Global L-Packets for GSp(2) and Theta Lifts

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Received: February 19, 2001
Revised: September 21, 2001

Communicated by Don Blasius

Abstract. Let F be a totally real number field. We define global L-packets for GSp(2) over F which should correspond to the elliptic tempered admissible homomorphisms from the conjectural Langlands group of F to the L-group of GSp(2) which are reducible, or irreducible and induced from a totally real quadratic extension of F. We prove that the elements of these global L-packets occur in the space of cusp forms on GSp(2) over F as predicted by Arthur's conjecture. This can be regarded as the GSp(2) analogue of the dihedral case of the Langlands-Tunnell theorem. To obtain these results we prove a nonvanishing theorem for global theta lifts from the similitude group of a general four dimensional quadratic space over F to GSp(2) over F.

2000 Mathematics Subject Classification: Primary 11F70; Secondary 11F27, 11R39.
Keywords and Phrases: L-packets, Arthur's conjecture, GSp(2), theta lifts

Introduction

Let F be a number field with adeles $\mathbb{A}$ and Weil group W_F, and let $\varphi : \mathrm{W}_F \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible continuous representation. There is a unique real number t such that the twist of φ by the canonical norm function on W_F raised to the t-th power has bounded image, so assume $\varphi(\mathrm{W}_F)$ is bounded; if φ factors through $\mathrm{Gal}(\overline{F}/F)$ this is automatic. For all places v of F, let π_v be the tempered irreducible admissible representation of $\mathrm{GL}(n, F_v)$ corresponding to the restriction φ_v under the local Langlands correspondence; then conjecturally $\otimes_v \pi_v$ is an irreducible unitary cuspidal automorphic representation (hereafter, cuspidal automorphic representation) of $\mathrm{GL}(n, \mathbb{A})$. This conjecture is known in

This work was partially supported by a NSA Young Investigators Grant.

some cases. For example, if $n = 2$ and the image of φ in $\mathrm{PGL}(2,\mathbb{C})$ is not the icosahedral group A_5, then the Langlands-Tunnell theorem asserts π is cuspidal automorphic.

Inspired by this, one can ask for a complete parameterization of the tempered cuspidal automorphic representations of $\mathrm{GL}(n)$ and of other groups ([Ko], Section 12; [LL]). Since there are tempered cuspidal automorphic representations of $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$ which do not correspond to any $\varphi : \mathrm{W}_{\mathbb{Q}} \to \mathrm{GL}(2,\mathbb{C})$, the Weil group is inadequate. Conjecturally, there exists a locally compact group L_F, called the Langlands group of F, which is an extension of W_F by a compact group and is formally similar to the Weil group; locally, if v is an infinite place of F, then $\mathrm{L}_{F_v} = \mathrm{W}_{F_v}$ and if v is finite, then $\mathrm{L}_{F_v} = \mathrm{W}_{F_v} \times \mathrm{SU}(2,\mathbb{R})$. Moreover, the tempered cuspidal automorphic representations of $\mathrm{GL}(n,\mathbb{A})$ should be in bijection with the n dimensional irreducible continuous complex representations of L_F with bounded image. For other connected reductive linear algebraic groups G over F the conjecture is more intricate, and involves L-packets attached to appropriate L-parameters $\mathrm{L}_F \to {}^L G$. In this paper we prove results about local and global theta lifts which yield parameterizations of some tempered cuspidal automorphic representations of $\mathrm{GSp}(2,\mathbb{A})$ in agreement with this conjecture.

To motivate the results we recall the conjecture, taking into account simplifications for $\mathrm{GSp}(2)$. Assume L_F exists. Then for $\mathrm{GSp}(2)$ one considers elliptic tempered admissible homomorphisms from L_F to ${}^L\,\mathrm{GSp}(2) = \widehat{\mathrm{GSp}(2)} \rtimes \mathrm{W}_F$. Concretely, since $\mathrm{GSp}(2)$ is split and one can fix an isomorphism between the dual group $\widehat{\mathrm{GSp}(2)}$ and $\mathrm{GSp}(2,\mathbb{C})$, such homomorphisms amount to continuous homomorphisms $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$ such that $\varphi(x)$ is semi-simple for all $x \in \mathrm{L}_F$ and $\varphi(\mathrm{L}_F)$ is bounded and not contained in the Levi subgroup of a proper parabolic subgroup of $\mathrm{GSp}(2,\mathbb{C})$. Since L_F should be an extension of W_F a basic example is a continuous homomorphism $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}(2,\mathbb{C})$ which is irreducible as a four dimensional complex representation. Fix such a $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$. The conjecture first asserts that for each place v of F one can associate to the restriction $\varphi_v : \mathrm{L}_{F_v} \to \mathrm{GSp}(2,\mathbb{C})$ a finite set $\Pi(\varphi_v)$ of irreducible admissible representations of $\mathrm{GSp}(2,F_v)$, the L-packet of φ_v. These packets should have a number of properties [B], but minimally we require that $\Pi(\varphi_v)$ consists of tempered representations, and if v is finite and φ_v is unramified, then $\Pi(\varphi_v)$ consists of a single representation unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_{F_v})$ with Satake parameter $\varphi_v(\mathrm{Frob}_v)$ where Frob_v is a Frobenius element at v; also, the common central character of the elements of $\Pi(\varphi_v)$ should correspond to $\lambda \circ \varphi_v$, where $\lambda : \mathrm{GSp}(2,\mathbb{C}) \to \mathbb{C}^\times$ is the similitude quasi-character. Define

$$\begin{aligned}\Pi(\varphi) &= \{\mathit{\Pi} = \otimes_v \mathit{\Pi}_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSp}(2,\mathbb{A})) : \mathit{\Pi}_v \in \Pi(\varphi_v) \text{ for all } v\} \\ &= \otimes_v \Pi(\varphi_v).\end{aligned}$$

Arthur's conjecture ([LL], [Ko], [A1], [A2]) now asserts that if $\mathit{\Pi} \in \Pi(\varphi)$ then

$\mathit{\Pi}$ occurs with multiplicity

$$m(\mathit{\Pi}) = \frac{1}{|\mathbb{S}(\varphi)|} \sum_{s \in \mathbb{S}(\varphi)} \langle s, \mathit{\Pi} \rangle$$

in the space of cusp forms on $\mathrm{GSp}(2, \mathbb{A})$ with central character $\lambda \circ \varphi$. Here, $\mathbb{S}(\varphi)$ is the connected component group $\pi_0(S(\varphi)/\mathbb{C}^\times)$, where $S(\varphi)$ is the centralizer of the image of φ, and $\langle \cdot, \cdot \rangle : \mathbb{S}(\varphi) \times \Pi(\varphi) \to \mathbb{C}$ is defined by

$$\langle s, \mathit{\Pi} \rangle = \prod_v \langle s_v, \mathit{\Pi}_v \rangle_v,$$

where s_v is the image of s under the natural map $\mathbb{S}(\varphi) \to \mathbb{S}(\varphi_v)$ and $\mathit{\Pi} = \otimes_v \mathit{\Pi}_v$; the $\langle \cdot, \cdot \rangle_v : \mathbb{S}(\varphi_v) \times \Pi(\varphi_v) \to \mathbb{C}$ should be functions such that $\langle \cdot, \mathit{\Pi}_v \rangle_v$ is the character of a finite dimensional complex representation of $\mathbb{S}(\varphi_v)$ which is identically 1 if $\mathit{\Pi}_v$ is unramified.

By looking at cases we can be more specific. Elliptic tempered admissible homomorphisms $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2, \mathbb{C})$ can be divided into three types: (A) those which are irreducible and induced as a representation; (B) those which are reducible as a representation; and (C) those which are irreducible and primitive as a representation, i.e., not induced. Our result is motivated by what the conjecture predicts for φ of the first two types.

Suppose φ is of type (A). Then one can show that φ is equivalent to $\varphi(\eta, \rho)$ for some η and ρ, where $\varphi(\eta, \rho) = \mathrm{Ind}_{\mathrm{L}_E}^{\mathrm{L}_F} \rho$, E is a quadratic extension of F, $\rho : \mathrm{L}_E \to \mathrm{GL}(2, \mathbb{C})$ is an irreducible continuous representation with bounded image such that ρ is not Galois invariant but $\det \rho$ is, and $\eta : \mathrm{L}_F \to \mathbb{C}^\times$ extends $\det \rho$; the symplectic form on $\varphi(\eta, \rho)$ (regarded as $\rho \oplus \rho$) is $\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \eta(h)\langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle$ where $\langle \cdot, \cdot \rangle$ is any fixed nondegenerate symplectic form on $\mathbb{C}^2$ (up to multiplication by nonzero scalars there is only one) and h is a representative for the nontrivial coset of $\mathrm{L}_E \setminus \mathrm{L}_F$. Evidently,

$$\lambda \circ \varphi(\eta, \rho) = \eta, \quad \mathbb{S}(\varphi(\eta, \rho)) = 1.$$

The conjecture thus predicts that every element $\mathit{\Pi}$ of $\Pi(\varphi) = \Pi(\varphi(\eta, \rho))$ should be cuspidal automorphic with $m(\mathit{\Pi}) = 1$; that is, $\Pi(\varphi)$ should be a stable global L-packet.

Type (B) parameters, however, will in general give unstable L-packets. Suppose φ is of type (B). Then $\varphi \cong \varphi(\rho_1, \rho_2)$, where $\varphi(\rho_1, \rho_2) = \rho_1 \oplus \rho_2$, $\rho_1, \rho_2 : \mathrm{L}_F \to \mathrm{GL}(2, \mathbb{C})$ are inequivalent irreducible continuous representations with bounded image and the same determinant, and the symplectic form on $\varphi(\rho_1, \rho_2)$ is $\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle$. We see that

$$\lambda \circ \varphi(\rho_1, \rho_2) = \det \rho_1 = \det \rho_2, \quad S_\varphi = \{ \begin{bmatrix} a \cdot I_2 & 0 \\ 0 & \pm a \cdot I_2 \end{bmatrix} : a \in \mathbb{C}^\times \}.$$

Thus,

$$\mathbb{S}(\varphi(\rho_1, \rho_2)) \cong Z_2.$$

Let $s \in \mathbb{S}(\varphi(\rho_1, \rho_2))$ be nontrivial. If $\Pi \in \Pi(\varphi) = \Pi(\varphi(\rho_1, \rho_2))$, the conjecture predicts

$$m(\Pi) = \frac{1}{2}(1 + \prod_v \langle s_v, \Pi_v \rangle_v).$$

Now for each v, $\mathbb{S}(\varphi_v) = 1$ or Z_2; and if $\mathbb{S}(\varphi_v) = Z_2$, then s_v is a nontrivial element of $\mathbb{S}(\varphi_v)$. Thus, if M is the number of times $\mathbb{S}(\varphi_v) = Z_2$ and Π_v induces the nontrivial character of $\mathbb{S}(\varphi_v)$, then

$$m(\Pi) = \frac{1}{2}(1 + (-1)^M).$$

By the conjecture, Π is cuspidal automorphic if and only if M is even; if so, $m(\Pi) = 1$. The conjecture thus provides exact predictions for φ of types (A) and (B).

But as precise as they are, these predictions concern conjectural objects. Globally, the hypothetical Langlands group underlies Arthur's conjecture; locally, the existence of L-packets is required. There are at least two approaches to the avoiding L_F and testing the conjecture. One natural alternative is to consider only L-parameters that factor through the Weil group or the Galois group. Another approach is to move matters, when possible, entirely to the automorphic side of the picture and render Arthur's conjecture into a statement involving only automorphic data. Base change and automorphic induction for $\mathrm{GL}(n)$ are important examples of such a shift. There is also a translation for parameters of type (A) and (B). The reason is that for φ of type (A), η corresponds to a Hecke character χ of $\mathbb{A}^\times$ by Abelian class field theory and ρ should correspond to a non-Galois invariant tempered cuspidal automorphic representation τ of $\mathrm{GL}(2, \mathbb{A}_E)$ whose central character factors through N_F^E via χ; for φ of type (B), ρ_1 and ρ_2 should correspond to a pair of inequivalent tempered cuspidal automorphic representations τ_1 and τ_2 of $\mathrm{GL}(2, \mathbb{A})$ with the same central character χ. Our first main result proves the automorphic version of Arthur's conjecture for φ of types (A) and (B).

To explain this automorphic analogue, suppose we are given, without reference to the global Langlands group, (A) a quadratic extension E of F and a non-Galois invariant tempered cuspidal automorphic representation τ of $\mathrm{GL}(2, \mathbb{A}_E)$ whose central character factors through N_F^E via a character χ, or (B) a pair of inequivalent tempered cuspidal automorphic representations τ_1 and τ_2 of $\mathrm{GL}(2, \mathbb{A})$ with common central character χ. Then we have a corresponding conjectural ρ or ρ_1 and ρ_2, a corresponding φ of type (A) or (B), and using φ, the statement of Arthur's conjecture. However, φ can be avoided entirely in arriving at a formulation of Arthur's conjecture starting from (A) or (B). This is due to two observations: first, the local L-parameters φ_v are defined via the local Langlands correspondence for $\mathrm{GL}(2)$ independent of the existence of φ; and second, the predictions of Arthur's conjecture for parameters of type (A) and (B) only involve local data.

To be specific, let v be a place of F, $E_v = F_v \otimes_F E$, and let τ_v be the irreducible admissible representation $\otimes_{w|v} \tau_w$ of $\mathrm{GL}(2, E_v)$, where w runs over the places of E lying over v (in case (B), $E_v = F_v \times F_v$, and $\tau_v = \tau_{1,v} \otimes \tau_{2,v}$). Then, as mentioned and using no conjecture, we can associate to χ_v and τ_v a canonical local L-parameter $\varphi(\chi_v, \tau_v) : \mathrm{L}_{F_v} \to \mathrm{GSp}(2, \mathbb{C})$. The automorphic version of Arthur's conjecture now presumes that we can further associate to χ_v and τ_v a local L-packet, satisfying certain basic requirements connected with $\varphi(\chi_v, \tau_v)$, and in the unstable case (B) a local pairing. This we do in Section 8: if F' is a local field of characteristic zero, E' is a quadratic extension of F' or $E' = F' \times F'$, and τ' is an infinite dimensional irreducible admissible representation of $\mathrm{GL}(2, E')$ with central character factoring through $\mathrm{N}_{F'}^{E'}$ via a quasi-character χ' (if F' is nonarchimedean of even residual characteristic we do also assume τ' is tempered; if F' is archimedean we assume $F' = \mathbb{R}$ and $E' = \mathbb{R} \times \mathbb{R}$), then we define a finite set $\Pi(\chi', \tau')$ of irreducible admissible representations of $\mathrm{GSp}(2, F')$. We show that this local L-packet has the desired essential properties: the common central character of the elements of $\Pi(\chi', \tau')$ is χ', the character corresponding to $\lambda \circ \varphi(\chi', \tau')$; if τ' is tempered, then $\varphi(\chi', \tau')$ and the elements of $\Pi(\chi', \tau')$ are tempered; and if τ' is unitary and E'/F' and τ' are unramified, then $\Pi(\chi', \tau')$ is a singleton whose Satake parameter is $\varphi(\chi', \tau')(\mathrm{Frob}_{F'})$ (if $E' = F' \times F'$, then we say that E'/F' is unramified). We also show $|\Pi(\chi', \tau')| = 1$ or 2 and $|\mathbb{S}(\varphi(\chi', \tau'))| = |\Pi(\chi', \tau')|$ at least if F' is not of even residual characteristic and E' is a field. Additionally, when $E' = F' \times F'$ we define a function $\langle \cdot, \cdot \rangle_{F'} : \mathbb{S}(\varphi(\chi', \tau')) \times \Pi(\chi', \tau') \to \mathbb{C}$, and show that for all $\mathit{\Pi} \in \Pi(\chi', \tau')$, $\langle \cdot, \mathit{\Pi} \rangle_{F'}$ is a character of $\mathbb{S}(\varphi(\chi', \tau'))$, and if $|\mathbb{S}(\varphi(\chi', \tau'))| = |\Pi(\chi', \tau')| = 2$ then both characters of $\mathbb{S}(\varphi(\chi', \tau'))$ arise in this way. The following theorem is now the automorphic version of Arthur's conjecture for parameters of type (A) and (B).

8.6 Theorem. *Let F be a totally real number field and let E be a totally real quadratic extension of F or $E = F \times F$. Let τ be a non-Galois invariant tempered cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A}_E)$ *whose central character factors through the norm N_F^E via a Hecke character χ of $\mathbb{A}^\times$. Thus, if $E = F \times F$, then τ is a pair τ_1, τ_2 of inequivalent tempered cuspidal automorphic representations of* $\mathrm{GL}(2, \mathbb{A})$ *sharing the same central character χ. Define the global L-packet:*

$$\begin{aligned}\Pi(\chi, \tau) &= \{\mathit{\Pi} = \otimes_v \mathit{\Pi}_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSp}(2, \mathbb{A})) : \mathit{\Pi}_v \in \Pi(\chi_v, \tau_v) \textit{ for all } v\} \\ &= \otimes_v \Pi(\chi_v, \tau_v).\end{aligned}$$

(1) *If E is a field, then every element of $\Pi(\chi, \tau)$ occurs with multiplicity one in the space of cusp forms on* $\mathrm{GSp}(2, \mathbb{A})$ *with central character χ.*
(2) *Suppose $E = F \times F$. Let $\mathit{\Pi} \in \Pi(\chi, \tau)$, and let $T_{\mathit{\Pi}}$ be the set of places v such that $\mathbb{S}(\varphi(\chi_v, \tau_v)) = Z_2$ and $\langle \cdot, \mathit{\Pi}_v \rangle_v$ is the nontrivial character of $\mathbb{S}(\varphi(\chi_v, \tau_v))$. If $|T_{\mathit{\Pi}}|$ is even, then $\mathit{\Pi}$ occurs with multiplicity one in the space of cusp forms on* $\mathrm{GSp}(2, \mathbb{A})$ *with central character χ. Conversely, if $\mathit{\Pi}$ occurs in the space of cusp forms on* $\mathrm{GSp}(2, \mathbb{A})$ *with central character χ, then $|T_{\mathit{\Pi}}|$ is even.*

We hope this result will be of some use to investigators of rank four motives, four dimensional symplectic Galois representations, Siegel modular forms or varieties of degree two, or Abelian surfaces. One way to think of this theorem is as an analogue for $\mathrm{GSp}(2)$ of the dihedral case of the Langlands-Tunnell theorem. Of course, some applications might require more information about the local L-packets of Theorem 8.6. For example, we still need to prove the sole dependence of the $\Pi(\chi_v, \tau_v)$ on the $\varphi(\chi_v, \tau_v)$, i.e., if $\varphi(\chi_v, \tau_v) \cong \varphi(\chi'_v, \tau'_v)$, then $\Pi(\chi_v, \tau_v) = \Pi(\chi'_v, \tau'_v)$. Also, detailed knowledge about the local L-packets at the ramified places and at infinity would be useful. We will return to these local concerns in a later work. The intended emphasis of this paper is, as much as possible, global.

As remarked, the proof of Theorem 8.6 uses theta lifts. Locally, χ and τ give irreducible admissible representations of $\mathrm{GO}(X, F_v)$ for various four dimensional quadratic spaces X over F_v; theta lifts of these define the local L-packets. Globally, χ and τ induce cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A})$ for various four dimensional quadratic spaces X over F. The automorphicity asserted in Theorem 8.6 is a consequence of our second main result, which gives a fairly complete characterization of global theta lifts from $\mathrm{GO}(X, \mathbb{A})$ to $\mathrm{GSp}(2, \mathbb{A})$ for four dimensional quadratic spaces X over F. In particular, it shows that the nonvanishing of the global theta lift to $\mathrm{GSp}(2, \mathbb{A})$ of a tempered cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A})$ is equivalent to the nonvanishing of all the involved local theta lifts; in turn, these local nonvanishings are equivalent to conditions involving distinguished representations.

8.3 Theorem. *Let F be a totally real number field, and let X be a four dimensional quadratic space over F. Let $d \in F^\times/F^{\times 2}$ be the discriminant of $X(F)$, and assume that the discriminant algebra E of $X(F)$ is totally real, i.e., either $d = 1$ or $d \neq 1$ and $E = F(\sqrt{d})$ is totally real. Let $\sigma \cong \otimes_v \sigma_v$ be a tempered cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A})$ with central character ω_σ. Let V_σ be the unique realization of σ in the space of cusp forms on $\mathrm{GO}(X, \mathbb{A})$ of central character ω_σ (Section 7). Then the following are equivalent:*

1. *The global theta lift $\Theta_2(V_\sigma)$ of V_σ to $\mathrm{GSp}(2, \mathbb{A})$ is nonzero.*
2. *For all places v of F, σ_v occurs in the theta correspondence with $\mathrm{GSp}(2, F_v)$.*
3. *For all places v of F, σ_v is not of the form π_v^- for some distinguished $\pi_v \in \mathrm{Irr}(\mathrm{GSO}(X, F_v))$ (Section 3).*

Let σ lie over the cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A})$ (Section 7), and let $s \in \mathrm{O}(X, F)$ be the element of determinant -1 from Lemma 6.1. If $s \cdot \pi \not\cong \pi$ and one of (1), (2) or (3) holds, then $\Theta_2(V_\sigma) \neq 0$, $\Theta_2(V_\sigma)$ is an irreducible unitary cuspidal automorphic representation of $\mathrm{GSp}(2, \mathbb{A})$ with central character ω_σ, and

$$\Theta_2(V_\sigma) \cong \otimes_v \theta_2(\sigma_v^\vee) = \otimes_v \theta_2(\sigma_v)^\vee,$$

where $\theta_2(\sigma_v)$ is the local theta lift of σ_v. For all v, $\theta_2(\sigma_v)$ is tempered.

In this theorem we make no assumptions about Howe duality at the even places: we prove a version of Howe duality for the case at hand in Section 1.

As mentioned, in the proof of Theorem 8.6 we use Theorem 8.3 to show that those elements of a global L-packet which should occur in the space of cusp forms really do. Theorem 8.6 also asserts that such elements occur with multiplicity one, and in the case of an unstable global L-packet, if Π is a cuspidal automorphic element, then $|T_\Pi|$ is even. To prove these two remaining claims the key step is to show that if Π is an element of a global L-packet and V is a subspace of the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ with $V \cong \Pi$, then V has a nonzero theta lift to $\mathrm{GO}(X,\mathbb{A})$ for some four dimensional quadratic space X with $\operatorname{disc} X = d$, where $E = F(\sqrt{d})$ with $d = 1$ if $E = F \times F$. This step is a consequence of a theorem of Kudla, Rallis and Soudry, which implies that if Π_1 is a cuspidal automorphic representation of $\mathrm{Sp}(2,\mathbb{A})$, V_1 is a realization of Π_1 in the space of cusp forms, and some twisted standard partial L-function $L^S(s,\Pi_1,\chi)$ has a pole at $s = 1$, then V_1 has a nonzero theta lift to $\mathrm{O}(X,\mathbb{A})$ for some four dimensional quadratic space X with $(\cdot, \operatorname{disc} X)_F = \chi$. Using this key result, multiplicity one follows from the Rallis multiplicity preservation principle and multiplicity one for $\mathrm{GO}(X,\mathbb{A})$ for four dimensional quadratic spaces X; our understanding of the involved local theta lifts and especially the relevant theta dichotomy also plays an important role. The proof of the evenness of $|T_\Pi|$ also uses the key step, local theory, and finally the fact that a quaternion algebra over F must be ramified at an even number of places.

Theorems 8.3 and 8.6 depend on many previous works. Locally, we use the papers [R1], [R2] and [R3] which dealt with the local nonarchimedean theta correspondence for similitudes, the nonarchimedean theta correspondence between $\mathrm{GO}(X,F)$ and $\mathrm{GSp}(2,F)$ for $\dim_F X = 4$, and tempered representations and the nonarchimedean theta correspondence, respectively. Globally, the critical nonvanishing results for theta lifts of this paper depend on the main result of [R4]. In turn, the essential idea of [R4] is based on an ingenious insight of [BSP]; [R4] also uses some strong results and ideas from [KR1] and [KR2]. The multiplicity one part of Theorem 8.6 uses one of the main results of [KRS], along with the multiplicity preservation principle of [Ra]. We use nonvanishing results for L-functions at $s = 1$ from [Sh] to satisfy the hypothesis of Corollary 1.2 of [R4]. Various results and ideas from [HST] are used in this paper. We would also like to mention as inspiration the papers of H. Yoshida [Y1] and [Y2] which first looked at theta lifts of automorphic forms on $\mathrm{GSO}(X,\mathbb{A}_\mathbb{Q})$ for $\dim_\mathbb{Q} X = 4$ to $\mathrm{GSp}(2,\mathbb{A}_\mathbb{Q})$. Using results from [HPS], the paper [V] also defined local discrete series L-packets for GSp(2) using theta lifts in the case of odd residual characteristic.

This paper is organized as follows. In Section 1 we consider the local theta correspondence for similitudes. The first main goal of this section is to extend the results of [R1] to the even residual characteristic and real cases. This requires that we prove a version of Howe duality in the even residual characteristic case: we do this for tempered representations when the underlying quadratic and symplectic bilinear spaces have the same dimension. The second main goal is to prove a case of S.S. Kudla's theta dichotomy conjecture, which is required for a complete theta lifting theory for similitudes for the relevant case. In Section 2 we review the basic theory of four dimensional quadratic spaces and their similitude groups. In particular, we define the four dimensional quadratic spaces $X_{D,d}$ of discriminant d over a field F not of characteristic two; here, D is a quaternion algebra over F and $d \in F^\times/F^{\times 2}$. Up to similitude, every four dimensional quadratic space over F is of the form $X_{D,d}$. The characterization of what irreducible representations of $\mathrm{GO}(X,F)$, $\dim_F X = 4$, occur in the theta correspondence with $\mathrm{GSp}(2,F)$ when F is a local field is given in Section 3. The case when F is nonarchimedean of odd residual characteristic was worked out in [R2] and the remaining cases are similar, but require additional argument. In Section 4 we define the local L-parameters and L-packets of Theorem 8.6; in fact, the L-parameters and L-packets are associated to irreducible admissible representations of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F)$ where $X_{\mathrm{M}_{2\times 2},d}$ is the four dimensional quadratic space from Section 2 over the local field F. The information is summarized in three tables which appear in the Appendix. Section 5 reviews the theory of global theta lifts for similitudes. Sections 6 and 7 explain the transition from cuspidal automorphic representations of a quaternion algebra over a quadratic extension to those of similitude groups of four dimensional quadratic spaces. Finally, in Section 8 we prove the main theorems.

I would like to thank N. Nygaard for suggesting this line of research, and S.S. Kudla, S. Rallis and J. Arthur for their interest and encouragement.

NOTATION. Let F be a field not of characteristic two. A quadratic space over F is a finite dimensional vector space X over F equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$. Let X be a quadratic space over F. In this and the next two paragraphs, also denote the F points of X by X; the same convention holds when we are considering quadratic spaces solely over a local field, as in Sections 1, 3 and 4. The discriminant $\operatorname{disc} X \in F^\times/F^{\times 2}$ of X is $(-1)^k \det X$ where $\dim X = 2k$ or $2k+1$. If $(X', (\cdot,\cdot)')$ is another quadratic space over F then a similitude from X to X' is an F linear map $t : X \to X'$ such that for some $\lambda \in F^\times$, $(tx, tx') = \lambda(x,x')$ for $x, x' \in X$; λ is uniquely determined, and we write $\lambda(t) = \lambda$. The group $\mathrm{GO}(X,F)$ is the set of $h \in \mathrm{GL}_F(X)$ which are similitudes from X to X. The group $\mathrm{O}(X,F)$ is the kernel of $\lambda : \mathrm{GO}(X,F) \to F^\times$, and $\mathrm{SO}(X,F)$ is the subgroup of $h \in \mathrm{O}(X,F)$ with $\det h = 1$. Assume $\dim X$ is even. Then $\mathrm{GSO}(X,F)$ is the kernel of sign : $\mathrm{GO}(X,F) \to \{\pm 1\}$ defined by $h \mapsto \det(h)/\lambda(h)^{\dim X/2}$; $\mathrm{SO}(X,F) = \mathrm{GSO}(X,F) \cap \mathrm{O}(X,F)$. Let n be a positive integer. Then $\mathrm{GSp}(n,F)$ is the group of $g \in \mathrm{GL}(2n,F)$ such that for some $\lambda \in F^\times$

$$ {}^t g \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} g = \lambda \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}; $$

λ is uniquely determined, and we write $\lambda(h) = \lambda$. The group $\mathrm{Sp}(n,F)$ is the kernel of $\lambda : \mathrm{GSp}(n,F) \to F^\times$. $\mathrm{M}_{2\times 2} = \mathrm{M}_{2\times 2}(F)$ is the quaternion algebra of 2×2 matrices over F with canonical involution $*$.

Suppose F is a nonarchimedean field of characteristic zero with integers $\mathfrak{O}_F$, prime ideal $\mathfrak{p}_F = \pi_F \mathfrak{O}_F \subset \mathfrak{O}_F$, Hilbert symbol $(\cdot,\cdot)_F$, and valuation $|\cdot| = |\cdot|_F$ such that if μ is an additive Haar measure on F, then $\mu(xA) = |x|\mu(A)$ for $x \in F$ and $A \subset F$. Let G be a group of td-type, as in [C]. Then $\mathrm{Irr}(G)$ is the set of equivalence classes of smooth admissible irreducible representations of G. If $\pi \in \mathrm{Irr}(G)$, then $\pi^\vee \in \mathrm{Irr}(G)$ is the contragredient of π and ω_π is the central character of π. The trivial representation of G is $\mathbf{1} = \mathbf{1}_G$. If H is a closed normal subgroup of G, $\pi \in \mathrm{Irr}(H)$ and $g \in G$, then $g \cdot \pi \in \mathrm{Irr}(H)$ has the same space as π and action defined by $(g \cdot \pi)(h) = \pi(g^{-1}hg)$. If G is the F-points of a connected reductive algebraic group defined over F, then $\pi \in \mathrm{Irr}(G)$ is tempered (square integrable) if and only if ω_π is unitary and every matrix coefficient of π lies in $L^{2+\epsilon}(G/Z(G))$ for all $\epsilon > 0$ (lies in $L^2(G/Z)$). Let X be a quadratic space over F. The quadratic character $\chi_X : F^\times \to \{\pm 1\}$ associated to X is $(\cdot, \operatorname{disc} X)_F$. We say $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F))$ ($\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ is tempered if all the irreducible components of $\sigma|_{\mathrm{SO}(X,F)}$ ($\sigma|_{\mathrm{GSO}(X,F)}$)) are tempered. A self-dual lattice L in X is a free $\mathfrak{O}_F$ submodule of rank $\dim X$ such that $L = \{x \in X : (x,y) \in \mathfrak{O}_F \text{ for all } y \in L\}$. D_{ram} is the division quaternion algebra over F with canonical involution $*$. If E/F is a quadratic extension the quadratic character of $F^\times$ associated to E/F is $\omega_{E/F}$.

Suppose $F = \mathbb{R}$. Let $|\cdot| = |\cdot|_\mathbb{R}$ be the usual absolute value on $\mathbb{R}$, and let $(\cdot,\cdot)_\mathbb{R}$ be the Hilbert symbol of $\mathbb{R}$. If X is quadratic space over $\mathbb{R}$, then the quadratic character $\chi_X : \mathbb{R}^\times \to \{\pm 1\}$ associated to X is $(\cdot, \operatorname{disc} X)_\mathbb{R}$. Let G be a real reductive group as in [Wal]. Let K be a maximal compact subgroup of G, and let $\mathfrak{g}$ be the Lie algebra G. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible $(\mathfrak{g}, K)$ modules. The trivial $(\mathfrak{g}, K)$ module will be denoted by $\mathbf{1} = \mathbf{1}_G$. If K_1 is a closed normal subgroup of K, π is a $(\mathfrak{g}, K_1)$ module and $s \in K$, then $s \cdot \pi$ is the $(\mathfrak{g}, K_1)$ module with the same space as π and action defined by $(s \cdot \pi)(k) = \pi(s^{-1}ks)$ for $k \in K_1$ and $(s \cdot \pi)(X) = \pi(\mathrm{Ad}(s)X)$ for $X \in \mathfrak{g}$. When G satisfies $G^{\mathrm{o}} = {}^{\mathrm{o}}(G^{\mathrm{o}})$ ([Wal], p. 48-9) the concepts of tempered and square integrable $(\mathfrak{g}, K)$ modules are defined in [Wal], 5.5.1; this includes $G = \mathrm{Sp}(n, \mathbb{R})$, $\mathrm{O}(p,q,\mathbb{R})$ and $\mathrm{SO}(p,q,\mathbb{R})$ for p and q not both 1. When $G^{\mathrm{o}} = {}^{\mathrm{o}}(G^{\mathrm{o}})$, then $\pi \in \mathrm{Irr}(G)$ is tempered (square integrable) if and only if π is equivalent to the underlying $(\mathfrak{g}, K)$ module of an irreducible unitary representation Π of G such that $g \mapsto \langle \Pi(g)v, w\rangle$ lies in $L^{2+\epsilon}(G)$ for all $v, w \in \pi$ and $\epsilon > 0$ (lies in $L^2(G)$ for all $v, w \in \pi$). When $G = \mathrm{GSp}(n,\mathbb{R})$, $\mathrm{GO}(p,q,\mathbb{R})$ or $\mathrm{GSO}(p,q,\mathbb{R})$ with p and q not both 1, then we say that $\pi \in \mathrm{Irr}(G)$ is tempered (square integrable) if π is equivalent to the underlying $(\mathfrak{g}, K)$ module of an irreducible unitary representation Π of G such that $g \mapsto \langle \Pi(g)v, w\rangle$ lies in $L^2(\mathbb{R}^\times \backslash G)$ for all $v, w \in \pi$ and $\epsilon > 0$ (lies in $L^2(\mathbb{R}^\times \backslash G)$ for all $v, w \in \pi$); this is equivalent to the irreducible constituents of $\pi|_{(\mathfrak{g}_1, K_1)}$ being tempered (square integrable), where $\mathfrak{g}_1$ is the Lie algebra and $K_1 \subset K$ is the maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{O}(p,q,\mathbb{R})$ or $\mathrm{SO}(p,q,\mathbb{R})$, respectively. D_{ram} is the division quaternion algebra over $\mathbb{R}$ with canonical involution $*$.

Suppose F is a number field with adeles $\mathbb{A}$ and finite adeles $\mathbb{A}_f$; set $F_\infty = F \otimes_\mathbb{Q} \mathbb{R}$. The Hilbert symbol of F is $(\cdot,\cdot)_F$. If X is quadratic space over F, then the quadratic Hecke character $\chi_X : \mathbb{A}^\times/F^\times \to \{\pm 1\}$ associated to X is $(\cdot, \operatorname{disc} X(F))_F$. Let G be a reductive linear algebraic group defined over F, let $\mathfrak{g}$ be the Lie algebra of $G(F_\infty)$, and let K be a maximal compact subgroup of $G(F_\infty)$. Then $\mathrm{Irr}_{\mathrm{admiss}}(G(\mathbb{A}))$ is the set of equivalence classes of irreducible admissible $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ modules. If $\pi \in \mathrm{Irr}_{\mathrm{admiss}}(G(\mathbb{A}))$ then the central character of π is ω_π and $\pi = \otimes_v \pi_v$ is tempered if π_v is tempered for all places v of F. A cuspidal automorphic representation of $G(\mathbb{A})$ is a $\pi \in \mathrm{Irr}_{\mathrm{admiss}}(G(\mathbb{A}))$ which is isomorphic to an irreducible submodule of the $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ module of cuspidal automorphic forms on $G(\mathbb{A})$ of central character ω_π; such a π is unitary.

1. The local theta correspondence for similitudes

In this section we recall and prove results about the local theta correspondence for similitudes. The paper [R1] dealt with the nonarchimedean odd residual characteristic case. Here we do the even residual characteristic and real cases and prove a very special, but adequate, case of S.S. Kudla's theta dichotomy conjecture. We also show that the theta correspondence for similitudes is independent of the additive character, compatible with contragredients, and respects unramified representations.

Fix the following notation. Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let n be a positive integer, and let X be a quadratic space of nonzero even dimension m over F. To simplify notation, denote the F points of X by X. Let $d = \operatorname{disc} X$. Fix a nontrivial unitary character ψ of F. The Weil representation $\omega = \omega_X = \omega_n = \omega_{X,n}$ of $\mathrm{Sp}(n, F) \times \mathrm{O}(X, F)$ defined with respect to ψ is the unitary representation on $L^2(X^n)$ given by

$$\begin{aligned} \omega(1, h)\varphi(x) &= \varphi(h^{-1}x), \\ \omega(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}, 1)\varphi(x) &= \chi_X(\det a)|\det a|^{m/2}\varphi(xa), \\ \omega(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, 1)\varphi(x) &= \psi(\frac{1}{2}\operatorname{tr}(bx, x))\varphi(x), \\ \omega(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 1)\varphi(x) &= \gamma\hat{\varphi}(x). \end{aligned}$$

Here, $\hat{\varphi}$ is the Fourier transform defined by

$$\hat{\varphi}(x) = \int_{X^n} \varphi(x')\psi(\operatorname{tr}(x, x'))\, dx'$$

with dx such that $\hat{\hat{\varphi}}(x) = \varphi(-x)$ for $\varphi \in L^2(X^n)$ and $x \in X^n$, and γ is a certain fourth root of unity depending only on the anisotropic component of X, n and ψ. If $h \in \mathrm{O}(X, F)$, $a \in \mathrm{GL}(n, F)$, $b \in \mathrm{M}_n(F)$ with ${}^t b = b$ and $x = (x_1, \dots, x_n), x' = (x'_1, \dots, x'_n) \in X^n$, we write $h^{-1}x = (h^{-1}x_1, \dots, h^{-1}x_n)$, $xa = (x_1, \dots, x_n)(a_{ij})$, $(x, x') = ((x_i, x'_j))$, $bx = b^t(x_1, \dots, x_n)$. Also, χ_X is the quadratic character of $F^\times$ defined by $\chi_X(t) = (t, d)_F$; χ_X depends only on the anisotropic component of X.

Suppose F is nonarchimedean. We will work with smooth representations of groups of td-type such as $\mathrm{Sp}(n, F)$ and $\mathrm{O}(X, F)$. We thus consider the restriction of ω to a smaller subspace. Let $\mathcal{S}(X^n)$ be the space of locally constant, compactly supported functions on X^n. Then ω preserves $\mathcal{S}(X^n)$. By ω we will usually mean ω acting on $\mathcal{S}(X^n)$; context will give the meaning. Let $\mathcal{R}_n(\mathrm{O}(X, F))$ be the set of elements of $\mathrm{Irr}(\mathrm{O}(X, F))$ which are nonzero quotients of ω, and define $\mathcal{R}_X(\mathrm{Sp}(n, F))$ similarly.

Suppose $F = \mathbb{R}$. In the analogy to the last case, we will work with Harish-Chandra modules of real reductive groups. This requires definitions. Fix $K_1 = \mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}(2n, \mathbb{R})$ as a maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$. The Lie algebra of $\mathrm{Sp}(n, \mathbb{R})$ is $\mathfrak{g}_1 = \mathfrak{sp}(n, \mathbb{R})$. Let X have signature (p, q). We parameterize the maximal compact subgroups of $\mathrm{O}(X, \mathbb{R})$ as follows. Let X^+ and X^- be positive and negative definite subspaces of X, respectively, such that $X = X^+ \perp X^-$. Then the maximal compact subgroup $J_1 = J_1(X^+, X^-)$ associated to (X^+, X^-) is the set of $k \in \mathrm{O}(X, \mathbb{R})$ such that $k(X^+) = X^+$ and $k(X^-) = X^-$. Of course, $J_1 = \mathrm{O}(X^+, \mathbb{R}) \times \mathrm{O}(X^-, \mathbb{R}) \cong \mathrm{O}(p, \mathbb{R}) \times \mathrm{O}(q, \mathbb{R})$. Fix one such $J_1 = J_1(X^+, X^-)$. The Lie algebra of $\mathrm{O}(X, \mathbb{R})$ is $\mathfrak{h}_1 = \mathfrak{o}(X, \mathbb{R})$. Let $\mathcal{S}(X^n) = \mathcal{S}_\psi(X^n)$ be the subspace of $L^2(X^n)$ of functions

$$p(x) \exp[-\frac{1}{2}|c|(\mathrm{tr}(x^+, x^+) - \mathrm{tr}(x^-, x^-))].$$

Here, $p : X^n \to \mathbb{C}$ is a polynomial function on X^n, and (x^+, x^+) and (x^-, x^-) are the $n \times n$ matrices with (i, j)-th entries (x_i^+, x_j^+) and (x_i^-, x_j^-) respectively, where $x_i = x_i^+ + x_i^-$, with $x_i^+ \in X^+$ and $x_i^- \in X^-$ for $1 \le i \le n$; $c \in \mathbb{R}^\times$ is such that $\psi(t) = \exp(ict)$ for $t \in \mathbb{R}$. Then $\mathcal{S}(X^n)$ is a $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1 \times J_1)$ module under the action of ω. By ω we will usually mean the $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1 \times J_1)$ module $\mathcal{S}(X^n)$. Let $\mathcal{R}_n(\mathrm{O}(X, \mathbb{R}))$ be the set of irreducible $(\mathfrak{g}_1, J_1)$ modules which are nonzero quotients of ω, and define $\mathcal{R}_X(\mathrm{Sp}(n, \mathbb{R}))$ similarly. For uniformity, write $\mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega, \pi \otimes \sigma)$ for $\mathrm{Hom}_{(\mathfrak{g}_1\times\mathfrak{h}_1, K_1\times J_1)}(\omega, \pi \otimes \sigma)$.

We have the following foundational result on the theta correspondence for isometries.

1.1 Theorem ([H], [W1]). *Suppose F is real or nonarchimedean of odd residual characteristic. The set*

$$\{(\pi, \sigma) \in \mathcal{R}_X(\mathrm{Sp}(n, F)) \times \mathcal{R}_n(\mathrm{O}(X, F)) : \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{Sp}(n, F))$ and $\mathcal{R}_n(\mathrm{O}(X, F))$, and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega, \pi \otimes \sigma) \le 1$$

for $\pi \in \mathcal{R}_X(\mathrm{Sp}(n, F))$ and $\sigma \in \mathcal{R}_n(\mathrm{O}(X, F))$.

When F is nonarchimedean of even residual characteristic partial results are known. For us the following unconditional result suffices. If F is nonarchimedean and $\sigma \in \mathrm{Irr}(\mathrm{O}(X, F))$ we say that σ is tempered if all the irreducible constituents of $\sigma|_{\mathrm{SO}(X,F)}$ are tempered.

1.2 Theorem. *Suppose F is nonarchimedean of even residual characteristic and $m = 2n$. Let $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ be the subsets of $\mathcal{R}_n(\mathrm{O}(X,F))$ and $\mathcal{R}_X(\mathrm{Sp}(n,F))$ of tempered elements, respectively. Then the statement of Theorem 1.1 holds with $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ replacing the sets $\mathcal{R}_X(\mathrm{Sp}(n,F))$ and $\mathcal{R}_n(\mathrm{O}(X,F))$, respectively.*

Proof. Let $\sigma \in \mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$. By 2) a), p. 69 of [MVW], there exists $\pi \in \mathcal{R}_X(\mathrm{Sp}(n,F))$ such that the homomorphism space of Theorem 1.1 is nonzero; π is tempered by (1) of Theorem 4.2 of [R3], and is unique by Theorem 4.4 of [R3]. To prove the map from $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ to $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ is injective and the homomorphism space has dimension at most one, let $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))_{\mathrm{temp}}$. By putting together the proofs of Proposition II.3.1 of [Ra] and Theorem 4.4 of [R3] one can show that there is a $\mathbb{C}$ linear injection

$$\bigoplus_{\substack{\sigma\in\mathrm{Irr}(\mathrm{O}(X,F))\\ \sigma \text{ unitary}}} \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma) \hookrightarrow \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n,F)}(\mathcal{S}(\mathrm{Sp}(n,F)),\pi\otimes\pi^\vee),$$

where $\mathcal{S}(\mathrm{Sp}(n,F))$ is the space of locally constant compactly supported functions on $\mathrm{Sp}(n,F)$ and the action of $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n,F)$ on $\mathcal{S}(\mathrm{Sp}(n,F))$ is defined by $((g,g')\cdot\phi)(x) = \phi(g^{-1}xg')$. The last space is one dimensional as

$$\pi\otimes\pi^\vee \cong \mathcal{S}(\mathrm{Sp}(n,F))/\bigcap_{\substack{f\in\mathrm{Hom}_{\mathrm{Sp}(n,F)}(\mathcal{S}(\mathrm{Sp}(n,F)),\pi\otimes U),\\ U \text{ a } \mathbb{C} \text{ vector space}}} \ker(f);$$

see the lemma on p. 59 of [MVW]. This proves the claims about injectivity and dimension. For surjectivity, let $\pi \in \mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$. As above, there exists $\sigma \in \mathcal{R}_n(\mathrm{O}(X,F))$ such that the homomorphism space is nonzero. An argument as in the proof of (1) of Theorem 4.2 of [R3] shows that σ must be tempered. See also [Mu]. □

It is worth noting the following from the proof of Theorem 1.2: Let $m = 2n$, $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ and $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F))$. If $\mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma) \neq 0$, then π is tempered if and only if σ is tempered.

When F is nonarchimedean of odd residual characteristic or $F = \mathbb{R}$, then the bijection from Theorem 1.1 and its inverse are denoted by

$$\theta : \mathcal{R}_X(\mathrm{Sp}(n,F)) \xrightarrow{\sim} \mathcal{R}_n(\mathrm{O}(X,F)), \quad \theta : \mathcal{R}_n(\mathrm{O}(X,F)) \xrightarrow{\sim} \mathcal{R}_X(\mathrm{Sp}(n,F));$$

if F is nonarchimedean of even residual characteristic we use the same notation for the bijections between $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ from Theorem 1.2.

Next we recall and prove a special case of a conjecture of S.S. Kudla on the theta correspondence for isometries. This conjecture has important implications for the theta correspondence for similitudes.

1.3 THETA DICHOTOMY CONJECTURE (S.S. KUDLA). *Assume F is nonarchimedean. Let m be a positive even integer, let $d \in F^\times/F^{\times 2}$, and let n be a positive integer such that $m \le 2n$. There exist at most two quadratic spaces Y and Y' over F of dimension m and discriminant d; assume both exist. Then $\mathcal{R}_Y(\mathrm{Sp}(n,F)) \cap \mathcal{R}_{Y'}(\mathrm{Sp}(n,F)) = \emptyset$.*

We can prove the conjecture when m is small in comparison to $2n$:

1.4 LEMMA. *Suppose that the notation is as in Conjecture 1.3, and assume the two quadratic spaces Y and Y' exist. If $m \le n+2$, then the theta dichotomy conjecture holds for m and n.*

Proof. Let Z be the quadratic space over F of dimension $2m$ with four dimensional anisotropic component. To prove the theta dichotomy conjecture for m and n it suffices to show that $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_Z(\mathrm{Sp}(n,F))$. This reduction is well known, but we recall the proof for the convenience of the reader. Assume $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_Z(\mathrm{Sp}(n,F))$, and suppose $\pi \in \mathcal{R}_Y(\mathrm{Sp}(n,F)) \cap \mathcal{R}_{Y'}(\mathrm{Sp}(n,F))$. To get a contradiction, let $g_0 \in \mathrm{GSp}(n,F)$ be such that $\lambda(g_0) = -1$. By the first theorem on p. 91 of [MVW], $g_0 \cdot \pi \cong \pi^\vee$. Thus, there is a nonzero $\mathrm{Sp}(n,F) \times \mathrm{Sp}(n,F)$ map from $\omega_{Y,n} \otimes g_0 \cdot \omega_{Y',n}$ to $\pi \otimes \pi^\vee$. By Lemma 1.6 below, $g_0 \cdot \omega_{Y',n} \cong \omega_{-Y',n}$, where $-Y'$ has the same space as Y' and form multiplied by -1. Also, $(\omega_{Y,n} \otimes \omega_{-Y',n})|_{\Delta\,\mathrm{Sp}(n,F)} \cong \omega_{Y\perp -Y',n}|_{\mathrm{Sp}(n,F)}$. Clearly, $Y \perp -Y' \cong Z$. Since $\mathrm{Hom}_{\mathrm{Sp}(n,F)}((\pi \otimes \pi^\vee)|_{\Delta\,\mathrm{Sp}(n,F)}, \mathbf{1}_{\mathrm{Sp}(n,F)}) \ne 0$, there now is a nonzero $\mathrm{Sp}(n,F)$ map from $\omega_{Z,n}$ to $\mathbf{1}_{\mathrm{Sp}(n,F)}$, i.e., $\mathbf{1}_{\mathrm{Sp}(n,F)} \in \mathcal{R}_Z(\mathrm{Sp}(n,F))$. We now show $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_Z(\mathrm{Sp}(n,F))$ for $m \le n+2$. Let i be the Witt index of Z, i.e., $i = m-2$, and write $V_i = Z$ to indicate that Z is the orthogonal direct sum of the four dimensional anisotropic quadratic space over F with i hyperbolic planes. We must show $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_{V_i}(\mathrm{Sp}(n,F))$ for $0 \le i \le n$; we do this by induction on n. The case $n=1$ follows from Lemma 7.3 of [R2] (see its proof, which is residual characteristic independent). Let $n > 1$, and assume the claim for $n-1$. Let $i \le n$, and assume $\mathbf{1}_{\mathrm{Sp}(n,F)} \in \mathcal{R}_{V_i}(\mathrm{Sp}(n,F))$. To find a contradiction we reduce dimensions using Jacquet functors and Kudla's filtration of the Jacquet module of the Weil representation. We use the notation of [R3]: write $\omega_{i,n} = \omega_{V_i,n}$. Since $\mathbf{1}_{\mathrm{Sp}(n,F)} \in \mathcal{R}_{V_i}(\mathrm{Sp}(n,F))$, by 2) a) of the Theorem on p. 69 of [MVW], there exists $\sigma \in \mathrm{Irr}(\mathrm{O}(V_i,F))$ and a nonzero $\mathrm{Sp}(n,F) \times \mathrm{O}(V_i,F)$ map

$$\omega_{i,n} \to \mathbf{1}_{\mathrm{Sp}(n,F)} \otimes \sigma.$$

Let N_1' be the unipotent radical of the standard maximal parabolic of $\mathrm{Sp}(n,F)$ with Levi factor isomorphic to $\mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F)$. Applying the normalized Jacquet functor with respect to N_1', which is exact, we obtain a nonzero $\mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F) \times \mathrm{O}(V_i,F)$ map

$$\mathrm{R}_{N_1'}(\omega_{i,n}) \to \mathrm{R}_{N_1'}(\mathbf{1}_{\mathrm{Sp}(n)}) \otimes \sigma = |\cdot|^{-n} \otimes \mathbf{1}_{\mathrm{Sp}(n-1,F)} \otimes \sigma.$$

Suppose first $i = 0$, so that V_0 is four dimensional and anisotropic. By Kudla's computation of the Jacquet functors of $\omega_{0,n}$, (see [R3] for a statement in our notation), $\mathrm{R}_{N_1'}(\omega_{0,n})|_{\mathrm{Sp}(n-1,F)} \cong \omega_{0,n-1}|_{\mathrm{Sp}(n-1,F)}$. Thus, $\mathbf{1}_{\mathrm{Sp}(n-1,F)} \in \mathcal{R}_{V_i}(\mathrm{Sp}(n-1,F))$. Since $i = 0 \le n-1$, by the induction hypothesis this is a contradiction.

Suppose $i > 0$. By Kudla's filtration (two step, in this case) of $\mathrm{R}_{N_1'}(\omega_{i,n})$ either there exists a nonzero $\mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F) \times \mathrm{O}(V_i,F)$ map

$$|\cdot|^{\dim V_i/2-n} \otimes \omega_{i,n-1} \to |\cdot|^{-n} \otimes \mathbf{1}_{\mathrm{Sp}(n-1,F)} \otimes \sigma,$$

or there exists a nonzero $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F) \times \mathrm{O}(V_{i-1},F)$ map

$$\xi_1\xi_1'\boldsymbol{\sigma}_1 \otimes \omega_{i-1,n-1} \to |\cdot|^{-n} \otimes \mathbf{1}_{\mathrm{Sp}(n-1,F)} \otimes \overline{\mathrm{R}}_{N_1}(\sigma);$$

here, ξ_1 and ξ_1' are quasi-characters of $\mathrm{GL}(1,F)$ and $\boldsymbol{\sigma}_1$ is a representation of $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F)$ whose precise definitions we will not need, $|\cdot|^{-n}$ is regarded as a quasi-character of $\mathrm{GL}(1,F)$, N_1 is the unipotent radical of the standard parabolic of $\mathrm{O}(V_i,F)$ with Levi factor isomorphic to $\mathrm{GL}(1,F) \times \mathrm{O}(V_{i-1},F)$, and $\overline{\mathrm{R}}_{N_1}(\sigma) = \mathrm{R}_{N_1}(\sigma^\vee)^\vee$. The first case is ruled out since $|\cdot|^{\dim V_i/2-n} \ne |\cdot|^{-n}$. Since the second case must therefore hold, we get $\mathrm{Hom}_{\mathrm{Sp}(n-1,F)}(\omega_{i-1,n-1}, \mathbf{1}_{\mathrm{Sp}(n-1,F)}) \ne 0$, i.e., $\mathbf{1}_{\mathrm{Sp}(n-1,F)} \in \mathcal{R}_{V_{i-1}}(\mathrm{Sp}(n-1,F))$. This contradicts the induction hypothesis since $i-1 \le n-1$. $\square$

The real analogue of the theta dichotomy conjecture is known. The assumption of the evenness of p and q in the following lemma is a consequence of the same assumptions in [M].

1.5 Lemma. *Suppose $F = \mathbb{R}$. Let m be a positive even integer and let n be a positive integer such that $m \le 2n$. Then the sets $\mathcal{R}_Y(\mathrm{Sp}(n,\mathbb{R}))$ as Y runs over the isometry classes quadratic spaces over $\mathbb{R}$ of dimension m and signature of the form (p,q) with p and q even are mutually disjoint.*

Proof. We argue as in the second paragraph of the proof of Lemma 1.8 of [AB]. Suppose Y and Y' are quadratic spaces of dimension m with signatures (p,q) and (p',q') with p,q,p' and q' even. Assume $\pi \in \mathcal{R}_Y(\mathrm{Sp}(n,\mathbb{R})) \cap \mathcal{R}_{Y'}(\mathrm{Sp}(n,\mathbb{R}))$. We must show that $Y \cong Y'$, i.e., $p = p'$ and $q = q'$. We have $\mathrm{Hom}_{(\mathfrak{g}_1,K_1)}(\omega_{Y,n},\pi) \ne 0$ and $\mathrm{Hom}_{(\mathfrak{g}_1,K_1)}(\omega_{Y',n},\pi) \ne 0$; as in the proof of Lemma 1.4 this implies $\mathrm{Hom}_{(\mathfrak{g}_1,K_1)}(\omega_{Z,n},(\pi \otimes \pi^\vee)|_{\Delta(\mathfrak{g}_1,K_1)}) \ne 0$, where $Z = Y \perp -Y'$, and $-Y'$ is the quadratic space with same space as Y' but with form multiplied by -1; Z has signature $(p+q',p'+q)$. Hence, $\mathbf{1}_{\mathrm{Sp}(n,\mathbb{R})} \in \mathcal{R}_Z(\mathrm{Sp}(n,\mathbb{R}))$. We now use [M] to complete the proof. The representation $\mathbf{1}_{\mathrm{Sp}(n,\mathbb{R})}$ has only one K_1-type, namely the trivial representation of K_1. As $\mathbf{1}_{\mathrm{Sp}(n,\mathbb{R})} \in \mathcal{R}_Z(\mathrm{Sp}(n,\mathbb{R}))$, the trivial representation of K_1 appears in the joint harmonics $H(K_1,\mathrm{O}(p+q',\mathbb{R}) \times \mathrm{O}(p'+q,\mathbb{R}))$ for this theta correspondence (see I.1 and the second paragraph of II.1 of [M]). By Corollaire I.4 of [M], which computes the representations of K_1 occurring in the joint harmonics, $p+q' = p'+q$; since $p+q = p'+q'$, we have $p = p'$ and $q = q'$. $\square$

We now recall the extended Weil representation which will be used to define the theta correspondence for similitudes; see [R1] for references. Define

$$R = R_X = R_n = R_{X,n} = \{(g,h) \in \mathrm{GSp}(n,F) \times \mathrm{GO}(X,F) : \lambda(g) = \lambda(h)\}.$$

The Weil representation ω of $\mathrm{Sp}(n,F)\times \mathrm{O}(X,F)$ on $L^2(X^n)$ extends to a unitary representation of R via

$$\omega(g,h)\varphi = |\lambda(h)|^{-\frac{mn}{4}}\omega(g_1,1)(\varphi\circ h^{-1}),$$

where

$$g_1 = g\begin{bmatrix}1&0\\0&\lambda(g)\end{bmatrix}^{-1}\in \mathrm{Sp}(n,F).$$

Evidently, the group of elements $(t,t)=(t\cdot 1, t\cdot 1)$ for $t\in F^\times$ is contained in the center of R, and we have $\omega(t,t)\varphi = \chi_X(t)^n\varphi$ for $\varphi\in L^2(X^n)$ and $t\in F^\times$. If F is nonarchimedean, then the extended Weil representation preserves $\mathcal{S}(X^n)$; when F is nonarchimedean, by ω we shall often mean ω acting on $\mathcal{S}(X^n)$.
Suppose $F=\mathbb{R}$; then ω extended to R also preserves $\mathcal{S}(X^n)$, but only at the level of Harish-Chandra modules. We need definitions. As a standard maximal compact subgroup K of $\mathrm{GSp}(n,\mathbb{R})$ take the group generated by K_1 and the order two element

$$k_0=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

The Lie algebra $\mathfrak{g}=\mathfrak{gsp}(n,\mathbb{R})$ of $\mathrm{GSp}(n,\mathbb{R})$ is the direct sum of its center $\mathbb{R}$ and $\mathfrak{g}_1=\mathfrak{sp}(n,\mathbb{R})$. If $p\neq q$, then any maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$ is a maximal compact subgroup of $\mathrm{O}(X,\mathbb{R})$, and we let J denote the subgroup J_1 from above. Suppose $p=q$. Then every maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$ contains a unique maximal compact subgroup of $\mathrm{O}(X,\mathbb{R})$ as a subgroup of index two, and any maximal compact subgroup of $\mathrm{O}(X,\mathbb{R})$ is contained in a unique maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$ as a subgroup of index two. As a maximal compact subgroup for $\mathrm{GO}(X,\mathbb{R})$ we take the maximal compact subgroup $J=J(X^+,X^-)$ containing $J_1=J_1(X^+,X^-)$. To get a coset representative j_0 for the nontrivial coset of J_1 in J, let $i: X^+\to X^-$ be an isomorphism of $\mathbb{R}$ vector spaces such that $(i(x^+),i(x^+)) = -(x^+,x^+)$ for $x^+\in X^+$ and using $X=X^+\perp X^-$ set

$$j_0=\begin{bmatrix}0&i^{-1}\\i&0\end{bmatrix}.$$

The Lie algebra $\mathfrak{h}=\mathfrak{go}(n,\mathbb{R})$ of $\mathrm{GO}(X,\mathbb{R})$ is the direct sum of its center $\mathbb{R}$ and $\mathfrak{h}_1=\mathfrak{o}(n,\mathbb{R})$. The group R is a real reductive group containing $(\mathrm{Sp}(n,\mathbb{R})\times \mathrm{O}(X,\mathbb{R}))\{(t,t): t\in\mathbb{R}^\times\}$ as an open subgroup of index one if $p\neq q$, and index two if $p=q$. As a maximal compact subgroup L of R we take $L=K_1\times J_1$ if $p\neq q$; if $p=q$, then we take L to be generated by $K_1\times J_1$ and (k_0,j_0). The Lie algebra $\mathfrak{r}$ of R is the set of pairs $(x,y)\in\mathfrak{g}\times\mathfrak{h}$ such that $x=z+x_1$ and $y=z+y_1$ for some $z\in\mathbb{R}, x_1\in\mathfrak{g}_1$ and $y_1\in\mathfrak{h}_1$. The space $\mathcal{S}(X^n)$ is evidently closed under the action of ω restricted to L and $\mathfrak{r}$. The $(\mathfrak{g}_1\times\mathfrak{h}_1, K_1\times J_1)$ module $\mathcal{S}(X^n)$ thus extends to an $(\mathfrak{r},L)$ module, which we will also denote by ω.

Before discussing the theta correspondence for similitudes it will be useful to describe the relationship between the extended Weil representations for similar quadratic spaces, and what happens to the extended Weil representation when the additive character is changed. For $\lambda \in F^\times$ and $g \in \mathrm{GSp}(n,F)$ write

$$g^{[\lambda]} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} g \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}^{-1}.$$

1.6 Lemma. *Let X' another quadratic space over F, and suppose $t : X \to X'$ is a similitude with similitude factor λ. Let ω' be the Weil representation of $R_{X',n}$ on $L^2(X'^n)$. Then*

$$\omega(g,h)(\varphi' \circ t) = [\omega'(g^{[\lambda]}, tht^{-1})\varphi')] \circ t$$

for $(g,h) \in R_{X,n}$ and $\varphi' \in L^2(X'^n)$.

Proof. By the formulas for the Weil representation the statement holds for $g \in \mathrm{Sp}(n,F)$ and $h = 1$, up to a factor $\alpha(g)$ in the fourth roots of unity μ_4. The function $\alpha : \mathrm{Sp}(n,F) \to \mu_4$ is a character. The only normal subgroups of $\mathrm{Sp}(n,F)$ are $\{\pm 1\}$ and $\mathrm{Sp}(n,F)$; α must be trivial. It is now easy to check that the formula holds for all $(g,h) \in R$. □

In the next result the dependence of ω on ψ is indicated by a subscript. Its proof is similar to that of Lemma 1.6.

1.7 Lemma. *Let ψ' be another nontrivial unitary character of F. Let $a \in F^\times$ be such that $\psi'(t) = \psi(at)$ for $t \in F$. Then there is an isomorphism*

$$(\omega_{\psi'}, L^2(X^n)) \xrightarrow{\sim} ((\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, 1) \cdot \omega_\psi, L^2(X^n))$$

of representations of R, where $\epsilon = a$ if F is nonarchimedean and $\epsilon = \mathrm{sign}(a)$ if $F = \mathbb{R}$. If F is nonarchimedean, the isomorphism is the identity map; if $F = \mathbb{R}$, the isomorphism sends φ' to φ, where $\varphi(x) = \varphi'(\sqrt{|a|}^{-1} x)$. This isomorphism maps $\mathcal{S}_{\psi'}(X^n)$ onto $\mathcal{S}_\psi(X^n)$ (the subscripts ψ and ψ' are relevant when $F = \mathbb{R}$).

With this preparation, we recall the theta correspondence for similitudes from [R1]. In analogy to the case of isometries, we ask when does $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ for $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ define the graph of a bijection between appropriate subsets of $\mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\mathrm{Irr}(\mathrm{GO}(X,F))$? In considering this, two initial observations come to mind. First, R only involves $\mathrm{GSp}(n,F)^+$, the subgroup of $\mathrm{GSp}(n,F)$ (of at most index two) of $g \in \mathrm{GSp}(n,F)^+$ with $\lambda(g) \in \lambda(\mathrm{GO}(X,F))$; thus, at first it might be better to look at representations of $\mathrm{GSp}(n,F)^+$ instead of $\mathrm{GSp}(n,F)$. Second, there should be a close relationship between $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ and $\mathrm{Hom}_{\mathrm{Sp}(n,F)\times \mathrm{O}(X,F)}(\omega, \pi_1 \otimes \sigma_1) \neq 0$ for π_1 and σ_1 irreducible constituents of

$\pi|_{\mathrm{Sp}(n,F)}$ and $\sigma|_{\mathrm{O}(X,F)}$, respectively. The basic result that builds on these remarks is Lemma 4.2 of [R1]. It asserts that if $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F)^+)$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ and $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$, then

$$\pi|_{\mathrm{Sp}(n,F)} = m \cdot \pi_1 \oplus \cdots \oplus m \cdot \pi_M, \quad \sigma|_{\mathrm{O}(X,F)} = m' \cdot \sigma_1 \oplus \cdots \oplus m' \cdot \sigma_M$$

with $\theta(\pi_i) = \sigma_i$ for $1 \leq i \leq M$ and $m = 1$ if and only if $m' = 1$. Here the $\pi_i \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ and $\sigma_i \in \mathrm{Irr}(\mathrm{O}(X,F))$ are mutually nonisomorphic. Actually, [R1] considers the nonarchimedean case of odd residual characteristic, but the same proof works if F has even residual characteristic, $\dim X = 2n$ and π and σ are tempered, so that Theorem 1.2 applies, or if $F = \mathbb{R}$; in this case $m = m' = 1$, as $[\mathrm{GSp}(n,\mathbb{R})^+ : \mathbb{R}^\times \mathrm{Sp}(n,\mathbb{R})], [\mathrm{GO}(X,\mathbb{R}) : \mathbb{R}^\times \mathrm{O}(X,\mathbb{R})] \leq 2$ (see Table 1 in the appendix for data on $\mathrm{GSp}(n,\mathbb{R})^+$). With this in place, [R1] shows that the condition $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ defines the graph of a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, where $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ is the set of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F)^+)$ such that $\pi|_{\mathrm{Sp}(n,F)}$ is multiplicity free and has an irreducible constituent in $\mathcal{R}_X(\mathrm{Sp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$ is similarly defined (again, this also holds if F has even residual characteristic or $F = \mathbb{R}$).

Finally, when $\mathrm{GSp}(n,F)^+$ is proper in $\mathrm{GSp}(n,F)$, [R1] shows $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ defines the graph of a bijection between suitable subsets of $\mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\mathrm{Irr}(\mathrm{GO}(X,F))$ provided $m \leq 2n$ and the relevant case of Conjecture 1.3 holds. The idea is that if $[\mathrm{GSp}(n,F) : \mathrm{GSp}(n,F)^+] = 2$, then X has a certain companion nonisometric quadratic space X' with the same dimension and discriminant (this determines X' if F is nonarchimedean; if $F = \mathbb{R}$, then X' is the quadratic space of signature (q,p)). When it holds and $m \leq 2n$, Conjecture 1.3 implies that together the two theta correspondences between $\mathrm{GSp}(n,F)^+$ and $\mathrm{GO}(X,F)$ and between $\mathrm{GSp}(n,F)^+$ and $\mathrm{GO}(X',F)$ give one theta correspondence between $\mathrm{GSp}(n,F)$ and $\mathrm{GO}(X,F)$ (which is the same as that between $\mathrm{GSp}(n,F)$ and $\mathrm{GO}(X',F)$, using $\mathrm{GO}(X,F) = \mathrm{GO}(X',F)$). Since [R1] explains this in somewhat different language, we recall the argument in the proof of the summary theorem below.

For the statement of the theorem we need some notation. If F is nonarchimedean, define $\mathcal{R}_n(\mathrm{GO}(X,F))$ and $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ as above, and let $\mathcal{R}_X(\mathrm{GSp}(n,F))$ be the set of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F))$ such that some irreducible constituent of $\pi|_{\mathrm{GSp}(n,F)^+}$ is contained in $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$. If $F = \mathbb{R}$, let $\mathcal{R}_n(\mathrm{GO}(X,\mathbb{R}))$ be the set of $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,\mathbb{R}))$ such that $\sigma|_{\mathrm{O}(X,\mathbb{R})}$ has an irreducible constituent in $\mathcal{R}_n(\mathrm{O}(X,\mathbb{R}))$, and let $\mathcal{R}_X(\mathrm{GSp}(n,\mathbb{R}))$ be the set of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,\mathbb{R}))$ such that $\pi|_{\mathrm{Sp}(n,\mathbb{R})}$ has an irreducible constituent in $\mathcal{R}_X(\mathrm{Sp}(n,\mathbb{R}))$. Here $\sigma|_{\mathrm{O}(X,\mathbb{R})}$ and $\pi|_{\mathrm{Sp}(n,\mathbb{R})}$ mean $\sigma|_{(\mathfrak{h}_1, J_1)}$ and $\pi|_{(\mathfrak{g}_1, K_1)}$, respectively. If $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ and F is nonarchimedean we say that σ is TEMPERED if all the irreducible constituents of $\sigma|_{\mathrm{GSO}(X,F)}$ are tempered; evidently, σ is tempered if and only if the irreducible constituents of $\sigma|_{\mathrm{O}(X,F)}$ are tempered and σ has unitary central character, and this happens if and only if the irreducible constituents of $\sigma|_{\mathrm{SO}(X,F)}$ are tempered and σ has unitary central character.

1.8 THEOREM. *Suppose first F is real or nonarchimedean of odd residual characteristic. Then*

$$\{(\pi,\sigma)\in\mathcal{R}_X(\mathrm{GSp}(n,F))\times\mathcal{R}_n(\mathrm{GO}(X,F)):\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, and

$$\dim_{\mathbb{C}}\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\leq 1$$

for $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$, in the following cases:

(1) *F is nonarchimedean and $d=1$;*
(2) *F is nonarchimedean, $d\neq 1$, and $m\leq n+2$;*
(3) *$F=\mathbb{R}$ and $p=q$;*
(4) *$F=\mathbb{R}$, $p\neq q$, p and q are even, and $p+q\leq 2n$.*

Now assume F is nonarchimedean of even residual characteristic and $m=2n$. As in Theorem 1.2, let the subscript temp denote the subset of tempered elements. Then the above statement holds with $\mathcal{R}_X(\mathrm{GSp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{GO}(X,F))_{\mathrm{temp}}$ in place of $\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, respectively, in the following cases:

(5) *$d=1$ and $m=2n$;*
(6) *$d\neq 1$ and $m=2n=n+2=4$.*

Proof. (1). Since $d=1$, $\mathrm{GSp}(n,F)^+=\mathrm{GSp}(n,F)$, and the statement follows from Theorem 4.4 of [R1].
(2). This is dealt with in [R1], but we shall briefly recall the argument for the purposes of explanation. In this case we have $[\mathrm{GSp}(n,F):\mathrm{GSp}(n,F)^+]=2$. Let $g\in\mathrm{GSp}(n,F)$ be a representative for the nontrivial coset of $\mathrm{GSp}(n,F)^+$ in $\mathrm{GSp}(n,F)$. As mentioned above, by Theorem 4.4 of [R1] the condition $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$ defines a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, and $\dim_{\mathbb{C}}\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\leq 1$ for $\pi'\in\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$. To prove the theorem in this case, we first claim that if $\pi'\in\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$ are such that $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$, then $g\cdot\pi'\not\cong\pi'$ (so that $\pi=\mathrm{Ind}_{\mathrm{GSp}(n,F)^+}^{\mathrm{GSp}(n,F)}\pi'$ is irreducible), and $\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\cong\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)$; also, if $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$ then $\pi|_{\mathrm{GSp}(n,F)^+}$ has two irreducible components. Let X' be the other quadratic space of dimension m and discriminant d nonisometric to X. We may assume that X' is obtained from X by multiplying the form on X by $\lambda(g)$; then $\mathrm{GO}(X',F)=\mathrm{GO}(X,F)$ and $R_{X',n}=R=R_{X,n}$. Let $\omega'=\omega_{X'}$; by Lemma 1.6, $g\cdot\omega\cong\omega'$. Now since $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$ we have $\mathrm{Hom}_R(g\cdot\omega,g\cdot\pi'\otimes\sigma)\neq 0$, and so $\mathrm{Hom}_R(\omega',g\cdot\pi'\otimes\sigma)\neq 0$. This gives $g\cdot\pi'\in\mathcal{R}_{X'}(\mathrm{GSp}(n,F)^+)$. If now $\pi'\cong g\cdot\pi'$, then $\mathcal{R}_X(\mathrm{Sp}(n,F))\cap\mathcal{R}_{X'}(\mathrm{Sp}(n,F))\neq\emptyset$ (see Lemma 4.2 of [R1]), contradicting Lemma 1.4. Thus, $g\cdot\pi'\not\cong\pi'$. Composing with the projection $\pi\to\pi'$ gives a map $\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\to\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)$; by arguments similar to those just given, this map is a $\mathbb{C}$ linear isomorphism. Let $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$, and suppose $\pi|_{\mathrm{GSp}(n,F)^+}=\pi'$ is irreducible. Then $\pi'\in\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$, and

so $g \cdot \pi' \not\cong \pi'$, a contradiction. This completes the proof of our claim. Using the claim, it is straightforward to prove the theorem in this case via analogous arguments.
(3) and (4). The arguments are similar to the nonarchimedean case in [R1]. In fact, they are easier since the indices of the various relevant subgroups are at most two. Thus, the analogues of the lemmas about induction and restriction from [GK] used in [R1] take on a simple form. For the convenience of the reader wishing to look closely at the arguments we present a table of data (See Table 1 in the Appendix). The $p = q$ and $p \neq q$ cases should be regarded as being analogous to the $d = 1$ and $d \neq 1$ nonarchimedean cases, respectively. In the table K^+ is a maximal compact subgroup of $\mathrm{GSp}(n, \mathbb{R})^+$.
(5) and (6). The arguments are similar to those for (1) and (2) as we have the inputs Theorem 1.2 and Lemma 1.4. The proofs of section 4 of [R1] are made in an abstracted context and thus residual characteristic independent; these arguments also go through with the restriction to tempered representations. The arguments in (2) for the case $[\mathrm{GSp}(n, F) : \mathrm{GSp}(n, F)^+] = 2$ also work with the restriction to tempered representations. The reader wishing to go through the details should note the remark after Theorem 1.2. □

The proof of Theorem 1.8 only used Lemmas 1.4 and 1.5 when F is nonarchimedean and $d \neq 1$ and $F = \mathbb{R}$ and $p \neq q$, respectively. However, Lemmas 1.4 and 1.5 have important applications when F is nonarchimedean and $d = 1$, and $F = \mathbb{R}$ and $p = q$: see Lemma 8.4 and the proof of Proposition 4.1.
We note that if $\pi \in \mathrm{Irr}(\mathrm{GSp}(n, F))$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X, F))$ and $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ then $\chi_X^n = \omega_\pi \omega_\sigma$ where ω_π and ω_σ are the central characters of π and σ, respectively. Here, if $F = \mathbb{R}$ then the central character of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n, \mathbb{R}))$ is defined by $\omega_\pi(e^z) = \exp(\pi(z))$ for $z \in \mathbb{R} \subset \mathfrak{g}$, and $\omega_\pi(-1) = \pi(-1)$, where $-1 \in K$; ω_σ is defined similarly.
The theta correspondence for similitudes from Theorem 1.8 is independent of the choice of character ψ.

1.9 Proposition. *Let ψ' be another nontrivial unitary character of F, and let $\omega_{\psi'}$ be the Weil representation of R on $\mathcal{S}_{\psi'}(X^n)$ corresponding to ψ' (the subscript ψ' in $\mathcal{S}_{\psi'}(X^n)$ is relevant when $F = \mathbb{R}$). Let $\sigma \in \mathrm{Irr}(\mathrm{GO}(X, F))$ and $\pi \in \mathrm{Irr}(\mathrm{GSp}(n, F))$. Then $\mathrm{Hom}_R(\omega_\psi, \pi \otimes \sigma) \neq 0$ if and only if $\mathrm{Hom}_R(\omega_{\psi'}, \pi \otimes \sigma) \neq 0$.*

Proof. This follows from Lemma 1.7. □

Assume we are in one of the cases of Theorem 1.8. We then denote the bijection between $\mathcal{R}_X(\mathrm{GSp}(n, F))$ and $\mathcal{R}_n(\mathrm{GO}(X, F))$ by θ:

$$\theta : \mathcal{R}_X(\mathrm{GSp}(n, F)) \xrightarrow{\sim} \mathcal{R}_n(\mathrm{GO}(X, F)),$$
$$\theta : \mathcal{R}_n(\mathrm{GO}(X, F)) \xrightarrow{\sim} \mathcal{R}_X(\mathrm{GSp}(n, F)).$$

If $\pi \in \mathcal{R}_X(\mathrm{GSp}(n, F))$ and $\sigma \in \mathcal{R}_n(\mathrm{GO}(X, F))$ then $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ if and only if $\theta(\pi) = \sigma$ and $\theta(\sigma) = \pi$; if F has even residual characteristic we use $\mathcal{R}_X(\mathrm{GSp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{GO}(X,F))_{\mathrm{temp}}$. If $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ we say that σ OCCURS IN THE THETA CORRESPONDENCE WITH $\mathrm{GSp}(n,F)$; similarly, if $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ we say that π occurs in the theta correspondence with $\mathrm{GO}(X,F)$. The above definition of θ is not quite compatible with the global definition; a contragredient must be introduced. If π is a cuspidal automorphic representation of $\mathrm{GSp}(X,\mathbb{A})$, the global theta lift $\Theta(\pi)$ is nonzero and cuspidal, and Theorem 1.8 applies at every place, then $\pi_v^\vee \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ for all places v of F, and $\Theta(\pi) = \otimes_v \theta(\pi_v^\vee)$ (See Section 5). However, we have the following proposition. It guarantees that if $\sigma_v = \theta(\pi_v^\vee)$ then $\theta(\sigma_v^\vee) = \pi_v$.

1.10 PROPOSITION. *Let $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ be unitary. Then $\mathrm{Hom}_R(\omega, \pi\otimes\sigma) \neq 0$ if and only if $\mathrm{Hom}_R(\omega,\pi^\vee\otimes\sigma^\vee) \neq 0$. Suppose one of (1)–(6) of Theorem 1.8 holds. Then $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ if and only if $\pi^\vee \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ and if $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$, then $\theta(\pi^\vee) = \theta(\pi)^\vee$. Similarly, $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ if and only if $\sigma^\vee \in \mathcal{R}_n(\mathrm{GO}(X,F))$, and if $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ then $\theta(\sigma^\vee) = \theta(\sigma)^\vee$. (If F has even residual characteristic, replace $\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$ by $\mathcal{R}_X(\mathrm{GSp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{GO}(X,F))_{\mathrm{temp}}$, respectively, in these statements.)*

Proof. Since π and σ are unitary, there exist $\mathbb{C}$ antilinear isomorphisms $\pi \xrightarrow{\sim} \pi^\vee$ and $\sigma \xrightarrow{\sim} \sigma^\vee$ intertwining the actions of $\mathrm{GSp}(n,F)$ and $\mathrm{GO}(X,F)$, respectively. It follows that there is a $\mathbb{C}$ antilinear isomorphism $\pi\otimes\sigma \xrightarrow{\sim} \pi^\vee\otimes\sigma^\vee$ intertwining the action of $\mathrm{GSp}(n,F)\times\mathrm{GO}(X,F)$. Let $\overline{\omega}$ be the representation of R on $\mathcal{S}(X^n)$ defined by $\overline{\omega}(r)\varphi = \overline{\omega(r)\overline{\varphi}}$ for $r \in R$ and $\varphi \in \mathcal{S}(X^n)$. Let $t : \omega \to \pi\otimes\sigma$ be a nonzero R map; then sending φ to $t(\overline{\varphi})$ gives a nonzero $\mathbb{C}$ antilinear R map $\overline{\omega} \to \pi\otimes\sigma$. Composing, we get a nonzero R map $\overline{\omega} \to \pi^\vee\otimes\sigma^\vee$. On the other hand, there is an R isomorphism

$$\overline{\omega} \cong (\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1)\cdot\omega.$$

This implies that there is a nonzero R map $\omega \to \pi^\vee\otimes\sigma^\vee$. The remaining claims of the proposition follow. □

Finally, using [H], we consider how the theta correspondence for similitudes treats unramified representations. This requires some definitions. Assume F is nonarchimedean, and let $\mathbb{H}$ be the hyperbolic plane over F. Then $X \cong \mathbb{H} \perp \cdots \perp \mathbb{H} \perp X_0$, where X_0 is an anisotropic quadratic space over F of dimension $0, 2$ or 4. In particular, if $\dim_F X_0 = 2$, then $d \neq 1$ and $X_0 \cong (E, \delta\,\mathrm{N}_F^E)$, for a quadratic extension E/F, where $\delta = 1$ or is a representative for the nontrivial coset of $F^\times/\mathrm{N}_F^E(E^\times)$. We say that X is UNRAMIFIED if either $\dim X_0 = 0$ or $\dim X_0 = 2$, E/F is unramified and $\delta = 1$. If X is unramified, then there exists a lattice $L \subset X$ which is self-dual, and if L' is any other self-dual lattice in X, then there exists $h \in \mathrm{SO}(X,F)$ such that $h(L) = L'$. If $\dim_F X_0 = 0$ or $\dim_F X_0 = 2$ and E/F is unramified, we define a maximal compact subgroup J of $\mathrm{GO}(X,F)$ in the following way. First, if X is unramified, we let J be the stabilizer in $\mathrm{GO}(X,F)$ of a fixed self-dual lattice L, i.e., J is the set of $k \in$

$\mathrm{GO}(X,F)$ such that $k(L)=L$. Next assume $\dim_F X_0=2$, E/F is unramified but $\delta\neq 1$. Then there exists a similitude h from X to the unramified quadratic space of the same dimension and discriminant with anisotropic component (E,N_F^E); we let J be the set of $k\in\mathrm{GO}(X,F)$ of the form $h^{-1}k'h$ were k' is in the maximal compact subgroup of $\mathrm{GO}(X',F)$ we have already defined. The definition of J depends on choices, but any two subgroups defined by different choices are conjugate. Let $K=\mathrm{GSp}(n,\mathfrak{O}_F)$.

1.11 PROPOSITION. *Suppose F is nonarchimedean of odd residual characteristic and X is such that J is defined. Let $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$ and $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$ and assume $\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\neq 0$. Then π is unramified with respect to K if and only if σ is unramified with respect to J.*

Proof. By Proposition 1.9 we may assume $\psi(\mathfrak{O}_F)=1$ but $\psi(\pi_F^{-1}\mathfrak{O}_F)\neq 1$; by Lemma 1.6 we may assume X is unramified. We have $K=\mathfrak{O}_F^\times K_1\cup k_0\mathfrak{O}_F^\times K_1$ and $J=\mathfrak{O}_F^\times J_1\cup j_o\mathfrak{O}_F^\times J_1$, where $K_1=K\cap\mathrm{Sp}(n,F)$, $J_1=J\cap\mathrm{O}(X,F)$, $\lambda(k_0)=\lambda(j_0)=\mu$, and μ is a representative for the nontrivial coset of $\mathfrak{O}_F^\times/\mathfrak{O}_F^{\times 2}$. For some irreducible component π' of $\pi|_{\mathrm{GSp}(n,F)^+}$, we have $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$. As $K\subset\mathrm{GSp}(n,F)^+$, it will suffice to show that σ is unramified with respect to J if and only if π' is unramified with respect to K. By the proof of Lemma 4.2 of [R1] we can write

$$\pi'|_{\mathrm{Sp}(n,F)}=\pi_1\oplus\cdots\oplus\pi_M,\quad \sigma|_{\mathrm{O}(X,F)}=\sigma_1\oplus\cdots\oplus\sigma_M$$

where the $\pi_i\in\mathrm{Irr}(\mathrm{Sp}(n,F))$ and the $\sigma_i\in\mathrm{Irr}(\mathrm{O}(X,F))$ are mutually nonisomorphic and $\sigma_i=\theta(\pi_i)$. Let V_i and W_i be the spaces of π_i and σ_i, respectively. Assume σ is unramified with respect to J. Let $w_1\in\sigma$ be nonzero and fixed by J. Since $\sigma|_{\mathrm{O}(X,F)}$ has exactly one irreducible constituent unramified with respect to J_1, we may assume, say, $w_1\in W_1$. Evidently, $\sigma(j_0)W_1=W_1$. By (b) of Theorem 7.1 of [H], $\pi_1=\theta(\sigma_1)$ is unramified with respect to K_1. Let $v_1\in V_1$ be nonzero and fixed by K_1. We will show that v_1 is in fact fixed by K, i.e., $\pi'(k_0)v_1=v_1$. As $\pi'|_{\mathrm{Sp}(n,F)}$ has exactly one irreducible constituent unramified with respect to K_1 we have $\pi'(k_0)V_1=V_1$. Since $V_1^{K_1}$ is one dimensional, $\pi'(k_0)v_1=\epsilon v_1$ for some $\epsilon\in\{\pm 1\}$. We must show $\epsilon=1$. Let $T:\omega\to\pi'\otimes\sigma$ be a nonzero R map, and let $p:\pi'\otimes\sigma\to\pi_1\otimes\sigma_1$ be projection. Let $T_1=p\circ T:\omega\to\pi_1\otimes\sigma_1$; this is a nonzero $\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)$ map. Let $\varphi\in\omega$ be such that $T_1(\varphi)=v_1\otimes w_1$; we may assume φ is fixed by $K_1\times J_1$. By the top of p. 107 of [MVW], there exists a locally constant compactly supported K_1 bi-invariant function $f:\mathrm{Sp}(n,F)\to\mathbb{C}$ such that

$$\varphi=\int_{\mathrm{Sp}(n,F)} f(g)\omega(g,1)\varphi_0\,dg;$$

here $\varphi_0\in\omega$ is a certain element fixed by $K_1\times J_1$. One can check that φ_0 is also fixed by (k_0,j_0). We have $T_1(\omega(k_0,j_0)\varphi)=\epsilon(v_1\otimes w_1)$. However,

$$\omega(k_0,j_0)\varphi=\int_{\mathrm{Sp}(n,F)} f(k_0^{-1}gk_0)\omega(g,1)\varphi_0\,dg.$$

Let $g \in \mathrm{Sp}(n, F)$. We claim $f(k_0^{-1} g k_0) = f(g)$. Write $g = kak'$ with $k, k' \in K_1$ and a a diagonal matrix. Since we may assume

$$k_0 = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix},$$

we have $k_0^{-1} g k_0 = k_0^{-1} k k_0 a k_0^{-1} k' k_0$; hence, $f(k_0^{-1} g k_0) = f(a) = f(g)$. Thus, $\omega(k_0, j_0)\varphi = \varphi$. This implies $v_1 \otimes w_1 = T_1(\varphi) = \epsilon(v_1 \otimes w_1)$, so that $\epsilon = 1$. The implication in the other direction has a similar argument. $\square$

2. Four dimensional quadratic spaces

In this section we recall background on four dimensional quadratic spaces X over a base field F and their similitude groups. We begin by characterizing the special similitude group $\mathrm{GSO}(X, F)$ of X via its even Clifford algebra. We also obtain canonical coset representatives for the nontrivial coset of $\mathrm{GSO}(X, F)$ in $\mathrm{GO}(X, F)$; these correspond to quaternion algebras over F contained in the even Clifford algebra over F, which in turn are in bijection with Galois actions on the even Clifford algebra. This leads to the concept of a quadratic quaternion algebra over F, an abstraction of the even Clifford algebra of a four dimensional quadratic space. We construct examples of four dimensional quadratic spaces from a given quadratic quaternion algebra over F and quaternion algebras over F contained in the quadratic quaternion algebra, or equivalently, Galois actions on the quadratic quaternion algebra. We prove that any four dimensional quadratic space over F is, up to similitude, one of these examples. We also describe the relationship between the examples that arise from a given quadratic quaternion algebra. To close the section, we consider four dimensional quadratic spaces over local and number fields. The material in this section is essentially well known. As some basic references we use [E], [Sch] and [Kn].

To begin, let F be a field not of characteristic two, and let $(X, (\cdot,\cdot))$ be a four dimensional quadratic space over F. For simplicity denote the F points of X by X. Set $d = \operatorname{disc} X$. Let x_1, x_2, x_3, x_4 be an orthogonal basis for X. Let C be the Clifford algebra of X, let $B = B(X)$ be the even Clifford algebra of X in C, let $E = E(X)$ be the center of B, and let $C_1 = C_1(X)$ be the subspace of C of odd elements. Then C, B, E and C_1 are 16, 8, 2 and 8 dimensional over F, respectively. The F algebra E is called the discriminant algebra of X and is reduced, i.e., has no nonzero nilpotent elements. Hence, E is either a field or is isomorphic to $F \times F$; these happen when $d \neq 1$ and $d = 1$, respectively. Let $\mathrm{Gal}(E/F) = \{1, \alpha\}$. Let N_F^E and T_F^E be the norm and trace from E to F defined by $\mathrm{N}_F^E(z) = z\alpha(z)$ and $\mathrm{T}_F^E(z) = z + \alpha(z)$, respectively. Let $*$ be the involution of C which takes a product of the x_i to the product of the same x_i in the reverse order. Clearly, $*$ preserves B and C_1. If $x \in B$, then $x \in E$ if and only if $x^* = x$. For $x \in C$, define $\mathrm{N}(x) = x^* x$. Then $\mathrm{N}(x) \in E$ for $x \in B$. We may regard X as contained in C_1. Evidently, X is the set of $x \in C_1$ such that $x^* = x$. For $x \in X$, $(x,x) = \mathrm{N}(x)$. Also, for $z \in E$ and $x \in C_1$ we have $xz = \alpha(z)x$.

To further describe the structure of B and E, suppose B is an arbitrary F algebra with center E and involution $*$ which is the identity on E. Then we say that B is a quadratic quaternion algebra over F if E is two dimensional over F and reduced, and there exists a quaternion algebra D over F contained in B such that the natural map $E \otimes_F D \to B$ given by $z \otimes x \mapsto zx$ is an isomorphism of E algebras and $*$ induces the canonical involution on D. Let B be a quadratic quaternion algebra over F with center E and involution $*$. We define the norm $\mathrm{N} : B \to E$ and trace $\mathrm{T} : B \to E$ by $\mathrm{N}(x) = xx^* = x^*x$ and $\mathrm{T}(x) = x + x^*$ respectively. We also define a symmetric E-bilinear form $(\cdot,\cdot) : B \times B \to E$ by $(x,y) = \mathrm{T}(xy^*)/2$. This form is nondegenerate, i.e., if $x \in B$ is nonzero, there exists $y \in B$ such that $(x,y) \neq 0$. The definition of a quadratic quaternion algebra B includes a particular quaternion algebra over F in B, but the next straightforward result shows that all the quaternion algebras over F in B have equal status.

2.1 Proposition. *Let B be a quadratic quaternion algebra over F with center E and involution $*$. Let D be any quaternion algebra over F in B. The natural map $E \otimes_F D \to B$ is an isomorphism of E algebras, and $*$ induces the canonical involution on D.*

Given a quadratic quaternion algebra B as above, in general there may be infinitely many nonisomorphic quaternion algebras D over F in B. However, if $E \cong F \times F$, then $B \cong D \times D$, and any quaternion algebra over F in B is isomorphic to D.

2.2 Proposition. *Let X be a four dimensional quadratic space over X. The F algebra $B(X)$ is a quadratic quaternion algebra over F.*

We characterize $\mathrm{GSO}(X,F)$. Write $B = B(X)$. Define a left action of $F^\times \times B^\times$ on C_1 by $\rho(t,g)x = t^{-1}gxg^*$. This action preserves X, and a computation shows that if $x \in X$ and $(t,g) \in F^\times \times B^\times$, then $\mathrm{N}(\rho(t,g)x) = t^{-2}\,\mathrm{N}_F^E(\mathrm{N}(g))\,\mathrm{N}(x)$; thus, $\rho(t,g) \in \mathrm{GO}(X,F)$, with similitude factor $t^{-2}\,\mathrm{N}_F^E(\mathrm{N}(g))$. In fact, if $(t,g) \in F^\times \times B^\times$, then $\rho(t,g) \in \mathrm{GSO}(X,F)$. For the following see for example V (4.6.1) of [Kn], p. 273.

2.3 Theorem. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Define an inclusion of $E^\times$ into $F^\times \times B^\times$ by $a \mapsto (\mathrm{N}_F^E(a), a)$. Then the following sequence is exact:*

$$1 \to E^\times \to F^\times \times B^\times \xrightarrow{\rho} \mathrm{GSO}(X,F) \to 1.$$

This theorem determines $\mathrm{GSO}(X,F)$. We also need to understand $\mathrm{GO}(X,F)$, and we now explain how to describe certain canonical coset representatives for the nontrivial coset of $\mathrm{GSO}(X,F)$ in $\mathrm{GO}(X,F)$. These coset representatives will correspond to choices of quaternion algebras over F in B. The following lemma is the key structural result for the construction of the coset representatives. It is an elaboration of a general result about Clifford algebras of even dimensional quadratic spaces (Chapter 9, Theorem 2.10 of [Sch], p. 332).

2.4 LEMMA. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Let D be a quaternion algebra over F contained in B. Let D' be the F algebra of elements of C which commute with all the elements of D. Then D' is a quaternion algebra over F and $X \cap D'$ is one dimensional and spanned by an anisotropic vector y, so that $D' = E + Ey$. The map $x' \otimes x \mapsto x'x$ determines an isomorphism $D' \otimes_F D \xrightarrow{\sim} C$ of F algebras. Conversely, if $y \in X$ is an anisotropic vector, then the set D of elements of B commuting with y is a quaternion algebra over F in B.*

The maps from the previous lemma are evidently inverses of each other; that is, there is a bijection

$$\text{Quaternion algebras over } F \text{ in } B \longleftrightarrow \text{Anisotropic lines in } X.$$

For the description of the nontrivial coset representatives of $\mathrm{GSO}(X,F)$ in $\mathrm{GO}(X,F)$ we also need the following. Suppose B is any quadratic quaternion algebra over F with center E with $\mathrm{Gal}(E/F) = \{1, \alpha\}$. Then a GALOIS ACTION ON B is an F-automorphism $a : B \to B$ such that $a^2 = 1$ and $a(zx) = \alpha(z)a(x)$ for $z \in E$ and $x \in B$. If a is a Galois action on B, then the fixed points of a are a quaternion algebra over F contained in B; conversely, if D is a quaternion algebra over F contained in B, and $a : B \to B$ is defined by $a(z \otimes x) = \alpha(z) \otimes x$, then a is a Galois action on B. These two maps are inverses of each other, and establish a bijection:

$$\text{Quaternion algebras over } F \text{ in } B \longleftrightarrow \text{Galois actions on } B.$$

Direct computation gives the following:

2.5 PROPOSITION. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Let D be a quaternion algebra over F contained in B, and let D' be as in Lemma 2.4. Let $\#$ be the involution of C obtained via the isomorphism $D' \otimes D \cong C$ from the tensor product of the canonical involutions on D' and D. Then $X^{\#} = X$; define $s : X \to X$ by $s(x) = -x^{\#}$. Then $s \in \mathrm{O}(X,F)$, $s^2 = 1$, and $\det s = -1$. Moreover, the following diagram commutes:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1 \\
 & & \alpha\big\downarrow & & 1\times a\big\downarrow & & \text{conj. by } s\big\downarrow & & \\
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1
\end{array}.$$

Here, a is the Galois action on B determined by D.

When F is a local field we shall deal with representations of $\mathrm{GSO}(X,F)$ distinguished with respect to subgroups $\mathrm{SO}(Y,F)$, where Y is a three dimensional subspace of X. The above development leads to a compatible characterization of such subgroups. For the exactness of the first sequence in the next result see for example [Kn], p. 264.

2.6 Proposition. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Let $y \in X$ be anisotropic, and set $Y = (F{\cdot}y)^{\perp}$ in X. Let D be the quaternion algebra over F in B corresponding to y. For $g \in D^{\times}$ and $x \in Y$, define $\rho(g)x = gxg^{-1}$. Then $\rho(g) \in \mathrm{SO}(Y, F)$ for $g \in D^{\times}$, the sequence*

$$1 \to F^{\times} \to D^{\times} \xrightarrow{\rho} \mathrm{SO}(Y, F) \to 1$$

is exact, there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & F^{\times} \arrow[r] \arrow[d] & D^{\times} \arrow[r, "\rho"] \arrow[d] & \mathrm{SO}(Y,F) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & E^{\times} \arrow[r] & F^{\times} \times B^{\times} \arrow[r, "\rho"] & \mathrm{GSO}(X,F) \arrow[r] & 1
\end{tikzcd}$$

where the inclusion of $D^{\times}$ in $F^{\times} \times B^{\times}$ is given by $g \mapsto (\mathrm{N}(g), g)$, and $\mathrm{SO}(Y, F)$ is included in $\mathrm{GSO}(X, F)$ by regarding $\mathrm{SO}(Y, F)$ as the stabilizer of y in $\mathrm{GSO}(X, F)$. Moreover, the element s from Proposition 2.5 corresponding to D is such that $s(y) = y$ and s is multiplication by -1 on Y, so that $s|_Y \in \mathrm{O}(Y, F)$, with $\det s|_Y = -1$.

It will be important to have some explicitly constructed four dimensional quadratic spaces, and we now reverse matters and construct such examples from a given quadratic quaternion algebra over F equipped with a Galois action. Let B be a quadratic quaternion algebra over F with center E with $\mathrm{Gal}(E/F) = \{1, \alpha\}$, involution $*$, and let $a : B \to B$ be a Galois action on B. Let D be the quaternion algebra over F in B corresponding to a, i.e., the fixed points of a. We let X_a be the set of $x \in B$ such that $a(x) = x^*$. Then X_a is a four dimensional vector space over F, and equipped with the symmetric bilinear form induced by the norm of B, X_a is a quadratic space over F. Define an explicit action of $F^{\times} \times B^{\times}$ on X_a by $\rho_a(t, g)x = t^{-1}gxa(g)^*$. Then $\rho_a(t, g) \in \mathrm{GSO}(X_a, F)$ for $(t, g) \in F^{\times} \times B^{\times}$. The relationship between the previous characterization of $\mathrm{GSO}(X_a, F)$ and the homomorphism ρ_a is given by the following proposition.

2.7 Proposition. *Let B be a quadratic quaternion algebra over F with center E with $\mathrm{Gal}(E/F) = \{1, \alpha\}$, involution $*$, and let $a : B \to B$ be a Galois action on B. Then the sequence*

$$1 \to E^{\times} \to F^{\times} \times B^{\times} \xrightarrow{\rho_a} \mathrm{GSO}(X_a, F) \to 1,$$

is exact, where the inclusion of $E^{\times}$ is defined by $z \mapsto (\mathrm{N}^E_F(z), z)$. There exists a unique F algebra isomorphism $B(X_a) \xrightarrow{\sim} B$ sending $E(X_a)$ onto E so that the diagram

$$\begin{tikzcd}
1 \arrow[r] & E(X_a)^{\times} \arrow[r] \arrow[d, "\wr"'] & F^{\times} \times B(X_a)^{\times} \arrow[r, "\rho"] \arrow[d, "\wr"'] & \mathrm{GSO}(X_a,F) \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & E^{\times} \arrow[r] & F^{\times} \times B^{\times} \arrow[r, "\rho_a"] & \mathrm{GSO}(X_a,F) \arrow[r] & 1
\end{tikzcd}$$

commutes. The map defined by $x \mapsto a(x) = x^*$ *maps* X_a *onto* X_a*, and is the element* $s \in \mathrm{O}(X_a, F)$ *from Proposition 2.5 associated to the quaternion algebra over* F *in* $B(X_a)$ *corresponding to* a*.*

Explicit quadratic quaternion algebras equipped with Galois actions may constructed as follows. Let E be a two dimensional reduced F algebra, so that E is either a quadratic extension of F, or $E \cong F \times F$. Let $\mathrm{Gal}(E/F) = \{1, \alpha\}$, and let D be a quaternion algebra over F with canonical involution $*$. Set $B_{D,E} = E \otimes_F D$, endow $B_{D,E}$ with the involution defined by $(z \otimes x)^* = z \otimes x^*$, and define $a = a(D,E) : B_{D,E} \to B_{D,E}$ by $a(z \otimes x) = \alpha(z) \otimes x$. Clearly $B_{D,E}$ is a quadratic quaternion algebra over F, and a is a Galois action on $B_{D,E}$; we will write $X_{D,E} = X_a$. To be even more concrete, let $d \in F^\times / F^{\times 2}$. If $d \neq 1$, let $E_d = F(\sqrt{d})$; if $d = 1$, let $E_d = F \times F$. We write $B_{D,d} = B_{D,E_d}$ and $X_{D,d} = X_{D,E_d}$. Evidently $\operatorname{disc} X_{D,d} = d$. Assume further $d = 1$. Then there is a canonical isomorphism $D \times D \xrightarrow{\sim} B_{D,1}$ of F algebras. With respect to this isomorphism, a is given by $a(x, x') = (x', x)$, and $*$ is given by $(x, x')^* = (x^*, x'^*)$. Thus, $X_{D,1}$ is the set of pairs (x, x^*) for $x \in D$, which can be identified with D. With respect to these identifications, $\rho_a(t, (g, g'))x = t^{-1} g x g'^*$ for $t \in F^\times$, $x \in D$, and $g, g' \in D^\times$.
Before turning to specific fields we address two natural questions. First, if X is an arbitrary four dimensional quadratic space over F, when can X be related to an $X_{D,E}$?

2.8 Proposition. *Let* X *be a four dimensional quadratic space over* F *and write* $B = B(X)$ *and* $E = E(X)$*. There exists a quaternion algebra* D *over* F *in* B *and a similitude* $T : X \to X_{D,E}$ *so that*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{id}}\downarrow & & \wr\downarrow & & \downarrow{\scriptstyle T\cdot T^{-1}} & & \\
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B_{D,E}^\times & \xrightarrow{\rho_{a(D,E)}} & \mathrm{GSO}(X_{D,E},F) & \longrightarrow & 1
\end{array}$$

commutes, and the element $s \in \mathrm{O}(X, F)$ *corresponding to* D *from Proposition 2.5 is mapped to the element of* $\mathrm{O}(X_{D,E}, F)$ *defined by* $x \mapsto a(x) = x^*$*, where* $a = a(D, E)$*. If* X *represents* 1*, then we may further choose* T *to be an isometry. Conversely, if* X *is isometric to* $X_{D,E}$ *for some* D*, then* X *represents* 1*.*

Given a quadratic quaternion algebra over F, what is the relationship between the X_a for different Galois actions a on the quadratic quaternion algebra? The main ingredient for the following is the Skolem-Noether theorem.

2.9 Proposition. *Let* B *be a quadratic quaternion algebra over* F *with center* E*, let* $\mathrm{Gal}(E/F) = \{1, \alpha\}$*, and let* a *and* a' *be Galois actions on* B*. There exists* $u \in B^\times$*, uniquely determined up to multiplication by elements of* $E^\times$*, such that* $a'(x) = u^{-1} a(x) u$ *for* $x \in B$*. We have* $u a(u) = u a'(u) \in F^\times$*. Let*

$\mu = ua(u) = ua'(u)$. Then u can be chosen so that $\mathrm{N}(u) = \mu$; choose such a u. The map $T : X_a \to X_{a'}$ given by $T(x) = xu$ is a well-defined similitude with similitude factor $\lambda(T) = \mu$. The diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho_a} & \mathrm{GSO}(X_a, F) & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle T\cdot T^{-1}} & & \\
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho_{a'}} & \mathrm{GSO}(X_{a'}, F) & \longrightarrow & 1
\end{array}$$

commutes.

To close this section we consider choices of F. Suppose F is nonarchimedean of characteristic zero. Let $d \in F^\times/F^{\times 2}$. Up to isometry, there are two four dimensional quadratic spaces of discriminant d; these are distinguished by their Hasse invariant. Both spaces represent 1. One space is isometric to $X_{\mathrm{M}_{2\times2},d}$, where $\mathrm{M}_{2\times2} = \mathrm{M}_{2\times2}(F)$ is the quaternion algebra of 2×2 matrices over F; the other is isometric to $X_{D_{\mathrm{ram}},d}$, where D_{ram} is the division quaternion algebra over F. These spaces have Hasse invariant $\epsilon(d)$ and $-\epsilon(d)$, respectively, where $\epsilon(d) = (-1,-d)_F$. If $d = 1$, then $X_{\mathrm{M}_{2\times2},1}$ is isometric to $\mathrm{M}_{2\times2}(F)$ equipped with the determinant, and $X_{D_{\mathrm{ram}},1}$ is isometric to D_{ram} equipped with the norm; see the remarks before Proposition 2.8. Suppose $d \neq 1$. Then $X_{\mathrm{M}_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ are both isotropic. Also, $B_{\mathrm{M}_{2\times2},d}$ and $B_{D_{\mathrm{ram}},d}$ are both isomorphic to $\mathrm{M}_{2\times2}(E_d)$. Explicitly, let δ be a representative for the nontrivial coset of $F^\times/\mathrm{N}_F^{E_d}(E_d^\times)$. Then we can take

$$D_{\mathrm{ram}} = \{\begin{bmatrix} e & f\delta \\ \alpha(f) & \alpha(e)\end{bmatrix} : e, f \in E_d\} \subset \mathrm{M}_{2\times2}(E_d).$$

The Galois actions $a = a(\mathrm{M}_{2\times2}, E_d)$ and $a' = a(D_{\mathrm{ram}}, E_d)$ on $\mathrm{M}_{2\times2}(E_d)$ corresponding to $\mathrm{M}_{2\times2}(F)$ and D_{ram} are given by

$$(2.1)\qquad a(\begin{bmatrix} e & f \\ g & h\end{bmatrix}) = \begin{bmatrix} \alpha(e) & \alpha(f) \\ \alpha(g) & \alpha(h)\end{bmatrix} \text{ and } a'(\begin{bmatrix} e & f \\ g & h\end{bmatrix}) = \begin{bmatrix} \alpha(h) & \delta\alpha(g) \\ \alpha(f)/\delta & \alpha(e)\end{bmatrix},$$

respectively, and $X_{\mathrm{M}_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ are the set of elements in $\mathrm{M}_{2\times2}(E_d)$

$$\begin{bmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \alpha(e)\end{bmatrix} \text{ and } \begin{bmatrix} f & -\delta e \\ \alpha(e) & g\end{bmatrix},$$

respectively, for $e \in E_d$ and $f, g \in F$. The element u from Proposition 2.9 can be taken to be

$$\sqrt{d}\begin{bmatrix} 0 & \delta \\ 1 & 0\end{bmatrix}.$$

Evidently, if the residual characteristic of F is odd and E_d/F is unramified, then $X_{\mathrm{M}_{2\times2},d}$ is unramified but $X_{D_{\mathrm{ram}},d}$ is not. The quadratic spaces $X_{\mathrm{M}_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ have isomorphic similitude groups, and from the point of view of the theta correspondence for similitudes, they are grouped together. The two quadratic spaces with discriminant 1, however, do not have isomorphic similitude groups and and are distinct from the point of view of the theta correspondence for similitudes. See the remarks before Theorem 1.8 and the proof of Theorem 1.8.

Suppose $F=\mathbb{R}$. Let $d \in \mathbb{R}^\times/\mathbb{R}^{\times 2}$. If $d=1$, then up to isometry there are three four dimensional quadratic spaces of discriminant 1, with signatures $(4,0)$, $(2,2)$ or $(0,4)$. The quadratic space with signature $(4,0)$ is $X_{D_{\mathrm{ram}},1}$; the ramified quaternion algebra D_{ram} over $\mathbb{R}$ is the Hamilton quaternion algebra. The quadratic space with signature $(2,2)$ is $X_{\mathrm{M}_{2\times 2},1}$ where $\mathrm{M}_{2\times 2}=\mathrm{M}_{2\times 2}(\mathbb{R})$. Finally, the quadratic space with signature $(0,4)$ is not of the form $X_{D,1}$. However, as predicted by Proposition 2.8, there is an intertwining similitude with the space $X_{D_{\mathrm{ram}},1}$: the quadratic space with signature $(0,4)$ can be taken to be $X_{D_{\mathrm{ram}},1}$ with form multiplied by -1. Then the intertwining similitude is just the identity function. If $d=-1$, then up to isometry there are two quadratics spaces of discriminant -1, with signatures $(1,3)$ or $(3,1)$. The quadratic space with signature $(3,1)$ is $X_{\mathrm{M}_{2\times 2},-1}$, while the quadratic space with signature $(1,3)$ is $X_{D_{\mathrm{ram}},-1}$. From the point of view of the theta correspondence for similitudes, the spaces with signature $(4,0)$ and $(0,4)$ are grouped together, the spaces with signature $(3,1)$ and $(1,3)$ are grouped together, and the space of signature $(2,2)$ is not grouped with another four dimensional quadratic space.

When $F=\mathbb{R}$ there are further exact sequences. Let X be a four dimensional quadratic space over $\mathbb{R}$, with even Clifford algebra B; let E be the center of B. We regard $F=\mathbb{R}, E$ and B as the Lie algebras of $F^\times=\mathbb{R}^\times, E^\times$ and $B^\times$, respectively. We take the Lie algebra $\mathfrak{gso}(X,\mathbb{R})$ of $\mathrm{GSO}(X,\mathbb{R})$ to be the subalgebra of $h \in \mathrm{End}_{\mathbb{R}} X$ for which there exists a $\lambda \in \mathbb{R}$ such that $(hx,x')+(x,hx')=\lambda(x,x')$ for $x,x' \in X$; then $\lambda=\mathrm{tr}(h)/2$. Define an action of $\mathbb{R}\times B$ on X by $\rho(r,h)x=-rx+hx+xh^*$, and an inclusion of E into $\mathbb{R}\times B$ by $b \mapsto (\mathrm{T}_F^E(b),b)$. By Theorem 2.3,

$$0 \to E \to \mathbb{R}\times B \xrightarrow{\rho} \mathfrak{gso}(X,\mathbb{R}) \to 0 \tag{2.2}$$

is an exact sequence of Lie algebras. Any two maximal compact subgroups of $\mathrm{GSO}(X,\mathbb{R})$ are conjugate. Let J_0 be a maximal compact subgroup of $\mathrm{GSO}(X,\mathbb{R})$. Then there exists a unique maximal compact subgroup K_B of $B^\times$ such that $\rho(\{\pm 1\}\times K_B)=J_0$. The normalizer of J_0 is $\mathbb{R}^\times J_0$, and J_0 is contained in a unique maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$. There is an exact sequence

$$1 \to K_B \cap E^\times \to \{\pm 1\}\times K_B \xrightarrow{\rho} J_0 \to 1. \tag{2.3}$$

Suppose that $y \in X$ is anisotropic and Y and D are as in Proposition 2.6. We take the Lie algebra of $\mathfrak{so}(Y,\mathbb{R})$ of $\mathrm{SO}(Y,\mathbb{R})$ to be the subalgebra of $h \in \mathrm{End}_{\mathbb{R}} Y$ such that $(hx,x')+(x,hx')=0$ for $x,x' \in Y$. We regard D as the Lie algebra of $D^\times$, and define an action of D on X by $\rho(h)x=hx-xh$. By Proposition

2.6 there is an exact sequence

$$0 \to \mathbb{R} \to D \to \mathfrak{so}(Y, \mathbb{R}) \to 0,$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & D & \xrightarrow{\rho} & \mathfrak{so}(Y,\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & \mathbb{R} \times B & \xrightarrow{\rho} & \mathfrak{gso}(X,\mathbb{R}) & \longrightarrow & 0 \end{array}$$

where D is included in $\mathbb{R} \times B$ via $h \mapsto (\mathrm{T}(h), h)$ and $\mathfrak{so}(Y, \mathbb{R})$ is included in $\mathfrak{gso}(X, \mathbb{R})$ by setting the elements of $\mathfrak{so}(Y, \mathbb{R})$ to be 0 on $\mathbb{R} \cdot y$. Any two maximal compact subgroups of $\mathrm{SO}(Y, \mathbb{R})$ are conjugate. Let J_Y be a maximal compact subgroup of $\mathrm{SO}(Y, \mathbb{R})$. Then there exists a unique maximal compact subgroup K_D of $D^\times$ such that $J_Y = \rho(K_D)$, and J_Y is contained in a unique maximal compact subgroup $J_0 = \rho(\{\pm 1\} \times K_B)$ of $\mathrm{GSO}(X, \mathbb{R})$. Also, $K_D \subset K_B$, the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & K_D & \xrightarrow{\rho} & J_Y & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_B \cap E^\times & \longrightarrow & \{\pm 1\} \times K_B & \xrightarrow{\rho} & J_0 & \longrightarrow & 1 \end{array}$$

commutes, and the element $s \in \mathrm{O}(X, \mathbb{R})$ from Proposition 2.6 normalizes J_Y and J_0. Conversely, if $J_0 = \rho(\{\pm 1\} \times K_B)$ is a maximal compact subgroup of $\mathrm{GSO}(X, \mathbb{R})$ there exists an anisotropic $y \in X$ and a maximal compact subgroup $J_Y = \rho(K_D) \subset \mathrm{SO}(Y, \mathbb{R})$ such that $J_Y \subset J_0$; in particular, the unique maximal compact subgroup of $\mathrm{GSO}(X, \mathbb{R})$ which contains J_0 is generated by J_0 and s. Finally, suppose F is a number field with adeles $\mathbb{A}$, X is a four dimensional quadratic space over F, B is the even Clifford algebra of X, and E is the center of B. Using Theorem 2.3 one can show that the sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho} \mathrm{GSO}(X, \mathbb{A}) \to 1$$

is exact; we identify $E^\times(\mathbb{A})$ and $\mathbb{A}_E^\times$. Similarly, if B is a quadratic quaternion algebra over F with center E, and a is a Galois action on B, then the sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho_a} \mathrm{GSO}(X_a, \mathbb{A}) \to 1$$

is exact. In addition, we have the following useful observation. Suppose D and D' are quaternion algebras over F, and E is a two dimensional reduced algebra over F. Let $S_{D,E}$ be the set of places v of F such that D_v is ramified and v splits in E; if $E \cong F \times F$, we will say that every place of F splits in E. Define $S_{D',E}$ similarly. Evidently, if $S_{D,E} = S_{D',E}$, then $B_{D,E} \cong B_{D',E}$ as E algebras. Thus, if $S_{D,E} = S_{D',E}$, then by Proposition 2.9 there exists an intertwining similitude from $X_{D,E}$ to $X_{D',E}$.

3. Local theta lifts for $\dim X = 2n = 4$

In this section we describe what irreducible representations of $\mathrm{GO}(X,F)$ occur in the theta correspondence with $\mathrm{GSp}(2,F)$ for X a four dimensional quadratic space over a local field F. This is needed to define local L-packets for $\mathrm{GSp}(2,F)$ in the next section. The description below involves distinguished representations, and was given in [R2] when F is a local field of characteristic zero with odd residual characteristic; we also do the even residual characteristic and real cases.

Fix the following notation. Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let X be a four dimensional quadratic space over F; write X for the F points of X. Let $d = \operatorname{disc} X$. As in Section 2, let B be the even Clifford algebra of X, and let E be the center of B. Let $s \in \mathrm{O}(X,F)$ be an element as in Proposition 2.5, so that $s^2 = 1$, $\det s = -1$, and s is a representative for the nontrivial coset of $\mathrm{GSO}(X,F)$ in $\mathrm{GO}(X,F)$. Suppose that $F = \mathbb{R}$. Fix a maximal compact subgroup K_B of $B^\times$, and let $J_0 = \rho(\{\pm 1\} \times K_B)$, a maximal compact subgroup of $\mathrm{GSO}(X,\mathbb{R})$. As explained in the penultimate paragraph of Section 2, we may assume that s normalizes J_0, so that the subgroup J generated by J_0 and s is a maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$. As usual, by $\mathrm{Irr}(B^\times)$ we mean the set of equivalence classes of irreducible (B, K_B) modules, where B is regarded as the Lie algebra of $B^\times$. If $\tau \in \mathrm{Irr}(B^\times)$, the central character $\omega_\tau : E^\times \to \mathbb{C}^\times$ of τ is defined by $\omega_\tau(e^z) = \exp(\tau(z))$ for $z \in E \subset B = \mathrm{Lie}(B^\times)$, and $\omega_\tau(\epsilon) = \tau(\epsilon)$ for $\epsilon \in E^\times \cap K_B$.

Using the exact sequences of Section 2, we can describe representations of $\mathrm{GSO}(X,F)$ in terms of representations of $B^\times$. Let $\mathrm{Irr}_f(F^\times \times B^\times)$ be the set of pairs (χ,τ), where $\tau \in \mathrm{Irr}(B^\times)$ is such that ω_τ is Galois invariant, and χ is a quasi-character of $F^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}^E_F$. The exact sequences from Theorem 2.3, (2.2) and (2.3) give a bijection

$$\mathrm{Irr}_f(F^\times \times B^\times) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{GSO}(X,F)), \quad (\chi,\tau) \mapsto \pi(\chi,\tau).$$

If F is nonarchimedean, $\pi(\chi,\tau)$ has the same space as τ, and is defined by $\pi(\chi,\tau)(\rho(t,g)) = \chi(t)^{-1}\tau(g)$. Suppose $F = \mathbb{R}$, and let $(\chi,\tau) \in \mathrm{Irr}_f(\mathbb{R}^\times \times B^\times)$. Since ω_τ is Galois invariant, it follows that there exists a unique $\mathbb{R}$ linear map $l_\tau : \mathbb{R} \to \mathbb{C}$ such that $\tau(z) = l_\tau(\mathrm{T}^E_{\mathbb{R}}(z))$ for $z \in E \subset \mathrm{Lie}(B^\times)$. We have $\chi(e^x) = \exp l_\tau(x)$ for $x \in \mathbb{R}$. Then $\pi(\chi,\tau)$ has the same space as τ, and $\pi(\chi,\tau)$ is defined by $\pi(\chi,\tau)(\rho(\epsilon,k)) = \chi(\epsilon)^{-1}\tau(k)$ for $\rho(\epsilon,k) \in J_0$, and by $\pi(\chi,\tau)(\rho(r,h)) = -l_\tau(r) + \tau(h)$ for $\rho(r,h) \in \mathfrak{gso}(X,\mathbb{R})$. The central character of $\pi(\chi,\tau)$ is χ.

In addition, if X is of the form X_a for some Galois action a on a quadratic quaternion algebra B (see Section 2), then it may be convenient to write $\pi = \pi(\chi,\tau)$ with respect to the first exact sequence from Proposition 2.7. By Proposition 2.7, the difference between using the exact sequences from Proposition 2.7 and Theorem 2.3 is inessential.

We describe representations of $\mathrm{GO}(X,F)$ via representations of $\mathrm{GSO}(X,F)$. Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$. If the induced representation of π to $\mathrm{GSO}(X,F)$ is irreducible, we say that π is REGULAR, and write π^+ for the induced representation. Here, if $F=\mathbb{R}$, $\mathrm{Ind}_{\mathrm{GSO}(X,\mathbb{R})}^{\mathrm{GO}(X,\mathbb{R})}\pi$ is the $(\mathfrak{go}(X,\mathbb{R}),J)=(\mathfrak{gso}(X,\mathbb{R}),J)$ module with space $\pi\oplus\pi$ and action

$$\pi^+(k)(w\oplus w')=\pi(k)w\oplus\pi(sks)w',\ k\in J_0,\quad \pi^+(s)(w\oplus w')=w'\oplus w,$$

and Lie algebra action

$$\pi^+(X)(w\oplus w')=\pi(X)w\oplus\pi(\mathrm{Ad}(s)X)w',\ X\in\mathfrak{gso}(X,\mathbb{R}).$$

If π is not regular, we say that π is INVARIANT. If π is invariant, then $s\cdot\pi\cong\pi$ and π extends to exactly two representations of $\mathrm{GO}(X,F)$; if $F=\mathbb{R}$, by $s\cdot\pi$ we mean the $(\mathfrak{gso}(X,\mathbb{R}),J_0)$ module with same space as π and action defined by $(s\cdot\pi)(k)w=\pi(sks)w$ for $k\in J_0$ and $w\in\pi$ and $(s\cdot\pi)(X)w=\pi(\mathrm{Ad}(s)X)w$ for $X\in\mathfrak{gso}(X,\mathbb{R})$ and $w\in\pi$. Before we can describe what representations of $\mathrm{GO}(X,F)$ occur in the theta correspondence with $\mathrm{GSp}(2,F)$ we must be able to adequately tell apart the two extensions of an invariant representation to $\mathrm{GO}(X,F)$. To do so we use distinguished representations.
Let $\pi\in\mathrm{Irr}(\mathrm{GSO}(X,F))$ be invariant. We say that π is DISTINGUISHED if there exists an anisotropic vector $y\in X$ such that $\mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi,\mathbf{1})\neq 0$, and if $d\neq 1$, then Y is isotropic. Here, $Y=(F\cdot y)^\perp$, as in Proposition 2.6, and $\mathbf{1}$ is the trivial representation of $\mathrm{SO}(Y,F)$, i.e., the representation with space $\mathbb{C}$ and trivial action. In the case $F=\mathbb{R}$ more comments are required. Let $y\in X$ be anisotropic, and let $Y=(F\cdot y)^\perp$. Let J_Y be a maximal compact subgroup of $\mathrm{SO}(Y,\mathbb{R})$. Then as mentioned in Section 2, J_Y is contained in a unique maximal compact subgroup J_0' of $\mathrm{GSO}(X,\mathbb{R})$. Since J_0' is conjugate to J_0, we may regard the $(\mathfrak{gso}(X,\mathbb{R}),J_0)$ module π as a $(\mathfrak{gso}(X,\mathbb{R}),J_0')$ module, and by restriction, as an $(\mathfrak{so}(Y,\mathbb{R}),J_Y)$ module. Then we say that π is distinguished if for some y, $\mathrm{Hom}_{(\mathfrak{so}(Y,\mathbb{R}),J_Y)}(\pi,\mathbf{1})\neq 0$, and if $d\neq 1$, then Y is isotropic. It is easy to verify that the nonvanishing of this homomorphism space does not depend on the choice of maximal compact subgroup of $\mathrm{SO}(Y,\mathbb{R})$ or element of $\mathrm{GSO}(X,\mathbb{R})$ used to conjugate J_0' into J_0 (use that the normalizer of J_0 is $\mathbb{R}^\times J_0$). Also, π is distinguished with respect to all anisotropic y if and only if it is distinguished with respect to one anisotropic y. If F is nonarchimedean, then this was pointed out in [R2]; if $F=\mathbb{R}$ it follows by a similar argument.

3.1 Proposition. *If $F=\mathbb{R}$ assume $d=1$. Let $\pi\in\mathrm{Irr}(\mathrm{GSO}(X,F))$. Assume π is invariant. Then for all anisotropic $y\in X$ such that $Y=(F\cdot y)^\perp$ is isotropic if $d\neq 1$,* $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi,\mathbf{1})\leq 1$.

Proof. This was proven in Proposition 4.1 of [R2] if F is nonarchimedean. Suppose $F=\mathbb{R}$. Since the homomorphism spaces for different anisotropic y are all isomorphic, it suffices to show this for one y. As $d=1$, we have $B\cong D\times D$ for some quaternion algebra D over $\mathbb{R}$. Identify B with $D\times D$, and let $y \in X$ be an anisotropic vector such that the line $\mathbb{R} \cdot y$ corresponds to ΔD, where ΔD consists of the $(x,x) \in B$ with $x \in D$ (see Section 2). Let K_D be a maximal compact subgroup of $D^\times$; then ΔK_D is a maximal compact subgroup of $\Delta D^\times$, $\Delta K_D \subset K_D \times K_D$, and $J_Y = \rho(K_D) \subset \rho(\{\pm 1\} \times K_D \times K_D)$. Write $\pi = \pi(\chi, \tau)$, with $\tau \cong \tau_1 \otimes \tau_2$, $\tau_1, \tau_2 \in \mathrm{Irr}(D^\times)$, $\omega_{\tau_1} = \omega_{\tau_2} = \chi$. Since π is invariant, $\tau_1 \cong \tau_2$. We have $\mathrm{Hom}_{\mathrm{SO}(Y,\mathbb{R})}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{(\Delta D, \Delta K_D)}(\tau, \chi \circ \mathrm{N}) \cong \mathrm{Hom}_{(\Delta D, \Delta K_D)}(\tau_1 \otimes \tau_1^\vee, \mathbf{1})$. This space is one dimensional. $\square$

As we shall see in the theorem below, we can now tell the two extensions of invariant representations apart to an extent sufficient for our purposes. Suppose $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$ is distinguished with respect to an anisotropic $y \in X$, with $d = 1$ if $F = \mathbb{R}$. Since $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1}) = 1$ by Proposition 3.1, it follows that for exactly one extension π^+ of π to $\mathrm{GO}(X,F)$ we have $\mathrm{Hom}_{\mathrm{O}(Y,F)}(\pi^+, \mathbf{1}) \neq 0$. Denote the other extension of π to $\mathrm{GO}(X,F)$ by π^-. The definitions of π^+ and π^- do not depend on the choice of y.
Before characterizing $\mathcal{R}_2(\mathrm{GO}(X,F))$ we require require two more results.

3.2 Lemma. *Let $F = \mathbb{R}$; assume $d = 1$. Then* $\mathrm{Hom}_{\mathrm{SO}(X,\mathbb{R})}(\omega, \pi_1) \neq 0$ *for* $\pi_1 \in \mathrm{Irr}(\mathrm{SO}(X,\mathbb{R}))$.

Proof. If the signature of X is $(2,2)$, this follows from (3.6.10) of [P]. If the signature of X is $(4,0)$ or $(0,4)$ this follows by (6.12) of [KV]. $\square$

3.3 Proposition. *The elements of* $\mathrm{Irr}(\mathrm{GO}(X,F))$ *have multiplicity free restrictions to* $\mathrm{O}(X,F)$.

Proof. If $F = \mathbb{R}$ then the restriction of any element of $\mathrm{Irr}(\mathrm{GO}(X,\mathbb{R}))$ is multiplicity free as $[\mathrm{GO}(X,\mathbb{R}) : \mathbb{R}^\times \mathrm{O}(X,\mathbb{R})] \leq 2$. If $d = 1$ and F is nonarchimedean then this is Lemma 7.2 of [HPS]. The case $d \neq 1$ and F nonarchimedean remains. If F is of odd residual characteristic then $[\mathrm{GO}(X,F) : F^\times \mathrm{O}(X,F)] = [\mathrm{N}_F^E(E^\times) : F^{\times 2}] = 2$ so the proposition follows from Lemma 2.1 of [GK]. We now give an argument for both the even and odd residual characteristic cases. There are two four dimensional quadratic spaces over F of discriminant d. By Proposition 2.9 there is a similitude between them; thus, it suffices to prove the result for one of them. We take $X = X_{\mathrm{M}_{2\times 2},d}$. Using Proposition 3.2 of [R2] it is easy to verify that the finite dimensional, i.e., one or two dimensional, elements of $\mathrm{Irr}(\mathrm{GO}(X,F))$ have multiplicity free restrictions to $\mathrm{O}(X,F)$. To complete the proof it will suffice to show that for infinite dimensional $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$, the representation $\sigma = \mathrm{Ind}_{\mathrm{GSO}(X,F)}^{\mathrm{GO}(X,F)} \pi$ (which may be reducible) has a multiplicity free restriction to $\mathrm{O}(X,F)$. Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$ and using the first exact sequence from Proposition 2.7 write $\pi = \pi(\chi, \tau)$ where $\tau \in \mathrm{Irr}(\mathrm{GL}(2,E))$ and χ is a quasi-character of $F^\times$ such that $\chi \circ \mathrm{N}_F^E = \omega_\tau$; here and below $E = E_d$. We make take s to be given by $s(x) = a(x)$, where a is the usual Galois action on $\mathrm{M}_{2\times 2}(E)$ as given in (2.1). Let V be the space of τ, i.e., the space of π. As a model for σ use $V \oplus V$ with

$$(3.1) \quad \sigma(h)(v \oplus v') = \pi(h)v \oplus \pi(shs)v',\ h \in \mathrm{GSO}(X,F), \quad \sigma(s)(v \oplus v') = v' \oplus v.$$

We begin with some remarks about the restriction of π to subgroups. Let $\psi_E : E \to \mathbb{C}^\times$ be a nontrivial quasi-character of E; we may assume ψ_E is $\mathrm{Gal}(E/F)$ invariant. Let N be the subgroup of $\mathrm{GSO}(X,F)$ of elements

$$n = \rho_a(1, \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix})$$

for $x \in E$. Since the space of Whittaker functionals on τ with respect to ψ_E is one dimensional, it follows that $\dim \mathrm{Hom}_N(\pi, \psi_E) = 1$, where ψ_E is the character of N defined by $\psi_E(n) = \psi_E(x)$. This fact allows us to prove the following statements just as in the proof of Theorem 4.3 of [R2]. Let H_0 be a closed normal subgroup of $\mathrm{GSO}(X,F)$ such that $F^\times H_0$ is open, $\mathrm{GSO}(X,F)/F^\times H_0$ is finite and Abelian, and $N \subset H_0$. Then the restriction $\pi|_{H_0}$ is multiplicity free: $\pi|_{H_0} = V_1 \oplus \cdots \oplus V_M$, where V_i, $1 \le i \le M$ are mutually nonisomorphic irreducible H_0 subspaces of π (see [GK] for general results about restrictions), and, say, $\dim_{\mathbb{C}} \mathrm{Hom}_N(V_1, \psi_E) = 1$ and $\dim_{\mathbb{C}} \mathrm{Hom}_N(V_i, \psi_E) = 0$ for $2 \le i \le M$. Suppose additionally $s \cdot \pi \cong \pi$, and let $\hat\pi$ be an extension of π to $\mathrm{GO}(X,F)$. Then $\hat\pi(s)V_1 = V_1$.

Now we show $\sigma|_{\mathrm{O}(X,F)}$ is multiplicity free. Suppose first there is no quasi-character β of $E^\times$ such that $\beta|_{F^\times} = 1$ and $\beta \otimes \tau \cong \tau \circ a$. Let W be a nonzero irreducible $\mathrm{O}(X,F)$ subspace of σ. Then either there is an irreducible $\mathrm{SO}(X,F)$ subspace U of (π, V) such that $W = U \oplus U$, or there is an irreducible $\mathrm{SO}(X,F)$ subspace U of (π, V) and $i : U \to U$ such that $i^2 = 1$, $i(\pi(h)u) = \pi(shs)i(u)$ for $h \in \mathrm{SO}(X,F)$ and $W = \{u \oplus i(u) : u \in U\}$. We assert the second case is impossible; suppose it holds. Then $\pi|_{\mathrm{SO}(X,F)}$ and $(s\cdot\pi)|_{\mathrm{SO}(X,F)}$ share an irreducible component. Since $\pi|_{\mathrm{SO}(X,F)}$ is multiplicity free by the last paragraph, by Lemma 2.4 of [GK] there is a quasi-character $\gamma : \mathrm{GSO}(X,F) \to \mathbb{C}^\times$ trivial on $F^\times \mathrm{SO}(X,F)$ such that $s \cdot \pi \cong \gamma \otimes \pi$. Since

$$1 \to F^\times \mathrm{SO}(X,F) \xrightarrow{\mathrm{inc}} \mathrm{GSO}(X,F) \xrightarrow{\lambda} \mathrm{N}_F^E(E^\times)/F^{\times 2} \to 1$$

is exact, $\gamma = \eta \circ \lambda$ for some quasi-character $\eta : \mathrm{N}_F^E(E^\times) \to \mathbb{C}^\times$ with $\eta^2 = 1$. Let $T : (\eta\circ\lambda) \otimes \pi \to s \cdot \pi$ be a $\mathrm{GSO}(X,F)$ isomorphism. Then for $g \in \mathrm{GL}(2,E)$ and $v \in V$,

$$\begin{aligned} T(\eta(\lambda(\rho_a(1,g))) \cdot \pi(\rho_a(1,g)v)) &= \pi(s\rho_a(1,g)s)T(v) \\ (\eta \circ \mathrm{N}_F^E)(\det g) T(\tau(g)v) &= \tau(a(g))T(v). \end{aligned}$$

This implies $(\eta \circ \mathrm{N}_F^E) \otimes \tau \cong \tau \circ a$, contradicting our assumption; note $(\eta \circ \mathrm{N}_F^E)|_{F^\times} = 1$. Thus, $W = U \oplus U$. Let W' be another nonzero irreducible $\mathrm{O}(X,F)$ subspace of σ and assume $W' \cong W$ as $\mathrm{O}(X,F)$ representations; to show $\sigma|_{\mathrm{O}(X,F)}$ is multiplicity free it will suffice to show W and W' are identical, i.e., $W = W'$. Write $W' = U' \oplus U'$, with U' an irreducible $\mathrm{SO}(X,F)$ subspace of (π, V). Consider the composition $U \to W \xrightarrow{\sim} W' \to U'$ where the first map sends u to $u \oplus 0$, the second is our fixed isomorphism $W \cong W'$, and the last map sends $u \oplus u'$ to u. This is an $\mathrm{SO}(X,F)$ map from $(U, \pi|_{\mathrm{SO}(X,F)})$ to $(U', \pi|_{\mathrm{SO}(X,F)})$. We claim it is nonzero; suppose not. Then the same composition with the last map replaced by the map sending $u \oplus u'$ to u' gives a nonzero $\mathrm{SO}(X,F)$ map from $(U, \pi|_{\mathrm{SO}(X,F)})$ to $(U', (s\cdot\pi)|_{\mathrm{SO}(X,F)})$. However, we just saw that $\pi|_{\mathrm{SO}(X,F)}$ and $(s\cdot\pi)|_{\mathrm{SO}(X,F)}$ have no common irreducible constituents. Thus, the first composition is nonzero, and U and U' are isomorphic irreducible subspaces of $\pi|_{\mathrm{SO}(X,F)}$. Since $\pi|_{\mathrm{SO}(X,F)}$ is multiplicity free, $U = U'$ and so $W = W'$.

Now suppose there is a quasi-character β of $E^\times$ such that $\beta|_{F^\times} = 1$ and $\beta \otimes \tau \cong \tau \circ a$. Let H_0 be the subgroup of $\mathrm{SO}(X,F)$ of $\rho_a(1,g)$ for $g \in \mathrm{Sl}(2,E)$, and let $H \subset \mathrm{O}(X,F)$ be generated by H_0 and s. To prove $\sigma|_{\mathrm{O}(X,F)}$ is multiplicity free it will suffice to prove $\sigma|_H$ is multiplicity free. For this, we replace σ with a more tractable representation via twisting. Since $\beta|_{F^\times} = 1$, there is a quasi-character μ of $E^\times$ such that $\beta(x) = \mu(x/a(x))$ for $x \in E^\times$. Letting $\tau' = \mu \otimes \tau$, we have $\tau' \circ a \cong \tau'$. Since ω_τ is Galois invariant, $\beta^2 = 1$, which implies μ^2 is Galois invariant. Let ν be a quasi-character of $F^\times$ such that $\mu^2 = \nu \circ \mathrm{N}_F^E$, and set $\chi' = \nu\chi$; then $\omega_{\tau'} = \chi' \circ \mathrm{N}_F^E$. Set $\pi' = \pi(\chi', \tau')$. Since $\tau' \circ a \cong \tau'$ we have $s \cdot \pi' \cong \pi'$. Let $\sigma' = \mathrm{Ind}_{\mathrm{GSO}(X,F)}^{\mathrm{GO}(X,F)} \pi'$, and use the same model for σ' as above, so the underlying space of σ' is $V \oplus V$. Now σ' may not be isomorphic to σ, but it is easy to see that the identity map between the models for σ and σ' gives an isomorphism $\sigma|_H \cong \sigma'|_H$. We are reduced to showing $\sigma'|_H$ is multiplicity free. As $s \cdot \pi' \cong \pi'$, we have $\sigma' \cong \pi_1' \oplus \pi_2'$, where π_1' and π_2' are the two extensions of π' to $\mathrm{GO}(X,F)$. Since the restrictions $\pi_1'|_{H_0} = \pi_2'|_{H_0} = \pi|_{H_0}$ are multiplicity free as $N \subset H_0$, it follows that $\pi_1'|_H$ and $\pi_2'|_H$ are multiplicity free. It will now suffice to show $\pi_1'|_H$ and $\pi_2'|_H$ do not share an irreducible component; suppose they do. By Lemma 2.4 of [GK], $\pi_1'|_H \cong \pi_2'|_H$. Let $R : \pi_1'|_H \to \pi_2'|_H$ be an H isomorphism. As indicated above, there is an irreducible H_0 subspace $V_1 \subset V$ such that $\pi_1'(s)V_1 = \pi_2'(s)V_1 = V_1$, i.e., V_1 is also an irreducible H subspace for $\pi_1'|_H$ and $\pi_2'|_H$. Since $\pi_1'|_H$ and $\pi_2'|_H$ are multiplicity free, we must have $R(V_1) = V_1$. Applying Schur's lemma to $R : V_1 \to V_1$, with V_1 regarded as an irreducible H_0 representation, there exists a nonzero scalar c such that $R(v) = cv$ for $v \in V_1$. This implies $\pi_1'(s)v = \pi_2'(s)v$ for $v \in V_1$. However, $\pi_2'(s)v = -\pi_1'(s)v$ for $v \in V$, a contradiction. $\square$

3.4 Theorem. *Let $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$. If F is nonarchimedean and $d \neq 1$, assume σ is infinite dimensional; if $F = \mathbb{R}$, assume $d = 1$. Then $\sigma \in \mathcal{R}_2(\mathrm{GO}(X,F))$ if and only if σ is not of the form π^- for some distinguished $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$.*

Proof. Suppose first F is nonarchimedean. Then this theorem was proven in [R2] in the case F has odd residual characteristic. To verify the theorem if F has even residual characteristic we proceed as follows. We note first that the background results of [R2] are valid in any residual characteristic; that is, the results of sections 2, 3 and 4 hold, and Lemma 6.1, Corollary 6,2, Lemma 6.3 and Lemma 6.4 also hold with the same proofs. We need to show that Lemmas

6.6 and 6.7 of [R2] also hold if F has even residual characteristic. Consider first the proof of Lemma 6.6. The first paragraph of the proof of Lemma 6.6 is independent of the residual characteristic. In the second paragraph, we used a result from [T] proven only in the case of odd residual characteristic; by [Sa], this also holds in the case of even residual characteristic. The third paragraph of the proof is also valid in even residual characteristic (in spite of the unnecessary mention there of odd residual characteristic). Next, we consider all remaining paragraphs but the last paragraph: these cover the case $d = 1$ and $\epsilon = -\epsilon(1)$, in the notation of [R2]. Letting D_{ram} be the ramified quaternion algebra over F, we are given $\tau, \tau' \in \mathrm{Irr}(D_{\mathrm{ram}}^\times)$ with $\omega_\tau = \omega_{\tau'}$, and we must show that there exists a quadratic extension $E \subset D_{\mathrm{ram}}$ of F of such that $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1}) \neq 0$, where $\mathrm{SO}(Z)$ is the subgroup $\{\rho(x, x^{*-1}) : x \in E^\times\}$ and $\pi = \pi(\tau, \tau')$. Embed $D_{\mathrm{ram}}^\times$ into $D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times$ via $x \mapsto (x, x^{*-1})$, and consider the restriction of $\tau \otimes \tau'$ to $D_{\mathrm{ram}}^\times$. Let τ'' be an irreducible component of $(\tau \otimes \tau')|_{D_{\mathrm{ram}}^\times}$; then $\omega_{\tau''} = 1$. By Proposition 18 of [W2], there exists a quadratic extension E of F in D_{ram} and a nonzero vector $v \in \tau''$ such that $\tau''(x)v = v$ for $x \in E^\times$. This implies that $\pi(h)v = v$ for $h \in \mathrm{SO}(Z)$, proving the required claim. The last paragraph of the proof of Lemma 6.6 is also valid in the case of even residual characteristic, thus completing the verification of Lemma 6.6 in this case. Next, we consider the proof of Lemma 6.7 of [R2]. To make the proof of Lemma 6.7 go through in the case of even residual characteristic it suffices to show that if K is a quadratic extension of F, $\tau \in \mathrm{Irr}(\mathrm{GL}(2, K))$ is Galois invariant with $\omega_\tau = \chi \circ \mathrm{N}_F^K$ and $\mathrm{Hom}_{\mathrm{GL}(2,F)}(\tau, \chi \circ \det) = 0$, then there exist quasi-characters ζ and ζ' of $K^\times$ extending χ such that $\epsilon(\tau \otimes \zeta^{-1}, 1/2, \psi_K) = \chi(-1)$ and $\epsilon(\tau \otimes \zeta'^{-1}, 1/2, \psi_K) = -\chi(-1)$. To show the existence of ζ, pick ζ extending χ such that ζ is very ramified (this can be done); then by 3 of Lemma 14 of [HST], $\epsilon(\tau \otimes \zeta^{-1}, 1/2, \psi_K) = \chi(-1)$. On the other hand, since $\mathrm{Hom}_{\mathrm{GL}(2,F)}(\tau, \chi \circ \det) = 0$, by the equivalence of 1 and 2 of Theorem 5.3 of [R2], there exists a quasi-character ζ' of $K^\times$ extending χ such that $\epsilon(\tau \otimes \zeta'^{-1}, 1/2, \psi_K) = -\chi(-1)$. (Note that the proof of the equivalence of 1 and 2 of Theorem 5.3 of [R2] works in any residual characteristic; the use of odd residual characteristic in the proof of Lemma 5.2 is easily seen to be unnecessary.)

Now suppose $F = \mathbb{R}$ and $d = 1$. Suppose $\sigma \in \mathcal{R}_2(\mathrm{GO}(X, \mathbb{R}))$. Then an argument as in Theorem 4.3 of [R2] shows that σ cannot be of the form π^- for some distinguished π. Conversely, suppose σ is not of the form π^- for some distinguished π. Then $\sigma \cong \pi^+$ for some regular π or distinguished π. Using Lemma 3.2, an argument as in Theorem 4.4 of [R2] shows that $\sigma \in \mathcal{R}_2(\mathrm{GO}(X, \mathbb{R}))$. $\square$

4. Definition of the local L-packets and parameters

Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let $d \in F^\times/F^{\times 2}$; assume $d = 1$ if $F = \mathbb{R}$. Let $X_{\mathrm{M}_{2\times 2}, d}$ be the four dimensional quadratic space over F defined after Proposition 2.7 and discussed after Proposition 2.9. We will parameterize $\mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2}, d}, F))$ as explained at the be-

ginning of Section 3. However, since we are dealing with the concrete quadratic spaces $X_{\mathrm{M}_{2\times 2},d}$ we will use the first exact sequence from Proposition 2.7; by Proposition 2.7, the difference is trivial. We let $s \in \mathrm{O}(X_{\mathrm{M}_{2\times 2},d}, F)$, $\det s = -1$, be defined by $s(x) = a(x)$, where a is the Galois action on $\mathrm{M}_{2\times 2}(E_d)$ defining $X_{\mathrm{M}_{2\times 2},d}$; see (2.1). Using the results of the last section, we will associate to every element $[\pi]$ of $\langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ a packet $\Pi([\pi])$ of elements of $\mathrm{Irr}(\mathrm{GSp}(2, F))$ and a $\mathrm{GSp}(2)$ L-parameter $\varphi([\pi]) : \mathrm{L}_F \to \mathrm{GSp}(2, \mathbb{C})$ over F, where L_F is the Langlands group of F (i.e., $\mathrm{W}_F \times \mathrm{SU}(2, \mathbb{R})$ if F is nonarchimedean and the Weil group W_F if $F = \mathbb{R}$). We expect that $\Pi([\pi])$ is the L-packet associated to $\varphi([\pi])$ under the conjectural Langlands correspondence. Some evidence is provided by Propositions 4.1, 4.2 and 4.3 below which give some basic properties of the $\Pi([\pi])$ and $\varphi([\pi])$. More work on this issue remains to be done: for example, are the packets $\Pi([\pi])$ disjoint, and if $\varphi([\pi])$ and $\varphi([\pi'])$ are equivalent, does it follow that $\Pi([\pi]) = \Pi([\pi'])$? We will return to this topic in a subsequent work; the thrust of this paper is global results.
To define the L-packets, we begin by noting that there is a surjective map

$$\mathrm{Irr}(\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)) \to \langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$$

which sends σ to the components of σ restricted to $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$. We will define the L-packet of elements of $\mathrm{Irr}(\mathrm{GSp}(2, F))$ associated to a point of $\langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ by considering the fiber over such a point, and applying the results of Section 3. For $\pi \in \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$ denote the element of $\langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ determined by π by $[\pi] = \{\pi, s\cdot\pi\}$. Let $[\pi] \in \langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$. We assume that π is infinite dimensional; if F is nonarchimedean of even residual characteristic, we assume additionally that π is tempered. Then how $[\pi]$ gives rise to irreducible representations of $\mathrm{GSp}(2, F)$ is described in Tables 2 and 3 of the Appendix. In the first step, using the results of Section 3, π gives rise to representations of various orthogonal similitude groups. This is summarized in the tables, but certain aspects deserve comment. If $d \neq 1$, then it may happen that π is invariant but not distinguished. Then the two extensions of π to $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$ are denoted by π_1 and π_2. When $d = 1$, then π is either regular or invariant and distinguished; in the first case π induces to give π^+, and in the second case π extends to give π^+ and π^-. Additionally, if $d = 1$ and π is essentially square integrable, then π gives an element $\pi^{\mathrm{JL}} \in \mathrm{Irr}(\mathrm{GSO}(X_{D_{\mathrm{ram}},1}, F))$ via the Jacquet-Langlands correspondence, and then analogously elements of $\mathrm{Irr}(\mathrm{GO}(X_{D_{\mathrm{ram}},1}, F))$. Here, D_{ram} is the ramified quaternion algebra over F, and π is ESSENTIALLY SQUARE INTEGRABLE if and only if $\pi = (\alpha\circ\lambda)\otimes\pi'$ for some quasi-character $\alpha : F^\times \to \mathbb{C}^\times$ and square integrable $\pi' \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F))$. To apply the Jacquet-Langlands correspondence, we write as in Section 3, $\pi = \pi(\chi, \tau)$ for $\tau = \tau_1 \otimes \tau_2 \in \mathrm{Irr}(\mathrm{GL}(2, F) \times \mathrm{GL}(2, F))$; recall that the exact sequence from Proposition 2.7 is in this case

$$1 \to F^\times \times F^\times \to F^\times \times \mathrm{GL}(2, F) \times \mathrm{GL}(2, F) \to \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F) \to 1.$$

We define $\pi^{\mathrm{JL}} = \pi(\chi, \tau^{\mathrm{JL}}) \in \mathrm{Irr}(\mathrm{GSO}(X_{D_{\mathrm{ram}},1}, F))$, where τ^{JL} is the irreducible representation of $D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times$ corresponding to τ under the Jacquet-Langlands correspondence (π being essentially square integrable means exactly τ_1 and τ_2 are essentially square integrable); the exact sequence from Proposition 2.7 for this is

$$1 \to F^\times \times F^\times \to F^\times \times D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times \to \mathrm{GSO}(X_{D_{\mathrm{ram}},1}, F) \to 1.$$

Next, using Theorem 3.4, the thus constructed representations of orthogonal similitude groups give representations of $\mathrm{GSp}(2, F)$ via theta correspondences; note that each theta correspondence used is covered by Theorem 1.8. In Tables 2 and 3 of the Appendix we indicate the appropriate theta correspondences with a subscript. We also indicate when representations do not have theta lifts. Finally, in the Table 4 of the Appendix the packets of representations associated to $[\pi]$ are defined using the representations constructed in Tables 2 and 3 of the Appendix. Note the introduction of the contragredient.
The next proposition describes a few basic properties of the L-packets $\Pi([\pi])$.

4.1 Proposition. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$. Assume π is infinite dimensional; if F is nonarchimedean of even residual characteristic, assume π is tempered. Then*

(1) *The common central character of the elements of $\Pi([\pi])$ is ω_π.*
(2) *If $d = 1$ then $|\Pi([\pi])| = 1$ unless π is essentially square integrable; in this case $|\Pi([\pi])| = 2$. If $d \neq 1$, then $|\Pi([\pi])| = 1$ unless π is invariant but not distinguished; in this case $|\Pi([\pi])| = 2$.*
(3) *If π is tempered, then all the elements of $\Pi([\pi])$ are tempered.*

Proof. (1) This follows from the remark on central characters after Theorem 1.8.
(2) Evidently, $|\Pi([\pi])| = 1$ except possibly if $d = 1$ and π is essentially square integrable, or $d \neq 1$ and π is invariant but not distinguished. If $d = 1$ and π is essentially square integrable, then $|\Pi([\pi])| = 2$ by Lemma 8.4 below. If $d \neq 1$ and π is invariant but not distinguished, then $|\Pi([\pi])| = 2$ because $\theta_{\mathrm{M}_{2\times 2},d}$ is a bijection.
(3) If F is nonarchimedean, this follows from (1) of Theorem 4.2 of [R3]. If $F = \mathbb{R}$, this follows from IV.3, p. 70 and III.2, p. 49 of [M]. $\square$

Next, we associate to each $[\pi] \in \langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ an L-parameter $\varphi([\pi]) : \mathrm{L}_F \to {}^L\mathrm{GSp}(2)$. Here L_F denotes the Langlands group of F, i.e., $\mathrm{L}_F = \mathrm{W}_F \times \mathrm{SU}(2, \mathbb{R})$ if F is nonarchimedean, and $\mathrm{L}_F = \mathrm{W}_F$ if F is archimedean ([Ko], Section 12); W_F is the Weil group of F. As is well known, the dual group $\widehat{\mathrm{GSp}(2)}$ of $\mathrm{GSp}(2, \overline{F})$ (${}^L\mathrm{GSp}(2)^0$ in the notation of [B]) is isomorphic to $\mathrm{GSp}(2, \mathbb{C})$, and we shall use such an isomorphism. But since $\mathrm{GSp}(2, \mathbb{C})$ has a non-inner automorphism, we need to be specific (the same issue arises for other groups, but for, say, GL(2) the choice is established). To do so, we will specify an isomorphism from the based root datum of ${}^L\mathrm{GSp}(2)$

to the based root datum of $\mathrm{GSp}(2,\mathbb{C})$. As a maximal split torus in $\mathrm{GSp}(2,\overline{F})$ we take the group T of elements $t = t(a,b,c) = \mathrm{diag}(a,b,a^{-1}c,b^{-1}c)$. The group X^* of characters of T is the free Abelian group with generators e_1, e_2 and e_3 defined by $e_1(t) = a, e_2(t) = b$ and $e_3(t) = c$. The group X_* of cocharacters of T is the free Abelian group with generators f_1, f_2 and f_3 defined by $f_1(x) = t(x,1,1)$, $f_2(x) = t(1,x,1)$ and $f_3(x) = t(1,1,x)$. The roots of $\mathrm{GSp}(2,\overline{F})$ with respect to T are $\{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2 - e_3, \alpha_1 + \alpha_2, 2\alpha_1+\alpha_2, -\alpha_1, -\alpha_2, -(\alpha_1+\alpha_2), -(2\alpha_1+\alpha_2)\}$. The coroots are $\{\alpha_1^\vee = \gamma_1 = f_1 - f_2, \alpha_2^\vee = \gamma_2 = f_2, (\alpha_1+\alpha_2)^\vee = \gamma_1 + 2\alpha_2, (2\alpha_1+\alpha_2)^\vee = \gamma_1+\gamma_2, (-\alpha_1)^\vee = -\gamma_1, (-\alpha_2)^\vee = -\gamma_2, (-(\alpha_1+\alpha_2))^\vee = -(\gamma_1+2\gamma_2), (-(2\alpha_1+\alpha_2))^\vee = -(\gamma_1+\gamma_2)\}$. As simple roots we take $\Delta^* = \{\alpha_1,\alpha_2\}$; then $\Delta_* = \Delta^{*\vee} = \{\gamma_1,\gamma_2\}$. We have similar notation for $\mathrm{GSp}(2,\mathbb{C})$, which we will indicate with the addition of a prime. Let $\Psi = (X^*,\Delta^*,X_*,\Delta_*)$; the dual of Ψ is $\Psi^\vee = (X_*,\Delta_*,X^*,\Delta^*)$; let $\Psi' = (X'^*,\Delta'^*,X'_*,\Delta'_*)$. Then an isomorphism $\Psi^\vee \xrightarrow{\sim} \Psi'$ amounts to an isomorphism $f : X_* \xrightarrow{\sim} X'^*$ of Abelian groups such that $f(\Delta_*) = \Delta'^*$ and the matrix of f with respect to our bases is symmetric. One can check that there are exactly two such isomorphisms f, with matrices

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & c \end{bmatrix}, \quad c = 0 \text{ or } 1.$$

As is done implicitly in [HST], we shall fix the isomorphism corresponding to the choice $c = 1$. Our fixed isomorphism of based root data $\Psi^\vee \xrightarrow{\sim} \Psi'$ determines a T conjugacy class of isomorphisms $\widehat{\mathrm{GSp}(2)} \xrightarrow{\sim} \mathrm{GSp}(2,\mathbb{C})$ ([Sp], Theorem 9.6.2); we fix one such isomorphism in the conjugacy class. Additionally, since the action of W_F on $\widehat{\mathrm{GSp}(2)}$ is trivial, ${}^L\,\mathrm{GSp}(2)$ is the direct product $\widehat{\mathrm{GSp}(2)} \times \mathrm{W}_F$. Thus, in considering L-parameters we may just as well look at maps into $\widehat{\mathrm{GSp}(2)}$, which we identify with $\mathrm{GSp}(2,\mathbb{C})$ (always via our fixed isomorphism). We define a $\mathrm{GSp}(2)$ L-PARAMETER over F to be a continuous homomorphism $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$ such that $\varphi(x)$ is semisimple for $x \in \mathrm{W}_F$, and if F is nonarchimedean then $\varphi|_{1\times\mathrm{SU}(2,\mathbb{R})}$ is a smooth representation. Let φ be a $\mathrm{GSp}(2)$ L-parameter over F. The SIMILITUDE QUASI-CHARACTER of φ is the quasi-character of L_F given by $\lambda\circ\varphi$, where $\lambda : \mathrm{GSp}(2,\mathbb{C}) \to \mathbb{C}^\times$ is the usual similitude homomorphism. If F is nonarchimedean, we say φ is UNRAMIFIED if $\varphi(\mathrm{SU}(2,\mathbb{R})) = 1$ and φ is trivial on the inertia subgroup of W_F. We say that φ is TEMPERED if $\varphi(\mathrm{L}_F)$ is bounded. If $\varphi' : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$ is another $\mathrm{GSp}(2)$ L-parameter over F we say that φ and φ' are EQUIVALENT if there exists $g \in \mathrm{GSp}(2,\mathbb{C})$ such that $g\varphi(x)g^{-1} = \varphi'(x)$ for all $x \in \mathrm{L}_F$. The CONNECTED COMPONENT GROUP of φ is the group $\mathbb{S}(\varphi) = \pi_0(S(\varphi)/\mathbb{C}^\times)$, where $S(\varphi)$ is the group of $g \in \mathrm{GSp}(2,\mathbb{C})$ such that $g\varphi(x) = \varphi(x)g$ for all $x \in \mathrm{L}_F$.

The parameter $\varphi([\pi])$ will be one of two kinds of examples of $\mathrm{GSp}(2)$ L-parameters over F. To define the first kind of example, suppose E/F is a quadratic extension and let $\rho : \mathrm{L}_E \to \mathrm{GL}(2,\mathbb{C})$ be a $\mathrm{GL}(2)$ L-parameter over E such that $\det\rho$ is Galois invariant. Let $\eta : \mathrm{L}_F \to \mathbb{C}^\times$ be a quasi-character extending $\det\rho$; there are two such quasi-characters. Let $V=\mathbb{C}^2$, and regard ρ as a representation on V. Define $\varphi(\eta,\rho):\mathrm{L}_F\to\mathrm{GSp}(W)$ by letting the space and action of $\varphi(\eta,\rho)$ be $W=\mathrm{Ind}_{\mathrm{L}_E}^{\mathrm{L}_F}\rho$ and defining a nondegenerate symplectic form on W by

$$\langle v_1\oplus v_2, v_1'\oplus v_2'\rangle=\eta(y)\langle v_1,v_1'\rangle+\langle v_2,v_2'\rangle.$$

Here, y is a fixed representative for the nontrivial coset of L_E in L_F, we identify the space of $\varphi(\eta,\rho)$ with $V\oplus V$ via the map $f\mapsto f(1)\oplus f(y)$, and we have fixed a nondegenerate symplectic form on V (note that up to multiplication by elements of $\mathbb{C}^\times$, there is only one nondegenerate symplectic form on a two dimensional complex vector space). Then $\varphi(\eta,\rho)$ is a GSp(2) L-parameter over F, and the similitude quasi-character of $\varphi(\eta,\rho)$ is $\lambda\circ\varphi(\eta,\rho)=\eta$. Suppose next that $\rho_1:\mathrm{L}_F\to\mathrm{GL}(2,\mathbb{C})$ and $\rho_2:\mathrm{L}_F\to\mathrm{GL}(2,\mathbb{C})$ are GL(2) L-parameters over F with $\det\rho_1=\det\rho_2$. Regard ρ_1 and ρ_2 as two dimensional representations of L_F on $V_1=\mathbb{C}^2$ and $V_2=\mathbb{C}^2$, respectively, and fix nondegenerate symplectic forms $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ on the spaces of ρ_1 and ρ_2, respectively. Define $\varphi(\rho_1,\rho_2):\mathrm{L}_F\to\mathrm{GSp}(W)$ by letting the space and action of $\varphi(\rho_1,\rho_2)$ be $W=\rho_1\oplus\rho_2$ and defining a nondegenerate symplectic form on the space of $\varphi(\rho_1,\rho_2)$ by

$$\langle v_1\oplus v_2, v_1'\oplus v_2'\rangle=\langle v_1,v_1'\rangle_1+\langle v_2,v_2'\rangle_2.$$

Then $\varphi(\rho_1,\rho_2)$ is a GSp(2) L-parameter over F, and the similitude quasi-character of $\varphi(\rho_1,\rho_2)$ is $\lambda\circ\varphi(\rho_1,\rho_2)=\det\rho_1=\det\rho_2$.

Now let $[\pi]\in\langle s\rangle\backslash\mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d},F))$. Write $\pi=\pi(\chi,\tau)$, with $\tau\in\mathrm{Irr}(\mathrm{GL}(2,E_d))$ and χ a quasi-character of $F^\times$ such that $\omega_\tau=\chi\circ\mathrm{N}_F^{E_d}$. Suppose first that $d\neq1$. Let $\rho:\mathrm{L}_{E_d}\to\mathrm{GL}(2,\mathbb{C})$ be the GL(2) L-parameter over E_d corresponding to τ, and let $\eta:\mathrm{L}_F\to\mathbb{C}^\times$ be the quasi-character of L_F corresponding to χ. Then η extends $\det\rho$ and the equivalence class of $\varphi(\eta,\rho)$ depends only on $[\pi]$ and not the choice of representative π. We set $\varphi([\pi])=\varphi(\eta,\rho)$. Suppose next $d=1$. Then $\mathrm{GL}(2,E_d)\cong\mathrm{GL}(2,F)\times\mathrm{GL}(2,F)$. Let $\tau\cong\tau_1\otimes\tau_2$, with $\tau_1,\tau_2\in\mathrm{Irr}(\mathrm{GL}(2,F))$ such that $\chi=\omega_{\tau_1}=\omega_{\tau_2}$. Let $\rho_1,\rho_2:\mathrm{L}_F\to\mathrm{GL}(2,\mathbb{C})$ be the GL(2) L-parameters over F corresponding to ρ_1 and ρ_2, respectively. Then $\det\rho_1=\det\rho_2$ and the equivalence class of $\varphi(\rho_1,\rho_2)$ depends only on $[\pi]$ and not the choice of representative π. We set $\varphi([\pi])=\varphi(\rho_1,\rho_2)$.

The following is an analogue of Proposition 4.1.

4.2 Proposition. *Let $\pi\in\mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d},F))$. Assume π is infinite dimensional; if F is nonarchimedean of even residual characteristic, assume π is tempered. Then*

1. *The similitude quasi-character of $\varphi([\pi])$ corresponds to ω_π.*
2. *If $d=1$ then $|\mathbb{S}_{\varphi([\pi])}|=1$ unless π is a essentially square integrable; in this case $\mathbb{S}(\varphi([\pi]))=Z_2$. If $d\neq1$ and F is not nonarchimedean of even residual characteristic, then $|\mathbb{S}(\varphi([\pi]))|=1$ unless π is invariant but not distinguished; in this case $\mathbb{S}(\varphi([\pi]))=Z_2$.*
3. *If π is tempered, then $\varphi([\pi])$ is tempered.*

Proof. (1). This follows from the definitions and above remarks.
(2) This follows by a case by case analysis following Tables 2 and 3 of the Appendix. We note in particular that by Theorem 5.3 of [R2] if $d \neq 1$, then $\pi = \pi(\chi, \tau)$ is distinguished if and only if τ is Galois invariant and τ is the base change of a $\tau_0 \in \mathrm{Irr}(\mathrm{GL}(2, F))$ such that $\omega_{\tau_0} = \omega_{E_d/F}\chi$.
(3) Assume π is tempered. Then ρ and η in the case $d \neq 1$, and ρ_1 and ρ_2 in the case $d = 1$, have bounded image. This implies that $\varphi([\pi])$ has bounded image. $\square$

4.3 Proposition. *Suppose F is nonarchimedean, E_d/F is unramified (if $d = 1$ by convention E_d/F is unramified) and $\pi = \pi(\chi, \tau) \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ is infinite dimensional with χ and τ unramified. If the residual characteristic of F is even, assume additionally π is tempered. Then*

(1) *$\varphi([\pi])$ is unramified, $|\Pi([\pi])| = 1$, and if the residual characteristic of F is odd, then the single element Π of $\Pi([\pi])$ is unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_F)$.*
(2) *([HST]) Let $\Pi([\pi]) = \{\Pi\}$. If π and Π are unitary (e.g., as in global applications, or π tempered), then Π is unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_F)$ and $\varphi([\pi])$ and Π correspond to the same conjugacy class in $\mathrm{GSp}(2, \mathbb{C})$.*

Proof. (1) Suppose $d = 1$. Evidently, $\varphi([\pi])$ is unramified. Write $\pi = \pi(\chi, \tau)$. As mentioned in Section 3 and the beginning of this section, instead of using the exact sequence of Theorem 2.3, let us use the more convenient sequence of Proposition 2.7, and let s be the representative for the nontrivial coset of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$ in $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},1}, F)$ from Proposition 2.7. Also, make the identification of $X_{\mathrm{M}_{2\times 2},1}$ with $\mathrm{M}_{2\times 2}(F)$ equipped with the determinant as remarked before Proposition 2.8. Then s is given by $s(x) = x^*$, with $*$ the canonical involution of matrices, and $\tau = \tau_1 \otimes \tau_2$ with $\tau_1, \tau_2 \in \mathrm{Irr}(\mathrm{GL}(2, F))$ and $\omega_{\tau_1} = \omega_{\tau_2} = \chi$. The lattice $\mathrm{M}_{2\times 2}(\mathfrak{O}_F) \subset X_{\mathrm{M}_{2\times 2},1}$ is self-dual, and the maximal compact subgroups J_0 and J of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$ and $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},1}, F)$ which are the stabilizers of $\mathrm{M}_{2\times 2}(\mathfrak{O}_F)$ are $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_F) \times \mathrm{GL}(2, \mathfrak{O}_F))$ and the subgroup generated by $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_F) \times \mathrm{GL}(2, \mathfrak{O}_F))$ and s, respectively. Since π is not essentially square integrable, $\Pi([\pi]) = \{\Pi = \theta_{\mathrm{M}_{2\times 2},1}(\pi^+)^\vee\}$ so that $|\Pi([\pi])| = 1$. By Proposition 1.11 to show Π is unramified it will suffice to show π^+ is unramified. Suppose $\tau_1 \not\cong \tau_2$. Then $\pi^+ = \mathrm{Ind}_{\mathrm{GSO}(X,F)}^{\mathrm{GO}(X,F)} \pi$. Using the model for π^+ as in (3.1), we see that if v is an unramified vector with respect to J_0, then $v \oplus v$ is an unramified vector for π^+. Suppose $\tau_1 \cong \tau_2$, so that π is distinguished and π^+ is the extension to $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},1}, F)$ of π defined in Section 3. Let $\tau = \tau_1$. It will suffice to show $\pi^+ = \pi(\chi, \tau \otimes \tau)^+$ is unramified. Define $T : \pi \to \pi$ by $T(v \otimes w) = w \otimes v$. To show π^+ is unramified it suffices to show $T = \pi^+(s)$. Let $Y = (F \cdot y)^\perp$, where $y \in X_{\mathrm{M}_{2\times 2},1}$ is the 2×2 identity matrix. Then $\mathrm{SO}(Y, F)$, identified as usual with the stabilizer of y in $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$, is the group of $\rho_a(1, g, g^{*-1})$ for $g \in \mathrm{GL}(2, F)$. To show $T = \pi^+(s)$ it will suffice to show $T\pi(shs) = \pi(h)T$ for $h \in \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$,

$T^2 = 1$, and $L \circ T = L$ for any nonzero element L of $\mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1})$. The first two statements follow from $s\rho_a(t,g,g')s = \rho_a(t,g',g)$ for $g, g' \in \mathrm{GL}(2,F)$. Let V be the space of τ. Fix a $\mathrm{GL}(2,F)$ isomorphism $R : (\omega_\tau^{-1} \otimes \tau, V) \to (\tau^\vee, V^\vee)$. Let $S = 1 \otimes R : V \otimes V \to V \otimes V^\vee$. We have a nonzero $\mathrm{GL}(2,F)$ invariant linear form $V \otimes V^\vee \to \mathbf{1}$ given by $v \otimes f \mapsto f(v)$. The composition of this with S gives us $L \in \mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1})$. Now $L \circ T = \epsilon L$ for some $\epsilon = \pm 1$. We must show $\epsilon = 1$. Let $v \in V$ be nonzero and fixed by $\mathrm{GL}(2, \mathfrak{O}_F)$. Then $L(v \otimes v) = L(T(v \otimes v)) = \epsilon L(v \otimes v)$. To see $L(v \otimes v) \neq 0$ and hence $\epsilon = 1$, let $V = \mathbb{C}v_0 \oplus W$ be a $\mathrm{GL}(2, \mathfrak{O}_F)$ decomposition, and define $f \in V^\vee$ by letting f be zero on W and setting $f(v) = 1$. Then $\tau^\vee(k)f = f$ for $k \in \mathrm{GL}(2, \mathfrak{O}_F)$. Evidently, $S(v \otimes v) = c(v \otimes f)$ for some $c \in \mathbb{C}^\times$ so that $L(v \otimes v) = cf(v) = c \neq 0$. Now suppose $d \neq 1$. Again, it is clear that $\varphi([\pi])$ is unramified. Write $\pi = \pi(\chi, \tau)$ again using Proposition 2.7. We will also use the notation after Proposition 2.9 regarding $X_{\mathrm{M}_{2\times 2},d}$. The representative s for the nontrivial coset of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$ in $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$ is given by $s(x) = x^* = a(x)$. The lattice $X_{\mathrm{M}_{2\times 2},d} \cap \mathrm{M}_{2\times 2}(\mathfrak{O}_{E_d})$ is self dual, and the maximal compact subgroups J_0 and J of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$ and $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$ which are the stabilizers of $X_{\mathrm{M}_{2\times 2},d} \cap \mathrm{M}_{2\times 2}(\mathfrak{O}_{E_d})$ are $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_{E_d}))$ and the subgroup generated by $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_{E_d}))$ and s, respectively. Since τ is unramified, $\tau \cong \mathrm{Ind}_P^{\mathrm{GL}(2,E_d)}(\mu_1 \otimes \mu_2)$, where B is the usual Borel subgroup of $\mathrm{GL}(2, E_d)$ and μ_1 and μ_2 are unramified and Galois invariant so that τ is Galois invariant. This implies $s \cdot \pi \cong \pi$. In the proof of Theorem 5.3 of [R2] it was shown that π is distinguished and that $L \in \mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1})$ is given by

$$L(f) = \int_{T \backslash \mathrm{GL}(2,F)} f(g_0^{-1} g)\chi(\det g)^{-1}\, dg$$

where T and g_0 are as in [R2]; here $Y = (F \cdot y)^\perp$, where y is the 2×2 identity matrix in $X_{\mathrm{M}_{2\times 2},d}$ and $\mathrm{SO}(Y,F)$ is the group of $\rho_a(\det g, g)$ for $g \in \mathrm{GL}(2,F)$. By definition, we have $\Pi([\pi]) = \{\mathit{\Pi} = \theta_{\mathrm{M}_{2\times 2},d}(\pi^+)^\vee\}$, so that $|\Pi([\pi])| = 1$. Again, by Proposition 1.11 to show that $\mathit{\Pi}$ is unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_F)$ when has F has odd residual characteristic it will suffice to show that π^+ is unramified. We proceed as in the case $d = 1$. Define $T : \pi \to \pi$ by $T(f) = f \circ a$. As in the $d = 1$ case, it will suffice to show $L \circ T = L$. For $f \in \pi$,

$$\begin{aligned}
(L \circ T)(f) &= \int_{T \backslash \mathrm{GL}(2,F)} f(a(g_0^{-1} g))\chi(\det g)^{-1}\, dg \\
&= \int_{T \backslash \mathrm{GL}(2,F)} f(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} g_0^{-1} g)\chi(\det g)^{-1}\, dg \\
&= \mu_1(-1) \int_{T \backslash \mathrm{GL}(2,F)} f(g_0^{-1} g)\chi(\det g)^{-1}\, dg \\
&= \mu_1(-1) L(f).
\end{aligned}$$

Since $\mu_1(-1) = 1$, we have $L \circ T = L$, as desired.

(2) This follows from Lemmas 10 and 11 of [HST] after one reconciles the definitions. ([HST] for example uses Whittaker models instead of distinguished representations to define extensions to $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$.) Lemma 1.6 and Proposition 2.9 are also useful for the comparison to [HST]. The reader should be aware that in Lemma 7 of [HST] the Langlands parameter should be $\mathrm{diag}(\chi_3(v), \chi_1(v)\chi_3(v), \chi_1(v)\chi_2(v)\chi_3(v), \chi_2(v)\chi_3(v))$, and in Lemma 10 of [HST] the L-parameter should be $\mathrm{diag}(\sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{\beta}, -\sqrt{\beta})$. $\square$

5. Global theta lifts for similitudes

In this section we review some foundational results on global theta lifts for similitudes ([HK], [HST]). We use the following definitions. Let F be a totally real number field with ring of integers $\mathfrak{O}$, and let X be a even dimensional quadratic space defined over F of positive dimension. For each infinite place v of F fix maximal compact subgroups $J_{1,v}$ and J_v of $\mathrm{O}(X, F_v)$ and $\mathrm{GO}(X, F_v)$, and let $\mathfrak{h}_{1,v}$ and $\mathfrak{h}_v$ be the Lie algebras of $\mathrm{O}(X, F_v)$ and $\mathrm{GO}(X, F_v)$, respectively, as in Section 1. Let $J_{1,\infty}$ and J_∞ be the products of $J_{1,v}$ and J_v, respectively, over the infinite places of F, and let $\mathfrak{h}_{1,\infty}$ and $\mathfrak{h}_\infty$ be the direct sums of the $\mathfrak{h}_{1,v}$ and $\mathfrak{h}_v$, respectively, over the infinite places of v. Let n be a positive integer. For each infinite place v of F let $K_{1,v}$ and K_v be the usual maximal compact subgroups of $\mathrm{Sp}(n, F_v)$ and $\mathrm{GSp}(n, F_v)$, and let $\mathfrak{g}_{1,v}$ and $\mathfrak{g}_v$ be the Lie algebras of $\mathrm{Sp}(n, F_v)$ and $\mathrm{GSp}(n, F_v)$, respectively, as in Section 1. Define $K_{1,\infty}$, K_∞, $\mathfrak{g}_{1,\infty}$ and $\mathfrak{g}_\infty$ as in the case of $\mathrm{O}(X)$ and $\mathrm{GO}(X)$. For v a place of F, define $R(F_v) \subset \mathrm{GSp}(n, F_v) \times \mathrm{GO}(X, F_v)$ as in Section 1. Let $R(F)$ and $R(\mathbb{A})$ be the set of pairs (g, h) in $\mathrm{GSp}(n, F) \times \mathrm{GO}(X, F)$ and $\mathrm{GSp}(n, \mathbb{A}) \times \mathrm{GO}(X, \mathbb{A})$, respectively, such that $\lambda(g) = \lambda(h)$. For v an infinite place of F, let L_v be the maximal compact subgroup of $R(F_v)$ as defined in Section 1, and let $\mathfrak{r}_v$ be Lie algebra of $R(F_v)$. Let L_∞ and $\mathfrak{r}_\infty$ be defined analogously to the last two cases. To define global theta lifts we need a global version of the Weil representation. Fix a nontrivial unitary character ψ of $\mathbb{A}/F$. For v a place of F, let ω_v be the Weil representation of $R(F_v)$ on $L^2(X(F_v)^n)$ defined with respect to ψ_v as in Section 1. Again, if v is a place of F then $\mathcal{S}(X(F_v)^n) \subset L^2(X(F_v)^n)$ is an $R(F_v)$ module if v is finite and an (r_v, L_v) module if v is infinite. Let $x_1, \dots, x_m$ be a vector space basis for $X(F)$ over F. Let $(g, h) \in R(\mathbb{A})$. Then for almost all finite v, $\omega_v(g_v, h_v)$ fixes the characteristic function of $\mathfrak{O}_v x_1 + \cdots + \mathfrak{O}_v x_m$. Let $\otimes_v \mathcal{S}(X(F_v)^n)$ be the algebraic restricted direct product over all the places of F of the complex vector spaces $\mathcal{S}(X(F_v)^n)$ with respect to the characteristic function of $\mathfrak{O}_v x_1 + \cdots + \mathfrak{O}_v x_m$ for v finite. We will denote the restricted algebraic direct product $\otimes_v \mathcal{S}(X(F_v)^n)$ by $\mathcal{S}(X(\mathbb{A})^n)$; then $\mathcal{S}(X(\mathbb{A})^n)$ is an $R(\mathbb{A}_f) \times (\mathfrak{r}_\infty, L_\infty)$ module, where $R(\mathbb{A}_f)$ has the obvious meaning. Let $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$ and $(g, h) \in R(\mathbb{A})$; assume $\varphi = \otimes_v \varphi_v$. The function $\omega(g, h)\varphi : X(\mathbb{A})^n \to \mathbb{C}$ given by $(\omega(g, h)\varphi)(x) = \prod_v (\omega_v(g_v, h_v)\varphi_v)(x_v)$ is well defined (note that for infinite v, $\omega_v(g_v, h_v)\varphi_v$ is a smooth function though it may not be in $\mathcal{S}(X(F_v)^n)$, so that it can be evaluated at a point). Using the universal property of the algebraic restricted direct product, this definition extends to all $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$: if $(g, h) \in R(\mathbb{A})$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$, then $\omega(g, h)\varphi$ may be regarded as a function on $X(\mathbb{A})^n$. In particular, the elements of $\mathcal{S}(X(\mathbb{A})^n)$ may be regarded as functions on $X(\mathbb{A})^n$.

Global theta lifts are now defined as follows. For $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$ and $(g,h) \in R(\mathbb{A})$, set

$$\theta(g,h;\varphi) = \sum_{x \in X(F)^n} \omega(g,h)\varphi(x).$$

This series converges absolutely and is left $R(F)$ invariant. Fix a right $\mathrm{O}(X,\mathbb{A})$ invariant quotient measure on $\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A})$. Let f be a cusp form on $\mathrm{GO}(X,\mathbb{A})$ of central character χ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. Let $\mathrm{GSp}(n,\mathbb{A})^+$ be the subgroup of $g \in \mathrm{GSp}(n,\mathbb{A})$ such that $\lambda(g) \in \lambda(\mathrm{GO}(X,\mathbb{A}))$. For $g \in \mathrm{GSp}(n,\mathbb{A})^+$ define

$$\theta_n(f,\varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A})} \theta(g,h_1h;\varphi)f(h_1h)\,dh_1,$$

where $h \in \mathrm{GO}(X,\mathbb{A})$ is any element such that $(g,h) \in R(\mathbb{A})$. This integral converges absolutely, does not depend on the choice of h, and the function $\theta_n(f,\varphi)$ on $\mathrm{GSp}(n,\mathbb{A})^+$ is left $\mathrm{GSp}(n,F)^+$ invariant. Moreover, $\theta_n(f,\varphi)$ extends uniquely to a $\mathrm{GSp}(n,F)$ left invariant function on $\mathrm{GSp}(n,\mathbb{A})$ with support in $\mathrm{GSp}(n,F)\,\mathrm{GSp}(n,\mathbb{A})^+$. This extended function, also denoted by $\theta_n(f,\varphi)$, is an automorphic form on $\mathrm{GSp}(n,\mathbb{A})$ of central character $\chi\chi_X^n = \chi(\cdot,\operatorname{disc} X(F))_F^n$. If V is a $\mathrm{GO}(X,\mathbb{A}_f) \times (h_\infty, J_\infty)$ subspace of the space of cusp forms on $\mathrm{GO}(X,\mathbb{A})$ of central character χ, then we denote by $\Theta_n(V)$ the $\mathrm{GSp}(n,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of the space of automorphic forms on $\mathrm{GSp}(n,\mathbb{A})$ of central character $\chi\chi_X^n$ generated by all the $\theta_n(f,\varphi)$ for $f \in V$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. Similarly, fix a right $\mathrm{Sp}(n,\mathbb{A})$ invariant quotient measure on $\mathrm{Sp}(n,F)\backslash \mathrm{Sp}(n,\mathbb{A})$, let F be a cusp form on $\mathrm{GSp}(n,\mathbb{A})$ of central character χ' and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. For $h \in \mathrm{GO}(X,\mathbb{A})$ define

$$\theta_X(F,\varphi)(h) = \int_{\mathrm{Sp}(n,F)\backslash \mathrm{Sp}(n,\mathbb{A})} \theta(g_1g,h;\varphi)F(g_1g)\,dg$$

where $g \in \mathrm{GSp}(n,\mathbb{A})$ is any element such that $(g,h) \in R(\mathbb{A})$. Again, this integral converges absolutely, does not depend on the choice of g, and the function $\theta_X(F,\varphi)$ is an automorphic form on $\mathrm{GO}(X,\mathbb{A})$ of central character $\chi'\chi_X^n$. If W is a $\mathrm{GSp}(n,\mathbb{A}_f)\times(\mathfrak{g}_\infty,K_\infty)$ subspace of the space of cusp forms on $\mathrm{GSp}(n,\mathbb{A})$ of central character χ', then we denote by $\Theta_X(W)$ the $\mathrm{GO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_\infty)$ subspace of the space of automorphic forms on $\mathrm{GO}(X,\mathbb{A})$ of central character $\chi'\chi_X^n$ consisting of the $\theta_X(F,\varphi)$ for $F \in W$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. We shall also occasionally consider global theta lifts of $\mathrm{O}(X,\mathbb{A}_f)\times(\mathfrak{h}_{1,\infty},J_{1,\infty})$ subspaces of the space of cusp forms on $\mathrm{O}(X,\mathbb{A})$ and of $\mathrm{Sp}(n,\mathbb{A}_f)\times(\mathfrak{g}_{1,\infty},K_{1,\infty})$ subspaces of the space of cusp forms on $\mathrm{Sp}(n,\mathbb{A})$. These have the obvious analogous definitions.

We will need to know how $\Theta_n(V)$ and $\Theta_X(W)$ behave if X is changed by a similitude. Let X' be another quadratic space over F, and suppose there is a similitude $t : X(F) \to X'(F)$ with similitude factor λ. Then for each place v of F, there is an isomorphism

$$\mathrm{GO}(X, F_v) \xrightarrow{\sim} \mathrm{GO}(X', F_v)$$

sending h to tht^{-1}. For each infinite v, let $J'_{1,v}$ and J'_v be the maximal compact subgroups of $\mathrm{O}(X', F_v)$ and $\mathrm{GO}(X', F_v)$ which are the images under the above isomorphism of $J_{1,v}$ and J_v, respectively. If v is infinite, then t also determines an isomorphism

$$\mathfrak{go}(X, F_v) \xrightarrow{\sim} \mathfrak{go}(X', F_v)$$

given by $h \mapsto tht^{-1}$. Via these two isomorphisms and definitions, for each v we obtain a bijection

$$\mathrm{Irr}(\mathrm{GO}(X, F_v)) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{GO}(X', F_v)),$$

and thus a bijection

$$\mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X, \mathbb{A})) \xrightarrow{\sim} \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X', \mathbb{A})).$$

If f is an automorphic form on $\mathrm{GO}(X, \mathbb{A})$, then $tf : \mathrm{GO}(X', \mathbb{A}) \to \mathbb{C}$ defined by $(tf)(h) = f(t^{-1}ht)$ is an automorphic form on $\mathrm{GO}(X', \mathbb{A})$. Under this map, cusp forms are mapped to cusp forms. Let the right $\mathrm{O}(X', \mathbb{A})$ invariant quotient measure on $\mathrm{O}(X', F)\backslash \mathrm{O}(X', \mathbb{A})$ be obtained from the fixed right $\mathrm{O}(X, \mathbb{A})$ invariant quotient measure on $\mathrm{O}(X, F)\backslash \mathrm{O}(X, \mathbb{A})$ via the isomorphism $h \mapsto tht^{-1}$.

5.1 LEMMA. *Let V be a $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ subspace of the space of cusp forms on $\mathrm{GO}(X, \mathbb{A})$ of central character χ, and let W be a $\mathrm{GSp}(n, \mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of the space of cusp forms on $\mathrm{GSp}(n, \mathbb{A})$ of central character χ'. Then $\Theta_n(V) = \Theta_n(tV)$ and $t\Theta_X(W) = \Theta_{X'}(W)$. Moreover, $\Theta_n(V)$ and $\Theta_X(W)$ do not depend on the choice of nontrivial unitary character ψ of $\mathbb{A}/F$.*

Proof. To show $\Theta_n(V) \subset \Theta_n(tV)$ it will suffice to show that if $f \in V$ and $\varphi = \otimes_v \varphi_v$, then $\theta_n^X(f, \varphi) \in \Theta_n(tV)$; here and below the superscript X will indicate the dependence on X. By Lemma 1.6, if $(g, h) \in R_{X,n}(\mathbb{A})$ then

$$\theta_n^X(g, h; \varphi) = \theta_n^{X'}(g^{[\lambda]}, tht^{-1}; \varphi \circ t^{-1}).$$

Let $g = g_0 g_1 \in \mathrm{GSp}(n, F)\,\mathrm{GSp}(n, \mathbb{A})^+$ with $g_0 \in \mathrm{GSp}(n, F)$ and $g_1 \in \mathrm{GSp}(n, \mathbb{A})^+$. A computation shows that

$$\theta_n^X(f, \varphi)(g) = \theta_n^X(f, \varphi)(g_1) = \theta_n^{X'}(tf, \varphi \circ t^{-1})(g_1^{[\lambda]}).$$

Write

$$g_1^{[\lambda]} = g' \begin{bmatrix} 1 & 0 \\ 0 & |\lambda|_\infty^{-1} \end{bmatrix}.$$

Here $|\lambda|_\infty$ is the element of $\mathbb{A}^\times$ which is 1 at the finite places and $|\lambda|_v$ at the infinite place v. Then $g' \in \mathrm{GSp}(n, \mathbb{A})^+$. Let $h' \in \mathrm{GO}(X, \mathbb{A})$ be such that $\lambda(h') = \lambda(g')$. We have

$$\begin{aligned}
&\theta_n^{X'}(tf, \varphi \circ t^{-1})(g_1^{[\lambda]}) = \theta_n^{X'}(tf, \varphi \circ t^{-1})(g' \begin{bmatrix} 1 & 0 \\ 0 & |\lambda|_\infty^{-1} \end{bmatrix}) \\
&= \int_{\mathrm{O}(X',F)\backslash \mathrm{O}(X',\mathbb{A})} \theta_n^{X'}(g' \begin{bmatrix} 1 & 0 \\ 0 & |\lambda|_\infty^{-1} \end{bmatrix}, h_1 h' \sqrt{|\lambda|_\infty}^{-1}; \varphi \circ t^{-1}) \\
&\qquad\qquad\qquad\qquad \cdot (tf)(h_1 h' \sqrt{|\lambda|_\infty}^{-1})\, dh_1 \\
&= \chi(\sqrt{|\lambda|_\infty}^{-1}) \theta_{X'}(tf, \varphi')(g'),
\end{aligned}$$

where

$$\varphi' = \prod_{v \text{ inf.}} |\lambda|_v^{n \dim X/4} \cdot (\varphi_f \circ t^{-1}) \otimes (\varphi_\infty \circ \sqrt{|\lambda|_\infty} t^{-1}).$$

Then $\varphi' \in \mathcal{S}(X'(\mathbb{A})^n)$, and

$$\begin{aligned}
\theta_n^X(f, \varphi)(g) &= \chi(\sqrt{|\lambda|_\infty}^{-1}) \theta_n^{X'}(tf, \varphi')(g') \\
&= \chi(\sqrt{|\lambda|_\infty}^{-1}) [\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}_f^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda)_\infty \end{bmatrix}^{-1} \theta_n^{X'}(tf, \varphi')](g).
\end{aligned}$$

Here, $\mathrm{sign}(\lambda)_\infty$ is the element of $\mathbb{A}^\times$ which is 1 at the finite places and at the infinite place v is the sign of λ in F_v. If $g \notin \mathrm{GSp}(n, F)\,\mathrm{GSp}(n, \mathbb{A})^+$, then also

$$g \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}_f^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda)_\infty \end{bmatrix}^{-1} \notin \mathrm{GSp}(n, F)\,\mathrm{GSp}(n, \mathbb{A})^+,$$

so that both sides of the last equality are by definition zero, and hence equal. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda)_\infty \end{bmatrix}^{-1} \in K_\infty,$$

it now follows that $\theta_n^X(f, \varphi) \in \Theta_n(tV)$, so that $\Theta_n(V) \subset \Theta_n(tV)$. Similarly, $\Theta_n(tV) \subset \Theta_n(V)$. The proof of $t\Theta_X(W) = \Theta_{X'}(W)$ and the independence of ψ are analogous. □

The next two results are due to Rallis [Ra] in the case of isometries. The first describes when a theta lift is cuspidal. The second result gives the structure of a theta lift of a space of cusp forms in the case the theta lift is cuspidal. The proofs are similar to or use the proofs in [Ra]. Section 1 is also a basic input for the proof of Proposition 5.3.

5.2 PROPOSITION (RALLIS). *Let $n \geq 1$ be an integer. Let f be a cusp form on* $\mathrm{GO}(X, \mathbb{A})$. *Suppose that $\theta_k(f,\varphi) = 0$ for all $0 \leq k \leq n-1$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^k)$. Then $\theta_n(f,\varphi)$ is cuspidal (though possibly zero) for all $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$.*

Here, $\theta_0(f,\varphi) = 0$ is taken to mean

$$0 = \int_{\mathrm{O}(X,F)\backslash\,\mathrm{O}(X,\mathbb{A})} f(h_1 h)\, dh_1 \tag{5.1}$$

for all $h \in \mathrm{GO}(X, \mathbb{A})$.

An analogous result holds for lifts from $\mathrm{GSp}(n)$ to $\mathrm{GO}(X)$. In this case, fix an even dimensional quadratic space X over F such that $X(F)$ is anisotropic. For an integer $k \geq 0$, let X_k be the orthogonal direct sum of X with k copies of the hyperbolic plane over F. Let f be a cusp form on $\mathrm{GSp}(n, \mathbb{A})$. Let $l \geq 0$ be an integer. If $l = 0$ and $\dim X = 0$, so that $X_l = 0$, then $\theta_{X_l}(f,\varphi)$ is not defined; if $l = 0$ and $\dim X > 0$ so that $X_l = X$, then $\theta_{X_l}(f,\varphi)$ is cuspidal for all $\varphi \in \mathcal{S}(X_l(\mathbb{A})^n)$ since the cuspidal condition is vacuous. Suppose $l \geq 1$. Suppose $\theta_{X_k}(f,\varphi) = 0$ for all $0 \leq k \leq l-1$ and $\varphi \in \mathcal{S}(X_k(\mathbb{A})^n)$; then $\theta_{X_l}(f,\varphi)$ is cuspidal (though possibly zero) for all $\varphi \in \mathcal{S}(X_l(\mathbb{A})^n)$. Here, if $\dim X = 0$ and $k = 0$ then the condition $\theta_{X_k}(f,\varphi) = 0$ is taken to be empty.

5.3 PROPOSITION (RALLIS; MULTIPLICITY PRESERVATION). *Let $2n = \dim X$. Let V be a $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ nonzero subspace of the space of cusp forms on $\mathrm{GO}(X, \mathbb{A})$ of central character χ. Assume that for each place v of F, $X(F_v)$ satisfies one of the conditions of (1)-(6) of Theorem 1.8. Assume that*

$$V = V_1 \oplus \cdots \oplus V_M,$$

where each V_i, $1 \leq i \leq M$, is a $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ subspace of V, and all the V_i are isomorphic to a single nonzero irreducible $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ representation σ. Let $\sigma \cong \otimes_v \sigma_v$, assume $\sigma_v|_{\mathrm{O}(X,F)}$ is multiplicity free for all v, and σ_v is tempered for $v \mid 2$. Suppose that $\Theta_n(V)$ is contained in the space of cusp forms on $\mathrm{GSp}(n, \mathbb{A})$ (necessarily of central character $\chi\chi_X^n = \chi(\cdot, \operatorname{disc} X(F))^n$), and that for any irreducible nonzero $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ subspace U of V we have $\Theta_n(U) \neq 0$. Then $\sigma_v \in \mathcal{R}_n(\mathrm{GO}(X, F_v))$ for all v,

$$\Theta_n(V) = \Theta_n(V_1) \oplus \cdots \oplus \Theta_n(V_M),$$

and each $\Theta_n(V_i)$, $1 \leq i \leq M$, is isomorphic to $\Pi = \otimes_v \theta(\sigma_v^\vee)$. An analogous result holds if the roles of $\mathrm{GSp}(n)$ and $\mathrm{GO}(X)$ are interchanged.

6. TEMPERED CUSPIDAL AUTOMORPHIC REPRESENTATIONS OF $B(\mathbb{A})^\times$ AND $\mathrm{GSO}(X, \mathbb{A})$

Let F be a totally real number field, and let X be a four dimensional quadratic space over F. As in Section 2, let B be the even Clifford algebra of $X(F)$, and let E be the center of B. Let $d = \operatorname{disc} X(F)$. In this section we describe the relationship between tempered cuspidal automorphic representations of $B^\times(\mathbb{A})$ and $\mathrm{GSO}(X, \mathbb{A})$. From Section 2, we have exact sequences

$$1 \to E^\times \to F^\times \times B^\times(F) \xrightarrow{\rho} \mathrm{GSO}(X, F) \to 1$$

and

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho} \mathrm{GSO}(X, \mathbb{A}) \to 1.$$

6.1 Lemma. *There exist $s \in \mathrm{O}(X, F)$, and for each infinite v, a maximal compact subgroup $J_{0,v}$ of $\mathrm{GSO}(X, F_v)$, such that $\det s = -1$, $s^2 = 1$, and $sJ_{0,v}s = J_{0,v}$ for all infinite v.*

Proof. Let $y \in X(F)$ be anisotropic, and let $Y \subset X$ be the three dimensional quadratic space over F such that $Y(F) = (F \cdot y)^\perp$. Let $s \in \mathrm{O}(X, F)$ be defined with respect to y as in Propositions 2.5 and 2.6. Then $\det s = -1$ and $s^2 = 1$. For each infinite v, choose a maximal compact subgroup $J_{Y,v}$ of $\mathrm{SO}(Y, F_v)$, and let $J_{0,v}$ be the unique maximal compact subgroup of $\mathrm{GSO}(X, F_v)$ containing $J_{Y,v}$ mentioned in the penultimate paragraph of Section 2. Then s normalizes $J_{Y,v}$ and $J_{0,v}$ for each infinite v. $\square$

For the remainder of this paper we fix the following choices of compact subgroups. Let s and the maximal compact subgroups $J_{0,v}$ of $\mathrm{GSO}(X, F_v)$ be as in Lemma 6.1. For each infinite place v of F, let $K_{B,v}$ be the unique maximal compact subgroup of $B^\times(F_v)$ such that $\rho(\{\pm 1\} \times K_{B,v}) = J_{0,v}$. Let $J_{0,\infty}$ be the product of the $J_{0,v}$ over the infinite places v of F, and let $\mathfrak{h}_\infty$ be the direct sum of the $\mathfrak{h}_v = \mathfrak{gso}(X, F_v) = \mathfrak{go}(X, F_v)$ over the infinite places v of F. Let $K_{B,\infty}$ be the product of the $K_{B,v}$ over the infinite places v of F and let B_∞ be the direct sum over the infinite places v of the Lie algebra $B(F_v)$ of $B^\times(F_v)$. We consider $B^\times(\mathbb{A}_f) \times (B_\infty, K_{B,\infty})$ and $\mathrm{GSO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_{0,\infty})$ modules.

We will use the following facts about the tempered cuspidal automorphic representations of $B^\times(\mathbb{A})$. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ be the set of tempered cuspidal automorphic representations τ of $B^\times(\mathbb{A})$. It is well known that $B^\times(\mathbb{A})$ has the multiplicity one property, i.e., the elements of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ of a fixed central character occur with multiplicity one in the space of cusp forms on $B^\times(\mathbb{A})$ of that central character. If $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B(\mathbb{A})^\times)$, then the unique space of cusp forms on $B^\times(\mathbb{A})$ isomorphic to τ will be denoted by V_τ. Also, $B^\times(\mathbb{A})$ has the strong multiplicity one property: if $\tau, \tau' \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ share the same central character and $\tau_v \cong \tau'_v$ for all but finitely many v, then $\tau \cong \tau'$, so that $V_\tau = V_{\tau'}$. In addition, the Jacquet-Langlands correspondence gives an injection of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ into $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2, \mathbb{A}_E))$. This map is constructed as follows. Suppose E is a field. Since B has center E, we may regard B as an algebra over E, and by Section 2, B is a quaternion algebra over E. There is a canonical isomorphism $B^\times(\mathbb{A}) \cong B^\times(\mathbb{A}_E)$, and thus a bijection $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A})) \xrightarrow{\sim} \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}_E))$. Composing with the Jacquet-Langlands map from $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}_E))$ to $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2, \mathbb{A}_E))$, we get an injection $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A})) \hookrightarrow \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2, \mathbb{A}_E))$ which we also call the

Jacquet-Langlands correspondence, and denote by $\tau \mapsto \tau^{\mathrm{JL}}$. If $E \cong F \times F$, we get a similar injection, with $\mathrm{GL}(2, \mathbb{A}_E)$ taken to be $\mathrm{GL}(2, \mathbb{A}) \times \mathrm{GL}(2, \mathbb{A})$.

Tempered cuspidal automorphic representations of $\mathrm{GSO}(X, \mathbb{A})$ and $B^\times(\mathbb{A})$ may be related as in the local case. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp,f}}(\mathbb{A}^\times \times B^\times(\mathbb{A}))$ be the set of pairs (χ, τ), where $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ and χ is a Hecke character of $\mathbb{A}^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}^E_F$. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ denote the set of tempered cuspidal automorphic representations of $\mathrm{GSO}(X, \mathbb{A})$. The above exact sequences give a bijection

$$\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp,f}}(\mathbb{A}^\times \times B^\times(\mathbb{A})) \xrightarrow{\sim} \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A})).$$

If $(\chi, \tau) \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp,f}}(\mathbb{A}^\times \times B^\times(\mathbb{A}))$, then $\pi(\chi, \tau) \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ corresponding to (χ, τ) consists of the space of functions $F : \mathrm{GSO}(X, \mathbb{A}) \to \mathbb{C}$ for which there exists $f \in \tau$ so that $F(\rho(t, g)) = \chi(t)^{-1} f(g)$. The central character of $\pi(\chi, \tau)$ is χ. If $d = 1$, so that $E \cong F \times F$ and $B^\times(\mathbb{A}) \cong D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$ (see Section 2), then every element $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ is of the form $\tau_1 \otimes \tau_2$ for some $\tau_1, \tau_2 \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(D^\times(\mathbb{A}))$, and the condition that ω_τ factors through N^E_F amounts to $\omega_{\tau_1} = \omega_{\tau_2}$. In this case ω_τ factors uniquely through N^E_F via $\chi = \omega_{\tau_1} = \omega_{\tau_2}$. Also, when dealing with a four dimensional quadratic space X_a over F defined by a Galois action a on a given quadratic quaternion algebra B over F with center E (Section 2), we will occasionally parameterize $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_a, \mathbb{A}))$ with respect to the explicit exact sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho_a} \mathrm{GSO}(X_a, \mathbb{A}) \to 1$$

derived from Proposition 2.7; by that proposition, the difference between the two parameterizations is insignificant.

Tempered cuspidal automorphic representations of $\mathrm{GSO}(X, \mathbb{A})$ inherit similar properties from those of $B^\times(\mathbb{A})$. The elements of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ have the multiplicity one property and the strong multiplicity one property. If $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ then the unique space of cusp forms on $\mathrm{GSO}(X, \mathbb{A})$ isomorphic to π will be denoted by V_π. If $\pi \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSO}(X, \mathbb{A}))$, then we denote by $s \cdot \pi$ the $\mathrm{GSO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_{0,\infty})$ module with the same space as π, but with twisted action $(s \cdot \pi)(h) = \pi(shs)$ for $h \in \mathrm{GSO}(X, \mathbb{A}_f) \times J_{0,\infty}$ and $(s \cdot \pi)(x) = \pi(\mathrm{Ad}(s)x)$ for $x \in \mathfrak{h}_\infty$. Let $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$. Then we denote by sV_π the space of cusp forms sf on $\mathrm{GSO}(X, \mathbb{A})$ defined by $(sf)(h) = f(shs)$ for $h \in \mathrm{GSO}(X, \mathbb{A})$ and $f \in V_\pi$. The map $f \mapsto sf$ from V_π with the twisted action $s \cdot \pi$ to sV_π with the usual action is an isomorphism; by multiplicity one, $s \cdot \pi \cong \pi$ if and only if $sV_\pi = V_\pi$.

7. From $\mathrm{GSO}(X, \mathbb{A})$ to $\mathrm{GO}(X, \mathbb{A})$

In this section F is a totally real number field and X is a four dimensional quadratic space over F. Let the notation be as in Section 6; following [HST], we explain how cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A})$ are obtained from those of $\mathrm{GSO}(X, \mathbb{A})$. For each infinite place v of F let $J_{0,v}$ be the maximal compact subgroup of $\mathrm{GSO}(X, F_v)$ defined in Section 6, and let J_v denote the maximal compact subgroup of $\mathrm{GO}(X, F_v)$ generated by $J_{0,v}$ and s, where s is as in Lemma 6.1. Let J_∞ be the product of the J_v over the infinite places of F. We consider $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ modules. Let $\mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GO}(X, \mathbb{A}))$ be the set of tempered cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A})$.

7.1 Theorem ([HST]). *The group* $\mathrm{GO}(X, \mathbb{A})$ *has the multiplicity one property; if* $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GO}(X, \mathbb{A}))$*, denote by* V_σ *the unique space of cusp forms isomorphic to* σ*. Let* $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GO}(X, \mathbb{A}))$*, and let* V_σ^0 *be the nonzero space of cusp forms on* $\mathrm{GSO}(X, \mathbb{A})$ *obtained by restricting the functions in* V_σ *to* $\mathrm{GSO}(X, \mathbb{A})$*. Either* V_σ^0 *is irreducible as a* $\mathrm{GSO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_{0,\infty})$ *module, or there exists* $\pi \in \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X, \mathbb{A}))$ *such that* $s \cdot \pi \ncong \pi$ *and* $V_\sigma^0 = V_\pi \oplus sV_\pi$ *(internal direct sum). Thus, there is a map*

$$\mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GO}(X, \mathbb{A})) \to \langle s \rangle \backslash \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X, \mathbb{A})),$$

and if $\sigma \mapsto [\pi] = \{\pi, s \cdot \pi\}$*, then*

$$\sigma_v \hookrightarrow \mathrm{Ind}_{\mathrm{GSO}(X, F_v)}^{\mathrm{GO}(X, F_v)} \pi_v \tag{7.1}$$

for all v*. The map* $\sigma \mapsto [\pi]$ *is surjective. If* $[\pi] \in \langle s \rangle \backslash \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X, \mathbb{A}))$ *and* $s \cdot \pi \ncong \pi$*, then the fiber over* $[\pi]$ *is the set of all* $\sigma \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X, \mathbb{A}))$ *such that* (7.1) *holds for all places* v *of* F*.*

Proof. See Section 1 of [HST]. □

8. Proofs of the main theorems

Let F be a totally real number field. In this final section we prove the main results Theorems 8.3 and 8.6 presented in the Introduction. Besides the general foundational work of Sections 1, 2 and 5, the main ingredients for Theorem 8.3 are the local results of Section 3 and the general nonvanishing result for global theta lifts from [R4]. Globally, the result from [R4] requires the nonvanishing of a certain L-function at $s = 1$; in the case at hand, this L-function turns out to be either a partial $\mathrm{GL}(2) \times \mathrm{GL}(2)$ L-function or a partial twisted Asai L-function, so that the nonvanishing at $s = 1$ follows from [Sh]. To prove Theorem 8.6 we actually first prove a different version, Theorem 8.5. In this version, using Section 4, a global L-packet $\Pi([\pi])$ of tempered irreducible admissible representations of $\mathrm{GSp}(2, \mathbb{A})$ is assigned to every element π of $\mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2}, d}, \mathbb{A}))$. When $s \cdot \pi \ncong \pi$, Theorem 8.5 determines exactly what elements of $\Pi([\pi])$ are cuspidal automorphic and shows that the cuspidal automorphic elements occur with multiplicity one. In addition to an understanding of the local situation, the main tool for showing cuspidality is Theorem 8.3. For multiplicity one, we use the Rallis multiplicity preservation principle in the context of similitudes (Proposition 5.3), along with the nonvanishing result for global theta lifts from $\mathrm{Sp}(2, \mathbb{A})$ from [KRS]. This result shows that if a twisted partial standard L-function of a cuspidal automorphic representation of $\mathrm{Sp}(2,\mathbb{A})$ has a pole at $s=1$, then it has a nonzero theta lift to the isometry group of a certain four dimensional quadratic space. Theorem 8.6 follows directly from Theorem 8.5.

We begin with a lemma which computes the standard partial L-function of an $\mathrm{O}(X,\mathbb{A})$ component of a cuspidal automorphic representation of $\mathrm{GO}(X,\mathbb{A})$ for a four dimensional quadratic space X over F. In the following lemma, $L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$ is the v-th Euler factor of the Asai L-function of τ^{JL} twisted by χ^{-1} ([HLR], p. 64–5); and $L_v(s,\tau_1^{\mathrm{JL}}\times\tau_2^{\mathrm{JL}\vee})$ is the v-th Euler factor of the usual Rankin–Selberg $\mathrm{GL}(2)\times\mathrm{GL}(2)$ L-function of τ_1^{JL} and $\tau_2^{\mathrm{JL}\vee}$; here, the superscript JL indicates the corresponding element under the Jacquet-Langlands correspondence (Section 6). Also, under the assumption that $X(F_v)$ is unramified (Section 1), when we say that an irreducible admissible representation of $\mathrm{GO}(X,F_v)$ (or of $\mathrm{O}(X,F_v)$ and $\mathrm{SO}(X,F_v)$) is unramified we mean with respect to the stabilizer in $\mathrm{GO}(X,F_v)$ (or in $\mathrm{O}(X,F_v)$ and $\mathrm{SO}(X,F_v)$, respectively) of a self-dual lattice in $X(F_v)$.

8.1 Lemma. *Let X be a four dimensional quadratic space over F, let B be the even Clifford algebra of $X(F)$, and let E be the center of B. Let $d=\operatorname{disc}X(F)$. Let $\sigma\in\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$, and assume that σ lies over $[\pi=\pi(\chi,\tau)]$ (See Sections 6 and 7). Let v be a finite place of F such that $X(F_v)$ and σ_v are unramified. Let $\sigma_{1,v}$ be the unramified component of $\sigma_v|_{\mathrm{O}(X,F_v)}$. Then the standard L-function of $\sigma_{1,v}$ is*

$$L(s,\sigma_{1,v})=\begin{cases}L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathit{Asai}) & \text{if } d\neq 1,\\ L_v(s,\tau_1^{\mathrm{JL}}\times\tau_2^{\mathrm{JL}\vee}) & \text{if } d=1 \text{ and } \tau\cong\tau_1\otimes\tau_2.\end{cases}$$

Proof. By definition, $L(s,\sigma_{1,v})$ (see Section 2 of [KR1]) is the standard L-function of any irreducible unramified component of $\sigma_{1,v}|_{\mathrm{SO}(X,F_v)}$. It will thus suffice to show that the standard L-function of any irreducible unramified component of $\sigma_v|_{\mathrm{SO}(X,F_v)}$ has the stated form; and since π_v is an irreducible component of $\sigma_v|_{\mathrm{GSO}(X,F_v)}$, it will be enough to show that the standard L-function of any irreducible unramified component of $\pi_v|_{\mathrm{SO}(X,F_v)}$ or $(s\cdot\pi_v)|_{\mathrm{SO}(X,F_v)}$ has the above form (s is as in Lemma 6.1). Since over a local nonarchimedean field a four dimensional quadratic space represents 1, by Proposition 2.8 there exists a quaternion algebra D over F_v contained in $B(F_v)$ and an isometry $T: X(F_v)\to X_{D,E_v}$ such that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times\times B^\times(F_v) & \xrightarrow{\rho} & \mathrm{GSO}(X,F_v) & \longrightarrow & 1\\
 & & \Big\downarrow{\scriptstyle\mathrm{id}} & & \Big\downarrow{\scriptstyle\wr} & & \Big\downarrow{\scriptstyle T\cdot T^{-1}} & & \\
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times\times B_{D,E_v}^\times & \xrightarrow{\rho_{a(D,E_v)}} & \mathrm{GSO}(X_{D,E_v},F_v) & \longrightarrow & 1
\end{array}$$

commutes, where $B^\times(F_v)\xrightarrow{\sim}B^\times_{D,E_v}$ is the isomorphism induced by the natural isomorphism $B(F_v)\cong E_v\otimes_{F_v}D$ of E_v algebras; $E_v=E(F_v)=F_v\otimes_F E$. Since

$X(F_v)$ is unramified, D is in particular split, i.e., there exists an isomorphism $D \xrightarrow{\sim} \mathrm{M}_{2\times 2}(F_v)$ of quaternion algebras over F_v. From this, we obtain an isomorphism $B_{D,E_v} \xrightarrow{\sim} \mathrm{M}_{2\times 2}(E_v)$ of E_v algebras and an isometry $t : X_{D,E_v} \xrightarrow{\sim} X_a$ so that

$$\begin{CD}
1 @>>> E_v^\times @>>> F_v^\times \times B_{D,E_v}^\times @>{\rho_{a(D,E_v)}}>> \mathrm{GSO}(X_{D,E_v}, F_v) @>>> 1\\
@. @V{\mathrm{id}}VV @V{\wr}VV @VV{t\cdot t^{-1}}V @.\\
1 @>>> E_v^\times @>>> F_v^\times \times \mathrm{GL}(2,E_v) @>{\rho_a}>> \mathrm{GSO}(X_a, F_v) @>>> 1
\end{CD}$$

commutes. Here, a is the Galois action on $\mathrm{M}_{2\times 2}(E_v)$ defined by the formula (2.1). Composing, we now have an isomorphism $i : B(F_v) \xrightarrow{\sim} \mathrm{M}_{2\times 2}(E_v)$ of E_v algebras and isometry $r : X(F_v) \xrightarrow{\sim} X_a$ such that

$$\begin{CD}
1 @>>> E_v^\times @>>> F_v^\times \times B^\times(F_v) @>{\rho}>> \mathrm{GSO}(X, F_v) @>>> 1\\
@. @V{\mathrm{id}}VV @V{\wr}VV @VV{r\cdot r^{-1}}V @.\\
1 @>>> E_v^\times @>>> F_v^\times \times \mathrm{GL}(2,E_v) @>{\rho_a}>> \mathrm{GSO}(X_a, F_v) @>>> 1
\end{CD}$$

commutes. Let π_v' be the representation of $\mathrm{GSO}(X_a, F_v)$ corresponding to π_v. By definition, $(\tau_v)^{\mathrm{JL}} = \tau_v \circ i$, and we have $\pi_v' = \pi(\chi_v, (\tau_v)^{\mathrm{JL}})$. Since the standard L-function of any unramified irreducible component of $\pi_v|_{\mathrm{SO}(X,F_v)}$ is the same as the standard L-function of any irreducible unramified component of $\pi_v'|_{\mathrm{SO}(X_a,F_v)}$, and the same holds for $(s\cdot\pi_v)|_{\mathrm{SO}(X,F_v)}$ and $(s\cdot\pi_v')|_{\mathrm{SO}(X_a,F_v)}$, it will now suffice to show that the the standard L-function of any irreducible unramified component of $\pi_v'|_{\mathrm{SO}(X_a,F_v)}$ or $(s\cdot\pi_v')|_{\mathrm{SO}(X_a,F_v)}$ has the above form. Assume first $d \neq 1$ (i.e., E is a field) and v stays prime in E; let w be the place of E lying over v. Then $E_w = E_v$. Since π_v' is unramified, so are χ_v and $(\tau^{\mathrm{JL}})_w = (\tau_v)^{\mathrm{JL}} \in \mathrm{Irr}(\mathrm{GL}(2,E_w))$. Let $(\tau^{\mathrm{JL}})_w = \mathrm{Ind}_P^{\mathrm{GL}(2,E_w)}(\mu_1\otimes\mu_2)$, where P is the usual upper triangular Borel subgroup of $\mathrm{GL}(2,E_w)$, induction is normalized, μ_1 and μ_2 are unramified quasi-characters of $E_w^\times$, and $\mu_1\otimes\mu_2$ is defined by

$$(\mu_1\otimes\mu_2)(\begin{bmatrix} a & b\\ 0 & c\end{bmatrix}) = \mu_1(a)\mu_2(c).$$

The space X_a was explicitly described in Section 2. With respect to the ordered basis

$$\begin{bmatrix} 0 & \sqrt{d}\\ 0 & 0\end{bmatrix}, \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix} \sqrt{d} & 0\\ 0 & -\sqrt{d}\end{bmatrix}, \begin{bmatrix} 0 & 0\\ -(2/d)\sqrt{d} & 0\end{bmatrix}$$

the symmetric bilinear form on X_a, which is given by the determinant, has the form

$$\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & -d & 0\\ 1 & 0 & 0 & 0\end{bmatrix}.$$

The stabilizer in $\mathrm{GSO}(X_a, F_v)$ of the isotropic subspace spanned by the first basis vector is a Borel subgroup P' of $\mathrm{GSO}(X_a, F_v)$, and $P' = \rho_\alpha(F_v^\times \times P)$. In particular, we have

$$\rho_\alpha(t, \begin{bmatrix} a & * \\ 0 & c \end{bmatrix}) = t^{-1} \begin{bmatrix} \mathrm{N}_{F_v}^{E_w}(a) & * & * \\ 0 & h & * \\ 0 & 0 & \mathrm{N}_{F_v}^{E_w}(c) \end{bmatrix},$$

with

$$h = \begin{bmatrix} a_1c_1 - a_2c_2d & (a_2c_1 - a_1c_2)d \\ a_2c_1 - a_1c_2 & a_1c_1 - a_2c_2d \end{bmatrix},$$

where $a = a_1 + a_2\sqrt{d}$ and $c = c_1 + c_2\sqrt{d}$. Here, the middle block h corresponds to multiplication by $a\alpha(c)$ on the two dimensional subspace spanned by the two middle basis vectors, using the obvious identification of this subspace with E_w. Recalling that $\chi_v \circ \mathrm{N}_{F_v}^{E_w} = \mu_1\mu_2$, a computation shows that

$$\pi'_v = \pi(\chi_v, (\tau^{\mathrm{JL}})_w) = \mathrm{Ind}_{P'}^{\mathrm{GSO}(X_a,F_v)} \mu,$$

where induction is normalized, and on the typical element of P' μ takes the value

$$\mu(\begin{bmatrix} a & * & * \\ 0 & h & * \\ 0 & 0 & \lambda a^{-1} \end{bmatrix}) = (\mu_2/\chi_v)(\lambda a^{-1})\mu_1(h),$$

where again we identify the elements of the middle block with $E_w^\times$ and $a, \lambda \in F_v^\times$. There is an $\mathrm{SO}(X_a, F_v)$ isomorphism

$$\pi'_v|_{\mathrm{SO}(X_a,F_v)} \xrightarrow{\sim} \mathrm{Ind}_{P'\cap \mathrm{SO}(X_a,F_v)}^{\mathrm{SO}(X_a,F_v)} \mu|_{P'\cap\mathrm{SO}(X_a,F_v)}$$

given by restriction of functions. We have

$$\mu|_{P'\cap\mathrm{SO}(X_a,F_v)}(\begin{bmatrix} a & * & * \\ 0 & h & * \\ 0 & 0 & a^{-1} \end{bmatrix}) = (\chi_v/\mu_2)(a)$$

since $\mathrm{N}_{F_v}^{E_w}(h) = 1$, so that $h \in \mathfrak{O}_w^\times$. By definition, the standard L-function of any irreducible unramified component of $\pi'_v|_{\mathrm{SO}(X_a,F_v)}$ is now

$$\begin{aligned} L(s, \chi_v/\mu_2)L(s, \mu_2/\chi_v)\zeta_{F_v}(2s) &= \det(1 - \chi(\pi_{F_v})^{-1}A|\pi_{F_v}|^s)^{-1} \\ &= L_v(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai}), \end{aligned}$$

where

$$A = \begin{bmatrix} \mu_1(\pi_{F_v}) & 0 & 0 & 0 \\ 0 & 0 & \mu_1(\pi_{F_v}) & 0 \\ 0 & \mu_2(\pi_{F_v}) & 0 & 0 \\ 0 & 0 & 0 & \mu_2(\pi_{F_v}) \end{bmatrix}.$$

For the last equality, see p. 64-65 of [HLR]. Since $s \cdot \pi'_v = \pi(\chi_v, (\tau^{\mathrm{JL}})_w \circ \alpha)$, a similar computation shows that the standard L-function of any irreducible unramified component of $(s \cdot \pi'_v)|_{\mathrm{SO}(X_a,F_v)}$ is also $L_v(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai})$.
Now suppose E is a field and v splits in E. Then F_v contains a square root of d; fix such a square root $\sqrt{d}$ in $F_v^\times$. Define an embedding of fields $i_1 : E \hookrightarrow F_v$ by sending a fixed square root of d in E to $\sqrt{d}$, and define another embedding $i_2 : E \hookrightarrow F_v$ by sending the fixed square root of d in E to $-\sqrt{d}$. We denote by w_1 and w_2 the places of E determined by i_1 and i_2, respectively. Then w_1 and w_2 are the two places of E lying over v, and via i_1 and i_2 we take F_v to be the completions E_{w_1} and E_{w_2} of E at w_1 and w_2, respectively. We also have an identification of $E_v = F_v \otimes_F E$ with $E_{w_1} \times E_{w_2}$ and hence with $F_v \times F_v$. Using the identification $E_v \cong F_v \times F_v$ we may identify $\mathrm{M}_{2\times 2}(E_v)$ with $\mathrm{M}_{2\times 2}(F_v) \times \mathrm{M}_{2\times 2}(F_v)$, $\mathrm{GL}(2, E_v)$ with $\mathrm{GL}(2, F_v) \times \mathrm{GL}(2, F_v)$ and a with the Galois action defined by $(x_1, x_2) \mapsto (x_2, x_1)$. Further, as explained after Proposition 2.7, we may identify X_a with $\mathrm{M}_{2\times 2}(F_v)$ and ρ_a with $\rho_a(t, (g_1, g_2))x = t^{-1} g_1 x g_2^*$. Using the canonical isomorphisms $B(F_v) \cong B(E_{w_1}) \times B(E_{w_2}) \cong D \times D$ write $\tau_v \cong \tau_{w_1} \otimes \tau_{w_2}$ with $\tau_{w_1}, \tau_{w_2} \in \mathrm{Irr}(D^\times)$; then $(\tau_v)^{\mathrm{JL}} \cong \tau_{w_1}^{\mathrm{JL}} \otimes \tau_{w_2}^{\mathrm{JL}}$. Let $\tau_{w_1}^{\mathrm{JL}} = \mathrm{Ind}_P^{\mathrm{GL}(2,F_v)}(\mu_1 \otimes \mu_2)$ and $\tau_{w_2}^{\mathrm{JL}} = \mathrm{Ind}_P^{\mathrm{GL}(2,F_v)}(\mu'_1 \otimes \mu'_2)$, with the notation analogous to the previous case. Note that $\chi = \mu_1\mu_2 = \mu'_1\mu'_2$. With respect to the ordered basis

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

the symmetric bilinear form on X_a has the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The stabilizer in $\mathrm{GSO}(X_a, F_v)$ of the isotropic flag

$$F_v \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \subset F_v \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + F_v \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is a Borel subgroup P', and $P' = \rho_\alpha(F_v^\times \times P \times P)$. We have

$$\rho_\alpha(t, \begin{bmatrix} a & * \\ 0 & c \end{bmatrix}, \begin{bmatrix} a' & * \\ 0 & c' \end{bmatrix}) = t^{-1} \begin{bmatrix} aa' & * & * & * \\ 0 & ac' & * & * \\ 0 & 0 & cc' & * \\ 0 & 0 & 0 & a'c \end{bmatrix}.$$

Using $\mu_1\mu_2 = \mu'_1\mu'_2$, a computation shows that

$$\pi'_v \cong \mathrm{Ind}_{P'}^{\mathrm{GSO}(X_a,F_v)} \mu$$

with

$$\mu(\begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & \lambda a^{-1} & * \\ 0 & 0 & 0 & \lambda b^{-1} \end{bmatrix}) = \mu_2(\lambda)(\mu_1'/\mu_2)(a)(\mu_2'/\mu_2)(b).$$

Again there is an $\mathrm{SO}(X_a, F_v)$ isomorphism

$$\pi_v'|_{\mathrm{SO}(X_a,F_v)} \xrightarrow{\sim} \mathrm{Ind}_{P'\cap \mathrm{SO}(X_a,F_v)}^{\mathrm{SO}(X_a,F_v)} \mu|_{P'\cap\mathrm{SO}(X_a,F_v)}$$

given by restriction of functions. We have

$$\mu|_{P'\cap \mathrm{SO}(X_a,F_v)}(\begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & a^{-1} & * \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}) = (\mu_1'/\mu_2)(a)(\mu_2'/\mu_2)(b).$$

We now have that the standard L-function of any irreducible unramified component of $\pi_v'|_{\mathrm{SO}(X_a,F_v)}$ is

$$\begin{aligned} L(s,\mu_1'/\mu_2)L(s,\mu_2/\mu_1')L(s,\mu_2'/\mu_2)L(s,\mu_2/\mu_2') &= \det(1-\chi(\pi_{F_v})^{-1}A|\pi_{F_v}|^s)^{-1} \\ &= L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai}), \end{aligned}$$

where

$$A = \begin{bmatrix} (\mu_1\mu_1')(\pi_{F_v}) & 0 & 0 & 0 \\ 0 & (\mu_2\mu_2')(\pi_{F_v}) & 0 & 0 \\ 0 & 0 & (\mu_1\mu_2')(\pi_{F_v}) & 0 \\ 0 & 0 & 0 & (\mu_2\mu_1')(\pi_{F_v}) \end{bmatrix};$$

here we have used $\chi_v = \mu_1\mu_2 = \mu_1'\mu_2'$. For the last equality, again see p. 64-65 of [HLR]. Since $s\cdot\pi_v' = \pi(\chi_v, \tau_{w_2}^{\mathrm{JL}} \otimes \tau_{w_1}^{\mathrm{JL}})$, a similar computation shows that the standard L-function of any irreducible unramified component of $(s\cdot\pi_v')|_{\mathrm{SO}(X_a,F_v)}$ is also $L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$.
The argument in the case $d=1$ is similar to the last case and will be omitted. □

To prove the nonvanishing part of the main result Theorem 8.3 we will use the following theorem, which follows from Corollary 1.2 of [R4]. In the following $L^S(s,\sigma_1)$ is the standard partial L-function of σ_1 (see Section 2 of [KR1]).

8.2 THEOREM ([R4]). *Let F be a totally real number field, and let X be a four dimensional quadratic space over F. Let $d \in F^\times/F^{\times 2}$ be the discriminant of $X(F)$, and assume that the discriminant algebra E of $X(F)$ is totally real, i.e., either $d=1$ or $d\neq 1$ and $E=F(\sqrt{d})$ is totally real. Let σ_1 be a tempered cuspidal automorphic representation of $\mathrm{O}(X,\mathbb{A})$ with $\sigma_1 \cong \otimes_v \sigma_{1,v}$, and let V_{σ_1} be a realization of σ_1 in the space of cusp forms on $\mathrm{O}(X,\mathbb{A})$. Assume $\sigma_{1,v}$*

occurs in the theta correspondence for $\mathrm{O}(X, F_v)$ *and* $\mathrm{Sp}(2, F_v)$ *for all places* v*. If* $L^S(s, \sigma_1)$ *does not vanish at* $s = 1$ *then* $\Theta_2(V_{\sigma_1}) \neq 0$*.*

Proof. This follows from Corollary 1.2 of [R4] (see also the following remark below). Note that by the assumption on E, at each infinite place v of F we have $d = 1$ in $F_v^\times / F_v^{\times 2}$, so that the signature of $X(F_v)$ is $(4, 0), (2, 2)$ or $(0, 4)$ and the signature assumptions from Corollary 1.2 of [R4] are satisfied. $\square$

We take the opportunity here to make a correction to [R4]. Namely, in Theorem 1.1 of [R4] hypothesis (2) should be replaced with the statement: for all places v, σ_v is tempered and if σ_v first occurs in the theta correspondence with $\mathrm{Sp}(n', F_v)$ with $2n' > \dim X$, then the first occurrence of σ_v is tempered; in Corollary 1.2 of [R4] σ_v should also be assumed to be tempered for infinite v; and finally in Lemma 2.1 of [R4] the assumption on σ (in both the nonarchimedean and real cases) should be that σ is tempered, and if σ first occurs in the theta correspondence with $\mathrm{Sp}(n', F)$ with $2n' > \dim X$, then the first occurrence of σ is tempered. The corrections thus also introduce temperedness assumptions at the infinite places entirely analogous to those at the finite places (note that in the corrections to Theorem 1.1 and Lemma 2.1 we have actually weakened the nonarchimedean assumption; this was mentioned in [R4], but not explicitly stated as part of Theorem 1.1 and Lemma 2.1). The omission of these temperedness assumptions at infinity was due to a misreading of [M], Corollaire IV.5 (ii). The only place where the result from [M] is used in [R4] is in the proof of Lemma 2.1 of [R4] where it is asserted that, in the terminology of that lemma, $\theta_{k+1}(\sigma) = L(\chi_X|\cdot|^{s_X(k+1)} \otimes \delta_2 \cdots \otimes \delta_t \otimes \tau)$. The argument for this is as follows. Assume σ first occurs in the theta correspondence with $\mathrm{Sp}(n', \mathbb{R})$ with $n' \leq \dim X/2$. Then σ occurs in the theta correspondence with $\mathrm{Sp}(\dim X/2, \mathbb{R})$ (Lemme I.9, p. 14, [M]) and $\theta_{\dim X/2}(\sigma) = \Psi_{\dim X/2}(\sigma)$ (Théorème IV.3, p. 70, [M]). Since σ is tempered, by the definition of $\Psi_{\dim X/2}(\sigma)$ (III.2, p. 49, [M]), $\theta_{\dim X/2}(\sigma) = \Psi_{\dim X/2}(\sigma)$ is also tempered. The Langlands data for $\theta_{k+1}(\sigma)$ is obtained from the Langlands data of $\theta_{\dim X/2}(\sigma)$ by adjoining the quasi-characters of $\mathbb{R}^\times$: $\chi_X|\cdot|^{s_X(k+1)}, \dots, \chi_X|\cdot|^{s_X(\dim X/2)+2}, \chi_X|\cdot|^{s_X(\dim X/2)+1}$ (Corollaire IV.5 (ii), p. 71, [M]). Since $\theta_{\dim X/2}(\sigma)$ is tempered, this implies $\theta_{k+1}(\sigma)$ has the claimed form. Next, assume σ first occurs in the theta correspondence with $\mathrm{Sp}(n', \mathbb{R})$ with $n' > \dim X/2$. Then $\theta_{n'}(\sigma)$ is tempered by assumption. Again, the Langlands data of $\theta_{k+1}(\sigma)$ is obtained from the Langlands data of $\theta_{n'}(\sigma)$ by adjoining the quasi-characters of $\mathbb{R}^\times$: $\chi_X|\cdot|^{s_X(k+1)}, \dots, \chi_X|\cdot|^{s_X(n')+2}, \chi_X|\cdot|^{s_X(n')+1}$ (Corollaire IV.5 (ii), p. 71, [M]). Again, since $\theta_{n'}(\sigma)$ is tempered, this implies $\theta_{k+1}(\sigma)$ has the claimed form. This completes the corrected argument for the new statement of Lemma 2.1 of [R4]. The corrected statements of Theorem 1.1 and Corollary 1.2 have exactly the same proofs as in [R4].

Proof of Theorem 8.3. (1) $\implies$ (2). Suppose $\Theta_2(V_\sigma) \neq 0$. Suppose $\Theta_2(V_\sigma)$ is contained in the space of cusp forms. Then by Proposition 5.3, (2) holds. Suppose $\Theta_2(V_\sigma)$ is not contained in the space of cusp forms. Since σ_v is infinite dimensional for at least one v (5.1) holds. By Proposition 5.2, $\Theta_1(V_\sigma)$ is contained in the space cusp forms; also, $\Theta_1(V_\sigma)$ is nonzero, for otherwise, by Proposition 5.2, $\Theta_2(V_\sigma)$ would be contained in the space of cusp forms. A standard argument as in the proof of Proposition 5.3 now shows that for all v, $\sigma_v \in \mathcal{R}_1(\mathrm{GO}(X, F_v))$. This implies (2) (Lemma 4.2 of [R1]; b), p. 67 of [MVW]; Lemme I.9 of [M]).

(2) $\iff$ (3). This is Theorem 3.4. Note that if $\operatorname{disc} X(F_v) \neq 1$, then $X(F_v)$ is isotropic, and so σ_v is infinite dimensional (as σ_v is tempered).

(2) $\Longrightarrow$ (1). Suppose (2) holds. Let σ lie over $[\pi]$ (Section 7). Restrict the functions in V_σ to $\mathrm{O}(X, \mathbb{A})$ to obtain the space of functions V_σ^1. Then V_σ^1 is nonzero and contained in the space of cusp forms on $\mathrm{O}(X, \mathbb{A})$; let W be an irreducible nonzero $\mathrm{O}(X, \mathbb{A}_f) \times (\mathfrak{h}_{1,\infty}, J_{1,\infty})$ component of V_σ^1, and denote the isomorphism class of W by σ_1. To show $\Theta_2(V_\sigma) \neq 0$ it will suffice to show $\Theta_2(W) \neq 0$ (for if $\theta_2(f, \varphi) \neq 0$ for some $f \in W$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$, then $\theta_2(F, \varphi)|_{\mathrm{Sp}(2,\mathbb{A})} = \theta_2(f, \varphi) \neq 0$ for any $F \in V_\sigma$ with $F|_{\mathrm{O}(X,\mathbb{A})} = f$). For this, we will use Theorem 8.2. We need to see that the hypotheses of Theorem 8.2 are satisfied. For all places v of F, $\sigma_{1,v}$ is an irreducible constituent of $\sigma_v|_{\mathrm{O}(X,F_v)}$. Since σ_v is tempered for all v, $\sigma_{1,v}$ is tempered for all v. Also, it is a basic consequence of (2) that $\sigma_{1,v} \in \mathcal{R}_2(\mathrm{O}(X, F_v))$ for all v (Lemma 4.2 of [R1]; see the discussion before Theorem 1.8). Finally, we need to see that the partial standard L-function $L^S(s, \sigma_1)$ of σ_1 does not vanish at $s = 1$. Writing $\pi = \pi(\chi, \tau)$, by Lemma 8.1 we have

$$L^S(s, \sigma_1) = \begin{cases} L^S(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai}) \text{ if } d \neq 1 \\ L^S(s, \tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee}) \text{ if } d = 1 \text{ and } \tau \cong \tau_1 \otimes \tau_2. \end{cases}$$

Showing the nonvanishing of $L^S(s, \sigma_1)$ at $s = 1$ is thus reduced to showing the nonvanishing of these two types of L-functions at $s = 1$. For the nonvanishing of $L^S(s, \tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee})$ at $s = 1$ see Theorem 5.2 of [Sh]. The nonvanishing of $L^S(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai})$ at $s = 1$ also follows from [Sh]. For an explanation of this, see p. 296–7 of [F]. Note that $L^S(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai})$ is of the form $L^S(s, \tau', \mathrm{Asai})$: there exists a Hecke character $\hat{\chi}$ of $\mathbb{A}_E^\times$ extending χ, and for such a $\hat{\chi}$ we have $L^S(s, \tau \otimes \hat{\chi}^{-1}, \mathrm{Asai}) = L^S(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai})$. By Theorem 8.2 we now have $\Theta_2(W) \neq 0$, and so $\Theta_2(V_\sigma) \neq 0$.

Now suppose that one of (1), (2) or (3) holds, and $s \cdot \pi \not\cong \pi$. By what we have already shown, $\Theta_2(V_\sigma) \neq 0$. We claim that $\Theta_2(V_\sigma)$ is contained in the space of cusp forms. Suppose not. Then as in the proof of (1) $\Longrightarrow$ (2), $\Theta_1(V_\sigma)$ is nonzero and contained in the space of cusp forms, and in particular $\sigma_v \in \mathcal{R}_1(\mathrm{GO}(X, F_v))$ for all v. By Theorem 7.4 of [R2] this implies $s \cdot \pi_v \cong \pi_v$ at least for all finite v of odd residual characteristic. However, by strong multiplicity one for $\mathrm{GSO}(X, \mathbb{A})$ (Section 6) and $s \cdot \pi \not\cong \pi$, we have $s \cdot \pi_v \not\cong \pi_v$ for infinitely many v, a contradiction. Thus, $\Theta_2(V_\sigma)$ is contained in the space of cusp forms. By Proposition 5.3, $\Theta_2(V_\sigma)$ is a cuspidal automorphic representation of $\mathrm{GSp}(2, \mathbb{A})$ of central character ω_σ and $\Theta_2(V_\sigma) = \otimes_v \theta_2(\sigma_v^\vee)$; by Proposition 1.10 this is also $\otimes_v \theta_2(\sigma_v)^\vee$. The proof that $\theta_2(\sigma_v)$ is tempered for all v is as in the proof of (3) of Proposition 4.1. $\square$

The next lemma was used in the proof of Proposition 4.1 to show that the two elements of an L-packet defined there are in fact distinct. It will also be used in the proof of Theorem 8.5.

8.4 Lemma. *Let v be a place of F. Let D_{ram} be the division quaternion algebra over F_v, and define the four dimensional quadratic spaces $X_{\mathrm{M}_{2\times 2},1}$ and $X_{D_{\mathrm{ram}},1}$ over F_v as in Section 2. Then $\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{GSp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v)) = \emptyset$.*

Proof. We will use the notation of Section 1. By Lemmas 1.4 and 1.5 it will suffice to show that if

$$\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{GSp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v)) \neq \emptyset$$

then

$$\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{Sp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{Sp}(2,F_v)) \neq \emptyset.$$

Suppose $\Pi \in \mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{GSp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v))$. Since Π is contained in $\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{GSp}(2,F_v))$, by definition $\Pi|_{\mathrm{Sp}(2,F_v)}$ is multiplicity free; let $\Pi|_{\mathrm{Sp}(2,F_v)} = W_1 \oplus \cdots \oplus W_M$ with the W_i, $1 \leq i \leq M$, mutually non-isomorphic irreducible $\mathrm{Sp}(2,F_v)$ subspaces of Π. Also by definition, some W_i, say W_1, is in $\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{Sp}(2,F_v))$. We assert that all the W_i are contained in $\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{Sp}(2,F_v))$. Let $g \in \mathrm{GSp}(2,F_v)$ be such that $\pi(g)W_1 = W_i$ (if $F_v \cong \mathbb{R}$ then $M = 1$ or 2 and we may take $g = k_0$ with k_0 as in Section 1). Since $W_1 \in \mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{Sp}(2,F_v))$ there exists a nonzero $\mathrm{Sp}(2,F_v)$ map $t : \omega_{X_{\mathrm{M}_{2\times 2},1}} \to W_1$. Let $h \in \mathrm{GO}(X_{\mathrm{M}_{2\times 2},1},F_v)$ be such that $(g,h) \in R_{X_{\mathrm{M}_{2\times 2},1}}$ (if $F_v \cong \mathbb{R}$ we take $h = j_0$ so that $(g,h) \in L$). Consider the composition

$$\omega_{X_{\mathrm{M}_{2\times 2},1}} \xrightarrow{\omega(g,h)^{-1}} \omega_{X_{\mathrm{M}_{2\times 2},1}} \xrightarrow{t} W_1 \xrightarrow{\pi(g)} W_i.$$

This is a nonzero $\mathrm{Sp}(2,F)$ map. Thus, $W_i \in \mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{Sp}(2,F_v))$. On the other hand, since $\Pi \in \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v))$ we have by definition that some irreducible component of $\Pi|_{\mathrm{Sp}(2,F_v)}$ is contained in $\mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{Sp}(2,F_v))$. We now have $\mathcal{R}_{X_{\mathrm{M}_{2\times 2},1}}(\mathrm{Sp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{Sp}(2,F_v)) \neq \emptyset$ as desired. $\square$

We come now to the definition and analysis of global L-packets for GSp(2). We begin by proving Theorem 8.5, a version of the main result Theorem 8.6. In this version, global L-packets for GSp(2) are associated to elements of $\mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}))$; Theorem 8.6 will follow easily from Theorem 8.5. As in Section 2, let $d \in F^\times/F^{\times 2}$, and let $E = E_d$ be $F(\sqrt{d})$ if $d \neq 1$ and $E = E_d = F \times F$ if $d = 1$. Assume E is totally real, i.e., in the case $d \neq 1$

assume E is totally real. Let $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}))$. The packet of irreducible admissible representations of $\mathrm{GSp}(2,\mathbb{A})$ corresponding to $[\pi]$ is defined to be

$$\Pi([\pi]) = \{\mathit{\Pi} = \otimes_v \mathit{\Pi}_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSp}(2,\mathbb{A})) : \mathit{\Pi}_v \in \Pi([\pi_v]) \text{ for all } v\}.$$

Here, $\Pi([\pi_v])$ is defined in Section 4. By Proposition 4.3, for almost all nonarchimedean v, $\Pi([\pi_v])$ consists of a single representation unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$. Thus,

$$\Pi([\pi]) = \otimes_v \Pi([\pi_v]).$$

Also, by (3) of Proposition 4.1, $\Pi([\pi_v])$ consists of tempered representations for all v. If S is any finite set of places such that for $v \notin S$, v is nonarchimedean and $\Pi([\pi_v])$ consists of a single representation unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$, then the cardinality of $\Pi([\pi])$ is:

$$|\Pi([\pi])| = \prod_{v\in S} |\Pi([\pi_v])| = 2^M, \text{ where } M = \sum_{v\in S}(|\Pi([\pi_v])| - 1).$$

For $\mathit{\Pi} = \otimes_v \mathit{\Pi}_v \in \Pi([\pi])$, let $T_{\mathit{\Pi}}$ be the set of places v of F such that v splits in E (as usual, if $d = 1$ so that $E = F \times F$ we say that every place of F splits in E) and $\mathit{\Pi}_v$ is of the form $\theta_{D_{\mathrm{ram}},1}(\pi_v^{\mathrm{JL}+})^\vee$ (so necessarily π_v is square integrable); see Section 4.

8.5 THEOREM. *Assume F is totally real, $d \in F^\times/F^{\times 2}$, and let $E = E_d$ be $F(\sqrt{d})$ if $d \neq 1$ and $F \times F$ if $d = 1$. Assume E is totally real, i.e., in the case $d \neq 1$ assume E is totally real. Let $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}))$ and assume $s \cdot \pi \not\cong \pi$.*

(1) *If $d \neq 1$, then all the elements of $\Pi([\pi])$ occur with multiplicity one in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ with central character ω_π.*
(2) *Assume $d = 1$. Let $\mathit{\Pi} \in \Pi([\pi])$. If $|T_{\mathit{\Pi}}|$ is even, then $\mathit{\Pi}$ occurs with multiplicity one in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ with central character ω_π. Conversely, if $\mathit{\Pi}$ occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ then $|T_{\mathit{\Pi}}|$ is even.*

Proof. Let $\mathit{\Pi} \in \Pi([\pi])$; if $d = 1$ assume $|T_{\mathit{\Pi}}|$ is even. We begin by showing that $\mathit{\Pi}$ occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π. To prove this we will construct a four dimensional quadratic space X over F and a $\sigma \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$ such that $\sigma_v \in \mathcal{R}_2(\mathrm{GO}(X,F_v))$ and $\theta_2(\sigma_v)^\vee = \mathit{\Pi}_v$ for all v; we will then apply Theorem 8.3 to show $\mathit{\Pi}$ is cuspidal automorphic. To start, let us set up some definitions involving π. As in Section 6, write $\pi = \pi(\chi,\tau)$; however instead of the abstract exact sequence of Theorem 2.3, let us use the concrete exact sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times \mathrm{GL}(2,\mathbb{A}_E) \xrightarrow{\rho_{a(\mathrm{M}_{2\times 2},E)}} \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}) \to 1$$

of Proposition 2.7; by this proposition there is no real distinction. Thus, $\pi = \pi(\chi,\tau)$, with $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2,\mathbb{A}_E))$ and χ a Hecke character of $\mathbb{A}^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}^E_F$. The X we will use will be of the form $X_{D,d}$. Specifically, let D be any quaternion algebra over F which is ramified at the places in T_Π, and which is unramified at any other of the places of F which split in E; note that if $d=1$, we use the evenness of $|T_\Pi|$ for the existence of D (again, our convention is that if $d=1$ so that $E = F \times F$ then every place of F is split in E). Evidently, if $d=1$, then D is uniquely determined, but if $d \neq 1$, then there will be infinitely many choices for D. Nevertheless, if we let $B = E \otimes_F D$, then B is uniquely determined (in the case $d \neq 1$, regarded as a quaternion algebra over E, B is split at any place of E lying over a nonsplit place of F), and $X_{D,d}$ is uniquely determined up to similitudes by Proposition 2.9. By Lemma 5.1, it thus follows that our construction will realize Π as a cuspidal automorphic representation in exactly one way in spite of the ambiguity in the choice of D when $d \neq 1$. To define the σ mentioned above, note that again by Proposition 2.7 we have an exact sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho_{a(D,E)}} \mathrm{GSO}(X_{D,d},\mathbb{A}) \to 1.$$

By the definition of T_Π, τ is in the image of the Jacquet-Langlands correspondence from $B^\times(\mathbb{A})$ discussed in Section 6; let $\tau^{\mathrm{JL}} \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ correspond to τ. Let $\pi' = \pi(\chi,\tau^{\mathrm{JL}})$; this is contained in $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{D,d},\mathbb{A}))$. We claim that for each place v there exists $\sigma_v \in \mathcal{R}_2(\mathrm{GO}(X_{D,d},F_v))$ such that

$$\sigma_v \hookrightarrow \mathrm{Ind}^{\mathrm{GO}(X_{D,d},F_v)}_{\mathrm{GSO}(X_{D,d},F_v)} \pi'_v$$

and $\theta(\sigma_v)^\vee = \Pi_v$. This is clear from the definition of $\Pi([\pi_v])$ and D if v is not a nonsplit place with $D(F_v)$ ramified; assume we are in this last case. Let w be the place of E lying over v. By Proposition 2.9 and the consideration of examples after this proposition, there exists an isomorphism $i : B(F_v) \xrightarrow{\sim} \mathrm{M}_{2\times 2}(E_w)$ of E_w algebras and a similitude $T : X_{D,d}(F_v) \to X_{\mathrm{M}_{2\times 2},d}(F_v)$ such that

$$\begin{tikzcd}
1 \arrow[r] & E_w^\times \arrow[r] \arrow[d, "\mathrm{id}"] & F_v^\times \times B^\times(F_v) \arrow[r] \arrow[d, "\mathrm{id}\times i"] & \mathrm{GSO}(X_{D,d},F_v) \arrow[r] \arrow[d, "T\cdot T^{-1}=j"] & 1 \\
1 \arrow[r] & E_w^\times \arrow[r] & F_v^\times \times \mathrm{GL}(2,E_w) \arrow[r] & \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F_v) \arrow[r] & 1
\end{tikzcd}$$

commutes. By the definition of $\Pi([\pi_v])$, there exists $\hat{\pi}_v \in \mathrm{Irr}(\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d},F_v))$ such that $\pi_v \hookrightarrow \hat{\pi}_v|_{\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F_v)}$ and $\theta(\hat{\pi}_v) = \Pi_v^\vee$, i.e.,

$$\mathrm{Hom}_{R_{X_{\mathrm{M}_{2\times 2},d}(F_v)}}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_v^\vee \otimes \hat{\pi}_v) \neq 0.$$

By Lemma 1.6, we obtain

$$\mathrm{Hom}_{R_{X_{D,d}(F_v)}}(\omega_{X_{D,d}(F_v)}, \Pi_v^\vee \otimes \sigma_v) \neq 0,$$

where $\sigma_v = \hat{\pi}_v \circ j$, so that $\theta(\sigma_v) = \Pi_v^\vee$. Since $\pi_v \circ j \hookrightarrow \sigma_v|_{\mathrm{GSO}(X_{D,d},F_v)}$ and $\pi_v \circ j = \pi'_v$ by the commutativity of the diagram, we get $\pi'_v \hookrightarrow \sigma_v|_{\mathrm{GSO}(X_{D,d},F_v)}$ as desired. Now Π_v is unramified for almost all finite v, and so by Proposition 1.11 σ_v is unramified for almost all finite v. We may form the restricted direct product $\sigma = \otimes_v \sigma_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X,\mathbb{A}))$. Since $s \cdot \pi \not\cong \pi$ we have $s \cdot \pi' \not\cong \pi'$. By Theorem 7.1 it follows that $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GO}(X,\mathbb{A}))$, and σ lies over $[\pi']$. By Theorem 8.3, $\Theta_2(V_\sigma)$ is cuspidal and $\Theta_2(V_\sigma) \cong \Pi$.

Having shown that Π occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π, we will now show that the multiplicity with which Π occurs is one. Our strategy will be to use the multiplicity preservation principle of Rallis (Proposition 5.3) along with the fact that for a four dimensional quadratic space X over F, $\mathrm{GO}(X,\mathbb{A})$ has the (weak) multiplicity one property (Theorem 7.1). Let W be the $\mathrm{GSp}(2,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π generated by the subspaces isomorphic to Π. Let U be an irreducible nonzero $\mathrm{GSp}(2,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of W. Then $U \cong \Pi$. To be in a position to apply Proposition 5.3 we must show that $\Theta_{X_{D,d}}(U)$ is nonzero and contained in the space of cusp forms on $\mathrm{GO}(X,\mathbb{A})$ of central character ω_π.

As a first step, we will prove that $\Theta_{X_{D',d}}(U)$ is nonzero and cuspidal for some quaternion algebra D' over F. In the following argument showing that $\Theta_{X_{D',d}}(U)$ is nonzero and cuspidal for some D' we ask the reader to take note that we only use that $\Pi \in \Pi([\pi])$; this will be germane in a subsequent part of the proof. We begin with a reduction to isometries. Restrict the functions in U to $\mathrm{Sp}(2,\mathbb{A})$. This space of restricted functions is nonzero and is an $\mathrm{Sp}(2,\mathbb{A}_f) \times (\mathfrak{g}_{1,\infty}, K_{1,\infty})$ subspace of the space of cusp forms on $\mathrm{Sp}(2,\mathbb{A})$; let U_1 be a nonzero $\mathrm{Sp}(2,\mathbb{A}_f) \times (\mathfrak{g}_{1,\infty}, K_{1,\infty})$ irreducible subspace of this space, and let Π_1 be the isomorphism class of U_1. As in the proof of (2) $\Longrightarrow$ (1) of Theorem 8.3, to show $\Theta_{X_{D',d}}(U) \neq 0$ for some D' it will suffice to show $\Theta_{X_{D',d}}(U_1) \neq 0$ for some D'. To prove this, we will use Theorem 7.1 of [KRS]. This application requires an understanding the behavior of the partial twisted standard L-function $L^S(s, \Pi_1, \chi_{X_{D,d}})$ at $s = 1$; we now compute this L-function. As $U \cong \Pi$, $\Pi_{1,v}$ is an irreducible component of $\Pi_v|_{\mathrm{Sp}(n,F_v)}$ for all v. Let S be a finite set of places of F such that for $v \notin S$, v is finite, $X_{\mathrm{M}_{2\times2},d}(F_v)$ is unramified (i.e., v is odd and v is unramified in E_d) and χ_v and τ_w for $w|v$ are unramified. For $v \notin S$, by Proposition 4.3 and its proof, $|\Pi([\pi_v])| = 1$, Π_v is the single element of $\Pi([\pi_v])$, Π_v is unramified and $\Pi_v = \theta_{\mathrm{M}_{2\times2},d}(\sigma'_v)^\vee = \theta_{\mathrm{M}_{2\times2},d}(\sigma'^\vee_v)$, with $\sigma'_v = \pi_v^+ \in \mathrm{Irr}(\mathrm{GO}(X_{\mathrm{M}_{2\times2},d}, F_v))$ unramified. Let $v \notin S$; we assert that there exists an unramified component $\sigma'_{1,v}$ of $\sigma'_v|_{\mathrm{O}(X_{\mathrm{M}_{2\times2},d},F_v)}$ such that $\Pi_{1,v} = \theta(\sigma'^\vee_{1,v})$. To see this let, as in Section 1, $\mathrm{GSp}(2,F_v)^+$ be the subgroup of $g \in \mathrm{GSp}(2,F_v)$ such that $\lambda(g) \in \lambda(\mathrm{GO}(X_{D,d},F_v))$; again, $\mathrm{GSp}(2,F_v)^+$ has index one or two in $\mathrm{GSp}(2,F_v)$. Let $\Pi_v|_{\mathrm{GSp}(2,F_v)^+} = \Pi_v^1 \oplus \cdots \oplus \Pi_v^M$, where the $\Pi_v^i \in \mathrm{Irr}(\mathrm{GSp}(2,F_v)^+)$ are mutually nonisomorphic and $M = 1$ or 2. We have by construction

$$\mathrm{Hom}_{R_{X_{\mathrm{M}_{2\times2},d}(F_v)}}(\omega_{X_{\mathrm{M}_{2\times2},d}(F_v)}, \Pi_v \otimes \sigma'^\vee_v) \neq 0.$$

This implies that for some i,

$$\mathrm{Hom}_{R_{X_{\mathrm{M}_{2\times 2},d}(F_v)}}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_v^i \otimes {\sigma_v'}^\vee) \neq 0.$$

By the proof of Proposition 1.11, Π_v^i is unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$ (which is contained in $\mathrm{GSp}(2,F_v)^+$). As $\Pi_v|_{\mathrm{Sp}(2,F_v)}$ has only one irreducible component unramified with respect to $\mathrm{Sp}(2,\mathfrak{O}_v)$, namely $\Pi_{1,v}$, it follows that $\Pi_{1,v}$ is an irreducible component of $\Pi_v^i|_{\mathrm{Sp}(2,F_v)}$. By Lemma 4.2 of [R1], there exists an irreducible component $\sigma_{1,v}'$ of σ_v' such that

$$\mathrm{Hom}_{\mathrm{Sp}(2,F_v)\times \mathrm{O}(X_{\mathrm{M}_{2\times 2},d},F_v)}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_{1,v} \otimes {\sigma_{1,v}'}^\vee) \neq 0.$$

By (b) of Theorem 7.1 of [H], $\sigma_{1,v}'$ is unramified. This proves our assertion. By Section 7 of [KR2] and Lemma 8.1 (or rather its proof), the twisted partial standard L-function of Π_1 now is

$$\begin{aligned} L^S(s,\Pi_1,\chi_{X_{D,d}}) &= \zeta_F^S(s)\prod_{v\notin S} L(s,\sigma_{1,v}') \\ &= \begin{cases} \zeta_F^S(s)L^S(s,\tau,\chi^{-1},\mathrm{Asai}) \text{ if } d\neq 1 \\ \zeta_F^S(s)L^S(s,\tau_1\times{\tau_2}^\vee) \text{ if } d=1 \text{ and } \tau\cong\tau_1\otimes\tau_2, \end{cases} \end{aligned}$$

where $\zeta_F^S(s)$ is the partial zeta function of F. We noted in the proof of Theorem 8.3 that L-functions of the type $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ or $L^S(s,\tau_1\times\tau_2^\vee)$ do not vanish at $s=1$; hence, $L^S(s,\Pi_1,\chi_{X_{D,d}})$ has a pole at $s=1$ (in fact, by Corollary 7.2.3 of [KR2] the pole must be simple).

Now we apply [KRS]. By Lemma 1.1 of [L], for some $f\in U_1$, f has a nonzero T-th Fourier coefficient with $\det T\neq 0$. Here, $T\in \mathrm{M}_2(F)$ is a symmetric matrix. Define a quadratic Hecke character χ of $\mathbb{A}^\times$ by $\chi_{X_{D,d}}=\chi_T\chi$, where we also write T for the two dimensional quadratic space defined by T. Since $L^S(s,\Pi_1,\chi_T\chi) = L^S(s,\Pi_1,\chi_{X_{D,d}})$ has a pole at $s=1$, by (i) and (ii) of Theorem 7.1 of [KRS], $\Theta_{X'}(U_1)\neq 0$, where $X'=X_T\perp X''$, with X'' some two dimensional quadratic space over F such that $\chi_{X''}=\chi$. We have

$$\chi_{X'}=\chi_{X_T}\cdot\chi_{X''}=\chi_{X_T}\cdot\chi=\chi_{X_T}^2\cdot\chi_{X_{D,d}}=\chi_{X_{D,d}}$$

which implies $\operatorname{disc} X'(F)=\operatorname{disc} X_{D,d}(F)=d$. By Proposition 2.8 and Lemma 5.1, we now know that $\Theta_{X_{D',d}}(U_1)\neq 0$ for some quaternion algebra D' over F. As mentioned, this implies $\Theta_{X_{D',d}}(U)\neq 0$.

Next, we claim $\Theta_{X_{D',d}}(U)$ is contained in the space of cusp forms on $\mathrm{GO}(X_{D',d},\mathbb{A})$ of central character ω_π; suppose not. Then by the remark after Proposition 5.2 there exists a two dimensional quadratic space X_0 over F such that $\Theta_{X_0}(U)$ is nonzero and is contained in the space of cusp forms of central character ω_π on $\mathrm{GO}(X_0,\mathbb{A})$. By a standard argument as in the proof of Proposition 5.3, for all but finitely many places v of F, $X_0(F_v)$ is unramified, Π_v is unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$, there exists a unitary $\rho_v \in \mathrm{Irr}(\mathrm{GO}(X_0,F_v))$ which is unramified with respect to the stabilizer in $\mathrm{GO}(X_0,F_v)$ of a self-dual lattice and

$$\mathrm{Hom}_{R_{X_0}(F_v)}(\omega_{X_0(F_v)}, \Pi_v^\vee \otimes \rho_v) \neq 0.$$

Let v be one such place. Let ρ_0 be an irreducible unramified component of $\rho_v|_{\mathrm{O}(X_0,F_v)}$. By Lemma 4.2 of [R1], there exists an irreducible component Π_0 of $\Pi_v^\vee|_{\mathrm{Sp}(2,F_v)}$ such that

$$\mathrm{Hom}_{\mathrm{Sp}(2,F_v)\times\mathrm{O}(X_0,F_v)}(\omega_{X_0(F_v)}, \Pi_0 \otimes \rho_0) \neq 0.$$

Now $\mathrm{SO}(X_0,F_v)$ is Abelian as $\dim X_0 = 2$; since ρ_0 is unitary, ρ_0 is therefore tempered. (Recall the definition of a tempered representation of $\mathrm{O}(X_0,F_v)$ preceding Theorem 1.2). Also, it is not difficult to show that $\rho_0 \in \mathcal{R}_1(\mathrm{O}(X_0,F_v))$ (in fact, the only element of $\mathrm{Irr}(\mathrm{O}(X_0,F_v))$ not contained in $\mathcal{R}_1(\mathrm{O}(X_0,F_v))$ is sign). Applying now Theorem 4.4 of [R3], we conclude that Π_0 is not tempered, contradicting the temperedness of Π_v (see (3) of Proposition 4.1). We have shown $\Theta_{X_{D',d}}(U) \neq 0$ is nonzero and cuspidal for some D'; as promised, the argument used only that the cuspidal automorphic representation Π is contained in $\Pi([\pi])$.

Now we will show that $\Theta_{X_{D,d}}(U)$ is nonzero and contained in the space of cusp forms of central character ω_π. By Lemma 5.1, it will suffice to show that there is a similitude between $X_{D,d}(F)$ and $X_{D',d}(F)$. Let $B' = E \otimes_F D'$. We assert that $B \cong B'$ as E algebras. As in the last paragraph of Section 2, let $S_{D,E}$ be the set of places v of F such that v splits in E and $D(F_v)$ is ramified; define $S_{D',E}$ similarly. As observed in Section 2, it will suffice to show that $S_{D,E} = S_{D',E}$. Let v be a place of F that splits in E. As v splits in E, $d = 1$ in $F_v^\times/F_v^{\times 2}$. By Proposition 5.3, since $\Theta_{X_{D',d}}(U)$ is nonzero and cuspidal, $\Pi_v \in \mathcal{R}_{X_{D',d}(F_v)}(\mathrm{GSp}(2,F_v))$; by construction, $\Pi_v \in \mathcal{R}_{X_{D,d}(F_v)}(\mathrm{GSp}(2,F_v))$. By Lemma 8.4 we must have $X_{D',d}(F_v) \cong X_{D,d}(F_v)$. This implies $D'(F_v) \cong D(F_v)$ so that D is ramified at v if and only if D' is ramified at v. This proves $S_{D,E} = S_{D',E}$. Since $B \cong B'$ as E algebras, by Proposition 2.9 there exists a similitude between $X_{D,d}(F)$ and $X_{D',d}(F)$.

We now apply Proposition 5.3 to conclude that the multiplicity of Π in W is the same as the multiplicity of $\Theta_{X_{D,d}}(U)$ in the space of cusp forms on $\mathrm{GO}(X_{D,d},\mathbb{A})$ of central character ω_π. By part of Theorem 7.1, this multiplicity is one.

To complete the proof we still must show that if $d = 1$, $\Pi \in \Pi([\pi])$ and Π occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$, then $|T_\Pi|$ is even. Let U be a realization of Π in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π. An argument just as above (which just used $\Pi \in \Pi([\pi])$ and nothing about the parity of $|T_\Pi|$) shows that $\Theta_{X_{D',1}}(U)$ is nonzero and cuspidal for some quaternion algebra D' over F. We claim that T_Π is exactly the set of places where D' is ramified; this will show that $|T_\Pi|$ is even. Let $v \in T_\Pi$. Then by the definition of T_Π, $\Pi_v \in \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v))$. On the other hand, since $\Theta_{X_{D',1}}(U)$

is nonzero and cuspidal, $\Pi_v \in \mathcal{R}_{X_{D'(F_v),1}(F_v)}(\mathrm{GSp}(2,F_v))$ (Proposition 5.3). By Lemma 8.4, $X_{D_{\mathrm{ram}},1}(F_v) \cong X_{D'(F_v),1}(F_v)$, which implies $D'(F_v)$ is ramified. Suppose next $D'(F_v)$ is ramified. Again, $\Pi_v \in \mathcal{R}_{X_{D'(F_v),1}(F_v)}(\mathrm{GSp}(2,F_v))$. By Lemma 8.4 and the definition of $\Pi([\pi_v])$, we must have $v \in T_\Pi$. $\square$

Finally, we prove Theorem 8.6. This result is essentially a restatement of Theorem 8.5, and will follow immediately from that theorem after we make some definitions.

First we make the definitions mentioned preceding the statement of Theorem 8.6 in the Introduction. Let F' be a local field of characteristic zero, and let E' be a quadratic extension of F' or $E' = F' \times F'$; if F' is archimedean, assume $F' = \mathbb{R}$ and $E' = \mathbb{R}\times\mathbb{R}$. If E' is a field, write $E' = F'(\sqrt{d})$; otherwise, let $d = 1$. Let $\tau' \in \mathrm{Irr}(\mathrm{GL}(2,E'))$ be infinite dimensional and assume the central character of τ' factors through $\mathrm{N}_{F'}^{E'}$ via χ'; if F' has even residual characteristic, assume additionally that τ' is tempered. By Proposition 2.7 the following sequence is exact:

$$1 \to E'^\times \to F'^\times \times \mathrm{GL}(2,E') \xrightarrow{\rho_{a(\mathrm{M}_{2\times 2},E')}} \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F') \to 1.$$

Using this exact sequence, define $\pi' = \pi(\chi',\tau') \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F'))$ as in Section 3. Define $\varphi(\chi',\tau') = \varphi([\pi'])$ and $\Pi(\chi',\tau') = \Pi([\pi'])$, where $\varphi([\pi'])$ and $\Pi([\pi'])$ are defined as in Section 4. If $E' = F' \times F'$, define

$$\langle\cdot,\cdot\rangle_{F'} : \mathbb{S}(\varphi(\chi',\tau')) \times \Pi(\chi',\tau') \to \mathbb{C}$$

as follows. If $|\mathbb{S}(\varphi(\chi',\tau'))| = |\Pi(\chi',\tau')| = 1$ set $\langle\cdot,\cdot\rangle_{F'}$ to be identically 1; if $|\mathbb{S}(\varphi(\chi',\tau'))| = |\Pi(\chi',\tau')| = 2$ (see Propositions 4.1 and 4.2) then define $\langle\cdot,\theta_{\mathrm{M}_{2\times 2},1}(\pi'^{+})^\vee\rangle_{F'} = 1$ and let $\langle\cdot,\theta_{D_{\mathrm{ram}},1}(\pi'^{\mathrm{JL}+})^\vee\rangle_{F'}$ to be the nontrivial character of $\mathbb{S}(\varphi(\chi',\tau')) = Z_2$ (see Table 4). The claims from the Introduction concerning these definitions follow from Propositions 4.1, 4.2 and 4.3.

Next, let E, τ and χ be as in the statement of Theorem 8.6. If E is a field, write $E = F(\sqrt{d})$; otherwise, let $d = 1$. By Proposition 2.7 the following sequence is exact:

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times \mathrm{GL}(2,\mathbb{A}_E) \xrightarrow{\rho_{a(\mathrm{M}_{2\times 2},E)}} \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}) \to 1.$$

Using this exact sequence, define $\pi = \pi(\chi,\tau) \in \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}))$ as in Section 6.

Proof of Theorem 8.6. This follows from the definitions involved and Theorem 8.5. $\square$

APPENDIX

	$p = q$	$p \neq q$
$\lambda(\mathrm{GO}(X,\mathbb{R}))$	$\mathbb{R}^\times$	$\mathbb{R}^\times_{>0}$
$[\mathrm{GSp}(n,\mathbb{R}) : \mathrm{GSp}(n,\mathbb{R})^+]$	1	2
$\mathrm{GSp}(n,\mathbb{R})^+$	$\mathrm{GSp}(n,\mathbb{R}) =$ $\mathbb{R}^\times(\mathrm{Sp}(n,\mathbb{R}) \cup \mathrm{Sp}(n,\mathbb{R})k_0)$	$\mathrm{Sp}(n,\mathbb{R})\mathbb{R}^\times$
K^+	$K = K_1 \cup K_1 k_0$	K_1
$\mathrm{GO}(X,\mathbb{R})$	$\mathbb{R}^\times(\mathrm{O}(X,\mathbb{R}) \cup \mathrm{O}(X,\mathbb{R})j_0)$	$\mathrm{O}(X,\mathbb{R})\mathbb{R}^\times$
J	$J_1 \cup J_1 j_0$	J_1
L	$(K_1 \times J_1) \cup (K_1 \times J_1)(k_0, j_0)$	$K_1 \times J_1$

TABLE 1

<table>
<tr><td colspan="2">$d \neq 1$</td></tr>
<tr><td rowspan="2">π regular or distinguished</td><td>π regular: $\pi \rightarrow \pi^+ \rightarrow \theta_{\mathrm{M}_{2\times2},d}(\pi^+)$</td></tr>
<tr><td>π distinguished: $\begin{array}{lll} & \nearrow & \pi^+ \rightarrow \theta_{\mathrm{M}_{2\times2},d}(\pi^+) \\ \pi & & \\ & \searrow & \pi^- \quad \text{does not lift to } \mathrm{GSp}(2,F) \end{array}$</td></tr>
<tr><td>π invariant but not distinguished</td><td>$\begin{array}{lll} & \nearrow & \pi_1 \rightarrow \theta_{\mathrm{M}_{2\times2},d}(\pi_1) \\ \pi & & \\ & \searrow & \pi_2 \rightarrow \theta_{\mathrm{M}_{2\times2},d}(\pi_2) \end{array}$</td></tr>
</table>

TABLE 2

d=1	
π not essentially square integrable	π regular: $\pi \to \pi^+ \to \theta_{\mathrm{M}_{2\times 2},1}(\pi^+)$
	π invariant and hence distinguished: $\begin{tikzcd} & \pi^+ \arrow[r] & \theta_{\mathrm{M}_{2\times 2},1}(\pi^+) \\ \pi \arrow[ur] \arrow[dr] & & \\ & \pi^- & \text{does not lift to GSp}(2,F) \end{tikzcd}$
π essentially square integrable	π regular: $\begin{tikzcd} \pi \arrow[r] \arrow[d] & \pi^+ \arrow[r] & \theta_{\mathrm{M}_{2\times 2},1}(\pi^+) \\ \pi^{\mathrm{JL}} \arrow[r] & \pi^{\mathrm{JL}+} \arrow[r] & \theta_{D_{\mathrm{ram}},1}(\pi^{\mathrm{JL}+}) \end{tikzcd}$
	π invariant and hence distinguished: $\begin{tikzcd} & \pi^+ \arrow[r] & \theta_{\mathrm{M}_{2\times 2},1}(\pi^+) \\ \pi \arrow[ur] \arrow[r] \arrow[d] & \pi^- & \text{does not lift to GSp}(2,F) \\ \pi^{\mathrm{JL}} \arrow[r] \arrow[dr] & \pi^{\mathrm{JL}+} \arrow[r] & \theta_{D_{\mathrm{ram}},1}(\pi^{\mathrm{JL}+}) \\ & \pi^{\mathrm{JL}-} & \text{does not lift to GSp}(2,F) \end{tikzcd}$

TABLE 3

d	$[\pi]$	$\Pi([\pi])$
1	π not essentially square integrable	$\{\theta_{\mathrm{M}_{2\times 2},1}(\pi^+)^\vee\}$
1	π essentially square integrable	$\{\theta_{\mathrm{M}_{2\times 2},1}(\pi^+)^\vee, \theta_{D_{\mathrm{ram}},1}(\pi^{\mathrm{JL}+})^\vee\}$
$\neq$1	π regular or invariant and distinguished	$\{\theta_{\mathrm{M}_{2\times 2},d}(\pi^+)^\vee\}$
$\neq$1	π invariant but not distinguished	$\{\theta_{\mathrm{M}_{2\times 2},d}(\pi_1)^\vee, \theta_{\mathrm{M}_{2\times 2},d}(\pi_2)^\vee\}$

TABLE 4

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