The Number of Independent Vassiliev Invariants in the Homfly and Kauffman Polynomials

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Abstract. We consider vector spaces $\mathcal{H}_{n,\ell}$ and $\mathcal{F}_{n,\ell}$ spanned by the degree-n coefficients in power series forms of the Homfly and Kauffman polynomials of links with ℓ components. Generalizing previously known formulas, we determine the dimensions of the spaces $\mathcal{H}_{n,\ell}$, $\mathcal{F}_{n,\ell}$ and $\mathcal{H}_{n,\ell}+\mathcal{F}_{n,\ell}$ for all values of n and ℓ. Furthermore, we show that for knots the algebra generated by $\bigoplus_n \mathcal{H}_{n,1}+\mathcal{F}_{n,1}$ is a polynomial algebra with $\dim(\mathcal{H}_{n,1}+\mathcal{F}_{n,1})-1=n+[n/2]-4$ generators in degree $n\geq 4$ and one generator in degrees 2 and 3.

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1 Introduction

Soon after the discovery of the Jones polynomial V ([Jon]), two 2-parameter generalizations of it were introduced: the Homfly polynomial H ([HOM]) and the Kauffman polynomial F ([Ka2]) of oriented links. Let $\mathcal{V}_{n,\ell}$ be the vector space of $\mathbb{Q}$-valued Vassiliev invariants of degree n of links with ℓ components. After a substitution of parameters, the polynomial H (resp. F) can be written as a power series in an indeterminate h, such that the coefficient of h^n is a polynomial-valued Vassiliev invariant p_n (resp. q_n) of degree n. Let $\mathcal{H}_{n,\ell}$ (resp. $\mathcal{F}_{n,\ell}$) be the vector space generated by the coefficients of p_n (resp. q_n) regarded as a subspace of $\mathcal{V}_{n,\ell}$. The dimensions of $\mathcal{H}_{n,\ell}$ and $\mathcal{F}_{n,\ell}$ have been determined in [Men] for $n\geq 0$ and $\ell=1$ and partial results were also known for $\ell>1$. We complete these formulas by calculating $\dim\mathcal{H}_{n,\ell}$, $\dim\mathcal{F}_{n,\ell}$ and $\dim(\mathcal{H}_{n,\ell}+\mathcal{F}_{n,\ell})$ for $n\geq 0$ and all pairs (n,ℓ).

THEOREM 1. *(1) For all $n, \ell \geq 1$ we have*

$$\dim \mathcal{H}_{n,\ell} = \min\left\{n, \left[\frac{n-1+\ell}{2}\right]\right\} = \begin{cases} n & \text{if } n < \ell, \\ \left[\frac{n-1+\ell}{2}\right] & \text{if } n \geq \ell. \end{cases}$$

(2) If $n \geq 4$, then

$$\dim \mathcal{F}_{n,\ell} = \begin{cases} n-1 & \text{if } \ell = 1, \\ 2n-1 & \text{if } \ell \geq 2 \text{ and } n \leq \ell, \\ n+\ell-1 & \text{if } \ell \geq 2 \text{ and } n \geq \ell. \end{cases}$$

The values of $\dim \mathcal{F}_{n,\ell}$ for $n \leq 3$ are given in the following table

(n,ℓ)	$(1,1)$	$(1,\geq 2)$	$(2,1)$	$(2,2)$	$(2,\geq 3)$	$(3,1)$	$(3,2)$	$(3,\geq 3)$
$\dim \mathcal{F}_{n,\ell}$	0	1	1	2	3	1	4	5

(3) For all $n, \ell \geq 1$ we have

$$\dim(\mathcal{H}_{n,\ell} \cap \mathcal{F}_{n,\ell}) = \min\{\dim \mathcal{H}_{n,\ell}, 2\}.$$

In the framework of Vassiliev invariants it is natural to consider the elements of $\bigoplus_{n,\ell}(\mathcal{H}_{n,\ell} \cap \mathcal{F}_{n,\ell})$ as *the common specializations of H and F*. It is known that a one-variable polynomial Y ([CoG], [Kn1], [Lik], [Lie], [Sul]) appears as a lowest coefficient in H and F. This is used in the proof of Theorem 1 to derive lower bounds for $\dim(\mathcal{H}_{n,\ell} \cap \mathcal{F}_{n,\ell})$. Let r_n^ℓ be the coefficient of h^n in the Jones polynomial $V(e^{h/2})$ and let y_n^ℓ be the coefficient of h^n in $Y(e^{h/2})$. Then we have $r_n^\ell, y_n^\ell \in \mathcal{H}_{n,\ell} \cap \mathcal{F}_{n,\ell}$. The following corollary to the proof of Theorem 1 says that the Jones polynomial V and the polynomial Y are *the only* common specializations of H and F in the sense above (compare [Lam] for common specializations in a different sense).

COROLLARY 2. *For all $n \geq 0, \ell \geq 1$ we have $\mathcal{H}_{n,\ell} \cap \mathcal{F}_{n,\ell} = \operatorname{span}\{r_n^\ell, y_n^\ell\}$.*

The main part of the proofs of Theorem 1 and Corollary 2 will not be given on the level of link invariants, but on the level of weight systems. A weight system of degree n is a linear form on a space $\bar{\mathcal{A}}_{n,\ell}$ generated by certain trivalent graphs with ℓ distinguished oriented circles and $2n$ vertices called *trivalent diagrams*. There exists a surjective map W from $\mathcal{V}_{n,\ell}$ to the space $\bar{\mathcal{A}}_{n,\ell}^* = \operatorname{Hom}(\bar{\mathcal{A}}_{n,\ell}, \mathbb{Q})$ of weight systems. The restriction of W to $\mathcal{H}_{n,\ell} + \mathcal{F}_{n,\ell}$ is injective. So we may study the spaces $\mathcal{H}'_{n,\ell} = W(\mathcal{H}_{n,\ell})$ and $\mathcal{F}'_{n,\ell} = W(\mathcal{F}_{n,\ell}) \subseteq \bar{\mathcal{A}}_{n,\ell}^*$ instead of $\mathcal{H}_{n,\ell}$ and $\mathcal{F}_{n,\ell}$. Using an explicit description of weight systems in $\mathcal{H}'_{n,\ell}$ and $\mathcal{F}'_{n,\ell}$ we derive upper bounds for $\dim \mathcal{H}'_{n,\ell}$ and $\dim \mathcal{F}'_{n,\ell}$. We obtain an upper bound for $\dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell})$ from a lower bound for $\dim(\mathcal{H}'_{n,\ell} \cap \mathcal{F}'_{n,\ell})$. We evaluate the weight systems in $\mathcal{H}'_{n,\ell}$ and $\mathcal{F}'_{n,\ell}$ on many trivalent diagrams which gives us lower bounds for $\dim \mathcal{H}'_{n,\ell}$, $\dim \mathcal{F}'_{n,\ell}$ and $\dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell})$. These lower bounds always coincide with the upper bounds. The resulting dimension formulas will imply Theorem 1.

For simplicity of notation we will drop the index ℓ when $\ell = 1$. The fact that the Jones polynomial and the square of the Jones polynomial appear by choosing special values of parameters of the Kauffman polynomial gives us quadratic relations between elements of $\bigoplus_{n=0}^{\infty} \mathcal{F}_{n,\ell}$. We will use the Hopf algebra structure of $\bar{\mathcal{A}} = \bigoplus_{n=0}^{\infty} \bar{\mathcal{A}}_n$ to show that we know all algebraic relations between elements of $\bigoplus_{n=0}^{\infty} \mathcal{H}_n + \mathcal{F}_n$:

Theorem 3. *The algebra generated by $\bigoplus_{n=0}^{\infty} \mathcal{H}_n + \mathcal{F}_n$ is a polynomial algebra with*

$$\max\{\dim(\mathcal{H}_n + \mathcal{F}_n) - 1, 1\} = \max\{n + [n/2] - 4, 1\}$$

generators in degree $n \geq 2$.

If knot invariants v_i satisfy $v_i(K_1) = v_i(K_2)$, then polynomials in the invariants v_i also cannot distinguish the knots K_1 and K_2. By Theorem 3 there is only one algebraic relation between elements $v_i \in \bigoplus_{n=1}^{m-1}(\mathcal{H}_n + \mathcal{F}_n)$ and elements of $\mathcal{H}_m + \mathcal{F}_m$ in each degree $m \geq 4$. This gives us a hint why it is possible to distinguish many knots by comparing their Homfly and Kauffman polynomials.

The plan of the paper is the following. In Section 2 we recall the definitions of the link polynomials H, F, V, Y, and we give the exact definitions of $\mathcal{H}_{n,\ell}$ and $\mathcal{F}_{n,\ell}$. Then we express relations between these polynomials in terms of Vassiliev invariants. In Section 3 we define $\bar{\mathcal{A}}_{n,\ell}$ and recall the connection between the Vassiliev invariants in $\mathcal{H}_{n,\ell} + \mathcal{F}_{n,\ell}$ and their weight systems in $\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell}$. In Section 4 we use a direct combinatorial description of the weight systems in $\mathcal{H}'_{n,\ell}$ and $\mathcal{F}'_{n,\ell}$ to derive upper bounds for $\dim \mathcal{H}'_{n,\ell}$ and $\dim \mathcal{F}'_{n,\ell}$. For the proof of lower bounds we state formulas for values of weight systems in $\mathcal{H}'_{n,\ell}$ and $\mathcal{F}'_{n,\ell}$ on certain trivalent diagrams in Section 5. We prove these formulas by making calculations in the Brauer algebra $\mathbf{Br}_k$. In Section 6 we complete the proofs of Theorem 1, Corollary 2 and Theorem 3 by using a module structure on the space of primitive elements $\mathcal{P}$ of $\bar{\mathcal{A}}$ over Vogel's algebra Λ ([Vog]).

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2 Vassiliev invariants and link polynomials

A singular link is an immersion of a finite number of oriented circles into $\mathbb{R}^3$ whose only singularities are transversal double points. A singular link without double points is called a link. We consider singular links up to orientation preserving diffeomorphisms of $\mathbb{R}^3$. The equivalence classes of this equivalence relation are called singular link types or by abuse of language simply singular links. A link invariant is a map from link types into a set. If v is a link invariant with values in an abelian group, then it can be extended recursively to an invariant of singular links by the local replacement rule $v(L_\times) = v(L_+) - v(L_-)$ (see Figure 1). A link invariant is called a *Vassiliev* invariant of degree n if it vanishes on all singular links with $n+1$ double points. Let $\mathcal{V}_{n,\ell}$ be the vector space of $\mathbb{Q}$-valued Vassiliev invariants of degree n of links with ℓ components.

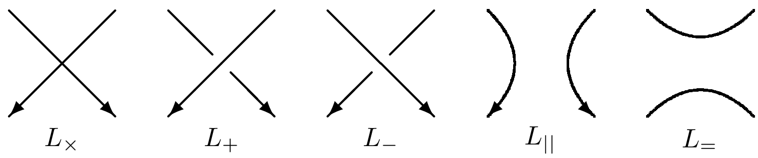


Figure 1: Local modifications (of a diagram) of a (singular) link

Let us recall the definitions of the link invariants H,F,V, and Y (see [HOM], [Ka2], [Jon], and Proposition 4.7 of [Lik]; the normalizations of H and V we will use are equivalent to the original definitions). For a link L, the Homfly polynomial $H_L(x,y) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ is given by

$$xH_{L_+}(x,y) - x^{-1}H_{L_-}(x,y) = yH_{L_{||}}(x,y), \tag{1}$$

$$H_{O^k}(x,y) = \left(\frac{x - x^{-1}}{y}\right)^k. \tag{2}$$

The links in Equation (1) are the same outside of a small ball and differ inside this ball as shown in Figure 1. The symbol O^k denotes the trivial link with $k \geq 1$ components.

A link diagram $L \subset \mathbb{R}^2$ is a generic projection of a link together with the information which strand is the overpassing strand at each double point of the projection. Call a crossing of a link diagram as in L_+ (see Figure 1) positive and a crossing as in L_- negative. Define the *writhe* $w(L)$ of a link diagram L as the number of positive crossings minus the number of negative crossings. Similar to the Homfly polynomial, the Dubrovnik version of the *Kauffman* polynomial $F_L(x,y) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ of a link diagram L is given by

$$xF_{L_+}(x,y) - x^{-1}F_{L_-}(x,y) = y\left(F_{L_{||}}(x,y) - x^{w(L_{=,or}) - w(L_{||})}F_{L_{=,or}}(x,y)\right), \tag{3}$$

$$F_{O^k}(x,y) = \left(\frac{x - x^{-1} + y}{y}\right)^k. \tag{4}$$

Here the link diagrams $L_+, L_-, L_{||}, L_=$ differ inside of a disk as shown in Figure 1 and coincide on the outside of this disk, and $L_{=,or}$ is the link diagram $L_=$

equipped with an arbitrary orientation of the components of the corresponding link. The symbol O^k denotes an arbitrary diagram of the trivial link with $k \geq 1$ components. The Homfly and the Kauffman polynomials are invariants of links. Let $|L|$ denote the number of components of a link L. For the links in Equation (1) we have $|L_+| = |L_-| = |L_{||}| \pm 1$. Since Equations (1) and (2) are sufficient to calculate H this implies $H_L(x,y) = (-1)^{|L|} H_L(x,-y)$ for every link L. The *Jones* polynomial V can be expressed in terms of the Homfly polynomial as

$$V_L(x) := H_L\left(x^2, x^{-1} - x\right) = (-1)^{|L|} H_L\left(x^2, x - x^{-1}\right) \in \mathbb{Z}[x^{\pm 1}].$$

It is easy to see that for every link L we have

$$\widetilde{H}_L(x,y) := y^{|L|} H_L(x,y) \in \mathbb{Z}[x^{\pm 1}, y] \ , \ \widetilde{F}_L(x,y) := y^{|L|} F_L(x,y) \in \mathbb{Z}[x^{\pm 1}, y]. \tag{5}$$

The link invariant Y is defined by

$$Y_L(x) = \widetilde{H}_L(x,0) \in \mathbb{Z}[x^{\pm 1}].$$

After substitutions of parameters we can express H and F as

$$H_L\left(e^{ch/2}, e^{h/2} - e^{-h/2}\right) = \sum_{j=0}^{\infty} \sum_{i=1}^{j+|L|} p_{i,j}^{|L|}(L) c^i h^j \in \mathbb{Q}[c][[h]], \tag{6}$$

$$F_L\left(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}\right) = \sum_{j=0}^{\infty} \sum_{i=1}^{j+|L|} q_{i,j}^{|L|}(L) c^i h^j \in \mathbb{Q}[c][[h]], \tag{7}$$

for the following reasons: Equation (5) implies that the sum over i is limited by $j+|L|$ in these expressions and one sees that no negative powers in h appear and that the sum over i starts with $i = 1$ directly by using the defining equations of H and F with the new parameters. For $j = 0$ we have $p_{i,0}^{|L|} = q_{i,0}^{|L|} = \delta_{i,|L|}$, where $\delta_{i,j}$ is 1 for $i = j$ and is 0 otherwise. It follows from Equations (1) and (3) that the link invariants $p_{i,n}^{\ell}$ and $q_{i,n}^{\ell}$ are in $\mathcal{V}_{n,\ell}$. Define

$$\mathcal{H}_{n,\ell} = \operatorname{span}\{p_{1,n}^{\ell}, p_{2,n}^{\ell}, \ldots, p_{n+\ell,n}^{\ell}\} \subseteq \mathcal{V}_{n,\ell}, \tag{8}$$

$$\mathcal{F}_{n,\ell} = \operatorname{span}\{q_{1,n}^{\ell}, q_{2,n}^{\ell}, \ldots, q_{n+\ell,n}^{\ell}\} \subseteq \mathcal{V}_{n,\ell}. \tag{9}$$

Define the invariants y_n^{ℓ}, r_n^{ℓ} of links with ℓ components by

$$Y_L\left(e^{h/2}\right) = \sum_{n=0}^{\infty} y_n^{|L|}(L) h^n \in \mathbb{Q}[[h]], \tag{10}$$

$$V_L\left(e^{h/2}\right) = \sum_{n=0}^{\infty} r_n^{|L|}(L) h^n \in \mathbb{Q}[[h]]. \tag{11}$$

In the following proposition we state the consequences of Propositions 4.7, 4.2, 4.5 of [Lik] for the versions of the Homfly and Kauffman polynomials from Equations (6) and (7).

Proposition 4. *For all $n \geq 0, \ell \geq 1$ we have*

$$
\begin{aligned}
&(1) \quad y_n^\ell = p_{n+\ell,n}^\ell = q_{n+\ell,n}^\ell,\\
&(2) \quad r_n^\ell = (-1)^\ell \sum_{i=1}^{n+\ell} 2^i p_{i,n}^\ell = (-1/2)^n \sum_{i=1}^{n+\ell} (-2)^i q_{i,n}^\ell,\\
&(3) \quad (-2)^n \sum_{i=1}^{n+\ell} 4^i q_{i,n}^\ell = \sum_{m=0}^{n} \sum_{i=1}^{m+\ell} \sum_{j=1}^{n-m+\ell} (-2)^{i+j} q_{i,m}^\ell q_{j,n-m}^\ell.
\end{aligned}
$$

Sketch of Proof. (1) The following formulas for Y can directly be derived from its definition:

$$
\begin{aligned}
&(a) \quad xY_{L_+}(x) - x^{-1}Y_{L_-}(x) = Y_{L_{||}}(x) \quad \text{if } |L_+| < |L_{||}|,\\
&(b) \quad xY_{L_+}(x) = x^{-1}Y_{L_-}(x) \quad \text{if } |L_+| > |L_{||}|,\\
&(c) \quad Y_{O^k}(x) = (x - x^{-1})^k.
\end{aligned}
$$

These relations are sufficient to calculate $Y_L(x)$ for every link L. The link invariant $Y_L'(x) := \widetilde{F}_L(x,0)$ satisfies the same Relations (a), (b), (c) as Y, hence we have $\widetilde{H}_L(x,0) = Y_L(x) = Y_L'(x) = \widetilde{F}_L(x,0)$. Now the formulas

$$
\widetilde{H}_L\left(e^{h/2},0\right) = \sum_{n=0}^{\infty} p_{n+|L|,n}^{|L|}(L)h^n \quad \text{and} \quad \widetilde{F}_L\left(e^{h/2},0\right) = \sum_{n=0}^{\infty} q_{n+|L|,n}^{|L|}(L)h^n
$$

imply Part (1) of the proposition.
(2) Let $< L > (A)$ be the Kauffman bracket (see [Ka1], [Ka3]) defined by

$<$  $> = A <$  $> + A^{-1} <$  $>$, $< O^k >= (-A^2 - A^{-2})^k$.

For a link diagram L define the link invariant $f_L(A)$ with values in $\mathbb{Z}[A^2, A^{-2}]$ by $f_L(A) = (-A^3)^{-w(L)} < L > (A)$, where $w(L)$ denotes the writhe of L. Then one can show that

$$
\begin{aligned}
F_L\left(e^{-3h/2}, e^{h/2} - e^{-h/2}\right) = f_L\left(-e^{-h/2}\right) &= f_L\left(e^{-h/2}\right) \quad \text{and}\\
(-1)^{|L|}H_L\left(e^h, e^{h/2} - e^{-h/2}\right) = V_L\left(e^{h/2}\right) &= f_L\left(e^{h/4}\right).
\end{aligned}
$$

This implies Part (2) of the proposition.

(3) With the notation of Part (2) of the proof we have

$$F_L(B^3, B - B^{-1}) = f_L(-A^{-1})^2 = F_L(A^{-3}, A - A^{-1})^2, \text{ where } B = A^{-2}.$$

Substituting $A = e^{h/2}$ and $B = e^{\hbar/2}$ with $\hbar = -2h$ and comparing with Equation (7) gives us Part (3) of the proposition. □

Parts (1) and (2) of Proposition 4 imply that $r_n^\ell, y_n^\ell \in \mathcal{H}_{n,\ell} \cap \mathcal{F}_{n,\ell}$. In other words, the polynomials V and Y are common specializations of H and F. This was the easy part of the proofs of Theorem 1 and Corollary 2. Part (3) of Proposition 4 will be used in the proof of Theorem 3.

3 Spaces of weight systems

We recall the following from [BN1]. A *trivalent diagram* is an unoriented graph with $\ell \geq 1$ disjointly embedded oriented circles such that every connected component of this graph contains at least one oriented circle, every vertex has valency three, and the vertices that do not lie on an oriented circle have a cyclic orientation. We consider trivalent diagrams up to homeomorphisms of graphs that respect the additional data. The degree of a trivalent diagram is defined as half of the number of its vertices. An example of a diagram on two circles of degree 8 is shown in Figure 2.

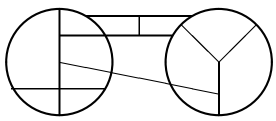


Figure 2: A trivalent diagram

In the picture the distinguished circles are drawn with thicker lines than the remaining part of the diagrams. Orientation of circles and vertices are assumed to be counterclockwise. Crossings in the picture do not correspond to vertices of a trivalent diagram. Let $\mathcal{A}_{n,\ell}$ be the $\mathbb{Q}$-vector space generated by trivalent diagrams of degree n on ℓ oriented circles together with the relations (STU), (IHX) and (AS) shown in Figure 3.

The diagrams in a relation are assumed to coincide everywhere except for the parts we have shown. Let $\bar{\mathcal{A}}_{n,\ell}$ be the quotient of $\mathcal{A}_{n,\ell}$ by the relation (FI), also shown in Figure 3. A *weight system* is a linear map from $\bar{\mathcal{A}}_{n,\ell}$ to a $\mathbb{Q}$-vector space.

A *chord diagram* is a trivalent diagram where every trivalent vertex lies on an oriented circle. It is easy to see that $\mathcal{A}_{n,\ell}$ is spanned by chord diagrams. If D is a chord diagram of degree n on ℓ oriented circles, then one can construct a singular link L_D with ℓ components such that the preimages of double points

(STU)-relation:

(IHX)-relation:

(AS)-relation (anti-symmetry):

(FI)-rel. (framing-independence):

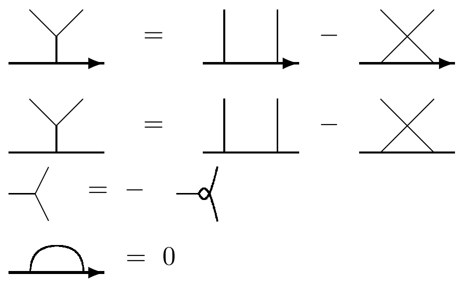


Figure 3: (STU), (IHX), (AS) and (FI)-relation

of L_D correspond to the points of D connected by a chord. The singular link L_D described above is not uniquely determined by D, but, if $v \in \mathcal{V}_{n,\ell}$, then the linear map $W(v) : \bar{\mathcal{A}}_{n,\ell} \longrightarrow \mathbb{Q}$ which sends D to $v(L_D)$ is well-defined. This defines a linear map $W : \mathcal{V}_{n,\ell} \longrightarrow \mathrm{Hom}(\bar{\mathcal{A}}_{n,\ell}, \mathbb{Q}) = \bar{\mathcal{A}}^*_{n,\ell}$. Let us define the spaces

$$\mathcal{H}'_{n,\ell} = W(\mathcal{H}_{n,\ell}) \qquad \text{and} \qquad \mathcal{F}'_{n,\ell} = W(\mathcal{F}_{n,\ell}) \subseteq \bar{\mathcal{A}}^*_{n,\ell}.$$

If $v_1 \in \mathcal{V}_{n,\ell}$ and $v_2 \in \mathcal{V}_{m,\ell}$, then the link invariant $v_1 v_2$ defined by $(v_1 v_2)(L) = v_1(L) v_2(L)$ is in $\mathcal{V}_{n+m,\ell}$. Weight systems are multiplied by using the algebra structure dual to the coalgebra structure of $\bigoplus_{n=0}^{\infty} \bar{\mathcal{A}}_{n,\ell}$ (see [BN1]). The following proposition is a well-known consequence of a theorem of Kontsevich (see Proposition 2.9 of [BNG] and Theorem 7.2 of [KaT], Theorem 10 of [LM3] or [LM1], [LM2]).

Proposition 5. *For all $\ell \geq 1$ there exists an isomorphism of algebras*

$$Z^* : \bigoplus_{n=0}^{\infty} \bar{\mathcal{A}}^*_{n,\ell} \longrightarrow \bigcup_{n=0}^{\infty} \mathcal{V}_{n,\ell}$$

such that for all $n \geq 0$ we have

$$Z^* \circ W_{|(\mathcal{H}_{n,\ell}+\mathcal{F}_{n,\ell})} = \mathrm{id}_{(\mathcal{H}_{n,\ell}+\mathcal{F}_{n,\ell})}.$$

This proposition reduces the study of $\mathcal{H}_{n,\ell}$ and $\mathcal{F}_{n,\ell}$ to that of $\mathcal{H}'_{n,\ell}$ and $\mathcal{F}'_{n,\ell}$: we have the following corollary.

Corollary 6. *For all $n \geq 0$ and $\ell \geq 1$ we have*

$$\begin{aligned} \dim \mathcal{H}'_{n,\ell} &= \dim \mathcal{H}_{n,\ell}, \\ \dim \mathcal{F}'_{n,\ell} &= \dim \mathcal{F}_{n,\ell}, \\ \dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell}) &= \dim(\mathcal{H}_{n,\ell} + \mathcal{F}_{n,\ell}). \end{aligned}$$

We will often use Corollary 6 without referring to it.

4 UPPER BOUNDS FOR $\dim \mathcal{H}'_{n,\ell}$ AND $\dim \mathcal{F}'_{n,\ell}$

Let us recall the explicit descriptions of $W(p^\ell_{i,j})$ and $W(q^\ell_{i,j})$ from [BN1]. Let D be a trivalent diagram. Cut it into pieces along small circles around each vertex. Then replace the simple parts as shown in Figure 4.

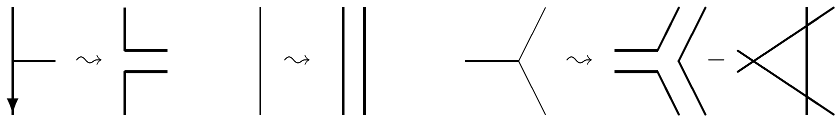


Figure 4: The map $W_{\mathfrak{gl}}$

Glue the substituted parts together. Sums of parts of diagrams are glued together after multilinear expansion. The result is a linear combination of unions of circles. Replace each circle by a formal parameter c and call the resulting polynomial $W_{\mathfrak{gl}}(D)$. It is well-known that this procedure determines a linear map $W_{\mathfrak{gl}} : \mathcal{A}_{n,\ell} \longrightarrow \mathbb{Q}[c]$ (see [BN1], Exercise 6.36). Proceeding with the replacement patterns shown in Figure 5, we get the linear map $W_{\mathfrak{so}} : \mathcal{A}_{n,\ell} \longrightarrow \mathbb{Q}[c]$.

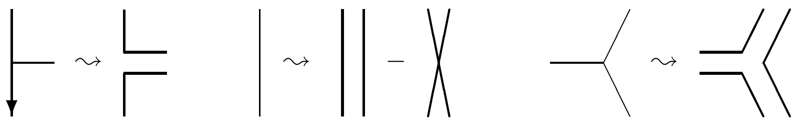


Figure 5: The map $W_{\mathfrak{so}}$

For a trivalent diagram D, define the linear combination of trivalent diagrams $\iota(D)$ by replacing each chord as shown in Figure 6. Connected components of $D \setminus S^{1\amalg\ell}$ with an internal trivalent vertex stay as they are.

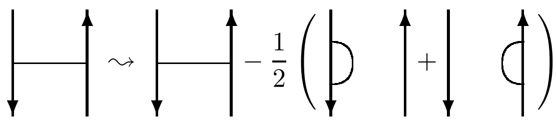


Figure 6: The deframing map ι

This definition determines a linear map $\iota : \bar{\mathcal{A}}_{n,\ell} \longrightarrow \mathcal{A}_{n,\ell}$, such that $\pi \circ \iota = \mathrm{id}$ where $\pi : \mathcal{A}_{n,\ell} \longrightarrow \bar{\mathcal{A}}_{n,\ell}$ denotes the canonical projection (compare [BN1], Exercise 3.16). By the following proposition ([BN1], Chapter 6.3) the weight systems $\overline{W}_{\mathfrak{gl}} = W_{\mathfrak{gl}} \circ \iota$ and $\overline{W}_{\mathfrak{so}} = W_{\mathfrak{so}} \circ \iota$ belong to the Homfly and Kauffman polynomials.

PROPOSITION 7. *For all $n \geq 0, i, \ell \geq 1$ the weight system $W(p_{i,n}^{\ell})$ (resp. $W(q_{i,n}^{\ell})$) is equal to the coefficient of c^i in $\overline{W}_{\mathfrak{gl}|\bar{\mathcal{A}}_{n,\ell}}$ (resp. $\overline{W}_{\mathfrak{so}|\bar{\mathcal{A}}_{n,\ell}}$).*

The direct description of $W(p_{i,n}^{\ell})$ and $W(q_{i,n}^{\ell})$ from the proposition above will simplify the computation of dimensions.

LEMMA 8. *(1) For all $n, \ell \geq 1$ we have*

$$\dim \mathcal{H}'_{n,\ell} \leq \begin{cases} n & \text{if } n < \ell, \\ \left[\frac{n-1+\ell}{2}\right] & \text{if } n \geq \ell. \end{cases}$$

(2) For all $n, \ell \geq 1$ we have

$$\dim \mathcal{F}'_{n,\ell} \leq \begin{cases} n-1 & \text{if } \ell = 1, \\ 2n-1 & \text{if } \ell \geq 2 \text{ and } n \leq \ell, \\ n+\ell-1 & \text{if } \ell \geq 2 \text{ and } n \geq \ell. \end{cases}$$

Proof. In the proof D will denote a chord diagram of degree $n \geq 1$ on ℓ circles.
(1) If $n \geq \ell$, then we get $[(n-1+\ell)/2]$ as an upper bound for $\dim \mathcal{H}'_{n,\ell}$ by the following observations:
(a) The polynomial $\overline{W}_{\mathfrak{gl}}(D)$ has degree $\leq n+\ell$ and vanishing constant term because the number of circles can at most increase by one with each replacement of a chord as shown in Figure 4, and there remains always at least one circle.
(b) The coefficients of $c^{n+\ell-1-2i}$ $(i = 0, 1, \dots)$ vanish because the number of circles changes by ± 1 with each replacement of a chord as shown in Figure 4.
(c) We have $\overline{W}_{\mathfrak{gl}}(D)(1) = 0$ because $W_{\mathfrak{gl}}(D')(1) = 1$ for each chord diagram D' and $\iota(D)$ is a linear combination of chord diagrams D' having 0 as sum of their coefficients.

If D is a chord diagram of degree $n < \ell$, then by similar arguments $\overline{W}_{\mathfrak{gl}}(D)$ is a linear combination of $c^{\ell-n}, c^{\ell-n+2}, \dots, c^{\ell+n}$ with $\overline{W}_{\mathfrak{gl}}(D)(1) = 0$. This implies the upper bound for $\dim \mathcal{H}'_{n,\ell}$.

(2) If $n \geq \ell$, then by the same arguments as above $\overline{W}_{\mathfrak{so}}(D)$ is a polynomial of degree $\leq n+\ell$ with vanishing constant term and $\overline{W}_{\mathfrak{so}}(D)(1) = 0$. This implies $\dim \mathcal{F}'_{n,\ell} \leq n+\ell-1$ in this case.
If $\ell = 1$, then for chord diagrams D' of degree n the value $W_{\mathfrak{so}}(D')(2)$ is constant because $\mathfrak{so}_2$ is an abelian Lie algebra (see [BN1]). This implies $\overline{W}_{\mathfrak{so}}(D)(2) = 0$ and hence $\dim \mathcal{F}'_{n,\ell} \leq n-1$ in this case.
If $\ell \geq 2$ and $n < \ell$, then the coefficient of $c^{\ell-n}$ in $\overline{W}_{\mathfrak{so}}(D)$ is 0 by the following argument: Assume that a chord diagram D' has the minimal possible number of $\ell - n$ connected components (in other words, if we contract the oriented circles of D' to points, then the resulting graph is a forest). Then we see that $W_{\mathfrak{so}}(D') = 0$ by using Figure 5. Hence $\overline{W}_{\mathfrak{so}}(D)$ is a linear combination of

$c^{\ell-n+1}, c^{\ell-n+2}, \ldots, c^{\ell+n}$ with $\overline{W}_{\mathfrak{so}}(D)(1) = 0$. This completes the proof of the upper bounds for $\dim \mathcal{F}'_{n,\ell}$. □

5 THE BRAUER ALGEBRA AND VALUES OF $\overline{W}_{\mathfrak{gl}}$ AND $\overline{W}_{\mathfrak{so}}$

In order to find lower bounds for $\dim \mathcal{H}'_{n,\ell}$, $\dim \mathcal{F}'_{n,\ell}$ and $\dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell})$, we shall evaluate the weight systems $\overline{W}_{\mathfrak{gl}}$ and $\overline{W}_{\mathfrak{so}}$ on sufficiently many trivalent diagrams. Let ω_k, L_k, C_k, T_k be the diagrams of degree k shown in Figure 7.

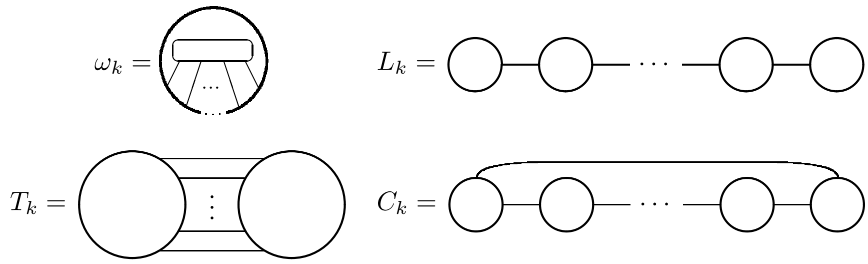


Figure 7: The diagrams ω_k, L_k, C_k, T_k

For technical reasons we extend this definition by setting $L_0 = C_0 = T_0 = S^1$ and $C_1 = L_1$. An important ingredient in the proofs of Theorems 1 and 3 is the following lemma.

LEMMA 9. *(1) For all $k \geq 2$ we have*

$$\begin{aligned} \overline{W}_{\mathfrak{gl}}(\omega_k) &= \begin{cases} c^{k+1} + c^3 - 2c & \text{if } k \text{ is even,} \\ c^{k+1} - c^2 & \text{if } k \text{ is odd, and} \end{cases} \\ \overline{W}_{\mathfrak{so}}(\omega_k) &= c(c-1)(c-2)R_k(c), \end{aligned}$$

where R_k is a polynomial with $R_k(0) \neq 0$. If $k = 2$, then $R_2 = 2$, and if $k \neq 3$, then $R_k(2) \neq 0$.

(2) For all $k \geq 1$ we have

$$\begin{array}{ll} \overline{W}_{\mathfrak{gl}}(L_k) = c(1-c^2)^k, & \overline{W}_{\mathfrak{so}}(L_k) = c^{k+1}(1-c)^k, \\ \overline{W}_{\mathfrak{gl}}(T_k) = (-c)^k(c^2-1), & \overline{W}_{\mathfrak{so}}(T_k) = c(c-1)Q_k(c), \\ & \overline{W}_{\mathfrak{so}}(C_k) = c(c-1)P_k(c), \end{array}$$

where P_k and Q_k are polynomials in c such that for $k \geq 2$ we have $P_k(0) \neq 0$, $Q_k(0) = 2^{k-1}$, and $Q_k(2) = (-2)^k$.

In the proof of the lemma we will determine the polynomials P_k, Q_k, and R_k explicitly, which will be helpful to us for calculations in low degrees. For the main parts of the proofs of Theorems 1 and 3 it will be sufficient to know the properties of these polynomials stated in the lemma. We do not need to know the value of $\overline{W}_{\mathfrak{gl}}(C_k)$.

In the proof of Lemma 9 we use the Brauer algebra ([Bra]) on k strands $\mathbf{Br}_k$. As a $\mathbb{Q}[c]$-module $\mathbf{Br}_k$ has a basis in one-to-one correspondence with involutions without fixed-points of the set $\{1,\ldots,k\}\times\{0,1\}$. We represent a basis element corresponding to an involution f graphically by connecting the points (i,j) and $f(i,j)$ by a curve in $\mathbb{R}\times[0,1]$. Examples are the diagrams u_-, x_+, x_-, $u_+ = d$, e, f, g, h in Figures 8 and 9.

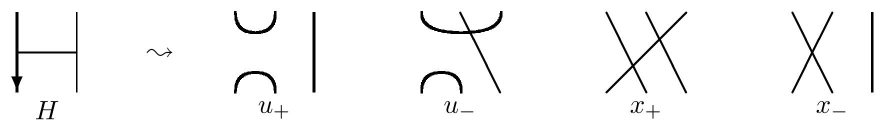


Figure 8: Elements of $\mathbf{Br}_3$ needed to calculate $W_{\mathfrak{so}}(\omega_k)$

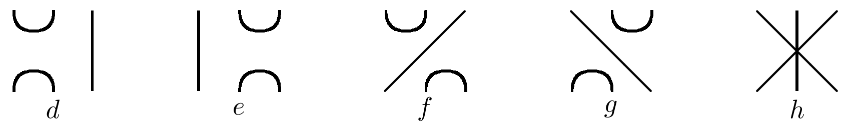


Figure 9: Diagrams needed to calculate $W_{\mathfrak{gl}}(\omega_k)$

The product of basis vectors a and b is defined graphically by placing a onto the top of b, by gluing the lower points $(i,0)$ of a to the upper points $(i,1)$ of b, and by introducing the relation that a circle is equal to the formal parameter c of the ground ring $\mathbb{Q}[c]$. We have a map $\mathrm{tr}:\mathbf{Br}_k\longrightarrow\mathbb{Q}[c]$, called trace, that is defined graphically by connecting the vertices $(i,0)$ and $(i,1)$ of a diagram by curves, and by replacing each circle by the indeterminate c. As an example, the trace of the diagram x_+u_- is shown in Figure 10.

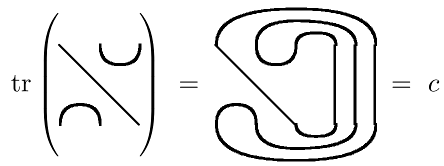


Figure 10: The trace of a diagram

The elements u_+, u_-, x_+, x_- arise among others when the replacement rules belonging to $W_{\mathfrak{so}}$ (see Figure 5) are applied to the part H (see Figure 8) of a trivalent diagram. Similarly, the elements d and h arise when we apply the replacement rules belonging to $W_{\mathfrak{gl}}$ (see Figure 4) to the part H of a trivalent diagram. We have $\iota(\omega_k) = \omega_k$ (see Figure 6) because the diagram ω_k contains no chords. The proof of the following lemma is now straightforward.

Lemma 10. *The following two formulas hold:*

$$\overline{W}_{\mathfrak{gl}}(\omega_k) = \mathrm{tr}\left((d-h)^k\right) \qquad \textit{and} \qquad \overline{W}_{\mathfrak{so}}(\omega_k) = \mathrm{tr}\left((u_+ - u_- + x_+ - x_-)^k\right).$$

Now we can prove Lemma 9 by making calculations in the Brauer algebra.

Proof of Lemma 9. (1) With the elements $u_\pm, x_\pm \in \mathbf{Br}_3$ shown in Figure 8 we define $u = u_+ - u_-$ and $x = x_+ - x_-$. It is easy to verify that

$$(u+x)u = (c-2)u \text{ and } x^3 = x^2 + 2x. \tag{12}$$

In view of the expression for x^3 it is clear that x^k can be expressed as a linear combination of x and x^2:

$$d_k x^2 + e_k x = x^k. \tag{13}$$

It can be shown by induction that the sequence of pairs $(d_k, e_k)_{k\geq 1}$ is given by $(d_1, e_1) = (0,1)$ and $(d_{k+1}, e_{k+1}) = (d_k + e_k, 2d_k)$. We deduce

$$\begin{aligned} & d_1 - e_1 = -1, d_{k+1} - e_{k+1} = d_k + e_k - 2d_k = e_k - d_k \\ & \Rightarrow d_k - e_k = (-1)^k, & (14) \\ & d_{k+1} + (-1)^k = (d_k + e_k) + (d_k - e_k) = 2d_k. & (15) \end{aligned}$$

By Equations (12) and (13) we have

$$\begin{aligned} (u+x)^k &= x^k + \sum_{i=0}^{k-1} (u+x)^i u x^{k-i-1} \\ &= d_k x^2 + e_k x + (c-2)^{k-1} u + \sum_{i=0}^{k-2} (c-2)^i (d_{k-i-1} u x^2 + e_{k-i-1} u x). \end{aligned} \tag{16}$$

It is easy to see that

$$\mathrm{tr}\left(x^2\right)/(c-1) = -\mathrm{tr}(x) = \mathrm{tr}(u) = -\mathrm{tr}(ux) = \mathrm{tr}\left(ux^2\right) = c^2 - c. \tag{17}$$

Applying the trace to Equation (16) yields by Lemma 10 and Equations (14) and (17):

$$\begin{aligned}
&\overline{W}_{\mathfrak{so}}(\omega_k)\\
= \quad & (c^2-c)\left[d_k(c-1)-e_k+(c-2)^{k-1}+\sum_{i=0}^{k-2}(c-2)^i(d_{k-i-1}-e_{k-i-1})\right]\\
= \quad & (c^2-c)\left[d_k(c-2)+(-1)^k-\sum_{i=0}^{k-1}(-1)^{k-i}(c-2)^i\right]\\
= \quad & (c^2-c)(c-2)\left[(d_k+(-1)^k)+\sum_{i=1}^{k-2}(-1)^{k-i}(c-2)^i\right].
\end{aligned}$$

Define the sequence $(a_k)_{k\geq 2}$ inductively by $a_2=2$ and $a_{k+1}=2a_k-4(-1)^k$. We have $a_2=d_2+(-1)^2$ and by definition of a_k, induction and Equation (15) also

$$a_{k+1}=2a_k-4(-1)^k=2(d_k+(-1)^k)-4(-1)^k=d_{k+1}+(-1)^{k+1}.$$

This implies $\overline{W}_{\mathfrak{so}}(\omega_k)=c(c-1)(c-2)R_k(c)$ with

$$R_k(c)=a_k+\sum_{i=1}^{k-2}(-1)^{k-i}(c-2)^i.$$

The properties of R_k stated in the lemma are satisfied because by a simple computation we have $R_k(2)=a_k>0$ for $k\neq 3$ and

$$R_k(0)=a_k+(-1)^k(2^{k-1}-2)\equiv 2 \bmod 4.$$

We only give a sketch of the proof of the formula for $\overline{W}_{\mathfrak{gl}}(\omega_k)$. Let d,e,f,g,h be the elements of $\mathbf{Br}_3$ shown in Figure 9. Then one can prove by induction on k that

$$(d-h)^{2k+1}=c^{2k}d-h+\sum_{i=0}^{k-1}c^{2i}(d+e)-c^{2i+1}(f+g).$$

Using Lemma 10 this formula allows to conclude by distinguishing whether k is even or odd.

(2) Let $a,b,\mathbf{1}$ be the elements of $\mathbf{Br}_2$ shown in Figure 11.
Then we have $ab=ba=a$, $a^2=ca$, $b^2=\mathbf{1}$, $\mathrm{tr}(a)=\mathrm{tr}(b)=c$, $\mathrm{tr}(\mathbf{1})=c^2$, and by convention $(a-b)^0=\mathbf{1}$. This implies for $k\geq 1$ that

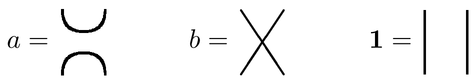


Figure 11: Diagrams in $\mathbf{Br}_2$

$$
\begin{aligned}
\overline{W}_{\mathfrak{so}}(T_k) &= \operatorname{tr}\left((a-b+\mathbf{1}-c\mathbf{1})^k\right)\\
&= \operatorname{tr}\left[\sum_{i=0}^{k}\binom{k}{i}(1-c)^{k-i}(a-b)^i\right]\\
&= \operatorname{tr}\left\{\sum_{i=0}^{k}\binom{k}{i}(1-c)^{k-i}\left[(-b)^i+\sum_{j=1}^{i}\binom{i}{j}c^{j-1}(-1)^{i-j}a\right]\right\}\\
&= \sum_{i=0}^{k}\binom{k}{i}(1-c)^{k-i}\left[\operatorname{tr}\left((-b)^i\right)+(c-1)^i-(-1)^i\right]\\
&= \sum_{i=0}^{k}\binom{k}{i}(1-c)^{k-i}\left[\operatorname{tr}\left((-b)^i\right)-(-1)^i\right]\\
&= \sum_{\substack{0\le i\le k\\ i \text{ even}}}\binom{k}{i}(1-c)^{k-i}(c^2-1)+\sum_{\substack{1\le i\le k\\ i \text{ odd}}}\binom{k}{i}(1-c)^{k-i}(1-c)\\
&= \sum_{i=0}^{k}\binom{k}{i}(1-c)^{k-i}(c-1)(-1)^i+c\sum_{\substack{0\le i\le k\\ i \text{ even}}}\binom{k}{i}(1-c)^{k-i}(c-1)\\
&= c(c-1)\left[-(-c)^{k-1}+\sum_{\substack{0\le i\le k\\ i \text{ even}}}\binom{k}{i}(1-c)^{k-i}\right].
\end{aligned}
$$

Now one checks the properties of Q_k using the last expression for $\overline{W}_{\mathfrak{so}}(T_k)$. The remaining formulas follow by easy computations. For example, $\overline{W}_{\mathfrak{so}}(L_k)$ is given by the value of $W_{\mathfrak{so}}$ on the diagrams in $\iota(L_k)$ where no chord connects two different circles. Furthermore, one can show for $k \geq 2$ that

$$\overline{W}_{\mathfrak{so}}(C_k) = \operatorname{tr}\left((\mathbf{1}-b)^k\right)+(1-c)\overline{W}_{\mathfrak{so}}(L_{k-1}) = c(c-1)\left(2^{k-1}-c^{k-1}(1-c)^{k-1}\right).$$

The property $P_k(0) \neq 0$ from the lemma is obvious from the formula above. $\square$

6 Completion of proofs using Vogel's algebra

In the case of diagrams on one oriented circle, the coalgebra structure of $\bar{\mathcal{A}} = \bigoplus_{n=0}^{\infty} \bar{\mathcal{A}}_n$ can be extended to a Hopf algebra structure (see [BN1]). The primitive elements $\mathcal{P}$ of $\bar{\mathcal{A}}$ are spanned by diagrams D such that $D \setminus S^1$ is connected, where S^1 denotes the oriented circle of D. Vogel defined an algebra Λ which acts on primitive elements (see [Vog]). The diagrams t and x_3 shown in Figure 12 represent elements of Λ.

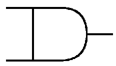
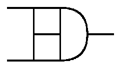
$t =$  $x_3 =$ 

Figure 12: Elements of Λ

The space of primitive elements $\mathcal{P}$ of $\bar{\mathcal{A}}$ becomes a Λ-module by inserting an element of Λ into a freely chosen trivalent vertex of a diagram of a primitive element. Multiplication by t increases the degree by 1 and multiplication by x_3 increases the degree by 3. An example is shown in Figure 13.

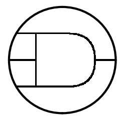
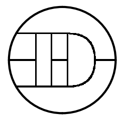
$x_3\omega_4 = x_3$  $=$ 

Figure 13: How $\mathcal{P}$ becomes a Λ–module

If D and D' are classes of trivalent diagrams with a distinguished oriented circle modulo (STU)-relations (see Figure 3), then their connected sum $D\#D'$ along these circles is well defined. We state in the following lemma how the weight systems $\overline{W}_{\mathfrak{gl}}$ and $\overline{W}_{\mathfrak{so}}$ behave under the operations described above: Part (1) of the lemma is easy to prove; for Part (2), see Theorem 6.4 and Theorem 6.7 of [Vog].

LEMMA 11. *(1) Let D and D' be chord diagrams each one having a distinguished oriented circle. Then the connected sum of D and D' satisfies*

$$\overline{W}_{\mathfrak{gl}}(D\#D') = \overline{W}_{\mathfrak{gl}}(D)\overline{W}_{\mathfrak{gl}}(D')/c \quad \textit{and} \quad \overline{W}_{\mathfrak{so}}(D\#D') = \overline{W}_{\mathfrak{so}}(D)\overline{W}_{\mathfrak{so}}(D')/c.$$

(2) For a primitive element $p \in \mathcal{P}$ we have:

$$\begin{aligned}
&\overline{W}_{\mathfrak{gl}}(tp) = c\overline{W}_{\mathfrak{gl}}(p),\\
&\overline{W}_{\mathfrak{so}}(tp) = \tilde{c}\overline{W}_{\mathfrak{so}}(p),\\
&\overline{W}_{\mathfrak{gl}}(x_3p) = (c^3 + 12c)\overline{W}_{\mathfrak{gl}}(p),\\
&\overline{W}_{\mathfrak{so}}(x_3p) = (\tilde{c}^3 - 3\tilde{c}^2 + 30\tilde{c} - 24)\overline{W}_{\mathfrak{so}}(p),
\end{aligned}$$

where $\tilde{c} = c - 2$.

We have the following formulas concerning spaces of weight systems restricted to primitive elements.

PROPOSITION 12. *For the restrictions of the weight systems to primitive elements of degree $n \geq 1$ we have*

$$\text{(1)} \qquad \dim \mathcal{H}'_{n|\mathcal{P}_n} = \dim \mathcal{H}'_n = [n/2],$$

$$\text{(2)} \qquad \dim \mathcal{F}'_{n|\mathcal{P}_n} = \max(n-2, [n/2]) = \begin{cases} [n/2] & \textit{if } n \leq 3, \\ n-2 & \textit{if } n \geq 3, \end{cases}$$

$$\text{(3)} \qquad \dim \left(\mathcal{H}'_{n|\mathcal{P}_n} \cap \mathcal{F}'_{n|\mathcal{P}_n} \right) = \min(2, [n/2]) = \begin{cases} [n/2] & \textit{if } n \leq 3, \\ 2 & \textit{if } n \geq 4. \end{cases}$$

The proof of Proposition 12 will be given in this section together with a proof of Theorem 1. The proof is divided into several steps.
If q is a polynomial, then we denote the degree of its lowest degree term by $\mathrm{ord}(q)$. Now we start to derive lower bounds for dimensions of spaces of weight systems.

Proof of Part (1) of Proposition 12. By Lemma 9 we have $\mathrm{ord}(\overline{W}_{\mathfrak{gl}}(\omega_k)) = 1$ for even k. By Lemma 11 we have $\overline{W}_{\mathfrak{gl}}(t^k p) = c^k \overline{W}_{\mathfrak{gl}}(p)$ for $p \in \mathcal{P}$. This implies

$$\dim \left(\overline{W}_{\mathfrak{gl}} \left(\mathrm{span}\{t^{n-2}\omega_2, t^{n-4}\omega_4, \ldots, t^{n-2[n/2]}\omega_{2[n/2]}\} \right) \right) = [n/2] \leq \dim \mathcal{H}'_{n|\mathcal{P}_n}.$$

Since this lower bound coincides with the upper bound from Lemma 8 we have $\dim \mathcal{H}'_{n|\mathcal{P}_n} = [n/2]$. □

Let $D_{ijk} = (L_i \# C_j) \# T_k$ (in this definition we choose arbitrary distinguished circles of L_i, C_j, $(L_i \# C_j)$ and for further use also for D_{ijk}). Let d_{ijk} be the number of oriented circles in D_{ijk} and define $D^\ell_{i,j,k} = D_{ijk} \amalg S^{1^{\amalg(\ell - d_{ijk})}}$ for $\ell \geq d_{ijk}$. We will make use of the formulas for $\overline{W}_{\mathfrak{gl}}(D^\ell_{i,0,k} \# \omega_m)$ and $\overline{W}_{\mathfrak{so}}(D^\ell_{i,j,k} \# \omega_m)$ implied by Lemmas 9 and 11 throughout the rest of this section.

Proof of Part (1) of Theorem 1. For all $n \geq 1$ we have $[n/2]$ primitive elements p_i such that the polynomials $g_i = \overline{W}_{\mathfrak{gl}}(p_i \amalg S^{1^{\amalg(\ell-1)}})$ are linearly independent and $c^\ell | g_i$ (see the proof of Part (1) of Proposition 12). Let $n < \ell$. The diagrams

$$D^\ell_{n,0,0}, D^\ell_{n-2,0,0} \# \omega_2, \ldots, D^\ell_{n-2[(n-1)/2],0,0} \# \omega_{2[(n-1)/2]} \qquad (18)$$

are mapped by $\overline{W}_{\mathfrak{gl}}$ to the values

$$c^{\ell-n}(1-c^2)^n, c^{\ell-n+2} f_2(c), \ldots, c^{\ell-1} f_{[(n+1)/2]}(c)$$

with polynomials f_i satisfying $f_i(0) = -2$ $(i = 2, \ldots, [(n+1)/2])$. So in this case we have found $[n/2] + [(n+1)/2] = n$ linearly independent values, which is the maximal possible number (see Lemma 8). If $n \geq \ell$, then we conclude in the same way using the following list of $k-n+1+[(n-1)/2] = k-[n/2]$ elements where $k = [(n+\ell-1)/2]$:

$$D^{\ell}_{2k-n,0,0}\#\omega_{2n-2k}, D^{\ell}_{2k-n-2,0,0}\#\omega_{2n-2k+2}, \ldots, D^{\ell}_{n-2[(n-1)/2],0,0}\#\omega_{2[(n-1)/2]}. \tag{19}$$

□

We will use the upper bounds for $\dim \mathcal{H}'_{n,\ell}$ and $\dim \mathcal{F}'_{n,\ell}$ together with the following lower bound for $\dim(\mathcal{H}'_{n,\ell}\cap\mathcal{F}'_{n,\ell})$ to get an upper bound for $\dim(\mathcal{H}'_{n,\ell}+\mathcal{F}'_{n,\ell})$. In the case $\ell = 1$ we will argue in a similar way for the restriction of weight systems to primitive elements.

Lemma 13. *For all $n, \ell \geq 1$ we have*

$$\begin{aligned} \dim\left(\mathcal{H}'_{n,\ell}\cap\mathcal{F}'_{n,\ell}\right) &\geq \min(\dim\mathcal{H}'_{n,\ell}, 2) \\ = \min(n, [(n-1+\ell)/2], 2) &= \dim(\mathrm{span}\{W(r_n^\ell), W(y_n^\ell)\}). \end{aligned}$$

For all $n \geq 1$ we have

$$\dim\left(\mathcal{H}'_{n|\mathcal{P}_n}\cap\mathcal{F}'_{n|\mathcal{P}_n}\right) \geq \min(\dim\mathcal{H}'_n, 2) = \min([n/2], 2).$$

Proof. Propositions 4 and 7 imply that the weight system $W(r_n^\ell) \in \mathcal{H}'_{n,\ell}\cap\mathcal{F}'_{n,\ell}$ is equal to $(-1)^\ell\overline{W}_{\mathfrak{gl}}(.)(2)_{|\bar{\mathcal{A}}_{n,\ell}}$ and the weight system $W(y_n^\ell) \in \mathcal{H}'_{n,\ell}\cap\mathcal{F}'_{n,\ell}$ is equal to the coefficient of $c^{\ell+n}$ in $\overline{W}_{\mathfrak{gl}|\bar{\mathcal{A}}_{n,\ell}}$. By the proof of Lemma 8 we have $\overline{W}_{\mathfrak{gl}}(D)(0) = \overline{W}_{\mathfrak{gl}}(D)(1) = 0$ and in the weight system $\overline{W}_{\mathfrak{gl}|\bar{\mathcal{A}}_{n,\ell}}$ the coefficients of $c^{\ell+n-1}, c^{\ell+n-3}, \ldots$ and the coefficients of $c^{\ell-n-1}, c^{\ell-n-2}, \ldots$ vanish. By Part (1) of Theorem 1 these are the only linear dependencies between the coefficients of $c^{\ell+n}, c^{\ell+n-1}, \ldots$ in the polynomial $\overline{W}_{\mathfrak{gl}|\bar{\mathcal{A}}_{n,\ell}}$. This implies for $\dim\mathcal{H}'_{n,\ell} = 1$ that the coefficient of $c^{\ell+n}$ in $\overline{W}_{\mathfrak{gl}|\bar{\mathcal{A}}_{n,\ell}}$ is not the trivial weight system and this implies for $\dim\mathcal{H}'_{n,\ell} \geq 2$ that $\overline{W}_{\mathfrak{gl}}(.)(2)_{|\bar{\mathcal{A}}_{n,\ell}}$ and the coefficient of $c^{\ell+n}$ in $\overline{W}_{\mathfrak{gl}|\bar{\mathcal{A}}_{n,\ell}}$ are linearly independent. By Part (1) of Proposition 12 we can argue in the same way with $\overline{W}_{\mathfrak{gl}|\mathcal{P}_n}$. This completes the proof. □

Define the weight system $w = (-2)^n\overline{W}_{\mathfrak{so}}(\cdot)(4) - 2(-2)^\ell\overline{W}_{\mathfrak{so}}(\cdot)(-2) \in \mathcal{F}'_{n,\ell}$. For $n \geq 4$ Lemmas 9 and 11 imply that

$$w\left(\omega_2\#(t^{n-4}\omega_2) \amalg S^{1^{\amalg\ell-1}}\right) = 18(-4)^n 4^{\ell-1} \neq 0. \tag{20}$$

Part (3) of Proposition 4 together with Propositions 5 and 7 implies that

$$0 \neq w \in \bigoplus_{i=1}^{n-1} \mathcal{F}'_{i,\ell}\mathcal{F}'_{n-i,\ell}. \tag{21}$$

For $\ell = 1$ Equation (21) implies $w(\mathcal{P}_n) = 0$. Therefore we have

$$\dim \mathcal{F}'_{n|\mathcal{P}_n} \leq \dim \mathcal{F}'_n - 1 \leq n - 2 \quad \text{for all } n \geq 4. \tag{22}$$

Since we have $\dim \mathcal{H}'_n = \dim \mathcal{H}'_{n|\mathcal{P}_n}$ by Part (1) of Proposition 12 we know that $w \notin \mathcal{H}'_n$ and therefore

$$\dim(\mathcal{H}'_n + \mathcal{F}'_n) \geq \dim(\mathcal{H}'_n + \mathcal{F}'_n)_{|\mathcal{P}_n} + 1 \quad \text{for all } n \geq 4. \tag{23}$$

Let $(\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}}) : \bar{\mathcal{A}}_{n,\ell} \longrightarrow \mathbb{Q}[c] \times \mathbb{Q}[c]$ be defined by

$$\left(\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}}\right)(D) = \left(\overline{W}_{\mathfrak{gl}}(D), \overline{W}_{\mathfrak{so}}(D)\right).$$

Then by Proposition 7 we have

$$\dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell}) = \dim\left((\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}})(\bar{\mathcal{A}}_{n,\ell})\right), \tag{24}$$

$$\dim\left(\mathcal{H}'_{n|\mathcal{P}_n} + \mathcal{F}'_{n|\mathcal{P}_n}\right) = \dim\left((\mathcal{H}'_n + \mathcal{F}'_n)_{|\mathcal{P}_n}\right) = \dim\left((\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}})(\mathcal{P}_n)\right) \tag{25}$$

We will use Equations (24) and (25) to derive lower bounds for $\dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell})$ and for $\dim(\mathcal{H}'_n + \mathcal{F}'_n)_{|\mathcal{P}_n}$. Now we can complete the proofs of Theorem 1 and Proposition 12.

Proof of Parts (2) and (3) of Proposition 12 and Theorem 1 for $\ell = 1$. Let

$$\Sigma_7 = \operatorname{span}\{\omega_7, t\omega_6, t^2\omega_5, t^3\omega_4, t^5\omega_2, x_3\omega_4\} \subset \mathcal{P}_7.$$

Define for $n > 7$:

$$\Sigma_n = \begin{cases} t\Sigma_{n-1} + \mathbb{Q}\omega_n & \text{if } n \text{ is odd,} \\ t\Sigma_{n-1} + \mathbb{Q}\omega_n + \mathbb{Q}x_3\omega_{n-3} & \text{if } n \text{ is even.} \end{cases}$$

By a calculation using Lemmas 9 and 11 we obtain

$$\dim\left((\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}})(\Sigma_7)\right) = 6.$$

In view of the proof of Lemma 8 we can define a polynomial-valued weight system by $\widetilde{W}_{\mathfrak{so}}(.) = \overline{W}_{\mathfrak{so}}(.)/(c(c-1))$. We used Lemma 9 and Lemma 11 to compute the degree 1 coefficients of the values of $\overline{W}_{\mathfrak{gl}}$ and $\widetilde{W}_{\mathfrak{so}}$ on elements of Σ_n stated in Table 1.

	$t\Sigma_{n-1}$	ω_n (n odd)	ω_n (n even)	$x_3\omega_{n-3}$ (n even)
coeff. of $\tilde{c}$ in $\widetilde{W}_{\mathfrak{so}}(\cdot)$:	0	$R_n(2)$	$R_n(2)$	$-24R_{n-3}(2)$
coeff. of c in $\overline{W}_{\mathfrak{gl}}(\cdot)$:	0	0	-2	0

Table 1: Degree 1 coefficients of $\overline{W}_{\mathfrak{gl}}$ and $\widetilde{W}_{\mathfrak{so}}$ on Σ_n

By Lemma 9 we have $R_k(2) \neq 0$ if $k \neq 3$. Then, by Table 1 and induction, we see that $\dim(\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}})(\Sigma_n) = [n/2] + n - 4$ for $n \geq 7$. By Equation (25), Lemmas 8 and 13, and Equation (22) we obtain

$$\begin{aligned}[n/2] + n - 4 + 2 \leq \dim\left(\mathcal{H}'_{n|\mathcal{P}_n} + \mathcal{F}'_{n|\mathcal{P}_n}\right) \quad &+ \quad \dim\left(\mathcal{H}'_{n|\mathcal{P}_n} \cap \mathcal{F}'_{n|\mathcal{P}_n}\right) = \\ = \dim \mathcal{H}'_{n|\mathcal{P}_n} + \dim \mathcal{F}'_{n|\mathcal{P}_n} \quad &\leq \quad [n/2] + n - 2.\end{aligned}$$

Thus equality must hold. This implies Parts (2) and (3) of Proposition 12 for $n \geq 7$. By Equation (23) we get $\dim(\mathcal{H}'_n + \mathcal{F}'_n) \geq [n/2] + n - 3$. Now we see by Lemmas 8 and 13 that

$$\begin{aligned}[n/2] + n - 3 + 2 \quad &\leq \quad \dim(\mathcal{H}'_n + \mathcal{F}'_n) + \dim(\mathcal{H}'_n \cap \mathcal{F}'_n) = \\ &= \quad \dim \mathcal{H}'_n + \dim \mathcal{F}'_n \leq [n/2] + n - 1\end{aligned}$$

which implies Part (2) and Part (3) of Theorem 1 for $n \geq 7$ and $\ell = 1$.
Let ψ be the element of degree 6 shown in Figure 14.

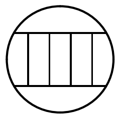
$\psi =$ 

Figure 14: A primitive element in degree 6

A calculation done by computer yields

$$\begin{aligned}\overline{W}_{\mathfrak{gl}}(\psi) &= c^7 + 13c^5 - 14c^3, \\ \widetilde{W}_{\mathfrak{so}}(\psi) &= \tilde{c}^5 - 3\tilde{c}^4 + 34\tilde{c}^3 - 36\tilde{c}^2 + 16\tilde{c}.\end{aligned}$$

Let $\Sigma_4 = \mathrm{span}\{\omega_4, t^2\omega_2\}$, $\Sigma_5 = t\Sigma_4 + \mathbb{Q}\omega_5$, and $\Sigma_6 = t\Sigma_5 + \mathbb{Q}\omega_6 + \mathbb{Q}\psi$. We obtain again $\dim(\overline{W}_{\mathfrak{gl}}, \overline{W}_{\mathfrak{so}})(\Sigma_n) = [n/2]+n-4$ which implies Parts (2) and (3) of Proposition 12 and Theorem 1 for $\ell = 1$ and $n \geq 4$ by the same argument as before. In degrees $n = 1, 2, 3$ we have $\dim\ \mathcal{P}_n = \dim\ \bar{\mathcal{A}}_n = \dim \mathcal{H}'_n = \dim \mathcal{F}'_n = [n/2]$. This completes the proof. ☐

Proof of Parts (2) and (3) of Theorem 1 for $\ell > 1$. Let $n \geq 4$ and $\ell > 1$. By the previous proof we have $n + [n/2] - 3$ elements $a_i \in \bar{\mathcal{A}}_n$ such that the values

$$\left(\overline{W}_{\mathfrak{gl}}(D_i), \overline{W}_{\mathfrak{so}}(D_i)\right) \in \mathbb{Q}[c] \times \mathbb{Q}[c]$$

of $D_i = a_i \amalg S^{1^{\amalg \ell - 1}}$ are linearly independent. Consider the following lists of elements:

If $n \leq \ell$, then we take the n elements
$$D^\ell_{0,n,0}, D^\ell_{1,n-1,0}, \dots, D^\ell_{n-3,3,0}, D^\ell_{0,0,n}, E^\ell_n := D^\ell_{0,0,n} - D^\ell_{0,0,n-2}\#\omega_2. \quad (26)$$
If $n \geq \ell + 1$, then we take the ℓ elements
$$D^\ell_{0,\ell-1,n-\ell+1}, D^\ell_{1,\ell-2,n-\ell+1}, \dots, D^\ell_{\ell-3,2,n-\ell+1}, D^\ell_{0,0,n}, E^\ell_n. \quad (27)$$

Let $\mathcal{M}_{n,\ell}$ be the list of elements D_i together with the elements from Equation (18) (resp. (19)) and Equation (26) (resp. (27)). We have

$$\operatorname{card}(\mathcal{M}_{n,\ell}) = \begin{cases} 3n - 3 & \text{if } n < \ell, \\ n + \ell - 3 + [(n + \ell - 1)/2] & \text{if } n \geq \ell. \end{cases} \quad (28)$$

The values of $\overline{W}_{\mathfrak{gl}}$ and $\overline{W}_{\mathfrak{so}}$ on elements of $\mathcal{M}_{n,\ell}$ have the properties stated in Table 2.

$\operatorname{ord}(\overline{W}_{\mathfrak{gl}}(D_i)) \geq \ell$	$\operatorname{ord}(\overline{W}_{\mathfrak{so}}(D_i)) \geq \ell, \overline{W}_{\mathfrak{so}}(D_i)(2) = 0$
$\operatorname{ord}(\overline{W}_{\mathfrak{gl}}(E^\ell_n)) \geq \ell$	$\operatorname{ord}(\overline{W}_{\mathfrak{so}}(E^\ell_n)) \geq \ell, \overline{W}_{\mathfrak{so}}(E^\ell_n)(2) \neq 0$
$\operatorname{ord}(\overline{W}_{\mathfrak{gl}}(D^\ell_{i,0,0}\#\omega_{n-i}))$ $= \ell - i$ $(i > 0,\ n - i \text{ even})$	$\operatorname{ord}(\overline{W}_{\mathfrak{so}}(D^\ell_{i,0,0}\#\omega_{n-i})) \geq \ell$
	$\operatorname{ord}(\overline{W}_{\mathfrak{so}}(D^\ell_{n-i,i,0})) = \ell + 1 - i\ (i \geq 3)$ $\operatorname{ord}(\overline{W}_{\mathfrak{so}}(D^\ell_{i,\ell-1-i,n-\ell+1})) = i + 1\ (i \leq \ell - 3)$ $\operatorname{ord}(\overline{W}_{\mathfrak{so}}(D^\ell_{0,0,n})) = \ell - 1$

Table 2: Properties of $\overline{W}_{\mathfrak{gl}}(e)$ and $\overline{W}_{\mathfrak{so}}(e)$ for $e \in \mathcal{M}_{n,\ell}$

The statements from this table are easily verified. For example, we have

$$\overline{W}_{\mathfrak{so}}(E^\ell_n) = c^{\ell-1}(c-1)h(c)$$

with $h(c) = Q_n(c) - 2(c-1)(c-2)Q_{n-2}(c)$. We have $h(0) = Q_n(0) - 4Q_{n-2}(0) = 0$ which implies

$$\operatorname{ord}\left(\overline{W}_{\mathfrak{so}}(E^\ell_n)\right) \geq \ell,$$

and $h(2) = Q_n(2) = (-2)^n$ which implies $\overline{W}_{\mathfrak{so}}(E_n^\ell)(2) \neq 0$. Now let

$$f = \sum_{e \in \mathcal{M}_{n,\ell}} \lambda(e) \left(\overline{W}_{\mathfrak{gl}}(e), \overline{W}_{\mathfrak{so}}(e)\right) = (f_1, f_2) \in \mathbb{Q}[c] \times \mathbb{Q}[c]$$

be a linear combination with $\lambda(e) \in \mathbb{Q}$. We want to show that $f = 0$ implies that all scalars $\lambda(e)$ are 0. For our arguments we will use the entries of Table 2 beginning at its bottom. The coefficients $\lambda(D^\ell_{n-i,i,0})$ (resp. $\lambda(D^\ell_{i,\ell-1-i,n-\ell+1})$) and $\lambda(D^\ell_{0,0,n})$ are 0 because they are multiples of

$$\frac{d^k f_2}{dc^k}(0), \ldots, \frac{d^{\ell-1} f_2}{dc^{\ell-1}}(0)$$

with $k = \max\{1, \ell - n + 1\}$. The coefficients $\lambda(D^\ell_{i,0,0} \# \omega_{n-i})$ must be 0 by a similiar argument for f_1. We get $\lambda(E^\ell_n) = 0$ because $\overline{W}_{\mathfrak{so}}(D_i)(2) = 0$ and $\overline{W}_{\mathfrak{so}}(E^\ell_n)(2) \neq 0$. The remaining coefficients $\lambda(D_i)$ are 0 because the values $(\overline{W}_{\mathfrak{gl}}(D_i), \overline{W}_{\mathfrak{so}}(D_i))$ are linearly independent. This implies $\dim(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell}) \geq \operatorname{card}(\mathcal{M}_{n,\ell})$. By Lemma 8 and Lemma 13 we have

$$\begin{aligned} \operatorname{card}(\mathcal{M}_{n,l}) + 2 \quad &\leq \quad \dim\left(\mathcal{H}'_{n,\ell} + \mathcal{F}'_{n,\ell}\right) + \dim\left(\mathcal{H}'_{n,\ell} \cap \mathcal{F}'_{n,\ell}\right) = \\ = \dim \mathcal{H}'_{n,\ell} + \dim \mathcal{F}'_{n,\ell} \quad &\leq \quad \begin{cases} 3n - 1 & \text{if } n < \ell, \\ n + \ell - 1 + [(n + \ell - 1)/2] & \text{if } n \geq \ell. \end{cases} \end{aligned} \tag{29}$$

Comparing with Equation (28) shows that equality must hold in Equation (29). This completes the proof of Parts (2) and (3) of the theorem for all $n \geq 4$.
In degrees $n = 1, 2, 3$ we used the diagrams shown in Table (3) (possibly together with some additional circles S^1) to determine $\dim \mathcal{F}'_{n,\ell}$.

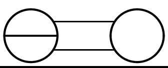
$n = 1$	L_1
$n = 2$	ω_2, C_2, L_2
$n = 3$	$\omega_3, \Omega_3 :=$  $, \omega_2 \# L_1, T_3, C_3$

Table 3: Diagrams used in low degrees

In the calculation we used the explicit formulas for the values of $\overline{W}_{\mathfrak{so}}$ from the proof of Lemma 9 together with $\overline{W}_{\mathfrak{so}}(\Omega_3) = 2c(c-1)(2-c)$. The number of linearly independent values coincides in all of these cases with the upper bound for $\dim \mathcal{F}'_{n,\ell}$ from Lemma 8 or with $\dim \bar{\mathcal{A}}_{n,\ell}$. For $\ell \geq 4$ and $a \in \bar{\mathcal{A}}_{3,3}$ we have

$$\operatorname{ord}\left(\overline{W}_{\mathfrak{gl}}\left(L_3 \amalg S^{1^{\amalg \ell - 4}}\right)\right) = \ell - 3 \quad \text{and} \quad \operatorname{ord}\left(\overline{W}_{\mathfrak{gl}}\left(a \amalg S^{1^{\amalg \ell - 3}}\right)\right) \geq \ell - 2.$$

Together with Lemmas 8 and 13 this implies

$$3+5-2 \geq \dim(\mathcal{H}'_{3,\ell}+\mathcal{F}'_{3,\ell}) \geq \dim \mathcal{F}'_{3,3}+1 = 6$$

and therefore $\dim(\mathcal{H}'_{3,\ell}+\mathcal{F}'_{3,\ell}) = 6$ and $\dim(\mathcal{H}'_{3,\ell}\cap\mathcal{F}'_{3,\ell}) = 2$ for $\ell \geq 4$. In the cases $n = 1, 2$, and in the case $n = 3$ and $\ell < 4$, we have $\dim \mathcal{H}'_{n,\ell} \leq 2$ and obtain $\dim(\mathcal{H}'_{n,\ell}\cap\mathcal{F}'_{n,\ell}) = \dim \mathcal{H}'_{n,\ell}$ by applying Lemma 13. This completes the proof. □

Corollary 2 can now be proven easily.

Proof of Corollary 2. Proposition 5, Lemma 13, and Part (3) of Theorem 1 imply

$$\begin{aligned}&\dim(\mathrm{span}\{r_n^\ell, y_n^\ell\}) = \dim(\mathrm{span}\{W(r_n^\ell), W(y_n^\ell)\})\\ &= \min(\dim \mathcal{H}'_{n,\ell}, 2) = \min(\dim \mathcal{H}_{n,\ell}, 2) = \dim(\mathcal{H}_{n,\ell}\cap\mathcal{F}_{n,\ell}).\end{aligned}$$

By Proposition 4 we have $r_n^\ell, y_n^\ell \in \mathcal{H}_{n,\ell}\cap\mathcal{F}_{n,\ell}$. This implies the statement $\mathrm{span}\{r_n^\ell, y_n^\ell\} = \mathcal{H}_{n,\ell}\cap\mathcal{F}_{n,\ell}$ of the corollary. □

Using Theorem 1 and Proposition 12 we can also prove Theorem 3.

Proof of Theorem 3. By Proposition 12 we have

$$\dim(\mathcal{H}'_n+\mathcal{F}'_n)_{|\mathcal{P}_n} = b_n := \begin{cases} [n/2] & n \leq 3 \\ n+[n/2]-4 & n \geq 4. \end{cases}$$

This implies that in the graded algebra A generated by $\bigoplus_{n=0}^{\infty}(\mathcal{H}'_n+\mathcal{F}'_n)$ we find a subalgebra $B \subseteq A$ which is a polynomial algebra with b_n generators in degree n. For $n \geq 4$ we find by Equation (21) a nontrivial element $w \in \mathcal{F}'_n$ lying in the algebra generated by $\bigoplus_{n=1}^{n-1}\mathcal{F}'_n$. This shows that A is generated by a_n elements in degree n with $a_n := \dim(\mathcal{H}'_n+\mathcal{F}'_n)-1$ for $n \geq 4$ and $a_n := \dim(\mathcal{H}'_n+\mathcal{F}'_n)$ for $n \leq 3$. By Theorem 1 we have $a_n = b_n$. Now $B \subseteq A$ implies $A = B$. By Proposition 5 the isomorphism Z^* maps A to the algebra generated by $\bigoplus_{n=0}^{\infty}(\mathcal{H}_n+\mathcal{F}_n)$. This completes the proof. □

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