Pseudokähler Forms on Complex Lie Groups

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Abstract. Let G be a semisimple complex group with real form $G_{\mathbb{R}}$. We define and study a pseudokähler form that is defined on a neighbhorhood of the identity in G and is invariant under left and right translation by $G_{\mathbb{R}}$.

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Summary

Let G be a complex semisimple algebraic group with real form $G_{\mathbb{R}}$, the fixed-point subgroup of an antiholomorphic involution $g \mapsto \overline{g}$. The group $G_{\mathbb{R}} \times G_{\mathbb{R}}$ acts of G by the rule ${}^{(r_1,r_2)}g = r_1 g r_2^{-1}$. In this paper, we give a construction of a $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant pseudokähler form on a neighborhood of $G_{\mathbb{R}}$ in G. We expect this result will find application in several related areas in complex geometry and representation theory. For example, future work of others will show that symplectic reduction (with respect to the imaginary part of the pseudokähler form) relates this open set in G with a neighborhood of a noncompact Riemannian symmetric space in its complexification, as studied by Akhiezer and Gindikin [AG].

As a first guess, one might attempt to construct such a pseudokähler form as follows: given left-invariant vector fields Z, W on G, define the Hermitian product of Z and W to be $\kappa(Z, \overline{W})$, where κ is the Killing form. However, this fails, since the corresponding 2-form (the imaginary part of the Hermitian form) is not closed. Instead, we take the following approach. We construct a pseudokähler form on a complex manifold $M \subset i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$. We then define $M' \subset M$ such that $f|_{M'} : M' \to G$ is a diffeomorphism onto an open subset of G. Here $f : i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}} \to G$ is the map $(iX, r) \overset{f}{\mapsto} e^{iX} \cdot r$. This allows us to push down the pseudokähler form on M' to the open set $f(M') \subset G$. The new form turns out to be closely related to the form $Z, W \mapsto \kappa(Z, \overline{W})$.

Our main results are as follows:

Let $G_{\mathbb{R}} \times G_{\mathbb{R}}$ act on $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ by the rule ${}^{(r_1,r_2)}(iX, r) = (i\mathrm{Ad}_{r_1} X, r_1 r r_2^{-1})$; then f equivariant. Define $M := \{(iX, r) : df \text{ is nonsingular at } (iX, r)\} \subset i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$. This makes M a complex manifold, with complex structure J induced from the complex structure on G. A useful description of M is:

Theorem 1. *$(iX, r) \notin M$ if and only if ad_X has an eigenvalue of $n\pi$ for some nonzero integer n. Equivalently, for $p := e^{iX}$, $(iX, r) \notin M$ exactly when either Ad_p has an eigenvalue of -1 or Ad_p fixes a vector in $\mathfrak{g}$ not fixed by ad_X. (Proof in §2.)* $\square$

Regard $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ as the cotangent bundle of $G_{\mathbb{R}}$. As such, there is a canonical real 1-form λ on $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ such that $\omega := d\lambda$ is an (exact) nondegenerate symplectic form. On the other hand, let $\phi : i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}} \to \mathbb{R}$, $(iX, r) \mapsto \kappa(X, X)$, which is a $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant function. These objects are related:

Theorem 2. *On the complex manifold $M \subset i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$, we have $2\lambda = d^c\phi$. (Proof in §4.)* $\square$

As an immediate corollary, we have:

Theorem 3. *$\omega = d\lambda = \frac{1}{2} dd^c\phi = -i\partial\overline{\partial}\phi$ is a $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant, J-invariant, nondegenerate, exact, real 2-form on M, and*

$$\langle A, B \rangle := \omega(JA, B) + i\omega(A, B) \qquad (A, B \in T_m M)$$

is a $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant pseudokähler form on M. $\square$

We seek to compute the pseudokähler form in terms of a reasonable collection of vector fields on M. Let $Z \in \mathfrak{g}$, that is to say, a tangent vector to G at the identity 1. As usual, we may identify Z with a left G-invariant vector field on G. Let $\widehat{Z}$ denote the vector field on M obtained by pulling back the left G-invariant vector field Z on G via the map f. These vector fields, which we call *canonical vector fields*, are the ones we shall use throughout for computations. We also need to define several linear transformations on $\mathfrak{g}$. Let $(iX, r) \in M$ and write $p := e^{iX}$. First, we define $A_p : \mathfrak{g} \to \mathfrak{g}$, $A_p := \left(\frac{I + \mathrm{Ad}_p}{2}\right)^{-1}$. (This makes sense, by Theorem 1.) We also define $F_{iX} : \mathfrak{g} \to \mathfrak{g}$ by $F_{iX}(Z) := \frac{d}{ds}\big|_{s=0} \log(pe^{sZ})$. (Here log denotes a local inverse for exp, returning a neighborhood of p in G to a neighborhood of iX in $\mathfrak{g}$.) Finally, define $E_{iX} := F_{iX} \circ A_p \circ \mathrm{Ad}_p$. Our result is:

Theorem 4. *For $Z, W \in \mathfrak{g}$ and $(iX, r) \in M$, we have that $\left\langle \widehat{Z}, \widehat{W} \right\rangle_{(iX,r)} = \kappa(E_{i\mathrm{Ad}_r^{-1} X} Z, \overline{W})$. (Proof in §5.)* $\square$

Theorem 4 is useful since E_{iX} is easy to understand: if ad_X is diagonalizable, then E_{iX} is also diagonalizable, has the same eigenspaces as ad_X, and its eigenvalues can be expressed in terms of the corresponding eigenvalues of ad_X. In particular, if X lies in a Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$, then one has a simple expression for $\langle\ ,\ \rangle$ at $(iX, 1)$ when expressed using canonical vector fields corresponding to elements of $\mathfrak{g}$ that are root vectors or vectors in $\mathfrak{t}$. We refer the reader to §6 for the precise statement. Additionally,

Theorem 5. *The signature of the pseudokähler form is constant on M, and is equal to the signature of the Hermitian form $Z, W \mapsto \kappa(Z, \overline{W})$ on $\mathfrak{g}$. (Proof in §6.)* $\square$

Trivially, if $M' \subset M$ is open and $G_{\mathbb{R}} \times G_{\mathbb{R}}$-stable, and if $f|_{M'}$ is injective, then the pseudokähler form on M' pushes down to a $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant pseudokähler form on $f(M')$, which is open in G. We produce such a set M':

Theorem 6. *Let $\psi : G \to GL(V)$, $\mathfrak{g} \to gl(V)$ be a finite-dimensional representation that is defined over $\mathbb{R}$ and is faithful modulo the center of G (e.g. the adjoint representation). Define $M' \subset i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$, where $(iX, r) \in M'$ if and only if for each eigenvalue λ of $\psi(X)$, $|\mathrm{Re}\,\lambda| < \pi/2$. Then*

1. *$M' \subset M$,*
2. *M' is $G_{\mathbb{R}} \times G_{\mathbb{R}}$-stable,*
3. *M' is open in $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$,*
4. *$f|_{M'}$ is injective. (Proof in §8.)* $\square$

In some applications, it is more convenient to replace the above canonical vector fields with tangent vectors that are either tangent ("orbital vectors") or transverse ("vertical vectors") to the $G_{\mathbb{R}} \times G_{\mathbb{R}}$-orbits. We set up the notation and compute the pseudokähler form using these vectors (§7). The imaginary part of the pseudokähler form is particularly easy, and from it, one easily computes the moment map:

Theorem 7. *Relative to the symplectic form ω, the moment map $\mu : i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}^* \times \mathfrak{g}_{\mathbb{R}}^* \simeq \mathfrak{g}_{\mathbb{R}} \times \mathfrak{g}_{\mathbb{R}}$ is given by $(iX, r) \mapsto (X, -\mathrm{Ad}_r^{-1} X)$. The moment map separates $G_{\mathbb{R}} \times G_{\mathbb{R}}$-orbits. We have $||\mu||^2 = 2\phi$.* $\square$

The present paper extends recent results of Gregor Fels. In [F], pseudokähler forms are defined on certain complex domains which turn out to be subsets of M. In §7, we verify that the restriction of $\langle\ ,\ \rangle$ to these domains coincides with Fels' definition. That paper uses orbital and vertical vector fields, and includes discussions of the moment map and CR-structures. I am grateful to G. F. for sharing a copy of his preprint with me. I also thank Alan Huckleberry for suggesting this problem to me and for helpful conversations.

§1. The Group Action

Let G be a connected complex semisimple algebraic group, endowed with a complex conjugation $g \mapsto \overline{g}$, defining a real form $G_{\mathbb{R}} \subset G$ (the fixed point subgroup of the complex conjugation). The real group $G_{\mathbb{R}} \times G_{\mathbb{R}}$ acts on G by the rule ${}^{(r_1,r_2)}g = r_1 g r_2^{-1}$. Let $\mathfrak{g} = T_e(G)$ denote the Lie algebra of G, with

Killing form κ. Given $Y \in \mathfrak{g}$, the left- and right-invariant vector fields on G generated by Y are denoted $g \mapsto dl_g Y$ and $g \mapsto dr_g Y$.

We can identify the cotangent bundle $T^*(G_{\mathbb{R}})$ with $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$; namely, from $X \in \mathfrak{g}_{\mathbb{R}}$ and $k \in G_{\mathbb{R}}$, we obtain the 1-form at k that sends $dr_k Y$ to $\kappa(X, Y)$, where $Y \in \mathfrak{g}_{\mathbb{R}}$. The action of $G_{\mathbb{R}} \times G_{\mathbb{R}}$ on $G_{\mathbb{R}}$ (by left/right translation) induces an action on $T^*(G_{\mathbb{R}})$, which in the above identification gives the following action of $G_{\mathbb{R}} \times G_{\mathbb{R}}$ on $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$:

$$^{(r_1,r_2)}(iX, r) = (\mathrm{Ad}_{r_1}(iX), r_1 r r_2^{-1}).$$

With this action, the map $f : i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}} \to G$, $(iX, r) \mapsto e^{iX} r$ is equivariant.

We shall often be particularly interested in the case when X is semisimple, and we now set up some notation. If X is semisimple, then it is contained in $\mathfrak{t}_{\mathbb{R}}$, where $\mathfrak{t}$ is a (complex) Cartan subalgebra of $\mathfrak{g}$ that is stable under complex conjugation. Also $p := e^{ix} \in T$, where T is the maximal (complex) $\mathbb{R}$-torus of G with $\mathrm{Lie}\,(T) = \mathfrak{t}$. We have a root system $\Phi(T, G)$ consisting of characters $\alpha : T \to \mathbb{C}^*$, with differentials $d\alpha : \mathfrak{t} \to \mathbb{C}$. (By abuse of notation, we write $-\alpha$ for the inverse of α.) Roots are real, imaginary, or complex according to whether $\overline{\alpha} = \alpha$, $-\alpha$, or neither. Imaginary roots arise in two ways, according to whether the set of real points of the corresponding root $sl(2)$ is isomorphic to $sl(2, \mathbb{R})$ or $su(2)$, and are respectively "noncompact imaginary" or "compact imaginary" roots. We have that $d\alpha(iX) \in i\mathbb{R}$ (resp. $\mathbb{R}$) if α is real (resp. imaginary) and hence $\alpha(p) = e^{d\alpha(iX)} \in U(1)$ (resp. $\mathbb{R}^{>0}$).

We recall some related facts (see [BF]). Let $Z(G) = \{g \in G : \overline{g} = g^{-1}\}$. It is a topologically closed, smooth, $\mathrm{Int}\, G_{\mathbb{R}}$-stable submanifold of G of real dimension equal to the complex dimension of G. The subset $B(G) := \{h \cdot \overline{h}^{-1} : h \in G\} \subset Z(G)$ coincides with the connected component of $Z(G)$ containing 1. Since $e^{iX} = e^{iX/2} \cdot \overline{e^{iX/2}}^{-1}$, we have $\exp(i\mathfrak{g}_{\mathbb{R}}) \subset B(G)$.

§2. The Complex Manifold M and Canonical Vector Fields

In this section, we define the "canonical vector fields," which are global vector fields on a dense open subset $M \subset i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ that are associated to elements of $\mathfrak{g}$. We define and provide a characterization of M (2.1). Given a point $(iX, r) \in M$ and $Z \in \mathfrak{g}$, we produce a curve through (iX, r) in M whose tangent vector at (iX, r) is the canonical tangent vector associated to Z (2.6).

We would like to define global vector fields on $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ by pulling back the left invariant vector fields on G via the map f; that is, given $Z \in \mathfrak{g}$, we would like to define a vector field $\widehat{Z} = Z^\wedge$ on $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ by the rule $\widehat{Z}_{(iX,r)} = (df)^{-1}(dl_{f(iX,r)} Z)$. This works precisely at the points where df is an isomorphism. We define $M = \{(iX, r) :$ ad_X has no eigenvalue of πn for any nonzero integer $n\} \subset i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$. Letting $p = e^{iX}$, we see that (iX, r) fails to be in M exactly when either Ad_p has an eigenvalue of -1, or Ad_p fixes a point in $\mathfrak{g}$ not fixed by ad_X. Below we prove:

THEOREM 2.1. *The differential of f is an isomorphism at (iX, r) precisely when $(iX, r) \in M$.* $\square$

Thus an element $Z \in \mathfrak{g}$ yields a globally-defined nonvanishing vector field $\widehat{Z}$ on M, which we call the *canonical vector field associated to Z*. We see, by taking a basis of $\mathfrak{g}$, that the tangent bundle of M is trivial. If we denote the complex structure on M by J, we have that $J(\widehat{Z}) = (iZ)^\wedge$, where i is the complex structure on the vector space $\mathfrak{g}$. The action of $G_\mathbb{R} \times G_\mathbb{R}$ on M induces an action on vector fields, and ${}^{(r_1,r_2)}(\widehat{Z}) = (\mathrm{Ad}_{r_2} Z)^\wedge$.
For *df* to be nonsingular at (iX, r), we need the exponential map $\exp : i\mathfrak{g}_\mathbb{R} \to B(G)$ to be nonsingular at iX, and we need the multiplication map $B(G) \times G_\mathbb{R} \to G$ to be nonsingular at (e^{iX}, r). Thus 2.1 follows from 2.2 and 2.3 below.

PROPOSITION 2.2. *(See* [V]*.) Given* $\exp : \mathfrak{g} \to G$ *and* $Y \in \mathfrak{g}$*, then the differential* $d\exp : T_Y(\mathfrak{g}) \to T_{e^Y}(G)$ *is given by*

$$d\exp : W \mapsto \left.\frac{d}{ds}\right|_{s=0} e^{Y+sW} = dl_{(e^Y)} \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_Y)^n}{(n+1)!} W$$

and is an isomorphism exactly when ad_Y *has no eigenvalue of* $2\pi in$*,* $n \in \mathbb{Z} \setminus \{0\}$*.* $\square$

(In particular, if $Y \in \mathfrak{g}$ has no eigenvalue of $2\pi in$ (n a nonzero integer), then there is a well-defined map $\log = \log_Y$ from a neighborhood of e^Y in G to a neighborhood of Y in $\mathfrak{g}$. If $Y = iX$ ($X \in \mathfrak{g}_\mathbb{R}$), then in addition, log maps a neighborhood of e^{iX} in $Z(G)$ to a neighborhood of iX in $i\mathfrak{g}_\mathbb{R}$.)

PROPOSITION 2.3. *The differential of the multiplication map* $Z(G) \times G_\mathbb{R} \to G$ *at* (p, r) *is an isomorphism if and only if* Ad_p *has no eigenvalue of* -1.

Before proving 2.3, we need to define an important linear operator on $\mathfrak{g}$. Let $X \in \mathfrak{g}_\mathbb{R}$ and let $p = e^{iX}$. Assume that Ad_p has no eigenvalue of -1. We define $A_p : \mathfrak{g} \to \mathfrak{g}$ by $A_p = \left(\frac{I+\mathrm{Ad}_p}{2}\right)^{-1}$. We will often use the following properties of A_p:

LEMMA 2.4.
1. $\overline{A_p} = \mathrm{Ad}_p \circ A_p = 2I - A_p$.
2. $\kappa(Z, \mathrm{Ad}_p W) = \kappa(\mathrm{Ad}_p^{-1} Z, W)$.
3. $\kappa(Z, A_p W) = \kappa(\mathrm{Ad}_p \circ A_p Z, W)$.
4. $[X, DW] = D[X, W]$, *where* $D = \mathrm{Ad}_p$ *or* A_p.
5. $[Z, A_p W] - [\mathrm{Ad}_p \circ A_p Z, W] = \frac{1}{2}(I - \mathrm{Ad}_p)[A_p Z, A_p W]$.
6. $\kappa(X, [Z, A_p W]) = \kappa(X, [\mathrm{Ad}_p \circ A_p Z, W])$.
7. $A_p W = W$ *if* $[X, W] = 0$*, and more generally,* $A_p W = \frac{2}{1+e^\mu}$ *if* $[iX, W] = \mu W$.

Proof. For (1), we compute that $\overline{A_p} = \left(\frac{I+\mathrm{Ad}_p^{-1}}{2}\right)^{-1} = \mathrm{Ad}_p \circ \left(\frac{\mathrm{Ad}_p+I}{2}\right)^{-1} = \mathrm{Ad}_p \circ A_p$. Moreover, $A_p + \mathrm{Ad}_p \circ A_p = A_p \circ (I + \mathrm{Ad}_p) = 2I$. (2) follows from the Ad-invariance of κ. To prove (3), let $Z' = A_p Z$ and $W' = A_p W$. Then the left side of (3) is $\kappa(\frac{I+\mathrm{Ad}_p}{2} Z', W') = \frac{1}{2}\kappa(Z', W') + \frac{1}{2}\kappa(\mathrm{Ad}_p Z', W')$, whereas the right side is $\kappa(\mathrm{Ad}_p Z', \frac{I+\mathrm{Ad}_p}{2} W') = \frac{1}{2}\kappa(\mathrm{Ad}_p Z', \mathrm{Ad}_p W') + \frac{1}{2}\kappa(\mathrm{Ad}_p Z', W')$. The proofs of (4), (5), and (6) are similar, using also that A_p and Ad_p fix X. (7) is immediate from the definition of A_p. $\square$

Lemma 2.5. *If $p \in Z(G)$, then $T_p(Z(G)) = \{dl_p Z : Z \in T_e(G)$ and $\overline{Z} = -\mathrm{Ad}_p(Z)\}$, and for such Z, $pe^{tZ} \in Z(G)$ for all $t \in \mathbb{R}$.*

Proof. Since $Z(G)$ is smooth, any tangent vector at p can be written as $dl_p Z$ for some $Z \in \mathfrak{g}$. *If $\overline{Z} = -\mathrm{Ad}_p Z$ then the curve pe^{tZ} is contained in* $Z(G)$ since $\overline{pe^{tZ}} = p^{-1}e^{t\overline{Z}}$ and $(pe^{tZ})^{-1} = e^{-tZ}p^{-1} = p^{-1}e^{-t\mathrm{Ad}_p Z} = p^{-1}e^{t\overline{Z}}$. Hence all such Z give tangent vectors in $Z(G)$. Note that since $\overline{p} = p^{-1}$, $Z \mapsto \overline{\mathrm{Ad}_p Z}$ gives a complex conjugation on the vector space $\mathfrak{g}$; the choice of Z above amounts to the pure imaginary elements of $\mathfrak{g}$ for this real structure. The lemma follows since $\dim_{\mathbb{R}}(Z(G)) = \dim_{\mathbb{C}} \mathfrak{g}$. $\square$

Proof of 2.3. Tangent vectors at pr which are in the image of the differential of the multiplication map at (p, r) are exactly those of the form $\frac{d}{dt}\big|_{t=0} pe^{tZ}e^{tY}r = \frac{d}{dt}\big|_{t=0} pe^{t(Z+Y)}r$, where $\overline{Z} = -\mathrm{Ad}_p Z$ and $\overline{Y} = Y$. Hence (Z, Y) is in the kernel of the differential exactly when $Z = -Y$, which is possible for Z exactly when Z is real, meaning $\mathrm{Ad}_p Z = -Z$. $\square$

Let $(iX, r) \in M$ and $Z \in \mathfrak{g}$. Since $t \mapsto e^{iX}re^{tZ}$ gives an integral curve (starting at $e^{iX}r$) for the left invariant vector field associated to Z, we can obtain (for t small) an integral curve at (iX, r) for $\widehat{Z}$, by locally inverting f. The resulting curve δ is described below. Unfortunately this curve is unwieldy for computations. Instead, we produce a simpler curve γ in $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$ which has tangent vector $\widehat{Z}$ at (iX, r) but not at other points on the curve. (This will be sufficient for applications.)

Proposition 2.6. *Let $(iX, r) \in M$, with $p = e^{iX}$, and let $Z \in \mathfrak{g}$. Define the following curves in M:*

$$\gamma_{iX,r,Z} : t \mapsto \left(\log\left(pe^{tA_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z}\right), e^{t(\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z)} \cdot r\right)$$

$$\delta_{iX,r,Z} : t \mapsto \left(\frac{1}{2}\log_{2iX}(pre^{tZ}e^{-t\overline{Z}}r^{-1}p), p(t)^{-1}pre^{tZ}\right)$$

$$p(t) := \exp\left(\frac{1}{2}\log_{2iX}(pre^{tZ}e^{-t\overline{Z}}r^{-1}p)\right)$$

Then $\frac{d}{dt}\big|_{t=0}\gamma = \widehat{Z}_{(iX,r)}$, and δ is an integral curve for $\widehat{Z}$ starting at (iX, r).

Proof. (a) Both curves have a value of (iX, r) at $t = 0$. (b) We must verify that the curves actually lie in $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$. For γ, one can use 2.5 to show that $pe^{tA_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z} \in Z(G)$, and using 2.4(1), it is easy to verify that $\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z \in \mathfrak{g}_{\mathbb{R}}$. For δ, one has that $pre^{tZ}e^{-t\overline{Z}}r^{-1}p \in Z(G)$ since its complex conjugate equals its inverse. The fact that $p(t)^{-1}pre^{tZ} \in G_{\mathbb{R}}$ turns out to be equivalent to $p(t)^2 = pre^{tZ}e^{-t\overline{Z}}r^{-1}p$, which is true by definition. (c) Proving that the curves have the correct derivatives follows from pushing them forward via f. We have that $f(\gamma(t)) = pe^{tA_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z}e^{t(\mathrm{Ad}_r - A_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z)}r$, whose derivative at $t = 0$ is $dl_p \circ dr_r \circ \mathrm{Ad}_r Z = dl_{pr}Z$, as required. Note that at other values of t, tangent vectors for this curve do not coincide with the left invariant vector field on G! However, we do have $f(\delta(t)) = pre^{tZ}$, as required. □

§3. The Differential of the Logarithm Map

Throughout this section, let $(iX, r) \in M$ and let $p = e^{iX}$. We define:

$$F_{iX} : \mathfrak{g} \to \mathfrak{g} \qquad F_{iX}(Z) = \frac{d}{ds}\bigg|_{s=0} \log_{iX}(pe^{sZ})$$

This section is devoted to listing properties of this map.
By definition $F_{iX} = d(\log_{iX} \circ l_p)$, with the differential taken at the identity. Near the identity element of G, the map $l_{p^{-1}} \circ \exp \circ \log_{iX} \circ l_p$ is (defined and) the identity function, so after taking differentials at the identity, we have

$$I = (dl_p)^{-1} \circ d\exp \circ d(\log_{iX} \circ l_p) = \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{iX})^n}{(n+1)!} \circ F_{iX} \qquad \text{by 2.2.}$$

Lemma 3.1. $F_{iX} = \lim_{c\to 0}(icI - \mathrm{ad}_{iX}) \circ (e^{icI - \mathrm{ad}_{iX}} - I)^{-1}$.

Proof.
Let $T = -\mathrm{ad}_{iX}$ and $\lambda \in \mathbb{C}$. We must prove that

$$\sum_{n=0}^{\infty} \frac{T^n}{(n+1)!} \circ \lim_{c\to 0}(icI - \mathrm{ad}_{iX}) \circ (e^{icI - \mathrm{ad}_{iX}} - I)^{-1} = I \in GL(\mathfrak{g}).$$

We compute that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{T^n}{(n+1)!} \circ \lim_{c\to 0}(icI - \mathrm{ad}_{iX}) \circ (e^{icI - \mathrm{ad}_{iX}} - I)^{-1} \\
&\qquad = \left(\sum_{n=0}^{\infty} \frac{T^n}{(n+1)!}\right) \circ \lim_{\lambda\to 0}(\lambda I + T) \circ (e^{\lambda I + T} - I)^{-1} \\
&\qquad = \lim_{\lambda\to 0}\left(\sum_{n=0}^{\infty} \frac{(\lambda I + T)^n}{(n+1)!}\right) \circ (\lambda I + T) \circ (e^{\lambda I + T} - I)^{-1} \\
&\qquad = \lim_{\lambda\to 0}\sum_{n=0}^{\infty} \frac{(\lambda I + T)^n}{(n+1)!} \circ (\lambda I + T) \circ (e^{\lambda I + T} - I)^{-1}
\end{aligned}$$

and if $S := \lambda I + T$, then $\sum_{n=0}^{\infty} \frac{S^n}{(n+1)!} \circ S \circ (e^S - I)^{-1} = \sum_{n=1}^{\infty} \frac{S^n}{n!} \circ (e^S - I)^{-1} = (e^S - I) \circ (e^S - I)^{-1} = I$. □

Lemma 3.2.

(1) $F_{iX}(W) = W$ *if* $[X,W]=0$, *and* $F_{iX}(W) = \frac{-\mu e^{\mu}}{1-e^{\mu}}W$ *if* $[iX,W]=\mu W$, $\mu \neq 0$.
(2) $\overline{F}_{iX} = \mathrm{Ad}_p^{-1} \circ F_{iX} = F_{-iX}$.
(3) $\kappa(Z, F_{iX}W) = \kappa(F_{-iX}Z, W)$ *for all* $Z, W \in \mathfrak{g}$.
(4) $\kappa(X, F_{iX}(W)) = \kappa(X,W)$ *for all* $W \in \mathfrak{g}$.
(5) $\mathrm{ad}_{iX} = F_{iX} \circ (I - \mathrm{Ad}_p^{-1})$.

Proof. (1,2) are easy. By substitution, (3) is equivalent to $\kappa(Z, d(l_{p^{-1}} \circ \exp)_{iX}W) = \kappa(d(l_p \circ \exp)_{-iX}Z, W)$. By 2.2, $\kappa(Z, d(l_{p^{-1}} \circ \exp)_{iX}W) = \kappa\left(Z, \sum \frac{(-\mathrm{ad}_{iX})^n}{(n+1)!}W\right)$, which equals $\kappa\left(\sum \frac{(\mathrm{ad}_{iX})^n}{(n+1)!}Z, W\right) = \kappa(d(l_p \circ \exp)_{-iX}Z, W)$ by the associativity of the Killing form. Then (4) follows from (1) and (3). For (5), we have $F_{iX} \circ (I - \mathrm{Ad}_p^{-1}) = F_{iX} \circ \left(-\sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{iX})^n}{n!} + I\right) = F_{iX} \circ \left(-\sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{iX})^{n+1}}{(n+1)!}\right) = F_{iX} \circ \left(\sum_{n=1}^{\infty} \frac{(-\mathrm{ad}_{iX})^n}{(n+1)!}\right) \circ \mathrm{ad}_{iX} = I \circ \mathrm{ad}_{iX}$. □

§4. The Liouville Form on M and its Exterior Derivative

We recall that the cotangent bundle to any real manifold possesses a canonical 1-form λ and that $\omega := d\lambda$ is a nondegenerate exact symplectic form (see [A], [CG]). We have identified $T^*(G_{\mathbb{R}})$ with $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}}$. Given a curve $(iX(t), r(t))$, then one can check that $\lambda\left(\frac{d}{dt}\big|_{t=0}(iX(t), r(t))\right) = \kappa\left(X(0), \frac{d}{dt}\big|_{t=0} r(t)r(0)^{-1}\right)$. It is easy to see that λ is $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant. We wish to obtain a formula for $\lambda(\widehat{Z})$; this conveniently expresses the restriction of λ to M. By 2.6, we have that $\lambda(\widehat{Z})_{(iX,r)} = \kappa(X, \frac{d}{dt}\big|_{t=0} e^{t(\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z)}) = \kappa(X, \mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z)$, where $p = e^{iX}$. This can be sharpened:

Proposition 4.1. *For all* $(iX, r) \in M$, $\lambda(\widehat{Z})_{(iX,r)} = \kappa(X, \mathrm{Ad}_r \mathrm{Re}\, Z)$.

Proof. We must show that $\kappa(X, \mathrm{Ad}_r i\mathrm{Im}\, Z) = \kappa(X, A_p \circ \mathrm{Ad}_r i\mathrm{Im}\, Z)$. This follows from 2.4(3), since $\mathrm{Ad}_p \circ A_p(X) = X$. □

On M, we have a complex structure J. Even prior to the explicit computation of $d\lambda$, we have the following important observation:

Theorem 4.2. *As a differential form on M, the 2-form $\omega := d\lambda$ is J-invariant.* □

This is a consequence of 4.4, for which we recall the customary notation. On any complex manifold, the exterior derivative is written as the sum $d = \partial + \overline{\partial}$. Let $d^c = i(\partial - \overline{\partial})$. From $d^2 = 0$, we know that $\partial\overline{\partial} = -\overline{\partial}\partial$, and hence $dd^c = 2i\overline{\partial}\partial = -d^c d$. One can show (e.g. using local coordinates) that if ϕ is a smooth function and X a vector field on M, then $d^c\phi(X) = d\phi(JX)$; by the derivation property, if μ is a 1-form, then $d^c\mu(X_1, X_2) = JX_1(\mu(X_2)) - JX_2(\mu(X_1)) - \mu(J[X_1, X_2])$.

We confirm that $dd^c(\phi)$ is J-invariant: by the product rule we have

$$\begin{aligned} dd^c\phi(X_1,X_2) &= X_1(d^c\phi(X_2)) - X_2(d^c\phi(X_1)) - d^c\phi([X_1,X_2]) \\ &= X_1(d\phi(JX_2)) - X_2(d\phi(JX_1)) - d\phi(J[X_1,X_2]), \end{aligned}$$

and hence

$$\begin{aligned} dd^c\phi(JX_1,JX_2) &= JX_1(d\phi(-X_2)) - JX_2(d\phi(-X_1)) - d\phi(J[JX_1,JX_2]) \\ &= -JX_1(d\phi(X_2) + JX_2(d\phi(X_1) + d\phi(J[X_1,X_2]) \\ &= -d^c d\phi(X_1,X_2) = dd^c\phi(X_1,X_2). \end{aligned}$$

Definition/Theorem 4.3. *Let $\phi : M \to \mathbb{R}$ be the $G_{\mathbb{R}} \times G_{\mathbb{R}}$-invariant function $\phi(iX,r) = \kappa(X,X)$. Then $2\lambda = d^c(\phi)$.*

Proof. By definition,

$$\begin{aligned} d^c\phi(\widehat{Z})_{(iX,r)} &\underset{\text{def of } d^c}{=} (J\widehat{Z}(\phi))_{(iX,r)} \underset{2.6}{=} \frac{d}{dt}\Big|_{t=0} (\phi(\gamma_{iX,r,jZ}(t))) \\ &\underset{2.6,\text{def of } \phi}{=} 2\kappa\left(-i\,\frac{d}{dt}\Big|_{t=0} \log(pe^{tA_p\circ \mathrm{Ad}_r i\mathrm{Im}\, iZ}), X\right) \\ &= 2\kappa(F_{iX}\circ A_p \circ \mathrm{Ad}_r \mathrm{Re}\, Z, X) \\ &\underset{2.4(3),3.2(4)}{=} 2\kappa(\mathrm{Ad}_r \mathrm{Re}\, Z, X) \underset{4.1}{=} 2\lambda(\widehat{Z})_{(iX,r)}. \quad \square \end{aligned}$$

Corollary 4.4. $\omega = d\lambda = \frac{1}{2}dd^c(\phi) = -i\partial\overline{\partial}(\phi)$. $\square$

§5. Computation of the Pseudokähler Form

Let $\omega = d\lambda$, a 2-form on $i\mathfrak{g}_{\mathbb{R}} \times G_{\mathbb{R}} \supset M$. Here we compute the restriction of ω to M, using the vector fields $\widehat{Z}$ (which are only defined on M). We use the formula $\omega(\widehat{Z},\widehat{W}) = \widehat{Z}(\lambda(\widehat{W})) - \widehat{W}(\lambda(\widehat{Z})) - \lambda([\widehat{Z},\widehat{W}])$.

We will require another linear operator on $\mathfrak{g}$. For $(iX,r) \in M$ and $p := e^{iX}$, let

$$E_{iX} = F_{iX} \circ \mathrm{Ad}_p \circ A_p = d(\log \circ l_p) \circ \mathrm{Ad}_p \circ A_p.$$

(Note that the three factors commute.) We collect some properties of E_{iX} and $F_{iX} \circ A_p$:

Lemma 5.1.

(1) $\overline{E}_{iX} = F_{iX} \circ \mathrm{Ad}_p^{-1} \circ A_p$.
(2) *For all $Z,W \in \mathfrak{g}$, $\kappa(E_{iX}Z,W) = \kappa(Z,\overline{E_{iX}}W)$.*
(3) $i\mathrm{Im}\, E_{iX} = \mathrm{ad}_{iX}$.
(4) *If $[X,W] = 0$ then $E_{iX}W = W$.*
(5) $\overline{F_{iX}\circ A_p} = F_{iX} \circ A_p$ *and for all $Z,W \in \mathfrak{g}$, we have $\kappa(F_{iX}\circ A_p\, Z,W) = \kappa(Z, F_{iX}\circ A_p\, W)$.*

Proof. $\overline{E}_{iX} = \overline{F_{iX} \circ (\mathrm{Ad}_p \circ A_p)} \underset{2.4(1),3.2(2)}{=} (F_{iX} \circ \mathrm{Ad}_p^{-1}) \circ A_p$, proving (1). Then (2) follows from (1), 2.4(1,2), and 3.2(2,3). To prove (3), we note $2i\mathrm{Im}\, E_{iX} = E_{iX} - \overline{E}_{iX} \underset{5.1(1)}{=} F_{iX} \circ A_p \circ \mathrm{Ad}_p - F_{iX} \circ A_p \circ \mathrm{Ad}_p^{-1} = F_{iX} \circ \mathrm{Ad}_p^{-1} \circ A_p \circ (\mathrm{Ad}_p^2 - I) = 2F_{iX} \circ (\mathrm{Ad}_p - I) \circ \mathrm{Ad}_p^{-1} \circ A_p \circ \left(\frac{\mathrm{Ad}_p + I}{2}\right) = 2F_{iX} \circ (I - \mathrm{Ad}_p)^{-1} \underset{3.2(5)}{=} 2\mathrm{ad}_{iX}$. Finally, (4) follows from 2.4(7) and 3.2(1), and (5) is similar to (1) and (2). $\square$

We return to the computation of $d\lambda$. First,

$$\begin{aligned}
&\widehat{Z}(\lambda\widehat{W})_{(iX,r)} \\
&\quad \underset{4.1}{=} \widehat{Z}\left(\kappa(X, \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\quad \underset{2.6}{=} \left.\frac{d}{dt}\right|_{t=0} \kappa\left(-i \log p e^{tA_p \circ \mathrm{Ad}_r i \mathrm{Im}\, Z}, \mathrm{Ad}_{e^{t(\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i \mathrm{Im}\, Z)}} \circ \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\quad = \kappa\left(-iF_{iX} \circ A_p \circ \mathrm{Ad}_r i \mathrm{Im}\, Z, \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\qquad + \kappa\left(X, \mathrm{ad}_{\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i \mathrm{Im}\, Z} \circ \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\quad = \kappa\left(F_{iX} \circ A_p \circ \mathrm{Ad}_r \mathrm{Im}\, Z, \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\quad - i\kappa\left(\mathrm{ad}_{iX}(\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i \mathrm{Im}\, Z), \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\quad = \kappa\left(F_{iX} \circ A_p \circ \mathrm{Ad}_r \mathrm{Im}\, Z, \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\qquad - i\kappa\left(F_{iX} \circ (I - \mathrm{Ad}_p^{-1})(\mathrm{Ad}_r Z - A_p \circ \mathrm{Ad}_r i \mathrm{Im}\, Z), \mathrm{Ad}_r \mathrm{Re}\, W\right) \text{ by } 3.2(5) \\
&\quad = \kappa\left(F_{iX} \circ A_p \circ \mathrm{Ad}_p^{-1} \circ \mathrm{Ad}_r \mathrm{Im}\, Z, \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\qquad - i\kappa\left(F_{iX} \circ (I - \mathrm{Ad}_p^{-1}) \circ \mathrm{Ad}_r Z, \mathrm{Ad}_r \mathrm{Re}\, W\right) \\
&\quad = \frac{-i}{4}\kappa\left(F_{iX} \circ A_p \circ \mathrm{Ad}_p^{-1} \circ \mathrm{Ad}_r(Z - \overline{Z}), \mathrm{Ad}_r(W + \overline{W})\right) \\
&\qquad + \frac{-i}{4}\kappa\left(F_{iX} \circ (I - \mathrm{Ad}_p^{-1}) \circ \mathrm{Ad}_r 2Z, \mathrm{Ad}_k(W + \overline{W})\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-\widehat{W}(\lambda\widehat{Z})_{(iX,r)} &= \frac{i}{4}\kappa\left(\mathrm{Ad}_r(Z + \overline{Z}), F_{iX} \circ A_p \circ \mathrm{Ad}_p^{-1} \circ \mathrm{Ad}_r(W - \overline{W})\right) \\
&\quad + \frac{i}{4}\kappa\left(\mathrm{Ad}_r(Z + \overline{Z}), F_{iX} \circ (I - \mathrm{Ad}_p^{-1}) \circ \mathrm{Ad}_r 2W\right) \\
&= \frac{i}{4}\kappa\left(F_{iX} \circ A_p \circ \mathrm{Ad}_p \circ \mathrm{Ad}_r(Z + \overline{Z}), \mathrm{Ad}_r(W - \overline{W})\right) \\
&\quad + \frac{i}{4}\kappa\left(F_{iX} \circ \mathrm{Ad}_p^{-1} \circ (I - \mathrm{Ad}_p) \circ \mathrm{Ad}_r(Z + \overline{Z}), \mathrm{Ad}_r 2W\right) \\
&\qquad \text{by } 5.1(5),\ 3.2(2,3).
\end{aligned}$$

Finally,

$$\begin{aligned}
-\lambda[\widehat{Z},\widehat{W}] &= -\lambda([Z,W]^\wedge) = -\kappa\left(X, \mathrm{Ad}_r \mathrm{Re}\,[Z,W]\right) \\
&= \frac{i}{2}\kappa\left(iX, \mathrm{Ad}_r[Z,W]\right) + \frac{i}{2}\kappa\left(iX, \mathrm{Ad}_r[\overline{Z},\overline{W}]\right) \\
&= \frac{i}{2}\kappa\left(F_{iX}\circ(I-\mathrm{Ad}_p^{-1})\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r W\right) \\
&\quad + \frac{i}{2}\kappa\left(F_{iX}\circ(I-\mathrm{Ad}_p^{-1})\circ\mathrm{Ad}_r \overline{Z}, \mathrm{Ad}_r \overline{W}\right).
\end{aligned}$$

Summing the terms and using 2.4(1), we find

$$\begin{aligned}
\omega(\widehat{Z},\widehat{W}) &= \frac{-i}{2}\kappa(F_{iX}\circ A_p\circ\mathrm{Ad}_p\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W}) \\
&\quad + \frac{i}{2}\kappa(F_{iX}\circ A_p\circ\mathrm{Ad}_p^{-1}\circ\mathrm{Ad}_r\overline{Z}, \mathrm{Ad}_r W) \\
&= \frac{-i}{2}\left(\kappa(E_{iX}\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W}) - \kappa(E_{iX}\circ\mathrm{Ad}_p^{-2}\circ\mathrm{Ad}_r\overline{Z}, \mathrm{Ad}_r W)\right) \\
&= \frac{1}{2i}\left(\kappa(E_{iX}\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W}) - \overline{\kappa(E_{iX}\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W})}\right. \\
&= \mathrm{Im}\left(\kappa(E_{iX}\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W}\right)
\end{aligned}$$

We have proved:

THEOREM 5.2. $\omega(\widehat{Z},\widehat{W}) = \mathrm{Im}\,\kappa\left(E_{iX}\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W}\right)$ *is an exact, nondegenerate, J-invariant, $G_{\mathbb{R}}\times G_{\mathbb{R}}$-invariant, real-valued 2-form on M, and coincides with the restriction to M of the standard (cotangent bundle) symplectic form on $T^*(G_{\mathbb{R}})$.* $\square$

Recall that on any complex manifold, a closed, nondegenerate, real 2-form ω for which the complex structure is an isometry yields a pseudokähler form, by the rule $\left\langle\widehat{Z},\widehat{W}\right\rangle = \omega(J\widehat{Z},\widehat{W}) + i\omega(\widehat{Z},\widehat{W})$. Here $\widehat{Z},\widehat{W}\mapsto\omega(J\widehat{Z},\widehat{W})$, the real part of $\left\langle\widehat{Z},\widehat{W}\right\rangle$, is a real, J-invariant, symmetric bilinear form (which need not be positive definite), and the imaginary part of $\langle\ ,\ \rangle$ is just ω.
In our situation, we have:

THEOREM 5.3. *The pseudokähler form associated to ω is*

$$\left\langle\widehat{Z},\widehat{W}\right\rangle_{(iX,r)} = \kappa(E_{iX}\circ\mathrm{Ad}_r Z, \mathrm{Ad}_r\overline{W}) = \kappa(E_{i\mathrm{Ad}_r^{-1}X} Z, \overline{W}). \quad \square$$

Note that 5.1(2) shows independently that $\langle\ ,\ \rangle$ is Hermitian.

§6. EVALUATION OF THE PSEUDOKÄHLER FORM ON A BASIS

Our next goal is to compute $\langle\ ,\ \rangle$ with respect to a natural basis of vector fields at (iX,r), in the (generic) case that X is semisimple. Without loss of generality, we assume that r is the identity element of $G_{\mathbb{R}}$. We use notation involving T and $\mathfrak{t}$ as in §1.

It is clear that E_{iX} preserves $\mathfrak{t}$ and each root space $\mathfrak{g}_\alpha$. Hence for our Hermitian form, $\mathfrak{t} \perp \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ unless $\overline{\beta} = -\alpha$. So, the only products we need compute are $\left\langle \widehat{Z}, \widehat{W} \right\rangle$ for $Z, W \in \mathfrak{t}$, and $\left\langle \widehat{Z}_\alpha, \widehat{\overline{Z_{-\alpha}}} \right\rangle$, where $Z_\alpha \in \mathfrak{g}_\alpha$.

By 2.4(1) and the definition of E_{iX}, it follows easily that:

Lemma 6.1. *$E_{iX}(Z) = Z$ if $Z \in \mathfrak{t}$ (or more generally, if $[X, Z] = 0$); and if $Z_\alpha \in \mathfrak{g}_\alpha$ with $d\alpha(iX) \neq 0$, then* $E_{iX}(Z_\alpha) = \dfrac{-\alpha(p^2) \cdot d\alpha(2iX)}{1 - \alpha(p^2)} Z_\alpha$. □

Lemma 6.2. *For $Z, W \in \mathfrak{t}$, we have*

$$\begin{aligned} \left\langle \widehat{Z}, \widehat{W} \right\rangle &= \kappa(Z, \overline{W}) \\ &= (\kappa(\operatorname{Re} Z, \operatorname{Re} W) + \kappa(\operatorname{Im} Z, \operatorname{Im} W)) \\ &\qquad + i\,(\kappa(\operatorname{Im} Z, \operatorname{Re} W) - \kappa(\operatorname{Re} Z, \operatorname{Im} W)) \\ &= \sum_{\alpha \in \Phi(T,G)} d\alpha(Z) \cdot d\alpha(\overline{W}). \quad \square \end{aligned}$$

From this, it is easy to describe the signature of $\langle\ ,\ \rangle$ on $\mathfrak{t}$: suppose the connected component of 1 in $T_{\mathbb{R}}$ is a product of n circles and m real lines (here $n + m$ is the complex dimension of T). *Then $\langle\ ,\ \rangle$ is negative-definite on the complexified Lie algebra of the circles and positive-definite on the lines, and these two subspaces of $\mathfrak{t}$ are perpendicular.* For: in computing signatures, we may assume that $Z \in \mathfrak{t}_{\mathbb{R}}$. If $Z \in \mathfrak{t}_{\mathbb{R}}$, then in the former case $d\alpha(Z) \in i\mathbb{R}$, and in the latter, $d\alpha(Z) \in \mathbb{R}$. Also for $Z \in \mathfrak{t}_{\mathbb{R}}$, $\langle Z, Z\rangle = \sum_{\alpha \in \Phi(T,G)} (d\alpha(Z))^2$. Also if $Z, W \in \mathfrak{t}_{\mathbb{R}}$ but are of "opposite types," the last lemma shows that $\langle Z, W\rangle \in i\mathbb{R} \cap \mathbb{R} = \{0\}$.

Now let $Z = Z_\alpha$ and $W = \overline{Z_{-\alpha}}$. Recall that by our definition of M, we have $\alpha(p) \neq -1$. Also, either $\alpha(p) \neq 1$, or $d\alpha(iX) = 0$ and $\alpha(p) = 1$.

Lemma 6.3. *If $\alpha(p) \neq 1$, then* $\left\langle \widehat{Z_\alpha}, \widehat{\overline{Z_{-\alpha}}} \right\rangle = \dfrac{-\alpha(p^2) \cdot d\alpha(2iX) \cdot \kappa(Z_\alpha, Z_{-\alpha})}{1 - \alpha(p^2)}$, *whereas if $d\alpha(iX) = 0$, then* $\left\langle \widehat{Z_\alpha}, \widehat{\overline{Z_{-\alpha}}} \right\rangle = \kappa(Z_\alpha, Z_{-\alpha})$. □

This shows that if α is not imaginary, then $\langle\ ,\ \rangle$ is isotropic on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\overline{\alpha}}$. Suppose that α is imaginary; we wish to see whether $\langle\ ,\ \rangle$ is positive- or negative-definite on $\mathfrak{g}_\alpha$. In the copy of $sl(2)$ ($sl(\alpha) \subset \mathfrak{g}$) corresponding to α, recall that we may pick a basis $\{H_\alpha, Z_\alpha, Z_{-\alpha}\}$ satisfying $[Z_\alpha, Z_{-\alpha}] = H_\alpha$, $[H_\alpha, Z_\alpha] = 2Z_\alpha$, and $[H_\alpha, Z_{-\alpha}] = -2Z_{-\alpha}$. By explicit computation, one sees that one can pick Z_α satisfying $\overline{Z_\alpha} = \epsilon Z_{-\alpha}$, where $\epsilon = 1$ (resp. -1) if $SL(\alpha)_{\mathbb{R}}$ is noncompact (resp. compact). Then $\left\langle \widehat{Z}_\alpha, \widehat{Z}_\alpha \right\rangle = \dfrac{-\epsilon \cdot \alpha(p^2) \cdot d\alpha(2iX) \cdot \kappa(Z_\alpha, Z_{-\alpha})}{1 - \alpha(p^2)}$, or simply $\epsilon\kappa(Z_\alpha, Z_{-\alpha})$ if $d\alpha(iX) = 0$. In the latter case, since $\kappa(Z_\alpha, Z_{-\alpha}) > 0$ we already see that $\langle\ ,\ \rangle$ is positive on $\mathfrak{g}_\alpha$ if α is noncompact and negative if α is compact. In the former case, we get the same information: since α is imaginary, we have $d\alpha(iX) \in \mathbb{R}$, and $\alpha(p) = e^{d\alpha(iX)} > 0$. Note then that $d\alpha(iX)/(1-\alpha(p)) < 0$; it then follows that the sign of $\left\langle \widehat{Z}_\alpha, \widehat{Z}_\alpha \right\rangle$ is ϵ. Summarizing the above, we have:

Theorem 6.4. *Suppose that $(iX, 1) \in M$ and $e^{iX} \in T$ for some maximal $\mathbb{R}$-torus T of G. Write $T = T_s \cdot T_a$, the decomposition of T into an almost direct product of split and anisotropic subtori. For each root α, let $\mathfrak{g}_\alpha$ be the root subspace of $\mathfrak{g}$. We identify elements of $\mathfrak{g}$ with the induced tangent vectors at $(iX, 1)$ coming from the canonical vector fields. Then under the Hermitian form $\langle\ ,\ \rangle$, $\mathfrak{t}$ is perpendicular to each root space; $\operatorname{Lie} T_a$ is perpendicular to $\operatorname{Lie} T_s$; $\mathfrak{g}_\alpha$ is perpendicular to $\mathfrak{g}_\beta$ unless $\beta = -\overline{\alpha}$; $\langle\ ,\ \rangle$ is positive definite on $\operatorname{Lie} T_s$ and negative definite on $\operatorname{Lie} T_a$; $\langle\ ,\ \rangle$ is isotropic on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\overline{\alpha}}$ if α is not imaginary; and $\langle\ ,\ \rangle$ is positive (resp. negative) definite on $\mathfrak{g}_\alpha$ if α is noncompact imaginary (resp. compact imaginary).* $\square$

A priori, the signature of $\langle\ ,\ \rangle$ is only constant on each connected component of M, but in fact more is true:

Corollary 6.5. *The signature of $\langle\ ,\ \rangle$ is constant on M.*

Proof. Since $(i\mathbf{0}, 1) \in M$, there exists a connected neighborhood U of $\mathbf{0}$ in $\mathfrak{g}_\mathbb{R}$ such that $iU \times G_\mathbb{R} \subset M$. It follows that $\langle\ ,\ \rangle$ has constant signature on $iU \times G_\mathbb{R}$ (note that while $G_\mathbb{R}$ need not be connected, this is irrelevant since $\langle\ ,\ \rangle$ is $G_\mathbb{R}$-invariant). We must show that the signature of $\langle\ ,\ \rangle$ on any connected component of M is the same as on $iU \times G_\mathbb{R}$. Without loss of generality we may choose $(iX, r) \in M$ such that X is regular semisimple in $\mathfrak{g}$, which is to say that $X \in \mathfrak{t}_\mathbb{R}$ for a (unique) Cartan subalgebra $t_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$. We may find $s \in \mathbb{R}^*$ sufficiently small that $sX \in U$, and of course $sX \in \mathfrak{t}_\mathbb{R}$ is still regular semisimple. However by 6.4, the signature at $(iY, r) \in M$ (for Y regular semisimple) depends only on attributes of the unique real Cartan subalgebra containing Y and of its root system. Hence the signature of $\langle\ ,\ \rangle$ at (iX, r) is the same as the signature of $\langle\ ,\ \rangle$ on $iU \times G_\mathbb{R}$. $\square$

Corollary 6.6. *The signature of $\langle\ ,\ \rangle$ equals the signature of the Hermitian form $Z, W \mapsto \kappa(Z, \overline{W})$ on $\mathfrak{g}$, which equals the signature of the (real) symmetric bilinear form $Z, W \mapsto \kappa(Z, W)$ on $\mathfrak{g}_\mathbb{R}$.*

Proof. By 6.5, it is enough to check the result at a single point of M, and at the point $(i\mathbf{0}, 1)$, $\langle Z, W\rangle = \kappa(Z, \overline{W})$. The second statement is a simple linear algebra fact. $\square$

7. Alternative Vector Fields and Comparison with Fels' Work

Let $(iX, r) \in M$. Since $G_\mathbb{R} \times G_\mathbb{R}$ acts on M, any pair $(Y_1, Y_2) \in \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R}$ produces a tangent vector at (iX, r) (indeed it produces an "orbital vector field" on all of $i\mathfrak{g}_\mathbb{R} \times G_\mathbb{R}$). It is easy to see that (Y_1, Y_2) and (Y_1', Y_2') produce the same tangent vector at (iX,r) if and only if $(Y_1',Y_2') = (Y_1+A, Y_2+\mathrm{Ad}_r^{-1}A)$ for some A in the centralizer in $\mathfrak{g}_\mathbb{R}$ of X. Given any V in this centralizer, we obtain a (non-orbital) tangent vector to M at (iX,r), namely $\frac{d}{dt}\big|_{t=0}(iX+itV,r)$ (however this need not be extendable to a vector field on M). By dimension count, any tangent vector to M at (iX,r) can be obtained from a combination of orbital and transverse vectors; namely, given (Y_1,Y_2,V) as above, we obtain the tangent vector $\frac{d}{dt}\big|_{t=0}\left(\mathrm{Ad}_{e^{tY_1}}(iX+itV), e^{tY_1}re^{-tY_2}\right)$, and every tangent vector can be obtained in this way. The relationship to canonical vectors is:

Proposition 7.1. *At (iX,r) (with $p = e^{iX}$), the tangent vector coming from the triple (Y_1,Y_2,V) coincides with the canonical vector $(\mathrm{Ad}_{pr}^{-1}Y_1 + i\mathrm{Ad}_r^{-1}V - Y_2)^\wedge$.*

Proof. This is equivalent to the (straightforward) proof that $\frac{d}{dt}\big|_{t=0}\left(e^{tY_1}e^{iX+itV}e^{-tY_1}\right)\cdot\left(e^{tY_1}ke^{-tY_2}\right) = dl_{pk}\left(\mathrm{Ad}_{pr}^{-1}Y_1 + i\mathrm{Ad}_r^{-1}V - Y_2\right)$. □

Fix $(iX,r)\in M$, and take (Y_1,Y_2,V_1) and (Y_3,Y_4,V_2) as above. Let $Z,W\in\mathfrak{g}$ be the corresponding canonical vectors (only valid for the point (iX,r)!). It follows from 2.4(2) and 5.1(1,2,4) that:

Theorem 7.2. *At (iX,r),*

$$\begin{aligned}\left\langle \widehat{Z},\widehat{W}\right\rangle &= \kappa(\overline{E_{iX}}Y_1,Y_3) - \kappa(F_{iX}\circ A_p\,Y_1,\mathrm{Ad}_r\,Y_4) - \kappa(Y_1,iV_2)\\ &\quad - \kappa(F_{iX}\circ A_p\circ\mathrm{Ad}_r\,Y_2,Y_3) + \kappa(E_{iX}\circ\mathrm{Ad}_r\,Y_2,\mathrm{Ad}_r\,Y_4) + \kappa(\mathrm{Ad}_r\,Y_2,iV_2)\\ &\qquad + \kappa(iV_1,Y_3) - \kappa(iV_1,\mathrm{Ad}_r\,Y_4) + \kappa(iV_1,-iV_2). \quad\square\end{aligned}$$

Using 5.1(1,3,5), we can separate easily the real and imaginary parts of $\left\langle \widehat{Z},\widehat{W}\right\rangle$:

Corollary 7.3.

$$\begin{aligned}&\mathrm{Re}\left\langle \widehat{Z},\widehat{W}\right\rangle\\ &= \kappa(F_{iX}\circ A_p\,\mathrm{Re}\,(\mathrm{Ad}_p\,Y_1),Y_3) + \kappa(F_{iX}\circ A_p\,\mathrm{Re}\,(\mathrm{Ad}_p\,\mathrm{Ad}_r\,Y_2),\mathrm{Ad}_r\,Y_4)\\ &\qquad - \kappa(F_{iX}\circ A_p\,Y_1,\mathrm{Ad}_r\,Y_4) - \kappa(F_{iX}\circ A_p\circ\mathrm{Ad}_r\,Y_2,Y_3) + \kappa(V_1,V_2),\end{aligned}$$

and

$$\begin{aligned}&\omega(\widehat{Z},\widehat{W})\\ &= \mathrm{Im}\left\langle \widehat{Z},\widehat{W}\right\rangle\\ &= \kappa(X,[Y_1,Y_3] - \mathrm{Ad}_r\,[Y_2,Y_4])\\ &\qquad + \kappa(V_1,Y_3-\mathrm{Ad}_r\,Y_4) - \kappa(Y_1-\mathrm{Ad}_r\,Y_2,V_2). \quad\square\end{aligned}$$

We recall some facts about moment maps (see [CG, Chapter 1], [HW]). Suppose that a Lie group K acts symplectically on a symplectic manifold (N,ω). There is a map sending smooth functions on N to symplectic (=locally Hamiltonian) vector fields on N. Since G acts symplectically, there is also a map sending each element of $\mathfrak{k}$ to a symplectic vector field. The action of K is said to be *Hamiltonian* if there is a Lie algebra homomorphism H from $\mathfrak{k}$ to smooth functions on N which makes a commutative triangle with the other two maps. The associated *moment map* $\mu : N \to \mathfrak{k}^*$ is the map sending $n \in N$ to the linear function on $\mathfrak{k}$ given by $x \mapsto H_x(n)$. If the manifold in question is a cotangent bundle, with canonical 1-form λ and $\omega := d\lambda$ and with K acting on the base space, then (N, ω) is Hamiltonian, with H sending $x \in \mathfrak{k}$ to the contraction of λ with the vector field coming from the infinitesimal action of x.

In our case, $N = i\mathfrak{g}_\mathbb{R} \times G_\mathbb{R} \simeq T^*(G_\mathbb{R})$ and $\mathfrak{k} = \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R}$. Given $(Y_1, Y_2) \in \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R}$, the induced tangent vector at (iX, r) is $\frac{d}{dt}\big|_{t=0} \left(i\mathrm{Ad}_{e^{tY_1}} X, e^{tY_1} r e^{-tY_2}\right)$, so by the remark at the beginning of §4, we have

$$\begin{aligned} H_{(Y_1,Y_2)} &= \kappa(X, \frac{d}{dt}\bigg|_{t=0} e^{tY_1} r e^{-tY_2} r^{-1}) \\ &= \kappa(X, Y_1 - \mathrm{Ad}_r Y_2) = \kappa(X, Y_1) - \kappa(\mathrm{Ad}_r^{-1} X, Y_2). \end{aligned}$$

We identify $\mathfrak{g}_\mathbb{R}$ with $\mathfrak{g}_\mathbb{R}^*$ via the Killing form. We have proved:

THEOREM 7.4. *Relative to the symplectic form ω, the moment map $\mu : i\mathfrak{g}_\mathbb{R} \times G_\mathbb{R} \to \mathfrak{g}_\mathbb{R}^* \times \mathfrak{g}_\mathbb{R}^* \simeq \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R}$ is given by $(iX, r) \mapsto (X, -\mathrm{Ad}_r^{-1} X)$.* $\square$

COROLLARY 7.5. *The image of the moment map is $\{(Y_1, Y_2)$: Y_1 and $-Y_2$ are $\mathrm{Ad}(G_\mathbb{R})$-conjugate$\}$.* $\square$

Another easy consequence of 7.4 is:

COROLLARY 7.6. *The moment map $\mu : i\mathfrak{g}_\mathbb{R} \times G_\mathbb{R} \to \mathfrak{g}_\mathbb{R}^* \times \mathfrak{g}_\mathbb{R}^* \simeq \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R}$ separates $G_\mathbb{R} \times G_\mathbb{R}$-orbits.* $\square$

The formula for $\omega(\widehat{Z}, \widehat{W})$ in (7.3) is essentially due to Gregor Fels [F]. Here we recall his construction in [F] of a pseudokähler form on certain complex manifolds and relate the construction to the one in this paper. (We have changed notation slightly from [F].)

Let $G_\mathbb{R} \subset G$ as usual, and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ that is stable under complex conjugation. Let $\mathfrak{t}'_\mathbb{R}$ denote the regular semisimple elements of $\mathfrak{t}_\mathbb{R}$. Define $N = \{(n, n) : n \in N_{G_\mathbb{R}}(T_\mathbb{R})\} \subset G_\mathbb{R} \times G_\mathbb{R}$. Since N acts on $\mathfrak{t}'_\mathbb{R}$, we have the usual twisted product $(G_\mathbb{R} \times G_\mathbb{R}) *^N i\mathfrak{t}'_\mathbb{R}$. It is easy to see that the map

$$\Theta : (G_\mathbb{R} \times G_\mathbb{R}) *^N i\mathfrak{t}'_\mathbb{R} \longrightarrow M$$

given by $[(r_1, r_2), iX] \mapsto \left(\mathrm{Ad}_{r_1}(iX), r_1 r_2^{-1}\right)$ is well-defined, injective, and $G_\mathbb{R} \times G_\mathbb{R}$-equivariant, with everywhere nonsingular differential. Moreover, the map forms one leg of a commutative triangle with the maps

$$\begin{aligned} (G_\mathbb{R} \times G_\mathbb{R}) *^N i\mathfrak{t}'_\mathbb{R} &\to G & \qquad M &\to G \\ [(r_1, r_2), iX] &\mapsto r_1 e^{iX} r_2^{-1} & \qquad (iX, k) &\mapsto e^{iX} k. \end{aligned}$$

It follows that there is a complex structure on $(G_{\mathbb{R}} \times G_{\mathbb{R}}) *^N i\mathfrak{t}'_{\mathbb{R}}$ that agrees with the ones on G and on M.
Given the point $v = [(r_1, r_2), iX] \in (G_{\mathbb{R}} \times G_{\mathbb{R}}) *^N i\mathfrak{t}'_{\mathbb{R}}$, one can construct (any) tangent vector as $\frac{d}{ds}\big|_{s=0}$ of the curve $s \mapsto [(r_1 e^{sY_1}, r_2 e^{sY_2}), iX + siY_3]$, where $Y_1, Y_2 \in \mathfrak{g}_{\mathbb{R}}$ and $Y_3 \in \mathfrak{t}_{\mathbb{R}}$. The J-invariant 2-form on $(G_{\mathbb{R}} \times G_{\mathbb{R}}) *^N i\mathfrak{t}'_{\mathbb{R}}$ constructed in [F] arises as $d\theta$, where θ is the 1-form which sends the above tangent vector to $\kappa(X, Y_1 - Y_2)$. However, one can show that Θ induces an identification between the 1-forms θ and λ, and hence Θ induces an identification between (the restriction of) the pseudokähler form in the present paper and the one in [F].

8. Proof of Theorem 6

Proof of (1). Write H for $GL(V)$. The map $\psi : G \to H$ induces an embedding $\psi : \mathfrak{g} \hookrightarrow \mathfrak{h}$. Since $\mathfrak{g}$ is reductive, we can choose a G-stable complement of $\mathfrak{g}$ in $\mathfrak{h}$ and obtain embeddings $\Gamma : GL(\mathfrak{g}) \hookrightarrow GL(\mathfrak{h})$ and $\Gamma : gl(\mathfrak{g}) \hookrightarrow gl(\mathfrak{h})$.
Suppose that $(iX, r) \in M'$; for any eigenvalue λ of $\psi(X)$, $|\mathrm{Re}\,\lambda| < \pi/2$. Since the eigenvalues of $\mathrm{ad}_{\psi(X)}$ are the pairwise differences of the eigenvalues of $\psi(X)$, we have that for any eigenvalue α of $\mathrm{ad}_{\psi(X)}$, $|\mathrm{Re}\,\alpha| < \pi$. However $\mathrm{ad}_{\psi(X)} = \Gamma(\mathrm{ad}_X)$, and Γ is an embedding, so we can say that for any eigenvalue β of ad_X, $|\mathrm{Re}\,\beta| < \pi$. In particular, this shows that $(iX, r) \in M$.
Proof of (2,3). These are trivial.
Proof of (4). Let $(iX_1, r_1), (iX_2, r_2) \in M'$ and suppose that $e^{iX_1} r_1 = e^{iX_2} r_2$. Applying the map $\eta : g \mapsto g\overline{g}^{-1}$, we have that $e^{2iX_1} = e^{2iX_2}$. Applying ψ, we have $e^{\psi(2iX_1)} = e^{\psi(2iX_2)}$. Let λ be any eigenvalue of $\psi(X_1)$ or $\psi(X_2)$. By assumption, $|\mathrm{Re}\,\lambda| < \pi/2$, hence for any eigenvalue α of $\psi(2iX_1)$ or $\psi(2iX_2)$, we have $|\mathrm{Im}\,\alpha| < \pi$. By a well-known property of the exponential map for linear groups (see[V, p. 111]), we may conclude that $\psi(2iX_1) = \psi(2iX_2)$. Since ψ is injective on the level of Lie algebras, we have $X_1 = X_2$, and $r_1 = r_2$. $\square$

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