mathrm{alg}}(H)$ by, respectively,

$$\begin{aligned} &a^*(\xi)\Omega = \xi, \\ &a^*(\xi)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \xi \otimes \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n, \end{aligned}$$

and

$$\begin{aligned} &a(\xi)\Omega = 0, \\ &a(\xi)\eta = \langle \xi, \eta \rangle \Omega, \\ &a(\xi)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^{n} q^{i-1} \langle \xi, \eta_i \rangle \eta_1 \otimes \ldots \otimes \hat{\eta}_i \otimes \ldots \otimes \eta_n, \end{aligned}$$

where as usual $\hat{\eta}_i$ means omit the i-th term. For $q \in (-1, 1)$, both operators can be extended to bounded operators on $\mathcal{F}_q(H)$, on which they are adjoints of each other [BS91]. They satisfy the commutation relations $a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) = \langle \xi, \eta \rangle \mathrm{Id}$. For $q = \pm 1$, we first need to compress the operators by the projection onto the symmetric / anti-symmetric Fock space, respectively, and the resulting operators differ from the usual ones by $\sqrt{n}$, but satisfy the usual commutation relations (thanks to a different inner product). For $q = 1$ the resulting operators are unbounded, but still adjoints of each other [RS75].

Denote by φ the vacuum vector state $\varphi[X] = \langle \Omega, X\Omega \rangle_q$.

2.4. Gauge operators. We now need to define differential second quantization. Consider the number operator, the differential second quantization of the identity operator. One choice, made in [Møl93], is to define it as the operator that has $H^{\otimes n}$ as an eigenspace with eigenvalue n. For a general self-adjoint operator T, this gives the true differential second quantization derived from the q-second quantization functor of [BKS97]. The resulting operators are self-adjoint, but do not satisfy nice commutation relations.

Another choice for the number operator is the operator that has $H^{\otimes n}$ as an eigenspace with eigenvalue $[n]_q$. For a general (bounded) operator T, the corresponding construction is

$$p(T)\Omega = 0,$$

$$p(T)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^{n} q^{i-1} \eta_1 \otimes \ldots \otimes (T\eta_i) \otimes \ldots \otimes \eta_n.$$

Similar operators were used in [Śni00], where stochastic calculus with respect to the corresponding processes was developed. They do have nice commutation properties, but are in general not symmetric.

Finally, another natural choice for the number operator is $\sum_i a^*(e_i)a(e_i)$, where $\{e_i\}$ is an orthonormal basis for H; the resulting operator is then independent of the choice of the basis. For a general bounded operator T, the corresponding construction is $\sum_i a^*(Te_i)a(e_i)$. It is easy to see that this sum converges strongly, to the following operator.

Definition 2.1. Let T be an operator on H with dense domain $\mathcal{D}$. The corresponding *gauge operator* $p(T)$ is an operator on $\mathcal{F}_q(H)$ with dense domain $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$ defined by

$$p(T)\Omega = 0,$$

$$p(T)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^{n} q^{i-1} (T\eta_i) \otimes \eta_1 \otimes \ldots \otimes \hat{\eta}_i \otimes \ldots \otimes \eta_n,$$

for $\eta_1, \eta_2, \ldots, \eta_n \in \mathcal{D}$.

Proposition 2.2. *If T is essentially self-adjoint on a dense domain $\mathcal{D}$ and $T(\mathcal{D}) \subset \mathcal{D}$, then $p(T)$ is essentially self-adjoint on a dense domain $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$.*

Proof. We first show that $p(T)$ is symmetric on $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$. Fix n, and denote by β_j the cycle in $\mathrm{Sym}(n)$ given by $\beta_j = (12\ldots j)$. For a permutation $\alpha \in \mathrm{Sym}(n)$, write $\alpha(\eta_1 \otimes \ldots \otimes \eta_n) = \eta_{\alpha(1)} \otimes \ldots \otimes \eta_{\alpha(n)}$. For $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n \in \mathcal{D}$,

$$\begin{aligned}
&\left\langle p(T)\eta_1 \otimes \ldots \otimes \eta_n, \xi_1 \otimes \ldots \otimes \xi_n \right\rangle_q \\
&= \sum_{j=1}^{n} q^{j-1} \left\langle \beta_j^{-1}(\eta_1 \otimes \ldots \otimes (T\eta_j) \otimes \ldots \otimes \eta_n), \xi_1 \otimes \ldots \otimes \xi_n \right\rangle_q \\
&= \sum_{j=1}^{n} \sum_{\alpha \in \mathrm{Sym}(n)} q^{j-1} q^{i(\alpha)} \left\langle \beta_j^{-1}(\eta_1 \otimes \ldots \otimes (T\eta_j) \otimes \ldots \otimes \eta_n), \alpha(\xi_1 \otimes \ldots \otimes \xi_n) \right\rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\substack{\alpha \in \mathrm{Sym}(n) \\ \alpha(1)=k}} q^{j-1} q^{i(\alpha)} \left\langle \beta_j^{-1}(\eta_1 \otimes \ldots \otimes \eta_n), \alpha(\xi_1 \otimes \ldots \otimes (T^*\xi_k) \otimes \ldots \otimes \xi_n) \right\rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\substack{\alpha \in \mathrm{Sym}(n) \\ \alpha(1)=k}} q^{j-1} q^{i(\alpha)} \left\langle \eta_1 \otimes \ldots \otimes \eta_n, (\beta_j \alpha \beta_k)\beta_k^{-1}(\xi_1 \otimes \ldots \otimes (T^*\xi_k) \otimes \ldots \otimes \xi_n) \right\rangle
\end{aligned}$$

Using the combinatorial lemma immediately following this proof, this expression is equal to

$$= \sum_{k=1}^{n} \sum_{\gamma \in \mathrm{Sym}(n)} q^{k-1} q^{i(\gamma)} \left\langle \eta_1 \otimes \ldots \otimes \eta_n, \gamma(\beta_k^{-1}(\xi_1 \otimes \ldots \otimes (T^* \xi_k) \otimes \ldots \otimes \xi_n)) \right\rangle$$
$$= \langle \eta_1 \otimes \ldots \otimes \eta_n, p(T^*) \xi_1 \otimes \ldots \otimes \xi_n \rangle_q .$$

Now we show that the operator $p(T)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$. For $q = 1$, the proof is contained in [RS75, X.6, Example 3]. For $q \in (-1, 1)$ we proceed similarly. Let $\mathcal{D}_n = \mathcal{D}^{\otimes n}$. Let $E.$ be the spectral measure of the closure $\bar{T}$ of T, and $C \in \mathbb{R}_+$. Let $\{\eta_i\}_{i=1}^n \subset (E_{[-C.C]} H) \cap \mathcal{D}$; then $\|T\eta_i\| \leq C \|\eta_i\|$. Let $\vec{\eta} = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n$. Then

$$\left\| p(T)^k \vec{\eta} \right\|_q^2 = \left\langle p(T)^k \vec{\eta}, P_n p(T)^k \vec{\eta} \right\rangle \leq \|P_n\| \, (n^k C^k \|\vec{\eta}\|)^2.$$

It was shown in [BS91] that $\|P_n\| \leq [n]_{|q|}! \leq n!$. We conclude that $\left\| p(T)^k \vec{\eta} \right\|_q \leq \sqrt{n!} n^k C^k \|\vec{\eta}\|$ and so

$$\limsup_{k \to \infty} \frac{1}{k} \left\| p(T)^k \vec{\eta} \right\|_q^{1/k} = 0.$$

Therefore $\vec{\eta}$ is an analytic vector for $p(T)$. The linear span of such vectors is invariant under $p(T)$ and is a dense subset of $\mathcal{D}_n$. Therefore by Nelson's analytic vector theorem, $p(T)$ is essentially self-adjoint on $\mathcal{D}_n$.

The rest of the argument proceeds as in [RS72, VIII.10, Example 2]. An operator A is essentially self-adjoint iff the range of $A \pm i$ is dense. Since $p(T)$ restricted to $H^{\otimes n}$ is essentially self-adjoint, this property holds for each such restriction, and then for the operator $p(T)$ itself, which therefore has to be essentially self-adjoint. $\square$

Lemma 2.3. *For a fixed k, every permutation $\gamma \in \mathrm{Sym}(n)$ appears in the collection*

$$\{\beta_j \alpha \beta_k : 1 \leq j \leq n, \alpha(1) = k\}$$

exactly once. Moreover, for such α, $i(\beta_j \alpha \beta_k) = i(\alpha) + j - k$.

Proof. It suffices to show the first property for the collection $\{\beta_j \alpha\}$. This collection contains at most $n!$ distinct elements. On the other hand, for $\gamma \in \mathrm{Sym}(n)$, let $j = \gamma^{-1}(k)$, and $\alpha = \beta_j^{-1} \gamma$; then j, α satisfy the conditions and $\beta_j \alpha = \gamma$.

For the second property, first take $\gamma \in \mathrm{Sym}(n)$ such that $\gamma(1) = 1$ and show that $i(\beta_j \gamma) = i(\gamma) + (j-1)$. Indeed, β_j only reverses the order of $(j-1)$ pairs (a, j) with $a < j$. β_j sends such a pair to $((a+1), 1)$, and since $\gamma(1) = 1$, γ preserves the order of such a pair.

We conclude that $i(\beta_j \alpha \beta_k) = i(\alpha \beta_k) + (j-1)$. Now we show that $i(\alpha \beta_k) = i(\alpha) - (k-1)$. Indeed, β_k only reverses the order of $(k-1)$ pairs (a, k) for $a < k$. The pre-image of such a pair under α is $(\alpha^{-1}(a), 1)$, and so α reverses its order. $\square$

These gauge operators themselves do not satisfy nice commutation relations. Nevertheless, we can still calculate their combinatorial cumulants. Another advantage of this definition is that it naturally generalizes to the "Yang-Baxter" commutation relations of [BS94]. However, in this more general context partition-dependent cumulants are not expressed in terms of the cumulant functional, so we do not pursue this direction in more detail.

For $q = 0$, $p(T)$ are precisely the gauge operators on the full Fock space as defined in [GSS92]. For $q = 1$, again we first need to compress $p(T)$ by the projection onto the symmetric Fock space, and the result is the usual differential second quantization. For $q = -1$, we first need to compress $p(T)$ by the projection onto the anti-symmetric Fock space, and the result is the anti-symmetric differential second quantization.

2.5. The processes. Let V be a Hilbert space, and let H be the Hilbert space $L^2(\mathbb{R}_+, dx) \otimes V$. Let $\xi \in V$, and let T be an essentially self-adjoint operator on a dense domain $\mathcal{D} \subset V$ so that $\mathcal{D}$ is equal to the linear span of $\{T^n \xi\}_{n=0}^{\infty}$ and moreover ξ is an analytic vector for T. Given a half-open interval $I \subset \mathbb{R}_+$, define $a_I(\xi) = a(\mathbf{1}_I \otimes \xi)$, $a_I^*(\xi) = a^*(\mathbf{1}_I \otimes \xi)$, $p_I(T) = p(\mathbf{1}_I \otimes T)$. Here $\mathbf{1}_I$ is the indicator function of the set I, considered both as a vector in $L^2(\mathbb{R}_+)$ and a multiplication operator on it. For $\lambda \in \mathbb{R}$, denote $p_I(\xi, T, \lambda) = a_I(\xi) + a_I^*(\xi) + p_I(T) + |I| \lambda$. Denote by a_t, a_t^*, p_t the appropriate objects corresponding to the interval $[0, t)$. We will call a process of the form $I \mapsto p_I(\xi, T, \lambda)$ a *q-Lévy process*. For $q = 1$ this is indeed a Lévy process.

Now fix a k-tuple $\{T_j\}_{j=1}^k$ of essentially self-adjoint operators on a common dense domain $\mathcal{D} \subset V$, $T_j(\mathcal{D}) \subset \mathcal{D}$, a k-tuple $\{\xi_j\}_{j=1}^k \subset \mathcal{D}$ of vectors, and $\{\lambda_j\}_{j=1}^k \subset \mathbb{R}$. We will make an extra assumption that

$$
\begin{aligned}
&\forall i, j \in [1 \ldots k], l \in \mathbb{N}, \vec{u} \in [1 \ldots k]^l, \\
&\mathbf{T}_{\vec{u}} \xi_i = T_{u(1)} T_{u(2)} \ldots T_{u(l)} \xi_i \text{ is an analytic vector for } T_j, \\
&\text{and } \mathcal{D} = \operatorname{span}\left(\left\{\mathbf{T}_{\vec{u}} \xi_i : i \in [1 \ldots k], l \in \mathbb{N}, \vec{u} \in [1 \ldots k]^l\right\}\right).
\end{aligned} \tag{1}
$$

Denote by $\mathbf{X}$ the k-tuple of processes $(X^{(1)}, \ldots, X^{(k)})$, where $X^{(j)}(I) = p_I(\xi_j, T_j, \lambda_j)$. In particular $\mathbf{X}(t) = \mathbf{X}([0, t))$. We call such a k-tuple a *multiple q-Lévy process*.

Remark 2.4. The assumption (1) is not essential for most of the paper. Most of the analysis could be done purely algebraically: see Remark 5.1. We will make this assumption to guarantee that we have a correspondence between self-adjoint processes and semigroups of measures, rather than between symmetric processes and semigroups of moment sequences.

3. Cumulants

3.1. Joint distribution. Since the processes in $\mathbf{X}$ do not necessarily commute, by their joint distribution we will mean the collection of their joint moments. We organize this information as follows.

Denote by $\mathbb{C}\langle \mathbf{x} \rangle = \mathbb{C}\langle x_1, x_2, \dots , x_k \rangle$ the algebra of polynomials in k formal noncommuting indeterminates with complex coefficients. Note that in a more abstract language, this is just the tensor algebra of the complex vector space V_0 with a distinguished basis $\{x_i\}_{i=1}^k$. While we take V_0 to be k-dimensional, the same arguments will work for an arbitrary V_0, as long as we use a more functorial definition of a process, namely for $f = \sum a_i x_i \in V_0$, we would define $T(f) = \sum a_i T_i, \xi(f) = \sum a_i \xi_i, \lambda(f) = \sum a_i \lambda_i$. See [Sch91] for a more detailed description of this approach.

Define a functional M on $\mathbb{C}\langle \mathbf{x} \rangle$ by the following action on monomials: $M(1, t; \mathbf{X}) = 1$, for a multi-index $\vec{u}$,

$$M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \varphi \left[\mathbf{X}^{(\vec{u})}(t) \right],$$

and extend linearly. We will call $M(\cdot, t; \mathbf{X})$ the *moment functional* of the process $\mathbf{X}$ at time t.

If we equip $\mathbb{C}\langle \mathbf{x} \rangle$ with a conjugation $*$ extending the conjugation on $\mathbb{C}$ so that each $x_i^* = x_i$, it is clear that M is a positive functional, i.e. $M(ff^*, t; \mathbf{X}) \geq 0$ for all $f \in \mathbb{C}\langle \mathbf{x} \rangle$.

For a partition $\pi \in \mathcal{P}(n)$ and a monomial $\mathbf{x}_{\vec{u}}$ of degree n, denote $M_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \prod_{B \in \pi} M(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X})$. These are the *combinatorial moments* of $\mathbf{X}$ at time t.

For a one-dimensional process, the functional $M(\cdot, t; X)$ can be extended to a probability measure μ_t such that $\mu_t(x^n) = M(x^n, t; X)$. Specifically, $\mu_t(S) = \varphi[E_S]$, where $E.$ is the spectral measure of $X(t)$.

3.2. MULTIPLE STOCHASTIC MEASURES AND CUMULANTS. For a set S and a partition $\pi \in \mathcal{P}(n)$, denote

$$S^n_\pi = \left\{ \vec{v} \in S^n : v(i) = v(j) \Leftrightarrow i \overset{\pi}{\sim} j \right\}$$

and

$$S^n_{\leq \pi} = \left\{ \vec{v} \in S^n : v(i) = v(j) \Rightarrow i \overset{\pi}{\sim} j \right\}.$$

Fix t. For $N \in \mathbb{N}$ and a subdivision of $[0, t)$ into N disjoint ordered half-open intervals $\mathcal{I} = \{I_1, I_2, \dots , I_N\}$, let $\delta(\mathcal{I}) = \max_{1 \leq i \leq N} |I_i|$. Denote X_i, a_i, a_i^*, p_i the appropriate objects for the interval I_i. Fix a monomial $\mathbf{x}_{\vec{u}} \in \mathbb{C}\langle x_1, x_2, \dots , x_k \rangle$ of degree n.

DEFINITION 3.1. The *stochastic measure* corresponding to the partition π, monomial $\mathbf{x}_{\vec{u}}$, and subdivision $\mathcal{I}$ is

$$\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I}) = \sum_{\vec{v} \in [1 \dots N]^n_\pi} \mathbf{X}^{(\vec{u})}_{\vec{v}}.$$

The stochastic measure corresponding to the partition π and the monomial $\mathbf{x}_{\vec{u}}$ is

$$\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \lim_{\delta(\mathcal{I}) \to 0} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})$$

if the limit, along the net of subdivisions of the interval $[0,t)$, exists. In particular, denote by $\Delta_n(\mathbf{x}_{\vec{u}},t;\mathbf{X},\mathcal{I}) = \mathrm{St}_{\hat{1}}(\mathbf{x}_{\vec{u}},t;\mathbf{X},\mathcal{I})$ and

$$\Delta_n(\mathbf{x}_{\vec{u}},t;\mathbf{X}) = \mathrm{St}_{\hat{1}}(\mathbf{x}_{\vec{u}},t;\mathbf{X})$$

the n-dimensional *diagonal measure.*

Definition 3.2. The *combinatorial cumulant* corresponding to the partition π and the monomial $\mathbf{x}_{\vec{u}}$ is

$$R_\pi(\mathbf{x}_{\vec{u}},t;\mathbf{X}) = \lim_{\delta(\mathcal{I})\to 0} \varphi\left[\mathrm{St}_\pi(\mathbf{x}_{\vec{u}},t;\mathbf{X},\mathcal{I})\right]$$

if the limit exists. In particular, denote by

$$R(\mathbf{x}_{\vec{u}},t;\mathbf{X}) = R_{\hat{1}}(\mathbf{x}_{\vec{u}},t;\mathbf{X}) = \lim_{\delta(\mathcal{I})\to 0} \varphi\left[\Delta_n(\mathbf{x}_{\vec{u}},t;\mathbf{X},\mathcal{I})\right]$$

the n-th joint cumulant of $\mathbf{X}$ at time t. Note that the functional $R(\cdot,t;\mathbf{X})$ can be linearly extended to all of $\mathbb{C}\langle\mathbf{x}\rangle$. We call this functional the *cumulant functional* of the process $\mathbf{X}$ at time t. For $t=1$ we call the corresponding functional the cumulant functional of the process $\mathbf{X}$.

We will omit the dependence on $\mathbf{X}$ in the notation if it is clear from the context. Clearly if $\mathrm{St}_\pi(\mathbf{x}_{\vec{u}},t)$ is well-defined, its expectation is equal to $R_\pi(\mathbf{x}_{\vec{u}},t)$.
By definition of S^n_π, for any $\mathcal{I}$

$$\mathbf{X}^{(\vec{u})}(t) = \sum_{\pi\in\mathcal{P}(n)} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}},t;\mathbf{X},\mathcal{I}). \tag{2}$$

If $\mathrm{St}_\pi(\mathbf{x}_{\vec{u}},t)$ are well-defined, then

$$\mathbf{X}^{(\vec{u})}(t) = \sum_{\pi\in\mathcal{P}(n)} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}},t;\mathbf{X}),$$

and so

$$M(\mathbf{x}_{\vec{u}},t;\mathbf{X}) = \sum_{\pi\in\mathcal{P}(n)} R_\pi(\mathbf{x}_{\vec{u}},t;\mathbf{X}); \tag{3}$$

in fact for this last property to hold it suffices that the combinatorial cumulants exist.
The following general algebraic notion of independence is due to Kümmerer.

Lemma 3.3. *A multiple q-Lévy process $\mathbf{X}(t)$ has* pyramidally independent *increments. That is, for a family of intervals $\left\{\{I_i\}_{i=1}^{n_1+n_3}, \{J_j\}_{j=1}^{n_2}\right\}$ in $\mathbb{R}_+$ such that for all i,j, $I_i\cap J_j=\emptyset$,*

$$\varphi\left[\left(\prod_{i=1}^{n_1} X^{(u(i))}(I_i)\right)\left(\prod_{j=1}^{n_2} X^{(v(j))}(J_j)\right)\left(\prod_{i=n_1+1}^{n_1+n_3} X^{(u(i))}(I_i)\right)\right] \\ = \varphi\left[\prod_{i=1}^{n_1+n_3} X^{(u(i))}(I_i)\right]\varphi\left[\prod_{j=1}^{n_2} X^{(v(j))}(J_j)\right].$$

We record the following facts we will use in the proof. Their own proof is immediate.

Lemma 3.4. *Choose two families of intervals* $\left\{ \{I_i\}_{i=1}^{n_1}, \{J_j\}_{j=1}^{n_2} \right\}$ *such that* $(\bigcup I_i) \cap (\bigcup J_j) = \emptyset$. *Let* $\mathbf{y} = \prod_{i=1}^{n_1} y_{I_i}^{(u(i))}$, *where each* $y_I^{(s)}$ *is one of* $a_I(\xi_s), a_I^*(\xi_s), p_I(T_s), |I| \lambda_s$. *Also let* $\vec{\eta}_1 \in \bigoplus_{j=0}^{n_1} (L^2(\bigcup I_i) \otimes V)^{\otimes j}$ *and* $\vec{\eta}_2 \in \bigoplus_{i=0}^{n_2} (L^2(\bigcup J_j) \otimes V)^{\otimes i}$, *where these two spaces are naturally embedded in* $\mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes V)$. *Then*

$$\mathbf{y}\vec{\eta}_2 = ((\mathbf{y} - \langle \Omega, \mathbf{y}\Omega \rangle_q)\Omega) \otimes \vec{\eta}_2 + \langle \Omega, \mathbf{y}\Omega \rangle_q \vec{\eta}_2$$

and

$$\langle \vec{\eta}_1, \vec{\eta}_2 \rangle_q = \langle \vec{\eta}_1, \Omega \rangle_q \langle \Omega, \vec{\eta}_2 \rangle_q .$$

Proof of Lemma 3.3. Fix a family of intervals $\left\{ \{I_i\}_{i=1}^{n_1+n_3}, \{J_j\}_{j=1}^{n_2} \right\}$ such that for all i, j, $I_i \cap J_j = \emptyset$. Denote

$$\begin{aligned}
\vec{\eta}_1 &= \left(\prod_{i=n_1}^{1} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)}) \right) \Omega && \in \bigoplus_{j=0}^{n_1} (L^2(\bigcup I_i) \otimes V)^{\otimes j}, \\
\vec{\eta}_2 &= \left(\prod_{i=1}^{n_2} p_{J_i}(\xi_{v(i)}, T_{v(i)}, \lambda_{v(i)}) \right) \Omega && \in \bigoplus_{j=0}^{n_2} (L^2(\bigcup J_i) \otimes V)^{\otimes j}, \\
\vec{\eta}_3 &= \left(\prod_{i=n_1+1}^{n_1+n_3} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)}) \right) \Omega && \in \bigoplus_{j=0}^{n_3} (L^2(\bigcup I_i) \otimes V)^{\otimes j}.
\end{aligned}$$

Then

$$\begin{aligned}
&\varphi \left[\left(\prod_{i=1}^{n_1} X^{(u(i))}(I_i) \right) \left(\prod_{j=1}^{n_2} X^{(v(j))}(J_j) \right) \left(\prod_{i=n_1+1}^{n_1+n_3} X^{(u(i))}(I_i) \right) \right] \\
&= \Bigg\langle \Big(\prod_{i=n_1}^{1} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)}) \Big) \Omega, \\
&\qquad \Big(\prod_{j=1}^{n_2} p_{J_j}(\xi_{v(j)}, T_{v(j)}, \lambda_{v(j)}) \prod_{i=n_1+1}^{n_1+n_3} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)}) \Big) \Omega \Bigg\rangle_q \\
&= \left\langle \vec{\eta}_1, \left(\prod_{j=1}^{n_2} p_{J_j}(\xi_{v(j)}, T_{v(j)}, \lambda_{v(j)}) \right) \vec{\eta}_3 \right\rangle_q \\
&= \left\langle \vec{\eta}_1, (\vec{\eta}_2 - \langle \Omega, \vec{\eta}_2 \rangle_q \Omega) \otimes \vec{\eta}_3 + \langle \Omega, \vec{\eta}_2 \rangle_q \vec{\eta}_3 \right\rangle_q \\
&= \langle \Omega, \vec{\eta}_2 \rangle_q \langle \vec{\eta}_1, \vec{\eta}_3 \rangle_q \\
&= \varphi \left[\prod_{j=1}^{n_2} X^{(v(j))}(J_j) \right] \varphi \left[\prod_{i=1}^{n_1+n_3} X^{(u(i))}(I_i) \right].
\end{aligned}$$

Proposition 3.5. *For a noncrossing partition σ,*

$$M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X})$$

if the combinatorial cumulants are well-defined.

Proof. A noncrossing partition is determined by the property that it contains a class that is an interval and the restriction of the partition to the complement of that class is still noncrossing. Using this fact and Lemma 3.3, we can conclude that for $\pi \in \mathcal{P}(n), \pi \leq \sigma$ and $\vec{v} \in [1 \ldots N]^n_\pi$,

$$\varphi\left[\mathbf{X}^{(\vec{u})}_{\vec{v}}\right] = \prod_{B \in \sigma} \varphi\left[\mathbf{X}^{(B:\vec{u})}_{(B:\vec{v})}\right].$$

Therefore

$$\varphi\left[\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})\right] = \prod_{B \in \sigma} \varphi\left[\mathrm{St}_{(B:\pi)}(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}, \mathcal{I})\right].$$

Thus if the combinatorial cumulants are well-defined,

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \prod_{B \in \sigma} R_{(B:\pi)}(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}),$$

and so

$$\sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} \prod_{B \in \sigma} R_{(B:\pi)}(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}).$$

If $\sigma = (B_1, B_2, \ldots, B_l)$, the left-hand-side of this equation is equal to

$$\prod_{i=1}^{l} \sum_{\pi_i \in \mathcal{P}(B_i)} R_{\pi_i}(\mathbf{x}_{(B_i:\vec{u})}, t; \mathbf{X}).$$

Combining this equation with equation (3), we obtain

$$M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}). \qquad \square$$

We emphasize that while σ is noncrossing, π need not be. Note that on the operator level we have for any $\sigma \in \mathcal{P}(n)$,

$$\sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I}) = \sum_{\vec{v} \in [1 \ldots N]^n_{\leq \sigma}} \mathbf{X}^{(\vec{u})}_{\vec{v}}.$$

Proposition 3.6. *For the monomial $\mathbf{x}_{\vec{u}}$ of degree n, the cumulant functional of the multiple q-Lévy process $\mathbf{X}$ is given by*

$$R(\mathbf{x}_{\vec{u}}, t) = \begin{cases} t\lambda_{u(1)} & \text{if } n = 1, \\ t\left\langle \xi_{u(1)}, \prod_{j=2}^{n-1} T_{u(j)} \xi_{u(n)} \right\rangle & \text{if } n \geq 2. \end{cases}$$

Proof. By definition,

$$R(\mathbf{x}_{\vec{u}}, t) = \lim_{\delta(\mathcal{I})\to 0} \varphi\left[\sum_{i=1}^{N}\prod_{j=1}^{n} p_i(\xi_{u(j)}, T_{u(j)}, \lambda_{u(j)})\right].$$

For $n = 1$,

$$\langle \Omega, p_i(\xi, T, \lambda)\Omega\rangle_q = |I_i|\,\lambda,$$

and so $R(x, t) = t\lambda$.

Now let $n \geq 2$. Decomposing each $p_i(\xi, T, \lambda)$ into the four defining summands, we see that

$$\varphi\left[\sum_{i=1}^{N}\prod_{j=1}^{n} p_i(\xi_{u(j)}, T_{u(j)}, \lambda_{u(j)})\right] = \sum_{S_1,S_2,S_3,S_4}\sum_{i=1}^{N}\left\langle \Omega, y_i^{(1)} y_i^{(2)}\ldots y_i^{(n)}\Omega\right\rangle_q. \tag{4}$$

Here the sum is taken over all decompositions of $[1\ldots n]$ into four disjoint subsets S_1, S_2, S_3, S_4, and for each choice of these subsets

$$y_i^{(j)} = \begin{cases} a_i(\xi_{u(j)}) & \text{if } j \in S_1, \\ a_i^*(\xi_{u(j)}) & \text{if } j \in S_2, \\ p_i(T_{u(j)}) & \text{if } j \in S_3, \\ |I_i|\,\lambda_{u(j)} & \text{if } j \in S_4. \end{cases}$$

The term corresponding to $S_1 = \{1\}, S_2 = \{n\}, S_3 = [2\ldots(n-1)], S_4 = \emptyset$ is equal to

$$\left\langle \mathbf{1}_{[0,t)} \otimes \xi_{u(1)}, (\mathbf{1}_{[0,t)} \otimes \prod_{j=2}^{n-1} T_{u(j)})(\mathbf{1}_{[0,t)} \otimes \xi_{u(n)})\right\rangle = t\left\langle \xi_{u(1)}, \prod_{j=2}^{n-1} T_{u(j)}\xi_{u(n)}\right\rangle.$$

We show that the limit of each of the remaining terms is 0. Indeed, $y_i^{(1)} y_i^{(2)}\ldots y_i^{(n)}\Omega \in H^{\otimes(|S_2|-|S_1|)}$, so if $|S_1| \neq |S_2|$ the corresponding term in (4) is 0 even for finite N. Otherwise denote by $b(S_1, S_2)$ the set of all bijections $S_1 \to S_2$. All the terms that are not 0 are of the form

$$\sum_{i=1}^{N}\left(\left(\prod_{j_4\in S_4}\lambda_{u(j_4)}\right)|I_i|^{|S_4|}\sum_{g\in b(S_1,S_2)}\sum_{\substack{S'_{j_1}\subset S_3: j_1\in S_1,\\ \bigcup_{j_1\in S_1} S'_{j_1} = S_3}} Q_{g,\{S_{j_1}, j_1\in S_1\}}(q)\,|I_i|^{|S_1|} \times \prod_{j_1\in S_1}\left\langle \xi_{u(j_1)}, \prod_{j_3\in S'_{j_1}} T_{u(j_3)}\xi_{u(g(j_1))}\right\rangle\right),$$

where each $Q_{g,\{S'_{j_1}: j_1\in S_1\}}(q)$ is a polynomial independent of i, and $|S_1| \geq 2$ or $|S_4| \geq 1, |S_1| \geq 1$; in both cases $|S_4| + |S_1| \geq 2$. Thus each of these terms is bounded by

$$C\sum_{i=1}^{N}|I_i|^{|S_1|+|S_4|} \leq C\delta(\mathcal{I})t^{|S_1|+|S_4|-1},$$

where C is a constant independent of the subdivision $\mathcal{I}$. Therefore such a term converges to 0 as $\delta(\mathcal{I}) \to 0$. $\square$

Construction 3.7 (An un-crossing map). Fix a partition π with l classes $B_1, \ldots, B_l$. In preparation for the next theorem, we need the following combinatorial construction. Define the map $F : \mathcal{P}(n) \to \mathcal{P}(n)$ as follows. If π is an interval partition, $F(\pi) = \pi$. Otherwise, let i be the largest index of a non-interval class B_i of π. Let $j_2 = \max\{s \in B_i : (s-1) \notin B_i\}$ and $j_1 = p(j_2)$. Let α be the power of a cycle permutation

$$((j_1+1)(j_1+2)\ldots b(B_i))^{b(B_i)-j_2+1}.$$

Then $F(\pi) = \alpha \circ \pi$, by which we mean $i \overset{\pi}{\sim} j \Leftrightarrow \alpha(i) \overset{F(\pi)}{\sim} \alpha(j)$. Also define $c_b(\pi) = |\{s : j_1 < b(B_s) < b(B_i)\}| - |\{s : j_1 < a(B_s) < b(B_i)\}|$. Then $\mathrm{rc}\,(\pi) = \mathrm{rc}\,(F(\pi)) + c_b(\pi)$. Indeed, for $B, C \in \pi, B, C \neq B_i$, $\mathrm{rc}\,(B,C) = \mathrm{rc}\,(\alpha(B), \alpha(C))$. The number of restricted crossings of B_i, B_j with $b_i \in B_i, b_j \in B_j$ and $p(b_i) < p(b_j) < b_i < b_j \leq j_1$ or $p(b_j) < p(b_i) < b_j < b_i \leq j_1$ is equal to the corresponding number for $\alpha(B_i), \alpha(B_j)$, while there are no restricted crossings for $b_i > j_2$ for B_i and $b_i > j_1$ for $\alpha(B_i)$. Finally, there are $c_b(\pi)$ restricted crossings of the form $p(j) < j_1 < j < j_2$ in π. See Figure 2 for an example.

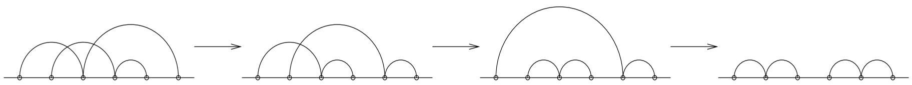

Figure 2. Iteration of F on a partition of 6 elements.

Clearly $F^n(\pi)$ is an interval partition. Therefore $\sum_{s=0}^{n} c_b(F^s \pi) = \mathrm{rc}\,(\pi)$.

Theorem 3.8. *The combinatorial cumulants can be expressed in terms of the cumulant functional: for $\pi \in \mathcal{P}(n)$ and $\mathbf{x}_{\vec{u}}$ a monomial of degree n,*

$$R_\pi(\mathbf{x}_{\vec{u}}, t) = q^{\mathrm{rc}(\pi)} \prod_{i=1}^{l} R(\mathbf{x}_{(B_i:\vec{u})}, t).$$

Proof. The same argument as in the previous proposition shows that

$$R_\pi(\mathbf{x}_{\vec{u}}, t) = \lim_{\delta(\mathcal{I})\to 0} \varphi \left[\sum_{\vec{v} \in [1\ldots N]^n_\pi} y^{(1)}_{v(1)} y^{(2)}_{v(2)} \ldots y^{(n)}_{v(n)} \right],$$

with

$$y_i^{(j)} = \begin{cases} |I_i|\,\lambda_{u(j)} & \text{if } (j) \text{ is a singleton in } \pi, \\ a_i(\xi_{u(j)}) & \text{if } j \text{ is the first element of its class in } \pi, \\ a_i^*(\xi_{u(j)}) & \text{if } j \text{ is the last element of its class in } \pi, \\ p_i(T_{u(j)}) & \text{otherwise.} \end{cases} \tag{5}$$

Fix $\vec{v}$. Let B be the class of π containing n. If B is an interval, then by Lemma 3.4

$$\left\langle \Omega, \left(\prod_{j=1}^{n} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q = \left\langle \Omega, \left(\prod_{j=1}^{a(B)-1} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q \left\langle \Omega, \left(\prod_{j=a(B)}^{n} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q .$$

Therefore

$$R_\pi(\mathbf{x}_{\vec{u}}, t) = R_{(B_1, \ldots, B_{l-1})}(\mathbf{x}_{([1\ldots n]\setminus B:\vec{u})}, t) R(\mathbf{x}_{(B:\vec{u})}, t).$$

Now suppose B is not an interval. Use the notation α, j_1, j_2, c_b of Construction 3.7. Denote

$$\eta(j_2) = \left(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \right) \Omega \in H$$

and

$$\vec{\eta}(j_1) = \left(\prod_{i=j_1+1}^{j_2-1} y_{v(i)}^{(i)} \right) \Omega \in H^{\otimes (c_b(\pi))}.$$

Note that $y_{v(j_1)}^{(j_1)}$ is either $a_{v(j_1)}(\xi_{u(j_1)})$ or $p_{v(j_1)}(T_{u(j_1)})$. Then

$$\begin{aligned} \left(\prod_{i=j_1}^{n} y_{v(i)}^{(i)} \right) \Omega &= \left(\prod_{i=j_1}^{j_2-1} y_{v(i)}^{(i)} \right) \eta(j_2) = y_{v(j_1)}^{(j_1)} \left(\left(\Big(\prod_{i=j_1+1}^{j_2-1} y_{v(i)}^{(i)} \Big) \Omega \right) \otimes \eta(j_2) \right) \\ &= y_{v(j_1)}^{(j_1)} (\vec{\eta}(j_1) \otimes \eta(j_2)) = q^{c_b(\pi)} (y_{v(j_1)}^{(j_1)} \eta(j_2)) \otimes \vec{\eta}(j_1) \end{aligned}$$

and

$$\begin{aligned} y_{v(j_1)}^{(j_1)} \left(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \prod_{i=j_1+1}^{j_2-1} y_{v(i)}^{(i)} \right) \Omega &= y_{v(j_1)}^{(j_1)} \left(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \right) \vec{\eta}(j_1) \\ &= y_{v(j_1)}^{(j_1)} \left(\left(\Big(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \Big) \Omega \right) \otimes \vec{\eta}(j_1) \right) \\ &= y_{v(j_1)}^{(j_1)} (\eta(j_2) \otimes \vec{\eta}(j_1)) = (y_{v(j_1)}^{(j_1)} \eta(j_2)) \otimes \vec{\eta}(j_1). \end{aligned}$$

Therefore

$$\left\langle \Omega, \left(\prod_{j=1}^{n} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q = q^{c_b(\pi)} \left\langle \Omega, \left(\prod_{j=1}^{n} y_{v(\alpha(j))}^{(\alpha(j))} \right) \Omega \right\rangle_q .$$

The right-hand-side contains precisely the product of y's corresponding to the partition $F(\pi)$. The result follows by iterating these two steps. $\square$

Remark 3.9 (Comments on Proposition 3.5). For $q = 0$, $R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = 0$ unless π is noncrossing. Then for $\sigma \in NC(n)$,

$$M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\pi \in NC(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}).$$

Therefore for $\pi \in NC(n)$,

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi}} \text{Möb}_{NC}(\sigma, \pi) M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}),$$

where Möb_{NC} is the Möbius function on the lattice of noncrossing partitions. For $q = 1$, if $\sigma \in \mathcal{P}(n)$, $\sigma = (B_1, B_2, \ldots, B_l)$, then

$$\begin{aligned} M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) &= \prod_{i=1}^{l} M(\mathbf{x}_{(B_i : \vec{u})}, t; \mathbf{X}) \\ &= \prod_{i=1}^{l} \sum_{\pi_i \in \mathcal{P}(B_i)} R_{\pi_i}(\mathbf{x}_{(B_i : \vec{u})}, t; \mathbf{X}) \\ &= \sum_{\pi \leq \sigma} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}). \end{aligned}$$

Therefore for $\pi \in \mathcal{P}(n)$,

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\sigma \in \mathcal{P}(n) \\ \sigma \leq \pi}} \text{Möb}_{\mathcal{P}}(\sigma, \pi) M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}),$$

where $\text{Möb}_{\mathcal{P}}$ is the Möbius function on the lattice of all partitions. Note that $\mathbf{X}(I)$ commute with $\mathbf{X}(J)$ for $I \cap J = \emptyset$ on the symmetric Fock space.
Thus for $q = 0, 1$, the cumulant functional at time 1 can be expressed through the moment functional at time 1. We will show how to do this for arbitrary q in the next section.

4. Characterization of generators

Denote by $R(f; \mathbf{X}) = R(f, 1; \mathbf{X})$ the cumulant functional.

Lemma 4.1. *The family of the moment functionals of a multiple q-Lévy process is determined by its cumulant functional. The functional $R(\cdot; \mathbf{X})$ on $\mathbb{C}\langle \mathbf{x} \rangle$ is the* generator *of the family of functionals $M(\cdot, t; \mathbf{X})$, that is,*

$$\left.\frac{d}{dt}\right|_{t=0} M(f, t; \mathbf{X}) = R(f; \mathbf{X}).$$

Proof. It suffices to prove these statements for a monomial $\mathbf{x}_{\vec{u}}$ of degree n. By equation (3), Theorem 3.8 and Proposition 3.6,

$$\begin{aligned} M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) &= \sum_{\pi \in \mathcal{P}(n)} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) \\ &= \sum_{\pi \in \mathcal{P}(n)} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} R(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}) \\ &= \sum_{\pi \in \mathcal{P}(n)} q^{\mathrm{rc}(\pi)} t^{|\pi|} \prod_{B \in \pi} R(\mathbf{x}_{(B:\vec{u})}, 1; \mathbf{X}), \end{aligned}$$

which implies the first statement. By differentiating this equality, we obtain

$$\frac{d}{dt}\bigg|_{t=0} M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = R_{\hat{1}}(\mathbf{x}_{\vec{u}}, 1; \mathbf{X}) = R(\mathbf{x}_{\vec{u}}; \mathbf{X}). \qquad \square$$

Definition 4.2. A functional ψ on $\mathbb{C}\langle \mathbf{x} \rangle$ is *conditionally positive* if its restriction to the subspace of polynomials with zero constant term is positive semi-definite.

We say that the functional ψ is *analytic* if for any i and any multi-index $\vec{u}$,

$$\limsup_{n \to \infty} \frac{1}{n} \psi[(\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}]^{1/2n} < \infty.$$

The following proposition is an analog of the Schoenberg correspondence for our context. Note that the formulation of the result does not involve q: the dependence on q is hidden in Theorem 3.8.

Proposition 4.3. *A functional ψ is analytic and conditionally positive if and only if it is the generator of the family of the moment functionals for some multiple q-Lévy process.*

Proof. The proof is practically identical to that of [GSS92], or indeed of [Sch91]. We provide an outline for the reader's convenience.

Suppose ψ is the generator of the family of moment functionals $M(\cdot, t; \mathbf{X})$ for a multiple q-Lévy process $\mathbf{X}(t) = p_t(\xi, \mathbf{T}, \lambda)$. From the fact that each of the moment functionals is positive and equals 1 on the constant 1 it follows by differentiating that the cumulant functional is conditionally positive. Since $\psi = R(\cdot; \mathbf{X})$, for $\mathbf{x}_{\vec{u}}$ of degree l

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \psi[(\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}]^{1/2n} &= \limsup_{n \to \infty} \frac{1}{n} R((\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}, t; \mathbf{X})^{1/2n} \\ &= \limsup_{n \to \infty} \frac{1}{n} \left\langle \xi_{u(l)}, \prod_{j=l-1}^{1} T_{u(j)} T_i^{2n} \prod_{j=1}^{l-1} T_{u(j)} \xi_{u(l)} \right\rangle^{1/2n} \\ &= \limsup_{n \to \infty} \frac{1}{n} \left\| T_i^n \prod_{j=1}^{l-1} T_{u(j)} \xi_{u(l)} \right\|^{1/n} < \infty \end{aligned}$$

since the vector $\prod_{j=1}^{l-1} T_{u(j)} \xi_{u(l)}$ is analytic for T_i.

Now suppose ψ is conditionally positive and analytic. Then it gives rise to a multiple q-Lévy process, as follows. Denote by $\delta_0(f)$ the constant term of $f \in \mathbb{C}\langle \mathbf{x} \rangle$. ψ induces a positive semi-definite inner product on the space $\mathbb{C}\langle \mathbf{x} \rangle$ by $\langle f, g \rangle_\psi = \psi[(f - \delta_0(f))^*(g - \delta_0(g))]$. Let $\mathcal{N}_\psi$ be the subspace of vectors of length 0 with respect to this inner product. Let V be the Hilbert space obtained by completing the quotient $(\mathbb{C}\langle \mathbf{x} \rangle)/\mathcal{N}_\psi$ with respect to this inner product, with the induced inner product. Denote by ρ the canonical mapping $\mathbb{C}\langle \mathbf{x} \rangle \to V$, let $\mathcal{D}$ be its image, and for $f, g \in \mathbb{C}\langle \mathbf{x} \rangle$ define the operator $\Gamma(f) : \mathcal{D} \to \mathcal{D}$ by

$$\Gamma(f)\rho(g) = \rho(fg) - \rho(f)\delta_0(g).$$

The operator Γ is well defined since, by the Cauchy-Schwartz inequality,

$$\left\|\Gamma(f)\rho(g)\right\|_\psi = \psi[(g - \delta_0(g))^* f^* f (g - \delta_0(g))] \leq \left\|\rho(g)\right\|_\psi \left\|f^* f(g - \delta_0(g))\right\|_\psi.$$

Clearly $\mathcal{D}$ is dense in V, invariant under $\Gamma(f)$, and $\Gamma(f)$ is symmetric on it if f is symmetric.

Put, for $i \in [1 \ldots k]$, $\lambda_i = \psi[x_i], \xi_i = \rho(x_i), T_i = \Gamma(x_i)$. Each T_i takes $\mathcal{D}$ to itself. By construction, $\Gamma(x_i)\rho(\mathbf{x}_{\vec{u}}) = \rho(x_i \mathbf{x}_{\vec{u}})$, and so

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \left\|T_i^n \rho(\mathbf{x}_{\vec{u}})\right\|_\psi^{1/n} &= \limsup_{n \to \infty} \frac{1}{n} \left\|x_i^n \mathbf{x}_{\vec{u}}\right\|_\psi^{1/n} \\ &= \limsup_{n \to \infty} \frac{1}{n} \psi[(\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}]^{1/2n} < \infty \end{aligned}$$

since the functional ψ is analytic. Therefore each of the vectors $\rho(\mathbf{x}_{\vec{u}})$ is analytic for T_i, and the linear span of these vectors is $\mathcal{D}$. In particular, T_i is essentially self-adjoint on $\mathcal{D}$.

Define the multiple q-Lévy process $\mathbf{X}$ by $X^{(i)}(t) = p_t(\xi_i, T_i, \lambda_i)$. Then

$$R(\mathbf{x}_{\vec{u}}; \mathbf{X}) = \psi[\mathbf{x}_{\vec{u}}].$$

Indeed, for $n = 1$

$$R(x_i; \mathbf{X}) = \lambda_i = \psi[x_i].$$

For $n \geq 2$,

$$\begin{aligned} R(\mathbf{x}_{\vec{u}}; \mathbf{X}) &= \left\langle \xi_{u(1)}, \prod_{j=2}^{n-1} T_{(u(j))} \xi_{u(n)} \right\rangle = \left\langle \rho(x_{u(1)}), \prod_{j=2}^{n-1} \Gamma(x_{u(j)}) \rho(x_{u(n)}) \right\rangle_\psi \\ &= \left\langle \rho(x_{u(1)}), \rho(\prod_{j=2}^{n} x_{u(j)}) \right\rangle_\psi = \psi[\prod_{j=1}^{n} x_{u(j)}] \\ &= \psi[\mathbf{x}_{\vec{u}}]. \end{aligned}$$

Therefore ψ is the generator of the moment functional family of $\mathbf{X}$. $\square$

4.1. Product states. For arbitrary q, the relation in the proof of Lemma 4.1 can be inverted.

Definition 4.4. Let Φ be any functional on $\mathbb{C}\langle \mathbf{x} \rangle$. Define the functional $\Psi = \log_q(\Phi)$ on monomials recursively by

$$\Psi(\mathbf{x}_{\vec{u}}) = \Phi(\mathbf{x}_{\vec{u}}) - \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \neq \hat{1}}} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} \Psi(\mathbf{x}_{(B:\vec{u})})$$

and extend linearly.

The definition has the form

$$\Psi(\mathbf{x}_{\vec{u}}) = \sum_{\sigma \in \mathcal{P}(n)} c(\sigma) \prod_{B \in \sigma} \Phi(\mathbf{x}_{(B:\vec{u})})$$

for some coefficient family $\{c(\sigma) : \sigma \in \mathcal{P}(k)\}$. For $q = 1$, Φ is the convolution exponential of Ψ [Sch91]. Lemma 4.1 and the discussion in Section 6 justify the notations $\Psi = \log_q(\Phi)$, $\Phi = \exp_q(\Psi)$. Note that this operation on functionals appears to bear no relation to the q-exponential power series.
It is clear that for any q-Lévy process, $R(\cdot, t; \mathbf{X}) = \log_q M(\cdot, t; \mathbf{X})$ and, moreover, that $M(\cdot, t; \mathbf{X}) = \exp_q(tR(\cdot; \mathbf{X}))$.

Definition 4.5. Let Φ_1 be a functional on $\mathbb{C}\langle x_1, x_2, \ldots, x_{k_1} \rangle$, Φ_2 a functional on $\mathbb{C}\langle x_1, x_2, \ldots, x_{k_2} \rangle$. Define their *product functional* $\Phi_1 \times_q \Phi_2$ on $\mathbb{C}\langle x_1, x_2, \ldots, x_{k_1+k_2} \rangle$ by the "mixed cumulants are 0" rule:

$$\log_q(\Phi_1 \times_q \Phi_2)(\mathbf{x}_{\vec{u}}) = \begin{cases} \log_q(\Phi_1)(\mathbf{x}_{\vec{u}}) & \text{if } \forall i, u(i) \leq k_1, \\ \log_q(\Phi_2)(\mathbf{x}_{\vec{u}}) & \text{if } \forall i, u(i) > k_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that it is more natural to think of this construction as taking the product of two one-parameter families of functionals,

$$\exp_q(t \log_q(\Phi_1)) \times_q \exp_q(t \log_q(\Phi_2)) = \exp_q(t \log_q(\Phi_1 \times_q \Phi_2)).$$

Denote

$$\begin{aligned} \mathcal{ID}_c(q, k) &= \{\Phi : \Phi = M(\cdot, 1; \mathbf{X}) \text{ for some } k\text{-dimensional } q\text{-Lévy process } \mathbf{X}\} \\ &= \left\{\Phi : \log_q(\Phi) \text{ is conditionally positive and analytic}\right\}. \end{aligned}$$

The notation stands for "combinatorially infinitely divisible".

Lemma 4.6. *For $\Phi_1 \in \mathcal{ID}_c(q, k_1), \Phi_2 \in \mathcal{ID}_c(q, k_2)$, their product functional is a state, that is, a positive functional that equals 1 on the identity element.*

Proof. It suffices to show that $\Phi_1 \times_q \Phi_2 \in \mathcal{ID}_c(q, k_1 + k_2)$. Let $\mathbf{X}_1, \mathbf{X}_2$ be the q-Lévy processes whose distributions at time 1 are Φ_1, Φ_2, respectively. Let $X^{(i,1)}(t) = p_t(\xi_{i,1}, T_{i,1}, \lambda_{i,1})$, $X^{(i,2)}(t) = p_t(\xi_{i,2}, T_{i,2}, \lambda_{i,2})$. Here $\xi_{i,1} \in V_1$, $T_{i,1}$ is an operator on V_1 with domain $\mathcal{D}_1$, $\xi_{i,2} \in V_2$, $T_{i,2}$ is an operator on V_2 with domain $\mathcal{D}_2$. Let $V = V_1 \oplus V_2$. Identify $\xi_{i,1}$ with $\xi_{i,1} \oplus 0$, $\xi_{i,2}$ with $0 \oplus \xi_{i,2}$, $T_{i,1}$ with $\left(\begin{smallmatrix} T_{i,1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $T_{i,2}$ with $\left(\begin{smallmatrix} 0 & 0 \\ 0 & T_{i,2} \end{smallmatrix}\right)$. It is easy to see that this identification does not change the cumulants or the moments of the processes $\mathbf{X}_1, \mathbf{X}_2$, and that condition (1) holds for the (k_1+k_2)-dimensional process $\mathbf{X} = (X^{(1,1)}, \ldots, X^{(k_1,1)}, X^{(1,2)}, \ldots, X^{(k_2,2)})$. Then $\Phi_1 \times_q \Phi_2$ is equal to $M(\cdot, 1; \mathbf{X})$. $\square$

For $q = 1$, the product state is the usual (tensor) product state, while for $q = 0$ it is the (reduced) free product state. Already for $q = -1$, the situation is unclear. The parity of $\operatorname{rc}(\pi)$ can differ from the parity of the number of left-reduced crossings of [Nic95] even for partitions all of whose classes have even order. Therefore even for $q = -1$, our cumulants are different from the q-cumulants of that paper. In particular, the results of [MN97] about graded independence do not apply. Note also that our product state construction is defined only on the polynomial algebras $\mathbb{C}\langle \mathbf{x} \rangle$, not on general algebras. So we do not obtain a universal product in the sense of [Spe97b].
A state Φ is tracial if for all a, b, $\Phi(ab) = \Phi(ba)$. For $q = 0, 1$, the product state of two tracial states is tracial [VDN92]. This property remains true for the q-Brownian motion (see below). However, the number of the restricted crossings of a partition is not invariant under cyclic permutations of the underlying set. For example, $\operatorname{rc}(((1,3,5)(2,4))) = 2$ while $\operatorname{rc}(((1,3)(2,4,5))) = 1$. So for general q, the product state of two tracial states need not be tracial.

5. THE ITÔ TABLE

In general we do not know how to calculate the partition-dependent stochastic measures $\mathrm{St}_\pi(\mathbf{X})$; indeed we don't expect a nice answer for a general process. In particular we don't expect that a functional Itô formula exists for q-Lévy processes. However, one ingredient of it is present, namely, we can calculate all the higher diagonal measures. These are higher variations of the processes, and appear in the functional Itô formula for the free Lévy processes [Ans01b].

REMARK 5.1 (Algebraic approach). Unless we are considering higher diagonal measures of a single one-dimensional process, for this section we also need a more general setup than the one we had before. First, we need to consider multiple processes whose components are of the form $X(t) = p_t(\xi, \eta, T, \lambda) = a_t(\xi) + a_t^*(\eta) + p_t(T) + t\lambda$. Second, we no longer can require T to be symmetric and λ to be real. The solution in [Sch91] is to require that T be a linear operator with domain $\mathcal{D}$, not necessarily dense, so that the restriction of T^* to $\mathcal{D}$ is a well-defined linear operator.
We describe briefly how to modify this paper for the algebraic context. The gauge operators are defined in the same way, and the multiple q-Lévy process are modified as in the previous paragraph, except that we drop the assumption (1). The moments and cumulants can be modified to include $*$-quantities, i.e. use words in both $\mathbf{X}$ and $\mathbf{X}^*$ in the definitions, and consider them as functionals on $\mathbb{C}\langle x_1, x_2, \ldots, x_k, x_1^*, x_2^*, \ldots, x_k^* \rangle$ with the obvious conjugation. All the relations between moments and cumulants, and between partition-dependent cumulants and the cumulant functional, remain the same, and it is clear how to

modify the formula for the cumulant functional in terms of ξ, η, T, λ. In the algebraic context, generators of the families of moment functionals for symmetric processes are precisely all the conditionally positive functionals.

For the Itô table, we first need a technical lemma.

Lemma 5.2. *For $f, g \in L^2(\mathbb{R}_+)$,*

$$\lim_{\delta(\mathcal{I})\to 0} \left| \sum_{i=1}^{N} \Big(\int_{I_i} f(x)dx \Big) \Big(\int_{I_i} g(y)dy \Big) \right| = 0, \tag{6a}$$

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^{N} \Big(\mathbf{1}_{I_i}(x) f(x) \Big) \Big(\int_{I_i} g(y)dy \Big) \right\|_2 = 0, \tag{6b}$$

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^{N} \Big(\mathbf{1}_{I_i}(x) f(x) \Big) \Big(\mathbf{1}_{I_i}(y) g(y) \Big) \right\|_2 = 0. \tag{6c}$$

Proof. We repeatedly use the Cauchy-Schwartz inequality for sequences and functions:

$$\begin{aligned}
\left| \sum_{i=1}^{N} \Big(\int_{I_i} f(x)dx \Big) \Big(\int_{I_i} g(y)dy \Big) \right| &\leq \sqrt{ \sum_{i=1}^{N} \Big(\int_{I_i} f(x)dx \Big)^2 \sum_{j=1}^{N} \Big(\int_{I_j} g(y)dy \Big)^2 } \\
&\leq \sqrt{ \sum_{i=1}^{N} |I_i| \int_{I_i} f^2(x)dx \sum_{j=1}^{N} |I_j| \int_{I_j} g^2(y)dy } \\
&\leq \delta(\mathcal{I}) \sqrt{ \sum_{i=1}^{N} \int_{I_i} f^2(x)dx \sum_{j=1}^{N} \int_{I_j} g^2(y)dy } \\
&\leq \delta(\mathcal{I}) \left\| f \right\|_2 \left\| g \right\|_2 .
\end{aligned}$$

$$\begin{aligned}
\left\| \sum_{i=1}^{N} \Big(\mathbf{1}_{I_i}(x) f(x) \Big) \Big(\int_{I_i} g(y)dy \Big) \right\|_2 &= \sqrt{ \sum_{i=1}^{N} \Big(\int_{I_i} f^2(x)dx \Big) \Big(\int_{I_i} g(y)dy \Big)^2 } \\
&\leq \sqrt{ \sum_{i=1}^{N} \int_{I_i} f^2(x)dx \, |I_i| \int_{I_i} g^2(y)dy } \\
&\leq \sqrt{\delta(\mathcal{I})} \left\| f \right\|_2 \left\| g \right\|_2 .
\end{aligned}$$

The last property requires a bit more work, since uniform estimates do not hold in this case. By the Cauchy-Schwartz inequality as above, we may assume that $f = g$; also without loss of generality we assume that $\left\| f \right\|_2 = 1$. Let $\mathcal{I} = (I_1, I_2, \ldots, I_M)$ be a subdivision of $[0, t)$, and $\varepsilon > 0$. For $N > \max(M, \frac{8}{\varepsilon^2})$ large enough, we can choose a subdivision $\mathcal{J}' = (J'_1, J'_2, \ldots, J'_N)$ so that all $\int_{J'_j} f^2(x)dx < \frac{2}{N}$ and no I_i is a subset of any J'_j. Let $\mathcal{J}$ be the smallest common refinement of $\mathcal{I}, \mathcal{J}'$. Then $\mathcal{J}$ consists of at most $M+N$ intervals J_j, and for each of them $\int_{J_j} f^2(x)dx < \frac{2}{N}$. Therefore

$$\begin{aligned}\left\|\sum_j \Big(\mathbf{1}_{J_j}(x)f(x)\Big)\Big(\mathbf{1}_{J_j}(y)f(y)\Big)\right\|_2 &= \sqrt{\sum_j \Big(\int_{J_j} f^2(x)dx\Big)\Big(\int_{J_j} f^2(y)dy\Big)} \\ &\leq \sqrt{\frac{4(M+N)}{N^2}} \\ &\leq \varepsilon.\end{aligned}$$

We conclude that $\left\|\sum_{i=1}^N \big(\mathbf{1}_{I_i}(x)f(x)\big)\big(\mathbf{1}_{I_i}(y)g(y)\big)\right\|_2$ converges to 0 along the net of subdivisions $\mathcal{I}$ as $\delta(\mathcal{I}) \to 0$. $\square$

PROPOSITION 5.3. *The Itô table for q-Lévy processes $X^{(i)}(t) = a_t(\xi_i)+a_t^*(\eta_i)+p_t(T_i)+t\lambda_i$ is*

$dX^{(1)}dX^{(2)}$	$da(\xi_2)$	$da^*(\eta_2)$	$dp(T_2)$	$\lambda_2 dt$
$da(\xi_1)$	0	$\langle \xi_1, \eta_2\rangle\, dt$	$da(T_2^*\xi_1)$	0
$da^*(\eta_1)$	0	0	0	0
$dp(T_1)$	0	$da^*(T_1\eta_2)$	$dp(T_1T_2)$	0
$\lambda_1 dt$	0	0	0	0

More precisely, the quadratic co-variation of these processes is

$$\Delta_2(x_1x_2, t; (X^{(1)}, X^{(2)})) = [X^{(1)}, X^{(2)}](t) = p_t(T_2^*\xi_1, T_1\eta_2, T_1T_2, \langle \xi_1, \eta_2\rangle).$$

Here the convergence in the definition of Δ is the pointwise convergence on the dense set $\mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes \mathcal{D})$.

Proof. We need to show that for $\vec{\zeta} \in \mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes \mathcal{D})$,

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \left(\sum_{u=1}^N (y_u^{((1),i)} y_u^{((2),j)}) - y^{(i,j)} \right) \vec{\zeta} \right\|_q = 0,$$

where $y^{((1),i)}, y^{((2),j)}$ are labels for rows, respectively, columns of the Itô table, and $y^{(i,j)}$ is the corresponding entry of the table. All of these are obtained by applying Lemma 5.2, possibly with one or both of f, g equal to $\mathbf{1}_{[0,t)}$. More precisely, we use equation (6a) for the product $da(\xi_1)da(\xi_2)$, equation (6b) for the products $da^*(\eta_1)da(\xi_2), dp(T_1)da(\xi_2), da(\xi_1)dp(T_2)$ and equation (6c) for the products $da^*(\eta_1)da^*(\eta_2), dp(T_1)da^*(\eta_2), da^*(\eta_1)dp(T_2), dp(T_1)dp(T_2)$.
We do the case $dp(T_1)da^*(\eta_2)$ as an example. The linear span of the vectors of the form $\vec{\zeta} = (f_1 \otimes \zeta_1) \otimes (f_2 \otimes \zeta_2) \otimes \ldots \otimes (f_n \otimes \zeta_n)$, for $f_1, f_2, \ldots, f_n \in$

$L^2(\mathbb{R}_+), \zeta_1, \zeta_2, \ldots, \zeta_n \in \mathcal{D}$, is dense in $\mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes \mathcal{D})$. For such a vector,

$$\begin{aligned}
\sum_{i=1}^{N} p_i(T_1) a_i^*(\eta_2) \vec{\zeta} &= \sum_{i=1}^{N} p_i(T_1)(\mathbf{1}_{I_i} \otimes \eta_2) \otimes \vec{\zeta} \\
&= \sum_{i=1}^{N} \Big((\mathbf{1}_{I_i} \mathbf{1}_{I_i} \otimes T_1 \eta_2) \otimes \vec{\zeta} \\
&\quad + \sum_{k=1}^{n} q^k (\mathbf{1}_{I_i} f_k \otimes T_1 \zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \otimes (f_1 \otimes \zeta_1) \otimes \ldots \otimes (f_n \otimes \zeta_n) \Big) \\
&= \sum_{i=1}^{N} a_i^*(T_1 \eta_2) \vec{\zeta} \\
&\quad + \sum_{i=1}^{N} \sum_{k=1}^{n} q^k (\mathbf{1}_{I_i} f_k \otimes T_1 \zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \otimes (f_1 \otimes \zeta_1) \otimes \ldots \otimes (f_n \otimes \zeta_n).
\end{aligned}$$

The first term is equal to $a_t^*(T_1 \eta_2) \vec{\zeta}$; we need to show that the second term tends to 0 as $\delta(\mathcal{I}) \to 0$. It suffices to do so for each fixed k. The operator P_n is bounded, so it suffices to show that

$$\lim_{\delta(\mathcal{I}) \to 0} \left\| \sum_{i=1}^{N} (\mathbf{1}_{I_i} f_k \otimes T_1 \zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \otimes (f_1 \otimes \zeta_1) \otimes \ldots \otimes (f_n \otimes \zeta_n) \right\| = 0,$$

where we are using the usual norm on $(L^2(\mathbb{R}_+) \otimes V)^{\otimes n}$. But for this it suffices to show that

$$\lim_{\delta(\mathcal{I}) \to 0} \left\| \sum_{i=1}^{N} (\mathbf{1}_{I_i} f_k \otimes T_1 \zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \right\| = 0,$$

and in fact only that $\lim_{\delta(\mathcal{I}) \to 0} \left\| \sum_{i=1}^{N} (\mathbf{1}_{I_i} f_k) \otimes \mathbf{1}_{I_i} \right\| = 0$. Now apply the lemma. $\square$

Remark 5.4. Note that the Itô table does not depend on q. The Itô table was known for $q = 1$ [HP84] (with a somewhat different set of convergence), $q = -1$ [AH84] and $q = 0$ [Spe91]; for the q-Brownian motion ($T = 0$) it was known for all q [Śni00]. In all of these cases it is only a facet of a well-defined theory of stochastic integration.

Corollary 5.5. *For a one-dimensional self-adjoint process $X(t) = p_t(\xi, T, \lambda)$ and $k \geq 2$,*

$$\Delta_k(t; X) = p_t(T^{k-1}\xi, T^k, \langle \xi, T^{k-2}\xi \rangle).$$

6. Single-variable analysis

Denote by $\mathcal{M}_c$ (for "combinatorial") the space of finite positive Borel measures on $\mathbb{R}$ all of whose moments are finite, and by $\mathcal{M}_c^1 \subset \mathcal{M}_c$ the subset of probability measures. For $\mu \in \mathcal{M}_c$ considered as a functional on $\mathbb{C}[x]$, denote its moments $\mu(x^n)$ by $m_n(\mu)$. For $\mu \in \mathcal{M}_c^1$ and $n \geq 1$, the q-cumulants $r_n(\mu) = (\log_q \mu)(x^n)$ are determined by

$$r_n(\mu) = m_n(\mu) - \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \neq \hat{1}}} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} r_{|B|}(\mu). \tag{7}$$

The expressions for the first few cumulants in terms of the moments and q are

$$\begin{aligned}
r_1 &= m_1, \\
r_2 &= m_2 - m_1^2, \\
r_3 &= m_3 - 3m_2 m_1 + 2m_1^3, \\
r_4 &= m_4 - 4m_3 m_1 - (2+q)m_2^2 + (10+2q)m_2 m_1^2 - (5+q)m_1^4, \\
r_5 &= m_5 - 5m_4 m_1 - (5+4q+q^2)m_3 m_2 + (15+4q+q^2)m_3 m_1^2 \\
&\quad + (15+12q+3q^2)m_2^2 m_1 - (35+20q+5q^2)m_2 m_1^3 + (14+8q+2q^2)m_1^5.
\end{aligned}$$

While these cumulants are well-defined for arbitrary $\mu \in \mathcal{M}_c^1$, our results apply only to a special class of them. For a sequence $\mathbf{r} = (r_0 = 0, r_1, r_2, \ldots)$ in $\mathbb{R}$, let $\psi_{\mathbf{r}}$ be the functional on $\mathbb{C}[x]$ defined by $\psi_{\mathbf{r}}(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n a_i r_i$. The functional $\psi_{\mathbf{r}}$ is analytic iff $\limsup_{n\to\infty} \frac{1}{n} r_{2(n+2)}^{1/2n} < \infty$. It is conditionally positive iff the functional $\psi_{(r_2, r_3, \ldots)}$ is positive semi-definite. These conditions imply [Shi96] that for $n \geq 0$, $r_{n+2} = m_n(\tau)$ for some $\tau \in \mathcal{M}_c$ that is uniquely determined by its moments. Denote by $\mathcal{M}_u$ (for "unique") the subspace of finite positive Borel measures in $\mathcal{M}_c$ that are of this form, i.e. for which $\limsup_{n\to\infty} \frac{1}{n} m_{2n}(\tau)^{1/2n} < \infty$. Equivalently, $\tau \in \mathcal{M}_u$ if its exponential moment-generating function $\int_{\mathbb{R}} \exp(\theta x) d\tau(x)$ is defined for θ in a neighborhood of 0.

Definition 6.1. Let $\tau \in \mathcal{M}_u$, and $\lambda \in \mathbb{R}$. Define $\mathrm{LH}_q^{-1}(\lambda, \tau)$ to be the probability measure in $\mathcal{M}_c^1$ determined by the cumulant sequence $r_1 = \lambda, r_n = m_{n-2}(\tau)$ for $n \geq 2$. Equivalently, $\mathrm{LH}_q^{-1}(\lambda, \tau)$ is the distribution at time 1 of the q-Lévy process $p_t(\xi, T, \lambda)$ such that the operator T has distribution τ with respect to the vector functional $\langle \xi, \cdot \xi \rangle$. Note that $\mathrm{LH}_q^{-1}(\lambda, \tau)$ is in fact in $\mathcal{M}_u^1$. Denote by $\mathcal{ID}_c(q)$ the image of the map LH_q^{-1}; clearly $\mathcal{ID}_c(q) = \mathcal{ID}_c(q, 1)$. Call a measure in $\mathcal{ID}_c(q)$ *q-infinitely divisible*.

It is clear that LH_q^{-1} is injective. We define $\mathrm{LH}_q : \mathcal{ID}_c(q) \to \mathbb{R} \times \mathcal{M}_u$ to be the inverse of LH_q^{-1}. This is an analog of the Lévy-Hinchin representation, or more precisely of the canonical representation; see Section 6.1.

Note that for the process $p_t(\xi, T, \lambda)$ in the definition above, we can identify the Hilbert space V with $L^2(\mathbb{R}, \tau)$, so that ξ corresponds to the constant function

1, and T corresponds to the operator of multiplication by the variable x. The Hilbert space H is then equal to $L^2(\mathbb{R}_+ \times \mathbb{R}, dx \otimes \tau)$.

DEFINITION 6.2. For $\mu, \nu \in \mathcal{ID}_c(q)$, define their q-convolution $\mu *_q \nu$ by the rule that $\mathrm{LH}_q(\mu *_q \nu) = \mathrm{LH}_q(\mu) + \mathrm{LH}_q(\nu)$.

LEMMA 6.3. *$(\mathcal{ID}_c(q), *_q)$ is an Abelian semigroup. In particular, the q-convolution of two positive measures is positive.*

Proof. The sum of two measures in $\mathcal{M}_u$ is in $\mathcal{M}_u$. □

LEMMA 6.4 (Relation to product states). *For $\mu_1, \mu_2 \in \mathcal{ID}_c(q)$,*

$$(\mu_1 *_q \mu_2)(x^n) = (\mu_1 \times_q \mu_2)((x_1 + x_2)^n).$$

Proof. Using the representation from the proof of Lemma 4.6, let $\xi = \xi_1 \oplus \xi_2 \in V$, $T = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right)$ an operator on V with domain $\mathcal{D}_1 \oplus \mathcal{D}_2$, $\lambda = \lambda_1 + \lambda_2$. Let V' be the closure of the span $\left(\left\{T^j \xi\right\}_{j=0}^{\infty}\right)$. Then $T\left(\operatorname{span}\left(\left\{T^j \xi\right\}_{j=0}^{\infty}\right)\right) \subset V'$. Define T' to be the restriction $T \restriction V'$. Then $X(t) = p_t(\xi, T', \lambda)$ is a q-Lévy process. Its distribution is equal to $\mu_1 *_q \mu_2$. Indeed, if we denote this distribution by μ, then

$$r_1(\mu) = \lambda = \lambda_1 + \lambda_2 = r_1(\mu_1) + r_1(\mu_2),$$

and for $n \geq 2$,

$$r_n(\mu) = \left\langle \xi, (T')^{n-2} \xi \right\rangle = \left\langle \xi_1, T_1^{n-2} \xi_1 \right\rangle + \left\langle \xi_2, T_2^{n-2} \xi_2 \right\rangle = r_n(\mu_1) + r_n(\mu_2).$$

But $\mu_1 \times_q \mu_2 = M(\cdot, 1; (X^{(1)}, X^{(2)}))$, and it is clear that

$$M(x^n, 1; X) = M((x_1 + x_2)^n, 1; (X^{(1)}, X^{(2)})).$$

□

6.1. THE BERCOVICI-PATA BIJECTION. One would not expect the q-cumulants to be defined precisely for all probability measures in $\mathcal{M}_c^1$, rather than for more general moment sequences. Indeed, such a construction would provide a continuous bijection Λ on $\mathcal{M}_c^1$ with the property that $r_n(q = 1, \mu) = r_n(q = 0, \Lambda(\mu))$. In particular, this would imply that $\Lambda(\mu * \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$, where $*$ is the usual convolution while $\boxplus$ is the additive free convolution. Such a map is not known, and indeed for the space of all probability measures it is known not to exist, since the analog of the Cramér theorem does not hold in free probability [BV95]. However, there is a remarkable bijection [BP99] between the usual and the free infinitely divisible measures. We now show that as long as we restrict ourselves to infinitely divisible measures in $\mathcal{M}_c^1$, this is precisely the map obtained by identifying the cumulants as above, and in particular our spaces $\mathcal{ID}_c(q)$ provide an interpolation between the usual and the free infinitely divisible measures in cases $q = 0$ and $q = 1$.

The bijection is defined as follows. Let σ be a finite positive Borel measure on $\mathbb{R}$ and $\gamma \in \mathbb{R}$. Denoting by $\mathcal{F}$ the Fourier transform, define $\mu_*^{\gamma,\sigma}$ to be the probability measure with the Lévy-Hinchin representation

$$\log \mathcal{F}_{\mu_*^{\gamma,\sigma}}(\theta) = i\gamma\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2} \right) \frac{1+x^2}{x^2} d\sigma(x).$$

Denoting by $\mathcal{R}$ the R-transform [VDN92, Voi00], define $\mu_{\boxplus}^{\gamma,\sigma}$ to be the probability measure with the free Lévy-Hinchin representation

$$\mathcal{R}_{\mu_{\boxplus}^{\gamma,\sigma}}(z) = \gamma + \int_{\mathbb{R}} \frac{z+x}{1-zx} d\sigma(x).$$

Then $\Lambda(\mu_*^{\gamma,\sigma}) = \mu_{\boxplus}^{\gamma,\sigma}$.

Lemma 6.5. *Let $\lambda \in \mathbb{R}, \tau \in \mathcal{M}_u$. For $d\sigma(x) = \frac{1}{1+x^2} d\tau(x)$ and $\gamma = \lambda - m_1(\sigma)$, $\mu_*^{\gamma,\sigma} = \mathrm{LH}_1^{-1}(\lambda, \tau)$ and $\mu_{\boxplus}^{\gamma,\sigma} = \mathrm{LH}_0^{-1}(\lambda, \tau)$.*

Proof. Since $\sigma \in \mathcal{M}_u$, $\mu_*^{\gamma,\sigma}$ has finite variance. Then

$$\begin{aligned}
\log \mathcal{F}_{\mu_*^{\gamma,\sigma}}(\theta) &= i\gamma\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x + \frac{i\theta x^3}{1+x^2} \right) \frac{1}{x^2} d\tau(x) \\
&= i\lambda\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \right) \frac{1}{x^2} d\tau(x)
\end{aligned}$$

is the canonical representation of $\log \mathcal{F}_{\mu_*^{\gamma,\sigma}}$. It has a convergent power series expansion

$$i\lambda\theta + \sum_{n=2}^{\infty} \frac{1}{n!} (i\theta)^n m_{n-2}(\tau).$$

It is well-known [Shi96] that the classical ($q=1$)-cumulants of μ are the coefficients in such a power series expansion of $\log \mathcal{F}_\mu$. Similarly,

$$\begin{aligned}
\mathcal{R}_{\mu_{\boxplus}^{\gamma,\sigma}}(z) &= \gamma + \int_{\mathbb{R}} \left(\frac{z}{1-zx} + \frac{x}{x^2+1} \right) d\tau(x) \\
&= \lambda + \int_{\mathbb{R}} \frac{z}{1-zx} d\tau(x),
\end{aligned}$$

and for $z = i\theta$, it has an expansion

$$\lambda + \sum_{n=2}^{\infty} (i\theta)^{n-1} m_{n-2}(\tau).$$

Here the sum in the last expression need not converge, so what we mean by it is that for $k \geq 2$,

$$\lim_{\theta \to 0} \frac{1}{(i\theta)^k} \left(\mathcal{R}_{\mu_{\boxplus}^{\gamma,\sigma}}(i\theta) - \lambda - \sum_{n=2}^{k} (i\theta)^{n-1} m_{n-2}(\tau) \right) = m_{k-1}(\tau).$$

Again, it is well-known [Spe97a] that the free ($q=0$)-cumulants of μ are the coefficients in such an expansion of $\mathcal{R}_\mu$. ☐

Lemma 6.6. *The mapping $(q, \lambda, \tau) \mapsto \mathrm{LH}_q^{-1}(\lambda, \tau)$ has the following properties.*

a. $\mathrm{LH}_q^{-1}(\lambda_1, \tau_1) *_q \mathrm{LH}_q^{-1}(\lambda_2, \tau_2) = \mathrm{LH}_q^{-1}(\lambda_1 + \lambda_2, \tau_1 + \tau_2)$.

b. *Denoting by D_c the dilation operator, $D_c(\mu)(S) = \mu(c^{-1}S)$,*

$$D_c(\mathrm{LH}_q^{-1}(\lambda, \tau)) = \mathrm{LH}_q^{-1}(c\lambda, c^2 D_c(\tau)).$$

c. *For any q, $\mathrm{LH}_q^{-1}(\lambda, 0) = \delta_\lambda$, and for any $\mu \in \mathcal{ID}_c(q)$, $\mu *_q \delta_\lambda = \mu * \delta_\lambda$.*

d. *For $q \in [-1, 1]$ and fixed λ, τ, the mapping $q \mapsto \mathrm{LH}_q^{-1}(\lambda, \tau)$ is weakly continuous.*

e. *For a fixed $q \in [-1, 1]$, the mapping $\mathrm{LH}_q^{-1} : \mathbb{R} \times \mathcal{M}_u \to \mathcal{ID}_c(q)$ is a homeomorphism in the weak topology.*

Proof. The first and the third properties are immediate. For the second one, we observe that $m_k(D_c(\mu)) = c^k m_k(\mu)$ and so $r_k(D_c(\mu)) = c^k r_k(D_c(\mu))$. The last two follow from the following fact [Dur91]. Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of finite measures in $\mathcal{M}_c$ that converges weakly to a finite measure $\mu \in \mathcal{M}_c$. Then for all k, $m_k(\mu_n) \to m_k(\mu)$. Conversely, let $\{\mu_n\}_{n=1}^\infty$ be a sequence of finite measures in $\mathcal{M}_c$ such that for any k, $m_k(\mu_n) \to m_k$. If the family $\{m_k\}_{k=0}^\infty$ are the moments of a unique finite positive measure μ, then $\mu_n \to \mu$ weakly. □

For $q = 0, 1$, it is known [BNTr00] that the map $(\gamma, \sigma) \mapsto \mathrm{LH}_q^{-1}(\gamma + m_1(\sigma), \frac{1}{1+x^2}\sigma)$ can be extended to a weak homeomorphism between the weak closures of $\mathbb{R} \times \mathcal{M}_u$ and $\mathcal{ID}_c(q)$.

Corollary 6.7. *Let $\tau \in \mathcal{M}_u, \lambda \in \mathbb{R}$. Fix three sequences $\{A(n)\}_{n=1}^\infty$, $\{B(n)\}_{n=1}^\infty \subset \mathbb{R}$, $\{N(1) < N(2) < \ldots\} \subset \mathbb{N}$. By limits of sequences of measures we will always mean weak limits.*

a. *Every measure in $\mathcal{ID}_c(q)$ arises as a limit*

$$\lim_{n\to\infty} (\underbrace{\mu_n *_q \mu_n *_q \ldots *_q \mu_n}_{N(n)\ times}) = \mathrm{LH}_q^{-1}(\lambda, \tau) \tag{8}$$

for some $\{\mu_n\}_{n=1}^\infty \subset \mathcal{ID}_c(q)$. The statement (8) *is equivalent to*

$$\lim_{n\to\infty} (N(n) m_1(\mu_n)) = \lambda, \qquad \lim_{n\to\infty} (N(n) x^2 \mu_n) = \tau.$$

b. *Let $\mu \in \mathcal{ID}_c(q)$. The statement*

$$\lim_{n\to\infty} (D_{B(n)^{-1}}(\underbrace{\mu *_q \ldots *_q \mu}_{N(n)\ times}) *_q \delta_{-A(n)}) = \mathrm{LH}_q^{-1}(\lambda, \tau)$$

is equivalent to

$$\lim_{n\to\infty} (\frac{N(n)}{B(n)} m_1(\mu) - A(n)) = \lambda, \qquad \lim_{n\to\infty} \frac{N(n)}{B(n)^2} = t, \qquad \tau = t\delta_0.$$

Hence only $\mathrm{LH}_q^{-1}(\lambda, t\delta_0)$ arise as such limits.

Proof. Denote by (λ_n, τ_n) the components of $\mathrm{LH}_q(\mu_n)$. From the preceding Lemma it follows that the statement (8) is equivalent to

$$\lim_{n\to\infty} (N(n)\lambda_n) = \lambda, \qquad \lim_{n\to\infty} (N(n)\tau_n) = \tau.$$

So to fulfill (8), it suffices to take $\mu_n = \mathrm{LH}_q^{-1}(\frac{1}{N(n)}\lambda, \frac{1}{N(n)}\tau)$.

Now we prove the equivalence. It is clear that $\lambda_n = r_1(\mu_n) = m_1(\mu_n)$. The family $\{\mu_n\}$ satisfies (8) iff, in addition, for all $k > 1$,

$$m_k(N(n)\tau_n) = N(n)m_k(\tau_n) = N(n)r_{k+2}(\mu_n) \overset{n\to\infty}{\longrightarrow} m_k(\tau).$$

This is equivalent to $r_{k+2}(\mu_n) = \frac{1}{N(n)}m_k(\tau) + o(\frac{1}{N(n)})$. By induction on k and using (7), this is equivalent to

$$m_k(x^2\mu_n) = m_{k+2}(\mu_n) = \frac{1}{N(n)}m_k(\tau) + o(\frac{1}{N(n)}),$$

i.e.

$$m_k(N(n)x^2\mu_n) \overset{n\to\infty}{\longrightarrow} m_k(\tau)$$

and

$$(N(n)x^2\mu_n) \overset{n\to\infty}{\longrightarrow} \tau.$$

The second statement follows from the first one with $\mu_n = D_{B(n)^{-1}}(\mu) *_q \delta_{-\frac{A(n)}{N(n)}}$. For $k \geq 2$,

$$m_k(\mu_n) = \frac{N(n)}{B(n)^k}m_k(\mu) \overset{n\to\infty}{\longrightarrow} m_{k-2}(\tau).$$

So $\lim_{n\to\infty} \frac{N(n)}{B(n)^2} = t$ for some t, and $m_k(\tau) = 0$ for $k \geq 0$, i.e. $\tau = t\delta_0$. So only shifted q-Gaussian distributions (see below) can arise as such a limit among the measures in $\mathcal{ID}_c(q)$. This means that the combinatorial framework is, in general, not adequate for identifying the domains of partial attraction. □

Remark 6.8. While the results of this section are of most interest in the one-dimensional case, there is no difficulty with the extension to k dimensions. That is, to every functional in $\mathcal{ID}_c(q,k)$ there corresponds a unique conditionally positive analytic functional, which can be identified with a pair of $\vec{\lambda} \in \mathbb{R}^k$ and a positive analytic functional on $\mathbb{C}\langle x_1, x_2, \ldots, x_k\rangle$. Using this bijection, we can define a convolution on $\mathcal{ID}_c(q,k)$, as well as a multi-dimensional extension of the bijection Λ.

Now we consider the q-Lévy processes in the simplest case of one-dimensional V. There are essentially two distinct situations, $T = 0$ and $T = 1$.

6.2. The q-Brownian motion.

Denote $\omega(\xi) = a(\xi) + a^*(\xi)$.

Definition 6.9. Let $V = \mathbb{C}$, $\xi = 1 \in V, T = 0, \lambda = 0$ and $\xi_t = \mathbf{1}_{[0,t)}$. Then the *$q$-Brownian motion* is the process $X(t) = p(\xi_t, 0, 0) = \omega(\xi_t)$. The distribution of $X(t)$ is the q-Gaussian distribution with parameter t, given by $\mathrm{LH}_q^{-1}(0, t\delta_0)$.

See, for example, [BKS97] for an explicit form of the q-Gaussian distribution.

Definition 6.10. q-Hermite polynomials are defined by the recursion relation

$$xH_{q,n}(x,t) = H_{q,n+1}(x,t) + [n]_q t H_{q,n-1}(x,t)$$

with initial conditions $H_{q,0}(x,t) = 1$, $H_{q,1}(x,t) = x$.

LEMMA 6.11. *The following chaos representation holds:*

$$H_{q,n}(X(t),t)\Omega = \xi_t^{\otimes n}.$$

Therefore the q-Gaussian distribution with parameter t is the orthogonalization measure of the q-Hermite polynomials with parameter t.

Proof. For $n = 0$, $1\Omega = \Omega$. For $n = 1$, $X(t)\Omega = \xi_t$. For $n \geq 2$ by induction

$$\begin{aligned} H_{q,n+1}(X(t),t)\Omega &= X(t)\xi_t^{\otimes n} - [n]_q t \xi_t^{\otimes (n-1)} \\ &= \xi_t^{\otimes (n+1)} + [n]_q t \xi_t^{\otimes (n-1)} - [n]_q t \xi_t^{\otimes (n-1)} \\ &= \xi_t^{\otimes n}. \end{aligned}$$

Since $\xi_t^{\otimes n}$ are orthogonal in $\mathcal{F}_q(H)$ for different n, the polynomials $H_{q,n}$ are orthogonal for different n with respect to the distribution of $X(t)$. □

For the q-Brownian motion, $\Delta_2(t) = t$ and $\Delta_k(t) = 0$ for $k > 2$. But in this case, we can in fact calculate all the partition-dependent stochastic measures. Temporarily denote by s_1, s_2 the numbers of singleton and 2-element classes of π, respectively. For a singleton (i), define its depth as

$$d(i) = \left|\{j | \exists a, b \in B_j : a < i < b\}\right|.$$

Define the singleton depth $\mathrm{sd}\,(\pi)$ to be the sum of depths of all the singletons of π. In the single-variable case, we will omit the polynomial from the notation for stochastic measures.

PROPOSITION 6.12. *The partition-dependent stochastic measures of the q-Brownian motion are*

$$\mathrm{St}_\pi(t;X) = \begin{cases} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2} H_{q,s_1}(X(t),t) & \text{if all the classes of } \pi \text{ contain at most 2 elements,} \\ 0 & \text{otherwise,} \end{cases}$$

where the defining limits are taken in the $L^p(\varphi)$ norm, for any $p \geq 1$ (where $\|X\|_p = \varphi\left[|X|^p\right]^{1/p}$).

The result is known for $q = 1$ [RW97] when a different mode of convergence is used, and for $q = 0$ [Ans00] when the limit is taken in the operator norm. The preceding proposition probably holds with the operator norm convergence as well.
Throughout, we will use the following explicit formula for the moments of the q-Brownian motion, implicitly contained already in [BS91]:

$$\varphi\left[\omega(\eta_1)\omega(\eta_2)\dots\omega(\eta_{2n})\right] = \sum_{\pi \in \mathcal{P}_2(2n)} q^{\mathrm{rc}(\pi)} \prod_{i=1}^n \left\langle \eta_{a(B_i)}, \eta_{b(B_i)} \right\rangle.$$

LEMMA 6.13. *If π has a class of at least three elements, then $\mathrm{St}_\pi(t;X) = 0$, where the limit is taken in the operator norm.*

Proof. For $\vec{v} \in [1 \ldots N]^n_\pi$ and $B \in \pi$, denote by $v(B)$ the value of v on any element of B. Denote by $\pi(\vec{v})$ the partition induced by $\vec{v}$, given by

$$i \overset{\pi(\vec{v})}{\sim} j \Leftrightarrow v(i) = v(j).$$

Assume $t > 1$ to simplify notation.

$$\|\mathrm{St}_\pi(t; X, \mathcal{I})\|_{2k}^{2k} = \left\| \sum_{\vec{v} \in [1 \ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) \right\|_{2k}^{2k},$$

which equals to

$$\begin{aligned}
&\varphi \left[\left(\left(\sum_{\vec{v} \in [1 \ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) \right) \left(\sum_{\vec{v} \in [1 \ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) \right)^* \right)^k \right] \\
&= \varphi \left[\sum_{\substack{\vec{v} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2k}) \\ \vec{v}_{2i+1} \in [1 \ldots N]^n_\pi, \vec{v}_{2i} \in [1 \ldots N]^n_{\pi^{op}}}} \mathbf{X}_{\vec{v}}(t) \right] \\
&= \sum_{\substack{\vec{v} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2k}) \\ \vec{v}_{2i+1} \in [1 \ldots N]^n_\pi, \vec{v}_{2i} \in [1 \ldots N]^n_{\pi^{op}}}} \sum_{\substack{\tau \in \mathcal{P}_2(2nk) \\ \tau \leq \pi(\vec{v})}} q^{\mathrm{rc}(\tau)} \prod_{B \in \tau} \left| I_{v(B)} \right| \\
&\leq \sum_{\tau \in \mathcal{P}_2(2nk)} q^{\mathrm{rc}(\tau)} \delta(\mathcal{I})^{k(n-2|\pi|)} t^{2k|\pi|} \\
&\leq Q_{2nk}(q) \delta(\mathcal{I})^{k(n-2|\pi|)} t^{2k|\pi|},
\end{aligned}$$

where $Q_{2n}(q) = \sum_{\tau \in \mathcal{P}_2(2n)} q^{\mathrm{rc}(\tau)}$. Therefore

$$\|\mathrm{St}_\pi(t; X, \mathcal{I})\|_{2k} \leq Q_{2nk}(q)^{1/2k} t^{|\pi|} \delta(\mathcal{I})^{(n-2|\pi|)/2}.$$

$Q_{2n}(q)$ is the $2n$-th moment of the q-Gaussian distribution. By [AB98], it is equal to $\sum_{\tau \in NC_2(2n)} \prod_{B \in \tau} [d(B)]_q$. Here $NC_2(2n)$ is the collection of noncrossing pair partitions on the set of $2n$ elements, and for $\tau \in NC(n)$ and an arbitrary class $B \in \tau$, we can define its depth in τ by $d(B) = |\{i : a(B_i) \leq B \leq b(B_i)\}|$; note that this differs by 1 from our definition of singleton depth above. For $q \in [-1, 1)$, $[k]_q \leq \frac{2}{1-q}$, and so the sum is bounded by $c_n \left(\frac{2}{1-q}\right)^n$, where c_n is the n-th Catalan number. Therefore $Q_{2nk}(q)^{1/2k} \leq 2^n \left(\frac{2}{1-q}\right)^{n/2}$. We conclude that

$$\|\mathrm{St}_\pi(t; X, \mathcal{I})\|_{2k} \leq 2^n \left(\frac{2}{1-q} \right)^{n/2} t^{|\pi|} \delta(\mathcal{I})^{(n-2|\pi|)/2}. \tag{9}$$

All the vectors ξ_t lie in the real subspace $L^2(\mathbb{R}_+, \mathbb{R})$ of $L^2(\mathbb{R}_+, \mathbb{C})$. The state φ is faithful on the algebra generated by $\left\{\omega(\xi) : \xi \in L^2(\mathbb{R}_+, \mathbb{R})\right\}$, in fact Ω is separating for this algebra [BS94]. Therefore the estimate (9) holds for the operator norm of $\mathrm{St}_\pi(t; X, \mathcal{I})$. So this norm converges to 0 as $\delta(\mathcal{I}) \to 0$. $\square$

Lemma 6.14. *Let π contain only classes of at most 2 elements. Suppose that one of the following conditions holds:*

a. *$B, C \in \pi$ are 2-element classes with $a(B) < a(C) = b(B) - 1 < b(C)$. Let α be the transposition $(a(C)b(B))$.*

b. *$B \in \pi$ is a 2-element class and $(j) \in \pi$ is a singleton with $a(B) < j = b(B) - 1$. Let α be the transposition $(j\ b(B))$.*

Then $\mathrm{St}_\pi = q\mathrm{St}_{\alpha\circ\pi}$, meaning

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \mathrm{St}_\pi(t; X, \mathcal{I}) - q\mathrm{St}_{\alpha\circ\pi}(t; X, \mathcal{I}) \right\|_p = 0,$$

for any $p \geq 1$.

Proof. We prove only the first case, the proof of the second case is similar. For a multi-index $\vec{v}$, denote $\alpha(\vec{v}) = (v(\alpha(1)), v(\alpha(2)), \dots, v(\alpha(n)))$.

$$\varphi\left[\left(\left(\sum_{\vec{v}\in[1\dots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) - q\mathbf{X}_{\alpha(\vec{v})}(t)\right)\left(\sum_{\vec{v}\in[1\dots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) - q\mathbf{X}_{\alpha(\vec{v})}(t)\right)^*\right)^k\right]$$

$$= \varphi\left[\sum_{S\subset[1\dots 2nk]} \sum_{\substack{\sigma\in\mathcal{P}(2nk)\\ \sigma\wedge 2k\hat{1}_n = \sum_{j=1}^{2nk}\pi_j(S)}} \sum_{\vec{v}\in[1\dots N]^{2nk}_\sigma} (-q)^{|S|}\mathbf{X}_{\vec{v}}(t)\right],$$

where

$$\pi_j(S) = \begin{cases} \alpha\circ\pi & \text{if } j\in S, j \text{ odd},\\ (\alpha\circ\pi)^{op} & \text{if } j\in S, j \text{ even},\\ \pi & \text{if } j\notin S, j \text{ odd},\\ \pi^{op} & \text{if } j\notin S, j \text{ even}.\end{cases}$$

First consider all the terms with $\sigma \in \mathcal{P}_2(2nk)$.

$$\varphi\left[\sum_{S\subset[1\dots 2nk]} \sum_{\substack{\sigma\in\mathcal{P}_2(2nk)\\ \sigma\wedge 2k\hat{1}_n = \sum_{j=1}^{2nk}\pi_j(S)}} \sum_{\vec{v}\in[1\dots N]^{2nk}_\sigma} (-q)^{|S|}\mathbf{X}_{\vec{v}}(t)\right] \tag{10}$$

$$= \sum_{S\subset[1\dots 2nk]} \sum_{\substack{\sigma\in\mathcal{P}_2(2nk)\\ \sigma\wedge 2k\hat{1}_n = \sum_{j=1}^{2nk}\pi_j(S)}} \sum_{\vec{v}\in[1\dots N]^{2nk}_\sigma} (-q)^{|S|} q^{\mathrm{rc}(\sigma)} \prod_{B\in\sigma} \left|I_{v(B)}\right|.$$

Since $\sigma \in \mathcal{P}_2(2nk)$, it is completely determined by the collection $\{\pi_j(S)\}$ and the partition σ_s induced by σ on the singleton classes of $\sum_{j=1}^{2nk} \pi_j(S)$. Note that there is a natural (order-preserving) identification of the singleton classes of π

and $\alpha \circ \pi$, so we can consider σ_s as a partition on the singletons of $k(\pi + \pi^{op})$. Denote by σ' the partition obtained from $k(\pi+\pi^{op})$ by identifying its singleton classes using σ_s.
It is easy to see that

$$\mathrm{rc}\,(\sigma) = \mathrm{rc}\,(\sigma_s) + \mathrm{rc}\left(\sum_{j=1}^{2nk} \pi_j(S)\right) + (2nk)\mathrm{sd}\,(\pi)\,.$$

In its turn, $\mathrm{rc}\left(\sum_{j=1}^{2nk} \pi_j(S)\right) = (2nk)\mathrm{rc}\,(\pi) - |S|$. Therefore, continuing expression (10),

$$= q^{(2nk)(\mathrm{rc}(\pi)+\mathrm{sd}(\pi))} \sum_{S \subset [1\ldots 2nk]} \sum_{\substack{\sigma \in \mathcal{P}_2(2nk) \\ \sigma \wedge 2k\hat{1}_n = \sum_{j=1}^{2nk} \pi_j(S)}} \sum_{\vec{v} \in [1\ldots N]^{2nk}_{\sigma}} (-1)^{|S|} q^{\mathrm{rc}(\sigma_s)} \prod_{B \in \sigma} \left|I_{v(B)}\right|$$

$$= q^{(2nk)(\mathrm{rc}(\pi)+\mathrm{sd}(\pi))} \sum_{\sigma_s} \sum_{\vec{v} \in [1\ldots N]^{2nk}_{\sigma'}} q^{\mathrm{rc}(\sigma_s)} \prod_{B \in \sigma'} \left|I_{v(B)}\right| \sum_{S \subset [1\ldots 2nk]} (-1)^{|S|}.$$

In this expression, the only dependence on S is in $(-1)^{|S|}$, and the sum $\sum_{S \subset [1\ldots 2nk]} (-1)^{|S|} = 0$.
Therefore the non-zero contributions come only from the terms with $\sigma \notin \mathcal{P}_2(2nk)$. The rest of the argument proceeds as in the previous lemma, and shows that $\left\|\mathrm{St}_\pi - q\mathrm{St}_{\alpha\circ\pi}\right\|_p = 0$. □

Proof of Proposition 6.12. Using the lemmas, it suffices to prove the proposition for an interval partition π whose classes have at most 2 elements. Moreover, by the same arguments as in the preceding lemmas it is easy to see that each 2-element class contributes a factor of t. It remains to show that

$$\mathrm{St}_{\hat{0}_n}(t; X) = H_{q,n}(X(t), t).$$

For $n = 1$, $\sum_{i=1}^N X_i(t) = X(t)$. For $n = 2$,

$$\sum_{i \neq j}^{N} X_i(t) X_j(t) = \Big(\sum_{i=1}^N X_i(t)\Big)^2 - \sum_{i=1}^N X_i^2(t) = X^2(t) - t = H_{q,2}(X(t), t).$$

For $n > 2$, it suffices to show that $\mathrm{St}_{\hat{0}_n}(t; X)$ satisfy the same recursion relations as the q-Hermite polynomials. Indeed,

$$X(t)\mathrm{St}_{\hat{0}_n}(t; X, \mathcal{I}) = \mathrm{St}_{\hat{0}_{(n+1)}}(t; X, \mathcal{I}) + \sum_{i=2}^{n+1} \mathrm{St}_{\pi_i}(t; X, \mathcal{I}),$$

where $\pi_i = ((1,i)(2)\ldots(\hat{\imath})\ldots(n+1)) \in \mathcal{P}(n+1)$. By the second case of Lemma 6.14 and using induction on n,

$$\mathrm{St}_{\pi_i}(t; X) = tq^{i-2}\mathrm{St}_{\hat{0}_{n-1}}(t; X).$$

Therefore

$$\mathrm{St}_{\hat{0}_{n+1}}(t; X) = X(t)\mathrm{St}_{\hat{0}_n}(t; X) - \sum_{i=2}^{n+1} tq^{i-2}\mathrm{St}_{\hat{0}_{n-1}}(t; X).$$

This implies by induction that $\mathrm{St}_{\hat{0}_{n+1}}(t;X)$ is well-defined, and

$$X(t)\mathrm{St}_{\hat{0}_n}(t;X) = \mathrm{St}_{\hat{0}_{n+1}}(t;X) + \sum_{i=2}^{n+1} tq^{i-2}\mathrm{St}_{\hat{0}_{n-1}}(t;X) \tag{11}$$
$$= \mathrm{St}_{\hat{0}_{n+1}}(t;X) + t[n]_q \mathrm{St}_{\hat{0}_{n-1}}(t;X).$$

□

REMARK 6.15 (A combinatorial corollary). Denote by $\mathcal{P}_{1,2}(n)$ the collection of all partitions in $\mathcal{P}(n)$ that have classes of only 1 or 2 elements, and by $s_1(\pi), s_2(\pi)$ the number of 1- and 2-elements classes, respectively. Then using equation (2), we have a combinatorial corollary of the preceding proposition:

$$x^n = \sum_{\pi\in\mathcal{P}_{1,2}(n)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} H_{q,s_1(\pi)}(x,t).$$

Using the Möbius function on $\mathcal{P}(n)$, this relation can be inverted, to obtain

$$H_{q,n}(x,t) = \sum_{\pi\in\mathcal{P}_{1,2}(n)} (-1)^{s_2(\pi)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} x^{s_1(\pi)},$$

which is a well-known expansion for q-Hermite polynomials. In particular,

$$\begin{aligned} X(t)^n\Omega &= \sum_{\pi\in\mathcal{P}_{1,2}(n)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} H_{q,s_1(\pi)}(X(t),t)\Omega \\ &= \sum_{\pi\in\mathcal{P}_{1,2}(n)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} \xi_t^{\otimes s_1(\pi)} \\ &= H_{q,n}(\xi_t,-t), \end{aligned}$$

where ξ_t is considered as an element of the tensor algebra, with the tensor multiplication.

6.3. THE q-POISSON PROCESS. The following representation is similar to but different from that of [SY00b].

DEFINITION 6.16. Let $V = \mathbb{C}$, $\xi = 1 \in V, T = \mathrm{Id}, \lambda = 1$ and $\xi_t = \mathbf{1}_{[0,t)}, T_t = \mathbf{1}_{[0,t)}$. The *$q$-Poisson process* is the process $X(t) = p(\xi_t, T_t, t)$. The distribution of $X(t)$ is the q-Poisson distribution with parameter t, given by $\mathrm{LH}_q^{-1}(t, t\delta_1)$.

We use the definitions of the q-Poisson distribution and the q-Poisson-Charlier polynomials that were introduced in [SY00a]. See that paper for an explicit formula for the q-Poisson distribution.

DEFINITION 6.17. q-Poisson-Charlier polynomials are defined by the recursion relations

$$xC_{q,n}(x,t) = C_{q,n+1}(x,t) + ([n]_q + t)C_{q,n}(x,t) + [n]_q tC_{q,n-1}(x,t) \tag{12}$$

with initial conditions $C_{q,0}(x,t) = 1$, $C_{q,1}(x,t) = x - t$.

Remark 6.18. Let $S_{k,n;q} = \sum_{\pi \in \Pi(n,k)} q^{\mathrm{rc}(\pi)}$, where $\Pi(n,k)$ is the set of partitions in $\mathcal{P}(n)$ with k classes. It is appropriate to call these q-Stirling numbers: they interpolate between the usual Stirling numbers for $q = 1$ and $\frac{1}{n-k+1}\binom{n}{k}\binom{n-1}{k-1}$ for $q = 0$. Then according to [Bia97] (cf. [NS94]), the generating function

$$\sum_{k,n \geq 0} S_{k,n;q} t^k z^n$$

has the continued fraction expansion

$$\cfrac{1}{1 - ([0]_q + t)z - \cfrac{[1]_q t z^2}{1 - ([1]_q + t)z - \cfrac{[2]_q t z^2}{\cdots}}}.$$

It is also the moment-generating function (in z) of the probability measure with q-cumulants $r_n = t$ for $n \geq 1$. The formula says precisely that the orthogonal polynomials with respect to that measure satisfy the 3-term recursion relation (12). These are then the orthogonal polynomials with respect to the q-Poisson distribution with parameter t. A more direct proof follows from the following lemma, which is almost verbatim from [SY00b].

Lemma 6.19. *The following chaos representation holds:*

$$C_{q,n}(X(t), t)\Omega = \xi_t^{\otimes n}.$$

Therefore the distribution of $X(t)$ is the orthogonalization measure of the q-Poisson-Charlier polynomials.

For the q-Poisson process, for $k > 0$, $\Delta_k(t) = X(t)$ independently of k. The situation with the more general stochastic measures is more complicated. In particular, it is *not* true that $\mathrm{St}_{\hat{0}_n}(t; X) = C_{q,n}(X(t), t)$, unlike in the classical and the free case [RW97, Ans00]. Nevertheless, the analog of equation (11), which is a form of q-Kailath-Segall formula for centered processes, does hold, as follows:

Lemma 6.20. *For $n \geq 0$,*

$$\begin{aligned} C_{q,n+1}(X(t), t) &= (X(t) - t) C_{q,n}(X(t), t) \\ &\quad + \sum_{j=1}^{n} (-1)^j [n]_q [n-1]_q \ldots [n-j+1]_q \Delta_{j+1}(t; X) C_{q,n-j}(X(t), t). \end{aligned}$$

Proof. We need to show that

(13)

$$C_{q,n+1}(x, t) = (x - t) C_{q,n}(x, t) + \sum_{j=1}^{n} (-1)^j [n]_q [n-1]_q \ldots [n-j+1]_q x C_{q,n-j}(x, t).$$

We will prove this by induction. The formula holds for $n = 0$. Suppose the formula true for $n-1$, i.e.

$$C_{q,n}(x,t) = (x-t)C_{q,n-1}(x,t) + \sum_{j=1}^{n-1}(-1)^j [n-1]_q [n-2]_q \ldots [n-j]_q x C_{q,n-j-1}(x,t).$$

Then

$$\begin{aligned}
&-[n]_q C_{q,n}(x,t) \\
&\quad = -[n]_q (x-t) C_{q,n-1}(x,t) + \sum_{j=1}^{n-1} (-1)^{j+1} [n]_q [n-1]_q \ldots [n-j]_q x C_{q,n-j-1}(x,t) \\
&\quad = [n]_q t C_{q,n-1}(x,t) + \sum_{j=0}^{n-1} (-1)^{j+1} [n]_q [n-1]_q \ldots [n-j]_q x C_{q,n-j-1}(x,t) \\
&\quad = [n]_q t C_{q,n-1}(x,t) + \sum_{j=1}^{n} (-1)^{j} [n]_q [n-1]_q \ldots [n-j+1]_q x C_{q,n-j}(x,t).
\end{aligned}$$

Add to it the recursion relation (12)

$$[n]_q C_{q,n}(x,t) + C_{q,n+1}(x,t) = (x-t)C_{q,n}(x,t) - [n]_q t C_{q,n-1}(x,t)$$

to obtain (13). □

7. von Neumann algebras

In this section we list some preliminary results on the algebras generated by the q-Lévy processes. Throughout the section we consider only $q \in (-1, 1)$.

Let $\mathbf{X}$ be a centered q-Lévy process with $X^{(i)} = p(\xi_i, T_i, 0)$, $i \in [1 \ldots k]$. We further assume that the Hilbert space V has a real Hilbert subspace $V_{\mathbb{R}}$ so that V is the complexification of $V_{\mathbb{R}}$. Then the Hilbert space H is the complexification of its real subspace $L^2(\mathbb{R}_+, \mathbb{R}, dx) \otimes V_{\mathbb{R}}$. So H has a natural conjugation $\bar{}$ defined on it. Assume that $\{\xi_i\}_{i=1}^k \subset V_{\mathbb{R}}$, and that for each i, $T_i(V_{\mathbb{R}}) \subset V_{\mathbb{R}}$ and T_i is the complexification of its restriction to $V_{\mathbb{R}}$. Denote by $\mathcal{B}(\mathcal{F}_q(H))$ the algebra of all bounded linear operators on $\mathcal{F}_q(H)$, and by $\mathcal{A}_{\mathbf{X}}$ its von Neumann subalgebra generated by $\left\{X^{(i)}(t) : i \in [1 \ldots k], t \in [0, \infty)\right\}$. As usual, if the operators comprising $\mathbf{X}$ are not bounded, we mean the algebra generated by their spectral projections.

First consider the multi-dimensional q-Brownian motion. Let $\{\xi_i\}_{i=1}^k$ be an orthonormal basis for V, let $V_{\mathbb{R}}$ be the real linear span of $\{\xi_i\}_{i=1}^k$, and all $T_i = 0$. Since the space of simple functions is dense in $L^2(\mathbb{R}_+)$, the resulting algebra is the same as the one obtained from the q-Gaussian functor. The algebra $\mathcal{A}$ is known to have the following properties [BS94, BKS97].

a. The vacuum vector Ω is a cyclic vector for $\mathcal{A}$.
b. The vacuum expectation φ is a trace on $\mathcal{A}$.
c. The vacuum vector Ω is a cyclic vector for the commutant $\mathcal{A}'$ of $\mathcal{A}$. Therefore it is a separating vector for $\mathcal{A}$, and the vacuum expectation φ is faithful on $\mathcal{A}$.

d. Define an anti-linear involution J on $\mathcal{F}_q(H)$ by

$$J(\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n) = \bar{\eta}_n \otimes \ldots \otimes \bar{\eta}_2 \otimes \bar{\eta}_1.$$

Then $\mathcal{A}' = J\mathcal{A}J$.

e. $\mathcal{A}$ is a factor. Therefore $\mathcal{A}$ is a II_1 factor in standard form.

We now investigate these properties for more general processes.

Lemma 7.1. *If* $\operatorname{span}(\{\xi_i : i \in [1 \ldots k]\})$ *is dense in* V*, the vacuum vector* Ω *is a cyclic vector for* $\mathcal{A}_{\mathbf{X}}$.

Proof. For a multi-index $\vec{u}$ of length n and a family of intervals $\{I_i\}$,

$$\prod_{i=1}^{n} X^{(u(i))}(I_i)\Omega = (\mathbf{1}_{I_1} \otimes \xi_{u(1)}) \otimes (\mathbf{1}_{I_2} \otimes \xi_{u(2)}) \otimes \ldots \otimes (\mathbf{1}_{I_n} \otimes \xi_{u(n)}) + \vec{\eta},$$

with $\vec{\eta} \in \bigoplus_{j=0}^{n-1} (L^2(\mathbb{R}_+) \otimes V)^{\otimes j}$. So if $\operatorname{span}(\{\xi_i : i \in [1 \ldots k]\})$ is dense in V, by induction on n we see that Ω is a cyclic vector for $\mathcal{A}_{\mathbf{X}}$. $\square$

Remark 7.2. We could also consider the algebra generated by the process and its higher diagonal measures determined in Section 5. We describe the construction in the one-dimensional case. Let $X = p(\xi, T, 0)$, and define

$$\Delta_n = p(T^{n-1}\xi, T^n, \langle \xi, T^{n-2}\xi \rangle).$$

Let $\mathcal{A}_{\mathbf{X},\Delta}$ be the von Neumann algebra generated by all the processes $\Delta_n(t)$ for $n \geq 1$. Then Ω is a cyclic vector for $\mathcal{A}_{\mathbf{X},\Delta}$. We may describe this construction in more detail elsewhere.

Lemma 7.3. *Let* $q = 0$*. If the cumulant functional* $R(\cdot; \mathbf{X})$ *is a trace on* $\mathbb{C}\langle \mathbf{x} \rangle$*, then* φ *is a trace on* $\mathcal{A}_{\mathbf{X}}$.

Proof. Let $\{I_i\}_{i=1}^{l}$ be a family of disjoint intervals. It suffices to show the trace property for the family of operators $\left\{X^{(u(i))}(I_{v(i)})\right\}_{i=1}^{n}$ for arbitrary multi-indices $\vec{u}, \vec{v}$. However, it is easy to see that

$$\begin{aligned}
\varphi&\left[\prod_{i=1}^{n} X^{(u(i))}(I_{v(i)})\right] \\
&= \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi(\vec{v})}} \prod_{\substack{B \in \sigma \\ B = (j(1), j(2), \ldots, j(l))}} \left| \bigcap_{j \in B} I_{v(j)} \right| \left\langle \xi_{j(1)}, T_{j(2)} \ldots T_{j(l-1)} \xi_{j(l)} \right\rangle \\
&= \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi(\vec{v})}} R_\sigma(\mathbf{x}_{\vec{u}}; \mathbf{X}) \prod_{B \in \sigma} \left| \bigcap_{j \in B} I_{v(j)} \right|.
\end{aligned}$$

If $R(\cdot; \mathrm{X})$ is a trace, this expression is symmetric under simultaneous cyclic permutations of the components of $\vec{u}$ and $\vec{v}$. $\square$

The hypothesis of Lemma 7.1 is rarely satisfied. It does hold for the q-Brownian motion, and it also holds for the q-Poisson process. For the remained of the section we investigate the latter.
Let $\{\xi_i\}_{i=1}^k$ be an orthonormal basis for V, with $V_{\mathbb{R}}$ the real linear span of $\{\xi_i\}_{i=1}^k$. Let T_i be the orthogonal projection on ξ_i. The process $\mathbf{X}$ with $X^{(i)} = p(\xi_i, T_i, 0)$ is the centered k-dimensional q-Poisson process. By Lemma 7.1, Ω is a cyclic vector for $\mathcal{A}_{\mathbf{X}}$; a related statement is contained in Lemma 6.19.
First let $q = 0$. Then by Lemma 7.3, φ is a trace on $\mathcal{A}_{\mathbf{X}}$. By the same arguments used in [BS94] for the q-Brownian motion, it is easy to see that Ω is separating for $\mathcal{A}_{\mathbf{X}}$, and $\mathcal{A}'_{\mathbf{X}} = J\mathcal{A}_{\mathbf{X}}J$. In fact, using a different representation of the process [NS96] it follows that $\mathcal{A}_{\mathbf{X}}$ is the reduced von Neumann algebra of the free group on infinitely many generators. The preceding discussion shows that it is given in standard form.
For $q \neq 0$, for simplicity we consider the 1-dimensional process. Then $H = L^2(\mathbb{R}_+)$. We extend the mapping $I \mapsto X(I)$ to the map on all of $H_{\mathbb{R}}$, namely for $f \in L^2(\mathbb{R}_+, \mathbb{R}, dx)$, $X(f) = a(f) + a^*(f) + p(M_f)$, where M_f is the (possibly unbounded) operator of multiplication by f. Then $\mathcal{A}_X$ is the von Neumann algebra generated by $\{X(f) : f \in H_{\mathbb{R}}\}$.

Proposition 7.4. *For the q-Poisson process X, Ω is a separating vector for $\mathcal{A}_X$.*

Proof. Define the *Wick map* $W : \mathcal{F}_{\mathrm{alg}}(H_{\mathbb{R}}) \to \mathcal{A}_X$ as follows. For $f, f_1, f_2, \ldots \in H_{\mathbb{R}}$, let $W(\Omega) = \mathrm{Id}$, $W(f) = X(f)$, inductively

$$\begin{aligned} W(f \otimes f_1 \otimes \ldots \otimes f_n) = X(f)&W(f_1 \otimes \ldots \otimes f_n) \\ &- \sum_{i=1}^n q^{i-1} \langle f, f_i \rangle W(f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n) \\ &- \sum_{i=1}^n q^{i-1} W(f f_i \otimes f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n), \end{aligned}$$

and extend $\mathbb{R}$-linearly. Clearly

$$W(f_1 \otimes \ldots \otimes f_n)\Omega = f_1 \otimes \ldots \otimes f_n. \tag{14}$$

For $f \in H_{\mathbb{R}}$, define the operator $X_r(f)$ with dense domain $\mathcal{F}_{\mathrm{alg}}(H_{\mathbb{R}})$ by

$$X_r(f) f_1 \otimes \ldots \otimes f_n = W(f_1 \otimes \ldots \otimes f_n) X(f)\Omega = W(f_1 \otimes \ldots \otimes f_n) f.$$

$X_r(f)$ commutes with $\mathcal{A}_X$ on its domain of definition. Indeed,

$$X(g) X_r(f)\Omega = X(g) f = W(g) f = X_r(f) g = X_r(f) X(g)\Omega.$$

Also,

$$X(g) X_r(f) f_1 \otimes \ldots \otimes f_n = X(g) W(f_1 \otimes \ldots \otimes f_n) f$$

and

$$\begin{aligned}
&X_r(f)X(g)f_1 \otimes \ldots \otimes f_n \\
&\quad = X_r(f)X(g)W(f_1 \otimes \ldots \otimes f_n)\Omega \\
&\quad = X_r(f)\Big[W(g \otimes f_1 \otimes \ldots \otimes f_n) + \sum_{i=1}^{n} q^{i-1} \langle g, f_i \rangle W(f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n) \\
&\qquad + \sum_{i=1}^{n} q^{i-1} W(g f_i \otimes f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n)\Big]\Omega \\
&\quad = X(g)W(f_1 \otimes \ldots \otimes f_n)f.
\end{aligned}$$

Next,

$$\begin{aligned}
X_r(f_n)X_r(f_{n-1}) \ldots X_r(f_2)X_r(f_1)\Omega &= W(\ldots W(W(f_1)f_2) \ldots f_{n-1})f_n \\
&= f_1 \otimes f_2 \otimes \ldots \otimes f_n + \vec{\eta},
\end{aligned}$$

with $\vec{\eta} \in \bigoplus_{i=0}^{n-1} H^{\otimes i}$. Therefore Ω is separating for $\mathcal{A}_X$. $\square$

As a consequence, the map W is in fact determined by the condition (14).

Lemma 7.5. *Assume $q \neq 0$. Then for the q-Poisson process X,*

a. *φ is not a trace on $\mathcal{A}_X$.*
b. *$\mathcal{A}_X$ and $J\mathcal{A}_X J$ do not commute.*

Proof. Let I_1, I_2 be two disjoint intervals. It is easy to see that

$$\varphi\left[X(I_1)X(I_2)X(I_1)X(I_2)X(I_1)\right] = q^2 \left|I_1\right| \left|I_2\right|,$$

while

$$\varphi\left[X(I_1)X(I_1)X(I_2)X(I_1)X(I_2)\right] = q \left|I_1\right| \left|I_2\right|.$$

Therefore φ is not a trace on $\mathcal{A}_X$.
Moreover, for an interval I, $(X(I)JX(I)J)(\eta_1 \otimes \eta_2 \otimes \eta_3)$ contains the term $\mathbf{1}_I \otimes \eta_1 \otimes \eta_2 \otimes \eta_3$ with coefficient q^3, while $(JX(I)JX(I))(\eta_1 \otimes \eta_2 \otimes \eta_3)$ contains no such term. So already on $H^{\otimes 3}$, $\mathcal{A}_X$ and $J\mathcal{A}_X J$ do not commute. $\square$

We conclude that even for the q-Poisson process, the Fock representation of the corresponding algebra provides little immediate information about the algebra. The subject certainly deserves further investigation.

References

[AB98] Luigi Accardi and Marek Bożejko, Interacting Fock spaces and Gaussianization of probability measures, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 1 (1998), no. 4, 663–670.

[AH84] D. B. Applebaum and R. L. Hudson, Fermion Itô's formula and stochastic evolutions, *Comm. Math. Phys.* 96 (1984), no. 4, 473–496.

[Ans00] Michael Anshelevich, Free stochastic measures via noncrossing partitions, *Adv. Math.* 155 (2000), no. 1, 154–179.

[Ans01a] Michael Anshelevich, Free stochastic measures via noncrossing partitions II, arXiv: math.OA/0102062, 2001. Accepted for publication by the *Pacific J. Math.*.

[Ans01b] Michael Anshelevich, Itô formula for free stochastic integrals, math.OA/0102063, 2001. Accepted for publication by the *J. Funct. Anal.*

[Bia97] Philippe Biane, Some properties of crossings and partitions, *Discrete Math.* 175 (1997), no. 1-3, 41–53.

[BKS97] Marek Bożejko, Burkhard Kümmerer, and Roland Speicher, q-Gaussian processes: non-commutative and classical aspects, *Comm. Math. Phys.* 185 (1997), no. 1, 129–154.

[BNTr00] O.E. Barndorff-Nielsen and S. Thorbjørnsen, Selfdecomposability and Lévy processes in free probability, Odense preprint no. 24, 2000.

[BP99] Hari Bercovici and Vittorino Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math. (2)* 149 (1999), no. 3, 1023–1060, With an appendix by Philippe Biane.

[BS91] Marek Bożejko and Roland Speicher, An example of a generalized Brownian motion, *Comm. Math. Phys.* 137 (1991), no. 3, 519–531.

[BS94] Marek Bożejko and Roland Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, *Math. Ann.* 300 (1994), no. 1, 97–120.

[BV95] Hari Bercovici and Dan Voiculescu, Superconvergence to the central limit and failure of the Cramér theorem for free random variables, *Probab. Theory Related Fields* 103 (1995), no. 2, 215–222.

[Dur91] Richard Durrett, *Probability*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1991.

[GSS92] Peter Glockner, Michael Schürmann, and Roland Speicher, Realization of free white noises, *Arch. Math. (Basel)* 58 (1992), no. 4, 407–416.

[HP84] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, *Comm. Math. Phys.* 93 (1984), no. 3, 301–323.

[vLM96] Hans van Leeuwen and Hans Maassen, An obstruction for q-deformation of the convolution product, *J. Phys. A* 29 (1996), no. 15, 4741–4748.

[MN97] James A. Mingo and Alexandru Nica, Crossings of set-partitions and addition of graded-independent random variables, *Internat. J. Math.* 8 (1997), no. 5, 645–664.

[Møl93] Jacob Schach Møller, Second quantization in a quon-algebra, *J. Phys. A* 26 (1993), no. 18, 4643–4652.

[Nic95] Alexandru Nica, A one-parameter family of transforms, linearizing convolution laws for probability distributions, *Comm. Math. Phys.* 168 (1995), no. 1, 187–207.

[NS94] Peter Neu and Roland Speicher, Spectra of Hamiltonians with generalized single-site dynamical disorder, *Z. Phys. B* 95 (1994), 101–111.

[NS96] Alexandru Nica and Roland Speicher, On the multiplication of free N-tuples of noncommutative random variables, *Amer. J. Math.* 118 (1996), no. 4, 799–837, With an appendix by Dan Voiculescu.

[RS72] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.

[RS75] Michael Reed and Barry Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

[RW97] Gian-Carlo Rota and Timothy C. Wallstrom, Stochastic integrals: a combinatorial approach, *Ann. Probab.* 25 (1997), no. 3, 1257–1283.

[Sch91] Michael Schürmann, Quantum stochastic processes with independent additive increments, *J. Multivariate Anal.* 38 (1991), no. 1, 15–35.

[Shi96] A. N. Shiryaev, *Probability*, second ed., Springer-Verlag, New York, 1996.

[Śni00] Piotr Śniady, On q-deformed quantum stochastic calculus, arXiv math-ph/0007035, 2000.

[Spe91] Roland Speicher, Stochastic integration on the full Fock space with the help of a kernel calculus, *Publ. Res. Inst. Math. Sci.* 27 (1991), no. 1, 149–184.

[Spe97a] Roland Speicher, Free probability theory and non-crossing partitions, *Sém. Lothar. Combin.* 39 (1997), Art. B39c, 38 pp. (electronic).

[Spe97b] Roland Speicher, On universal products, *Free probability theory (Waterloo, ON, 1995)*, Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 257–266.

[Spe98] Roland Speicher, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, *Mem. Amer. Math. Soc.* 132 (1998), no. 627, x+88.

[SY00a] Naoko Saitoh and Hiroaki Yoshida, A q-deformed Poisson distribution based on orthogonal polynomials, *J. Phys. A* 33 (2000), no. 7, 1435–1444.

[SY00b] Naoko Saitoh and Hiroaki Yoshida, q-deformed Poisson random variables on q-Fock space, *J. Math. Phys.* 41 (2000), no. 8, 5767–5772.

[VDN92] D. V. VOICULESCU, K. J. DYKEMA, AND A. NICA, *Free random variables*, CRM Monograph Series, vol. 1, American Mathematical Society, Providence, RI, 1992, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[Voi00] DAN VOICULESCU, Lectures on free probability theory, *Lectures on probability theory and statistics (Saint-Flour, 1998)*, Springer, Berlin, 2000, pp. 279–349.

Michael Anshelevich
Department of Mathematics
University of California
Berkeley, CA 94720
manshel@math.berkeley.edu