On the Inner Daniell-Stone and Riesz Representation Theorems

Heinz König

Received: October 19, 1999
Revised: May 25, 2000

Communicated by Alfred K. Louis

Abstract. The paper deals with the context of the inner Daniell-Stone and Riesz representation theorems, which arose within the new development in measure and integration in the book 1997 and subsequent work of the author. The theorems extend the traditional ones, in case of the Riesz theorem to arbitrary Hausdorff topological spaces. The extension enforces that the assertions attain different forms. The present paper wants to exhibit special situations in which the theorems retain their familiar appearance.

1991 Mathematics Subject Classification: 28A12, 28A25, 28C05, 28C15
Keywords and Phrases: Inner extensions, inner premeasures, Radon measures, Radon premeasures, positive (=isotone) linear functionals, inner sources, inner preintegrals, Radon preintegrals, tightness, theorem of Kisyński

In the recent book [3] on measure and integration (cited as MI) and in subsequent papers [4]-[10] the present author attempted to restructure the area of the basic extension and representation procedures and results, and to develop the implications on various issues in measure and integration and beyond. One main point was to extend the Riesz representation theorem in terms of Radon measures on locally compact Hausdorff topological spaces, one of the most famous and important theorems in abstract analysis, to arbitrary Hausdorff topological spaces. The resultant theorem in MI section 16 was a direct specialization of the new inner type Daniell-Stone representation theorem in terms of abstract measures in MI section 15. This is in quite some contrast to the traditional situation, where the Daniell-Stone theorem does not furnish the Riesz theorem.

However, the two new theorems look different from their traditional versions, because of the inherent so-called *tightness* conditions. The conditions of this type came up in the characterization of Radon premeasures due to Kisyński [2], and dominated the subsequent extension and representation theories ever since. They are an unavoidable consequence of the transition from rings of subsets to lattices, and from lattice subspaces of functions to lattice cones, a transition which forms the basis of the theories in question. It is of course desirable to exhibit comprehensive special situations in which the relevant tightness conditions become automatic facts, as it has been done in the second part of MI section 7 in the extension theories for set functions.

The present paper wants to obtain some such situations. Section 1 recalls the context. Then section 2 considers the Daniell-Stone theorem, while section 3 specializes to the Riesz theorem. At last the short section 4 uses the occasion to comment on related recent work of Zakharov and Mikhalev [13]-[16].

1. Inner Preintegrals

We adopt the terms of MI but shall recall the less familiar ones. The extension and representation theories in MI come in three parallel versions. They are marked $\bullet = \star\sigma\tau$, where $\star$ is to be read as *finite*, σ as *sequential* or countable, and τ as *nonsequential* or arbitrary (or as the respective adverbs).

Let X be a nonvoid set. For a nonvoid set system $\mathfrak{S}$ in X we define $\mathfrak{S}^\bullet$ and $\mathfrak{S}_\bullet$ to consist of the unions and intersections of its nonvoid $\bullet$ subsystems. If $\varnothing \in \mathfrak{S}$ then for an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ we define the *outer* and *inner* $\bullet$ *envelopes* $\varphi^\bullet, \varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\begin{aligned}\varphi^\bullet(A) &= \inf\{\sup_{S\in\mathfrak{M}} \varphi(S) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid} \bullet \text{ with } \mathfrak{M} \uparrow\supset A\},\\ \varphi_\bullet(A) &= \sup\{\inf_{S\in\mathfrak{M}} \varphi(S) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid} \bullet \text{ with } \mathfrak{M} \downarrow\subset A\},\end{aligned}$$

in the obvious terms of MI and with the usual convention $\inf\varnothing := \infty$. For a nonvoid function class $E \subset [0,\infty]^X$ on X we define $E^\bullet$ and $E_\bullet$ to consist of the pointwise suprema and infima of its nonvoid $\bullet$ subclasses. If $0 \in E$ then for an isotone functional $I : E \to [0,\infty]$ with $I(0) = 0$ we define the *outer* and *inner* $\bullet$ *envelopes* $I^\bullet, I_\bullet : [0,\infty]^X \to [0,\infty]$ to be

$$\begin{aligned}I^\bullet(f) &= \inf\{\sup_{u\in M} I(u) : M \subset E \text{ nonvoid} \bullet \text{ with } M \uparrow\geqq f\},\\ I_\bullet(f) &= \sup\{\inf_{u\in M} I(u) : M \subset E \text{ nonvoid} \bullet \text{ with } M \downarrow\leqq f\}.\end{aligned}$$

In the sequel we restrict ourselves to the *inner* theories, but note that in MI and in [7]-[9] the outer ones are presented as well. Also it is explained that in some more abstract frame at least the outer and inner extension theories for set functions are identical. For concrete purposes the inner approach turns out to be the more important one. But this approach requires that one starts with *finite* set functions $\varphi : \mathfrak{S} \to [0,\infty[$, and likewise with $E \subset [0,\infty[^X$ and $I : E \to [0,\infty[$.

Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$ and $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$. We define an *inner* $\bullet$ *extension* of φ to be an extension $\alpha : \mathfrak{A} \to [0,\infty]$

of φ which is a content on a ring, such that also $\mathfrak{S}_\bullet \subset \mathfrak{A}$ and

α is inner regular $\mathfrak{S}_\bullet$, and
$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (which is void for $\bullet = \star$).

Then we define φ to be an *inner* $\bullet$ *premeasure* iff it admits inner $\bullet$ extensions. The *inner* $\bullet$ *main theorem* MI 6.31 characterizes those φ which are inner $\bullet$ premeasures, and then describes all inner $\bullet$ extensions of φ. The theorem is in terms of the *inner* $\bullet$ *envelopes* $\varphi_\bullet$ of φ defined above and of their so-called satellites, and with *inner* $\bullet$ *tightness* as the essential condition. We shall not repeat the main theorem, as it has been done in [7] section 1 and [8] section 1, but instead quote an implication which will be referred to in the sequel.

Recollection 1.1 (for $\bullet = \sigma\tau$). *Let $\mathfrak{S}$ be a lattice and $\mathfrak{A}$ be a σ algebra in X with $\varnothing \in \mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{A} \subset \mathrm{A}\sigma(\mathfrak{S}\top\mathfrak{S}_\bullet)$* (where $\top$ denotes the *transporter*). *Then there is a one-to-one correspondence between the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ and the measures $\alpha : \mathfrak{A} \to [0,\infty]$ such that*

$\alpha|\mathfrak{S} < \infty$ and hence $\alpha|\mathfrak{S}_\bullet < \infty$,
α is inner regular $\mathfrak{S}_\bullet$, and
$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous.

The correspondence is $\alpha = \varphi_\bullet|\mathfrak{A}$ and $\varphi = \alpha|\mathfrak{S}$.

For the next step we recall from MI section 11 the integral of Choquet type called the *horizontal* integral. Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$. We form the function classes

$$\begin{aligned} \mathrm{LM}(\mathfrak{S}) &: \text{ the } f \in [0,\infty]^X \text{ such that } [f > t] \in \mathfrak{S} \text{ for all } t > 0, \\ \mathrm{UM}(\mathfrak{S}) &: \text{ the } f \in [0,\infty]^X \text{ such that } [f \geqq t] \in \mathfrak{S} \text{ for all } t > 0. \end{aligned}$$

Let $\varphi : \mathfrak{S} \to [0,\infty]$ be an isotone set function with $\varphi(\varnothing) = 0$. We define the integral $⨍ f d\varphi \in [0,\infty]$ with respect to φ

$$\begin{aligned} &\text{for } f \in \mathrm{LM}(\mathfrak{S}) \quad \text{to be} \quad ⨍ f d\varphi = \int_{0\leftarrow}^{\to\infty} \varphi([f > t])dt, \\ &\text{for } f \in \mathrm{UM}(\mathfrak{S}) \quad \text{to be} \quad ⨍ f d\varphi = \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq t])dt, \end{aligned}$$

both times as an improper Riemann integral of a monotone function with values in $[0,\infty]$. It is well-defined since for $f \in \mathrm{LM}(\mathfrak{S}) \cap \mathrm{UM}(\mathfrak{S})$ the two last integrals are equal. If $\mathfrak{S}$ is a σ algebra then $\mathrm{LM}(\mathfrak{S}) = \mathrm{UM}(\mathfrak{S})$ consists of the functions $f \in [0,\infty]^X$ which are measurable $\mathfrak{S}$ in the usual sense, and in case of a measure $\varphi : \mathfrak{S} \to [0,\infty]$ then $⨍ f d\varphi$ is the usual integral $\int f d\varphi$.

After this we introduce the class of functionals which are to be represented. Let $E \subset [0,\infty[^X$ be a lattice cone in the pointwise operations, which is meant to include $0 \in E$. We recall from [9] that the remainder of the present section can be preserved even when E need not be stable under addition. We form the set systems

$$\begin{aligned} \mathfrak{t}(E) &:= \{A \subset X : \chi_A \in E\}, \\ \geqq(E) &:= \{[f \geqq t] : f \in E \text{ and } t > 0\}, \end{aligned}$$

which are lattices with $\varnothing \in \mathfrak{t}(E) \subset \geqq(E)$. E is called *Stonean* iff $f \in E \Rightarrow f \wedge t$, $f - f \wedge t = (f-t)^+ \in E$ for all $t > 0$. We recall from MI 15.2 or [9] 3.2 that for E Stonean

$$\mathfrak{t}(E_\bullet) = \geqq(E_\bullet) = (\geqq(E))_\bullet \supset \geqq(E) \supset \mathfrak{t}(E) \quad \text{for } \bullet = \sigma\tau.$$

Next let $I : E \to [0, \infty[$ be an isotone and positive-linear functional, which implies that $I(0) = 0$. We define the *inner sources* of I to be the isotone set functions $\varphi : \geqq(E) \to [0, \infty[$ with $\varphi(\varnothing) = 0$ which fulfil $I(f) = ⨍ f d\varphi$ for all $f \in E$.
Then we define I to be an *inner* $\bullet$ *preintegral* if it admits inner sources which are inner $\bullet$ premeasures. We note an immediate consequence of the above 1.1 which characterizes the inner $\bullet$ preintegrals via representation in terms of certain measures.

RECOLLECTION 1.2 (for $\bullet = \sigma\tau$). *Let $E \subset [0, \infty[^X$ be a lattice cone and $\mathfrak{A}$ be a σ algebra in X with $(\geqq(E))_\bullet \subset \mathfrak{A} \subset \mathrm{A}\sigma\big(\geqq(E) \top (\geqq(E))_\bullet\big)$. Let $I : E \to [0, \infty[$ be isotone and positive-linear. Then there is a one-to-one correspondence between the inner sources $\varphi : \geqq(E) \to [0, \infty[$ of I which are inner $\bullet$ premeasures, and the measures $\alpha : \mathfrak{A} \to [0, \infty]$ which fulfil $I(f) = \int f d\alpha$ for all $f \in E$ and hence $\alpha | \geqq(E) < \infty$, and are such that*

α is inner regular $(\geqq(E))_\bullet$, and
$\alpha | (\geqq(E))_\bullet$ is downward $\bullet$ continuous.

The correspondence is $\alpha = \varphi_\bullet | \mathfrak{A}$ and $\varphi = \alpha | \geqq(E)$.

We come to the fundamental inner $\bullet$ Daniell-Stone theorem MI 15.9 (for $\bullet = \sigma\tau$), which is an intrinsic characterization of the inner $\bullet$ preintegrals; an extended version is [9] 5.8. The theorem is in terms of the inner $\bullet$ envelopes $I_\bullet$ of I defined above and of their satellites $I_\bullet^v : [0, \infty]^X \to [0, \infty[$ for $v \in E$, defined to be

$$I_\bullet^v(f) = \sup\{\inf_{u \in M} I(u) : M \subset E \text{ nonvoid} \bullet \text{with } M \downarrow\leqq f \text{ and } u \leqq v \,\forall u \in M\}.$$

THEOREM 1.3 (for $\bullet = \sigma\tau$). *Let $E \subset [0, \infty[^X$ be a Stonean lattice cone and $I : E \to [0, \infty[$ be isotone and positive-linear. Then the following are equivalent.*

1) *I is an inner $\bullet$ preintegral.*

2) *I is downward $\bullet$ continuous; and*

$$I(v) - I(u) \leqq I_\bullet(v-u) \quad \textit{for all } u \leqq v \textit{ in } E.$$

3) *I is $\bullet$ continuous at 0; and*

$$I(v) - I(u) \leqq I_\bullet^v(v-u) \quad \textit{for all } u \leqq v \textit{ in } E.$$

In this case $\varphi := I^\star(\chi_.) | \geqq(E)$ is the unique inner source of I which is an inner $\bullet$ premeasure. It fulfils $\varphi_\bullet = I_\bullet(\chi_.)$, and even $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0, \infty]^X$.

We conclude the section with another characterization of the inner $\bullet$ preintegrals. It is of interest because it relates this class to the simpler class of inner

$\star$ premeasures. The proof does not depend on the above inner $\bullet$ Daniell-Stone theorem, but uses some basic results from [9].

Theorem 1.4 (for $\bullet = \sigma\tau$). *Let $E \subset [0,\infty[^X$ be a Stonean lattice cone and $I : E \to [0,\infty[$ be isotone and positive-linear. Then the following are equivalent.*
1) I is an inner $\bullet$ preintegral.
2) I is $\bullet$ continuous at 0; and $\varphi := I^\star(\chi_.)|{\geqq}(E)$ is an inner $\bullet$ premeasure.
3) I is $\bullet$ continuous at 0; and $\phi := I^\star(\chi_.)|({\geqq}(E))_\bullet$ is an inner $\star$ premeasure.
In this case $I_\bullet(\chi_.) = \varphi_\bullet = \phi_\star$.

Proof. 1)$\Rightarrow$2) follows at once from [9] 4.2. 2)$\Rightarrow$1) From [9] 2.3 we see that I is truncable in the sense of that paper. Then [9] 2.12 implies that φ is an inner source of I. Thus I is an inner $\bullet$ preintegral.

1)2)$\Rightarrow$3) Let $\alpha : \mathfrak{A} \to [0,\infty]$ be an inner $\bullet$ extension of φ. i) From MI 6.18 we have $\alpha = \varphi_\bullet|\mathfrak{A}$, and hence from MI 6.5.iii) that $\alpha|({\geqq}(E))_\bullet$ is downward $\bullet$ continuous. ii) From [9] 4.2 we see that I is downward $\bullet$ continuous, and hence from [9] 3.5.1.Inn)2.Inn) that $I^\star|E_\bullet$ is downward $\bullet$ continuous. Because of $({\geqq}(E))_\bullet = \mathfrak{t}(E_\bullet)$ therefore $\phi = I^\star(\chi_.)|({\geqq}(E))_\bullet$ is downward $\bullet$ continuous. iii) On ${\geqq}(E)$ we have $\alpha = \varphi_\bullet = \varphi = I^\star(\chi_.) = \phi$. Since $\alpha|({\geqq}(E))_\bullet$ and ϕ are both downward $\bullet$ continuous by i)ii) it follows that $\alpha|({\geqq}(E))_\bullet = \phi$. Thus α is an inner $\star$ extension of ϕ, and hence ϕ is an inner $\star$ premeasure.

3)$\Rightarrow$2) From [9] 3.6.3) we see that ϕ is $\bullet$ continuous at $\varnothing$, and hence from MI 6.31 that ϕ is an inner $\bullet$ premeasure. Now each inner $\bullet$ extension of ϕ is also an inner $\bullet$ extension of φ. Therefore φ is an inner $\bullet$ premeasure.

It remains to prove $I_\bullet(\chi_.) = \varphi_\bullet = \phi_\star$ under 1)2)3). From [9] 4.2 we know that $I_\bullet(\chi_.) = \varphi_\bullet$. Then $\varphi_\bullet = \phi_\star$ on $({\geqq}(E))_\bullet$, because from [9] 3.5.1.Inn) and $({\geqq}(E))_\bullet = \mathfrak{t}(E_\bullet)$ we have $\varphi_\bullet = I_\bullet(\chi_.) = I^\star(\chi_.) = \phi = \phi_\star$. Since both $\varphi_\bullet$ and $\phi_\star$ are inner regular $({\geqq}(E))_\bullet$ it follows that $\varphi_\bullet = \phi_\star$ partout. $\square$

2. The Inner Daniell-Stone Theorem

The present results will be for $\bullet = \sigma\tau$ as before. We start with a consequence of 1.3 which consists of two parts. The first part has an immediate proof.

Theorem 2.1. *Let $E \subset [0,\infty[^X$ be a Stonean lattice cone.* 1) *Assume that $v - u \in E_\bullet$ for all $u \leqq v$ in E. Then an isotone and positive-linear functional $I : E \to [0,\infty[$ is an inner $\bullet$ preintegral iff it is $\bullet$ continuous at* 0.
2) *Assume that $v - u \in (E_\bullet)^\sigma$ for all $u \leqq v$ in E. Then an isotone and positive-linear functional $I : E \to [0,\infty[$ is an inner $\bullet$ preintegral iff it is $\bullet$ continuous at* 0 *and upward σ continuous.*

A special case of 1) is the situation that $v - u \in E$ for all $u \leqq v$ in E. After MI 14.6-7 it is equivalent to assume that $E = H^+$ for the Stonean lattice subspace $H = E - E \subset \mathbb{R}^X$ (*Stonean* in the usual sense). So this special case furnishes the traditional Daniell-Stone theorem in the versions $\bullet = \sigma\tau$. However, unlike the present procedure the traditional proofs do not lead to measures $\alpha : \mathfrak{A} \to [0,\infty]$ with $I(f) = \int f d\alpha$ for all $f \in E$ which have the fundamental additional *inner* properties recorded in 1.2 above. The reason is that those proofs are based on *outer* procedures. In order to arrive at the present inner type entities one has to mount the so-called *essential* construction on top of them. This is a formidable detour. We have clarified all this in [7] section 5.

Proof of 1). To be shown is that the assumption implies the tightness condition in 1.3.3). Fix $u \leqq v$ in E, and let $M \subset E$ be nonvoid $\bullet$ with $M \downarrow v-u$ and $h \leqq v$ for all $h \in M$. For $h \in M$ then $h \geqq v-u$ and hence $I(h) \geqq I(v)-I(u)$. It follows that $I(v)-I(u) \leqq \inf\{I(h): h \in M\} \leqq I_\bullet^v(v-u)$. □

Proof of 2). We show that the assumption combined with I upward σ continuous implies the tightness condition in 1.3.3). Fix $u \leqq v$ in E, and then a sequence $(f_l)_l$ in $E_\bullet$ with $f_l \uparrow v-u$. i) For each $l \in \mathbb{N}$ there exists an $M(l) \subset E$ nonvoid $\bullet$ such that $M(l) \downarrow f_l$ and $h \leqq v$ for all $h \in M(l)$. We note that then the

$$N(l) := \{h_1 \vee \cdots \vee h_l : h_k \in M(k) \text{ for } k = 1, \cdots, l\} \subset E \quad \text{for } l \in \mathbb{N}$$

do the same. Thus we can assume that for each $g \in M(l+1)$ there is an $f \in M(l)$ such that $f \leqq g$. ii) Now fix $\varepsilon > 0$, and then $u_l \in M(l)$ for $l \in \mathbb{N}$ such that

$$I(u_l) \leqq c_l + \frac{\varepsilon}{2^l} \quad \text{with } c_l := \inf\{I(h): h \in M(l)\}.$$

Then the $v_l := u_1 \vee \cdots \vee u_l \in E$ fulfil $v_l \geqq u_l \geqq f_l$ and hence $v_l \uparrow \geqq v-u$. We show via induction that

$$I(v_l) \leqq c_l + \varepsilon\big(1 - \frac{1}{2^l}\big) \quad \text{for } l \in \mathbb{N}.$$

The case $l = 1$ is clear. For the induction step $1 \leqq l \Rightarrow l+1$ we note from i) that $v_l \wedge u_{l+1} \geqq u_l \wedge u_{l+1}$ is $\geqq$ some member of $M(l)$, so that $I(v_l \wedge u_{l+1}) \geqq c_l$. Thus from $v_{l+1} + v_l \wedge u_{l+1} = v_l + u_{l+1}$ it follows that

$$\begin{aligned} I(v_{l+1}) &= I(v_l) + I(u_{l+1}) - I(v_l \wedge u_{l+1}) \\ &\leqq c_l + \varepsilon\big(1 - \frac{1}{2^l}\big) + c_{l+1} + \frac{\varepsilon}{2^{l+1}} - c_l = c_{l+1} + \varepsilon\big(1 - \frac{1}{2^{l+1}}\big). \end{aligned}$$

iii) From ii) we obtain on the one hand $c_l \leqq I_\bullet^v(v-u)$ and hence $\lim_{l\to\infty} I(v_l) \leqq I_\bullet^v(v-u) + \varepsilon$. On the other hand $(u+v_l) \wedge v \uparrow v$ because $v \geqq (u+v_l) \wedge v \geqq (u+f_l) \wedge v = u + f_l \uparrow v$, and hence

$$I(u) + \lim_{l\to\infty} I(v_l) \geqq \lim_{l\to\infty} I((u+v_l) \wedge v) = I(v).$$

It follows that $I(v) - I(u) \leqq I_\bullet^v(v-u)$. □

In 2.1.2) the condition that I be upward σ continuous cannot be dispensed with. This will be seen after 3.10 below.

In the sequel we shall exhibit a class of Stonean lattice cones $E \subset [0,\infty[^X$ for which the assumption of 2.1.2) is fulfilled. We need some preparations.

Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$. We define $\mathrm{S}(\mathfrak{S})$ to consist of the positive-linear combinations of the characteristic functions χ_S of the $S \in \mathfrak{S}$. We know from MI 11.4 that $\mathrm{S}(\mathfrak{S})$ consists of the functions $X \to [0,\infty[$ with finite value set which are in $\mathrm{LM}(\mathfrak{S})$, and the same with $\mathrm{UM}(\mathfrak{S})$. We define $f \in [0,\infty]^X$ to be *enclosable* $\mathfrak{S}$ iff $f \leqq u$ for some $u \in \mathrm{S}(\mathfrak{S})$. This means of course that f be $< \infty$ and bounded above, and $= 0$ outside some member of $\mathfrak{S}$. At last we define $\mathrm{LMo}(\mathfrak{S})$ and $\mathrm{UMo}(\mathfrak{S})$ to consist of those members of $\mathrm{LM}(\mathfrak{S})$ and $\mathrm{UM}(\mathfrak{S})$ which are enclosable $\mathfrak{S}$.

After this we recall the assertion MI 22.1 on monotone approximation: *For each $f \in \mathrm{LM}(\mathfrak{S}) \cup \mathrm{UM}(\mathfrak{S})$ there exists a sequence $(f_n)_n$ in $\mathrm{S}(\mathfrak{S})$ such that $f_n \uparrow f$ pointwise, and in supnorm on $[f \leqq c]$ for each $0 < c < \infty$.* We shall need the counterpart for downward monotone approximation.

Lemma 2.2. *For each $f \in \mathrm{LMo}(\mathfrak{S}) \cup \mathrm{UMo}(\mathfrak{S})$ there exists a sequence $(f_n)_n$ in $\mathrm{S}(\mathfrak{S})$ such that $f_n \downarrow f$ pointwise and in supnorm.*

Proof. Assume that $f \leqq c$ with $0 < c < \infty$ and that $f = 0$ outside $S \in \mathfrak{S}$. For the subdivision $\mathfrak{t} : 0 = t(0) < t(1) < \cdots < t(r) = c$ we form $\delta(\mathfrak{t}) = \max\{t(l) - t(l-1) : l = 1, \cdots, r\}$. We define $u_{\mathfrak{t}} \in \mathrm{S}(\mathfrak{S})$ to be

$$u_{\mathfrak{t}} = \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f>t(l)]} \quad \text{when } f \in \mathrm{LMo}(\mathfrak{S}),$$

$$u_{\mathfrak{t}} = \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f\geqq t(l)]} \quad \text{when } f \in \mathrm{UMo}(\mathfrak{S}),$$

and $v_{\mathfrak{t}} \in \mathrm{S}(\mathfrak{S})$ to be

$$v_{\mathfrak{t}} = \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f>t(l-1)][*]} \quad \text{when } f \in \mathrm{LMo}(\mathfrak{S}),$$

$$v_{\mathfrak{t}} = \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f\geqq t(l-1)][*]} \quad \text{when } f \in \mathrm{UMo}(\mathfrak{S}),$$

where $[*]$ is to mean that for $l = 1$ one has to take S instead of $[f > 0]$ when $f \in \mathrm{LMo}(\mathfrak{S})$ and instead of $[f \geqq 0]$ when $f \in \mathrm{UMo}(\mathfrak{S})$. From the basic lemma MI 11.6 one verifies that $u_{\mathfrak{t}} \leqq f \leqq v_{\mathfrak{t}}$ and $v_{\mathfrak{t}} \leqq u_{\mathfrak{t}} + \delta(\mathfrak{t})\chi_A$, and moreover that $\mathfrak{t} \mapsto u_{\mathfrak{t}}$ is isotone and $\mathfrak{t} \mapsto v_{\mathfrak{t}}$ is antitone with respect to refinement in $\mathfrak{t}$. Now take for $n \in \mathbb{N}$ the subdivision $\mathfrak{t} : t(l) = cl2^{-n}$ for $l = 0, 1, \cdots, 2^n$ with $\delta(\mathfrak{t}) = c2^{-n}$. Then the assertions are all clear. □

The final preparation will be on the monotone approximation of differences.

Lemma 2.3. *Let $\mathfrak{S}$ and $\mathfrak{K}$ be lattices in X with $\varnothing \in \mathfrak{S} \subset \mathfrak{K}$ such that $B \setminus A \in \mathfrak{K}^\sigma$ for all $A \subset B$ in $\mathfrak{S}$. Then for each pair of functions $u \leqq v < \infty$ in $\mathrm{UM}(\mathfrak{S})$ there exists a sequence $(f_n)_n$ of functions in $\mathrm{S}(\mathfrak{K})$ enclosable $\mathfrak{S}$ such that $f_n \uparrow v - u$.*

Proof. The first part of the proof assumes that $u \leqq v \leqq c$ with $0 < c < \infty$ and that $v = 0$ outside some $S \in \mathfrak{S}$. 1) We fix $\mathfrak{t} : 0 = t(0) < t(1) < \cdots < t(r) = c$ with $\delta(\mathfrak{t}) = \max\{t(l) - t(l-1) : l = 1, \cdots, r\}$ as before and define $u_{\mathfrak{t}}, v_{\mathfrak{t}} \in \mathrm{S}(\mathfrak{S})$ to be

$$u_{\mathfrak{t}} = \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[u\geqq t(l)]},$$

$$v_{\mathfrak{t}} = \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[v\geqq t(l)]},$$

so that $u_{\mathfrak{t}} \leqq v_{\mathfrak{t}}$ are $= 0$ outside S. From MI 11.6 we have $u_{\mathfrak{t}} \leqq u \leqq u_{\mathfrak{t}} + \delta(\mathfrak{t})\chi_S$ and $v_{\mathfrak{t}} \leqq v \leqq v_{\mathfrak{t}} + \delta(\mathfrak{t})\chi_S$, and hence

$$\begin{aligned} v - u \ &\geqq\ v_{\mathfrak{t}} - u_{\mathfrak{t}} - \delta(\mathfrak{t})\chi_S \geqq v - u - 2\delta(\mathfrak{t})\chi_S, \\ v - u \ &\geqq\ (v_{\mathfrak{t}} - u_{\mathfrak{t}} - \delta(\mathfrak{t})\chi_S)^+ = (v_{\mathfrak{t}} - u_{\mathfrak{t}} - \delta(\mathfrak{t}))^+ \geqq v - u - 2\delta(\mathfrak{t})\chi_S. \end{aligned}$$

2) For fixed $l = 1, \cdots, r$ there is a sequence $(K(l,n))_n$ in $\mathfrak{K}$ such that $K(l,n) \uparrow [v \geqq t(l)] \setminus [u \geqq t(l)]$. We form the functions

$$h_n := \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{K(l,n)} \in \mathrm{S}(\mathfrak{K}) \quad \text{for } n \in \mathbb{N},$$

so that $h_n = 0$ outside S. Then $h_n \uparrow v_{\mathfrak{t}} - u_{\mathfrak{t}}$. Therefore the functions $g_n := (f_n - \delta(\mathfrak{t}))^+ \in \mathrm{S}(\mathfrak{K})$ are $= 0$ outside S as well, and $g_n \uparrow (v_{\mathfrak{t}} - u_{\mathfrak{t}} - \delta(\mathfrak{t}))^+ =: g$ with $v - u \geqq g \geqq v - u - 2\delta(\mathfrak{t})\chi_S$. 3) From 1)2) we obtain for each $l \in \mathbb{N}$ a sequence $(g_n^l)_n$ in $\mathrm{S}(\mathfrak{K})$ with $g_n^l = 0$ outside S such that $g_n^l \uparrow$ some g^l with $v - u \geqq g^l \geqq v - u - \frac{1}{l}\chi_S$. We define $f_n := g_n^1 \vee \cdots \vee g_n^n \in \mathrm{S}(\mathfrak{K})$ for $n \in \mathbb{N}$, so that $f_n = 0$ outside S. Then $f_n \uparrow$ some $f \leqq v - u$. From $f_n \geqq g_n^l$ for $n \geqq l$ we obtain $f \geqq g^l \geqq v - u - \frac{1}{l}\chi_S$ for $l \in \mathbb{N}$. Therefore $f = v - u$. 4) Thus the result of the first part is a sequence $(f_n)_n$ in $\mathrm{S}(\mathfrak{K})$ with $f_n = 0$ outside S such that $f_n \uparrow v - u$.

The second part of the proof will obtain the full assertion. Thus let $u \leqq v < \infty$ in $\mathrm{UM}(\mathfrak{S})$. We fix a pair of numerical sequences $0 < a_l < b_l < \infty$ with $a_l \downarrow 0$ and $b_l \uparrow \infty$. We form

$$u_l := (u - a_l)^+ \wedge (b_l - a_l) \quad \text{and} \quad v_l := (v - a_l)^+ \wedge (b_l - a_l),$$

so that $u_l, v_l \in \mathrm{UM}(\mathfrak{S})$ with $u_l \leqq v_l \leqq b_l - a_l$ which are $= 0$ outside $[v \geqq a_l]$. 1) We claim that $v_l - u_l \leqq v_{l+1} - u_{l+1}$, which can also be written $u_{l+1} - u_l \leqq v_{l+1} - v_l$. Thus the claim is that the function $\vartheta : [0, \infty[\to \mathbb{R}$, defined to be

$$\vartheta(x) = (x - a_{l+1})^+ \wedge (b_{l+1} - a_{l+1}) - (x - a_l)^+ \wedge (b_l - a_l) \quad \text{for } 0 \leqq x < \infty,$$

is monotone increasing. To see this note that $0 < a_{l+1} \leqq a_l < b_l \leqq b_{l+1} < \infty$. Since ϑ is continuous it remains to show that it is monotone increasing in each of the closed subintervals of $[0, \infty[$ thus produced. Now one verifies that

$$\begin{aligned} \text{on } [0, a_{l+1}] \ &:\ \vartheta(x) = 0, \\ \text{on } [a_{l+1}, a_l] \ &:\ \vartheta(x) = x - a_{l+1}, \\ \text{on } [a_l, b_l] \ &:\ \vartheta(x) = a_l - a_{l+1}, \\ \text{on } [b_l, b_{l+1}] \ &:\ \vartheta(x) = x - a_{l+1} - b_l + a_l, \\ \text{on } [b_{l+1}, \infty[\ &:\ \vartheta(x) = b_{l+1} - a_{l+1} - b_l + a_l. \end{aligned}$$

Thus the assertion follows. 2) From the first part of the proof we obtain for each fixed $l \in \mathbb{N}$ a sequence $(f_n^l)_n$ in $\mathrm{S}(\mathfrak{K})$ with $f_n^l = 0$ outside $[v \geqq a_l]$ such that $f_n^l \uparrow v_l - u_l$. We define $f_n := f_n^1 \vee \cdots \vee f_n^n \in \mathrm{S}(\mathfrak{K})$ for $n \in \mathbb{N}$, so that $f_n = 0$ outside $[v \geqq a_n]$. Then $f_n \uparrow$ some $f \in [0, \infty]^X$. We claim that $f = v - u$, which will complete the proof. From 1) we see that $v_l - u_l \uparrow v - u$. Thus on the one hand $f_n \leqq (v_1 - u_1) \vee \cdots \vee (v_n - u_n) = v_n - u_n$ and hence $f \leqq v - u$. On the other hand $f_n \geqq f_n^l$ for $n \geqq l$ and hence $f \geqq v_l - u_l$ for $l \in \mathbb{N}$, so that $f \geqq v - u$. The assertion follows. □

Combination 2.4. *Let $E \subset [0,\infty[^X$ be a Stonean lattice cone and $\mathfrak{S}$ be a lattice in X with $\geqq(E) \subset \mathfrak{S} \subset \geqq(E_\bullet)$. Then* 1) $\mathfrak{S}_\bullet = \geqq(E_\bullet)$ *and* $\mathrm{UMo}(\mathfrak{S}_\bullet) \subset \mathrm{E}_\bullet \subset \mathrm{UM}(\mathfrak{S}_\bullet)$. 2) *Assume that $B \setminus A \in (\mathfrak{S}_\bullet)^\sigma$ for all $A \subset B$ in $\mathfrak{S}$. Then $v - u \in (E_\bullet)^\sigma$ for all $u \leqq v$ in E.*

Proof. 1) We know that $\mathfrak{t}(E_\bullet) = \geqq(E_\bullet) = (\geqq(E))_\bullet$. Therefore $\mathfrak{S}_\bullet \subset \geqq(E_\bullet) = \mathfrak{t}(E_\bullet)$, that is $\chi_T \in E_\bullet$ for all $T \in \mathfrak{S}_\bullet$. Since $E_\bullet$ is a cone it follows that $\mathrm{S}(\mathfrak{S}_\bullet) \subset E_\bullet$. Thus 2.2 implies that $\mathrm{UMo}(\mathfrak{S}_\bullet) \subset \mathrm{E}_\bullet$. In the other direction $\geqq(E) \subset \mathfrak{S}$ implies that $\geqq(E_\bullet) = (\geqq(E))_\bullet \subset \mathfrak{S}_\bullet$ or $E_\bullet \subset \mathrm{UM}(\mathfrak{S}_\bullet)$. 2) For $u \leqq v$ in $E \subset \mathrm{UM}(\mathfrak{S})$ we obtain from 2.3 a sequence $(f_n)_n$ in $\mathrm{S}(\mathfrak{S}_\bullet) \subset E_\bullet$ such that $f_n \uparrow v - u$. Thus $v - u \in (E_\bullet)^\sigma$. □

We combine 2.4.2) with the above 2.1.2) to obtain the other main result of the present section.

Theorem 2.5. *Let $E \subset [0,\infty[^X$ be a Stonean lattice cone and $\mathfrak{S}$ be a lattice in X with $\geqq(E) \subset \mathfrak{S} \subset \geqq(E_\bullet)$. Assume that $\mathfrak{S}$ satisfies $B \setminus A \in (\mathfrak{S}_\bullet)^\sigma$ for all $A \subset B$ in $\mathfrak{S}$. Then an isotone and positive-linear functional $I : E \to [0,\infty[$ is an inner $\bullet$ preintegral iff it is $\bullet$ continuous at* 0 *and upward σ continuous.*

3. The Riesz Representation Theorem

The present section assumes a Hausdorff topological space X with its obvious set systems $\mathrm{Op}(X)$ and $\mathrm{Cl}(X)$, $\mathrm{Comp}(X) =: \mathfrak{K}$ and its σ algebra $\mathrm{Bor}(X) =: \mathfrak{B}$. We start with a little historical sketch on Radon measures.

A Borel measure $\alpha : \mathfrak{B} \to [0,\infty]$ is called *Radon* iff $\alpha|\mathfrak{K} < \infty$ and α is inner regular $\mathfrak{K}$. When in particular X is locally compact then all these measures are *locally finite* in the obvious sense. There is a related notion, which in earlier presentations sometimes even cut out the present one. Let a Borel measure $\beta : \mathfrak{B} \to [0,\infty]$ be called *associate Radon* iff $\beta|\mathfrak{K} < \infty$ and β is inner regular $\mathfrak{K}$ at $\mathrm{Op}(X)$ and outer regular $\mathrm{Op}(X)$. Then Schwartz [12] pp.12-15 established a one-to-one correspondence between the locally finite Radon measures $\alpha : \mathfrak{B} \to [0,\infty]$ and the associate Radon measures $\beta : \mathfrak{B} \to [0,\infty]$, which is unique both under $\alpha|\mathfrak{K} = \beta|\mathfrak{K}$ and under $\alpha|\mathrm{Op}(X) = \beta|\mathrm{Op}(X)$. Thus he was led to include local finiteness in the definition of Radon measures, but this could be abandoned since.

After this a set function $\phi : \mathfrak{K} \to [0,\infty[$ is called a *Radon premeasure* iff it can be extended to some Radon measure, and then of course to the unique one $\alpha := \phi_\star|\mathfrak{B}$. It is an obvious problem to characterize those set functions $\phi : \mathfrak{K} \to [0,\infty[$ which are Radon premeasures. There appeared two such characterizations at about the same time, in 1968 in Kisyński [2] and in 1969 in Bourbaki [1] section 3 théorème 1 p.43 (the latter restricted to local finiteness).

Kisyński Theorem 3.1. *For an isotone set function $\phi : \mathfrak{K} \to [0,\infty[$ the following are equivalent.* 1) *ϕ is a Radon premeasure.* 2) *ϕ is supermodular* *with $\phi(\varnothing) = 0$; and*

$$\phi(B) - \phi(A) \leqq \phi_\star(B \setminus A) \quad \textit{for all } A \subset B \textit{ in } \mathfrak{K}.$$

Bourbaki Theorem 3.2. *For an isotone set function $\phi : \mathfrak{K} \to [0, \infty[$ the following are equivalent.* 1) *ϕ is a Radon premeasure.* 2) *$\phi(A \cup B) \leqq \phi(A) + \phi(B)$ for all $A, B \in \mathfrak{K}$, with $=$ when $A \cap B = \varnothing$; and ϕ is downward τ continuous.*

These two characterizations are so different that they must come from different conceptions. In fact, it turned out that Kisyński had captured the adequate concept in order to prepare the transition from topological to abstract measure and integration, which then started in no time as described in the introduction to MI. At present the above 3.1 is contained in MI 9.1, which is a simple consequence of the inner $\bullet$ main theorem MI 6.31. Moreover MI 9.1 asserts for each $\bullet = \star\sigma\tau$ that $\phi : \mathfrak{K} \to [0, \infty[$ is a Radon premeasure iff it is an inner $\bullet$ premeasure, and that in this case all three $\phi_\bullet$ are equal. The reason for this coincidence are the two properties of the lattice $\mathfrak{K}$ that $\mathfrak{K} = \mathfrak{K}_\tau$ and that $\mathfrak{K}$ is τ compact (recall that a set system in an abstract set is called $\bullet$ compact iff each of its nonvoid $\bullet$ subsystems $\mathfrak{M}$ with $\mathfrak{M} \downarrow \varnothing$ has $\varnothing \in \mathfrak{M}$).

Then in 3.2 the implication 1)$\Rightarrow$2) is contained in the inherent fact that the inner τ premeasures are downward τ continuous. However, the implication 2)$\Rightarrow$1) and thus the characterization asserted in 3.2 appears to be limited to the topological context in the strict sense. The remark below wants to serve as an illustration.

Remark 3.3. *Let $\mathfrak{S}$ be a lattice in an abstract set with $\varnothing \in \mathfrak{S}$ which fulfils $\mathfrak{S} = \mathfrak{S}_\tau$ and is τ compact. Let $\phi : \mathfrak{S} \to [0, \infty[$ be isotone and modular with $\phi(\varnothing) = 0$, and downward τ continuous. Then ϕ need not be an inner $\bullet$ premeasure for any $\bullet = \star\sigma\tau$.* Our example is the simplest possible one: Let X have more than one element, and fix an $a \in X$. Define $\mathfrak{S}$ to consist of $\varnothing$ and of the finite $S \subset X$ with $a \in S$. Then let $\phi : \mathfrak{S} \to [0, \infty[$ be $\phi(\varnothing) = 0$ and $\phi(S) = \#(A \setminus \{a\})$ for the other $S \in \mathfrak{S}$. It is obvious that $\mathfrak{S}$ and ϕ are as required. Moreover $\phi_\star(A) = 0$ when $a \notin A$ and $\phi_\star(A) = \#(A \setminus \{a\})$ when $a \in A$. Now assume that $\alpha : \mathfrak{A} \to [0, \infty]$ is an extension of ϕ which is a content on a ring and inner regular $\mathfrak{S}$. Thus $\alpha = \phi_\star | \mathfrak{A}$. For all $A \in \mathfrak{S}$ with $a \in A$ then

$$\phi(A) = \alpha(A) = \alpha(\{a\}) + \alpha(A \setminus \{a\}) = \phi_\star(\{a\}) + \phi_\star(A \setminus \{a\}) = 0,$$

which is a contradiction. $\square$

After this excursion we turn to the Riesz representation theorem as obtained in MI section 16, not without notice that the extension of the inner $\bullet$ Daniell-Stone theorem in [9] 5.8 produces of course an extended Riesz theorem. But for the present issue this would not contribute much.

Let $E \subset [0, \infty[^X$ be a lattice cone. Since we want to represent our functionals on E in terms of Radon measures, it is adequate after 1.2 to assume that E satisfies $(\geqq(E))_\bullet = \mathfrak{K}$ for some $\bullet = \sigma\tau$. In particular then $\geqq(E) \subset \mathfrak{K}$, so that the members of E are upper semicontinuous and bounded above. Since the traditional Riesz theorem is for the lattice subspace $\mathrm{CK}(X)$ of the continuous functions $X \to \mathbb{R}$ which vanish outside certain compact subsets of X, that is for the lattice cone $\mathrm{CK}^+(X)$, it is adequate to assume that E be $\subset \mathrm{USCK}^+(X)$, defined to consist of the upper semicontinuous functions $X \to [0,\infty[$ which vanish outside certain compact subsets of X. In the previous notation we have $\mathrm{USCK}^+(X) = \mathrm{UMo}(\mathfrak{K})$. Henceforth the lattice cones $E \subset \mathrm{USCK}^+(X)$ with $(\geqq(E))_\bullet = \mathfrak{K}$ will be called $\bullet$ *rich*; it will become clear that the situation $\bullet = \tau$ is the more important one. From 2.4.1) one obtains the remark below.

Remark 3.4. *If $E \subset \mathrm{USCK}^+(X)$ is a $\bullet$ rich Stonean lattice cone then $E_\bullet = \mathrm{USCK}^+(X)$.*

Examples 3.5. 1) The lattice cone $E = \mathrm{CK}^+(X)$ is τ rich iff X is locally compact. This is a standard fact; see for example MI 16.3. 2) If the lattice cone $E \subset \mathrm{USCK}^+(X)$ satisfies $\mathfrak{S} \subset \geqq(E_\bullet)$ for some lattice $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S} \subset \mathfrak{S}_\bullet = \mathfrak{K}$ then E is $\bullet$ rich. In fact,we have $\mathfrak{S} \subset \geqq(E_\bullet) = (\geqq(E))_\bullet \subset \mathfrak{K}$.

In the sequel let $E \subset \mathrm{USCK}^+(X)$ be a $\bullet$ rich lattice cone. We define an isotone and positive-linear functional $I : E \to [0,\infty[$ to be a *Radon preintegral* iff there exists a Radon measure $\alpha : \mathfrak{B} \to [0,\infty]$ such that $I(f) = \int f d\alpha$ for all $f \in E$. Equivalent is of course that there exists a Radon premeasure $\phi : \mathfrak{K} \to [0,\infty[$ such that $I(f) = ⨍ f d\phi$ for all $f \in E$, and these α and ϕ correspond to each other via $\alpha = \phi_\star|\mathfrak{B}$ and $\phi = \alpha|\mathfrak{K}$.

We turn to the connection with the previous representation theories. We start with 1.2, where $\mathfrak{A}$ can be chosen to be $\mathfrak{B}$.

Proposition 3.6 (for $\bullet = \sigma\tau$). *Let $E \subset \mathrm{USCK}^+(X)$ be a $\bullet$ rich lattice cone and $I : E \to [0,\infty[$ be isotone and positive-linear. Then I is a Radon preintegral iff it is an inner $\bullet$ preintegral. In this case the inner sources $\varphi : \geqq(E) \to [0,\infty[$ of I which are inner $\bullet$ premeasures correspond to the above α and ϕ via $\varphi = \alpha| \geqq(E) = \phi| \geqq(E)$, and $\alpha = \varphi_\bullet|\mathfrak{B}$ and $\phi = \varphi^\star|\mathfrak{K} = \varphi_\bullet|\mathfrak{K}$.*

If in particular E is *Stonean* then first of all the Dini consequence MI 16.4 asserts that all these $I : E \to [0,\infty[$ are τ continuous at 0. Thus 1.3 and 1.4 furnish the Riesz representation theorem in the version which follows.

Theorem 3.7 (for $\bullet = \sigma\tau$). *Let $E \subset \mathrm{USCK}^+(X)$ be a $\bullet$ rich Stonean lattice cone and $I : E \to [0,\infty[$ be isotone and positive-linear. Then the following are equivalent.* 1) *I is a Radon preintegral.*

2) *I is an inner $\bullet$ preintegral.*
3) *$I(v) - I(u) \leqq I_\bullet^v(v-u)$ for all $u \leqq v$ in E.*
4) *$\varphi := I^\star(\chi_.)| \geqq(E)$ is an inner $\bullet$ premeasure.*
5) *$\phi := I^\star(\chi_.)|\mathfrak{K}$ is a Radon premeasure.*

In this case φ is the unique inner source of I which is an inner $\bullet$ premeasure, and ϕ is the unique Radon premeasure which represents I; likewise $\alpha := I_\bullet(\chi_.)|\mathfrak{B}$ is the unique Radon measure which represents I. We have $I_\bullet(\chi_.) = \varphi_\bullet = \phi_\star = \alpha_\star$.

Our ultimate aim is the specialization of 2.1 and 2.5. We have $\bullet = \sigma\tau$ as before, but this time the case $\bullet = \sigma$ is contained in $\bullet = \tau$.

THEOREM 3.8. *Let $E \subset \mathrm{USCK}^+(X)$ be a τ rich Stonean lattice cone.* 1) *Assume that $v - u \in \mathrm{USCK}^+(X)$ for all $u \leqq v$ in E. Then each isotone and positive-linear $I : E \to [0, \infty[$ is a Radon preintegral.*

2) *Assume that $v - u \in (\mathrm{USCK}^+(X))^\sigma$ for all $u \leqq v$ in E. Then an isotone and positive-linear $I : E \to [0, \infty[$ is a Radon preintegral iff it is upward σ continuous.*

In view of 3.5.1) the first assertion 3.8.1) contains the traditional Riesz representation theorem. Thus we have the traditional Daniell-Stone and Riesz theorems both under the same roof (and at the same time, as pointed out after 2.1, the former one enriched to a usable assertion).

THEOREM 3.9. *Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S} \subset \mathfrak{S}_\tau = \mathfrak{K}$ and $E \subset \mathrm{UMo}(\mathfrak{S}) \subset \mathrm{USCK}^+(\mathrm{X})$ be a τ rich Stonean lattice cone. Assume that $B \setminus A \in \mathfrak{K}^\sigma$ for all $A \subset B$ in $\mathfrak{S}$. Then an isotone and positive-linear functional $I : E \to [0, \infty[$ is a Radon preintegral iff it is upward σ continuous.*

Proof. We have in fact $\geqq(E) \subset \mathfrak{S} \subset \geqq(E_\tau)$. Thus the assertion follows from 2.5, and likewise from 3.8.2) and hence from 2.1.2) via 2.4.2). $\square$

We conclude with two illustrative examples, in that we specialize 3.9 to $\mathfrak{S} := \mathfrak{K}$ and to $\mathfrak{S} := \mathfrak{K} \cap (\mathrm{Op}(X))_\sigma$ (=:the compact G_δ subsets).

EXAMPLE 3.10. Assume that $B \setminus A \in \mathfrak{K}^\sigma$ for all $A \subset B$ in $\mathfrak{K}$, that is that $\mathfrak{K}$ is *upward σ full* in the sense of MI section 7. Let $E \subset \mathrm{USCK}^+(X)$ be a τ rich Stonean lattice cone; in particular one can take $E = \mathrm{USCK}^+(X)$ itself. *Then an isotone and positive-linear $I : E \to [0, \infty[$ is a Radon preintegral iff it is upward σ continuous.*

We turn to the counterexample announced after 2.1. As before assume that $B \setminus A \in \mathfrak{K}^\sigma$ for all $A \subset B$ in $\mathfrak{K}$. Consider an isotone and modular set function $\phi : \mathfrak{K} \to [0, \infty[$ with $\phi(\varnothing) = 0$ which is not a Radon premeasure; there is an example with $X = [0, 1]$ in [10] 1.4. In view of MI 11.11 then $I(f) = ⨍ f d\phi$ for all $f \in E = \mathrm{USCK}^+(X)$ defines an isotone and positive-linear $I : E \to [0, \infty[$ which is not a Radon preintegral. Thus we see that in 3.9 and 2.5, and likewise in 3.8.2) and 2.1.2), the condition that I be upward σ continuous cannot be dispensed with.

EXAMPLE 3.11. Assume that the lattice $\mathfrak{S} = \mathfrak{K} \cap (\mathrm{Op}(X))_\sigma$ fulfils $\mathfrak{S}_\tau = \mathfrak{K}$. Let $E \subset \mathrm{UMo}(\mathfrak{S}) \subset \mathrm{USCK}^+(\mathrm{X})$ be a τ rich Stonean lattice cone; in particular one can take $E = \mathrm{UMo}(\mathfrak{S})$ itself in view of MI 11.1.3). *Then an isotone and positive-linear $I : E \to [0, \infty[$ is a Radon preintegral iff it is upward σ continuous.*

4. COMPARISON WITH ANOTHER APPROACH

The present final section wants to relate the previous one to recent work of Zakharov and Mikhalev [13]-[16]; see also the conference abstracts [11][17]. This work has the aim to transfer one basic feature within the Riesz representation theorem to arbitrary Hausdorff topological spaces. It does in fact not even contain the Riesz theorem itself, but rather wants, in the words of the authors, to find a class of linear functionals which via integration is in one-to-one correspondence with the class of Radon measures on the space. Nonetheless this less ambitious aim is called the *General* Riesz-Radon problem.

We retain the terms of the last section. The approach of the authors is via the simple lattice cone $\mathrm{S}(\mathfrak{K})$, but in terms of a certain lattice *subspace*. For a lattice $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$ define $\mathrm{D}(\mathfrak{S}) := \mathrm{S}(\mathfrak{S}) - \mathrm{S}(\mathfrak{S})$, that is to consist of the real-linear combinations of the χ_S for $S \in \mathfrak{S}$. We form the supnorm closure $H(X) := \overline{\mathrm{D}(\mathrm{Cl}(X))}$ in the space of all (Borel measurable) bounded functions $X \to \mathbb{R}$. The members of $H(X)$ are the *metasemicontinuous* functions in the sense of the papers under view; but the definition of the authors is more complicated and involves the so-called *Aleksandrov* set system. Then define $K(X) \subset H(X)$ to consist of the members of $H(X)$ which vanish outside certain compact subsets of X. $H(X)$ and $K(X)$ are lattice subspaces. With the $\mathfrak{K}(A) := \{K \in \mathfrak{K} : K \subset A\}$ for $A \in \mathfrak{K}$ one has

$$\mathrm{S}(\mathfrak{K}) \subset \mathrm{D}(\mathfrak{K}) \subset K(X) = \bigcup_{A \in \mathfrak{K}} \overline{\mathrm{D}(\mathfrak{K}(A))} \subset H(X).$$

The authors consider the isotone and linear functionals $I : K(X) \to \mathbb{R}$. We continue with our own reconstruction. From an obvious manipulation combined with the old Kisyński theorem 3.1 we obtain the assertion which follows.

Assertion 4.1. *Let $I : K(X) \to \mathbb{R}$ be isotone and linear. Then there exists a Radon measure $\alpha : \mathfrak{B} \to [0, \infty]$ such that $I(f) = \int f d\alpha$ for all $f \in K(X)$* (and hence of course a unique one) *iff the set function $\phi := I(\chi_{\cdot})|\mathfrak{K}$ is a Radon premeasure. In view of the Kisyński theorem* 3.1 *this means that*

$$I(\chi_{B \setminus A}) \leqq \sup\{I(\chi_K) : K \in \mathfrak{K} \textit{ with } K \subset B \setminus A\} \quad \textit{for all } A \subset B \textit{ in } \mathfrak{K}.$$

Proof. 1) For fixed $A \in \mathfrak{K}$ the restriction $I|\overline{\mathrm{D}(\mathfrak{K}(A))}$ is an isotone linear functional on the linear subspace $\overline{\mathrm{D}(\mathfrak{K}(A))} \subset K(X)$. One has $\chi_A \in \overline{\mathrm{D}(\mathfrak{K}(A))}$. For $f \in \overline{\mathrm{D}(\mathfrak{K}(A))}$ therefore $|f| \leqq \|f\|\chi_A$ implies that $|I(f)| \leqq \|f\| I(\chi_A)$. Thus $I|\overline{\mathrm{D}(\mathfrak{K}(A))}$ is supnorm continuous. 2) Let $\alpha : \mathfrak{B} \to [0, \infty]$ be a Radon measure. The relation $I(f) = \int f d\alpha$ for all $f \in K(X)$ means that for each fixed $A \in \mathfrak{K}$ one has $I(f) = \int f d\alpha$ for all $f \in \overline{\mathrm{D}(\mathfrak{K}(A))}$, that is after 1) for all $f \in \mathrm{D}(\mathfrak{K}(A))$, that is for all $f \in \mathrm{S}(\mathfrak{K}(A))$. Thus one ends up with $I(f) = \int f d\alpha$ for all $f \in \mathrm{S}(\mathfrak{K})$, which says that $\phi(K) := I(\chi_K) = \alpha(K)$ for all $K \in \mathfrak{K}$. $\square$

After this one notes with surprise that Zakharov-Mikhalev [13]-[16] did not characterize the representable functionals $I : K(X) \to \mathbb{R}$ by the simple Kisyński type condition of 4.1, but by means of a much more complicated equivalent condition. In fact, their condition consists of the two parts

1) I is σ continuous under monotone pointwise convergence; and
2) each sequence $(A(l))_l$ in $\mathrm{R}(\mathfrak{K})$ which decreases $A(l) \downarrow A \subset X$ satisfies

$$\lim_{l\to\infty} I(\chi_{A(l)}) \leqq \sup\{I(\chi_K) : K \in \mathfrak{K} \text{ with } K \subset A\},$$

once more in simplified form, with $\mathrm{R}(\mathfrak{K})$ the ring generated by $\mathfrak{K}$.

The comparison with 4.1 makes clear that this equivalent condition is inadequate in depth in both parts.

This adds to the fact that the basic set-up in the papers under view, that is the limitation to $\mathrm{S}(\mathfrak{K}) \subset \mathrm{D}(\mathfrak{K}) \subset K(X)$, appears to be much too narrow. Thus one more surprise is the sheer extent of the papers. To be sure, there are other conclusions, but the equivalence described above forms their central result.

Acknowledgement

Part of the present work has been done while the author visited the Mathematical Research Institute Oberwolfach in September/October 1999. He wants to express his warmest thanks to the Institute and its director, Professor Matthias Kreck.

References

[1] N. Bourbaki, *Intégration*, Chap. IX. Hermann 1969.
[2] J. Kisyński, On the generation of tight measures. Studia Math. 30 (1968), 141-151.
[3] H. König, *Measure and Integration: An Advanced Course in Basic Procedures and Applications*. Springer 1997.
[4] H. König, Image measures and the so-called image measure catastrophe. Positivity 1 (1997), 255-270.
[5] H. König, What are signed contents and measures? Math.Nachr. 204 (1999), 101-124.
[6] H. König, The product theory for inner premeasures. Note di Mat. 17 (1997), 235-249.
[7] H. König, Measure and integration: Comparison of old and new procedures. Arch. Math. 72 (1999), 192-205.
[8] H. König, Measure and integration: Mutual generation of outer and inner premeasures. Annales Univ. Saraviensis Ser. Math. 9 (1998), 99-122.
[9] H. König, Measure and integration: Integral representations of isotone functionals. Annales Univ.Saraviensis Ser. Math. 9 (1998), 123-153.
[10] H. König, Upper envelopes of inner premeasures. To appear in Annales Inst. Fourier.
[11] A. V. Mikhalev and V. K. Zakharov, Solution of Riesz-Radon Problem. In: Intern. Congress Math. Berlin 1998, Abstracts of Short Comm. and Poster Sess., pp. 141-142.
[12] L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*. Oxford Univ. Press 1973.

[13] V. K. Zakharov and A. V. Mikhalev, The Radon problem for regular measures on an arbitrary Hausdorff space (in Russian). Fund. Prikl. Mat. 3 (1997), 801-808.

[14] V. K. Zakharov and A. V. Mikhalev, Integral representation for Radon measures on an arbitrary Hausdorff space (in Russian). Fund. Prikl. Mat. 3 (1997), 1135-1172.

[15] V. K. Zakharov and A.V. Mikhalev, The problem of integral representation for Radon measures on a Hausdorff space (in Russian). Doklady Akad. Nauk 360 (1998), 13-15. English translation in Doklady Math. 57 (1998), 337-339.

[16] V. K. Zakharov and A. V. Mikhalev, The problem of general Radon representation for an arbitrary Hausdorff space (in Russian). Izvestiya Akad. Nauk 63 (1999), 37-82. English translation in Izvestiya Math. 63 (1999), 881-921.

[17] V. K. Zakharov, A solution of the general Riesz-Radon problem for a Hausdorff space. In: Intern.Congress Math. Berlin 1998, Abstracts of Short Comm. and Poster Sess., p.166.

Heinz König
Universität des Saarlandes,
Mathematik, D-66041
Saarbrücken, Germany
hkoenig@math.uni-sb.de