A Classification Theorem for Nuclear Purely Infinite Simple C*-Algebras[1]

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Abstract. Starting from Kirchberg's theorems announced at the operator algebra conference in Genève in 1994, namely $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for separable unital nuclear simple A and $\mathcal{O}_\infty \otimes A \cong A$ for separable unital nuclear purely infinite simple A, we prove that KK-equivalence implies isomorphism for nonunital separable nuclear purely infinite simple C^*-algebras. It follows that if A and B are unital separable nuclear purely infinite simple C^*-algebras which satisfy the Universal Coefficient Theorem, and if there is a graded isomorphism from $K_*(A)$ to $K_*(B)$ which preserves the K_0-class of the identity, then $A \cong B$.

Our main technical results are, we believe, of independent interest. We say that two asymptotic morphisms $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ from A to B are asymptotically unitarily equivalent if there exists a continuous unitary path $t \mapsto u_t$ in the unitization B^+ such that $\|u_t \varphi_t(a) u_t^* - \psi_t(a)\| \to 0$ for all a in A. We prove the following two results on deformations and unitary equivalence. Let A be separable, nuclear, unital, and simple, and let D be unital. Then any asymptotic morphism from A to $K \otimes \mathcal{O}_\infty \otimes D$ is asymptotically unitarily equivalent to a homomorphism, and two homotopic homomorphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$ are necessarily asymptotically unitarily equivalent. We also give some nonclassification results for the nonnuclear case.

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0 Introduction

We prove that the isomorphism class of a separable nuclear unital purely infinite simple C^*-algebra satisfying the Rosenberg-Schochet Universal Coefficient Theorem is completely determined by its K-theory. More precisely, let A and

B be separable nuclear unital purely infinite simple C^*-algebras which satisfy the Universal Coefficient Theorem, and suppose that there is a graded isomorphism $\alpha : K_*(A) \to K_*(B)$ such that $\alpha([1_A]) = [1_B]$ in $K_0(B)$. Then there is an isomorphism $\varphi : A \to B$ such that $\varphi_* = \alpha$. This theorem follows from a result asserting that whenever A and B are separable nuclear unital purely infinite simple C^*-algebras (not necessarily satisfying the Universal Coefficient Theorem) which are KK-equivalent via a class in KK-theory which respects the classes of the identities, then there is an isomorphism from A to B whose class in KK-theory is the given one.

As intermediate results, we prove some striking facts about homomorphisms and asymptotic morphisms from a separable nuclear unital simple C^*-algebra to the tensor product of a unital C^*-algebra and the Cuntz algebra $\mathcal{O}_\infty$. If A and D are any two C^*-algebras, we say that two homomorphisms $\varphi, \psi : A \to D$ are *asymptotically unitarily equivalent* if there is a continuous unitary path $t \mapsto u_t$ in $\widetilde{D}$ such that $\lim_{t\to\infty} u_t\varphi(a)u_t* = \psi(a)$ for all $a \in A$. (Here $\widetilde{D} = D$ if D is unital, and $\widetilde{D}$ is the unitization D^+ if D is not unital.) Note that asymptotic unitary equivalence is a slightly strengthened form of approximate unitary equivalence, and is an approximate form of unitary equivalence. Our results show that if A is separable, nuclear, unital, and simple, and D is separable and unital, then $KK^0(A, D)$ can be computed as the set of asymptotic unitary equivalence classes of full homomorphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$, with direct sum as the operation. Note that we use something close to unitary equivalence, and that there is no need to use asymptotic morphisms, no need to take suspensions, and (essentially because $\mathcal{O}_\infty$ is purely infinite) no need to form formal differences of classes. We can furthermore replace A by $K \otimes \mathcal{O}_\infty \otimes A$, in which case the Kasparov product reduces exactly to composition of homomorphisms. These results can be thought of as a form of unsuspended E-theory. (Compare with [16], but note that we don't even need to use asymptotic morphisms.) There are also perturbation results: any asymptotic morphism is in fact asymptotically unitarily equivalent (with a suitable definition) to a homomorphism.

We also present what is now known about how badly the classification fails in the nonnuclear case. There are separable purely infinite simple C^*-algebras A with $\mathcal{O}_\infty \otimes A \not\cong A$ (Dykema–Rørdam), there are infinitely many nonisomorphic separable exact purely infinite simple C^*-algebras A with $\mathcal{O}_\infty \otimes A \cong A$ and $K_*(A) = 0$ (easily obtained from results of Haagerup and Cowling–Haagerup), and for given K-theory there are uncountably many nonisomorphic separable nonexact purely infinite simple C^*-algebras with that K-theory.

Classification of C^*-algebras started with Elliott's classification [19] of AF algebras up to isomorphism by their K-theory. It received new impetus with his successful classification of certain C^*-algebras of real rank zero with nontrivial K_1-groups. We refer to [21] for a recent comprehensive list of work in this area. The initial step toward classification in the infinite case was taken in [8], and was quickly followed by a number of papers [48], [49], [33], [34], [22], [50], [35], [7], [51], [32], [36]. In July 1994, Kirchberg announced [27] a breakthrough: proofs that if A is a separable nuclear unital purely infinite simple C^*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes A \cong A$. (The proofs, closely following Kirchberg's original methods, are in [29].) This quickly led to two more papers [44], [52]. Here, we use Kirchberg's results to nearly solve the classification problem for separable nuclear unital purely infinite simple C^*-algebras; the only difficulty that remains is the Universal Coefficient Theorem. The method is a great generalization of that of [44], in which we replace homomorphisms by asymptotic morphisms and approximate unitary equivalence by asymptotic unitary equivalence. We also need a form of unsuspended E-theory, as alluded to above. The most crucial step is done in Section 2, where we show that, in a particular context, homotopy implies asymptotic unitary equivalence. We suggest reading [44] to understand the basic structure of Section 2.

Kirchberg has in [28] independently derived the same classification theorem we have. His methods are somewhat different, and mostly independent of the proofs in [29]. He proves that homotopy implies a form of unitary equivalence in a different context, and does so by eventually reducing the problem to a theorem of this type in Kasparov's paper [26]. By contrast, the main machinery in our proof is simply the repeated use of Kirchberg's earlier results as described above.

This paper is organized as follows. In Section 1, we present some important facts about asymptotic morphisms, and introduce asymptotic unitary equivalence. In Section 2, we prove our main technical results: under suitable conditions, homotopic asymptotic morphisms are asymptotically unitarily equivalent and asymptotic morphisms are asymptotically unitarily equivalent to homomorphisms. These results are given at the end of the section. In Section 3, we prove the basic form (still using asymptotic morphisms) of our version of unsuspended E-theory. Finally, Section 4 contains the classification theorem and some corollaries, as well as the nicest forms of the intermediate results discussed above. It also contains the nonclassification results.

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Throughout this paper, $U(D)$ denotes the unitary group of a unital C^*-algebra D, and $U_0(D)$ denotes the connected component of $U(D)$ containing 1. We will use repeatedly and without comment Cuntz's result that

$K_1(D) = U(D)/U_0(D)$ for a unital purely infinite simple C^*-algebra D, as well as his corresponding result that $K_0(D)$ is the set of Murray-von Neumann equivalence classes of nonzero projections [14]. We similarly use Kasparov's KK-theory [26], and we recall here (and do not mention again) that every separable nonunital purely infinite simple C^*-algebra has the form $K \otimes D$ for a unital purely infinite simple C^*-algebra D [61].

1 Asymptotic morphisms and asymptotic unitary equivalence

The basic objects we work with in this paper are asymptotic morphisms. In the first subsection, we state for convenient reference some of the facts we need about asymptotic morphisms, and establish notation concerning them. In the second subsection, we define and discuss full asymptotic morphisms; fullness is used as a nontriviality condition later in the paper. In the third subsection, we introduce asymptotic unitary equivalence of asymptotic morphisms. This relation is the appropriate version of unitary equivalence in the context of asymptotic morphisms, and will play a fundamental role in Sections 2 and 3.

1.1 Asymptotic morphisms and asymptotic unitary equivalence

Asymptotic morphisms were introduced by Connes and Higson [11] for the purpose of defining E-theory, a simple construction of KK-theory (at least if the first variable is nuclear). In this subsection, we recall the definition and some of the basic results on asymptotic morphisms, partly to establish our notation and partly for ease of reference. We also prove a few facts that are well known but seem not to have been published. We refer to [11], and the much more detailed paper [54], for the details of the rest of the development of E-theory.

If X is a compact Hausdorff Hausdorff space, then $C(X, D)$ denotes the C^*-algebra of all continuous functions from X to D, while if X is locally compact Hausdorff Hausdorff, then $C_0(X, D)$ denotes the C^*-algebra of all continuous functions from X to D which vanish at infinity, and $C_{\mathrm{b}}(X, D)$ denote the C^*-algebra of all bounded continuous functions from X to D.

We begin by recalling the definition of an asymptotic morphism.

1.1.1 Definition. Let A and D be C^*-algebras, with A separable. An *asymptotic morphism* $\varphi : A \to D$ is a family $t \to \varphi_t$ of functions from A to D, defined for $t \in [0, \infty)$, satisfying the following conditions:

(1) For every $a \in A$, the function $t \mapsto \varphi_t(a)$ is continuous from $[0, \infty)$ to D.

(2) For every $a, b \in A$ and $\alpha, \beta \in \mathbf{C}$, the limits

$$\lim_{t\to\infty} (\varphi_t(\alpha a + \beta b) - \alpha\varphi_t(a) - \beta\varphi_t(b)),$$

$$\lim_{t\to\infty}(\varphi_t(ab)-\varphi_t(a)\varphi_t(b)), \quad \text{and} \quad \lim_{t\to\infty}(\varphi_t(a^*)-\varphi_t(a)^*)$$

are all zero.

1.1.2 Definition. ([11]) Let φ and ψ be asymptotic morphisms from A to D.

(1) We say that φ and ψ are *asymptotically equal* (called "equivalent" in [11]) if for all $a \in A$, we have $\lim_{t\to\infty}(\varphi_t(a)-\psi_t(a))=0$.

(2) We say that φ and ψ are *homotopic* if there is an asymptotic morphism $\rho : A \to C([0,1], D)$ whose restrictions to $\{0\}$ and $\{1\}$ are φ and ψ respectively. In this case, we refer to $\alpha \mapsto \rho^{(\alpha)} = \mathrm{ev}_\alpha \circ \rho$ (where $\mathrm{ev}_\alpha : C([0,1], D) \to D$ is evaluation at α) as a homotopy from φ to ψ, or as a continuous path of asymptotic morphisms from φ to ψ.

The set of homotopy classes of asymptotic morphisms from A to D is denoted $[[A, D]]$, and the homotopy class of an asymptotic morphism φ is denoted $[[\varphi]]$.

It is easy to check that asymptotic equality implies homotopy ([54], Remark 1.11).

1.1.3 Definition. Let $\varphi, \psi : A \to K \otimes D$ be asymptotic morphisms. The direct sum $\varphi \oplus \psi$, well defined up to unitary equivalence (via unitaries in $M(K \otimes D)$), is defined as follows. Choose any isomorphism $\delta : M_2(K) \to K$, let $\overline{\delta} : M_2(K \otimes D) \to K \otimes D$ be the induced map, and define

$$(\varphi \oplus \psi)_t(a) = \overline{\delta}\left(\begin{pmatrix} \varphi_t(a) & 0 \\ 0 & \psi_t(a) \end{pmatrix}\right).$$

Note that any two choices for δ are unitarily equivalent (and hence homotopic).

The individual maps φ_t of an asymptotic morphism are not assumed bounded or even linear.

1.1.4 Definition. Let $\varphi : A \to D$ be an asymptotic morphism.

(1) We say that φ is *completely positive contractive* if each φ_t is a linear completely positive contraction.

(2) We say that φ is *bounded* if each φ_t is linear and $\sup_t \|\varphi_t\|$ is finite.

(3) We say that φ is *selfadjoint* if $\varphi_t(a^*) = \varphi_t(a)^*$ for all t and a.

Unless otherwise specified, homotopies of asymptotic morphisms from A to D satisfying one or more of these conditions will be assumed to satisfy the same conditions as asymptotic morphisms from A to $C([0,1], D)$.

Note that if φ is bounded, then the formula $\psi_t(a) = \frac{1}{2}(\varphi_t(a) + \varphi_t(a^*)^*)$ defines a selfadjoint bounded asymptotic morphism which is asymptotically equal to φ. We omit the easy verification that ψ is in fact an asymptotic morphism.

1.1.5 Lemma. ([54], Lemma 1.6.) Let A and D be C^*-algebras, with A separable and nuclear. Then every asymptotic morphism from A to D is asymptotically equal to a completely positive contractive asymptotic morphism. Moreover, the obvious map defines a bijection between the sets of homotopy classes of completely positive contractive asymptotic morphisms and arbitrary asymptotic morphisms. (Homotopy classes are as in the convention in Definition 1.1.4.)

1.1.6 Lemma. Let $\varphi : A \to D$ be an asymptotic morphism. Define $\varphi^+ : A^+ \to D^+$ by $\varphi_t(a + \lambda \cdot 1) = \varphi_t(a) + \lambda \cdot 1$ for $a \in A$ and $\lambda \in \mathbf{C}$. Then φ^+ is an asymptotic morphism from A^+ to D^+, and is completely positive contractive, bounded, or selfadjoint whenever φ is.

The proof of this is straightforward, and is omitted.

The following result is certainly known, but we know of no reference.

1.1.7 Proposition. Let A be a C^*-algebra which is given by exactly stable (in the sense of Loring [37]) generators and relations (G, R), with both G and R finite. Let D be a C^*-algebra. Then any asymptotic morphism from A to D is asymptotically equal to a continuous family of homomorphisms from A to D (parametrized by $[0, \infty)$). Moreover, if $\varphi^{(0)}$ and $\varphi^{(1)}$ are two homotopic asymptotic morphisms from A to D, such that each $\varphi_t^{(0)}$ and each $\varphi_t^{(1)}$ is a homomorphism, then there is a homotopy $\alpha \mapsto \varphi^{(\alpha)}$ which is asymptotically equal to the given homotopy and such that each $\varphi_t^{(\alpha)}$ is a homomorphism.

Note that it follows from Theorem 2.6 of [38] that exact stability of (G, R) depends only on A, not on the specific choices of G and R.

Proof of Proposition 1.1.7: Theorem 2.6 of [38] implies that the algebra A is semiprojective in the sense of Blackadar [4]. (Also see Definition 2.3 of [38].) We will use semiprojectivity instead of exact stability.

We prove the first statement. Let $\varphi : A \to D$ be an asymptotic morphism. Then φ defines in a standard way (see Section 1.2 of [54]) a homomorphism $\psi : A \to C_{\mathrm{b}}([0, \infty), D)/C_0([0, \infty), D)$. Let

$$I_n(D) = \{f \in C_{\mathrm{b}}([0, \infty), D) : f(t) = 0 \text{ for } t \geq n\}.$$

Then $C_0([0, \infty), D) = \overline{\bigcup_{n=1}^{\infty} I_n(D)}$. Semiprojectivity of A provides an n and a homomorphism $\sigma : A \to C_{\mathrm{b}}([0, \infty), D)/I_n(D)$ such that the composite of σ and the quotient map

$$C_{\mathrm{b}}([0, \infty), D)/I_n(D) \to C_{\mathrm{b}}([0, \infty), D)/C_0([0, \infty), D)$$

is ψ. Now σ can be viewed as a continuous family of homomorphisms σ_t from A to D, parametrized by $[n, \infty)$. Define $\sigma_t = \sigma_n$ for $0 \leq t \leq n$. This gives the required continuous family of homomorphisms.

The proof of the statement about homotopies is essentially the same. We use $C_{\mathrm{b}}([0, 1] \times [0, \infty), D)$ in place of $C_{\mathrm{b}}([0, \infty), D)$,

$$J = \{f \in C_0([0, 1] \times [0, \infty), D) : f(\alpha, t) = 0 \text{ for } \alpha = 0, 1\}$$

in place of $C_0([0,\infty),D)$, and $J \cap I_n([0,1],D)$ in place of $I_n(D)$. We obtain $\varphi_t^{(\alpha)}$ for all t greater than or equal to some t_0, and for all t when $\alpha = 0$ or 1. We then extend over $(0,1) \times [0,t_0)$ via a continuous retraction

$$[0,1] \times [0,\infty) \to ([0,1] \times [t_0,\infty)) \cup (\{0\} \times [0,\infty)) \cup (\{1\} \times [0,\infty)).$$

□

We refer to [11] (and to [54] for more detailed proofs) for the definition of $E(A,B)$ as the abelian group of homotopy classes of asymptotic morphisms from $K \otimes SA$ to $K \otimes SB$, for the construction of the composition of asymptotic morphisms (well defined up to homotopy), and for the construction of the natural map $KK^0(A,B) \to E(A,B)$ and the fact that it is an isomorphism if A is nuclear. We do state here for reference the existence of the tensor product of asymptotic morphisms. For the proof, see Section 2.2 of [54].

1.1.8 Proposition. ([11]) Let A_1, A_2, B_1, and B_2 be separable C^*-algebras, and let $\varphi^{(i)} : A_i \to B_i$ be asymptotic morphisms. Then there exists an asymptotic morphism $\psi : A_1 \otimes A_2 \to B_1 \otimes B_2$ (maximal tensor products) such that $\psi_t(a_1 \otimes a_2) - \varphi_t^{(1)}(a_1) \otimes \varphi_t^{(2)}(a_2) \to 0$ as $t \to \infty$, for all $a_1 \in A_1$ and $a_2 \in A_2$. Moreover, ψ is unique up to asymptotic equality.

1.2 Full asymptotic morphisms

In this subsection, we define full asymptotic morphisms. Fullness will be used as a nontriviality condition on asymptotic morphisms in Section 3. It will also be convenient (although not, strictly speaking, necessary) in Section 2.

We make our definitions in terms of projections, because the behavior of asymptotic morphisms on projections can be reasonably well controlled. We do not want to let the asymptotic morphism $\varphi : C_0(\mathbf{R}) \to C_0(\mathbf{R})$, defined by $\varphi_t(f) = tf$, be considered to be full, since it is asymptotically equal to the zero asymptotic morphism, but in the absence of projections it is not so clear how to rule it out. Fortunately, in the present paper this issue does not arise.

We start with a useful definition and some observations related to the evaluation of asymptotic morphisms on projections.

1.2.1 Definition. Let A and D be C^*-algebras, with A separable. Let $p \in A$ be a projection, and let $\varphi : A \to D$ be an asymptotic morphism. A *tail projection* for $\varphi(p)$ is a continuous function $t \mapsto q_t$ from $[0,\infty)$ to the projections in D which, thought of as an asymptotic morphism $\psi : \mathbf{C} \to D$ via $\psi_t(\lambda) = \lambda q_t$, is asymptotically equal to the asymptotic morphism $\psi'_t(\lambda) = \lambda \varphi_t(p)$.

1.2.2 Remark. (1) Tail projections always exist: Choose a suitable t_0, apply functional calculus to $\frac{1}{2}(\varphi_t(p) + \varphi_t(p)^*)$ for $t \geq t_0$, and take the value at t for $t \leq t_0$ to be the value at t_0. (Or use Proposition 1.1.7.)

(2) If φ is an asymptotic morphism from A to D, then a tail projection for $\varphi(p)$, regarded as an asymptotic morphism from $\mathbf{C}$ to D, is a representative of the product homotopy class of φ and the asymptotic morphism from $\mathbf{C}$ to A given by p.

(3) A *homotopy* of tail projections is defined in the obvious way: it is a continuous family of projections $(\alpha, t) \to q_t^{(\alpha)}$ with given values at $\alpha = 0$ and $\alpha = 1$.

(4) If φ is an asymptotic morphism, then it makes sense to say that a tail projection is (or is not) full (that is, generates a full hereditary subalgebra), since fullness depends only on the homotopy class of a projection.

1.2.3 Lemma. Let A and D be C^*-algebras, with A separable. Let $\varphi : A \to D$ be an asymptotic morphism, and let p_1 and p_2 be projections in A. If p_1 is Murray-von Neumann equivalent to a subprojection of p_2, then a tail projection for $\varphi(p_1)$ is Murray-von Neumann equivalent to a subprojection of a tail projection for $\varphi(p_2)$.

Proof: Let $t \mapsto q_t^{(1)}$ and $t \mapsto q_t^{(2)}$ be tail projections for $\varphi(p_1)$ and $\varphi(p_2)$ respectively. Let v be a partial isometry with $v^*v = p_1$ and $vv^* \leq p_2$. Using asymptotic multiplicativity and the definition of a tail projection, we have

$$\lim_{t\to\infty} \left(\varphi_t(v)^*\varphi_t(v) - q_t^{(1)}\right) = 0 \quad \text{and} \quad \lim_{t\to\infty} \left(q_t^{(2)}\varphi_t(v)\varphi_t(v)^* q_t^{(2)} - q_t^{(2)}\right) = 0.$$

It follows that for t sufficiently large, $q_t^{(1)}$ is Murray-von Neumann equivalent to a subprojection of $q_t^{(2)}$, with the Murray-von Neumann equivalence depending continuously on t. It is easy to extend it from an interval $[t_0, \infty)$ to $[0, \infty)$. ◻

1.2.4 Lemma. Let A and D be as in Definition 1.2.1, let $\alpha \mapsto \varphi^{(\alpha)}$ be a homotopy of asymptotic morphisms from A to D, and let p_0, $p_1 \in A$ be homotopic projections. Let $q^{(0)}$ and $q^{(1)}$ be tail projections for $\varphi^{(0)}(p_0)$ and $\varphi^{(1)}(p_1)$ respectively. Then $q^{(0)}$ is homotopic to $q^{(1)}$ in the sense of Remark 1.2.1 (3).

Proof: This can be proved directly, but also follows by combining Remark 1.2.2 (2), Proposition 1.1.7, and the fact that products of homotopy classes of asymptotic morphisms are well defined. ◻

1.2.5 Definition. Let A be a separable C^*-algebra which contains a full projection, and let D be any C^*-algebra. Then an asymptotic morphism $\varphi : A \to D$ is *full* if there is a full projection $p \in A$ such that some (equivalently, any) tail projection for $\varphi(p)$ is full in D.

This definition rejects, not only the identity map of $C_0(\mathbf{R})$, but also the identity map of $C_0(\mathbf{Z})$. (The algebra $C_0(\mathbf{Z})$ has no full projections.) However, it will do for our purposes.

Note that, by Lemma 1.2.3, if a tail projection for $\varphi(p)$ is full, then so is a tail projection for $\varphi(q)$ whenever p is Murray-von Neumann equivalent to a subprojection of q.

We now list the relevant properties of full asymptotic morphisms. We omit the proofs; they are mostly either immediate or variations on the proof of Lemma 1.2.3.

1.2.6 Lemma. (1) Fullness of an asymptotic morphism depends only on its homotopy class.

(2) If $\varphi, \psi : A \to D$ are asymptotic morphisms, and if φ is full, then so is the asymptotic morphism $\varphi \oplus \psi : A \to M_2(D)$.

(3) Let B be separable, and have a full projection, and further assume that given two full projections in B, each is Murray-von Neumann equivalent to a subprojection of the other. Then any asymptotic morphism representing the product of full asymptotic morphisms from A to B and from B to D is again full.

The extra assumption in part (3) is annoying, but we don't see an easy way to avoid it. This suggests that we don't quite have the right definition. However, in this paper B will almost always have the form $K \otimes \mathcal{O}_\infty \otimes D$ with D unital. Lemma 2.1.8 (1) below will ensure that the assumption holds in this case.

1.3 Asymptotic unitary equivalence

Approximately unitarily equivalent homomorphisms have the same class in Rørdam's KL-theory (Proposition 5.4 of [51]), but need not have the same class in KK-theory. (See Theorem 6.12 of [51], and note that $KL(A, B)$ is in general a proper quotient of $KK^0(A, B)$.) Since the theorems we prove in Section 3 give information about KK-theory rather than about Rørdam's KL-theory, we introduce and use the notion of asymptotic unitary equivalence instead. We give the definition for asymptotic morphisms because we will make extensive technical use of it in this context, but, for reasons to be explained below, it is best suited to homomorphisms.

1.3.1 Definition. Let A and D be C^*-algebras, with A separable. Let $\varphi, \psi : A \to D$ be two asymptotic morphisms. Then φ is *asymptotically unitarily equivalent* to ψ if there is a continuous family of unitaries $t \mapsto u_t$ in $\widetilde{D}$, defined for $t \in [0, \infty)$, such that

$$\lim_{t \to \infty} \|u_t \varphi_t(a) u_t^* - \psi_t(a)\| = 0$$

for all $a \in A$. We say that two homomorphisms $\varphi, \psi : A \to D$ are *asymptotically unitarily equivalent* if the corresponding constant asymptotic morphisms with $\varphi_t = \varphi$ and $\psi_t = \psi$ are asymptotically unitarily equivalent.

1.3.2 Lemma. Asymptotic unitary equivalence is the equivalence relation on asymptotic morphisms generated by asymptotic equality and unitary equivalence in the exact sense (that is, $u_t \varphi_t(a) u_t^* = \psi_t(a)$ for all $a \in A$).

Proof: The only point needing any work at all is transitivity of asymptotic unitary equivalence, and this is easy. □

1.3.3 Lemma. Let A and D be C^*-algebras, with A separable.

(1) Let $\varphi, \psi : A \to K \otimes D$ be asymptotically unitarily equivalent asymptotic morphisms. Then φ is homotopic to ψ.

(2) Let $\varphi, \psi : A \to K \otimes D$ be asymptotically unitarily equivalent homomorphisms. Then φ is homotopic to ψ via a path of homomorphisms.

Proof: (1) Let $t \mapsto u_t \in (K \otimes D)^+$ be an asymptotic unitary equivalence. Modulo the usual isomorphism $M_2(K) \cong K$, the asymptotic morphisms φ and ψ are homotopic to the asymptotic morphisms $\varphi \oplus 0$ and $\psi \oplus 0$ from A to $M_2(K \otimes D)$. Choose a continuous function $(\alpha, t) \mapsto v_{\alpha,t}$ from $[0,1] \times [0, \infty)$ to $U(M_2((K \otimes D)^+))$ such that $v_{0,t} = 1$ and $v_{1,t} = u_t \oplus u_t^*$ for all t. Define a homotopy of asymptotic morphisms by $\rho_t^{(\alpha)}(a) = v_{\alpha,t}(\varphi_t(a) \oplus 0)v_{\alpha,t}^*$. Then $\rho^{(0)} = \varphi \oplus 0$ and $\rho^{(1)}$ is asymptotically equal to $\psi \oplus 0$. So φ is homotopic to ψ.

(2) Apply the proof of part (1) to the constant paths $t \mapsto \varphi$ and $t \mapsto \psi$. Putting $t = 0$ gives homotopies of homomorphisms from φ to $\rho_0^{(1)}$ and from ψ to $\psi \oplus 0$. The remaining piece of our homotopy is taken to be defined for $t \in [0, \infty]$, and is given by $t \mapsto \rho_t^{(1)}$ for $t \in [0, \infty)$ and $\infty \mapsto \psi \oplus 0$. □

1.3.4 Corollary. Two asymptotically unitarily equivalent asymptotic morphisms define the same class in E-theory.

If the domain is nuclear, this corollary shows that asymptotically unitarily equivalent asymptotic morphisms define the same class in KK-theory. Asymptotic unitary equivalence thus rectifies the most important disadvantage of approximate unitary equivalence for homomorphisms. Asymptotic unitary equivalence, however, also has its problems, connected with the extension to asymptotic morphisms. The construction of the product of asymptotic morphisms requires reparametrization of asymptotic morphisms, as in the following definition.

1.3.5 Definition. Let A and D be C^*-algebras, and let $\varphi : A \to D$ be an asymptotic morphism. A *reparametrization* of φ is an asymptotic morphism from A to D of the form $t \mapsto \varphi_{f(t)}$ for some continuous nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} f(t) = \infty$.

Other versions are possible: one could replace "nondecreasing" by "strictly increasing", or omit this condition entirely. The version we give is the most convenient for our purposes.

It is not in general true that an asymptotic morphism is asymptotically unitarily equivalent to its reparametrizations. (Consider, for example, the asymptotic morphism $\varphi : C(S^1) \to \mathbf{C}$ given by $\varphi_t(f) = f(\exp(it))$.) The product is thus not defined on asymptotic unitary equivalence classes of asymptotic morphisms. (The product is defined on asymptotic unitary equivalence classes when one factor is a homomorphism. We don't prove this fact because we don't need it, but see the last part of the proof of Lemma 2.3.5.) In fact, if an asymptotic morphism is asymptotically unitarily equivalent to its reparametrizations, then it is asymptotically unitarily equivalent to a homomorphism, and this will play an important role in our proof. The observation that this is true is due to Kirchberg. It replaces a more complicated argument in the earlier version of this paper, which involved the use throughout of "local asymptotic morphisms", a generalization of asymptotic morphisms in which there is another parameter. We start the proof with a lemma.

1.3.6 Lemma. Let A and D be C^*-algebras, and let $\varphi : A \to D$ be an asymptotic morphism. Suppose φ is asymptotically unitarily equivalent to all its reparametrizations. Then for any $\varepsilon > 0$ and any finite set $F \subset A$ there is $M \in [0, \infty)$ such that for any compact interval $I \subset \mathbf{R}$ and any continuous nondecreasing functions $f,\, g : I \to [M, \infty)$, there is a continuous unitary path $t \mapsto v_t$ in $\widetilde{D}$ satisfying $\|v_t \varphi_{f(t)}(a) v_t^* - \varphi_{g(t)}(a)\| < \varepsilon$ for all $t \in I$ and $a \in F$.

Proof: Suppose the lemma is false. We can obviously change I at will by reparametrizing, so there are $\varepsilon > 0$ and $F \subset A$ finite such that for all $M \in [0, \infty)$ and all compact intervals $I \subset \mathbf{R}$ there are continuous nondecreasing functions $f,\, g : I \to [M, \infty)$ for which no continuous unitary path $t \mapsto v_t$ in $\widetilde{D}$ gives $\|v_t \varphi_{f(t)}(a) v_t^* - \varphi_{g(t)}(a)\| < \varepsilon$ for $t \in I$ and $a \in F$. Choose f_1 and g_1 for $M = M_1 = 1$ and $I = I_1 = [1, 1 + \frac{1}{2}]$. Given f_n and g_n, choose f_{n+1} and g_{n+1} as above for $M = M_{n+1} = 1 + \max(f_n(n + \frac{1}{2}), g_n(n + \frac{1}{2}))$ and $I = I_{n+1} = [n + 1, n + 1 + \frac{1}{2}]$. By induction, we have $M_n \geq n$. Let $f, g : [0, \infty) \to [0, \infty)$ be the unique continuous functions which are linear on the intervals $[n+\frac{1}{2}, n+1]$ and satisfy $f|_{[n,n+\frac{1}{2}]} = f_n$ and $g|_{[n,n+\frac{1}{2}]} = g_n$. Since f and g are nondecreasing and satisfy $f(t),\, g(t) \geq n$ for $t \geq n$, the functions $t \mapsto \varphi_{f(t)}$ and $t \mapsto \varphi_{g(t)}$ are asymptotic morphisms which are reparametrizations of φ. By hypothesis, both are asymptotically unitarily equivalent to φ, and are therefore also asymptotically unitarily equivalent to each other. Let $t \mapsto v_t$ be a unitary path in $\widetilde{D}$ which implements this asymptotic unitary equivalence. Choose T such that for $a \in F$ and $t > T$ we have $\|v_t \varphi_{f(t)}(a) v_t^* - \varphi_{g(t)}(a)\| < \varepsilon/2$. Restricting to $[n, n + \frac{1}{2}]$ for some $n > T$ gives a contradiction to the choice of M and ε. This proves the lemma. □

1.3.7 Proposition. Let A be a separable C^*-algebra, and let $\varphi : A \to D$ be a bounded asymptotic morphism. Suppose that φ is asymptotically unitarily equivalent to all its reparametrizations. Then φ is asymptotically unitarily equivalent to a homomorphism. That is, there exist a homomorphism $\omega : A \to D$ and a continuous path $t \mapsto v_t$ of unitaries in $\widetilde{D}$ such that for every $a \in A$, we have $\lim_{t\to\infty} v_t \varphi_t(a) v_t^* = \omega(a)$.

Recall from Definition 1.1.4 (2) that bounded asymptotic morphisms are assumed in particular to be linear.

Proof of Proposition 1.3.7: Choose finite sets $F_0 \subset F_1 \subset \cdots \subset A$ whose union is dense in A. Choose a sequence $t_0 < t_1 < \cdots$, with $t_n \to \infty$, such that

$$\|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\|, \ \|\varphi_t(a^*) - \varphi_t(a)^*\| < 1/2^n$$

for $a, b \in F_n$ and $t \geq t_n$, and also such that, as in the previous lemma, for any compact interval $I \subset \mathbf{R}$ and any continuous nondecreasing functions $f, g : I \to [t_n, \infty)$, there is a continuous unitary path $t \mapsto v_t$ in $\widetilde{D}$ satisfying $\|v_t \varphi_{f(t)}(a) v_t^* - \varphi_{g(t)}(a)\| < 2^{-n-1}$ for all $t \in I$ and $a \in F_n$. For $n \geq 0$ let $t \mapsto u_t^{(n)}$ be the unitary path associated with the particular choices $I = [t_n, t_{n+1}]$, $f(t) = t$, and $g(t) = t_n$. Set $\widetilde{u}_t^{(n)} = (u_{t_n}^{(n)})^* u_t^{(n)}$. We have $\|(u_{t_n}^{(n)})^* \varphi_{t_n}(a) u_{t_n}^{(n)} - \varphi_{t_n}(a)\| < 2^{-n-1}$ for $a \in F_n$, so $\|\widetilde{u}_t^{(n)} \varphi_t(a) (\widetilde{u}_t^{(n)})^* - \varphi_{t_n}(a)\| < 2^{-n}$ for $t \in [t_n, t_{n+1}]$ and $a \in F_n$. Also note that $\widetilde{u}_{t_n}^{(n)} = 1$. Now define a continuous unitary function $[0, \infty) \to \widetilde{D}$ by

$$v_t = \widetilde{u}_{t_1}^{(0)} \cdot \widetilde{u}_{t_2}^{(1)} \cdots \widetilde{u}_{t_n}^{(n-1)} \cdot \widetilde{u}_t^{(n)}$$

for $t_n \leq t \leq t_{n+1}$.

We claim that $\omega(a) = \lim_{t \to \infty} v_t \varphi_t(a) v_t^*$ exists for all $a \in A$. Since $\sup_{t \in [0, \infty)} \|\varphi_t\| < \infty$, it suffices to check this on the dense subset $\bigcup_{k=0}^{\infty} F_k$. So let $a \in F_k$. We prove that the net $t \mapsto v_t \varphi_t(a) v_t^*$ is Cauchy. Let $m \geq k$, and let $t \geq t_m$. Choose n such that $t_n \leq t \leq t_{n+1}$. Then

$$\begin{aligned}
&\|v_t \varphi_t(a) v_t^* - v_{t_m} \varphi_{t_m}(a) v_{t_m}^*\| \\
&\quad = \left\| \left[\widetilde{u}_{t_{m+1}}^{(m)} \cdot \widetilde{u}_{t_{m+2}}^{(m+1)} \cdots \widetilde{u}_{t_n}^{(n-1)} \cdot \widetilde{u}_t^{(n)} \right] \right. \\
&\qquad\qquad \left. \varphi_t(a) \left[\widetilde{u}_{t_{m+1}}^{(m)} \cdot \widetilde{u}_{t_{m+2}}^{(m+1)} \cdots \widetilde{u}_{t_n}^{(n-1)} \cdot \widetilde{u}_t^{(n)} \right]^* - \varphi_{t_m}(a) \right\| \\
&\quad \leq \left\| \left(\widetilde{u}_t^{(n)} \right) \varphi_t(a) \left(\widetilde{u}_t^{(n)} \right)^* - \varphi_{t_n}(a) \right\| \\
&\qquad\qquad + \sum_{j=m}^{n-1} \left\| \left(\widetilde{u}_{t_{j+1}}^{(j)} \right) \varphi_{t_{j+1}}(a) \left(\widetilde{u}_{t_{j+1}}^{(j)} \right)^* - \varphi_{t_j}(a) \right\| \\
&\quad \leq \sum_{j=m}^{n} \frac{1}{2^j} < \frac{1}{2^{m-1}}.
\end{aligned}$$

Therefore, if $r, t \geq t_m$, we obtain

$$\|v_r \varphi_r(a) v_r^* - v_t \varphi_t(a) v_t^*\| < 1/2^{m-2}.$$

So we have a Cauchy net, which must converge. The claim is now proved.

Since φ_t is multiplicative and *-preserving to within 2^{-n} on F_n for $t \geq t_n$, it follows that ω is exactly multiplicative and *-preserving on each F_n. Since $\|\omega\| \leq \sup_{t \in [0, \infty)} \|\varphi_t\| < \infty$, it follows that ω is a homomorphism. $\square$

In the rest of this section, we prove some useful facts about asymptotic unitary equivalence.

1.3.8 Lemma. Let $\varphi : A \to D$ be an asymptotic morphism, with A unital. Then there is a projection $p \in D$ and an asymptotic morphism $\psi : A \to D$ which is asymptotically unitarily equivalent to φ and satisfies $\psi_t(1) = p$ and $\psi_t(a) \in pDp$ for all $t \in [0, \infty)$ and and $a \in A$.

Proof: Let $t \mapsto q_t$ be a tail projection for $\varphi(1)$, as in Definition 1.2.1. Standard results yield a continuous family of unitaries $t \mapsto u_t$ in $\widetilde{D}$ such that $u_0 = 1$ and $u_t q_t u_t^* = q_0$ for all $t \in [0, \infty)$. Define $p = q_0$ and define $\rho_t(a) = u_t q_t \varphi_t(a) q_t u_t^*$ for $t \in [0, \infty)$ and $a \in A$. Note that the definition of an asymptotic morphism implies that $(t, a) \mapsto q_t \varphi_t(a) q_t$ is asymptotically equal to φ, and hence is an asymptotic morphism. Thus ρ is an asymptotic morphism which is asymptotically unitarily equivalent to φ.

The only problem is that $\rho_t(1)$ might not be equal to p. We do know that $\rho_t(1) \to p$ as $t \to \infty$. Choose a closed subspace A_0 of A which is complementary to $\mathbf{C} \cdot 1$, and for $a \in A_0$ and $\lambda \in \mathbf{C}$ define $\psi_t(a + \lambda \cdot 1) = \rho_t(a) + \lambda p$. □

1.3.9 Lemma. Let $\varphi, \psi : A \to K \otimes D$ be asymptotic morphisms, with A and D unital. Suppose that there is a continuous family of unitaries $t \mapsto u_t$ in the multiplier algebra $M(K \otimes D)$ such that $\lim_{t\to\infty} \|u_t \varphi_t(a) u_t^* - \psi_t(a)\| = 0$ for all $a \in A$. Then φ is asymptotically unitarily equivalent to ψ.

Proof: We have to show that u_t can be replaced by $v_t \in (K \otimes D)^+$.

Applying the previous lemma twice, and making the corresponding modifications to the given u_t, we may assume that $\varphi_t(1)$ and $\psi_t(1)$ are projections p and q not depending on t, and that we always have $\varphi_t(a) \in pDp$ and $\psi_t(a) \in qDq$.

We now want to reduce to the case $p = q$. The hypothesis implies that there is t_0 such that $\|u_{t_0} p u_{t_0}^* - q\| < 1/2$. Therefore there is a unitary w in $(K \otimes D)^+$ such that $w u_{t_0} p u_{t_0}^* w^* = q$. Now if p, $q \in K \otimes D$ are projections which are unitarily equivalent in $M(K \otimes D)$, then standard arguments show they are unitarily equivalent in $(K \otimes D)^+$. Therefore conjugating φ by $w u_{t_0}$ changes neither its asymptotic unitary equivalence class nor the validity of the hypotheses. We may thus assume without loss of generality that $p = q$.

Now choose t_1 such that $t \geq t_1$ implies $\|u_t p u_t^* - p\| < 1$. Define a continuous family of unitaries by

$$c_t = 1 - p + p u_t p (p u_t^* p u_t p)^{-1/2} \in (K \otimes D)^+$$

for $t \geq t_1$. (Functional calculus is evaluated in $p(K \otimes D)p$.) For any $d \in p(K \otimes D)p$, we have

$$\begin{aligned} \|c_t d c_t^* - u_t d u_t^*\| &= \|pdp - c_t^* u_t p d p u_t^* c_t\| \\ &\leq 2\|d\|\,\|p - c_t^* u_t p\| \leq 2\|d\|(\|u_t p - p u_t\| + \|p - c_t^* p u_t p\|). \end{aligned}$$

The first summand in the last factor goes to 0 as $t \to \infty$. Substituting definitions, the second summand becomes $\|p - (p u_t^* p u_t p)^{1/2}\|$, which does the same.

Since $\varphi_t(a) \in p(K \otimes D)p$ for all $a \in A$, and since (using Lemma 1.2 of [54] for the first)

$$\limsup_{t\to\infty} \|\varphi_t(a)\| \leq \|a\| \quad \text{and} \quad \lim_{t\to\infty} \|u_t\varphi_t(a)u_t^* - \psi_t(a)\| = 0,$$

it follows that $\lim_{t\to\infty} \|c_t\varphi_t(a)c_t^* - \psi_t(a)\| = 0$ as well. This is the desired asymptotic unitary equivalence. □

2 Asymptotic morphisms to tensor products with $\mathcal{O}_\infty$.

The purpose of this section is to prove two things about asymptotic morphisms from a separable nuclear unital simple C^*-algebra A to a C^*-algebra of the form $K \otimes \mathcal{O}_\infty \otimes D$ with D unital: homotopy implies asymptotic unitary equivalence, and each such asymptotic morphism is asymptotically unitarily equivalent to a homomorphism. The basic method is the absorption technique used in [35] and [44], and in fact this section is really just the generalization of [44] from homomorphisms and approximate unitary equivalence to asymptotic morphisms and asymptotic unitary equivalence.

There are three subsections. In the first, we collect for reference various known results involving Cuntz algebras (including in particular Kirchberg's theorems on tensor products) and derive some easy consequences. In the second subsection, we replace approximate unitary equivalence by asymptotic unitary equivalence in the results of [48] and [35]. In the third, we carry out the absorption argument and derive its consequences.

The arguments involving asymptotic unitary equivalence instead of approximate unitary equivalence are sometimes somewhat technical. However, the essential outline of the proof is the same as in the much easier to read paper [44].

2.1 Preliminaries: Cuntz algebras and Kirchberg's stability theorems

In this subsection, we collect for convenient reference various results related to Cuntz algebras. Besides Rørdam's results on approximate unitary equivalence and Kirchberg's basic results on tensor products, we need material on unstable K-theory and hereditary subalgebras of tensor products with $\mathcal{O}_\infty$ and on exact stability of generating relations of Cuntz algebras.

We start with Rørdam's work [48]; we also use this opportunity to establish our notation. The first definition is used implicitly by Rørdam, and appears explicitly in the work of Ringrose.

We will generally let $s_1, s_2, \ldots, s_m$ be the standard generators of $\mathcal{O}_m$, and analogously for $\mathcal{O}_\infty$.

2.1.1 Definition. ([47], [46]) Let A be a unital C^*-algebra. Then its *(C^*) exponential length* $\mathrm{cel}(A)$ is

$$\sup_{u \in U_0(A)} \inf \left\{ \sum_{k=1}^{n} \|h_k\| : n \in \mathbf{N},\ h_1, \ldots, h_n \in A \text{ selfadjoint}, \right.$$
$$\left. u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_n) \right\}.$$

In preparation for the following theorem, and to establish notation, we make the following remark, most of which is in [48], 3.3.

2.1.2 Remark. Let B be a unital C^*-algebra, and let $m \geq 2$.

(1) If $\varphi, \psi : \mathcal{O}_m \to B$ are unital homomorphisms, then the element $u = \sum_{j=1}^{m} \psi(s_j)\varphi(s_j)^*$ is a unitary in B such that $u\varphi(s_j) = \psi(s_j)$ for $1 \leq j \leq m$.

(2) If $\varphi : \mathcal{O}_m \to B$ is a unital homomorphism, then the formula

$$\lambda_\varphi(a) = \sum_{j=1}^{m} \varphi(s_j) a \varphi(s_j)^*$$

defines a unital endomorphism λ_φ (or just λ when φ is understood) of B.

(3) If φ and λ are as in (2), and if $u \in B$ has the form $u = v\lambda(v^*)$ for some unitary $v \in B$, then $v\varphi(s_j)v^* = u\varphi(s_j)$ for $1 \leq j \leq m$.

2.1.3 Theorem. Let B be a unital C^*-algebra such that $\mathrm{cel}(B)$ is finite and such that the canonical map $U(B)/U_0(B) \to K_1(B)$ is an isomorphism. Let $m \geq 2$, and let $\varphi, \psi : \mathcal{O}_m \to B$, $\lambda : B \to B$, and $u \in U(B)$ be as in Remark 2.1.2 (1) and (2). Then the following are equivalent:

(1) $[u] \in (m-1)K_1(B)$.

(2) For every $\varepsilon > 0$ there is $v \in U(B)$ such that $\|u - v\lambda(v^*)\| < \varepsilon$.

(3) $[\varphi] = [\psi]$ in $KK^0(\mathcal{O}_m, B)$.

(4) The maps φ and ψ are approximately unitarily equivalent.

Proof: For m even, this is Theorem 3.6 of [48]. In Section 3 of [48], it is also proved that (1) is equivalent to (3) and (2) is equivalent to (4) for arbitrary m, and Theorem 4.2 of [44] implies that (3) is equivalent to (4) for arbitrary m. □

We will not actually need to use the equivalence of (3) and (4) for odd m.

That $\mathrm{cel}(D)$ is finite for purely infinite simple C^*-algebras D was first proved in [42]. We will, however, apply this theorem to algebras D of the form $\mathcal{O}_\infty \otimes B$ with B an arbitrary unital C^*-algebra. Such algebras are shown in Lemma 2.1.7 (2) below to have finite exponential length. Actually, to prove the classification theorem, it suffices to know that there is a universal upper bound on $\mathrm{cel}(C(X) \otimes B)$ for B purely infinite and simple. This follows from Theorem 1.2 of [62].

We now state the fundamental results of Kirchberg on which our work depends. These were stated in [27]; proofs appear in [29].

2.1.4 Theorem. ([27]; [29], Theorem 3.8) Let A be a separable nuclear unital simple C^*-algebra. Then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

2.1.5 Theorem. ([27]; [29], Theorem 3.15) Let A be a separable nuclear unital purely infinite simple C^*-algebra. Then $\mathcal{O}_\infty \otimes A \cong A$.

We now derive some consequences of Kirchberg's results.

2.1.6 Corollary. Every separable nuclear unital purely infinite simple C^*-algebra is approximately divisible in the sense of [6].

Proof: It suffices to show that $\mathcal{O}_\infty$ is approximately divisible. Let $\varphi : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ be an isomorphism, as in the previous theorem. Define $\psi : \mathcal{O}_\infty \to \mathcal{O}_\infty$ by $\psi(a) = \varphi(1 \otimes a)$. Then ψ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty}$ by Theorem 3.3 of [35]. That is, there are unitaries $u_n \in \mathcal{O}_\infty$ such that $u_n \varphi(1 \otimes a) u_n^* \to a$ for all $a \in \mathcal{O}_\infty$. Let $B \subset \mathcal{O}_\infty$ be a unital copy of $M_2 \oplus M_3$. Then for large enough n, the subalgebra $u_n \varphi(B \otimes 1) u_n^*$ of $\mathcal{O}_\infty$ commutes arbitrarily well with any finite subset of $\mathcal{O}_\infty$. ◻

2.1.7 Lemma. Let D be any unital C^*-algebra. Then:

(1) The canonical map $U(\mathcal{O}_\infty \otimes D)/U_0(\mathcal{O}_\infty \otimes D) \to K_1(\mathcal{O}_\infty \otimes D)$ is an isomorphism.

(2) $\mathrm{cel}(\mathcal{O}_\infty \otimes D) \leq 3\pi$.

Proof: We first prove surjectivity in (1). Let $\eta \in K_1(\mathcal{O}_\infty \otimes D)$. Choose n and $u \in U(M_n \otimes \mathcal{O}_\infty \otimes D)$ such that $[u] = \eta$. Let e_{ij} be the standard matrix units in M_n. Define (nonunital) homomorphisms

$$\varphi : \mathcal{O}_\infty \otimes D \to M_n \otimes \mathcal{O}_\infty \otimes D \quad \text{and} \quad \psi : M_n \otimes \mathcal{O}_\infty \otimes D \to \mathcal{O}_\infty \otimes D$$

by

$$\varphi(a) = e_{11} \otimes a \quad \text{and} \quad \psi(e_{ij} \otimes b) = (s_i \otimes 1) b (s_j^* \otimes 1).$$

Then φ_* is the standard stability isomorphism

$$K_1(\mathcal{O}_\infty \otimes D) \to K_1(M_n \otimes \mathcal{O}_\infty \otimes D).$$

Also, $\psi \circ \varphi(a) = (s_1 \otimes 1) a (s_1^* \otimes 1)$ for $a \in \mathcal{O}_\infty \otimes D$. Since $s_1 \otimes 1$ is an isometry, this implies that $\psi \circ \varphi$ is the identity on K-theory. Therefore $\psi_* = \varphi_*^{-1}$. Consequently

$$\eta = \varphi_*^{-1}([u]) = \psi_*([u]) = [\psi(u) + 1 - \psi(1)],$$

showing that η is the class of a unitary in $\mathcal{O}_\infty \otimes D$.

Now let $u \in U(\mathcal{O}_\infty \otimes D)$ satisfy $[u] = 0$ in $K_1(\mathcal{O}_\infty \otimes D)$. We prove that u can be connected to the identity by a path of length at most $3\pi + \varepsilon$. This will simultaneously prove (2) and injectivity in (1).

Using approximate divisibility of $\mathcal{O}_\infty$ and approximating u by finite sums of elementary tensors, we can find nontrivial projections $e \in \mathcal{O}_\infty$ with $\|u(e \otimes 1) - (e \otimes 1)u\|$ arbitrarily small. If this norm is small enough, we can find a unitary $v \in K \otimes \mathcal{O}_\infty \otimes D$ which commutes with $e \otimes 1$ and is connected to u by a unitary path of length less than $\varepsilon/2$. Write $v = v_1 + v_2$ with

$$v_1 \in U(e\mathcal{O}_\infty e \otimes D) \quad \text{and} \quad v_2 \in U((1-e)\mathcal{O}_\infty(1-e) \otimes D).$$

Choose a partial isometry $s \in \mathcal{O}_\infty$ with $s^*s = 1 - e$ and $ss^* \leq e$. The proof of Corollary 5 of [42] shows that v can be connected to the unitary

$$\begin{aligned} w &= v\left[(e - ss^*) \otimes 1 + (s \otimes 1)v_2(s \otimes 1)^* + v_2^*\right] \\ &= 1 - e \otimes 1 + v_1\left[(s \otimes 1)v_2(s \otimes 1)^* + (e - ss^*) \otimes 1\right] \end{aligned}$$

by a path of length π.

Since $\mathcal{O}_\infty$ is purely infinite, there is an embedding of $K \otimes e\mathcal{O}_\infty e$ in $\mathcal{O}_\infty$ which extends the obvious identification of $e_{11} \otimes e\mathcal{O}_\infty e$ with $e\mathcal{O}_\infty e$. It extends to a unital homomorphism $\varphi : (K \otimes e\mathcal{O}_\infty e \otimes D)^+ \to \mathcal{O}_\infty \otimes D$ whose range contains w, and such that $[\varphi^{-1}(w)] = 0$ in $K_1(K \otimes e\mathcal{O}_\infty e \otimes D)$. Thus $\varphi^{-1}(w) \in U_0((K \otimes e\mathcal{O}_\infty e \otimes D)^+)$. Theorem 3.8 of [43] shows that the C^* exponential rank of any stable C^*-algebra is at most $2 + \varepsilon$. An examination of the proof, and of the length of the path used in the proof of Corollary 5 of [42], shows that in fact any stable C^*-algebra has exponential length at most 2π. Thus, in particular, $\varphi^{-1}(w)$ can be connected to 1 by a unitary path of length $2\pi + \varepsilon/2$. It follows that u can be connected to 1 by a unitary path of length at most $3\pi + \varepsilon$. □

A somewhat more complicated argument shows that in fact $\mathrm{cel}(\mathcal{O}_\infty \otimes D) \leq 2\pi$. Details will appear elsewhere [45].

2.1.8 Lemma. Let D be a unital C^*-algebra. Then:

(1) Given two full projections in $K \otimes \mathcal{O}_\infty \otimes D$, each is Murray-von Neumann equivalent to a subprojection of the other.

(2) If two full projections in $K \otimes \mathcal{O}_\infty \otimes D$ have the same K_0-class, then they are homotopic.

Proof: Taking direct limits, we reduce to the case that D is separable. Then $\mathcal{O}_\infty \otimes D$ is approximately divisible by Corollary 2.1.6. It follows from Proposition 3.10 of [6] that two full projections in $K \otimes \mathcal{O}_\infty \otimes D$ with the same K_0-class are Murray-von Neumann equivalent. Now (2) follows from the fact that Murray-von Neumann equivalence implies homotopy in the stabilization of a unital C^*-algebra.

Part (1) requires slightly more work. Let $\mathcal{P}$ be the set of all projections $p \in \mathcal{O}_\infty \otimes D$ such that there are two orthogonal projections $q_1, q_2 \leq p$, both Murray-von Neumann equivalent to 1. One readily verifies that $\mathcal{P}$ is nonempty and satisfies the conditions (Π_1)-(Π_4) on page 184 of [14]. Therefore, by [14], the group $K_0(\mathcal{O}_\infty \otimes D)$ is exactly the set of Murray-von Neumann equivalence classes of projections in $\mathcal{P}$. Since projections in $\mathcal{P}$ are full, Proposition 3.10 of [6] now implies that every full projection is in $\mathcal{P}$. Clearly (1) holds for projections in $\mathcal{P}$. We obtain (1) in general by using the pure infiniteness of $\mathcal{O}_\infty$ to show that every full projection in $K \otimes \mathcal{O}_\infty \otimes D$ is Murray-von Neumann equivalent to a (necessarily full) projection in $\mathcal{O}_\infty \otimes D$. □

Next, we turn to exact stability. For $\mathcal{O}_m$, we need only the following standard result:

2.1.9 Proposition. ([35], Lemma 1.3 (1)) For any integer m, the defining relations for $\mathcal{O}_m$, namely $s_j^* s_j = 1$ and $\sum_{k=1}^m s_k s_k^* = 1$ for $1 \leq j \leq m$, are exactly stable.

We will also need to know about the standard extension E_m of $\mathcal{O}_m$ by the compact operators. Recall from [13] that E_m is the universal C^*-algebra on generators $t_1, \ldots, t_m$ with relations $t_j^* t_j = 1$ and $(t_j t_j^*)(t_k t_k^*) = 0$ for $1 \leq j, k \leq m$, $j \neq k$. Its properties are summarized in [35], 1.1. In particular, we have $\varinjlim E_m \cong \mathcal{O}_\infty$ using the standard inclusions.

Exact stability of the generating relations for E_m is known, but we need the following stronger result, which can be thought of as a finite version of exact stability for $\mathcal{O}_\infty$. Essentially, it says that if elements approximately satisfy the defining relations for E_m, then they can be perturbed in a functorial way to exactly satisfy these relations, and that the way the first k elements are perturbed does not depend on the remaining $m - k$ elements.

Recently, Blackadar has proved that in fact $\mathcal{O}_\infty$ is semiprojective in the usual sense [5].

2.1.10 Proposition. For each $\delta \geq 0$ and $m \geq 2$, let $E_m(\delta)$ be the universal unital C^*-algebra on generators $t_{j,\delta}^{(m)}$ for $1 \leq j \leq m$ and relations

$$\|(t_{j,\delta}^{(m)})^* t_{j,\delta}^{(m)} - 1\| \leq \delta \quad \text{and} \quad \left\| \left(t_{j,\delta}^{(m)} (t_{j,\delta}^{(m)})^* \right) \left(t_{k,\delta}^{(m)} (t_{k,\delta}^{(m)})^* \right) \right\| \leq \delta$$

for $j \neq k$, and let $\kappa_\delta^{(m)} : E_m(\delta) \to E_m$ be the homomorphism given by sending $t_{j,\delta}^{(m)}$ to the corresponding standard generator $t_j^{(m)}$ of E_m. Then there are $\delta(2) \geq \delta(3) \geq \cdots > 0$, nondecreasing functions $f_m : [0, \delta(m)] \to [0, \infty)$ with $\lim_{\delta \to 0} f_m(\delta) = 0$ for each m, and homomorphisms $\varphi_\delta^{(m)} : E_m \to E_m(\delta)$ for $0 \leq \delta \leq \delta(m)$, satisfying the following properties:

(1) $\kappa_\delta^{(m)} \circ \varphi_\delta^{(m)} = \mathrm{id}_{E_m}$.

(2) $\|\varphi_\delta^{(m)}(t_j^{(m)}) - t_{j,\delta}^{(m)}\| \leq f_m(\delta)$.

(3) If $0 \leq \delta \leq \delta' \leq \delta(m)$, then the composite of $\varphi_{\delta'}^{(m)}$ with the canonical map from $E_m(\delta')$ to $E_m(\delta)$ is $\varphi_{\delta}^{(m)}$.

(4) Let $\iota_{\delta}^{(m)} : E_m(\delta) \to E_{m+1}(\delta)$ be the map given by $\iota_{\delta}^{(m)}(t_{j,\delta}^{(m)}) = t_{j,\delta}^{(m+1)}$ for $1 \leq j \leq m$. Then for $0 \leq \delta \leq \delta(m+1)$ and $1 \leq j \leq m$, we have $\iota_{\delta}^{(m)}(\varphi_{\delta}^{(m)}(t_j^{(m)})) = \varphi_{\delta}^{(m+1)}(t_j^{(m+1)})$.

Proof: The proof of exact stability of E_m, as sketched in the proof of Lemma 1.3 (2) of [35], is easily seen to yield homomorphisms satisfying the conditions demanded here. □

2.1.11 PROPOSITION. Let D be a unital purely infinite simple C^*-algebra. Then any two unital homomorphisms from $\mathcal{O}_\infty$ to D are homotopic. Moreover, if $\varphi, \psi : \mathcal{O}_\infty \to D$ are unital homomorphisms such that $\varphi(s_j) = \psi(s_j)$ for $1 \leq j \leq m$, then there is a homotopy $t \mapsto \rho_t$ such that $\rho_t(s_j) = \varphi(s_j)$ for $1 \leq j \leq m$ and all t.

Proof: We prove the second statement; the first is the special case $m = 0$.

We construct a continuous path $t \mapsto \rho_t$ of unital homomorphisms from $\mathcal{O}_\infty$ to D, defined for $t \in [m, \infty)$ and satisfying the following conditions:

(1) For $k \geq m$, for $t \in [k, \infty)$, and for $1 \leq j \leq k$, we have $\rho_t(s_j) = \psi(s_j)$.

(2) $\rho_m = \varphi$.

Given such a path, $\rho_t(s_j) \to \psi(s_j)$ for all j. Since the s_j generate $\mathcal{O}_\infty$ as a C^*-algebra, standard arguments show that $\rho_t(a) \to \psi(a)$ for all $a \in \mathcal{O}_\infty$. We have therefore constructed the required homotopy.

It remains to carry out the construction. We construct the paths $t \mapsto \rho_t$, for $t \in [m, n]$, by induction on n. The initial step is thus simply to take $\rho_m = \varphi$. For the induction step, it suffices to do the following. Assume we are given $t \mapsto \rho_t$, for $t \in [m, n]$, and satisfying $\rho_n(s_j) = \psi(s_j)$ for $1 \leq j \leq n$. We then extend $t \mapsto \rho_t$ over $t \in [n, n+1]$ so that $\rho_t(s_j) = \psi(s_j)$ for $1 \leq j \leq n$ and $t \in [n, n+1]$, and in addition $\rho_{n+1}(s_{n+1}) = \psi(s_{n+1})$.

Let $p = \sum_{j=1}^{n} \rho_n(s_j)\rho_n(s_j)^*$, which is a projection in D. Then define

$$e_0 = \rho_n(s_{n+1})\rho_n(s_{n+1})^* \quad \text{and} \quad e_1 = \psi(s_{n+1})\psi(s_{n+1})^*.$$

Both e_0 and e_1 are proper projections in the purely infinite simple C^*-algebra $(1-p)D(1-p)$ with K_0-class equal to $[1_D]$, so they are homotopic. It follows that there is a unitary path $s \mapsto u_s$ in $(1-p)D(1-p)$ such that $u_0 = 1-p$ and $u_1 e_0 u_1^* = e_1$. For $s \in [0, 1/3]$, define $\rho_{n+s}(s_j) = \rho_n(s_j)$ for $1 \leq j \leq n$ and $\rho_{n+s}(s_j) = u_{3s}\rho_n(s_j)$ for $j \geq n+1$. This yields a homotopy of homomorphisms $\rho_{n+s} : \mathcal{O}_\infty \to D$, with ρ_n as already given, and such that the isometries $\rho_{n+1/3}(s_{n+1})$ and $\psi(s_{n+1})$ have the same range projection, namely e_1, although they themselves are probably not equal.

By a similar argument, we extend the homotopy over $[n+1/3, n+2/3]$ in such a way that $\rho_{n+s}(s_j)$ is constant for $s \in [n+1/3, n+2/3]$ and $1 \leq j \leq n+1$, and so that $\rho_{n+2/3}(s_{n+2})$ and $\psi(s_{n+2})$ also have the same range projection, say f.

Now e_1 and f are Murray-von Neumann equivalent, so we can identify $(e_1+f)D(e_1+f)$ with $M_2(e_1 D e_1)$. Since

$$w_1 = \begin{pmatrix} \psi(s_{n+1})\rho_{n+2/3}(s_{n+1})^* & 0 \\ 0 & [\psi(s_{n+1})\rho_{n+2/3}(s_{n+1})^*]^* \end{pmatrix} \in U_0(M_2(e_1 D e_1)),$$

there is a continuous path of unitaries $s \mapsto w_s$ in $M_2(e_1 D e_1)$, with $w_0 = 1$ and w_1 as given. For $s \in [2/3, 1]$, we now define $\rho_{n+s}(s_j) = \rho_n(s_j)$ for $j \neq n+1,\ n+2$, and $\rho_{n+s}(s_j) = w_{3s-2}\rho_{n+2/3}(s_j)$ for $j = n+1,\ n+2$. This is again a homotopy, and gives $\rho_{n+1}(s_j) = \psi(s_j)$ for $1 \leq j \leq n+1$, as desired. The induction step is complete. $\square$

2.1.12 Corollary. Let D be any unital C^*-algebra, and let $p \in K \otimes \mathcal{O}_\infty \otimes D$ be a projection. Then $\mathcal{O}_\infty \otimes p(K \otimes \mathcal{O}_\infty \otimes D)p \cong p(K \otimes \mathcal{O}_\infty \otimes D)p$.

Proof: We may replace p by any Murray-von Neumann equivalent projection. So without loss of generality $p \leq e \otimes 1 \otimes 1$ for some projection $e \in K$. Using the pure infiniteness of $\mathcal{O}_\infty$, we can in fact require that e be a rank one projection. That is, we may assume $p \in \mathcal{O}_\infty \otimes D$.

By Theorem 2.1.5, there is an isomorphism $\delta : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$. Using it, we need only consider projections $p \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D$. By the previous proposition and Theorem 2.1.5, $a \mapsto 1 \otimes \delta(a)$ is homotopic to $\mathrm{id}_{\mathcal{O}_\infty \otimes \mathcal{O}_\infty}$. Therefore such a projection p is homotopic to $q = 1 \otimes (\delta \otimes \mathrm{id}_D)(p)$, and hence also Murray-von Neumann equivalent to q. Now

$$q(\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D)q \cong \mathcal{O}_\infty \otimes [(\delta \otimes \mathrm{id}_D)(p)][\mathcal{O}_\infty \otimes D][(\delta \otimes \mathrm{id}_D)(p)],$$

which is unchanged by tensoring with $\mathcal{O}_\infty$ by Theorem 2.1.5. $\square$

2.1.13 Corollary. Let D be a unital C^*-algebra. Then the hypotheses on B in Theorem 2.1.3 are satisfied for any unital corner of $K \otimes \mathcal{O}_\infty \otimes D$.

Proof: Combine the previous corollary and Lemma 2.1.7. $\square$

2.2 Asymptotic unitary equivalence of homomorphisms from Cuntz algebras

In this subsection, we strengthen the main technical theorems of [48] (restated here as Theorem 2.1.3) and of [35], replacing approximate unitary equivalence by asymptotic unitary equivalence in the conclusions. We use the strong versions to obtain variants of several other known results in which we replace sequences of homomorphisms by continuous paths.

The first lemma contains the essential point in the strengthening of Theorem 2.1.3. Its proof uses the original theorem in a sort of bootstrap argument. The remaining results lead up to the strengthening of the main theorem of [35]. They are proved by modifying the proofs there.

The first two lemmas are stated together, because the proofs are very similar.

2.2.1 Lemma. (Compare with Theorem 2.1.3.) *Let D_0 be a unital C^*-algebra, and let $D = \mathcal{O}_\infty \otimes D_0$. Let $m \geq 2$, and let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$, for $t \in [0, \infty)$, be two continuous paths of unital homomorphisms from $\mathcal{O}_m$ to D. Suppose that the unitary $u_0 = \sum_{j=1}^{m} \psi_0(s_j)\varphi_0(s_j)^*$ satisfies $[u_0] \in (m-1)K_1(D)$. Then φ and ψ are asymptotically unitarily equivalent.*

2.2.2 Lemma. (Compare with Proposition 1.7 of [35].) *Let D be a unital purely infinite simple C^*-algebra, with $[1] = 0$ in $K_0(D)$. Let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$, for $t \in [0, \infty)$, be two continuous paths of unital homomorphisms from $\mathcal{O}_\infty$ to D. Then $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ are asymptotically unitarily equivalent.*

We will actually only need Lemma 2.2.1 for $m = 2$.

The two lemmas are actually valid for unital asymptotic morphisms (with a suitable modification of the definition of u_0 in Lemma 2.2.2), as can be seen by applying Proposition 1.1.7. We don't need this generality, so we don't state it formally.

The proofs of the two lemmas are rather technical. We do the first (which is easier) in detail, and then describe the modifications needed for the second. Before proving them, we outline the basic idea. Given two continuous paths as in Lemma 2.2.1, we can start by applying Theorem 2.1.3 to the homomorphisms from $\mathcal{O}_m$ to $C([0,1]) \otimes D$ obtained by restricting the paths to $t \in [0,1]$, with a certain error tolerance. The result is a continuous unitary path $t \mapsto v_t^{(1)}$, for $t \in [0,1]$, such that $(v_t^{(n)})^* \psi_t(s_j) v_t^{(n)}$ is close to $\varphi_t(s_j)$. Extend $t \mapsto v_t^{(1)}$ over $[0, \infty)$ by taking it to be constant on $[1, \infty)$. Now replace $t \mapsto \psi_t$ by $t \mapsto \gamma_t^{(1)} = (v_t^{(1)})^* \psi_t(\cdot) v_t^{(1)}$. Next, we want to repeat this over $[1,2]$, with a smaller error tolerance. Because there will be infinitely many steps, requiring in the end a product of infinitely many unitaries, there is a potential convergence problem. Thus, we insist here that the new unitary $v_t^{(1)}$ be equal to 1 for $t \in [0,1]$. In order to arrange this, we must have homomorphisms from $\mathcal{O}_m$ to $C([1,2]) \otimes D$ which agree when restricted to $\{1\} \subset [1,2]$. But $\gamma_1^{(1)}$ is only close to φ_1. However, if $\gamma_1^{(1)}$ is close enough to φ_1 on the generators, then we can use exact stability of the relations to construct a new homomorphism $\sigma \colon \mathcal{O}_m \to C([1,2]) \otimes D$ which is close to $t \mapsto \gamma_t^{(1)}$ for $t \in [1,2]$ and does agree with φ_1 at $t = 1$. We apply Theorem 2.1.3 to this homomorphism, and proceed as before. Now repeat on $[2,3]$, etc.

In the actual proof, it is technically convenient to reduce to the case in which $t \mapsto \varphi_t$ is constant, and to start over $\{0\}$ rather than over $[0,1]$.

The proof of Lemma 2.2.2 is essentially the same, but it is complicated by the fact that there are infinitely many generators. We deal with only finitely many of them over each interval $[n, n+1]$, but more and more as n increases.

Proof of Lemma 2.2.1: Corollary 2.1.13 shows that both D and $C([0,1], D)$ satisfy the hypotheses of Theorem 2.1.3.

By transitivity of asymptotic unitary equivalence, it suffices to show that $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ are both asymptotically unitarily equivalent to some constant path. Thus, without loss of generality $t \mapsto \varphi_t$ is a constant path $\varphi_t = \varphi$ for all t. Let $\lambda : D \to D$ be λ_φ as in Remark 2.1.2 (2).

Let $f : [0, \delta] \to [0, \infty)$ be a function associated with the exact stability of $\mathcal{O}_m$ (Proposition 2.1.9) in the same way the functions f_m of Proposition 2.1.10 are associated with the exact stability of E_m.

Choose $\varepsilon_0' > 0$ with $f(\varepsilon_0') < 1$. Choose $\varepsilon_0 > 0$ with $\varepsilon_0 < 1/2$, and also so small that if $\omega : \mathcal{O}_m \to A$ is a unital homomorphism, and $a_1, \dots, a_m \in A$ satisfy $\|a_j - \omega(s_j)\| < \varepsilon_0$, then the a_j satisfy the relations for $\mathcal{O}_m$ to within ε_0', that is,

$$\|a_j^* a_j - 1\| < \varepsilon_0' \quad \text{and} \quad \left\| \sum_{k=1}^{m} a_k a_k^* - 1 \right\| < \varepsilon_0'$$

for $1 \le j \le m$. Set $u_0 = \sum_{j=1}^{m} \psi_0(s_j)\varphi(s_j)^*$; this is the same as the u_0 in the statement of the lemma, so its K_1-class is in $(m-1)K_1(D)$. Theorem 2.1.3 therefore yields a unitary $v_0^{(0)} \in D$ such that $\|u_0 - v_0^{(0)}\lambda(v_0^{(0)})^*\| < \varepsilon_0$. Define $v_t^{(0)} = v_0^{(0)}$ for all t, and define $\gamma_t^{(0)} : \mathcal{O}_m \to D$ by $\gamma_t^{(0)}(a) = (v_t^{(0)})^* \psi_t(a) v_t^{(0)}$. Using Remark 2.1.2, we calculate:

$$\begin{aligned} \|\varphi(s_j) - \gamma_0^{(0)}(s_j)\| &= \|v_0^{(0)}\varphi(s_j)(v_0^{(0)})^* - \psi_0(s_j)\| \\ &= \|v_0^{(0)}\lambda(v_0^{(0)})^*\varphi(s_j) - u_0\varphi(s_j)\| < \varepsilon_0 \end{aligned}$$

for $1 \le j \le m$.

We now construct, by induction on n, numbers $\varepsilon_n, \varepsilon_n' > 0$ and continuous paths $t \mapsto v_t^{(n)}$ of unitaries in D and $t \mapsto \gamma_t^{(n)}$ of unital homomorphisms from $\mathcal{O}_m$ to D, for $t \in [0, \infty)$, such that ε_0, ε_0', $v_t^{(0)}$, and $\gamma_t^{(0)}$ are as already chosen, and:

(1) $\gamma_t^{(n)}(a) = (v_t^{(n)})^* \gamma_t^{(n-1)}(a) v_t^{(n)}$ for $a \in \mathcal{O}_m$ and $t \in [0, \infty)$.

(2) If $n \ge 1$, then $v_t^{(n)} = 1$ for $t \le n - 1$.

(3) If $n \ge 1$, then $\|\varphi(s_j) - \gamma_t^{(n)}(s_j)\| < 2^{-n+1}$ for $t \in [n-1, n]$, and if $n \ge 0$ then $\|\varphi(s_j) - \gamma_t^{(n)}(s_j)\| < \varepsilon_n$ for $t = n$.

(4) $f(\varepsilon_n') < 2^{-n}$.

(5) Whenever $\omega : \mathcal{O}_m \to A$ is a unital homomorphism, and $a_1, \dots, a_m \in A$ satisfy $\|a_j - \omega(s_j)\| < \varepsilon_n$, then the a_j satisfy the relations for $\mathcal{O}_m$ to within ε_n'.

(6) $\varepsilon_n < 2^{-(n+1)}$.

Suppose that ε_n, ε_n', $v_t^{(n)}$, and $\gamma_t^{(n)}$ have been chosen. Choose ε_{n+1}' and then ε_{n+1} as in (4), (5), and (6).

For $\alpha \in [0,1]$, define

$$a_j(\alpha) = (1-\alpha)(\varphi(s_j) - \gamma_n^{(n)}(s_j)) + \gamma_{n+\alpha}^{(n)}(s_j).$$

Then $\|a_j(\alpha) - \gamma_{n+\alpha}^{(n)}(s_j)\| < \varepsilon_n$ for $1 \leq j \leq m$ and $\alpha \in [0,1]$. Conditions (4) and (5), and the choice of f, provide a unital homomorphism $\sigma : \mathcal{O}_m \to C([0,1], D)$ such that $\|\sigma(s_j) - a_j\| < 2^{-n}$ for $1 \leq j \leq m$. Define $\sigma_\alpha : \mathcal{O}_m \to D$ by $\sigma_\alpha(a) = \sigma(a)(\alpha)$ for $\alpha \in [0,1]$ and $a \in \mathcal{O}_m$. Then

$$\|\sigma_\alpha(s_j) - \gamma_{n+\alpha}^{(n)}(s_j)\| < \varepsilon_n + 2^{-n}.$$

Functoriality of the approximating homomorphisms (the analog of (3) of Proposition 2.1.10) guarantees that $\sigma_0 = \varphi$ and $\sigma_1 = \gamma_{n+1}^{(n)}$.

Define a unitary $z \in C([0,1], D)$ by $z_\alpha = \sum_{j=1}^m \sigma_\alpha(s_j)\varphi(s_j)^*$ for $\alpha \in [0,1]$. Note that $z_0 = 1$, so $z \in U_0(C([0,1], D))$. Theorem 2.1.3 provides a unitary $\alpha \mapsto y_\alpha$ in $C([0,1], D)$ such that $\|z_\alpha - y_\alpha \lambda(y_\alpha)^*\| < \varepsilon_{n+1}/2$ for $\alpha \in [0,1]$. Putting $\alpha = 0$, using $z_0 = 1$, and rearranging terms, we obtain $\|y_0^* \lambda(y_0) - 1\| < \varepsilon_{n+1}/2$. Now define

$$v_t^{(n+1)} = \begin{cases} 1 & t \leq n \\ y_{t-n} y_0^* & n \leq t \leq n+1 \\ y_1 y_0^* & n+1 \leq t \end{cases}$$

and define $\gamma_t^{(n+1)}(a) = (v_t^{(n+1)})^* \gamma_t^{(n)}(a) v_t^{(n+1)}$.

It remains only to verify condition (3) in the induction hypothesis. For $\alpha \in [0,1]$,

$$\begin{aligned} \|z_\alpha - v_{n+\alpha}^{(n+1)} \lambda(v_{n+\alpha}^{(n+1)})^*\| &\leq \|z_\alpha - y_\alpha \lambda(y_\alpha)^*\| + \|y_\alpha\| \, \|1 - y_0^* \lambda(y_0)\| \, \|\lambda(y_\alpha)^*\| \\ &< \varepsilon_{n+1}/2 + \varepsilon_{n+1}/2 = \varepsilon_{n+1}. \end{aligned}$$

Therefore, for $t \in [n, n+1]$, Remark 2.1.2 yields

$$\begin{aligned} &\|\varphi(s_j) - \gamma_t^{(n+1)}(s_j)\| = \|v_t^{(n+1)} \varphi(s_j)(v_t^{(n+1)})^* - \gamma_t^{(n)}(s_j)\| \\ &\quad \leq \|v_t^{(n+1)} \lambda(v_t^{(n+1)})^* \varphi(s_j) - z_{t-n}\varphi(s_j)\| + \|\sigma_{t-n}(s_j) - \gamma_t^{(n)}(s_j)\| \\ &\quad < \varepsilon_{n+1} + \varepsilon_n + 2^{-n} < 2^{-(n+2)} + 2^{-(n+1)} + 2^{-n} < 2^{-n+1}. \end{aligned}$$

Furthermore, if $t = n+1$, then actually $\sigma_{t-n}(s_j) = \gamma_t^{(n)}(s_j)$, and we obtain $\|\varphi(s_j) - \gamma_t^{(n+1)}(s_j)\| < \varepsilon_{n+1}$. This completes the induction.

To complete the proof, we now define $v_t = \lim_{n \to \infty} v_t^{(0)} v_t^{(1)} \cdots v_t^{(n)}$ for $t \in [0, \infty)$. Note that the limit exists and defines a continuous unitary path

$t \mapsto v_t$, since $v_t^{(n+1)}$, $v_t^{(n+2)}$, ... are all equal to 1 on $[0, n)$. Furthermore, for $t \in [n, n+1]$, we have

$$\|\varphi(s_j) - v_t^* \psi_t(s_j) v_t\| = \|\varphi(s_j) - \gamma_t^{(n+1)}(s_j)\| < 2^{-n+1}.$$

This implies that φ and $t \mapsto \psi_t$ are asymptotically unitarily equivalent. □

For the proof of Lemma 2.2.2, we need the following lemma.

2.2.3 Lemma. Let D be a unital purely infinite simple C^*-algebra with $[1] = 0$ in $K_0(D)$. Let $m < n$, and identify E_m with the subalgebra of $\mathcal{O}_n$ generated by $s_1, \ldots, s_m$. Let $\varphi : E_m \to D$ be an injective unital homomorphism. Then there exists a unital homomorphism $\widetilde{\varphi} : \mathcal{O}_n \to D$ such that $\widetilde{\varphi}|_{E_m} = \varphi$. Moreover, if we are already given a unital homomorphism $\psi : \mathcal{O}_n \to D$, then $\widetilde{\varphi}$ can be chosen to satisfy $[\widetilde{\varphi}] = [\psi]$ in $KK^0(\mathcal{O}_n, D)$.

Proof: This is essentially contained in the proof of Proposition 1.7 of [35], using the equivalence of conditions (1) and (3) in Theorem 2.1.3. □

Proof of Lemma 2.2.2: We describe how to modify the proof of Lemma 2.2.1 to obtain this result.

First, note that $U(D)/U_0(D) \to K_1(D)$ is an isomorphism because D is purely infinite simple. Furthermore, $\mathrm{cel}(C([0,1], D)) \leq 5\pi/2 < \infty$ by Theorem 1.2 of [62]. (It turns out that we only need this result for $D = \mathcal{O}_\infty$, so we could use Corollary 2.1.13 here instead.) Thus, the conditions on D in Lemma 2.2.1 are satisfied.

As in the proof of Lemma 2.2.1, we may assume that $t \mapsto \varphi_t$ is a constant path $\varphi_t = \varphi$ for all t.

Let the functions f_m be the ones associated with the exact stability of E_m as in Proposition 2.1.10.

The proof uses an induction argument similar to that of the proof of Lemma 2.2.1, except that at the n-th stage we work with extensions to $\mathcal{O}_{2n}$ of $\varphi|_{E_n}$ and $\psi_t|_{E_n}$. To avoid confusion, we let $s_1, s_2, \ldots$ be the standard generators of $\mathcal{O}_\infty$, with the first n of them generating E_n, and we let $s_1^{(2n)}, \ldots, s_{2n}^{(2n)}$ be the standard generators of $\mathcal{O}_{2n}$, with E_k, for $k < 2n$, being identified with the subalgebra generated by the first k of them.

We start the construction at $n = 2$ so as not to have to worry about E_0 and E_1.

In the preliminary step, we choose $\varepsilon_2 > 0$ and $\varepsilon_2' > 0$ so that $\varepsilon_2 < 1/8$, $f_4(\varepsilon_2') < 1/4$, and whenever $\omega : E_2 \to A$ is a unital homomorphism, and $a_1, a_2 \in A$ satisfy $\|a_j - \omega(s_j)\| < \varepsilon_2$, then the a_j satisfy the relations for E_2 to within ε_2'. Use Lemma 2.2.3 to choose unital homomorphisms $\widetilde{\varphi}^{(2)}, \widetilde{\psi}_2^{(2)} : \mathcal{O}_4 \to D$ such that $\widetilde{\varphi}^{(2)}|_{E_2} = \varphi|_{E_2}$, $\widetilde{\psi}_2^{(2)}|_{E_2} = \psi_2|_{E_2}$, and $[\widetilde{\varphi}^{(2)}] = [\widetilde{\psi}_2^{(2)}]$ in $KK^0(\mathcal{O}_4, D)$. Set

$$u = \sum_{j=1}^{4} \widetilde{\psi}_2^{(2)}(s_j^{(4)}) \widetilde{\varphi}^{(2)}(s_j^{(4)})^*.$$

From (2) implies (3) in Theorem 2.1.3, we obtain a unitary $v_2^{(2)} \in D$ such that

$$\|u - v_2^{(2)} \lambda_{\varphi^{(2)}} (v_2^{(2)})^*\| < \varepsilon_2.$$

Define $v_t^{(2)} = v_2^{(2)}$ for $t \in [2, \infty)$, and define $\gamma_t^{(2)} : \mathcal{O}_\infty \to D$ by $\gamma_t^{(2)}(a) = (v_t^{(2)})^* \psi_t(a) v_t^{(2)}$. As in the proof of Lemma 2.2.1, a calculation shows that

$$\|\widetilde{\varphi}^{(2)}(s_j^{(4)}) - (v_2^{(2)})^* \widetilde{\psi}_2^{(2)}(s_j^{(4)}) v_2^{(2)}\| < \varepsilon_2$$

for $1 \leq j \leq 4$. It follows that

$$\|\varphi(s_j) - \gamma_2^{(2)}(s_j)\| < \varepsilon_2$$

for $1 \leq j \leq 2$.

In the induction step, we now require that $t \in [2, \infty)$, that ε_2, ε_2', $\gamma_t^{(2)}$, and $v_t^{(2)}$ be as already given, that $\gamma_t^{(n)} : \mathcal{O}_\infty \to D$, and that:

(1) $\gamma_t^{(n)}(a) = (v_t^{(n)})^* \gamma_t^{(n-1)}(a) v_t^{(n)}$ for $a \in \mathcal{O}_\infty$ and $t \in [2, \infty)$.

(2) If $n \geq 3$, then $v_t^{(n)} = 1$ for $t \leq n$.

(3) If $n \geq 3$, then $\|\varphi(s_j) - \gamma_t^{(n)}(s_j)\| < 2^{-n+1}$ for $t \in [n-1, n]$ and $1 \leq j \leq n-1$, and if $n \geq 2$ then $\|\varphi(s_j) - \gamma_t^{(n)}(s_j)\| < \varepsilon_n$ for $t = n$ and $1 \leq j \leq n$.

(4) $f_n(\varepsilon_n') < 2^{-n}$.

(5) Whenever $\omega : E_n \to A$ is a unital homomorphism, and $a_1, \ldots, a_n \in A$ satisfy $\|a_j - \omega(s_j)\| < \varepsilon_n$, then the a_j satisfy the relations for E_n to within ε_n'.

(6) $\varepsilon_n < 2^{-(n+1)}$.

For the proof of the inductive step, we first choose ε_{n+1}' and ε_{n+1} to satisfy (4), (5), and (6). Then construct, as in the proof of Lemma 2.2.1, a continuous path of homomorphisms $\sigma_\alpha : E_n \to D$ such that $\sigma_0 = \varphi|_{E_n}$, $\sigma_1 = \gamma_{n+1}^{(n)}|_{E_n}$, and

$$\|\sigma_\alpha(s_j) - \gamma_{n+\alpha}^{(n)}(s_j)\| < \varepsilon_n + 2^{-n}$$

for $1 \leq j \leq n$.

We now claim that there is a unitary path $\alpha \mapsto w_\alpha$ in D such that $w_0 = 1$, $w_\alpha \sigma_\alpha(s_j) = \sigma_0(s_j)$ for $\alpha \in [0, 1]$ and $1 \leq j \leq n$, and $w_1 \gamma_{n+1}^{(n)}(s_{n+1}) = \varphi(s_{n+1})$. To prove this, start by defining $q_\alpha = \sum_{j=1}^n \sigma_\alpha(s_j)\sigma_\alpha(s_j)^*$. Then set $w_\alpha' = \sum_{j=1}^n \sigma_0(s_j)\sigma_\alpha(s_j)^*$, which is a partial isometry from q_α to q_0 such that $w_\alpha' \sigma_\alpha(s_j) = \sigma_0(s_j)$ for $1 \leq j \leq n$. Next, define

$$p_1 = \gamma_{n+1}^{(n)}(s_{n+1})\gamma_{n+1}^{(n)}(s_{n+1})^* \quad \text{and} \quad p_0 = \varphi(s_{n+1})\varphi(s_{n+1})^*.$$

Since $\gamma_{n+1}^{(n)}|_{E_n} = \sigma_1$ and $\varphi|_{E_n} = \sigma_0$, we see that p_1 and p_0 are proper subprojections of $1 - q_1$ and $1 - q_0$ respectively, both with the same class (namely $[1] = 0$) in $K_0(D)$. Standard methods therefore yield a unitary path $\alpha \mapsto c_\alpha$ in D such that $c_0 = 1$, $c_\alpha q_\alpha c_\alpha^* = q_0$, and $c_1 p_1 c_1^* = p_0$. Then $\varphi(s_{n+1})\gamma_{n+1}^{(n)}(s_{n+1})^* c_1^*$ is a unitary in $p_0 D p_0$, so there is a unitary $d \in (1 - q_0 - p_0)D(1 - q_0 - p_0)$ such that

$$\varphi(s_{n+1})\gamma_{n+1}^{(n)}(s_{n+1})^* c_1^* + d \in U_0((1 - q_0)D(1 - q_0)),$$

and a unitary path $\alpha \mapsto w''_\alpha$ in $(1 - q_0)D(1 - q_0)$ such that

$$w''_0 = 1 \quad \text{and} \quad w''_1 = \varphi(s_{n+1})\gamma_{n+1}^{(n)}(s_{n+1})^* c_1^* + d.$$

Set $w_\alpha = w'_\alpha + w''_\alpha c_\alpha$; this is the path that proves the claim.

Use Lemma 2.2.3 to choose a unital homomorphism $\widetilde{\varphi}^{(n+1)} : \mathcal{O}_{2n+2} \to D$ such that $\widetilde{\varphi}^{(n+1)}|_{E_{n+1}} = \varphi|_{E_{n+1}}$. Define unital homomorphisms $\widetilde{\sigma}_\alpha : \mathcal{O}_{2n+2} \to D$ by

$$\widetilde{\sigma}_\alpha(s_j^{(2n+2)}) = w_\alpha^* \widetilde{\varphi}^{(n+1)}(s_j^{(2n+2)})$$

for $1 \leq j \leq 2n + 2$. Then

$$\widetilde{\sigma}_0 = \widetilde{\varphi}^{(n+1)}, \quad \widetilde{\sigma}_\alpha|_{E_n} = \sigma_\alpha, \quad \text{and} \quad \widetilde{\sigma}_1|_{E_{n+1}} = \gamma_{n+1}^{(n)}|_{E_{n+1}}.$$

Define z and choose y as in the proof of Lemma 2.2.1, using $\mathcal{O}_{2n+2}$ in place of $\mathcal{O}_m$, $\widetilde{\sigma}$ in place of σ, $\widetilde{\varphi}^{(n+1)}$ in place of φ, and $\lambda = \lambda_{\widetilde{\varphi}^{(n+1)}}$. Define $v_t^{(n+1)}$ and $\gamma_t^{(n+1)}$ as there. The same computations as there show that

$$\begin{aligned} &\|\varphi(s_j) - \gamma_t^{(n+1)}(s_j)\| \\ &\quad = \quad \|\widetilde{\varphi}^{(n+1)}(s_j^{(2n+2)}) - (v_t^{(n+1)})^* \widetilde{\sigma}_{t-n}(s_j^{(2n+2)}) v_t^{(n+1)}\| < 2^{-n+1} \end{aligned}$$

for $1 \leq j \leq n$ and $t \in [n, n + 1]$, and

$$\|\varphi(s_j) - \gamma_{n+1}^{(n+1)}(s_j)\| = \|\widetilde{\varphi}^{(n+1)}(s_j^{(2n+2)}) - (v_t^{(n+1)})^* \widetilde{\sigma}_1(s_j^{(2n+2)}) v_t^{(n+1)}\| < \varepsilon_{n+1}$$

for $1 \leq j \leq n + 1$. This completes the induction step.

Define $v_t = \lim_{n\to\infty} v_t^{(2)} v_t^{(3)} \cdots v_t^{(n)}$. Calculations analogous to those in the proof of Lemma 2.2.1 show that $t \mapsto v_t$ is a continuous unitary path in D, and that for $n \geq 2$ we have

$$\|\varphi(s_j) - v_t^* \psi_t(s_j) v_t\| < 2^{-n+1}$$

for $t \in [n, n + 1]$ and $1 \leq j \leq n$. This implies that

$$\lim_{t\to\infty} \left(\varphi(a) - v_t^* \psi_t(a) v_t \right) = 0$$

for all $a \in \mathcal{O}_\infty$. ◻

2.2.4 LEMMA. There exists a continuous family $t \mapsto \varphi_t$ of unital endomorphisms of $\mathcal{O}_\infty$, for $t \in [0, \infty)$, which is asymptotically central in the sense that

$$\lim_{t \to \infty} \Big(\varphi_t(b)a - a\varphi_t(b) \Big) = 0$$

for all $a, b \in \mathcal{O}_\infty$.

Proof: Let A_n be the tensor product of n copies of $\mathcal{O}_\infty$, and define $\mu_n : A_n \to A_{n+1}$ by $\mu_n(a) = a \otimes 1$. Set $A = \varinjlim A_n$, which is just $\bigotimes_1^\infty \mathcal{O}_\infty$. Theorem 2.1.5 implies that $A_n \cong \mathcal{O}_\infty$, so Theorem 3.5 of [35] implies that $A \cong \mathcal{O}_\infty$. (Actually, that $A \cong \mathcal{O}_\infty$ is shown in the course of the proof of Theorem 2.1.5. See [29].) It therefore suffices to construct a continuous asymptotically central inclusion of $\mathcal{O}_\infty$ in A rather than in $\mathcal{O}_\infty$.

Let $\nu_n : A_n \to A$ be the inclusion. Proposition 2.1.11 provides a homotopy $\alpha \mapsto \psi_\alpha$ of unital homomorphisms $\psi_\alpha : \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ such that $\psi_0(a) = a \otimes 1$ and $\psi_1(a) = 1 \otimes a$. For $n \geq 1$ and $t \in [n, n+1]$, we write $t = n + \alpha$ and define

$$\varphi_t(a) = \nu_{n+2}(1 \otimes 1 \otimes \cdots \otimes 1 \otimes \psi_\alpha(a)),$$

where the factor 1 appears n times in the tensor product. The two definitions of $\varphi_n(a)$ agree, so $t \mapsto \varphi_t$ is continuous. We clearly have $\lim_{t \to \infty} (\varphi_t(b)a - a\varphi_t(b)) = 0$ for $b \in \mathcal{O}_\infty$ and $a \in \bigcup_{n=1}^\infty \nu_n(A_n)$, and a standard argument then shows this is true for all $a \in A$. ◻

The notation introduced in the following definition is the same as in [34], [35], and [44].

2.2.5 DEFINITION. Let A be any unital C^*-algebra, and let D be a purely infinite simple C^*-algebra. Let $\varphi, \psi : A \to D$ be two homomorphisms, and assume that $\varphi(1) \neq 0$ and $[\psi(1)] = 0$ in $K_0(D)$. We define a homomorphism $\varphi \widetilde{\oplus} \psi : A \to D$, well defined up to unitary equivalence, by the following construction. Choose a projection $q \in D$ such that $0 < q < \varphi(1)$ and $[q] = 0$ in $K_0(D)$. Since D is purely infinite and simple, there are partial isometries v and w such that

$$vv^* = \varphi(1) - q, \quad v^*v = \varphi(1), \quad ww^* = q, \quad \text{and} \quad w^*w = \psi(1).$$

Now define $(\varphi \widetilde{\oplus} \psi)(a) = v\varphi(a)v^* + w\psi(a)w^*$ for $a \in A$.

The proof of the following lemma could be simplified considerably by using semiprojectivity of $\mathcal{O}_\infty$ ([5]) and Proposition 1.1.7. Since [5] remains (to our knowledge) unpublished, we retain the original proof.

2.2.6 LEMMA. (Compare with Proposition 2.3 of [35].) Let D be a unital purely infinite simple C^*-algebra, and let $q \in D$ be a projection with $[q] = 0$ in $K_0(D)$. Let $\varphi : \mathcal{O}_\infty \to D$ and $\psi : \mathcal{O}_\infty \to qDq$ be unital homomorphisms. Then φ is asymptotically unitarily equivalent to $\varphi \widetilde{\oplus} \psi$.

Proof: Let $t \mapsto \gamma_t$ be a continuously parametrized asymptotically central inclusion of $\mathcal{O}_\infty$ in $\mathcal{O}_\infty$, as in Lemma 2.2.4. Let $e \in \mathcal{O}_\infty$ be a nonzero projection with $[e] = 0$ in $K_0(\mathcal{O}_\infty)$, and set $e_t = \gamma_t(e)$. Choose a continuous unitary path $t \mapsto u_t$ such that $u_t e_t u_t^* = e_0$.

Let the functions $f_m : [0, \delta(m)] \to [0, \infty)$ be as in Proposition 2.1.10. Choose numbers $\varepsilon_2 > \varepsilon_3 > \cdots 0$ and $\varepsilon'_2 > \varepsilon'_3 > \cdots 0$ such that:

(1) $\varepsilon'_m < \delta(m)$ and $f_m(\varepsilon'_m) < 1/m$.

(2) Whenever $\omega : E_m \to A$ is a unital homomorphism, and $a_1, \ldots, a_m \in A$ satisfy $\|a_j - \omega(s_j)\| < \varepsilon_m$, then the a_j satisfy the relations for E_m to within ε'_m.

(3) $\varepsilon_m < 1/m$.

Next, use the asymptotic centrality of $t \mapsto e_t$ to choose $t_2 < t_3 < \cdots$, with $t_m \to \infty$ as $m \to \infty$, such that

$$\left\| s_j - \Big[e_t s_j e_t + (1 - e_t) s_j (1 - e_t) \Big] \right\| < \varepsilon_m$$

for $1 \leq j \leq m$ and $t \geq t_m$. Define

$$a_j(t) = u_t \Big[e_t s_j e_t + (1 - e_t) s_j (1 - e_t) \Big] u_t^* \in e_0 \mathcal{O}_\infty e_0 \oplus (1 - e_0) \mathcal{O}_\infty (1 - e_0).$$

Conditions (1) and (2), and Proposition 2.1.10, then yield continuous paths $t \mapsto \sigma_t^{(m)}$ of homomorphisms from E_m to $e_0 \mathcal{O}_\infty e_0 \oplus (1 - e_0) \mathcal{O}_\infty (1 - e_0)$, defined for $t \geq t_m$, such that $\|\sigma_t^{(m)}(s_j) - a_j(t)\| < 1/m$ for $1 \leq j \leq m$, and $\sigma_t^{(m+1)}|_{E_m} = \sigma_t^{(m)}$ for $t \geq t_{m+1}$.

Define

$$\alpha_t^{(m)} : E_m \to e_0 \mathcal{O}_\infty e_0 \quad \text{and} \quad \beta_t^{(m)} : E_m \to (1 - e_0) \mathcal{O}_\infty (1 - e_0)$$

by

$$\alpha_t^{(m)}(a) = e_0 \sigma_t^{(m)}(a) e_0 \quad \text{and} \quad \beta_t^{(m)}(a) = (1 - e_0) \sigma_t^{(m)}(a) (1 - e_0).$$

Note that $\alpha_{t_m}^{(m)}$ is homotopic to $\alpha_{t_{m+1}}^{(m+1)}|_{E_m}$; since $\alpha_{t_{m+1}}^{(m+1)}|_{E_m}$ is injective, it follows that $\alpha_{t_m}^{(m)}$ is injective. Since $e_0 \mathcal{O}_\infty e_0$ is purely infinite simple, it is easy to extend $\alpha_{t_m}^{(m)}$ to a homomorphism $\alpha_{t_m} : \mathcal{O}_\infty \to e_0 \mathcal{O}_\infty e_0$. Proposition 2.1.11 provides homotopies $t \mapsto \alpha_t$ of homomorphisms from $\mathcal{O}_\infty$ to $e_0 \mathcal{O}_\infty e_0$, defined for $t \in [t_m, t_{m+1}]$, such that $\alpha_t|_{E_m} = \alpha_t^{(m)}$ and such that α_{t_m} and $\alpha_{t_{m+1}}$ are as already given. Putting these homotopies together, and defining $\alpha_t = \alpha_{t_2}$ for $t \in [0, t_2]$, we obtain a continuous path $t \mapsto \alpha_t$ of unital homomorphisms from $\mathcal{O}_\infty$ to $e_0 \mathcal{O}_\infty e_0$, defined for $t \in [0, \infty)$, such that $\alpha_t|_{E_m} = \alpha_t^{(m)}$ whenever $t \geq t_m$. Similarly, there is a continuous path $t \mapsto \beta_t$ of unital homomorphisms from $\mathcal{O}_\infty$ to $(1-e_0)\mathcal{O}_\infty(1-e_0)$, defined for $t \in [0, \infty)$, such that $\beta_t|_{E_m} = \beta_t^{(m)}$ whenever $t \geq t_m$. Define $\sigma_t : \mathcal{O}_\infty \to \mathcal{O}_\infty$ by $\sigma_t(a) = \alpha_t(a) + \beta_t(a)$.

For $t \geq t_m$ and $1 \leq j \leq m$, we have $u_t^* \sigma_t(s_j) u_t = u_t^* \sigma_t^{(m)}(s_j) u_t$, and

$$\|u_t^* \sigma_t^{(m)}(s_j) u_t - s_j\| \leq \|\sigma_t^{(m)}(s_j) - a_j(t)\| + \|u_t^* a_j(t) u_t - s_j\| < 1/m + \varepsilon_m < 2/m.$$

Therefore $\lim_{t \to \infty} \|u_t^* \sigma_t^{(m)}(s_j) u_t - s_j\| = 0$ for all j. Thus $t \mapsto \sigma_t$ is asymptotically unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty}$. So φ is asymptotically unitarily equivalent to $t \mapsto \varphi \circ \sigma_t$.

Let $f < \varphi(e_0)$ be a nonzero projection with $[f] = 0$ in $K_0(D)$. Let $w_1, w_2 \in D$ be partial isometries satisfying

$$w_1^* w_1 = 1, \;\; w_1 w_1^* = 1 - f, \quad \text{and} \quad w_1(1 - \varphi(e_0)) = (1 - \varphi(e_0)) w_1 = 1 - \varphi(e_0)$$

and

$$w_2^* w_2 = q \quad \text{and} \quad w_2 w_2^* = f.$$

The homomorphism $\varphi \widetilde{\oplus} \psi$ is only defined up to unitary equivalence, and we can take it to be

$$(\varphi \widetilde{\oplus} \psi)(x) = w_1 \varphi(x) w_1^* + w_2 \psi(x) w_2^*.$$

We make the same choices when defining $(\varphi \circ \sigma_t) \widetilde{\oplus} \psi$. Writing $\varphi \circ \sigma_t = \varphi \circ \alpha_t + \varphi \circ \beta_t$, with

$$\varphi \circ \alpha_t : \mathcal{O}_\infty \to \varphi(e_0) D \varphi(e_0) \quad \text{and} \quad \varphi \circ \beta_t : \mathcal{O}_\infty \to \varphi(1 - e_0) D \varphi(1 - e_0),$$

this choice gives

$$(\varphi \circ \sigma_t) \widetilde{\oplus} \psi = [(\varphi \circ \alpha_t) \widetilde{\oplus} \psi] + \varphi \circ \beta_t.$$

By Lemma 2.2.2, $t \mapsto (\varphi \circ \alpha_t) \widetilde{\oplus} \psi$ is asymptotically unitarily equivalent to $\varphi \circ \alpha_t$. Therefore, with $\sim$ denoting asymptotic unitary equivalence, we have

$$\varphi \widetilde{\oplus} \psi \sim (\varphi \circ \alpha_t) \widetilde{\oplus} \psi + \varphi \circ \beta_t \sim \varphi \circ \alpha_t + \varphi \circ \beta_t \sim \varphi.$$

This is the desired result. ◻

2.2.7 Proposition. (Compare with Theorem 3.3 of [35].) Let D be a unital purely infinite simple C^*-algebra, and let $\varphi, \psi : \mathcal{O}_\infty \to D$ be two unital homomorphisms. Then φ is asymptotically unitarily equivalent to ψ.

Proof: Let $e = 1 - s_1 s_1^* - s_2 s_2^* \in \mathcal{O}_\infty$, and let $f = \varphi(e) \in D$. Define $\overline{\varphi} : \mathcal{O}_\infty \to fDf$ by $\overline{\varphi}(s_j) = \varphi(s_{j+2}) f$. Let $w \in M_2(D)$ be a partial isometry with $w^* w = 1 \oplus f$ and $w w^* = q \oplus 0$ for some $q \in D$. We regard $w(\varphi \oplus \overline{\varphi})(-)w^*$ and $w(\psi \oplus \overline{\varphi})(-)w^*$ as homomorphisms from $\mathcal{O}_\infty$ to qDq. Furthermore, $[q] = 0$ in $K_0(D)$, so

$$\varphi \widetilde{\oplus} w(\psi \oplus \overline{\varphi})(-)w^* \quad \text{and} \quad \psi \widetilde{\oplus} w(\varphi \oplus \overline{\varphi})(-)w^*$$

are defined; they are easily seen to be unitarily equivalent. Using Lemma 2.2.6 for the other two steps, we therefore obtain asymptotic unitary equivalences

$$\varphi \sim \varphi \widetilde{\oplus} w(\psi \oplus \overline{\varphi})(-)w^* \sim \psi \widetilde{\oplus} w(\varphi \oplus \overline{\varphi})(-)w^* \sim \psi.$$ ◻

2.2.8 Corollary. Let A be any unital C^*-algebra such that $\mathcal{O}_\infty \otimes A \cong A$. Then there exists an isomorphism $\beta : \mathcal{O}_\infty \otimes A \to A$ such that the homomorphism $a \mapsto \beta(1 \otimes a)$ is asymptotically unitarily equivalent to id_A.

Proof: We first prove this for $A = \mathcal{O}_\infty$. Theorem 2.1.5 implies that $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$; let $\beta_0 : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ be an isomorphism. Then $a \mapsto \beta_0(1 \otimes a)$ and $\mathrm{id}_{\mathcal{O}_\infty}$ are two unital homomorphisms from $\mathcal{O}_\infty$ to $\mathcal{O}_\infty$, so they are asymptotically unitarily equivalent by Proposition 2.2.7. Let $t \mapsto u_t$ be a unitary path such that $\lim_{t \to \infty} (\beta_0(1 \otimes a) - u_t a u_t^*) = 0$ for all $a \in \mathcal{O}_\infty$.

Now let A be as in the hypotheses. We may as well prove the result for $\mathcal{O}_\infty \otimes A$ instead of A. Take $\beta = \beta_0 \otimes \mathrm{id}_A$; then $a \mapsto \beta(1 \otimes a)$ is asymptotically unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty \otimes A}$ via the unitary path $t \mapsto u_t \otimes 1$. □

2.3 When homotopy implies asymptotic unitary equivalence

In this subsection, we will prove that if A is a separable nuclear unital simple C^*-algebra and D_0 is unital, then two homotopic asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes D_0$ are asymptotically unitarily equivalent. We will furthermore prove that an asymptotic morphism from A to $K \otimes \mathcal{O}_\infty \otimes D_0$ is asymptotically unitarily equivalent to a homomorphism. The method of proof of the first statement will generalize the methods of [44]. We will obtain the second via a trick.

The following two definitions will be convenient. The first is used, both here and in Section 3, to simplify terminology, and the second is the analog of Definition 2.1 of [44].

2.3.1 Definition. Let A, D, and Q be C^*-algebras, with A and Q separable and with Q also unital and nuclear. Let $\varphi : A \to D$ be an asymptotic morphism. A *standard factorization* of φ through $Q \otimes A$ is an asymptotic morphism $\psi : Q \otimes A \to D$ such that $\varphi_t(a) = \psi_t(1 \otimes a)$ for all t and all $a \in A$. An *asymptotic standard factorization* of φ through $Q \otimes A$ is an asymptotic morphism $\psi : Q \otimes A \to D$ such that φ is asymptotically unitarily equivalent to the asymptotic morphism $(t, a) \mapsto \psi_t(1 \otimes a)$.

2.3.2 Definition. Let A, D, and φ be as in the previous definition. An *(asymptotically) trivializing factorization* of φ is a (asymptotic) standard factorization with $Q = \mathcal{O}_2$. In this case, we say that φ is *(asymptotically) trivially factorizable.*

2.3.3 Lemma. (Compare [44], Lemma 2.2.) Let A be separable, nuclear, unital, and simple, let D_0 be a unital C^*-algebra, and let $D = \mathcal{O}_\infty \otimes D_0$. Then any two full asymptotically trivially factorizable asymptotic morphisms from A to $K \otimes D$ are asymptotically unitarily equivalent.

Proof: It suffices to prove this for full asymptotic morphisms $\varphi, \psi : A \to K \otimes D$ with trivializing factorizations $\varphi', \psi' : \mathcal{O}_2 \otimes A \to K \otimes D$. Note that φ' and ψ' are again full, and it suffices to prove that φ' is asymptotically unitarily equivalent to ψ'. By Theorem 2.1.4, we have $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$, and Proposition 1.1.7 then implies that φ' and ψ' are asymptotically equal to continuous families φ'' and ψ'' of homomorphisms.

We now have two continuous families of full projections $t \mapsto \varphi''_t(1)$ and $t \mapsto \psi''_t(1)$ in $K \otimes D$, parametrized by $[0, \infty)$. Standard methods show that each family is unitarily equivalent to a constant projection. Moreover, the projections $\varphi''_0(1)$ and $\psi''_0(1)$ have trivial K_0 classes, so are homotopic by Lemma 2.1.8 (2). Therefore they are unitarily equivalent. Combining the unitaries involved and conjugating by the result, we can assume $\varphi''_t(1)$ and $\psi''_t(1)$ are both equal to the constant family $t \mapsto p$ for a suitable full projection p. Now replace $K \otimes D$ by $p(K \otimes D)p$, and apply Lemma 2.2.1; its hypotheses are satisfied by Corollary 2.1.12. ◻

2.3.4 Corollary. (Compare [44], Lemma 2.3.) Under the hypotheses of Lemma 2.3.3, the direct sum of two full asymptotically trivially factorizable asymptotic morphisms $\varphi, \psi : A \to K \otimes D$ is again full and asymptotically trivially factorizable.

Proof: Since asymptotic unitary equivalence respects direct sums, the previous lemma implies we may assume $\varphi = \psi$. We may further assume that φ actually has a trivializing factorization $\varphi' : \mathcal{O}_2 \otimes A \to K \otimes D$. Then $\varphi \oplus \psi$ has the standard factorization $\mathrm{id}_{M_2} \otimes \varphi'$ through $(M_2 \otimes \mathcal{O}_2) \otimes A$, and this is a trivializing factorization because $M_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Fullness follows from Lemma 1.2.6 (2). ◻

We also need asymptotically standard factorizations through $\mathcal{O}_\infty \otimes A$. The special properties required in the following lemma will be used in the proof of Theorem 2.3.7.

2.3.5 Lemma. Let A be a separable unital nuclear C^*-algebra, let D_0 be unital, and let $D = \mathcal{O}_\infty \otimes D_0$. Let $\varphi : A \to K \otimes D$ be an asymptotic morphism. Then φ has an asymptotic standard factorization through $\mathcal{O}_\infty \otimes A$. In fact, φ is asymptotically unitarily equivalent to an asymptotic morphism of the form $\psi_t(a) = \delta \circ (\mathrm{id}_{\mathcal{O}_\infty} \otimes \widetilde{\varphi}_t)(1 \otimes a)$, in which $\delta : \mathcal{O}_\infty \otimes K \otimes D \to K \otimes D$ is an isomorphism, $\widetilde{\varphi}$ is completely positive contractive and asymptotically equal to φ, and $\mathrm{id}_{\mathcal{O}_\infty} \otimes \widetilde{\varphi}_t$ is defined to be the tensor product of completely positive maps and is again completely positive contractive.

Proof: Lemma 1.1.5 provides a completely positive contractive asymptotic morphism $\widetilde{\varphi}$ which is asymptotically equal to φ. Then $\mathrm{id}_{\mathcal{O}_\infty} \otimes \widetilde{\varphi}_t$ is the minimal tensor product of two completely positive contractive linear maps, and is therefore bounded and completely positive by Proposition IV.4.23 (i) of [58]. Looking at the proof of that proposition and of Theorem IV.3.6 of [58], we see that such a tensor product is in fact contractive. Thus, $\|\mathrm{id}_{\mathcal{O}_\infty} \otimes \widetilde{\varphi}_t\| \leq 1$ for all t. One checks that $t \mapsto (\mathrm{id}_{\mathcal{O}_\infty} \otimes \widetilde{\varphi}_t)(b)$ is continuous for b in the algebraic tensor product of $\mathcal{O}_\infty$ and A. It follows that continuity holds for all $b \in \mathcal{O}_\infty \otimes A$. Similarly, one checks that $t \mapsto \mathrm{id}_{\mathcal{O}_\infty} \otimes \widetilde{\varphi}_t$ is asymptotically multiplicative, so is an asymptotic morphism.

Use Corollary 2.2.8 to find an isomorphism $\delta_0 : \mathcal{O}_\infty \otimes D \to D$ such that $d \mapsto \delta_0(1 \otimes d)$ is asymptotically unitarily equivalent to id_D. This induces an isomorphism $\delta : \mathcal{O}_\infty \otimes K \otimes D \to K \otimes D$, and a unitary path $t \mapsto u_t^{(0)} \in M(K \otimes D)$ such that $\|u_t^{(0)} \delta(1 \otimes d)(u_t^{(0)})^* - d\| \to 0$ for all $d \in K \otimes D$. By Lemma 1.3.9, there is a unitary path $t \mapsto u_t \in (K \otimes D)^+$ such that $\|u_t \delta(1 \otimes d) u_t^* - d\| \to 0$ for all $d \in K \otimes D$.

We prove that the ψ that results from these choices is in fact asymptotically unitarily equivalent to $\widetilde{\varphi}$; this will prove the lemma. Choose finite subsets $F_1 \subset F_2 \subset \cdots$ whose union is dense in A. For each n, note that the set $S_n = \{\widetilde{\varphi}_t(a) : a \in F_n,\ t \in [0, n]\}$ is compact in D, so that there is r_n with $\|u_t \delta(1 \otimes d) u_t^* - d\| < 2^{-n}$ for all $d \in S_n$ and $t \geq r_n$. For $\alpha \in [0, 1]$ define $f(n+\alpha) = (1-\alpha) r_n + \alpha r_{n+1}$. Then define unitaries $v_t \in (K \otimes D)^+$ by $v_t = u_{f(t)}$. For $t \in [n, n+1]$ and $a \in F_n$, this gives (using $f(t) \geq r_n$)

$$\|v_t \psi_t(a) v_t^* - \widetilde{\varphi}_t(a)\| = \|u_{f(t)} \delta(1 \otimes \widetilde{\varphi}_t(a)) u_{f(t)}^* - \widetilde{\varphi}_t(a)\| < 2^{-n}.$$

Thus ψ is in fact asymptotically unitarily equivalent to $\widetilde{\varphi}$. □

2.3.6 Lemma. (Compare [44], Proposition 3.3.) Assume the hypotheses of Lemma 2.3.3. Let $\varphi, \psi : A \to K \otimes D$ be full asymptotic morphisms with ψ asymptotically trivially factorizable. Then $\varphi \oplus \psi$ is asymptotically unitarily equivalent to φ.

Proof: By Lemma 2.3.5, we may assume that φ has a standard factorization through $\mathcal{O}_\infty \otimes A$, say $\varphi' : \mathcal{O}_\infty \otimes A \to K \otimes D$. Using Lemma 1.3.8 on φ' and on an asymptotically trivializing factorization for ψ, we may assume without loss of generality that there are projections $p, q \in K \otimes D$ such that φ' is a unital asymptotic morphism from A to $p(K \otimes D)p$ and ψ is an asymptotically trivially factorizable unital asymptotic morphism from A to $q(K \otimes D)q$.

Choose a nonzero projection $e \in \mathcal{O}_\infty$ with trivial K_0 class. Let $t \mapsto f_t$ be a tail projection for $\varphi'(e \otimes 1)$. Choose a continuous unitary family $t \mapsto u_t$ in $p(K \otimes D)p$ such that $u_t f_t u_t^* = f_0$ for all t. Define bounded asymptotic morphisms

$$\sigma : (1 - e)\mathcal{O}_\infty(1 - e) \otimes A \to (p - f_0)(K \otimes D)(p - f_0)$$

and

$$\tau : e\mathcal{O}_\infty e \otimes A \to f_0(K \otimes D) f_0$$

by

$$\sigma_t(x) = u_t(p - f_t)\varphi'_t(x)(p - f_t)u_t^* \quad \text{and} \quad \tau_t(x) = u_t f_t \varphi'_t(x) f_t u_t^*.$$

These are in fact asymptotic morphisms, because $\lim_{t \to \infty} \|f_t - \varphi'_t(e \otimes 1)\| = 0$. Then define asymptotic morphisms

$$\widetilde{\sigma} : A \to (p - f_0)(K \otimes D)(p - f_0) \quad \text{and} \quad \widetilde{\tau} : A \to f_0(K \otimes D) f_0$$

by

$$\widetilde{\sigma}_t(a) = \sigma_t((1-e)\otimes a)) \quad \text{and} \quad \widetilde{\tau}_t(a) = \tau_t(e \otimes a).$$

It follows that

$$\lim_{t\to\infty} \|u_t \varphi'_t(1\otimes a)u_t^* - \widetilde{\sigma}_t(a) - \widetilde{\tau}_t(a)\| = 0$$

for all $a \in A$, so φ is asymptotically unitarily equivalent to $\widetilde{\sigma} \oplus \widetilde{\tau}$. Since $[e] = 0$ in $K_0(\mathcal{O}_\infty)$, there is a unital homomorphism $\nu : \mathcal{O}_2 \to e\mathcal{O}_\infty e$, and the formula $\widetilde{\tau}_t(a) = (\tau_t \circ (\nu \otimes \mathrm{id}_A))(1 \otimes a)$ shows that $\widetilde{\tau}$ has a trivializing factorization. Furthermore, $\widetilde{\tau}$ is full because φ' is. So $\widetilde{\tau} \oplus \psi$ is full and asymptotically trivially factorizable by Corollary 2.3.4, and therefore asymptotically unitarily equivalent to $\widetilde{\tau}$ by Lemma 2.3.3. The asymptotic unitary equivalence of φ and $\widetilde{\sigma} \oplus \widetilde{\tau}$ now implies that $\varphi \oplus \psi$ is asymptotically unitarily equivalent to φ. ◻

We now come to the main technical theorem of this section.

2.3.7 Theorem. Let A be separable, nuclear, unital, and simple. Let D_0 be a unital C^*-algebra, and let $D = \mathcal{O}_\infty \otimes D_0$. Then two full asymptotic morphisms from A to $K \otimes D$ are asymptotically unitarily equivalent if and only if they are homotopic.

This result is a continuous analog of Theorem 3.4 of [44], which gives a similar result for approximate unitary equivalence. In the proof of that theorem, to get approximate unitary equivalence to within ε on a finite set F, it was necessary to approximately absorb a large direct sum of asymptotically trivially factorizable homomorphisms—a direct sum which had to be larger for smaller ε and larger F. In the proof of the theorem stated here, we must continuously interpolate between approximate absorption of ever larger numbers of asymptotic morphisms. The resulting argument is rather messy. We try to make it easier to follow by isolating two pieces as lemmas. For the first of these, recall from Definition 1.1.4 (2) that bounded asymptotic morphisms are assumed in particular to be linear.

2.3.8 Lemma. Let A and D be C^*-algebras, with A separable. Let $\alpha \mapsto \varphi^{(\alpha)}$ be a bounded homotopy of asymptotic morphisms from A to D. Then there exists a continuous function $f : [0, \infty) \to (0, \infty)$ such that for every $a \in A$, we have

$$\lim_{t\to\infty} \left(\sup_{|\alpha_1-\alpha_2|\leq 1/f(t)} \|\varphi_t^{(\alpha_1)}(a) - \varphi_t^{(\alpha_2)}(a)\| \right) = 0.$$

Proof: Choose finite sets $F_0 \subset F_1 \subset \cdots \subset A$ whose union is dense in A.

For each n and each fixed $a \in A$, the map $(t, \alpha) \mapsto \varphi_t^{(\alpha)}(a)$ is uniformly continuous on $[0, n] \times [0, 1]$. So there is $\delta_n > 0$ such that

$$\sup\{\|\varphi_t^{(\alpha_1)}(a) - \varphi_t^{(\alpha_2)}(a)\| : t \in [0, n],\ |\alpha_1 - \alpha_2| \leq \delta_n,\ a \in F_n\} < 2^{-n}.$$

We may clearly assume $\delta_1 \geq \delta_2 \geq \cdots$. Let $t \mapsto \delta(t)$ be a continuous function such that $0 < \delta(t) \leq \delta_n$ for $t \in [n-1, n]$.

We claim that if $a \in \bigcup_{n=0}^{\infty} F_n$, then

$$\lim_{t \to \infty} \left(\sup_{|\alpha_1 - \alpha_2| \leq \delta(t)} \|\varphi_t^{(\alpha_1)}(a) - \varphi_t^{(\alpha_2)}(a)\| \right) = 0.$$

To see this, let $a \in F_m$. For $n \geq m+1$, $t \in [n-1, n]$, and $|\alpha_1 - \alpha_2| \leq \delta(t)$, we have in particular $|\alpha_1 - \alpha_2| \leq \delta_n$, so that $\|\varphi_t^{(\alpha_1)}(a) - \varphi_t^{(\alpha_2)}(a)\| \leq 2^{-n}$.

The statement of the lemma, using $f(t) = 1/\delta(t)$, follows from the claim by a standard argument, since φ is bounded and $\bigcup_{n=0}^{\infty} F_n$ is dense in A. $\square$

2.3.9 Lemma. Let A and Q be C^*-algebras, with Q unital and nuclear. Let $N \geq 2$, let $e_0, e_1, \ldots, e_N \in Q$ be mutually orthogonal projections which sum to 1, and let $w \in Q$ be a unitary such that $we_0w^* \leq e_1$, $we_jw^* \leq e_j + e_{j+1}$ for $1 \leq j \leq N-1$, and $we_Nw^* \leq e_N + e_0$. Let $a_0, \ldots, a_N, b_0, \ldots, b_N \in A$. Then in $Q \otimes A$ we have

$$\begin{aligned} & \left\| (w \otimes 1) \left(\sum_{j=0}^{N} e_j \otimes a_j \right) (w \otimes 1)^* - \sum_{j=0}^{N} e_j \otimes b_j \right\| \\ \leq \ & \max\{\|a_N - b_0\|, \|a_0 - b_1\| + \|a_1 - b_1\|, \|a_1 - b_2\| + \|a_2 - b_2\|, \\ & \ldots, \|a_{N-1} - b_N\| + \|a_N - b_N\|\}. \end{aligned}$$

Proof: Let

$$x = (w \otimes 1) \left(\sum_{j=0}^{N} e_j \otimes a_j \right) (w \otimes 1)^* = \sum_{j=0}^{N} we_jw^* \otimes a_j \quad \text{and} \quad y = \sum_{j=0}^{N} e_j \otimes b_j.$$

Observe that if we take the indices mod $N+1$, then e_k is orthogonal to we_jw^* whenever $k \neq j, j+1$, and also if $j = k = 0$. Therefore we can calculate

$$\begin{aligned} x - y &= \left(\sum_{i=0}^{N} e_i \otimes 1 \right) (x - y) \left(\sum_{k=0}^{N} e_k \otimes 1 \right) \\ &= \sum_{j=0}^{N} \Big[(e_j \otimes 1)(we_jw^* \otimes a_j + we_{j-1}w^* \otimes a_{j-1})(e_j \otimes 1) - e_j \otimes b_j \Big] \\ &\quad + \sum_{j=0}^{N} \Big[e_j(we_jw^*)e_{j+1} + e_{j+1}(we_jw^*)e_j \Big] \otimes b_j. \end{aligned}$$

We now claim that the second term in the last expression is zero. The projections we_kw^* are orthogonal and add up to 1, and e_{j+1} is orthogonal to all of them except for $k = j$ and $k = j+1$. Therefore $e_{j+1} \leq we_j w^* + we_{j+1}w^*$. Also, $e_j we_{j+1} w^* = 0$, so we obtain

$$e_j(we_jw^*)e_{j+1} = e_j(we_jw^* + we_{j+1}w^*)e_{j+1} = e_je_{j+1} = 0.$$

Similarly, $e_{j+1}(we_jw^*)e_j = 0$. So the claim is proved.

It remains to estimate the first term. Since the summands are orthogonal, the norm of this term is bounded by the maximum of the norms of the summands. Using again $e_j \leq we_{j-1}w^* + we_jw^*$, we obtain

$$\begin{aligned} &\|(e_j \otimes 1)(we_jw^* \otimes a_j + we_{j-1}w^* \otimes a_{j-1})(e_j \otimes 1) - e_j \otimes b_j\| \\ &\quad\leq \quad \|a_{j-1} - a_j\| + \|(e_j \otimes 1)(we_jw^* \otimes a_j + we_{j-1}w^* \otimes a_j)(e_j \otimes 1) - e_j \otimes b_j\| \\ &\quad= \quad \|a_{j-1} - a_j\| + \|e_j \otimes (a_j - b_j)\| \leq \|a_{j-1} - a_j\| + \|a_j - b_j\|. \end{aligned}$$

If $j = 0$, then $j - 1 = N$. We then have also $e_0we_0w^* = 0$, so $e_0 \leq we_Nw^*$, whence

$$\begin{aligned} \|(e_0 \otimes 1)(we_0w^* \otimes a_0 + we_Nw^* \otimes a_N)(e_0 \otimes 1) - e_0 \otimes b_0\| \\ = \|e_0 \otimes (a_N - b_0)\| \leq \|a_N - b_0\|. \end{aligned}$$

This proves the lemma. □

Proof of Theorem 2.3.7: That asymptotic unitary equivalence implies homotopy is Lemma 1.3.3 (1). We therefore prove the reverse implication.

Using Lemma 2.3.5, we may without loss of generality assume our homotopy has the form $\widetilde{\varphi}_t^{(\alpha)}(a) = \delta(1_{\mathcal{O}_\infty} \otimes \varphi_t^{(\alpha)}(a))$, where $\delta : \mathcal{O}_\infty \otimes K \otimes D \to K \otimes D$ is a homomorphism and φ is a completely positive contractive asymptotic morphism from A to $C([0,1], K \otimes D)$. It then suffices to prove the theorem for the homotopy of asymptotic morphisms from A to $\mathcal{O}_\infty \otimes K \otimes D$ given by $\overline{\varphi}_t^{(\alpha)}(a) = 1 \otimes \varphi_t^{(\alpha)}(a)$. (We get an asymptotic unitary equivalence of $\widetilde{\varphi}^{(0)}$ and $\widetilde{\varphi}^{(1)}$ by applying δ.)

The next step is to do some constructions in $\mathcal{O}_\infty$ and $\mathcal{O}_2$. Choose a projection $e \in \mathcal{O}_\infty$ with $e \neq 1$ and $[e] = [1]$ in $K_0(\mathcal{O}_\infty)$. Choose a unital homomorphism $\gamma : \mathcal{O}_2 \to (1-e)\mathcal{O}_\infty(1-e)$. Define isometries $\widetilde{s_j} \in \mathcal{O}_\infty$ by $\widetilde{s}_j = \gamma(s_j)$. Let $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ be the standard shift $\lambda(c) = s_1cs_1^* + s_2cs_2^*$. Since any two unital endomorphisms of $\mathcal{O}_2$ are homotopic (by Remark 2.1.2 (1) and the connectedness of the unitary group of $\mathcal{O}_2$), there is a homotopy $\alpha \mapsto \omega_\alpha$ of endomorphisms of $\mathcal{O}_2$ with $\omega_0 = \mathrm{id}_{\mathcal{O}_2}$ and $\omega_1 = \lambda$.

We will now suppose that we are given continuous functions

$$\alpha_n : [n-1, \infty) \to [0,1]$$

for $n \geq 1$ such that

$$\alpha_{n+1}(n) = \alpha_n(n) \tag{1}$$

for all n, and a continuous function $F : [0, \infty) \to (0, \infty)$. (These will be chosen below.) Then we define $\psi_t : \mathcal{O}_2 \otimes A \to \mathcal{O}_\infty \otimes K \otimes D$ by

$$\psi_t(c \otimes a) = \sum_{k=1}^{n} \widetilde{s}_2^{k-1}\widetilde{s}_1\gamma(c)(\widetilde{s}_2^{k-1}\widetilde{s}_1)^* \otimes \varphi_t^{(\alpha_k \circ F(t))}(a)$$
$$+ \widetilde{s}_2^n\gamma(\omega_{F(t)-n}(c))(\widetilde{s}_2^n)^* \otimes \varphi_t^{(\alpha_{n+1} \circ F(t))}(a) \quad \text{for} \quad F(t) \in [n, n+1]. \qquad (2)$$

(This is an orthogonal sum since the projections

$$\widetilde{s}_1\widetilde{s}_1^*, \ \widetilde{s}_2\widetilde{s}_1(\widetilde{s}_2\widetilde{s}_1)^*, \ldots, \ \widetilde{s}_2^{n-1}\widetilde{s}_1(\widetilde{s}_2^{n-1}\widetilde{s}_1)^*, \ \widetilde{s}_2^n(\widetilde{s}_2^n)^*$$

are mutually orthogonal.) As in the proof of Lemma 2.3.5, each ψ_t is well defined, linear, and contractive, and $t \mapsto \psi_t(b)$ is continuous for b in the algebraic tensor product of $\mathcal{O}_2$ and A (using (1) when $F(t) \in \mathbf{N}$), and so for all $b \in \mathcal{O}_2 \otimes A$.

We now claim that ψ, as defined by (2), is actually an asymptotic morphism from $\mathcal{O}_2 \otimes A$ to $\mathcal{O}_\infty \otimes K \otimes D$. It only remains to prove asymptotic multiplicativity. By linearity and finiteness of $\sup_t \|\psi_t\|$, it suffices to do this on elementary tensors. Since γ, ω_α, and the maps $c \mapsto \widetilde{s}_2^{k-1}\widetilde{s}_1 c(\widetilde{s}_2^{k-1}\widetilde{s}_1)^*$ and $c \mapsto \widetilde{s}_2^n c(\widetilde{s}_2^n)^*$ are homomorphisms (and so contractive), a calculation gives, for $F(t) \in [n, n+1]$,

$$\lim_{t\to\infty} \|\psi_t((c_1 \otimes a_1)(c_2 \otimes a_2)) - \psi_t(c_1 \otimes a_1)\psi_t(c_2 \otimes a_2)\|$$
$$\leq \lim_{t\to\infty} \|c_1c_2\| \left(\sup_{\alpha\in[0,1]} \left\| \varphi_t^{(\alpha)}(a_1a_2) - \varphi_t^{(\alpha)}(a_1)\varphi_t^{(\alpha)}(a_2) \right\| \right) = 0.$$

Define $\iota : A \to \mathcal{O}_2 \otimes A$ by $\iota(a) = 1 \otimes a$. Then $\psi \circ \iota$ is an asymptotic morphism from A to $\mathcal{O}_\infty \otimes K \otimes D$. By definition, it has a trivializing factorization, so Lemma 2.3.6 implies that $\overline{\varphi}^{(\alpha)} \oplus (\psi \circ \iota)$ is asymptotically unitarily equivalent to $\overline{\varphi}^{(\alpha)}$. The theorem will therefore be proved if we can choose the functions F and α_n in such a way that $\overline{\varphi}^{(0)} \oplus (\psi \circ \iota)$ is asymptotically unitarily equivalent to $\overline{\varphi}^{(1)} \oplus (\psi \circ \iota)$.

Before actually choosing F and the α_n, we construct, in terms of F, the unitary path we will use for the desired asymptotic unitary equivalence. Let τ be an automorphism of $M_2(\mathcal{O}_\infty)$ which sends $1 \oplus 0$ to $e \oplus 0$ and $0 \oplus c$ to $c \oplus 0$ for all $c \in (1-e)\mathcal{O}_\infty(1-e)$. Let $\widetilde{\tau}$ be the obvious induced automorphism of $M_2(\mathcal{O}_\infty \otimes K \otimes D)$. It suffices to prove asymptotic unitary equivalence of $\widetilde{\tau} \circ (\overline{\varphi}^{(0)} \oplus (\psi \circ \iota))$ and $\widetilde{\tau} \circ (\overline{\varphi}^{(1)} \oplus (\psi \circ \iota))$. Furthermore, these two asymptotic morphisms take values in $\mathcal{O}_\infty \otimes K \otimes D$, embedded as the upper left corner, so we only work there. This results in the identification

$$\widetilde{\tau} \circ (\overline{\varphi}^{(\alpha)} \oplus (\psi \circ \iota)) = e \otimes \varphi^{(\alpha)}(-) + (\psi \circ \iota).$$

We further note that, by Lemma 1.3.9, it suffices to construct a continuous family of unitaries in the multiplier algebra $M(\mathcal{O}_\infty \otimes K \otimes D)$.

With these identifications and reductions, our unitary path will take the form $u_t = v(F(t)) \otimes 1$ for a suitable unitary path $r \mapsto v(r)$ in $\mathcal{O}_\infty$, defined for $r \in [0, \infty)$. The construction of v requires further notation.

Define projections in $\mathcal{O}_\infty$ by $p_k = \widetilde{s}_2^{k-1} \widetilde{s}_1 (\widetilde{s}_2^{k-1} \widetilde{s}_1)^*$ and $q_n = \widetilde{s}_2^n (\widetilde{s}_2^n)^*$. Then

$$p_1 + p_2 + \cdots + p_n + q_n + e = 1 \quad \text{and} \quad p_{n+1} + q_{n+1} = q_n.$$

Choose projections $f_k < p_k$ with $[f_k] = 1$ in $K_0(\mathcal{O}_\infty)$. Note that $f_{n+1} < q_n$. Then there are partial isometries v_k with

$$v_0^* v_0 = e, \quad v_0 v_0^* = f_1, \quad v_k^* v_k = f_k, \quad \text{and} \quad v_k v_k^* = f_{k+1},$$

and w_n with

$$w_n^* w_n = f_{n+1} \quad \text{and} \quad w_n w_n^* = e.$$

Using the connectedness of the unitary group of $(f_{n+1}+f_{n+2})\mathcal{O}_\infty(f_{n+1}+f_{n+2})$, choose a continuous path $\alpha \mapsto y_n(\alpha)$ of partial isometries from $f_{n+1} + f_{n+2}$ to $f_{n+2} + e$ such that $y_n(0) = w_n + f_{n+2}$ and $y_n(1) = v_{n+1} + w_{n+1}$. Then define

$$v(n+\alpha) = (p_1 - f_1) + \cdots + (p_{n+1} - f_{n+1}) + (q_{n+1} - f_{n+2}) + v_0 + v_1 + \cdots + v_n + y_n(\alpha)$$

for $n \in \mathbf{N}$ and $\alpha \in [0, 1]$. There are two definitions at each integer, but they agree, so v is a continuous path of unitaries. Furthermore, one immediately verifies that for fixed $r \in [n, n+1]$, the unitary $w = v(r)$ and sequence of projections

$$e_0 = e, \ e_1 = p_1, \ e_2 = p_2, \ldots, e_n = p_n, \ e_{n+1} = q_n \tag{3}$$

satisfy the hypotheses in Lemma 2.3.9.

Now take f to be as in Lemma 2.3.8, and set $F(t) = f(t) + 2$. Define $\alpha_0 : [0, \infty) \to [0, 1]$ by $\alpha_0(r) = 0$ for all r, and choose the functions $\alpha_n : [n-1, \infty) \to [0, 1]$ to be continuous, to satisfy (1), and such that $\alpha_{n+1}(r) = 1$ for $r \in [n, n+1]$ and

$$|\alpha_{k+1}(r) - \alpha_k(r)| \le 1/(n-1) \quad \text{for } r \in [n, n+1] \text{ and } 0 \le k \le n.$$

Take ψ and u to be defined using these choices of F and the α_n. Let $t \in [0, \infty)$. Set $r = F(t)$ and choose $n \in \mathbf{N}$ such that $r \in [n, n+1]$. Let $w = v(r)$ and let $e_0, \ldots, e_{n+1}$ be as in (3). For $a \in A$, we then have

$$\begin{aligned} &\left\| u_t \left[e \otimes \varphi^{(0)}(a) + \psi_t(1 \otimes a) \right] u_t^* - \left[e \otimes \varphi^{(1)}(a) + \psi_t(1 \otimes a) \right] \right\| \\ &= \Bigg\| (w \otimes 1) \left[\sum_{k=0}^{n+1} e_k \otimes \varphi_t^{(\alpha_k(r))}(a) \right] (w \otimes 1)^* \\ &\qquad - \left[e_0 \otimes \varphi_t^{(1)}(a) + \sum_{k=1}^{n+1} e_k \otimes \varphi_t^{(\alpha_k(r))}(a) \right] \Bigg\|. \end{aligned}$$

Apply Lemma 2.3.9 with $a_k = b_k = \varphi_t^{(\alpha_k(r))}(a)$ for $1 \leq k \leq n+1$, and with $a_0 = \varphi_t^{(\alpha_0(r))}(a)$ and $b_0 = \varphi_t^{(1)}(a) = \varphi_t^{(\alpha_{n+1}(r))}(a) = a_{n+1}$. It follows that the expression above is at most

$$\begin{aligned} &\max(0, \|a_0 - a_1\|, \ldots, \|a_n - a_{n+1}\|) \\ &= \max\{\|\varphi_t^{(\alpha_k(r))}(a) - \varphi_t^{(\alpha_{k+1}(r))}(a)\| : 0 \leq k \leq n\} \\ &\leq \sup\{\|\varphi_t^{(\alpha_1)} - \varphi_t^{(\alpha_2)}\| : |\alpha_1 - \alpha_2| \leq 1/(n-1)\}. \end{aligned}$$

Since $n-1 \geq r-2 = f(t)$, we have $1/(n-1) \leq 1/f(t)$, and this last expression converges to 0 as $t \to \infty$. Thus we have shown that

$$e \otimes \varphi^{(0)}(-) + (\psi \circ \iota) \quad \text{and} \quad e \otimes \varphi^{(1)}(-) + (\psi \circ \iota)$$

are asymptotically unitarily equivalent. This completes the proof. ◻

2.3.10 Corollary. Let A be separable, nuclear, unital, and simple, let D_0 be unital, and let $D = \mathcal{O}_\infty \otimes D_0$. Then any full asymptotic morphism $\varphi : A \to K \otimes D$ is asymptotically unitarily equivalent to a homomorphism.

Proof: It is obvious that an asymptotic morphism is homotopic to all of its reparametrizations. The result therefore follows from Theorem 2.3.7 and Proposition 1.3.7. ◻

2.3.11 Remark. The hypothesis of fullness can be removed in Theorem 2.3.7 (and in Corollary 2.3.10) in the following way. Let $\alpha \mapsto \varphi^{(\alpha)}$ be a homotopy of asymptotic morphisms from A to $K \otimes D$, with $D = \mathcal{O}_\infty \otimes D_0$. Applying Lemma 1.3.8, we can assume $\alpha \mapsto \varphi^{(\alpha)}$ is a homotopy of unital (hence full) asymptotic morphisms from A to $D' = p(K \otimes D)p$ for a suitable projection p. The algebra D' is stable under tensoring with $\mathcal{O}_\infty$ by Corollary 2.1.12. So we can apply the result already proved to asymptotic morphisms from A to $K \otimes D'$. Then embed $K \otimes D'$ in $K \otimes D$.

3 Unsuspended E-theory for simple nuclear C^*-algebras

In [16], Dădărlat and Loring proved that for certain C^*-algebras A, one can obtain the groups $KK^0(A, B)$ via "unsuspended E-theory": $KK^0(A, B) \cong [[K \otimes A, K \otimes B]]$ (notation from Definition 1.1.2) for all separable B. The terminology comes from the omission of the suspension that is normally required. The conditions on A are quite restrictive, and in particular fail for trivial reasons as soon as A has even one nonzero projection.

In this section, we want to take A to be separable, nuclear, unital, and simple. To make enough room, we assume B is a tensor product $\mathcal{O}_\infty \otimes D$ with D unital. We then discard the class of the zero asymptotic morphism (the source of the difficulty with projections). We are able to prove, with the help of Kirchberg's results as stated in Section 2.1 and also using Theorem 2.3.7, that we do in fact get $KK^0(A,B)$ as a set of suitable homotopy classes of asymptotic morphisms from $K \otimes A$ to $K \otimes B$. (Corollary 2.3.10 implies that we can even use asymptotic unitary equivalence classes of homomorphisms. See Section 4.1.)

In the first subsection, we construct for fixed A a middle exact homotopy invariant functor from separable C^*-algebras to abelian groups in a manner analogous to the definition of $K_0(D)$, but using asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes D^+$ in place of projections in $K \otimes D^+$. The fact that the target algebra is infinite means that, as for K_0 of a purely infinite simple C^*-algebra, we do not need to take formal differences of classes. We do, however, need to introduce the unitization of the target algebra for essentially the same reason that it is necessary in the definition of K_0. In the second subsection, we then show that this functor is naturally isomorphic to $KK^0(A,-)$.

3.1 The groups $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ and $\widetilde{E}_A(D)$

Let A be separable, nuclear, unital, and simple. In this subsection we construct a functor $[[A, K \otimes \mathcal{O}_\infty \otimes -]]_+$ on unital C^*-algebras and the corresponding functor $\widetilde{E}_A(-)$ on general C^*-algebras (obtained via the unitization). We then prove that $\widetilde{E}_A$ is a cohomology theory on separable C^*-algebras in the usual sense. This information is needed in order to apply the uniqueness theorems for KK-theory in the next subsection.

3.1.1 Definition. Let A be separable and unital, and assume each ideal of A is generated by its projections. Let B have an approximate identity of projections. Then $[[A,B]]_+$ denotes the set of homotopy classes of full asymptotic morphisms from A to B.

3.1.2 Proposition. Let A be simple, separable, unital, and nuclear. For any unital C^*-algebra D, give $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ the addition operation that it receives from being a subset of $[[A, K \otimes \mathcal{O}_\infty \otimes D]]$. Then $[[A, K \otimes \mathcal{O}_\infty \otimes -]]_+$ is a functor from separable unital C^*-algebras and homotopy classes of unital asymptotic morphisms to abelian groups. The zero element is the class of any full asymptotic morphism from A to $K \otimes \mathcal{O}_\infty \otimes D$ with a standard factorization (see Definition 2.3.1) through $\mathcal{O}_2 \otimes A$.

Proof: Lemma 1.2.6 (2) shows that $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ is closed under the addition in $[[A, K \otimes \mathcal{O}_\infty \otimes D]]$. Therefore $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ is an abelian semigroup, provided it is not empty.

According to Theorem 2.3.7, homotopy is the same relation as asymptotic unitary equivalence in this set. So we can use them interchangeably.

For functoriality, let E be another unital C^*-algebra, and let $\varphi : D \to E$ be a unital asymptotic morphism. Let $\overline{\varphi} = \mathrm{id}_{K \otimes \mathcal{O}_\infty} \otimes \varphi$ (see Proposition 1.1.8) be the induced asymptotic morphism from $K \otimes \mathcal{O}_\infty \otimes D$ to $K \otimes \mathcal{O}_\infty \otimes E$. It is full because if $e \in K$ is any nonzero projection, then $e \otimes 1 \otimes 1$ is a full projection in $K \otimes \mathcal{O}_\infty \otimes D$ which is sent to the full projection $e \otimes 1 \otimes 1$ in $K \otimes \mathcal{O}_\infty \otimes E$. Lemmas 1.2.6 (2) and 2.1.8 (1) now imply that $\eta \mapsto [[\varphi]] \cdot \eta$ sends full asymptotic morphisms to full asymptotic morphisms.

We now construct an identity element. Theorem 2.1.4 provides an isomorphism $\nu : \mathcal{O}_2 \otimes A \to \mathcal{O}_2$. Let $\tau : \mathcal{O}_2 \to \mathcal{O}_\infty$ be an injective homomorphism (sending 1 to a nonzero projection in $\mathcal{O}_\infty$ with trivial K_0-class), and define a full homomorphism $\zeta : A \to \mathcal{O}_\infty$ by $\zeta(a) = (\tau \circ \nu)(1 \otimes a)$. Composing it with the full homomorphism $x \mapsto e \otimes x \otimes 1$ from $\mathcal{O}_\infty$ to $K \otimes \mathcal{O}_\infty \otimes D$, where $e \in K$ is any nonzero projection, we obtain a full asymptotic morphism from A to $K \otimes \mathcal{O}_\infty \otimes D$ which has a standard factorization through $\mathcal{O}_2 \otimes A$.

Lemma 2.3.3 implies that any other full asymptotic morphism with a trivializing factorization is asymptotically unitarily equivalent to ζ. This class acts as the identity by Lemma 2.3.6.

Finally, we must construct additive inverses. Let $\eta \in [[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$. By Lemma 2.3.5, we can take $\eta = [[\varphi]]$, where φ has a standard factorization through $\mathcal{O}_\infty \otimes A$, say $\varphi_t(a) = \psi_t(1 \otimes a)$ for some asymptotic morphism $\psi : \mathcal{O}_\infty \otimes A \to K \otimes \mathcal{O}_\infty \otimes D$. Choose a projection $f \in \mathcal{O}_\infty$ with $[f] = -1$ in $K_0(\mathcal{O}_\infty)$. Define $\overline{\psi}_t = \psi_t|_{f\mathcal{O}_\infty f \otimes A}$, and define $\overline{\varphi} : A \to K \otimes \mathcal{O}_\infty \otimes D$ by $\overline{\varphi}_t(a) = \varphi_t(f \otimes a)$. Choose a unital homomorphism

$$\nu : \mathcal{O}_2 \to \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix} M_2(\mathcal{O}_\infty) \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}.$$

Then $(\mathrm{id}_{M_2} \otimes \psi) \circ \nu$ provides a standard factorization of $\varphi \oplus \overline{\varphi}$ through $\mathcal{O}_2 \otimes A$. Note that $\varphi \oplus \overline{\varphi}$ is full because φ is, so it is asymptotically unitarily equivalent to ζ by Lemma 2.3.3. This shows that $[[\overline{\varphi}]]$ is the inverse of η. ◻

3.1.3 Definition. If D is any C^*-algebra, then we denote by $D^\#$ the C^*-algebra $K \otimes \mathcal{O}_\infty \otimes D^+$. We use the analogous notation for homomorphisms. If D is separable, we define $\widetilde{E}_A(D)$ to be the kernel of the map $[[A, D^\#]]_+ \to [[A, K \otimes \mathcal{O}_\infty]]_+$ induced by the unitization map $D^+ \to \mathbf{C}$.

3.1.4 Proposition. Let A be separable, nuclear, unital, and simple. Then $\widetilde{E}_A$ is a functor from separable C^*-algebras and homotopy classes of asymptotic morphisms to abelian groups.

Proof: This follows from Proposition 3.1.2 and the fact that unitizations and tensor products of asymptotic morphisms are well defined (Lemma 1.1.6 and Proposition 1.1.8). ◻

3.1.5 Remark. It is obvious that if D_1 and D_2 are unital, then there is a natural isomorphism

$$[[A, K \otimes \mathcal{O}_\infty \otimes (D_1 \oplus D_2)]]_+ \cong [[A, K \otimes \mathcal{O}_\infty \otimes D_1]]_+ \oplus [[A, K \otimes \mathcal{O}_\infty \otimes D_2)]]_+.$$

It follows that for unital D, there is a natural isomorphism

$$\widetilde{E}_A(D) \cong [[A, K \otimes \mathcal{O}_\infty \otimes D]]_+.$$

We will sometimes denote by φ_* the map $[[A, D_1]]_+ \to [[A, D_2]]_+$ or the map $\widetilde{E}_A(D_1) \to \widetilde{E}_A(D_2)$ induced by a (full) homomorphism $\varphi : D_1 \to D_2$.

3.1.6 Lemma. Let A be separable, nuclear, unital, and simple. Let

$$0 \longrightarrow J \xrightarrow{\mu} D \xrightarrow{\pi} D/J \longrightarrow 0$$

be a short exact sequence of separable C^*-algebras. Then the sequence

$$\widetilde{E}_A(J) \xrightarrow{\mu_*} \widetilde{E}_A(D) \xrightarrow{\pi_*} \widetilde{E}_A(D/J)$$

is exact in the middle.

Proof: It is immediate that $\pi_* \circ \mu_* = 0$.

For the other half, we introduce the maps $\chi_D : D^\# \to K \otimes \mathcal{O}_\infty$ and $\iota_D : K \otimes \mathcal{O}_\infty \to D^\#$ associated with the unitization maps $D^+ \to \mathbf{C}$ and $\mathbf{C} \to D^+$. Define $\chi_{D/J}$, $\iota_{D/J}$, etc. similarly. Now let $\eta \in \ker(\pi_*)$, and choose a full asymptotic morphism $\varphi : A \to D^\#$ whose class is η. By definition, we have $[[\pi^\# \circ \varphi]] = 0$ in $[[A, (D/J)^\#]]_+$. Choose a full homomorphism $\zeta : A \to K \otimes \mathcal{O}_\infty$ with a standard factorization through $\mathcal{O}_2 \otimes A$, as in the proof of Proposition 3.1.2. Theorem 2.3.7 then implies that $\pi^\# \circ \varphi$ is asymptotically unitarily equivalent to $\iota_{D/J} \circ \zeta$, so there is a unitary path $t \to u_t$ in $((D/J)^\#)^+$ such that $u_t(\pi^\# \circ \varphi_t)(a)u_t^* \to (\iota_{D/J} \circ \zeta)(a)$ for all $a \in A$.

Without changing homotopy classes, we may replace φ by $\varphi \oplus 0$ and ζ by $\zeta \oplus 0$. This also replaces $\pi^\# \circ \varphi$ and $\iota_{D/J} \circ \zeta$ by their direct sums with the zero asymptotic morphism. We then replace u_t by $u_t \oplus u_t^*$. We may thus assume without loss of generality that u is in the identity component of the unitary group of $C_{\mathrm{b}}([0, \infty), ((D/J)^\#)^+)$. Therefore there is $v \in U_0(C_{\mathrm{b}}([0, \infty), (D^\#)^+))$ whose image is u. Then $\pi^\#(v_t) = u_t$ for all t, whence

$$\lim_{t \to 0} \pi^\#(v_t \varphi_t(a) v_t^* - (\iota_D \circ \zeta)(a)) = 0$$

for all $a \in A$.

Let $\sigma : (D/J)^\# \to D^\#$ be a continuous (nonlinear) cross section for $\pi^\#$ satisfying $\sigma(0) = 0$. (See [1].) Define $\psi_t : A \to D^\#$ by

$$\psi_t(a) = v_t \varphi_t(a) v_t^* - (\sigma \circ \pi^\#) \left(v_t \varphi_t(a) v_t^* - (\iota_D \circ \zeta)(a) \right).$$

This yields an asymptotic morphism asymptotically equal to $t \mapsto v_t \varphi_t(-) v_t^*$, and hence asymptotically unitarily equivalent to φ. Furthermore, $\pi^\#(\psi_t(a) - (\iota_D \circ \zeta)(a)) = 0$ for all t and a. It follows that $\psi_t(a) \in J^\#$ and that $\chi_J(\psi_t(a)) = \zeta(a)$. So ψ is in fact a full asymptotic morphism from A to $J^\#$ such that $[[\chi_J \circ \psi]] = 0$, from which it follows that ψ defines a class $[[\psi]] \in \widetilde{E}_A(J)$. Clearly $\mu_*([[\psi]]) = \eta$. This shows that $\ker(\pi_*) \subset \mathrm{Im}(\mu_*)$. □

3.1.7 Corollary. Let A be separable, nuclear, unital, and simple. Let

$$0 \longrightarrow J \xrightarrow{\mu} D \xrightarrow{\pi} D/J \longrightarrow 0$$

be a split short exact sequence of separable C^*-algebras. Then there is a natural split exact sequence

$$0 \longrightarrow \widetilde{E}_A(J) \xrightarrow{\mu_*} \widetilde{E}_A(D) \xrightarrow{\pi_*} \widetilde{E}_A(D/J) \longrightarrow 0.$$

Proof: This is Proposition 4.1 (b) of [15], noting that the proof of Part (b) there doesn't use stability. Indeed, since $\widetilde{E}_A$ is middle exact (the previous lemma) and homotopy invariant, Lemma 5 in Section 7 of [26] provides a long exact sequence

$$\cdots \xrightarrow{(S\mu)_*} \widetilde{E}_A(SD) \xrightarrow{(S\pi)_*} \widetilde{E}_A(S(D/J)) \longrightarrow \widetilde{E}_A(J) \xrightarrow{\mu_*} \widetilde{E}_A(D) \xrightarrow{\pi_*} \widetilde{E}_A(D/J).$$

The desired conclusion can now be immediately obtained using the splitting map. $\square$

3.1.8 Remark. It should be pointed out that we need much less than the full strength of Theorem 2.3.7 here. Only knowing that homotopy implies asymptotic unitary equivalence for full asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes C([0,1])$, it is possible to prove middle exactness in the first stage of the Puppe sequence, namely

$$\widetilde{E}_A(C\pi) \longrightarrow \widetilde{E}_A(D) \longrightarrow \widetilde{E}_A(D/J).$$

This sequence can be extended to the left as in the proof of Proposition 2.6 of [56]. Proposition 3.2 of [16] can then be used to show that $\widetilde{E}_A$ is split exact.

We now prove stability of $\widetilde{E}_A$ under formation of tensor products with both K and $\mathcal{O}_\infty$.

3.1.9 Lemma. Let A be separable, nuclear, unital, and simple, and let D be a separable C^*-algebra. Then the map $d \mapsto 1 \otimes d$, from D to $\mathcal{O}_\infty \otimes D$, induces an isomorphism $\widetilde{E}_A(D) \to \widetilde{E}_A(\mathcal{O}_\infty \otimes D)$.

Proof: By naturality, Proposition 3.1.7, and the Five Lemma, it suffices to prove this for unital D. By Remark 3.1.5, we have to prove that $d \mapsto 1 \otimes d$ induces an isomorphism $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+ \to [[A, K \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D]]_+$. This follows from Theorem 2.1.5 and Proposition 2.1.11, since these results imply that the map $x \mapsto x \otimes 1$, from $\mathcal{O}_\infty$ to $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$, is homotopic to an isomorphism. $\square$

The other stability result requires the following lemma. We really want an increasing continuously parametrized approximate identity of projections, but of course such a thing does not exist. The quasiincreasing version in the lemma is good enough.

3.1.10 Lemma. Let D be a unital purely infinite simple C^*-algebra, and let $e_0 \in K \otimes D$ be a nonzero projection. Then there exists a continuous family

$t \mapsto e_t$ of projections in $K \otimes D$ such that, for every $b \in K \otimes D$, we have

$$\lim_{t\to\infty}(e_t b - b) = \lim_{t\to\infty}(b e_t - b) = \lim_{t\to\infty}(e_t b e_t - b) = 0,$$

such that e_0 is the given projection, and such that $e_s \geq e_t$ for $s \geq t+1$.

Proof: Choose a nonzero projection $p \in K \otimes D$ such that $[p] = 0$ in $K_0(D)$. We start by constructing a family $t \mapsto f_t$ in $K \otimes pDp$. Note that

$$[\mathrm{diag}(1_{pDp}, 0, 0)] = [\mathrm{diag}(1_{pDp}, 1_{pDp}, 0)] = 0$$

in $K_0(M_3(pDp))$. Therefore there is a homotopy $t \mapsto q_t$ of projections in $M_3(pDp)$ such that

$$q_0 = \mathrm{diag}(1, 0, 0) \quad \text{and} \quad q_1 = \mathrm{diag}(1, 1, 0).$$

Now define

$$f_{n+s} = 1_{M_{n+1}(pDp)} \oplus q_s \oplus 0 \in K \otimes pDp$$

for $n = 0, 1, \ldots$ and $s \in [0, 1]$. The family f_t is clearly continuous. It satisfies $f_0 = p \oplus p$. We have $f_t \geq 1_{M_{n+1}(pDp)}$ for $t \geq n$, so $t \mapsto f_t$ really is an approximate identity. Finally, $f_t \leq 1_{M_{n+3}(pDp)}$ for $t \leq n$, so $f_s \geq f_t$ for $s \geq t+4$. We can replace 4 by 1 in this last statement by a reparametrization.

To get the general case, choose a projection $r \in pDp$ with $[r] = -[e_0]$ in $K_0(D)$. Then $f_t \geq p \geq r$ for all t, so $t \mapsto f_t - r$ is a continuously parametrized approximate identity of projections for $(1-r)(K \otimes pDp)(1-r)$. (Here 1 is the identity of $(K \otimes pDp)^+$.) There is an isomorphism

$$\varphi : K \otimes D \to (1-r)(K \otimes pDp)(1-r),$$

and since $[f_0 - r] = [e_0]$ in $K_0(D)$, we can require that $\varphi(e_0) = f_0 - r$. Now set $e_t = \varphi^{-1}(f_t - r)$. Then clearly $e_t b - b$, $b e_t - b \to 0$ as $t \to \infty$. It follows that

$$\|e_t b e_t - b\| \leq \|e_t b - b\| \, \|e_t\| + \|b e_t - b\| \to 0$$

as well. □

3.1.11 Lemma. Let A be separable, nuclear, unital, and simple, let D be separable, and let $e \in K$ be a rank one projection. Then the map $d \mapsto e \otimes d$, from D to $K \otimes D$, induces an isomorphism $\widetilde{E}_A(D) \to \widetilde{E}_A(K \otimes D)$.

Proof: By Lemma 3.1.9, we may use $\mathcal{O}_\infty \otimes D$ in place of D, and as in its proof we may assume D is unital.

Let $s \in \mathcal{O}_\infty$ be a proper isometry, and define $\gamma : \mathcal{O}_\infty \otimes D \to \mathcal{O}_\infty \otimes D$ by $\gamma(a) = (s \otimes 1)a(s \otimes 1)^*$. We claim that $\gamma_* : \widetilde{E}_A(\mathcal{O}_\infty \otimes D) \to \widetilde{E}_A(\mathcal{O}_\infty \otimes D)$ is an isomorphism. It follows from Remark 3.1.5 and Definition 3.1.3 that this map can be thought of as composition with $\mathrm{id}_{K \otimes \mathcal{O}_\infty} \otimes \gamma$ from $[[A, K \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D]]_+$ to itself, even though γ is not unital. (The discrepancy is an orthogonal sum with an asymptotic morphism which up to homotopy has a trivializing factorization. Note that the composition with γ is still full.) Now $K \otimes ss^* \mathcal{O}_\infty ss^*$ and $K \otimes (1 - ss^*)\mathcal{O}_\infty(1 - ss^*)$ are both isomorphic to $K \otimes \mathcal{O}_\infty$, so we may as well consider the map from $[[A, K \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D]]_+$ to $[[A, M_2(K \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D)]]_+$ induced by inclusion in the upper right corner. Let $\tau : M_2(K) \to K$ be an isomorphism. Then $a \mapsto \tau(a \oplus 0)$ is homotopic to id_K and $b \mapsto \tau(b) \oplus 0$ is homotopic to $\mathrm{id}_{M_2(K)}$. So our map has an inverse given by composition with $\tau \otimes \mathrm{id}_{\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D}$.

We next require a construction involving $\mathcal{O}_\infty$ and $K \otimes \mathcal{O}_\infty$. Define $\varphi : \mathcal{O}_\infty \to K \otimes \mathcal{O}_\infty$ by $\varphi(x) = e \otimes x$. Let $t \mapsto e_t$ be a continuously parametrized approximate identity for $K \otimes \mathcal{O}_\infty$ which satisfies the properties of the previous lemma and has $e_0 = e \otimes 1$. Let $t \mapsto u_t$ be a continuous family of unitaries in $(K \otimes \mathcal{O}_\infty)^+$ such that $u_0 = 1$ and $u_t e_t u_t^* = e_0$ for all t. Define $\psi_t^{(0)} : K \otimes \mathcal{O}_\infty \to K \otimes \mathcal{O}_\infty$ by $\psi_t^{(0)}(a) = u_t e_t a e_t u_t^*$. One immediately checks that $\psi^{(0)}$ is an asymptotic morphism whose values are in $(e \otimes 1)(K \otimes \mathcal{O}_\infty)(e \otimes 1)$, so that there is an asymptotic morphism $t \mapsto \psi_t$ from $K \otimes \mathcal{O}_\infty$ to $\mathcal{O}_\infty$ such that $\varphi \circ \psi_t = \psi_t^{(0)}$ for all t.

The composite asymptotic morphisms $\varphi \circ \psi$ and $\psi \circ \varphi$ can be computed without reparametrization, because φ is a homomorphism. Now $\varphi \circ \psi = \psi^{(0)}$, which is asymptotically unitarily equivalent to $(t, a) \mapsto e_t a e_t$, which in turn is asymptotically equal to $\mathrm{id}_{K \otimes \mathcal{O}_\infty}$. So $\varphi \circ \psi$ is homotopic to $\mathrm{id}_{K \otimes \mathcal{O}_\infty}$. Also, $\psi \circ \varphi$ is clearly homotopic to a map of the form $x \mapsto sxs^*$ for a proper isometry $s \in \mathcal{O}_\infty$.

We now observe that $\mathrm{id}_{K \otimes \mathcal{O}_\infty} \otimes (\varphi \otimes \mathrm{id}_D)^+$ and $\mathrm{id}_{K \otimes \mathcal{O}_\infty} \otimes (\psi \otimes \mathrm{id}_D)^+$ define full asymptotic morphisms from $(\mathcal{O}_\infty \otimes D)^\#$ to $(K \otimes \mathcal{O}_\infty \otimes D)^\#$ and back. The composite from $(K \otimes \mathcal{O}_\infty \otimes D)^\#$ to itself is homotopic to the identity, and therefore induces the identity map on $\widetilde{E}_A(K \otimes \mathcal{O}_\infty \otimes D)$. Composition on the right with the composite from $(\mathcal{O}_\infty \otimes D)^\#$ to itself is a map of the form γ_* as considered at the beginning of the proof, and is thus an isomorphism from $\widetilde{E}_A(\mathcal{O}_\infty \otimes D)$ to itself. It follows that φ_* is an isomorphism. ◻

3.2 The isomorphism with KK-theory

In this subsection, we prove that if A is separable, nuclear, unital, and simple, and D is separable, then the natural map from $\widetilde{E}_A(D)$ to $KK^0(A, D)$ is an isomorphism. Combined with Remark 3.1.5, this gives for unital D a form of "unsuspended E-theory" as in [16], in which we need only discard the zero asymptotic morphism.

We will use the universal property of KK-theory with respect to split exact, stable, and homotopy invariant functors on separable C^*-algebras [24]. (We use this instead of the related property of E-theory because it is more convenient for the proof of Lemma 3.2.3 below.)

3.2.1 Notation. In this subsection, we denote by $\mathcal{S}$ the category of separable C^*-algebras and homomorphisms and by $\mathcal{KK}$ the category of separable C^*-algebras with morphisms $KK^0(A,B)$ for C^*-algebras A and B. If $\eta \in KK^0(A,B)$ and $\lambda \in KK^0(B,C)$, we denote their product by $\lambda \times \eta \in KK^0(A,C)$. We further denote by k the functor from $\mathcal{S}$ to $\mathcal{KK}$ which sends a homomorphism to the class it defines in KK-theory.

3.2.2 Lemma. Let A be separable, nuclear, unital, and simple. Then there is a functor $\widehat{E}_A$ from $\mathcal{KK}$ to the category of abelian groups such that $\widehat{E}_A \circ k = \widetilde{E}_A$.

This simply means that one can make sense of $\widetilde{E}_A(\eta) : \widetilde{E}_A(D) \to \widetilde{E}_A(F)$ not only when η is an asymptotic morphism from D to F, but also when η is merely an element of $KK^0(D,F)$.

Proof of Lemma 3.2.2: The result is immediate from Theorem 4.5 of [24], since $\widetilde{E}_A$ is a stable (Lemma 3.1.11), split exact (Corollary 3.1.7), and homotopy invariant (Proposition 3.1.4) functor from separable C^*-algebras to abelian groups. □

We want to show that $\widetilde{E}_A(D)$ is naturally isomorphic to $KK^0(A,D)$. Our argument is based on an alternate proof of the main theorem of [16] suggested by the referee of that paper; we are grateful to Marius Dădărlat for telling us about it. The argument requires the construction of certain natural transformations. (The argument used in Section 4 of [16] presumably also works.)

Before starting the construction, we prove a lemma on the functors $\widehat{F}$ of Higson [24] (as used in the previous lemma).

3.2.3 Lemma. Let F and G be stable, split exact, and homotopy invariant functors from $\mathcal{S}$ to the category of abelian groups, and let $\widehat{F}$ and $\widehat{G}$ be the unique extensions to functors from $\mathcal{KK}$ of Theorem 4.5 of [24]. (In particular, F or G could be $\widetilde{E}_A$ and $\widehat{F}$ or $\widehat{G}$ could be $\widehat{E}_A$, as in Lemma 3.2.2.) If α is a natural transformation from F to G, then α is also a natural transformation from $\widehat{F}$ to $\widehat{G}$.

Proof: Let $\mu \in KK^0(A,B)$. By Lemma 3.6 of [24], we can choose a representative cycle (in the sense of Definition 2.1 of [24]) of the form $\Phi = (\varphi_+, \varphi_-, 1)$, where $\varphi_+, \varphi_- : A \to M(K \otimes B)$ are homomorphisms such that $\varphi_+(a) - \varphi_-(a) \in K \otimes B$ for $a \in A$. The homomorphism $\widehat{F}(\mu)$ is then the composite

$$F(A) \xrightarrow{F(\widehat{\varphi}_+) - F(\widehat{\varphi}_-)} F(A_\Phi) \xrightarrow{F(\pi)} F(K \otimes B) \xrightarrow{F(\varepsilon)^{-1}} F(B),$$

for a certain C^*-algebra A_Φ, certain homomorphisms π, $\widehat{\varphi}_+$, and $\widehat{\varphi}_-$, and with $\varepsilon(a) = 1 \otimes a$. (See Definition 3.4 and the proofs of Theorems 3.7 and 4.5 in [24].) From this expression, it is obvious that naturality with respect to homomorphisms implies naturality with respect to classes in KK-theory. □

3.2.4 Definition. Let A be separable, nuclear, unital, and simple. We regard $KK^0(A, -)$ and $\widehat{E}_A$ as functors from $\mathcal{KK}$ to abelian groups. (On morphisms, the first of these sends $\eta \in KK^0(D_1, D_2)$ to Kasparov product with η.) We now define natural transformations

$$\alpha : KK^0(A, -) \to \widehat{E}_A \quad \text{and} \quad \beta : \widehat{E}_A \to KK^0(A, -).$$

To define α_D, let $e \in K$ be a rank one projection, let $\iota_A : A \to K \otimes \mathcal{O}_\infty \otimes A$ be the map $\iota_A(a) = e \otimes 1 \otimes a$, and let $[[\iota_A]] \in \widetilde{E}_A(A)$ denote its class in $[[A, K \otimes \mathcal{O}_\infty \otimes A]]_+ \cong \widetilde{E}_A(A)$. (Recall that A is unital, so that Remark 3.1.5 applies.) Now let $\eta \in KK^0(A, D)$. Then $\widehat{E}_A(\eta)$ is a homomorphism from $\widetilde{E}_A(A)$ to $\widetilde{E}_A(D)$. Define

$$\alpha_D(\eta) = \widehat{E}_A(\eta)([[\iota_A]]) \in \widetilde{E}_A(D).$$

To define β_D, let $\chi_D : D^\# \to K \otimes \mathcal{O}_\infty$ be the standard map (as in the proof of Lemma 3.1.6). Starting with $\eta \in \widetilde{E}_A(D) \subset [[A, D^\#]]$, choose a full asymptotic morphism $\varphi : A \to D^\#$ with $[[\chi_D]] \cdot [[\varphi]] = 0$ which represents η. Now recall (Corollary 9 (b) of [11]; Section 5 of [54]) that for A and B separable and A (K-)nuclear, there is a canonical isomorphism $E(A, B) \cong KK^0(A, B)$. Further recall that there is a canonical isomorphism $KK^0(SA, SB) \cong KK^0(A, B)$ (Theorem 7 of Section 5 of [26]). Form the second suspension

$$\begin{aligned}[][S^2\varphi]] \in [[S^2 A, S^2 D^\#]] &= E(SA, SD^\#) \\ \cong \quad KK^0(SA, SD^\#) &\cong KK^0(A, \mathcal{O}_\infty \otimes D^+),\end{aligned}$$

and regard $[[S^2\varphi]]$ as an element of $KK^0(A, \mathcal{O}_\infty \otimes D^+)$. Since $[[S^2\chi_D]] \cdot [[S^2\varphi]] = 0$, split exactness of $KK^0(A, -)$ implies that $[[S^2\varphi]]$ is actually in $KK^0(A, \mathcal{O}_\infty \otimes D)$. In this last expression, we can use the KK-equivalence of $\mathcal{O}_\infty$ and $\mathbf{C}$, given by the unital homomorphism $\mathbf{C} \to \mathcal{O}_\infty$, to drop $\mathcal{O}_\infty$. We thus obtain an element $\beta_D(\eta) \in KK^0(A, D)$.

3.2.5 Lemma. The maps α_D and β_D of the previous definition are in fact natural transformations.

Proof: It is easy to check that both α and β are natural with respect to homomorphisms, so naturality with respect to classes in KK-theory follows from Lemma 3.2.3. ◻

3.2.6 Theorem. Let A be separable, nuclear, unital, and simple. Then for every separable D, the maps α_D and β_D of Definition 3.2.4 are mutually inverse isomorphisms.

Proof: It is convenient to prove this first under the assumptions that $\mathcal{O}_\infty \otimes A \cong A$ and $\mathcal{O}_\infty \otimes D \cong D$. It then follows that the map $a \mapsto 1 \otimes a$ from A to $\mathcal{O}_\infty \otimes A$ is homotopic to an isomorphism, and similarly for D. (This is true for $\mathcal{O}_\infty$ by Theorem 2.1.5 and Proposition 2.1.11. Therefore it is true for $\mathcal{O}_\infty \otimes A$ and $\mathcal{O}_\infty \otimes D$, hence for A and D.) Thus, A and $\mathcal{O}_\infty \otimes A$ are naturally homotopy equivalent, and therefore also naturally equivalent in $\mathcal{KK}$ as well.

Similar considerations apply to D. Thus, $\widetilde{E}_A(D)$ becomes just $[[A, K \otimes D]]_+$. The natural transformations above are then given by

$$\alpha_D(\eta) = \widehat{E}_A(\eta)([[\mathrm{id}_A]])$$

(with id_A being the obvious map from A to $K \otimes A$), and

$$\beta_D([[\varphi]]) = [[S^2\varphi]] \in [[S^2 A, K \otimes S^2 D]] \cong KK^0(A, D).$$

Letting 1_A denote the class in $KK^0(A, A)$ of the identity map, we then immediately verify that

$$\alpha_A(1_A) = [[\mathrm{id}_A]] \quad \text{and} \quad \beta_A([[\mathrm{id}_A]]) = 1_A.$$

We now show that these two facts imply the theorem for unital D. Let $\eta \in KK^0(A, D)$. Then $\eta = 1_A \times \eta$, and naturality implies that

$$\beta_D(\alpha_D(1_A \times \eta)) = \beta_D(\widehat{E}_A(\eta)(\alpha_A(1_A))) = \beta_A(\alpha_A(1_A)) \times \eta = 1_A \times \eta.$$

So $\beta_D \circ \alpha_D = \mathrm{id}$. For the other direction, let $\mu \in \widehat{E}_A(D)$. Using Corollary 2.3.10, represent μ as the class of a full homomorphism $\varphi : A \to K \otimes D$. Let $\eta = [[S^2\varphi]]$ be the KK-class determined by $[[\varphi]]$. Then, identifying $K \otimes K$ with K as necessary, we have

$$\mu = \varphi_*([[\mathrm{id}_A]]) = \widehat{E}_A(\eta)([[\mathrm{id}_A]]).$$

The same argument as above now shows that

$$(\alpha_D \circ \beta_D)\left(\widehat{E}_A(\eta)([[\mathrm{id}_A]])\right) = \widehat{E}_A(\eta)([[\mathrm{id}_A]]).$$

So $\alpha_D \circ \beta_D = \mathrm{id}$ also.

The result for nonunital algebras follows from naturality, split exactness, and the Five Lemma.

To remove the assumption that $\mathcal{O}_\infty \otimes D \cong D$, use Lemma 3.1.9.

Finally, we remove the assumption that $\mathcal{O}_\infty \otimes A \cong A$. Let $\delta_0 : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ be an isomorphism (from Theorem 2.1.5), and let $\delta : \mathcal{O}_\infty \otimes K \otimes \mathcal{O}_\infty \to K \otimes \mathcal{O}_\infty$ be the obvious corresponding map. Define $i_D : \widetilde{E}_A(D) \to \widetilde{E}_{\mathcal{O}_\infty \otimes A}(D)$ by

$$i_D([[\eta]]) = [[\delta \otimes \mathrm{id}_{D^+}]] \cdot [[\mathrm{id}_{\mathcal{O}_\infty} \otimes \eta]].$$

Let $j_D : KK^0(A, D) \to KK^0(\mathcal{O}_\infty \otimes A, D)$ be the isomorphism induced by the KK-equivalence of $\mathbf{C}$ and $\mathcal{O}_\infty$. Both i and j are natural transformations. Using Theorem 2.1.5 and Proposition 2.1.11, we can rewrite $j_{\mathcal{O}_\infty \otimes D}(\mu)$ as $(\delta_0 \otimes \mathrm{id}_D)_*(1_{\mathcal{O}_\infty} \otimes \mu)$. This formula and Remark 3.1.5 imply that $i_D \circ \alpha_D = \alpha_D \circ j_D$ when D is unital and $\mathcal{O}_\infty \otimes D \cong D$. The previous paragraph and the definition of $\widetilde{E}_A(D)$ in terms of $[[A, D^\#]]_+$ now imply that $i_D \circ \alpha_D = \alpha_D \circ j_D$ for all D. A related argument shows that also $j_D \circ \beta_D = \beta_D \circ i_D$ for all D.

It now suffices to prove that i_D is an isomorphism for all D. By naturality, split exactness, and the Five Lemma, it suffices to do so for unital D. In this case, we have

$$i_D : [[A, K \otimes \mathcal{O}_\infty \otimes D]]_+ \to [[\mathcal{O}_\infty \otimes A, K \otimes \mathcal{O}_\infty \otimes D]]_+$$

given by $i_D([[\eta]]) = [[\delta \otimes \mathrm{id}_D]] \cdot [[\mathrm{id}_{\mathcal{O}_\infty} \otimes \eta]]$. Define a map k_D in the opposite direction by restriction to $1 \otimes A \subset \mathcal{O}_\infty \otimes A$. We prove that $k_D = i_D^{-1}$.

Let $\widetilde{\delta}(x) = \delta(1 \otimes x)$. Proposition 2.1.11 implies that there is a homotopy $\widetilde{\delta} \simeq \mathrm{id}_{K \otimes \mathcal{O}_\infty}$. It is easy to check directly that $k_D \circ i_D$ sends $[[\eta]]$ to $[[(\widetilde{\delta} \otimes \mathrm{id}_D) \circ \eta]]$, so $k_D \circ i_D$ is the identity. For the reverse composition, let τ_A be the inclusion of $A = 1 \otimes A$ in $\mathcal{O}_\infty \otimes A$, and let $\varphi : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ be the flip $\varphi(x \otimes y) = y \otimes x$. Then $\varphi \simeq \mathrm{id}_{\mathcal{O}_\infty \otimes \mathcal{O}_\infty}$ by Proposition 2.1.11 and Theorem 2.1.5. Therefore, for $[[\eta]] \in [[\mathcal{O}_\infty \otimes A, K \otimes \mathcal{O}_\infty \otimes D]]_+$, we have

$$\begin{aligned} & (\delta \otimes \mathrm{id}_D) \circ (\mathrm{id}_{\mathcal{O}_\infty} \otimes (\eta \circ \tau_A)) \\ \simeq \quad & (\delta \otimes \mathrm{id}_D) \circ (\mathrm{id}_{\mathcal{O}_\infty} \otimes \eta) \circ (\varphi \otimes \mathrm{id}_A) \circ (\mathrm{id}_{\mathcal{O}_\infty} \otimes \tau_A) = (\widetilde{\delta} \otimes \mathrm{id}_D) \circ \eta \simeq \eta. \end{aligned}$$

This shows that $i_D \circ k_D$ is the identity. □

3.2.7 Remark. We used Corollary 2.3.10 in this proof because we had it available. It is, however, not necessary for the argument. Using methods similar to, but a bit more complicated than, the proof of Lemma 3.2.3, one can show that if F as there is in fact a functor on homotopy classes of asymptotic morphisms, then $F([[\varphi]])$ is equal to $\widehat{F}$ applied to the KK-theory class given by φ.

3.2.8 Theorem. Let A be a separable unital nuclear simple C^*-algebra. Then for separable unital C^*-algebras D, the set of homotopy classes of full asymptotic morphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$ is naturally isomorphic to $KK^0(A, D)$ via the map sending an asymptotic morphism to the KK-class it determines.

Proof: This follows from Theorem 3.2.6 and Remark 3.1.5. □

4 Theorems on KK-theory and classification

In this section, we present our main results. The first subsection contains the alternate descriptions of KK-theory in terms of homotopy classes and asymptotic unitary equivalence classes of homomorphisms, in case the first variable is separable, nuclear, unital, and simple. We also give here a proof that homotopies of automorphisms of separable nuclear unital purely infinite simple C^*-algebras can in fact be chosen to be isotopies. The second subsection contains the classification theorem and its corollaries. The third subsection contains the nonclassification results.

4.1 Descriptions of KK-theory

Probably the most striking of our descriptions of KK-theory is the following:

4.1.1 Theorem. For a separable unital nuclear simple C^*-algebra A and a separable unital C^*-algebra D, the obvious maps define natural isomorphisms of abelian groups between the following three objects:

(1) The set of asymptotic unitary equivalence classes of full homomorphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$, with the operation given by direct sum (Definition 1.1.3).

(2) The set of homotopy classes of full homomorphisms from A to $K \otimes \mathcal{O}_\infty \otimes D$, with the operation given by direct sum as above.

(3) The group $KK^0(A, D)$.

Proof: For the purposes of this proof, denote the set in (1) by $KU(A, D)$ and the set in (2) by $KH(A, D)$. The map from $KH(A, D)$ to $KK^0(A, D)$ is the one from Theorem 3.2.8. By this theorem, we can use $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ in place of $KK^0(A, D)$.

Lemma 1.3.3 (2) implies that the map from $KU(A, D)$ to $KH(A, D)$ is well defined, and it is then clearly surjective. Injectivity is immediate from Theorem 2.3.7. Thus this map is an isomorphism. Theorem 3.2.8 implies that the map from $KH(A, D)$ to $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ is injective, while Corollary 2.3.10 implies that the map from $KU(A, D)$ to $[[A, K \otimes \mathcal{O}_\infty \otimes D]]_+$ is surjective. Therefore these maps are in fact both isomorphisms. It now follows that $KU(A, D)$ and $KH(A, D)$ are both abelian groups. □

We now want to give a stable version of this theorem, in which the Kasparov product will reduce exactly to composition of homomorphisms. We need the following lemma. The hypotheses allow one continuous path of homomorphisms, and require unitaries in $U_0((K \otimes D)^+)$, for use in the next subsection.

4.1.2 Lemma. Let A be separable, nuclear, unital, and simple, let D_0 be separable and unital, and let $D = \mathcal{O}_\infty \otimes D_0$. Let $t \mapsto \varphi_t$, for $t \in [0, \infty)$, be a continuous path of full homomorphisms from $K \otimes A$ to $K \otimes D$, and let $\psi : K \otimes A \to K \otimes D$ be a full homomorphism. Assume that $[\varphi_0] = [\psi]$ in $KK^0(A, D)$. Then there is an asymptotic unitary equivalence from φ to ψ which consists of unitaries in $U_0((K \otimes D)^+)$.

Proof: Let $\{e_{ij}\}$ be a system of matrix units for K. Identify A with the subalgebra $e_{11} \otimes A$ of $K \otimes A$. Define $\varphi_t^{(0)}$ and $\psi^{(0)}$ to be the restrictions of φ_t and ψ to A. Then $[\varphi_0^{(0)}] = [\psi^{(0)}]$ in $KK^0(A, D)$. It follows from Theorem 4.1.1 that $\varphi_0^{(0)}$ is homotopic to $\psi^{(0)}$. Therefore $\varphi^{(0)}$ and $\psi^{(0)}$ are homotopic as asymptotic morphisms, and Theorem 2.3.7 provides an asymptotic unitary equivalence $t \mapsto u_t$ in $U((K \otimes D)^+)$ from $\varphi^{(0)}$ to $\psi^{(0)}$. Let $c \in U((K \otimes D)^+)$ be a unitary with

$c\psi^{(0)}(1) = \psi^{(0)}(1)c = \psi^{(0)}(1)$ and such that c is homotopic to u_0^{-1}. Then c commutes with every $\psi^{(0)}(a)$. Replacing u_t by cu_t, we obtain an asymptotic unitary equivalence, which we again call $t \mapsto u_t$, from $\varphi^{(0)}$ to $\psi^{(0)}$ which is in $U_0((K \otimes D)^+)$.

Define $\overline{e}_{ij} = e_{ij} \otimes 1$. Then in particular $u_t \varphi_t(\overline{e}_{11}) u_t^* \to \psi(\overline{e}_{11})$ as $t \to \infty$. Therefore there is a continuous path $t \to z_t^{(1)} \in U_0((K \otimes D)^+)$ such that $z_t^{(1)} \to 1$ and $z_t^{(1)} u_t \varphi_t(\overline{e}_{11}) u_t^* (z_t^{(1)})^* = \psi(\overline{e}_{11})$ for all t. We still have $z_t^{(1)} u_t \varphi_t(e_{11} \otimes a) u_t^* (z_t^{(1)})^* \to \psi(e_{11} \otimes a)$ for $a \in A$.

For convenience, set $f_{ijt} = z_t^{(1)} u_t \varphi_t(\overline{e}_{ij}) u_t^* (z_t^{(1)})^*$, for all t and for $1 \leq i, j \leq 2$. For each fixed t, the f_{ijt} are matrix units, and $f_{11t} = \psi(\overline{e}_{11})$. Set $w_t = \psi(\overline{e}_{21}) f_{12t} + 1 - f_{22t} \in U((K \otimes D)^+)$. Then one checks that $w_t f_{ijt} w_t^* = \psi(\overline{e}_{ij})$ for all t and for $1 \leq i, j \leq 2$. Choose $c \in U((K \otimes D)^+)$ with

$$c\psi(\overline{e}_{11} + \overline{e}_{22}) = \psi(\overline{e}_{11} + \overline{e}_{22})c = \psi(\overline{e}_{11} + \overline{e}_{22}) \quad \text{and} \quad cw_1 \in U_0((K \otimes D)^+).$$

Set $z_t^{(2)} = cw_t$ for $t \geq 1$ and extend $z_t^{(2)}$ over $[0, 1]$ to be continuous, unitary, and satisfy $z_0^{(2)} = 1$. This gives $z_t^{(2)} = 1$ for $t = 0$, $z_t^{(2)} \psi(\overline{e}_{11}) = \psi(\overline{e}_{11}) z_t^{(2)} = \psi(\overline{e}_{11})$ for all t, and

$$z_t^{(2)} \left[z_t^{(1)} u_t \varphi_t(\overline{e}_{ij}) u_t^* (z_t^{(1)})^* \right] (z_t^{(2)})^* = \psi(\overline{e}_{ij})$$

for $t \geq 1$ and $1 \leq i, j \leq 2$.

We continue inductively, obtaining by the same method a sequence of continuous paths $t \mapsto z_t^{(n)}$ such that $z_t^{(n+1)} = 1$ for $t \leq n - 1$,

$$z_t^{(n+1)} \left(\sum_{j=1}^{n} \psi(\overline{e}_{jj}) \right) = \left(\sum_{j=1}^{n} \psi(\overline{e}_{jj}) \right) z_t^{(n+1)} = \sum_{j=1}^{n} \psi(\overline{e}_{jj})$$

for all t, and

$$z_t^{(n+1)} \left[\left(z_t^{(1)} z_t^{(2)} \cdots z_t^{(n)} \right) u_t \varphi_t(\overline{e}_{ij}) u_t^* \left(z_t^{(1)} z_t^{(2)} \cdots z_t^{(n)} \right)^* \right] (z_t^{(n+1)})^* = \psi(\overline{e}_{ij})$$

for $t \geq n$ and $1 \leq i, j \leq n + 1$.

Now define

$$z_t = \left(\lim_{n \to \infty} z_t^{(1)} z_t^{(2)} \cdots z_t^{(n)} \right) u_t.$$

In a neighborhood of each t, all but finitely many of the $z_t^{(k)}$ are equal to 1, so this limit of products yields a continuous path of unitaries in $U_0((K \otimes D)^+)$. Moreover, $z_t \varphi_t(\overline{e}_{ij}) z_t^* = \psi(\overline{e}_{ij})$ whenever $t \geq i, j$, so that $\lim_{t \to \infty} z_t \varphi_t(\overline{e}_{ij}) z_t^* = \psi(\overline{e}_{ij})$ for all i and j, while

$$\lim_{t \to \infty} z_t \varphi_t(e_{11} \otimes a) z_t^* = \lim_{t \to \infty} z_t^{(1)} u_t \varphi_t(e_{11} \otimes a) u_t^* (z_t^{(1)})^* = \psi(e_{11} \otimes a)$$

for all $a \in A$. Since the $\overline{e}_{ij}$ and $e_{11} \otimes a$ generate $K \otimes A$, this shows that $t \mapsto z_t$ is an asymptotic unitary equivalence. ◻

4.1.3 Theorem. For a separable unital nuclear simple C^*-algebra A and a separable unital C^*-algebra D, the obvious maps and the isomorphism $KK^0(A, D) \to KK^0(K \otimes \mathcal{O}_\infty \otimes A, K \otimes \mathcal{O}_\infty \otimes D)$ define natural isomorphisms of abelian groups between the following three objects:

(1) The set of asymptotic unitary equivalence classes of full homomorphisms from $K \otimes \mathcal{O}_\infty \otimes A$ to $K \otimes \mathcal{O}_\infty \otimes D$, with the operation given by direct sum (as in Theorem 4.1.1).

(2) The set of homotopy classes of full homomorphisms from $K \otimes \mathcal{O}_\infty \otimes A$ to $K \otimes \mathcal{O}_\infty \otimes D$, with the operation given by direct sum as above.

(3) The group $KK^0(A, D)$.

Moreover, if B is another a separable unital nuclear simple C^*-algebra, then the Kasparov product $KK^0(A, B) \times KK^0(B, D) \to KK^0(A, D)$ is given in the groups in (1) and (2) by composition of homomorphisms.

Proof: The last statement will follow immediately from the rest of the theorem, since if two KK-classes are represented by homomorphisms, then their product is represented by the composition.

For the rest of the theorem, first note that the map $KK^0(A, D) \to KK^0(K \otimes \mathcal{O}_\infty \otimes A, K \otimes \mathcal{O}_\infty \otimes D)$ is a natural isomorphism because it is induced by the KK-equivalence $\mathbf{C} \to K \otimes \mathcal{O}_\infty$, given by $1 \mapsto e \otimes 1$ for some rank one projection $e \in K$, in each variable.

Now observe that the previous lemma implies that the map from the set in (1) to $KK^0(A, D)$ is injective. Moreover, the map from the set in (1) to the set in (2) is well defined by Lemma 1.3.3 (2), and is then obviously surjective. It therefore suffices to prove that the map from the set in (2) to $KK^0(A, D)$ is surjective, that is, that every class in $KK^0(A, D)$ is represented by a homomorphism from $K \otimes \mathcal{O}_\infty \otimes A$ to $K \otimes \mathcal{O}_\infty \otimes D$. It follows from Theorem 4.1.1 that every such class is represented by a homomorphism from A to $K \otimes \mathcal{O}_\infty \otimes D$, and we obtain a homomorphism from $K \otimes \mathcal{O}_\infty \otimes A$ to $K \otimes \mathcal{O}_\infty \otimes D$ by tensoring with $\mathrm{id}_{K \otimes \mathcal{O}_\infty}$ and composing with the tensor product of id_D and an isomorphism $K \otimes \mathcal{O}_\infty \otimes K \otimes \mathcal{O}_\infty \to K \otimes \mathcal{O}_\infty$ which is the identity on K-theory. □

We finish this section with one other application. Following terminology from differential topology, we define an *isotopy* to be a homotopy $t \mapsto \varphi_t$ in which each φ_t is an isomorphism.

4.1.4 Theorem. Let A be a separable nuclear unital purely infinite simple C^*-algebra.

(1) If $U(A)$ is connected, then two automorphisms of A with the same class in $KK^0(A, A)$ are isotopic.

(2) Any two automorphisms of $K \otimes A$ with the same class in $KK^0(A, A)$ are isotopic.

Proof: For (2), take $D = A$ in Lemma 4.1.2, note that $\mathcal{O}_\infty \otimes A \cong A$ (Theorem 2.1.5), and note that an asymptotic unitary equivalence with unitaries in $U_0((K \otimes A)^+)$ gives an isotopy, not just a homotopy.

For (1), let φ and ψ be automorphisms of A with the same class in $KK^0(A, A)$. Let $e \in K$ be a rank one projection. Apply (2) to $\mathrm{id}_K \otimes \varphi$ and $\mathrm{id}_K \otimes \psi$. Thus, there is a unitary path $t \mapsto u_t$ in $(K \otimes A)^+$ with $u_t \varphi(e \otimes a) u_t^* \to \psi(e \otimes a)$ for $a \in A$. In particular, $u_t(e \otimes 1)u_t^* \to (e \otimes 1)$. Replacing u_t by $v_t u_t$ for a suitable unitary path $t \mapsto v_t$, we may therefore assume that $u_t(e \otimes 1)u_t^* = e \otimes 1$ for all t. Cut down by $e \otimes 1$, and observe that the hypotheses imply that $(e \otimes 1)u_0(e \otimes 1)$ is homotopic to 1. Now finish as in the proof of (2). □

4.2 The classification theorem

The following theorem is the stable version of the main classification theorem. Everything else will be an essentially immediate corollary.

In the proof, it is easy to get the existence of the isomorphism; this is just the by now well known Elliott approximate intertwining argument. We need a more complicated version of this argument to make sure that the isomorphism we construct has the right class in KK-theory: we construct a suitable homotopy at the same time that we construct the isomorphism. One might hope to prove that if A and B are separable C^*-algebras, and if $\varphi_0 : A \to B$ and $\psi_0 : B \to A$ are homomorphisms such that $\psi_0 \circ \varphi_0$ is asymptotically unitarily equivalent to id_A and $\varphi_0 \circ \psi_0$ is asymptotically unitarily equivalent to id_B, then there is an isomorphism $\varphi : A \to B$ which is asymptotically unitarily equivalent to φ_0, or at least is homotopic to φ_0. Unfortunately, we have not been able to prove this; the arguments in the proof below don't seem to quite give such a result.

4.2.1 Theorem. Let A and B be separable nuclear unital purely infinite simple C^*-algebras, and suppose that there is an invertible element $\eta \in KK^0(A, B)$. Then there is an isomorphism $\varphi : K \otimes A \to K \otimes B$ such that $[\varphi] = \eta$ in $KK^0(A, B)$.

Proof: Theorems 3.2.8 and 2.1.5 provide a full asymptotic morphism $\alpha : A \to K \otimes B$ whose class in $KK^0(A, B)$ is η. By Corollary 2.3.10, we may in fact take α to be a homomorphism. Let $\mu : K \otimes K \to K$ be an isomorphism, and set $\varphi_0 = (\mu \otimes \mathrm{id}_B) \circ (\mathrm{id}_K \otimes \alpha)$. Then φ_0 is a nonzero (hence full) homomorphism from $K \otimes A$ to $K \otimes B$ whose class in $KK^0(A, B)$ is also η. Similarly, there is a full homomorphism $\psi_0 : K \otimes B \to K \otimes A$ whose class in $KK^0(B, A)$ is η^{-1}. It follows from Theorems 4.1.3 and 2.1.5 that $\psi_0 \circ \varphi_0$ is homotopic to $\mathrm{id}_{K \otimes A}$ and $\varphi_0 \circ \psi_0$ is homotopic to $\mathrm{id}_{K \otimes B}$.

We now construct homomorphisms $\varphi^{(n)} : K \otimes A \to K \otimes B$, $\psi^{(n)} : K \otimes B \to K \otimes A$, homotopies $\alpha \mapsto \widetilde{\varphi}_\alpha^{(n)}$ (for $\alpha \in [0, 1]$) of homomorphisms from $K \otimes A$ to $K \otimes B$, and finite subsets $F_n \subset K \otimes A$ and $G_n \subset K \otimes B$ such that the following conditions are satisfied:

(1) $\varphi^{(0)} = \varphi_0$.

(2) Each $\varphi^{(n)}$ is of the form $a \mapsto v\varphi_0(a)v^*$ for some suitable $v \in U_0((K \otimes B)^+)$, and similarly each $\psi^{(n)}$ is of the form $b \mapsto u\varphi_0(b)u^*$ for some suitable $u \in U_0((K \otimes A)^+)$.

(3) $F_0 \subset F_1 \subset \cdots$ and $\bigcup_{n=0}^{\infty} F_n$ is dense in $K \otimes A$, and similarly $G_0 \subset G_1 \subset \cdots$ and $\bigcup_{n=0}^{\infty} G_n$ is dense in $K \otimes B$.

(4) $\varphi^{(n)}(F_n) \subset G_n$ and $\psi^{(n)}(G_n) \subset F_{n+1}$.

(5) $\|\psi^{(n)} \circ \varphi^{(n)}(a) - a\| < 2^{-n}$ for $a \in F_n$ and $\|\varphi^{(n+1)} \circ \psi^{(n)}(b) - b\| < 2^{-n}$ for $b \in G_n$.

(6) $\|\widetilde{\varphi}_\alpha^{(n+1)}(a) - \widetilde{\varphi}_\alpha^{(n)}(a)\| < 2^{-n}$ for $a \in F_n$ and $\alpha \in [0,1]$.

(7) $\widetilde{\varphi}_\alpha^{(n)} = \varphi_0$ for $\alpha \geq 1 - 2^{-n}$ and $\widetilde{\varphi}_0^{(n)} = \varphi^{(n)}$.

This will yield the following approximately commutative diagram:

$$
\begin{tikzcd}
F_0 & F_1 & & & & F_{n-1} & F_n & \\
\cap & \cap & & & & \cap & \cap & \\
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi^{(0)}"'] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi^{(1)}"'] & {} & \cdots & {} \arrow[r, "\mathrm{id}_A"] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi^{(n-1)}"] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\varphi^{(n)}"] & {} \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\psi^{(0)}"] & B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\psi^{(1)}"] & {} & \cdots & {} \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\psi^{(n-2)}"] & B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\psi^{(n-1)}"'] & B \arrow[r, "\mathrm{id}_B"'] \arrow[ur] & {} \\
\cup & \cup & & & & \cup & \cup & \\
G_0 & G_1 & & & & G_{n-1} & G_n &
\end{tikzcd}
$$

The diagram will remain approximately commutative if we replace each $\varphi^{(n)}$ by $\widetilde{\varphi}_\alpha^{(n)}$ (with $\alpha \in [0,1]$ fixed) and delete the diagonal arrows.

The proof is by induction on n. We start by choosing finite sets

$$F_0^{(0)} \subset F_1^{(0)} \subset \cdots \subset K \otimes A \quad \text{and} \quad G_0^{(0)} \subset G_1^{(0)} \subset \cdots \subset K \otimes B$$

such that $\overline{\bigcup_{n=0}^{\infty} F_n^{(0)}} = K \otimes A$ and $\overline{\bigcup_{n=0}^{\infty} G_n^{(0)}} = K \otimes B$. For the initial step of the induction, we take $F_0 = F_0^{(0)}$, $\varphi^{(0)} = \widetilde{\varphi}_\alpha^{(0)} = \varphi_0$, and $G_0 = G_0^{(0)} \cup \varphi^{(0)}(F_0)$. We then assume we are given F_k, $\varphi^{(k)}$, G_k, and $\varphi_\alpha^{(k)}$ for $0 \leq k \leq n$ and $\psi^{(k)}$ for $0 \leq k \leq n-1$, and we construct $\psi^{(n)}$, F_{n+1}, $\varphi^{(n+1)}$, G_{n+1}, and $\alpha \mapsto \widetilde{\varphi}_\alpha^{(n+1)}$. That is, we are given the diagram above through the column containing F_n and G_n, as well as the corresponding homotopies $\widetilde{\varphi}^{(k)}$, and we construct the next rectangle (consisting of two triangles) and the corresponding homotopy $\widetilde{\varphi}^{(n+1)}$.

Define $\sigma : K \otimes A \to C([0,1]) \otimes K \otimes A$ by $\sigma(a)(\alpha) = \psi_0(\widetilde{\varphi}_\alpha^{(n)}(a))$. Note that σ is homotopic to $a \mapsto 1 \otimes \psi_0(\varphi_0(a))$, and so has the same class in KK-theory as $a \mapsto 1 \otimes a$. Lemma 4.1.2 provides a unitary path $(\alpha, t) \mapsto u_{\alpha,t} \in U_0((K \otimes A)^+)$ such that

$$\lim_{t\to\infty} \sup_{\alpha\in[0,1]} \|u_{\alpha,t}\psi_0(\widetilde{\varphi}_\alpha^{(n)}(a))u_{\alpha,t}^* - a\| = 0$$

for all $a \in K \otimes A$. Next, define an asymptotic morphism τ from $K \otimes B$ to $C([0,1]) \otimes K \otimes B$ by $\tau_t(b)(\alpha) = \varphi_0(u_{\alpha,t}\psi_0(b)u_{\alpha,t}^*)$. Then τ is homotopic to $b \mapsto 1 \otimes \varphi_0(\psi_0(b))$, and so has the same class in KK-theory as $b \mapsto 1 \otimes b$. Again by Lemma 4.1.2, there is a unitary path $(\alpha, t) \mapsto v_{\alpha,t} \in U_0((K \otimes B)^+)$ such that

$$\lim_{t\to\infty} \sup_{\alpha\in[0,1]} \|v_{\alpha,t}\varphi_0(u_{\alpha,t}\psi_0(b)u_{\alpha,t}^*)v_{\alpha,t}^* - b\| = 0$$

for all $b \in K \otimes B$.

Since $\widetilde{G} = G_n \cup \bigcup_{\alpha\in[0,1]} \widetilde{\varphi}_\alpha^{(n)}(F_n)$ is a compact subset of $K \otimes B$, we can choose T so large that

$$\|v_{\alpha,t}\varphi_0(u_{\alpha,t}\psi_0(b)u_{\alpha,t}^*)v_{\alpha,t}^* - b\| < 2^{-(n+1)}$$

for all $b \in \widetilde{G}$ and $t \geq T$. Increasing T if necessary, we can also require

$$\|u_{\alpha,t}\psi_0(\widetilde{\varphi}_\alpha^{(n)}(a))u_{\alpha,t}^* - a\| < 2^{-(n+1)}$$

for all $a \in F_n$ and $t \geq T$. Now define

$$\psi^{(n)}(b) = u_{0,T}\psi_0(b)u_{0,T}^* \quad \text{and} \quad \varphi^{(n+1)}(a) = v_{0,T}\varphi_0(a)v_{0,T}^*,$$

and

$$F_{n+1} = F_{n+1}^{(0)} \cup F_n \cup \psi^{(n)}(G_n) \quad \text{and} \quad G_{n+1} = G_{n+1}^{(0)} \cup G_n \cup \varphi^{(n+1)}(F_{n+1}).$$

The relevant parts of conditions (2)–(4) are then certainly satisfied. For (5), we have in fact

$$\|\psi^{(n)} \circ \varphi^{(n)}(a) - a\| = \|u_{0,T}\psi_0(\widetilde{\varphi}_0^{(n)}(a))u_{0,T}^* - a\| < 2^{-(n+1)}$$

for $a \in F_n$ by the choice of T, and similarly

$$\|\varphi^{(n+1)} \circ \psi^{(n)}(b) - b\| = \|v_{0,T}\varphi_0(u_{0,T}\psi_0(b)u_{0,T}^*)v_{0,T}^* - b\| < 2^{-(n+1)}$$

for $b \in G_n$.

Now choose a continuous function $f : [0, 1 - 2^{-(n+1)}) \to [T, \infty)$ such that $f(\alpha) = T$ for $0 \leq \alpha \leq 1 - 2^{-n}$ and $f(\alpha) \to \infty$ as $\alpha \to 1 - 2^{-(n+1)}$. Define $\alpha \mapsto \widetilde{\varphi}_\alpha^{(n+1)}$ by

$$\widetilde{\varphi}_\alpha^{(n+1)}(a) = \begin{cases} v_{\alpha,f(\alpha)}\varphi_0(a)v_{\alpha,f(\alpha)}^* & 0 \leq \alpha < 1 - 2^{-(n+1)} \\ \varphi_0(a) & 1 - 2^{-(n+1)} \leq \alpha \leq 1. \end{cases}$$

We first have to show that the functions $\alpha \mapsto \widetilde{\varphi}_\alpha^{(n+1)}(a)$ are continuous at $1-2^{-(n+1)}$ for $a \in K \otimes A$. Set $\alpha_0 = 1-2^{-(n+1)}$, and consider α with $1-2^{-n} \leq \alpha < 1-2^{-(n+1)}$. By the induction hypothesis, we then have $\widetilde{\varphi}_\alpha^{(n)}(a) = \varphi_0(a)$. For $a \in K \otimes A$, set $b = \varphi_0(a)$; then

$$\begin{aligned} &\|\widetilde{\varphi}_\alpha^{(n+1)}(a) - \widetilde{\varphi}_{\alpha_0}^{(n+1)}(a)\| \\ &\quad \leq \quad \|a - u_{\alpha,f(\alpha)}\psi_0(\widetilde{\varphi}_\alpha^{(n)}(a))u_{\alpha,f(\alpha)}^*\| \\ &\qquad\qquad + \|v_{\alpha,f(\alpha)}\varphi_0(u_{\alpha,f(\alpha)}\psi_0(b)u_{\alpha,f(\alpha)}^*)v_{\alpha,f(\alpha)}^* - b\|. \end{aligned}$$

The requirement that $f(\alpha) \to \infty$ as $\alpha \to 1-2^{-(n+1)}$, together with the condition of uniformity in α in the limits used in the choices of $u_{\alpha,t}$ and $v_{\alpha,t}$, implies that both terms on the right converge to 0. So the required continuity holds.

The relevant part of condition (7) is satisfied by definition, so it remains only to check (6). We may assume $\alpha < 1-2^{-(n+1)}$. So let $a \in F_n$. Then $b = \widetilde{\varphi}_\alpha^{(n)}(a) \in \widetilde{G}$. So

$$\begin{aligned} &\|\widetilde{\varphi}_\alpha^{(n+1)}(a) - \widetilde{\varphi}_\alpha^{(n)}(a)\| \\ &\quad \leq \sup_{\alpha\in[0,1],t\geq T} \|v_{\alpha,t}\varphi_0(a)v_{\alpha,t}^* - \widetilde{\varphi}_\alpha^{(n)}(a)\| \\ &\quad \leq \sup_{\alpha\in[0,1],t\geq T} \|a - u_{\alpha,t}\psi_0(\widetilde{\varphi}_\alpha^{(n)}(a))u_{\alpha,t}^*\| \\ &\qquad\qquad + \sup_{\alpha\in[0,1],t\geq T} \|v_{\alpha,t}\varphi_0(u_{\alpha,t}\psi_0(b)u_{\alpha,t}^*)v_{\alpha,t}^* - b\| \\ &\quad < 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}. \end{aligned}$$

This proves (6), and finishes the inductive construction. Note that the set $\bigcup_{n=0}^\infty F_n$ is dense in $K \otimes A$ because it contains the dense subset $\bigcup_{n=0}^\infty F_n^{(0)}$, and similarly $\bigcup_{n=0}^\infty G_n$ is dense in $K \otimes B$.

We now define $\varphi : K \otimes A \to K \otimes B$ by $\varphi(a) = \lim_{n\to\infty} \varphi^{(n)}(a)$, and define $\psi : K \otimes B \to K \otimes A$ and the homotopy $\widetilde{\varphi} : K \otimes A \to C([0,1]) \otimes K \otimes B$ analogously. As in Section 2 of [20], these limits all exist and define homomorphisms; moreover, $\psi \circ \varphi = \mathrm{id}_{K\otimes A}$, $\varphi \circ \psi = \mathrm{id}_{K\otimes B}$, $\widetilde{\varphi}_0 = \varphi$, and $\widetilde{\varphi}_1 = \varphi_0$. So φ is an isomorphism from $K \otimes A$ to $K \otimes B$ which is homotopic to φ_0 and therefore satisfies $[\varphi] = \eta$ in $KK^0(A,B)$. ◻

4.2.2 Corollary. Let A and B be separable nuclear unital purely infinite simple C^*-algebras, and suppose that there is an invertible element $\eta \in KK^0(A,B)$ such that $[1_A] \times \eta = [1_B]$. Then there is an isomorphism $\varphi : A \to B$ such $[\varphi] = \eta$ in $KK^0(A,B)$.

Proof: The previous theorem provides an isomorphism $\alpha : K \otimes A \to K \otimes B$ such that $[\alpha] = \eta$ in $KK^0(A,B)$. Choose a rank one projection $e \in K$. Then $[\alpha(e \otimes 1_A)] = [1_A] \times \eta = [e \otimes 1_B]$ in $K_0(B)$. Since $K \otimes B$ is purely infinite simple, it follows that there is a unitary $u \in (K \otimes B)^+$ such that $u\alpha(e \otimes 1_A)u^* = e \otimes 1_B$. Define $\varphi(a) = u\alpha(e \otimes a)u^*$, regarded as an element of $(e \otimes 1_B)(K \otimes B)(e \otimes 1_B) = B$. ◻

The remaining corollaries require some hypotheses on the Universal Coefficient Theorem. (See [53].) The following terminology is convenient.

4.2.3 Definition. Let A and D be separable nuclear C^*-algebras. We say that the pair (A, D) *satisfies the Universal Coefficient Theorem* if the sequence

$$0 \longrightarrow \mathrm{Ext}_1^{\mathbf{Z}}(K_*(A), K_*(D)) \longrightarrow KK^0(A, D) \longrightarrow \mathrm{Hom}(K_*(A), K_*(D)) \longrightarrow 0$$

of Theorem 1.17 of [53] is defined and exact. (Note that the second map is always defined, and the first map is the inverse of a map that is always defined.) We further say that A satisfies the Universal Coefficient Theorem if (A, D) does for every separable C^*-algebra D.

4.2.4 Theorem. Let A and B be separable nuclear purely infinite simple C^*-algebras which satisfy the Universal Coefficient Theorem. Assume that A and B are either both unital or both nonunital. If there is a graded isomorphism $\alpha : K_*(A) \to K_*(B)$ which (in the unital case) satisfies $\alpha_*([1_A]) = [1_B]$, then there is an isomorphism $\varphi : A \to B$ such that $\varphi_* = \alpha$.

Proof: The proof of Proposition 7.3 of [53] shows that there is a KK-equivalence $\eta \in KK^0(A, B)$ which induces α. Now use Theorem 4.2.1 or Corollary 4.2.2 as appropriate. □

This theorem gives all the classification results of [48], [49], [34], [22], [35], [51], [36], [44], and [52]. Of course, we have used the main technical theorem of [48], as well as substantial material from [35], in the proof. We do not obtain anything new about the Rokhlin property of [8]; indeed, our results show that the C^*-algebras of [51] are classifiable as long as they are purely infinite and simple, regardless of whether the Rokhlin property is satisfied. On the other hand, the Rokhlin property has been verified in many cases; see [30] and [31].

We finish this section by giving some further corollaries. Let $\mathcal{C}$ be the "classifiable class" given in Definition 5.1 of [22], and let $\mathcal{N}$ denote the bootstrap category of [53], for which the Universal Coefficient Theorem was shown to hold (Theorem 1.17 of [53]).

4.2.5 Theorem. Let G_0 and G_1 be countable abelian groups, and let $g \in G_0$. Then:

(1) There is a separable nuclear unital purely infinite simple C^*-algebra algebra $A \in \mathcal{N}$ such that

$$(K_0(A), [1_A], K_1(A)) \cong (G_0, g, G_1).$$

(2) There is a separable nuclear nonunital purely infinite simple C^*-algebra $A \in \mathcal{N}$ such that

$$(K_0(A), K_1(A)) \cong (G_0, G_1).$$

Proof: The construction of Theorem 5.6 of [22] gives algebras which are easily seen to be in $\mathcal{N}$. □

4.2.6 Corollary. Every C^*-algebra in $\mathcal{C}$ is in $\mathcal{N}$. Every purely infinite simple C^*-algebra in $\mathcal{N}$, and more generally every separable nuclear purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem, is in $\mathcal{C}$.

Proof: The first part follows immediately from the previous theorem, since it follows from the definition of $\mathcal{C}$ that any $A \in \mathcal{C}$ must be isomorphic to the C^*-algebra of that theorem with the same K-theory. The second part follows from Theorem 4.2.4, since Theorem 1.17 of [53] states that every C^*-algebra in $\mathcal{N}$ satisfies the Universal Coefficient Theorem. □

4.2.7 Corollary. Let $A \in \mathcal{C}$, and let B be a separable nuclear unital simple C^*-algebra which satisfies the Universal Coefficient Theorem. (In particular, B could be a unital simple C^*-algebra in $\mathcal{N}$.) Then $A \otimes B \in \mathcal{C}$.

Proof: The C^*-algebra $A \otimes B$ is separable, nuclear, unital, and simple, and Theorem 7.7 of [53] (and the remark after this theorem) shows that it satisfies the Universal Coefficient Theorem. Furthermore, A is approximately divisible by Corollary 2.1.6, and it follows from the remark after Theorem 1.4 of [6] that $A \otimes B$ is approximately divisible. Clearly $A \otimes B$ is infinite, so it is purely infinite by Theorem 1.4 (a) of [6]. The result now follows from the previous corollary. □

4.2.8 Corollary. The class $\mathcal{C}$ is closed under tensor products.

4.2.9 Corollary. For any m, $n \geq 2$, we have $\mathcal{O}_m \otimes \mathcal{O}_n \in \mathcal{C}$. In particular, if $m-1$ and $n-1$ are relatively prime, then $\mathcal{O}_m \otimes \mathcal{O}_n \cong \mathcal{O}_2$.

4.2.10 Corollary. Let A_1 and A_2 be two higher dimensional noncommutative toruses of the same dimension, and let B be any simple Cuntz-Krieger algebra. Then $A_1 \otimes B \cong A_2 \otimes B$.

Proof: The Künneth formula [55] shows that $A_1 \otimes B$ and $A_2 \otimes B$ have the same K-theory. □

4.2.11 Theorem. Let A be a separable nuclear unital purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem. Let A^{op} be the opposite algebra, that is, A with the multiplication reversed but all other operations the same. Then $A \cong A^{\mathrm{op}}$.

Proof: The identity map from A to A^{op} is an antiisomorphism which induces an isomorphism on K-theory sending $[1_A]$ to $[1_{A^{\mathrm{op}}}]$. Also, the pair (A^{op}, B) (for any separable B) always satisfies the Universal Coefficient Theorem, because (A, B^{op}) does. □

By way of contrast, we note that Connes has shown [10] that there is a type III factor not isomorphic to its opposite algebra. It is also known (although apparently not published) that there are nonsimple separable nuclear (even type I) C^*-algebras not isomorphic to their opposite algebras.

4.3 Nonclassification

In this subsection, we give some results which show how badly the classification theorem fails if the algebras are not nuclear. The results are mostly either proved elsewhere or follow fairly easily from results proved by other people. There are three main results. First, nonnuclear separable purely infinite simple C^*-algebras need not be approximately divisible in the sense of [6], but whenever A is a purely infinite simple C^*-algebra, then $\mathcal{O}_\infty \otimes A$ is an approximately divisible purely infinite simple C^*-algebra with exactly the same K-theoretic invariants. Second, there are infinitely many mutually nonisomorphic approximately divisible separable exact unital purely infinite simple C^*-algebras A satisfying $K_*(A) = 0$. Finally, given arbitrary countable abelian groups G_0 and G_1, and $g \in G_0$, there are uncountably many mutually nonisomorphic approximately divisible separable unital purely infinite simple C^*-algebras A satisfying $K_j(A) \cong G_j$ with $[1] \mapsto g_0$. Unfortunately these algebras are not exact, and it remains unknown whether the same is true with the additional requirement of exactness.

The first result is taken straight from a paper of Dykema and Rørdam.

4.3.1 Theorem. ([18], Theorem 1.4) There exists a separable unital purely infinite simple C^*-algebra which is not approximately divisible.

4.3.2 Remark. In fact, there exists a separable unital purely infinite simple C^*-algebra A which is not approximately divisible and such that $K_*(A) = 0$.

One way to see this is to modify the proof of Proposition 1.3 of [18] so as to ensure that $K_*(A_n) \to K_*(B)$ is injective for all n. This is done by enlarging the set X_{n+1} in the proof so as to include appropriate partial isometries (implementing equivalences between projections) and paths of unitaries (implementing triviality of classes of unitaries in K_1). See the proof of Theorem 4.3.11 below for this argument in a related context.

The second result is a fairly easy consequence of a computation of Cowling and Haagerup and of unpublished work of Haagerup. The key invariant is described in the following definition. I am grateful to Uffe Haagerup for explaining the properties of this invariant and where to find proofs of them.

4.3.3 Definition. (Haagerup [23]; also see Section 6 of [12].) Let A be a C^*-algebra. Define $\Lambda(A)$ to be the infimum of numbers C such that there is a net of finite rank operators $T_\alpha : A \to A$ for which $\|T_\alpha(a) - a\| \to 0$ for all $a \in A$ and the completely bounded norms satisfy $\sup_\alpha \|T_\alpha\|_{\mathrm{cb}} \leq C$. Note that $\Lambda(A) = \infty$ if no such C exists, that is, if A does not have the completely bounded approximation property.

There is a similar definition for von Neumann algebras, in which $T_\alpha(a)$ is required to converge to a in the weak operator topology. (See [23] and Section 6 of [12].) There is also a definition of $\Lambda(G)$ for a locally compact group G, using completely bounded norms of multipliers of G which converge to 1 uniformly on compact sets; see [23] and Section 1 of [12]. We do not formally state the definitions, but we recall the following theorems from [23] (restated as Propositions 6.1 and 6.2 of [12]):

4.3.4 Theorem. Let Γ be a discrete group, and let $C^*_{\mathrm{r}}(\Gamma)$ and $W^*(\Gamma)$ be its reduced C^*-algebra and von Neumann algebra respectively. Then $\Lambda(\Gamma) = \Lambda(C^*_{\mathrm{r}}(\Gamma)) = \Lambda(W^*(\Gamma))$.

4.3.5 Theorem. Let G be a second countable locally compact group, and let Γ be a lattice in G. Then $\Lambda(\Gamma) = \Lambda(G)$.

In Section 6 of [12], Cowling and Haagerup exhibit type II_1 factors M_n with $\Lambda(M_n) = 2n-1$. Using the same results on groups, we exhibit simple C^*-algebras with the same values of Λ.

4.3.6 Proposition. Let Γ_n^0 be as in Corollary 6.6 of [12]. Then $A_n = C^*_{\mathrm{r}}(\Gamma_n^0)$ is a simple separable unital C^*-algebra which satisfies $\Lambda(A_n) = 2n-1$.

We recall that Γ_n^0 is the quotient by its center of a particular lattice Γ_n in the simple Lie group $\mathrm{Sp}(n,1)$.

Proof of Proposition 4.3.6: It is shown in the proof of Corollary 6.6 of [12] that $\Lambda(\Gamma_n^0) = 2n-1$. (This follows from the computation $\Lambda(\mathrm{Sp}(n,1)) = 2n-1$, which is the main result of [12], together with Theorem 4.3.5 above and Proposition 1.3 (c) of [12].) Therefore $\Lambda(A_n) = 2n-1$ by Theorem 4.3.4. Clearly A_n is separable and unital. Simplicity of A_n follows from Theorem 1 of [2], applied to the quotient of $\mathrm{Sp}(n,1)$ by its center, because (as observed in the introduction to [2]) lattices satisfy the density hypothesis of that theorem. $\square$

The algebras A_n are not purely infinite, and their K-theory seems to be unknown. So we will tensor them with $\mathcal{O}_2$. For this, we need the following result.

4.3.7. Lemma. Let A be any C^*-algebra, and let B be unital and nuclear. Then $\Lambda(A \otimes B) = \Lambda(A)$.

For von Neumann algebras, it is known [57] that $\Lambda(M \otimes N) = \Lambda(M)\Lambda(N)$. We presume, especially in view of Remark 3.5 of [57], that the analogous statement is true for C^*-algebras as well. However, the special case in the lemma is sufficient here.

Proof of Lemma 4.3.7: If $S : A_1 \to A_2$ and $T : B_1 \to B_2$ are completely bounded, then the map $S \otimes_{\min} T : A_1 \otimes_{\min} B_1 \to A_2 \otimes_{\min} B_2$ is completely bounded, and satisfies $\|S \otimes_{\min} T\|_{\mathrm{cb}} = \|S\|_{\mathrm{cb}} \|T\|_{\mathrm{cb}}$ by Theorem 10.3 of [40]. In Definition 4.3.3, one need only consider elements a of a dense subset, and so it follows that $\Lambda(A \otimes_{\min} B) \leq \Lambda(A)\Lambda(B)$ for any C^*-algebras A and B. For B nuclear, we have $\Lambda(B) = 1$, so $\Lambda(A \otimes B) \leq \Lambda(A)$.

For the reverse inequality, let $R_\alpha : A \otimes B \to A \otimes B$ be finite rank operators such that $\|R_\alpha(x) - x\| \to 0$ for all $x \in A \otimes B$. Choose any state ω on B, and define $T_\alpha : A \to A$ by $T_\alpha(a) = (\mathrm{id}_B \otimes \omega) \circ R_\alpha(a \otimes 1)$. Theorem 10.3 of [40] implies that $\|T_\alpha\|_{\mathrm{cb}} \leq \|R_\alpha\|_{\mathrm{cb}}$. Also, clearly $\|T_\alpha(a) - a\| \to 0$ for all $a \in A$. So $\Lambda(A) \leq \Lambda(A \otimes B)$. ◻

4.3.8 Theorem. There exist infinitely many mutually nonisomorphic separable exact unital purely infinite simple C^*-algebras B satisfying $K_*(B) = 0$ and $\mathcal{O}_\infty \otimes B \cong B$. In particular, these algebras are approximately divisible in the sense of [6].

Proof: Let $A_n = C^*_{\mathrm{r}}(\Gamma^0_n)$ as in Proposition 4.3.6. Set $B_n = \mathcal{O}_2 \otimes A_n$. Clearly B_n is separable and unital. Furthermore, B_n is purely infinite simple by the proof of Corollary 4.2.7. We have $\mathcal{O}_\infty \otimes B_n \cong B_n$ because $\mathcal{O}_\infty \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. The algebras B_n are mutually nonisomorphic because $\Lambda(B_n) = 2n - 1$, by the previous lemma and Proposition 4.3.6.

It remains to check exactness. The proof of Corollary 3.12 of [17] shows that if $\Lambda(A)$ is finite, then A has the slice map property (as defined, for example, in Remark 9 of [59], where it is called Property S), and this property implies exactness (see, for example, Section 2.5 of [60]). ◻

Our third result is based on the theorem of Junge and Pisier that for $n \geq 3$ the collection of n-dimensional operator spaces is not separable in the completely bounded analog of the Banach-Mazur distance.

4.3.9 Definition. ([25]) Let E and F be operator spaces of the same finite dimension. Then

$$d_{\mathrm{cb}}(E, F) = \inf\{\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} : T \text{ is a linear bijection from } E \text{ to } F\},$$

and $\delta_{\mathrm{cb}}(E, F) = \log(d_{\mathrm{cb}}(E, F))$.

4.3.10 Theorem. (Theorem 2.3 of [25]) Let OS_n be the set of all complete isometry classes of n-dimensional operator spaces. Let $n \geq 3$. Then $(\mathrm{OS}_n, \delta_{\mathrm{cb}})$ is an inseparable metric space.

4.3.11 Theorem. Let G_0 and G_1 be countable abelian groups, and let $g \in G_0$. Then there exist uncountable many mutually nonisomorphic separable unital purely infinite simple C^*-algebras A, each with $K_0(A) \cong G_0$ in such a way that $[1] \mapsto g$ and $K_1(A) \cong G_1$, and each satisfying $\mathcal{O}_\infty \otimes A \cong A$.

Proof: If A is a separable C^*-algebra, then the set of (complete isometry classes of) n-dimensional operator subspaces of A is separable (by Proposition 2.6 (a) of [25]). By the previous theorem, it therefore suffices to show that if E is a finite dimensional operator space then there exists a C^*-algebra B having the properties claimed in the theorem and such that E is completely isometric to a subspace of B.

Since E is a finite dimensional operator space, it is a subspace of a separable C^*-algebra A. Represent A on a separable Hilbert space H with infinite multiplicity, and follow this representation with the quotient map from $L(H)$

to the Calkin algebra Q. This gives a completely isometric embedding of E in Q. For convenience, we identify E with its image. Let $u \in Q$ be the image of the unilateral shift; note that $[u]$ generates $K_1(Q)$ and that $K_0(Q) = 0$. Let $B_0 = C^*(E, 1, u) \subset Q$. We now construct by induction an increasing sequence $B_0 \subset B_1 \subset B_2 \subset \cdots \subset Q$ of separable C^*-algebras such that B_{2n+1} is simple and such that every nonzero projection in B_{2n-1} is Murray-von Neumann equivalent to 1 in B_{2n}, every selfadjoint element of B_{2n-1} is a limit of selfadjoint elements of B_{2n} with finite spectrum, and every unitary in $U(B_{2n-1}) \cap U_0(Q)$ is homotopic to 1 in B_{2n}.

Given B_{2n}, we choose B_{2n+1} to be any separable simple C^*-algebra with $B_{2n} \subset B_{2n+1} \subset Q$. Such a subalgebra exists by Proposition 2.2 of [3] and the simplicity of Q. Given B_{2n-1}, we note that it suffices to have the required elements of B_{2n} only for countable dense subsets S_1 of the nonzero projections in B_{2n-1}, S_2 of the selfadjoint elements in B_{2n-1}, and S_3 of the unitaries in $U(B_{2n-1}) \cap U_0(Q)$. For each $p \in S_1$, since p is Murray-von Neumann equivalent to 1 in Q, we can choose an isometry $v \in Q$ such that $v^*v = 1$ and $vv^* = p$. Let T_1 be the set of all these as p runs through S_1. For each $a \in S_2$, since Q has real rank zero, there is a sequence (b_n) in Q consisting of selfadjoint elements with finite spectrum such that $b_n \to a$. Let T_2 be the set of all terms of all such sequences as a runs through S_2. For each $u \in S_3$, since $u \in U_0(Q)$, there is a unitary path $t \mapsto v(t)$ in Q with $v(0) = 1$ and $v(1) = u$. Let T_3 consist of all $v(t)$ for $t \in [0, 1] \cap \mathbf{Q}$ as u runs through S_3. Then take B_{2n} to be the C*-subalgebra of Q generated by B_{2n-1} and $T_1 \cup T_2 \cup T_3$. This subalgebra is separable because B_{2n-1} is separable and $T_1 \cup T_2 \cup T_3$ is countable.

Now set $B = \overline{\bigcup_{n=0}^{\infty} B_n}$. Then B is simple because it is the direct limit of the simple C^*-algebras B_{2n+1}. From the construction of B_{2n}, it is clear that B is unital and separable, contains the operator space E, has real rank zero, that all nonzero projections in B are Murray-von Neumann equivalent to 1, and that $U(B) \cap U_0(Q) \subset U_0(B)$. The third and fourth properties imply that B is purely infinite and $K_0(B) = 0$. The last property implies that $K_1(B) \to K_1(Q)$ is injective. But this map is also surjective, since B_0 contains a unitary whose class generates $K_1(Q)$. So $K_1(B) \cong \mathbf{Z}$.

Taking $A = \mathcal{O}_\infty \otimes B$ (which has the same K-theory by the Künneth formula [55]), we obtain the statement of the theorem for the special case $G_0 = 0$, $g = 0$, and $G_1 = \mathbf{Z}$. For the general case, choose (by Theorem 4.2.5) a separable nuclear unital purely infinite simple C^*-algebra D satisfying the Universal Coefficient Theorem and such that $K_0(D) \cong G_1$ and $K_1(D) \cong G_0$. (We don't actually need D to be purely infinite here, but it must be in the bootstrap category of [55].) Then $D \otimes B$ is purely infinite and simple, and has the right K-theory by the Künneth formula, except that $[1] = 0$. Choose a projection $p \in D \otimes B$ such that the isomorphism $K_0(D \otimes B) \cong G_0$ sends $[p]$ to g. Then the C^*-algebra $A = \mathcal{O}_\infty \otimes p(D \otimes B)p$ satisfies all the conditions of the theorem and contains the given operator space E. ◻

4.3.12 Remark. Simplicity and pure infiniteness of $\overline{\bigcup_{n=0}^{\infty} B_n}$ in the proof above can also be arranged by the method of the proof of Proposition 1.3 of [18]. Versions of the construction here have been used many times before.

4.3.13 Remark. The invariant used here, the set of finite dimensional operator spaces contained in A, does not distinguish between any two separable exact purely infinite simple C^*-algebras. (Any separable exact C^*-algebra embeds in $\mathcal{O}_2$ by Theorem 2.8 of [29], and $\mathcal{O}_2$ embeds in any purely infinite simple C^*-algebra.) Therefore, for given K-theory, at most one of the C^*-algebras proved above to be nonisomorphic can be exact.

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