Some Boundedness Results for Fano-Like Moishezon Manifolds

Laurent Bonavero and Shigeharu Takayama

Received: February 2, 2000

Communicated by Thomas Peternell

Abstract. We prove finiteness of the number of smooth blow-downs on Fano manifolds and boundedness results for the geometry of non projective Fano-like manifolds. Our proofs use properness of Hilbert schemes and Mori theory.

2000 Mathematics Subject Classification: 14J45, 14E30, 14E05, 32J18.
Keywords and Phrases: Fano manifolds, non projective manifold, smooth blow-down, Mori theory, Hilbert scheme.

Introduction

In this Note, we say that a compact complex manifold X is a *Fano-like manifold* if it becomes Fano after a finite sequence of blow-ups along smooth connected centers, *i.e* if there exist a Fano manifold $\tilde{X}$ and a finite sequence of blow-ups along smooth connected centers $\pi\ : \tilde{X} \to X$. We say that a Fano-like manifold X is *simple* if there exists a smooth submanifold Y of X (Y may not be connected) such that the blow-up of X along Y is Fano. If Z is a projective manifold, we call *smooth blow-down of Z (with an s-dimensional center)* a map π and a manifold Z' such that $\pi\ : Z \to Z'$ is the blow-up of Z' along a smooth connected submanifold (of dimension s). We say that a smooth blow-down of Z is projective (resp. non projective) if Z' is projective (resp. non projective).

It is well-known that any Moishezon manifold becomes projective after a finite sequence of blow-ups along smooth centers. Our aim is to bound the geometry of *Moishezon manifolds becoming Fano after one blow-up along a smooth center*, i.e the geometry of *simple non projective Fano-like manifolds.*

Our results in this direction are the following, the simple proof of Theorem 1 has been communicated to us by Daniel Huybrechts.

Theorem 1. *Let Z be a Fano manifold of dimension n. Then, there is only a finite number of smooth blow-downs of Z.*

Let us recall here that the assumption Z Fano is essential : there are projective smooth surfaces with infinitely many -1 rational curves, hence with infinitely many smooth blow-downs.

Since there is only a finite number of deformation types of Fano manifolds of dimension n (see [KMM92] and also [Deb97] for a recent survey on Fano manifolds) and since smooth blow-downs are stable under deformations [Kod63], we get the following corollary (see section 1 for a detailed proof) :

Corollary 1. *There is only a finite number of deformation types of simple Fano-like manifolds of dimension n.*

The next result is essentially due to Wiśniewski ([Wis91], prop. (3.4) and (3.5)). Before stating it, let us define

$$d_n = \max\{(-K_Z)^n \mid Z \text{ is a Fano manifold of dimension}\, n\}$$

and

$$\rho_n = \max\{\rho(Z) := \operatorname{rk}(\operatorname{Pic}(Z)/\operatorname{Pic}^0(Z)) \mid Z \text{ is a Fano manifold of dimension}\, n\}.$$

The number ρ_n is well defined since there is only a finite number of deformation types of Fano manifolds of dimension n and we refer to [Deb97] for an explicit bound for d_n.

Theorem 2. *Let X be an n-dimensional simple non projective Fano-like manifold, Y a smooth submanifold such that the blow-up $\pi : \tilde{X} \to X$ of X along Y is Fano, and E the exceptional divisor of π. Then*

(i) *if each component of Y has Picard number equal to one, then each component of Y has ample conormal bundle in X and is Fano. Moreover*

$$\deg_{-K_{\tilde{X}}}(E) := (-K_{\tilde{X}|E'})^{n-1} \leq (\rho_n - 1)d_{n-1};$$

(ii) *if Y is a curve, then (each component of) Y is a smooth rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-1}$.*

Finally, we prove here the following result :

Theorem 3. *Let Z be a Fano manifold of dimension n and index r. Suppose there is a non projective smooth blow-down of Z with an s-dimensional center. Then*

$$r \leq (n-1)/2 \;\text{ and }\; s \geq r.$$

Moreover,

(i) *if $r > (n-1)/3$, then $s = n - 1 - r$;*
(ii) *if $r < (n-1)/2$ and $s = r$, then $Y \simeq \mathbb{P}^r$.*

Recall that the index of a Fano manifold Z is the largest integer m such that $-K_Z = mL$ for L in the Picard group of Z.

Remarks.

a) For a Fano manifold X of dimension n and index r with second Betti number greater than or equal to 2, it is known that $2r \leq n+2$ [Wi91], with equality if and only if $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

b) Fano manifolds of even dimension (resp. odd dimension n) and middle index (resp. index $(n+1)/2$) with $b_2 \geq 2$ have been intensively studied, see for example [Wis93]. Our Theorem 3 shows that there are no non projective smooth blow-down of such a Fano manifold, without using any explicit classification.
c) The assumption that there is a *non projective* smooth blow-down of Z is essential in Theorem 3 (i) : the Fano manifold obtained by blowing-up $\mathbb{P}^{2r-1}$ along a $\mathbb{P}^{r-2}$ has index r.

1. Proof of Theorem 1 and Corollary 1. An example.

1.1. Proof of Theorem 1. *Thanks to D. Huybrechts for the following proof.* Let Z be a Fano manifold and $\pi\ : Z \to Z'$ a smooth blow-down of Z with an s-dimensional connected center. Let f be a line contained in a non trivial fiber of π. Then, the Hilbert polynomial $P_{-K_Z}(m) := \chi(f, m(-K_Z)_{|f})$ is determined by s and n since $-K_Z \cdot f = n-s-1$ and f is a smooth rational curve. Since $-K_Z$ is ample, the Hilbert scheme Hilb_{-K_Z} of curves in Z having P_{-K_Z} as Hilbert polynomial is a projective scheme, hence has a finite number of irreducible components. Since each curve being in the component $\mathcal{H}$ of Hilb_{-K_Z} containing f is contracted by π, there is only a finite number of smooth blow-downs of Z with an s-dimensional center. $\square$

1.2. Proof of Corollary 1. Let us first recall ([Deb97] section 5.2) that there exists an integer $\delta(n)$ such that every Fano n-fold can be realized as a smooth submanifold of $\mathbb{P}^{2n+1}$ of degree at most $\delta(n)$. Let us denote by T a closed irreducible subvariety of the disjoint union of Chow varieties of n-dimensional subvarieties of $\mathbb{P}^{2n+1}$ of degree at most $\delta(n)$, and by $\pi : \mathcal{X}_T \to T$ the universal family.
Step 1 : Stability of smooth blow-downs. Fix t_0 in the smooth locus T_{smooth} of T and suppose that $X_{t_0} := \pi^{-1}(t_0)$ is a Fano n-fold and there exists a smooth blow-down of X_{t_0} (denote by E_{t_0} the exceptional divisor, P its Hilbert polynomial with respect to $\mathcal{O}_{\mathbb{P}^{2n+1}}(1)$). Let S be the component of the Hilbert scheme of $(n-1)$-dimensional subschemes of $\mathbb{P}^{2n+1}$ with Hilbert polynomial P and $u\ : \mathcal{E}_S \to S$ the universal family. Finally, let I be the following subscheme of $T \times S$:

$$I = \{(t,s) \,|\, u^{-1}(s) \subset X_t\}$$

and $p\ : I \to T$ the proper algebraic map induced by the first projection. Thanks to the analytic stability of smooth blow-downs due to Kodaira (see [Kod63], Theorem 5), the image $p(I)$ contains an analytic open neighbourhood of t_0 hence it also contains a Zariski neighbourhood of t_0. Moreover, since exceptional divisors are rigid, the fiber $p^{-1}(t)$ is a single point for t in a Zariski neighbourhood of t_0. Finally, we get algebraic stability of smooth blow-downs (the $\mathbb{P}^r$-fibered structure of exceptional divisor is also analytically stable - [Kod63], Theorem 4 - hence algebraically stable by the same kind of argument).

Step 2 : Stratification of T by the number of smooth blow-downs. For any integer $k \geq 0$, let us define

$$U_k(T) = \{t \in T_{smooth} \,|\, X_t \text{ is a Fano manifold and there exists at least } k \text{ smooth blow-downs of } X_t\};$$

and $U_{-1}(T) = T_{smooth}$. Thanks to Step 1, $U_k(T)$ is Zariski open in T, and thanks to Theorem 1,

$$\bigcap_{k \geq -1} U_k(T) = \emptyset.$$

Since $\{U_k(T)\}_{k \geq -1}$ is a decreasing sequence of Zariski open sets, by noetherian induction, we get that there exists an integer k such that $U_k(T) = \emptyset$ and we can thus define

$$k(T) := \max\{k \geq -1 \,|\, U_k(T) \neq \emptyset\},\ U(T) := U_{k(T)}(T).$$

Finally, we have proved that $U(T)$ is a non empty Zariski open set of T_{smooth} such that for every $t \in U(T)$, Z_t is a Fano n-fold with exactly $k(T)$ smooth blow-downs ($k(T) = -1$ means that for every $t \in T_{smooth}$, X_t is not a Fano manifold).
Now let $T_0 = T$, and T_1 be any closed irreducible component of $T_0 \setminus U(T_0)$. We get $U(T_1)$ as before and denote by T_2 any closed irreducible component of $T_1 \setminus U(T_1)$, and so-on. Again by noetherian induction, this process terminates after finitely many steps and we get a finite stratification of T such that each strata corresponds to an algebraic family of Fano n-folds with the same number of smooth blow-downs.

Step 3 : Conclusion. Since there is only a finite number of irreducible components in the Chow variety of Fano n-folds, each being finitely stratified by Step 2, we get a finite number of deformation types of simple Fano-like n-folds. □

As it has been noticed by Kodaira, it is essential to consider only *smooth* blow-downs. A -2 rational smooth curve on a surface is, in general, not stable under deformations of the surface.

1.3. An example. Before going further, let us recall the following well known example. Let Z be the projective 3-fold obtained by blowing-up $\mathbb{P}^3$ along a smooth curve of type $(3,3)$ contained in a smooth quadric $\mathcal{Q}$ of $\mathbb{P}^3$. Let π denotes the blow-up $Z \to \mathbb{P}^3$. Then Z is a Fano manifold of index one and there are at least three smooth blow-downs of Z : π, which is projective, and two non projective smooth blow-downs consisting in contracting the strict transform $\mathcal{Q}'$ of the quadric $\mathcal{Q}$ along one of its two rulings (the normal bundle of $\mathcal{Q}'$ in Z is $\mathcal{O}(-1,-1)$).

Lemma 1. *There are exactly three smooth blow-downs of Z.*

Proof : the Mori cone $\mathrm{NE}(Z)$ is a 2-dimensional closed cone, one of its two extremal rays being generated by the class of a line f_π contained in a non trivial fiber of π, the other one, denoted by $[R]$, by the class of one of the two rulings of $\mathcal{Q}'$ (the two rulings are numerically equivalent, the corresponding extremal contraction consists in contracting $\mathcal{Q}'$ to a singular point in a projective variety, hence is not a smooth blow-down). If E is the exceptional divisor of π, we have

$$E \cdot [f_\pi] = -1,\ E \cdot [R] = 3,\ \mathcal{Q}' \cdot [f_\pi] = 1,\ \mathcal{Q}' \cdot [R] = -1.$$

Now suppose there exists a smooth blow-down τ of Z with a 1-dimensional center, which is not one of the three previously described. Let L be a line contained in a non trivial fiber of τ, then since $-K_Z \cdot [L] = 1$, we have $[L] = a[f_\pi] + b[R]$ for some strictly positive numbers such that $a + b = 1$. Since we have moreover

$$\mathcal{Q}' \cdot [L] = a - b = 2a - 1 \in \mathbb{Z} \text{ and } E \cdot [L] = 3b - a = 3 - 4a,$$

we get $a = b = 1/2$. Therefore $\mathcal{Q}' \cdot [L] = 0$ hence L is disjoint from $\mathcal{Q}'$ (it can not be contained in $\mathcal{Q}'$ since $\mathcal{Q}'_{|\mathcal{Q}'} = \mathcal{O}(-1,-1)$). It implies that there are two smooth blow-downs of Z with disjoint exceptional divisors, which is impossible since $\rho(Z) = 2$.
Finally, if there is a smooth blow-down $\tau : Z \to Z'$ of Z with a 0-dimensional center, then Z' is projective and τ is a Mori extremal contraction, which is again impossible since we already met the two Mori extremal contractions on Z. □

2. Non projective smooth blow-downs on a center with Picard number 1. Proof of Theorem 2.

The proof of Theorem 2 we will give is close to Wiśniewski's one but we give two intermediate results of independant interest.

2.1. On the normal bundle of the center. Let us recall that a smooth submanifold A in a complex manifold W is contractible to a point (i.e. there exists a complex space W' and a map $\mu : W \to W'$ which is an isomorphism outside A and such that $\mu(A)$ is a point) if and only if $N^*_{A/W}$ is ample (Grauert's criterion [Gra62]).
The following proposition was proved by Campana [Cam89] in the case where Y is a curve and $\dim(X) = 3$.

Proposition 1. *Let X be a non projective manifold, Y a smooth submanifold of X such that the blow-up $\pi : \tilde{X} \to X$ of X along Y is projective. Then, for each connected component Y' of Y with $\rho(Y') = 1$, the conormal bundle $N^*_{Y'/X}$ is ample.*

Before the proof, let us remark that Y is projective since the exceptional divisor of π is.

Proof of Proposition 1 : (following Campana) we can suppose that Y is connected. Let E be the exceptional divisor of π and f a line contained in a non trivial fiber of π. Since $E \cdot f = -1$, there is an extremal ray R of the Mori cone $\overline{\mathrm{NE}}(\tilde{X})$ such that $E \cdot R < 0$. Since $E \cdot R < 0$, R defines an extremal ray of the Mori cone $\overline{\mathrm{NE}}(E)$ which we still denote by R (even if $\overline{\mathrm{NE}}(E)$ is not a subcone of $\overline{\mathrm{NE}}(\tilde{X})$ in general !). Since $\rho(Y) = 1$, we have $\rho(E) = 2$, hence $\overline{\mathrm{NE}}(E)$ is a 2-dimensional closed cone, one of its two extremal rays being generated by f. Then :

- either R is not generated by f and $E_{|E}$ is strictly negative on $\overline{\mathrm{NE}}(E)\setminus\{0\}$. In that case, $-E_{|E} = \mathcal{O}_E(1)$ is ample by Kleiman's criterion, which means that $N^*_{Y/X}$ is ample.
- or, R is generated by f. In that case, the Mori contraction $\varphi_R : \tilde{X} \to Z$ factorizes through π :

$$\begin{tikzcd} \tilde{X} \arrow[r, "\varphi_R"] \arrow[d, "\pi"'] & Z \\ X \arrow[ur, "\psi"'] & \end{tikzcd}$$

where $\psi : X \to Z$ is an isomorphism outside Y. Since the variety Z is projective and X is not, ψ is not an isomorphism and since $\rho(Y) = 1$, Y is contracted to a point by ψ, hence $N^*_{Y/X}$ is ample by Grauert's criterion. □

Let us prove the following consequence of Proposition 1:

PROPOSITION 2. *Let X be a non projective manifold, Y a smooth submanifold of X such that the blow-up $\pi : \tilde{X} \to X$ of X along Y is projective with $-K_{\tilde{X}}$ numerically effective (nef). Then, each connected component Y' of Y with $\rho(Y') = 1$ is a Fano manifold.*

PROOF : we can suppose that Y is connected. Let E be the exceptional divisor of π. Since $-E_{|E}$ is ample by Proposition 1, the adjunction formula $-K_E = -K_{\tilde{X}|E} - E_{|E}$ shows that $-K_E$ is ample, hence E is Fano. By a result of Szurek and Wiśniewski [SzW90], Y is itself Fano. □

2.2. PROOF OF THEOREM 2. For the first assertion, we only have to prove that

$$\deg_{-K_{\tilde{X}}}(E) \leq (\rho_n - 1)d_{n-1}.$$

Let Y' be a connected component of Y and $E' = \pi^{-1}(Y')$. Then, since $-E_{|E'}$ is ample :

$$\deg_{-K_{\tilde{X}}}(E') = (-K_{\tilde{X}|E'})^{n-1} = (-K_{E'} + E_{|E'})^{n-1} \leq (-K_{E'})^{n-1} \leq d_{n-1}.$$

Now, if m is the number of connected components of Y, then

$$\rho_n \geq \rho(\tilde{X}) = m + \rho(X) \geq m + 1.$$

Putting all together, we get

$$\deg_{-K_{\tilde{X}}}(E) \leq (\rho_n - 1)d_{n-1},$$

which ends the proof of the first point.
We refer to [Wis91] prop. (3.5) for the second point. □

3. On the dimension of the center of non projective smooth blow-downs. Proof of Theorem 3.

Theorem 3 is a by-product of the more precise following statement and of Proposition 3 below :

Theorem 4. *Let Z be a Fano manifold of dimension n and index r, $\pi : Z \to Z'$ be a non projective smooth blow-down of Z, $Y \subset Z'$ the center of π. Let f be a line contained in a non trivial fiber of π, then*

(i) *if f generates an extremal ray of* $\mathrm{NE}(Z)$*, then* $\dim(Y) \geq (n-1)/2$.
(ii) *if f does not generate an extremal ray of* $\mathrm{NE}(Z)$*, then* $\dim(Y) \geq r$. *Moreover, if* $\dim(Y) = r$*, then Y is isomorphic to* $\mathbb{P}^r$.

In both cases (i) and (ii), Y contains a rational curve.

The proof relies on Wiśniewski's inequality (see [Wis91] and [AnW95]), which we recall now for the reader's convenience : let $\varphi : X \to Y$ be a Fano-Mori contraction (i.e $-K_X$ is φ-ample) on a projective manifold X, $\mathrm{Exc}(\varphi)$ its exceptional locus and

$$l(\varphi) := \min\{-K_X \cdot C\,;\, C \text{ rational curve contained in} \,\mathrm{Exc}(\varphi)\}$$

its length, then for every non trivial fiber F :

$$\dim \mathrm{Exc}(\varphi) + \dim(F) \geq \dim(X) - 1 + l(\varphi).$$

Proof of Theorem 4. The method of proof is taken from Andreatta's recent paper [And99] (see also [Bon96]).
First case : suppose that a line f contained in a non trivial fiber of π generates an extremal ray R of $\mathrm{NE}(Z)$. Then the Mori contraction $\varphi_R : Z \to W$ factorizes through π :

$$\begin{tikzcd} Z \arrow[r, "\varphi_R"] \arrow[d, "\pi"'] & W \\ Z' \arrow[ur, "\psi"'] & \end{tikzcd}$$

where ψ is an isomorphism outside Y. In particular, the exceptional locus E of π is equal to the exceptional locus of the extremal contraction φ_R.
Let us now denote by ψ_Y the restriction of ψ to Y, $s = \dim(Y)$, π_E and $\varphi_{R,E}$ the restriction of π and φ_R to E. Since Z' is not projective, ψ_Y is not a finite map. Since φ_R is birational, W is $\mathbb{Q}$-Gorenstein, hence K_W is $\mathbb{Q}$-Cartier and $K_{Z'} = \psi^* K_W$. Therefore, $K_{Z'}$ is ψ-trivial, hence $K_Y + \det N^*_{Y/Z'}$ is ψ_Y-trivial. Moreover, $\mathcal{O}_E(1) = -E_{|E}$ is $\varphi_{R,E}$-ample by Kleiman's criterion, hence $N^*_{Y/Z'}$ is ψ_Y-ample. Finally, ψ_Y is a Fano-Mori contraction, of length greater or equal to $n - s = \mathrm{rk}(N^*_{Y/Z'})$. Together with Wiśniewski's inequality applied on Y, we get that for every non trivial fiber F of ψ_Y

$$2s \geq \dim(F) + \dim \mathrm{Exc}(\psi_Y) \geq n - s + s - 1$$

hence $2s \geq n-1$. Moreover, $\mathrm{Exc}(\psi_Y)$ is covered by rational curves, hence Y contains a rational curve.

Second case : suppose that a line f contained in a non trivial fiber of π does not generate an extremal ray R of $\mathrm{NE}(Z)$. In that case, since $E \cdot f = -1$, there is an extremal ray R of $\mathrm{NE}(Z)$ such that $E \cdot R < 0$. In particular, the exceptional locus $\mathrm{Exc}(R)$ of the extremal contraction φ_R is contained in E, and since f is not on R, we get for any fiber F of φ_R :

$$\dim(F) \leq s = \dim(Y).$$

By the adjunction formula, $-K_E = -K_{Z|E} - E_{|E}$, the length $l_E(R)$ of R as *an extremal ray of E* satisfies

$$l_E(R) \geq r + 1,$$

where r is the index of Z. Together with Wiśniewski's inequality applied on E, we get :

$$r + 1 + (n-1) - 1 \leq s + \dim(\mathrm{Exc}(R)) \leq s + n - 1.$$

Finally, we get $r \leq s$, and since the fibers of φ_R are covered by rational curves, there is a rational curve in Y. Suppose now (up to the end) that $r = s$. Then E is the exceptional locus of the Mori extremal contraction φ_R. Moreover, $K_Z + r(-E)$ is a good supporting divisor for φ_R, and since every non trivial fiber of φ_R has dimension r, φ_R is a smooth projective blow-down. In particular, the restriction of π to a non trivial fiber $F \simeq \mathbb{P}^r$ induces a finite surjective map $\pi : F \simeq \mathbb{P}^r \to Y$ hence $Y \simeq \mathbb{P}^r$ by a result of Lazarsfeld [Laz83].
This ends the proof of Theorem 4. □

The proof of Theorem 4 does not use the hypothesis Z Fano in the first case. We therefore have the following :

Corollary 2. *Let Z be a projective manifold of dimension n, $\pi : Z \to Z'$ be a non projective smooth blow-down of Z, $Y \subset Z'$ the center of π. Let f be a line contained in a non trivial fiber of π and suppose f generates an extremal ray of $\overline{\mathrm{NE}}(Z)$. Then* $\dim(Y) \geq (n-1)/2$. *Moreover, if* $\dim(Y) = (n-1)/2$, *then Y is contractible on a point.*

We finish this section by the following easy proposition, which combined with Theorem 4 implies Theorem 3 of the Introduction :

Proposition 3. *Let Z be a Fano manifold of dimension n and index r, $\pi : Z \to Z'$ be a smooth blow-down of Z, $Y \subset Z'$ the center of π. Then $n - 1 - \dim(Y)$ is a multiple of r.*

Proof. Write

$$-K_Z = rL \quad \text{and} \quad -K_Z = -\pi^* K_{Z'} - (n - 1 - \dim(Y))E$$

where E is the exceptional divisor of π. Let f be a line contained in a fiber of π. Then $rL \cdot f = n - 1 - \dim(Y)$, which ends the proof. □

Proof of Theorem 3. Let Z be a Fano manifold of dimension n and index r and suppose there is a non projective smooth blow-down of Z with an s-dimensional center. By Proposition 3, there is a strictly positive integer k such that $n - 1 - kr = s$. By Theorem 4, either $n - 1 - kr \geq (n-1)/2$ or

$n-1-kr \geq r$. In both cases, it implies that $r \leq (n-1)/2$ and therefore $s \geq r$. If $r > (n-1)/3$, since $n-1 \geq (k+1)r > (k+1)(n-1)/3$, we get $k=1$ and $s = n-1-r$. □

4. Rational curves on simple Moishezon manifolds.

The arguments of the previous section can be used to deal with the following well-known question : *does every non projective Moishezon manifold contain a rational curve ?* The answer is positive in dimension three (it is due to Peternell [Pet86], see also [CKM88] p. 49 for a proof using the completion of Mori's program in dimension three).

Proposition 4. *Let Z be a projective manifold, $\pi : Z \to Z'$ be a non projective smooth blow-down of Z. Then Z' contains a rational curve.*

Proof. With the notations of the previous section, it is clear in the first case where a line f contained in a non trivial fiber of π generates an extremal ray R of $\overline{\mathrm{NE}}(Z)$ (in that case, the center of π contains a rational curve). In the second case, since f is not extremal and K_Z is not nef, there is a Mori contraction φ on Z such that any rational curve contained in a fiber of φ is mapped by π to a non constant rational curve in Z'. □

References

[And99] M. Andreatta. Moishezon manifolds. *Math. Zeitschrift, 230, 713-726 (1999).*

[AnW95] M. Andreatta, J. Wiśniewski. A view on contractions of higher-dimensional varieties. *Algebraic geometry—Santa Cruz, 153–183, (1995) Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, (1997).*

[Bon96] L. Bonavero. Sur des variétés de Moishezon dont le groupe de Picard est de rang un. *Bull. Soc. Math. Fr. 124, No.3, 503-521 (1996).*

[Cam89] F. Campana. Critère de projectivité de la transformation élémentaire d'une variété projective de dimension 3. *C. R. Acad. Sci., Paris, Ser. I 309, No.14, 863-866 (1989).*

[CKM88] H. Clemens, J. Kollár, S. Mori. Higher dimensional complex geometry. *Société Mathématique de France, Astérisque 166 (1988).*

[Deb97] O. Debarre. Variétés de Fano. *Séminaire Bourbaki. Volume 1996/97. Exposés 820–834. Paris, Société Mathématique de France, Astérisque 245, 197-221, Exp. No.827 (1997).*

[Gra62] H. Grauert. Ueber Modifikationen und exzeptionelle analytische Mengen. *Math. Ann. 146, 331-368 (1962).*

[Kod63] K. Kodaira. On stability of compact submanifolds of complex manifolds. *Amer. J. Math., 85, 79-94 (1963).*

[KMM92] J. Kollár, Y. Miyaoka, S. Mori. Rational connectedness and boundedness of Fano manifolds. *J. Diff. Geom. 36, 765-769 (1992).*

[Laz83] R. Lazarsfeld. Some applications of the theory of positive vector bundles. *Complete intersections, Lect. 1st Sess. C.I.M.E., Acireale/Italy 1983, Lect. Notes Math. 1092, 29-61 (1984).*

[Pet86] T. Peternell. Rational curves on a Moishezon 3-fold. *Complex Analysis and Algebraic Geometry. Springer LN 1194, 133-144 (1986).*

[SzW90] M. Szurek, J. Wiśniewski. Fano bundles over $\mathbb{P}^3$ and Q^3. *Pac. J. Math. 141, No.1, 197-208 (1990).*

[Wis91] J. Wiśniewski. On contractions of extremal rays of Fano manifolds. *J. Reine Angew. Math. 417, 141-157 (1991).*

[Wi91] J. Wiśniewski. On Fano manifolds of large index. *Manuscripta Math. 70, no. 2, 145–152 (1991).*

[Wis93] J. Wiśniewski. A report on Fano manifolds of middle index and $b_2 \geq 2$. *Mathematica Gottingensis, Schriftenreihe des Sonderforschungsbereichs Geometrie und Analysis. Heft 16 (1993).*

Laurent Bonavero
Institut Fourier, UMR 5582
Université de Grenoble 1
BP 74.
38402 Saint Martin d'Hères
France
bonavero@ujf-grenoble.fr

Shigeharu Takayama
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka, 560-0043
Japan
taka@math.sci.osaka-u.ac.jp