Einstein Metrics and Stability for Flat Connections on Compact Hermitian Manifolds, and a Correspondence with Higgs Operators in the Surface Case

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Abstract. A flat complex vector bundle (E, D) on a compact Riemannian manifold (X, g) is stable (resp. polystable) in the sense of Corlette [C] if it has no D-invariant subbundle (resp. if it is the D-invariant direct sum of stable subbundles). It has been shown in [C] that the polystability of (E, D) in this sense is equivalent to the existence of a so-called harmonic metric in E. In this paper we consider flat complex vector bundles on compact Hermitian manifolds (X, g). We propose new notions of g-(poly-)stability of such bundles, and of g-Einstein metrics in them; these notions coincide with (poly-)stability and harmonicity in the sense of Corlette if g is a Kähler metric, but are different in general. Our main result is that the g-polystability in our sense is equivalent to the existence of a g-Hermitian-Einstein metric. Our notion of a g-Einstein metric in a flat bundle is motivated by a correspondence between flat bundles and Higgs bundles over compact surfaces, analogous to the correspondence in the case of Kähler manifolds [S1], [S2], [S3].

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1 Introduction.

Let X be an n-dimensional compact complex manifold. If X admits a *Kähler* metric g, then it is known by work of in particular Simpson [S1],[S2],[S3] that there exists an canonical identification of the moduli space of polystable (or semisimple) flat bundles on X with the moduli space of g-polystable Higgs-bundles with vanishing Chern classes on X. This identification has been used in showing that certain groups are not fundamental groups of compact Kähler manifolds. The construction uses the existence of *canonical* metrics, called *g-harmonic* in the case of flat bundles, and *g-Einstein* in the case of Higgs bundles.

For flat bundles, the equivalence of semisimplicity and the existence of a g-harmonic metric holds on compact *Riemannian* manifolds [C]. Furthermore, the equivalence of g-polystability and the existence of a g-Einstein metrics for Higgs bundles should generalize to the case of *Hermitian* manifolds as in the case of holomorphic vector bundles, using *Gauduchon* metrics. Nevertheless, an identification as above cannot be expected for general compact Hermitian manifolds, since it should imply restrictions on the fundamental group, but every finitely presented group is the fundamental group of a 3-dimensional compact complex manifold by a theorem of Taubes [T].

In the case of compact complex *surfaces*, however, things are different. We show that for an integrable Higgs bundle (E, d'') with vanishing real Chern numbers and of g-degree 0 with g-Einstein metric h on a compact complex surface X with Hermitian metric g, there is an canonically associated flat connection D in E, again of g-degree 0, such that h is what we call a *g-Einstein* metric for (E, D), and that the converse is also true. Furthermore, this correspondence preserves isomorphism types and hence descends to a bijection between moduli spaces.

The notion of a g-Einstein metric in a flat bundle makes sense in higher dimension, too, is equivalent to g-harmonicity in the case of a Kähler metric, but different in general, and we show that the existence of such a metric in a flat bundle (E, D) is equivalent to the g-polystability of this bundle in the sense that E is the direct sum of D-invariant g-stable flat subbundles. Here we call a flat bundle (E, D) g-stable if every D-invariant subbundle has *g-slope* larger(!) than the g-slope of (E, D). g-stability of a flat bundle is equivalent to its stability (in the sense of Corlette) in the Kähler case, but a weaker condition in general: A stable bundle is always g-stable, but the tangent bundles of certain Inoue surfaces are examples of g-stable bundles which are not stable.

We expect that for a non-Kähler surface with Hermitian metric g, there is a natural bijection between the moduli space of g-polystable Higgs bundles, with vanishing Chern numbers and g-degree, and the moduli space of g-polystable flat bundles with vanishing g-degree. In the last section we consider the special case of *line* bundles on surfaces. Here the stability is trivial, and the existence of Einstein metrics is easy to show, so we get indeed the expected natural bijection between moduli spaces of line bundles of degree 0. We further show how this can be extended (in a non-natural way) to the moduli spaces of line bundles of arbitrary degree; this extension argument works in fact for bundles of arbitrary rank once the correspondence for degree 0 has been established.

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2 PRELIMINARIES.

Let X be a compact n-dimensional complex manifold, and $E \longrightarrow X$ a differentiable $\mathbb{C}^r$-vector bundle on X. We fix the following

NOTATIONS:

$A^p(X)$ (resp. $A^{p,q}(X)$) is the space of differentiable p-forms (forms of type (p,q)) on X.

$A^p(E)$, $A^{p,q}(E)$ are the spaces of differential forms with values in E.

$\mathcal{A}(E)$ is the space of linear connections D in E. For a connection $D \in \mathcal{A}(E)$ we write $D = D' + D''$, where D' is of type (1,0) and D'' of type (0,1).

$\mathcal{A}(E,h) \subset \mathcal{A}(E)$ is the subspace of h-unitary connections d in E, where h is a Hermitian metric in E. We write $d = \partial + \bar{\partial}$, where ∂ is of type (1,0) and $\bar{\partial}$ of type (0,1).

$\mathcal{A}_f(E) := \{ D \in \mathcal{A}(E) \mid D^2 = 0 \}$ is the subset of *flat* connections.

$\bar{\mathcal{A}}(E)$ is the space of semiconnections $\bar{\partial}$ of type (0,1) in E (i.e. $\bar{\partial}$ is the (0,1)-part of some $D \in \mathcal{A}(E)$).

$\mathcal{H}(E) := \{ \bar{\partial} \in \bar{\mathcal{A}}(E) \mid \bar{\partial}^2 = 0 \}$ is the subset of *integrable* semiconnections or *holomorphic structures* in E.

$\mathcal{A}''(E) := \bar{\mathcal{A}}(E) \oplus A^{1,0}(\mathrm{End}E) = \{ d'' = \bar{\partial} + \theta \mid \bar{\partial} \in \bar{\mathcal{A}}(E),\ \theta \in A^{1,0}(\mathrm{End}E) \}$ is the space of *Higgs operators* in E.

$\mathcal{H}''(E) := \{ d'' \in \mathcal{A}''(E) \mid (d'')^2 = 0 \}$ is the subset of *integrable* Higgs operators.

Often the same symbol is used for a connection, semiconnection, Higgs operator etc. in E and the induced operator in $\mathrm{End}E$.

Two connections $D_1, D_2 \in \mathcal{A}(E)$ are *isomorphic*, $D_1 \cong D_2$, if there exists a differentiable automorphism f of E such that $f \circ D_1 = D_2 \circ f$, which is equivalent to $D(f) = 0$, where D is the connection in $\mathrm{End}E$ induced by D_1 and D_2, i.e. $D(f) = D_2 \circ f - f \circ D_1$. In the same way the isomorphy of semiconnections resp. Higgs operators is defined.

If a Hermitian metric h in E is given, then a superscript * means adjoint with respect to h.

For $D = D' + D''$ there are unique semiconnections δ'_h,δ''_h of type (1,0), (0,1) respectively such that $D' + \delta''_h$ and $\delta'_h + D''$ are h-unitary connections. Define $\delta_h := \delta'_h + \delta''_h$; then $d_h := \frac{1}{2}(D + \delta_h)$ is h-unitary, and $\Theta_h := D - d_h = \frac{1}{2}(D - \delta_h)$ is a h-selfadjoint 1-form with values in $\mathrm{End}E$. Let $d_h = \partial_h + \bar{\partial}_h$ be the decomposition in the parts of type (1,0) and (0,1), and let θ_h be the $(1,0)$-part of Θ_h; then it holds

$$D = d_h + \Theta_h = \partial_h + \bar{\partial}_h + \theta_h + \theta_h^*.$$

The map

$$I_h : \mathcal{A}(E) \longrightarrow \mathcal{A}''(E), I_h(D) := d''_h := \bar{\partial}_h + \theta_h \in \mathcal{A}''(E)$$

is bijective; the inverse is given as follows. For $d'' = \bar{\partial} + \theta \in \mathcal{A}''(E)$ let ∂_h be the unique semiconnection of type $(1,0)$ such that the connection $d_h := \partial_h + \bar{\partial}$ is h-unitary, and define $\Theta := \theta + \theta^*$. Then

$$I_h^{-1}(d'') = D_h := d_h + \Theta \in \mathcal{A}(E).$$

Remark 2.1 *i) In general, if $D_1, D_2 \in \mathcal{A}(E)$ are isomorphic, then $I_h(D_1)$ and $I_h(D_2)$ are not isomorphic, and vice versa.*
ii) $D_h = d_h + \theta + \theta^*$ is not h-unitary unless $\theta = 0$, but the connections $d_h - \theta + \theta^*$ and $d_h + \theta - \theta^*$ are.
iii) Any metric h' in E is of the form $h' = f \cdot h$, i.e. $h'(s,t) = h(f(s),t)$, where f is a h-selfadjoint and positive definite. For a connection D it is easy to show that the operator $\delta_{h\cdot f}$ associated to D and $f \cdot h$ is given by $\delta_{h\cdot f} = f^{-1} \circ \delta_h \circ f = \delta_h + f^{-1} \circ \delta_h(f)$, so it holds

$$\begin{aligned} d''_{f\cdot h} &= d''_h + \frac{1}{2} f^{-1} \circ \delta''_h(f) - f^{-1} \circ \delta'_h(f) \\ &= d''_h + \frac{1}{2} f^{-1} \circ \bar{\partial}_h(f) - \frac{1}{2} f^{-1} \circ \theta^*_h(f) - \frac{1}{2} f^{-1} \circ \partial_h(f) + \frac{1}{2} f^{-1} \circ \theta_h(f). \end{aligned}$$

Conversely, for a given Higgs operator d'' one verifies

$$D_{f\cdot h} = D_h + f^{-1} \circ \partial_h(f) + f^{-1} \circ \theta(f).$$

In particular, if f is constant then the two maps I_h and $I_{f\cdot h}$ coincide.

Definition 2.2 *i) $G_h := (d''_h)^2$ is called the pseudocurvature of D with respect to h.*
ii) $F_h := D_h^2$ is called the curvature of d'' with respect to h.

Remark 2.3 *i) Obviously it holds: $I_h(D)$ is an integrable Higgs operator if and only if $G_h = 0$, and $I_h^{-1}(d'')$ is a flat connection if and only if $F_h = 0$.*
ii) For $i = 1, 2$, let E_i be a differentiable complex vector bundle on X with Hermitian metric h_i and connection D_i. Let h be the induced metric and D the induced connection in $\mathrm{Hom}(E_1, E_2)$. Denote by $G_{i,h}$ resp. G_h the pseudocurvature of D_i resp. D with respect to h_i resp. h. Then for $f \in A^0(\mathrm{Hom}(E_1, E_2))$ it holds $G_h(f) = G_{2,h} \circ f - f \circ G_{1,h}$.
Similarly, the curvature F_h of the Higgs operator induced in $\mathrm{Hom}(E_1, E_2)$ by Higgs operators d''_i in the E_i is given by $F_h(f) = F_{2,h} \circ f - f \circ F_{1,h}$.
iii) If D is a connection, then D^2 is the curvature of d''_h with respect to h, and if d'' is a Higgs operator, then $(d'')^2$ is the pseudocurvature of D_h with respect to h. This trivially follows from the bijectivity of I_h.

Lemma 2.4 *i) For $D \in \mathcal{A}(E)$ let $D = d_h + \Theta_h = \partial_h + \bar\partial_h + \theta_h + \theta_h^*$ be the decomposition induced by h as above. If D is flat, then it holds $\delta_h^2 = 0$, $d_h(\Theta_h) = 0$, i.e. $\partial_h(\theta_h) = \bar\partial_h(\theta_h^*) = \partial_h(\theta_h^*) + \bar\partial_h(\theta_h) = 0$, and furthermore $d_h^2 = -\Theta_h \wedge \Theta_h$.*
ii) For $d'' = \bar\partial + \theta \in \mathcal{A}''(E)$ let ∂_h, d_h and D_h be as above, and write $d'_h := \partial_h + \theta^*$.
If d'' is integrable, then it holds $(d'_h)^2 = 0$, i.e. $\partial_h^2 = \partial_h(\theta^*) = \theta^* \wedge \theta^* = 0$, $d_h^2 = [\partial_h, \bar\partial]$, and hence $F_h = d_h^2 + [\theta, \theta^*] + \partial_h(\theta) + \bar\partial(\theta^*)$.

Proof: i) For $D = D' + D'' \in \mathcal{A}_f(E)$ it holds

$$\begin{aligned} 0 &= \partial\partial h(s,t) \\ &= h((D')^2(s),t) - h(D'(s), \delta_h''(t)) + h(D'(s), \delta_h''(t)) + h(s, (\delta_h'')^2(t)) \\ &= h(s, (\delta_h'')^2(t)) \end{aligned}$$

for all $s, t \in A^0(E)$, i.e. $(\delta_h'')^2 = 0$. Similarly one sees $(\delta_h')^2 = 0 = \delta_h'\delta_h'' + \delta_h''\delta_h'$, yielding $\delta_h^2 = 0$. We conclude

$$d_h(\Theta_h) = \frac{1}{4}[D + \delta_h, D - \delta_h] = 0,$$

and

$$0 = D^2 = (d_h + \Theta_h)^2 = d_h^2 + d_h(\Theta_h) + \Theta_h \wedge \Theta_h = d_h^2 + \Theta_h \wedge \Theta_h.$$

ii) For $d'' = \bar\partial + \theta \in \mathcal{H}''(E)$ and $d_h = \partial_h + \bar\partial$ it is well known that $\partial_h^2 = 0$, and hence $d_h^2 = [\partial_h, \bar\partial]$. Furthermore, for all $s, t \in A^0(E)$ it holds

$$\begin{aligned} & h(\partial_h(\theta^*)(s), t) \\ &= h(\partial_h \circ \theta^*(s), t) + h(\theta^* \circ \partial_h(s), t) \\ &= \partial h(\theta^*(s), t) + h(\theta^*(s), \bar\partial(t)) - h(\partial_h(s), \theta(t)) \\ &= \partial h(s, \theta(t)) + h(s, \theta \circ \bar\partial(t)) - h(\partial_h(s), \theta(t)) \\ &= h(\partial_h(s), \theta(t)) + h(s, \bar\partial \circ \theta(t)) + h(s, \theta \circ \bar\partial(t)) - h(\partial_h(s), \theta(t)) \\ &= h(s, \bar\partial(\theta)(t)) = 0, \end{aligned}$$

and

$$h(\theta^* \wedge \theta^*(s), t) = -h(s, \theta \wedge \theta(t)) = 0;$$

this shows $\partial_h(\theta^*) = 0 = \theta^* \wedge \theta^*$. □

Now let g be a Hermitian metric in X, and denote by ω_g the associated $(1,1)$-form on X, by Λ_g the contraction by ω_g, and by $*_g$ the associated Hodge-$*$-operator.
Recall that in the conformal class of g there exists a *Gauduchon* metric $\tilde g$, i.e. a metric satisfying $\bar\partial\partial(\omega_{\tilde g}^{n-1}) = 0$; $\tilde g$ is unique up to a constant positive factor if $n \geq 2$ ([G] p. 502, [LT] Theorem 1.2.4).

There is a natural way to define a map

$$\deg_g : \mathcal{H}(E) \longrightarrow \mathbb{R},$$

called *g-degree*, with the following properties (see [LT] sections 1.3 and 1.4):
- If g is a Gauduchon metric, and $\bar\partial \in \mathcal{H}(E)$ is a holomorphic structure, then $\deg_g(\bar\partial)$ is given as follows: Choose any Hermitian metric h in E, and let d be the *Chern connection* in (E, ∂) induced by h, i.e. the unique h-unitary connection in E with $(0,1)$-part $\bar\partial$. Then

$$\deg_g(\bar\partial) := \frac{i}{2\pi}\int\limits_X \mathrm{tr}(d^2)\wedge\omega_g^{n-1} = \frac{i}{2n\pi}\int\limits_X \mathrm{tr}\Lambda_g d^2\cdot\omega_g^n = \frac{i}{2n\pi}\int\limits_X \mathrm{tr}\Lambda_g[\bar\partial,\partial]\cdot\omega_g^n.$$

- If g is arbitrary, then there is a unique Gauduchon metric $\tilde g$ in the conformal class of g such that $\deg_g = \deg_{\tilde g}$.
The *g-slope* of $\bar\partial$ is

$$\mu_g(\bar\partial) := \frac{\deg_g(\bar\partial)}{r},$$

where r is the rank of E.
If $D = D' + D''$ is a flat connection, then it holds $(D'')^2 = 0$, so D'' is a holomorphic structure. We define the g-degree and g-slope of D as

$$\deg_g(D) := \deg_g(D''), \mu_g(D) := \mu_g(D'').$$

Similarly, for an integrable Higgs operator $d'' = \bar\partial + \theta$ it holds $\bar\partial^2 = 0$, and we define

$$\deg_g(d'') := \deg_g(\bar\partial), \mu_g(d'') := \mu_g(\bar\partial).$$

Observe that in all three cases the g-degrees (resp. slopes) of isomorphic operators are the same.

Remark 2.5 *Suppose that g is a Kähler metric, i.e. $d(\omega_g) = 0$. Then the g-degree is a topological invariant of the bundle E, completely determined by the first real Chern class $c_1(E)_{\mathbb{R}} \in H^2(X,\mathbb{R})$. In particular, since all real Chern classes of a flat bundle vanish, it holds $\deg_g(D) = 0$ for every flat connection D in E. On the other hand, if e.g. X is a surface admitting no Kähler metric and g is Gauduchon, then every real number is the g-degree of a flat line bundle on X ([LT] Proposition 1.3.13).*

Lemma 2.6 *If g is a Gauduchon metric, then for any metric h in E it holds:
i) If D is a flat connection, then*

$$\deg_g(D) = -\frac{i}{n\pi}\int\limits_X \mathrm{tr}\Lambda_g G_h\cdot\omega_g^n,$$

where G_h is the pseudocurvature of d'' with respect to h.
ii) If d'' is an integrable Higgs operator, then

$$\deg_g(d'') = \frac{i}{2n\pi}\int\limits_X \mathrm{tr}\Lambda_g F_h \cdot \omega_g^n,$$

where F_h is the curvature of d'' with respect to h.

PROOF: i) Observe that $\Lambda_g G_h = \Lambda_g \bar\partial_h(\theta_h)$. The Chern connection in (E, D'') induced by h is $D'' + \partial_h - \theta_h = D - 2\theta_h$, and it holds

$$\mathrm{tr}\Lambda_g(D-2\theta_h)^2 = -2\mathrm{tr}\Lambda_g((\bar\partial+\theta^*)(\theta) = -2\mathrm{tr}\Lambda_g(G_h+[\theta,\theta^*]) = -2\mathrm{tr}\Lambda_g(G_h),$$

so the claim follows by integration.
ii) Lemma 2.4 implies $\mathrm{tr}\Lambda_g F_h = \mathrm{tr}\Lambda_g d_h''^2$; again the claim follows by integration. □

3 EINSTEIN METRICS AND STABILITY FOR FLAT BUNDLES.

We fix a Hermitian metric g in X; the associated volume form is $\mathrm{vol}_g := \frac{1}{n!}\omega_g^n$, and the g-volume of X is $\mathrm{Vol}_g(X) := \int\limits_X \mathrm{vol}_g$. We further fix a Hermitian metric h in E, and denote by $|.|$ the pointwise norm on forms with values in E (and associated bundles) defined by h and g.
Let $D \in \mathcal{A}_f(E)$ be a flat connection in E, and write $D = d+\Theta = \partial+\bar\partial+\theta+\theta^*$ as in section 1. Let $d_h'' = I_h(D) = \bar\partial + \theta \in \mathcal{A}''(E)$ be the Higgs operator associated to D, and $G_h = (d_h'')^2$ its pseudocurvature. From $\Lambda_g G_h = \Lambda_g\bar\partial_h(\theta_h)$ and Lemma 2.4 we deduce

$$(i\Lambda_g G_h)^* = -i\Lambda_g((\bar\partial(\theta))^*) = -i\Lambda_g\partial(\theta^*) = i\Lambda_g\bar\partial(\theta) = i\Lambda_g G_h,$$

so $i\Lambda_g G_h$ is selfadjoint with respect to h.

REMARK 3.1 *It also holds $i\Lambda_g G_h = \frac{i}{2}\Lambda_g(\bar\partial(\Theta) - \partial(\Theta))$, which in the case of a Kähler metric g equals $\frac{1}{2}d^*(\Theta)$, where d^* is the L^2-adjoint of $d = \partial + \bar\partial$.*

DEFINITION 3.2 *h is called a g-Einstein metric in (E, D) if $i\Lambda_g G_h = c\cdot \mathrm{id}_E$ with a real constant c, which is called the Einstein constant.*

LEMMA 3.3 *Let h be a g-Einstein metric in (E, D), and $\tilde g = \varphi\cdot g$ conformally equivalent to g. Then there exists a $\tilde g$-Einstein metric $\tilde h$ in (E, D) which is conformally equivalent to h.*

PROOF: $\tilde g = \varphi \cdot g$ implies $\Lambda_{\tilde g} = \frac{1}{\varphi}\cdot\Lambda_g$. From Remark 2.1 iii) it follows that for $f \in \mathcal{C}^\infty(X,\mathbb{R})$ it holds $G_{e^f\cdot h} = G_h - \frac{1}{4}\bar\partial\partial(f)\cdot \mathrm{id}_E$. Hence the condition $i\Lambda_g G_h = c\cdot\mathrm{id}_E$ implies $i\Lambda_{\tilde g}G_{e^f\cdot h} = (\frac{c}{\varphi} - \frac{1}{4}P(f))\cdot\mathrm{id}_E$, where $P := i\Lambda_{\tilde g}\bar\partial\partial$. Since $\mathcal{C}^\infty(X,\mathbb{R}) = \mathrm{im}P\oplus\mathbb{R}$ ([LT] Corollary 2.9), there exists an f such that $\frac{c}{\varphi} - \frac{1}{4}P(f)$ is constant. □

Lemma 3.4 *If* $i\Lambda_g G_h = c \cdot \mathrm{id}_E$ *with* $c \in \mathbb{R}$, *then it holds:*
i) $c = -\frac{\pi}{(n-1)! \cdot \mathrm{Vol}_g(X)} \cdot \mu_g(D)$ *if* g *is Gauduchon.*
ii) $\deg_g(D) = 0$ *if and only if* $c = 0$.

Proof: i) is an immediate consequence of Lemma 2.6.
ii) If g is Gauduchon, then this follows from i). If g is arbitrary, then let $\tilde{g} = \varphi \cdot g$ be the Gauduchon metric in its conformal class such that $\deg_g = \deg_{\tilde{g}}$. Now we have

$$i\Lambda_g G_h = 0 \Longleftrightarrow i\Lambda_{\tilde{g}} G_h = 0 \Longleftrightarrow \deg_{\tilde{g}}(D) = 0 \Longleftrightarrow \deg_g(D) = 0.$$

□

Remark 3.5 *i) If two flat connections* D_1,D_2 *are isomorphic via the automorphism* f *of* E, *i.e. if* $D_2 \circ f - f \circ D_1 = 0$, *and if* h *is a* g*-Einstein metric in* (E, D_1), *then* $f_* h$ *is a* g*-Einstein metric in* (E, D_2) *with the same Einstein constant.*
ii) By Remark 2.3, a necessary condition for $d''_h = I_h(D)$ *to be an integrable Higgs operator is that* h *is a* g*-Einstein metric for* D *with Einstein constant* $c = 0$, *so in particular* $\deg_g(D) = 0$. *On the other hand it holds* $d^2 = -\Theta \wedge \Theta$ *(Lemma 2.4), and, if* d''_h *is integrable,* $\theta \wedge \theta = 0$ *implying* $\theta^* \wedge \theta^* = 0$. *This gives* $\mathrm{tr}(d^2) = -\mathrm{tr}[\theta, \theta^*] = 0$, *which implies* $\deg_g(d''_h) = 0$.
iii) For complex vector bundles on compact Riemannian manifolds (X, g), *Corlette defines a* g*-harmonic metric for a flat connection by the condition* $d^*(\Theta) = 0$ *([C]). If* X *is complex and* g *is a Kähler metric, then the* g*-degree of any flat connection vanishes, so in this context* g*-harmonic is the same as* g*-Einstein (see Remarks 2.5 and 3.1), but in general the two notions are different.*

Now we prove a useful Vanishing Theorem.

Proposition 3.6 *Let* D *be a flat connection in* E, *and* h *a* g*-Einstein metric in* (E, D) *with Einstein constant* c.
If $c > 0$, *then the only section* $s \in A^0(E)$ *with* $D(s) = 0$ *is* $s = 0$.
If $c = 0$, *then for every section* $s \in A^0(E)$ *with* $D(s) = 0$ *it holds* $\bar{\partial}(s) = \theta(s) = 0$ *and* $\partial(s) = \theta^*(s) = 0$, *so in particular* $d''_h(s) = 0$.

Proof: $D(s) = 0$ is equivalent to

$$\partial(s) = -\theta(s), \bar{\partial}(s) = -\theta^*(s); \tag{1}$$

this implies

$$\bar{\partial}\partial h(s,s) = -h(\bar{\partial} \circ \theta(s), s) - h(\theta(s), \theta(s)) + h(\bar{\partial}(s), \bar{\partial}(s)) - h(s, \partial \circ \theta^*(s)). \tag{2}$$

The assumption that h is g-Einstein means $i\Lambda_g \bar{\partial}(\theta) = i\Lambda_g G_h = c \cdot \mathrm{id}_E$, which is equivalent to $i\Lambda_g \partial(\theta^*) = -c \cdot \mathrm{id}_E$ since $(i\Lambda_g \bar{\partial}(\theta))^* = -i\Lambda_g(\bar{\partial}(\theta)^*) = -i\Lambda_g \partial(\theta^*)$; these relations can be rewritten as

$$i\Lambda_g \bar{\partial} \circ \theta = -i\Lambda_g \theta \circ \bar{\partial} + c \cdot \mathrm{id}_E, \quad i\Lambda_g \partial \circ \theta^* = -i\Lambda_g \theta^* \circ \partial - c \cdot \mathrm{id}_E. \tag{3}$$

Using (1) and (3) we get

$$\begin{aligned} i\Lambda_g h(\bar\partial\circ\theta(s),s) &= -i\Lambda_g h(\theta\circ\bar\partial(s),s)+c\cdot|s|^2 = i\Lambda_g h(\bar\partial(s),\theta^*(s))+c\cdot|s|^2 \\ &= -i\Lambda_g h(\bar\partial(s),\bar\partial(s))+c\cdot|s|^2 = |\bar\partial(s)|^2+c\cdot|s|^2, \end{aligned}$$

and similarly

$$i\Lambda_g h(s,\partial\circ\theta^*(s)) = |\theta(s)|^2+c\cdot|s|^2,$$

so (2) implies

$$i\Lambda_g\bar\partial\partial h(s,s) = -2\left(|\bar\partial(s)|^2+|\theta(s)|^2+c\cdot|s|^2\right).$$

Since the image of the operator $i\Lambda_g\bar\partial\partial$ on real functions contains no non-zero functions of constant sign ([LT] Lemma 7.2.7), this gives $s=0$ in the case $c>0$, and if $c=0$ we get $\bar\partial(s)=\theta(s)=0$, implying $\partial(s)=\theta^*(s)=0$ because of (1). □

The following corollary will be used later in the context of moduli spaces.

Corollary 3.7 *For $i=1,2$ let $D_i\in\mathcal{A}_f(E)$ be a flat connection, h_i a g-Einstein metric in (E,D_i), and $d_i'':=I_{h_i}(D_i)\in\mathcal{A}''(E)$ the associated Higgs operator. If D_1 and D_2 are isomorphic via the automorphism f of E, then d_1'' and d_2'' are isomorphic via f, too.*

Proof: Let h be the metric in End$E=E^*\otimes E$ induced by the dual metric of h_1 in E^* and h_2 in E, and D the connection in End$E$ defined by $D(f)=D_2\circ f-f\circ D_1$ for all $f\in A^0(\mathrm{End}E)$. Then D is flat of g-degree 0 since D_1 and D_2 are flat of equal degree, and h is a g-Einstein metric in $(\mathrm{End}E,D)$ with Einstein constant $c=0$ (compare Remark 2.3). Furthermore, the Higgs operator d'' in End$E$ defined by $d''(f)=d_2''\circ f-f\circ d_1''$ equals $I_h(D)$. Hence Proposition 3.6 implies that an automorphism f of E with $D(f)=0$ also satisfies $d''(f)=0$. □

If $F\subset E$ is a D-invariant subbundle of E, then it is obvious that flatness of D implies flatness of $D|_F$, and hence the following definition makes sense.

Definition 3.8 *A flat connection D in E is called g-(semi)stable iff for every proper D-invariant subbundle $0\neq F\subset E$ it holds $\mu_g(D|_F)>\mu_g(D)$ $(\mu_g(D|_F)\geq\mu_g(D))$. D is called g-polystable iff $E=E_1\oplus E_2\oplus\ldots\oplus E_k$ is a direct sum of D-invariant and g-stable subbundles E_i with $\mu_g(D|_{E_i})=\mu_g(D)$ for $i=1,2,\ldots,k$.*

Remark 3.9 *i) Let D be a flat connection in E, and $0\neq F\subset E$ a proper D-invariant subbundle. Then g-stability of D implies $\mu_g(D|_F)>\mu_g(D)$ and hence the g-instability of the holomorphic structure D'' in E (in the sense of e.g. [LT]) since F is a D''-holomorphic subbundle of E.*
ii) Suppose that g is a Kähler metric; then $\deg_g(D)=0$ for every flat connection D (Remark 2.5). Hence a flat connection D in E is

- always g-semistable,
- g-stable if and only if E has no proper non-trivial D-invariant subbundle,
- g-polystable if E is a direct sum of D-invariant g-stable subbundles.
This means that g-(poly-)stability on a Kähler manifold coincides with (poly-)stability in the sense of Corlette [C].
iii) It is obvious that stability in the sense of Corlette always implies g-stability, but at the end of this section we will give an example of a g-stable bundle which is not stable in the sense of Corlette.

DEFINITION 3.10 *A flat connection D in E is simple if the only D-parallel endomorphisms f, i.e. those with $D_{\mathrm{End}}(f) = D \circ f - f \circ D = 0$, are the homotheties $f = a \cdot \mathrm{id}_E,\ a \in \mathbb{C}$.*

Let D be a flat connection in E, $0 \neq F \subset E$ a D-invariant subbundle, and $Q := {}^{E}/_{F}$ the quotient with natural projection $\pi : E \longrightarrow Q$. Then D induces a flat connection D_Q in Q such that $D_Q \circ \pi = \pi \circ D$. In particular, F is a holomorphic subbundle of (E, D''), and D''_Q is the induced holomorphic structure in Q. Since the g-degree of a flat connection D by definition equals the g-degree of the associated holomorphic structure D'', it follow $\deg_g(D) = \deg_g(D_1) + \deg_g(D_Q)$. Hence as in the case of holomorphic bundles one verifies (compare [K] Chapter V)

PROPOSITION 3.11 *i) A flat connection D in E is g-(semi)stable if and only if for every D-invariant proper subbundle $0 \neq F \subset E$ with quotient $Q = {}^{E}/_{F}$ it holds $\mu_g(D_Q) < \mu_g(D)$ (resp. $\mu_g(D_Q) \leq \mu_g(D)$.)*
ii) Let (E_1, D_1) and (E_2, D_2) be g-stable flat bundles over X with $\mu_g(D_1) = \mu_g(D_2)$. If $f \in A^0(\mathrm{Hom}(E_1, E_2))$ satisfies $D_2 \circ f = f \circ D_1$, then either $f = 0$ or f is an isomorphism.
iii) A g-stable flat connection D in E is simple.

Next we prove the first half of the main result of this section.

PROPOSITION 3.12 *Let D be a flat connection in E, and h a g-Einstein metric in (E, D) with Einstein constant c; then D is g-semistable. If D is not g-stable, then D is g-polystable; more precisely, $E = E_1 \oplus E_2 \oplus \ldots \oplus E_k$ is a h-orthogonal direct sum of D-invariant g-stable subbundles such that $\mu_g(D|_{E_i}) = \mu_g(D)$ for $i = 1, 2, \ldots, k$. Furthermore, $h|_{E_i}$ is a g-Einstein metric in $(E_i, D|_{E_i})$ with Einstein constant c for all i, and the direct sum is invariant with respect to the Higgs operator $d''_h = I_h(D)$.*

PROOF: First we consider the case when g is a Gauduchon metric. Let $0 \neq F \subset E$ be a D-invariant proper subbundle of rank s; then $E = F \oplus F^{\perp}$, where $F^{\perp}$ is the h-orthogonal complement of F. With respect to this decomposition, we write operators as 2×2 matrices, so D has the form

$$D = \begin{pmatrix} D_1 & A \\ 0 & D_2 \end{pmatrix},$$

where $D_1 = D|_F$ and D_2 is a flat connection in $F^\perp$. We use notations as in section 2; it is easy to see that the operator δ associated to D by h has the form

$$\delta = \begin{pmatrix} \delta_1 & 0 \\ A^* & \delta_2 \end{pmatrix},$$

where the δ_i are the operators associated to the D_i by h. Similarly it holds

$$\bar\partial = \frac{1}{2}(D'' + \delta'') = \frac{1}{2}\begin{pmatrix} D_1'' + \delta_1'' & A'' \\ A'^* & D_2'' + \delta_2'' \end{pmatrix} = \begin{pmatrix} \bar\partial_1 & \frac{1}{2}A'' \\ \frac{1}{2}A'^* & \bar\partial_2 \end{pmatrix},$$

and

$$\theta = \frac{1}{2}(D' - \delta') = \begin{pmatrix} D_1' - \delta_1' & A' \\ -A''^* & D_2' - \delta_2' \end{pmatrix} = \begin{pmatrix} \theta_1 & \frac{1}{2}A' \\ -\frac{1}{2}A''^* & \theta_2 \end{pmatrix},$$

where A' resp. A'' is the part of A of type $(1,0)$ resp. $(0,1)$. This implies

$$\begin{aligned} \bar\partial(\theta) &= [\bar\partial, \theta] \\ &= \begin{pmatrix} \bar\partial_1(\theta_1) + \frac{1}{4}(A' \wedge A'^* - A'' \wedge A''^*) & * \\ * & \bar\partial_2(\theta_2) + \frac{1}{4}(A'^* \wedge A' - A''^* \wedge A'') \end{pmatrix}, \end{aligned}$$

hence

$$\begin{aligned} c \cdot \mathrm{id}_E &= i\Lambda_g G_h \\ &= \begin{pmatrix} i\Lambda_g G_{1,h} + \frac{i}{4}\Lambda_g(A' \wedge A'^* - A'' \wedge A''^*) & * \\ * & i\Lambda_g G_{2,h} + \frac{i}{4}\Lambda_g(A'^* \wedge A' - A''^* \wedge A'') \end{pmatrix}, \end{aligned} \tag{4}$$

and thus

$$sc = \mathrm{tr}(i\Lambda_g G_{1,h} + \frac{i}{4}\Lambda_g(A' \wedge A'^* - A'' \wedge A''^*)) = i\mathrm{tr}\Lambda_g G_{1,h} + \frac{1}{4}|A|^2 .$$

Using Lemma 2.6 and Lemma 3.4 we conclude

$$\mu_g(D_1) = -\frac{i}{sn\pi}\int_X \mathrm{tr}\Lambda_g G_{1,h} \cdot \omega_g^n \geq -\frac{c(n-1)!}{\pi}\mathrm{Vol}_g(X) = \mu_g(D); \tag{5}$$

this prove that D is g-semistable.

If D is not g-stable, then there exists a subbundle F as above such that equality holds in (5), which implies $A = 0$. This means not only that $F^\perp$ is D-invariant, too, with $D|_{F^\perp} = D_2$, but also that

$$i\Lambda_g G_{1,h} = c \cdot \mathrm{id}_F, i\Lambda_g G_{2,h} = c \cdot \mathrm{id}_{F^\perp}$$

by (4). Hence the restriction of h to F resp. $F^\perp$ is g-Einstein for D_1 resp. D_2, and it holds $\mu_g(D_1) = \mu_g(D) = \mu_g(D_2)$ by Lemma 3.4. Furthermore, the D-invariance of F means that the inclusion $i : F \hookrightarrow E$ is parallel with respect to the flat connection in $\mathrm{Hom}(F, E)$ induced by D_1 and D. Using Remark 2.3

and Proposition 3.6 as in the proof of Corollary 3.7, we conclude that i is also parallel with respect to the associated Higgs operator, i.e. that F is d''_h-invariant; the same argument works for $F^\perp$. If D_1 and D_2 are stable, then we are done; otherwise the proof is finished by induction on the rank.

Now let g be arbitrary, let $\tilde{g}$ be the Gauduchon metric in its conformal class with $\deg_g = \deg_{\tilde{g}}$, and let $\tilde{h}$ be a $\tilde{g}$-Einstein metric in the conformal class of h, which exists by Lemma 3.3; then the theorem holds for $\tilde{g}$ and $\tilde{h}$. Since g and $\tilde{g}$ define the same degree and slope, and hence stability, it follows that D is $\tilde{g}$-semistable. If D is not g-stable, then there exists a D-invariant proper subbundle F as above with $\mu_{\tilde{g}}(D_1) = \mu_g(D_1) = \mu_g(D) = \mu_{\tilde{g}}(D)$. Note that the h-orthogonal complement $F^\perp$ of F is also the $\tilde{h}$-orthogonal complement, since h and $\tilde{h}$ are conformally equivalent. Hence, using $\tilde{g}$ and $\tilde{h}$ we conclude as above that $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$ with respect the decomposition $E = F \oplus F^\perp$; now we can proceed as in the Gauduchon case. □

Another consequence of Proposition 3.6 is

Proposition 3.13 *Let D be a simple flat connection in E. If a g-Einstein metric in (E, D) exists, then it is unique up to a positive scalar.*

Proof: Let h_1,h_2 be g-Einstein metrics in (E, D), and $c \in \mathbb{R}$ the Einstein constant. There are differentiable automorphisms f and k of E, selfadjoint with respect to both h_1 and h_2, such that $f = k^2$ and $h_2(s,t) = h_1(f(s), t) = h_1(k(s), k(t))$ for all $s, t \in A^0(E)$. Since D is simple it suffices to show $D(f) = 0$. We define a new flat connection $\tilde{D} := k \circ D \circ k^{-1}$. In what follows, operators δ, d, Θ etc. with a subscript i are associated to D by the metric h_i, without a subscript they are associated to $\tilde{D}$ by h_1. One verifies

$$\delta_2 = f^{-1} \circ \delta_1 \circ f, \delta = k^{-1} \circ \delta_1 \circ k = k \circ \delta_2 \circ k^{-1},$$

implying

$$d = \frac{1}{2}(\tilde{D} + \delta) = k \circ d_2 \circ k^{-1}, \Theta = \frac{1}{2}(\tilde{D} - \delta) = k \circ \Theta_2 \circ k^{-1}$$

and hence

$$i\Lambda_g G_{h_1} = i\Lambda_g \bar{\partial}(\theta) = ik \circ \Lambda_g \bar{\partial}_2(\theta_2) \circ k^{-1} = ik \circ \Lambda_g G_{2,h_2} \circ k^{-1} = c \cdot \mathrm{id}_E,$$

so h_1 is a g-Einstein metric in $(E, \tilde{D})$. It follows that h_1 induces a g-Einstein metric with Einstein constant 0 for the flat connection $\tilde{D}_{\mathrm{End}}(.) = . \circ D - \tilde{D} \circ .$ in End$E$. By definition it holds $\tilde{D}_{\mathrm{End}}(k) = 0$, so Proposition 3.6 implies $\tilde{d}_{\mathrm{End}}(k) = 0$. Since $\tilde{\delta}_{\mathrm{End}} = 2\tilde{d}_{\mathrm{End}} - \tilde{D}_{\mathrm{End}}$, it follows

$$0 = \tilde{\delta}_{\mathrm{End}}(k) = k \circ \delta_1 - \delta \circ k = k \circ \delta_1 - k^{-1} \circ \delta_1 \circ k^2 = k^{-1} \circ (f \circ \delta_1 - \delta_1 \circ f) \ ,$$

implying $\delta_{1,\mathrm{End}}(f)=0$, where $\delta_{1,\mathrm{End}}$ is the operator on End$E$ induced by D and h_1. But this is equivalent to $\delta'_{1,\mathrm{End}}(f)=0$ and $\delta''_{1,\mathrm{End}}(f)=0$, and taking adjoints with respect to h_1 we get

$$0=(\delta'_{1,\mathrm{End}}(f))^*=D''_{\mathrm{End}}(f), 0=(\delta''_{1,\mathrm{End}}(f))^*=D'_{\mathrm{End}}(f),$$

i.e. $D_{\mathrm{End}}(f)=0$. □

Let (E,D), $(\tilde{E},\tilde{D})$ be flat bundles with g-Einstein metrics h, $\tilde{h}$. Let $E=\bigoplus_{i=1}^{k}E_i$ and $\tilde{E}=\bigoplus_{i=1}^{l}\tilde{E}_i$ be the orthogonal, invariant splittings given by Proposition 3.12. We write $D_i:=D|_{E_i}$, $\tilde{D}_i:=\tilde{D}|_{\tilde{E}_i}$, $h_i:=h|_{E_i}$, $\tilde{h}_i:=\tilde{h}|_{\tilde{E}_i}$. Using Propositions 3.11 and 3.13 one verifies

COROLLARY 3.14 *If there exists an isomorphism $f\in A^0(\mathrm{Hom}(E,\tilde{E}))$ satisfying $f\circ D=\tilde{D}\circ f$, then it holds $k=l$, and, after renumbering of the summands if necessary, there are isomorphisms $f_i\in A^0(\mathrm{Hom}(E_i,\tilde{E}_i))$ such that $f_i\circ D_i=\tilde{D}_i\circ f$ and $f_*(h_i)=\tilde{h}_i$.*

The following result is the converse of Proposition 3.12.

PROPOSITION 3.15 *Let (E,D) a g-stable flat bundle over X. Then there exists a g-Einstein metric for (E,D).*

SKETCH OF PROOF: The proof is very similar to the one for the existence of a g-Hermitian Einstein metric in a g-stable holomorphic vector bundle as given in Chapter 3 of [LT]. Therefore we will be brief, leaving it to the reader to fill in the necessary details.
First observe that by Lemma 3.3 we may assume that g is a Gauduchon metric. For any metric h in E it holds

$$G_h=\bar{\partial}_h(\theta_h)=\frac{1}{4}[D''+\delta''_h,D'-\delta'_h]=-\frac{1}{4}[D'',\delta'_h]+\frac{1}{4}[D',\delta''_h]$$

since $D^2=\delta_h^2=0$. Observe that $[D'',\delta'_h]$ resp. $[D',\delta''_h]$ is the curvature of the h-unitary connection $D''+\delta'_h$ resp. $D'+\delta''_h$.
Fix a metric h_0 in E, and let $\delta=\delta'+\delta''$, $d=\partial+\bar{\partial}$, $\Theta=\theta+\theta^*$ be the operators associated to $D=D'+D''$ and h_0 as in section 2. Consider for an h_0-selfadjoint positive definite endomorphism f of E and $\varepsilon\in[0,1]$ the differential equation

$$L_\varepsilon(f):=K^0-\frac{i}{4}\Lambda_g D''(f^{-1}\circ\delta'(f))+\frac{i}{4}\Lambda_g D'(f^{-1}\circ\delta''(f))-\varepsilon\cdot\log(f)=0, \quad (6)$$

where $K^0:=i\Lambda_g\bar{\partial}(\theta)-c\cdot\mathrm{id}_E=-\frac{i}{4}\Lambda_g([D'',\delta']-[D',\delta''])-c\cdot\mathrm{id}_E$, and c is the constant associated to a possible g-Einstein metric for (E,D). The metric $f\cdot h_0$, defined by $f\cdot h_0(s,t):=h_0(f(s),t)$ for sections s,t in E, is g-Einstein if and only if $L_0(f)=0$.

The term $T_1 := i\Lambda_g D''(f^{-1} \circ \delta'(f))$ (associated to the unitary connection $d_1 := \delta' + D''$) in equation (6) is of precisely the same type as the term $T_0 := i\Lambda_g \bar\partial(f^{-1} \circ \partial_0(f))$ (associated to the unitary connection $d_0 = \partial_0 + \bar\partial$) in equation $(**)$ on page 62 in [LT], and the term $T_2 := -i\Lambda_g D'(f^{-1} \circ \delta''(f))$ (associated to the unitary connection $d_2 := D' + \delta''$) is *almost* of this type; e.g. the trace of all three terms equals $i\Lambda_g \bar\partial\partial(\mathrm{tr}(\log f)$, and the symbols of the differential operators $\frac{d}{df}\hat T_i$, where $\hat T_i(f) := f \circ T_i(f)$, are equal, too. Therefore most of the arguments in [LT] can easily be adapted to show first that for a simple flat connection D equation (6) has solutions f_ε for all $\varepsilon \in (0,1]$, which satisfy $\det f_\varepsilon \equiv 1$, and which converge to a solution f of $L_0(f) = 0$ if the L^2-norms of the f_ε are uniformly bounded. (There are two places where one has to argue in a slightly different way: In the proof of the analogue of [LT] Lemma 3.3.1, one uses the Laplacian $\Delta_D = D^* \circ D$ instead of $\Delta_{\bar\partial}$, and in the proof of the analogue of [LT] Proposition 3.3.5 the *sum* $\Delta_{d_1} + \Delta_{d_2}$ of the two Laplacians associated to d_1 and d_2 instead of just one.)
Then, under the assumptions that $\mathrm{rk}E \geq 2$ and that the L^2-norms of the f_ε are unbounded, one shows that for suitable $\varepsilon_i \longrightarrow 0$, $\rho(\varepsilon_i) \longrightarrow 0$, the limit

$$\pi := \mathrm{id}_E - \lim_{\sigma \longrightarrow 0} \left(\lim_{i \longrightarrow \infty} \rho(\varepsilon_i) \cdot f_{\varepsilon_i} \right)^\sigma$$

exists weakly in L_1^2, and satisfies in L^1 $\pi = \pi^* = \pi^2$ and

$$(\mathrm{id}_E - \pi) \circ D(\pi) = 0. \tag{7}$$

This implies $(\mathrm{id}_E - \pi) \circ D''(\pi) = 0$, so π defines a weakly holomorphic subbundle $\mathcal{F}$ of the holomorphic bundle (E, D'') by a theorem of Uhlenbeck and Yau (see [UY], [LT] Theorem 3.4.3). $\mathcal{F}$ is a coherent subsheaf of (E, D''), a holomorphic subbundle outside an analytic subset $S \subset X$ of codimension at least 2, and π is smooth on $X \setminus S$. Therefore (7) implies that $\mathcal{F}|_{X\setminus S}$ is in fact a D-invariant subbundle of $E|_{X\setminus S}$, which extends to a D-invariant subbundle F of E by the Lemma below. Again using arguments as is [LT], one finally shows that F violates the stability condition for (E, D). □

Lemma 3.16 *Let X be a differentiable manifold, E a differentiable vector bundle over X, and D a flat connection in E. Let $S \subset X$ be a subset such that $X \setminus S$ is open and dense in X, and with the following property: For every point $x \in S$ and every open neighborhood U of x in X there exists an open neighborhood $x \in U' \subset U$ such that $U' \setminus S$ is path-connected.*
Then every D-invariant subbundle $\mathcal{F}$ of $E|_{X\setminus S}$ extends to a D-invariant subbundle F of E.

Proof: For every $x \in S$ choose an open neighborhood $x \in U \subset X$ such that $U \setminus S$ is path connected and $(E|_U, D) \cong (U \times V, d)$, where V is a vector space and d the trivial flat connection. Since $\mathcal{F}$ is D-invariant and $U \setminus S$ is path connected, it holds

$$(\mathcal{F}|_{U\setminus S}, D) \cong ((U \setminus S) \times W, d),$$

where $W \subset V$ is a constant subspace. Define F over U by $F|_U :\cong U \times W$; then the topological condition on S implies that this is well defined on S, and hence gives a D-invariant extension F of $\mathcal{F}$ over X. $\square$

The following main result of this section is a direct consequence of Propositions 3.12 and 3.15.

THEOREM 3.17 *A flat connection D in E admits a g-Einstein metric if and only if it is g-polystable.*

As for stable vector bundles and Hermitian-Einstein metrics, the gauge theoretic interpretation of our results is as follows. The group

$$\mathcal{G}^{\mathbb{C}} := A^0(GL(E))$$

of differentiable automorphisms of E acts on $\mathcal{A}(E)$ by $D \cdot f = f^{-1} \circ D \circ f$, so

$$\mathcal{A}(E)/_{\mathcal{G}^{\mathbb{C}}}$$

is the moduli space of isomorphism classes of connections in E. Observe that flatness, simplicity and g-stability are preserved under this action. Fix a metric h in E; then it holds:

COROLLARY 3.18 *The following two statements for a flat connection D are equivalent:*
i) D is g-stable.
ii) D is simple, and there is a connection D_0 in the $\mathcal{G}^{\mathbb{C}}$-orbit through D such that h is g-Einstein for D_0.

The essential uniqueness of a g-Einstein metric (Proposition 3.13) implies that the connection D_0 in ii) is unique up to the action of the subgroup

$$\mathcal{G} := A^0(U(E,h)) \subset \mathcal{G}^{\mathbb{C}}$$

of h-unitary automorphisms. This means that the moduli space

$$\mathcal{M}_f^{\mathrm{st}}(E) = \{\ D \in \mathcal{A}_f(E) \mid D \text{ is } g- \text{ stable }\}/_{\mathcal{G}^{\mathbb{C}}}$$

of isomorphism classes of g-stable flat connections in E coincides with the quotient

$$\{\ D \in \mathcal{A}_f(E) \mid D \text{ is simple and } h \text{ is } g-\text{Einstein for } D\ \}/_{\mathcal{G}}.$$

EXAMPLE: We now give the promised example of a flat bundle which is g-stable, but not stable in the sense of Corlette.

An Inoue surface of type $S_N^{\pm}$ is the quotient of $\mathbb{H}\times\mathbb{C}$ by an affine transformation group G generated by

$$\begin{aligned} g_0(w,z) &:= (\alpha w, \pm z + t), \\ g_i(w,z) &:= (w + a_i, z + b_i w + c_i),\ i = 1,2, \\ g_3(w,z) &:= (w, z + c_3), \end{aligned}$$

with certain constants $\alpha, a_i, b_i, c_3 \in \mathbb{R},\ \ c_1, c_2 \in \mathbb{C}$ (see [P] p. 160). Since the second Betti number of $S_N^{\pm}$ vanishes, the degree map

$$\deg_g : \mathrm{Pic}(S_N^{\pm}) \longrightarrow \mathbb{R}$$

associated to a Gauduchon metric g is, up to a positive factor, independent of the chosen metric g. In particular, *all* Hermitian metrics g define the same notion of g-stability ([LT] Remark 1.4.4 iii)).
The trivial flat connection d on $\mathbb{H} \times \mathbb{C}$ induces a flat connection D in the tangent bundle $E := T_{S_N^{\pm}}$. A D-invariant sub-line bundle of E is in particular a holomorphic subbundle, so it defines a holomorphic foliation of $S_N^{\pm}$. According to [B] Théorème 2, there is precisely one such foliation, namely the one induced by the G-invariant vertical foliation (i.e. with leaves $\{w\} \times \mathbb{C}$) of $\mathbb{H} \times \mathbb{C}$. The corresponding trivial line bundle L_0 on $\mathbb{H} \times \mathbb{C}$ is d-invariant, so it descends to a unique D-invariant subbundle L of E; this shows that E is not stable in the sense of Corlette. Observe that L has factors of automorphy $\chi(g_i) = \pm 1,\ i = 0,1,2,3$, so the standard flat metric in L_0 defines a metric h in L such that the associated Chern connection in $(L, D''|_L)$ is flat; this implies $\mu_g(D|_L) = \deg_g(D|_L) = 0$. On the other hand, the g-degree, and hence the g-slope, of E is negative by [P] Proposition 4.7; this implies the g-stability of E since L is the only D-invariant proper subbundle of E.

4 Einstein metrics and stability for Higgs bundles.

Again we fix Hermitian metrics g in X and h in E.
Let $d'' = \bar\partial + \theta \in \mathcal{A}_i''(E)$ be an integrable Higgs operator,

$$D_h = I_h^{-1}(d'') = d + \Theta = \partial + \bar\partial + \theta + \theta^* \in \mathcal{A}(E)$$

the connection associated to d'' as in section 2, and $F_h = D_h^2$ its curvature.

Definition 4.1 *h is called a g-Einstein metric in (E, d'') if and only if $K_h := i\Lambda_g F_h = c \cdot \mathrm{id}_E$ with a real constant c, the Einstein constant.*

Lemma 4.2 *Let h be a g-Einstein metric in (E, d''), and $\tilde g = \varphi \cdot g$ conformally equivalent to g. Then there exists a $\tilde g$-Einstein metric $\tilde h$ in (E, d'') which is conformally equivalent to h.*

Proof: From Remark 2.1 iii) it follows that for $f \in \mathcal{C}^\infty(X,\mathbb{R})$ it holds $F_{e^f\cdot h} = F_h + \bar\partial\partial(f)\cdot \mathrm{id}_E$. Using this, the proof is analogous to that of Lemma 3.3. □

Notice that since d'' is integrable it holds (compare Lemma 2.4)

$$K_h = i\Lambda_g(d^2 + [\theta,\theta^*]) = i\Lambda_g([\partial,\bar\partial] + [\theta,\theta^*])$$

where $d = \partial + \bar\partial$. An immediate consequence of Lemma 2.6 and Lemma 4.2 is (compare the proof of Lemma 3.4)

Lemma 4.3 *If $i\Lambda_g F_h = c\cdot \mathrm{id}_E$ with $c\in\mathbb{R}$, then it holds:*
i) $c = \frac{2\pi}{(n-1)!\cdot \mathrm{Vol}_g(X)}\cdot \mu_g(d'')$ if g is Gauduchon.
ii) $\deg_g(d'') = 0$ if and only if $c = 0$.

Remark 4.4 *(compare Remark 3.5)*
i) If two integrable Higgs operators d''_1,d''_2 are isomorphic via the automorphism f of E, i.e. if $d''_2\circ f - f\circ d''_1 = 0$, and if h is a g-Einstein metric in (E,d''_1), then f_*h is a g-Einstein metric in (E,d''_2), and the associated Einstein constants are equal.
ii) By Remark 2.3, a necessary condition for $D_h = I_h(d'')$ to be a flat connection is h to be Einstein with Einstein constant $c = 0$, so in particular $\deg_g(d'') = 0$. On the other hand, the Chern connection in (E, D''_h) is $\partial - \theta + \bar\partial + \theta^*$, so the g-degree of D_h is obtained by integrating $\mathrm{tr}\Lambda_g[\bar\partial+\theta^*, \partial-\theta]$ which equals $\mathrm{tr}\Lambda_g[\bar\partial,\partial]$ since d'' is integrable (Lemma 2.4 ii)). If D_h is flat, we furthermore have $d^2 = -\Theta\wedge\Theta$ (Lemma 2.4 i)), implying $\mathrm{tr}[\bar\partial,\partial] = 0$ and hence $\deg_g(D_h) = 0$.

In analogy with the case of Hermitian-Einstein metrics in holomorphic vector bundles, the following vanishing theorem holds.

Proposition 4.5 *Let h be a g-Einstein metric in (E,d'') with Einstein constant c.*
If $c < 0$, then the only section $s\in A^0(E)$ with $d''(s) = 0$ is $s = 0$.
If $c = 0$, then for every section $s \in A^0(E)$ with $d''(s) = 0$ it holds $D_h(s) = 0$.

Proof: For $s\in A^0(E)$, $d''(s) = 0$ is equivalent to $\bar\partial(s) = 0 = \theta(s)$. This implies

$$c\cdot|s|^2 = c\cdot h(s,s) = h(K_h(s),s) = i\Lambda_g\left(h(\bar\partial\partial(s),s) + h(\theta^*(s),\theta^*(s))\right). \quad (8)$$

We have

$$i\Lambda_g\bar\partial\partial h(s,s) = i\Lambda_g\left(h(\bar\partial\partial(s),s) - h(\partial(s),\partial(s))\right)$$

since $\bar\partial(s) = 0$, and using (8) we get

$$i\Lambda_g\bar\partial\partial h(s,s) = c\cdot|s|^2 - |\partial(s)|^2 - |\theta^*(s)|^2.$$

Now the claim follows as in the proof of Proposition 3.6. □

The proof of the following corollary is analogous to that of Corollary 3.7.

Corollary 4.6 *For $i=1,2$ let $d_i'' \in \mathcal{A}_i''(E)$ be an integrable Higgs operators, h_i a g-Einstein metric in (E,d_i''), and $D_i := I_{h_i}^{-1}(d_i'') \in \mathcal{A}(E)$ the associated connection. If d_1'' and d_2'' are isomorphic via the automorphism f of E, then D_1 and D_2 are isomorphic via f, too.*

Let $d'' = \bar\partial + \theta$ be an integrable Higgs operator in E. A coherent subsheaf $\mathcal{F}$ of the holomorphic bundle $(E,\bar\partial)$ is called a *Higgs-subsheaf* of (E,d'') iff it is d''-invariant. For the definition of the g-degree and g-slope of a coherent sheaf see [LT].

Definition 4.7 *An integrable Higgs operator d'' in E is called g-(semi)stable iff for every coherent Higgs-subsheaf $\mathcal{F}$ of (E,d'') with $0 < \mathrm{rk}\mathcal{F} < \mathrm{rk}E$ it holds $\mu_g(\mathcal{F}) < \mu_g(E)$ ($\mu_g(\mathcal{F}) \le \mu_g(E)$). d'' is called g-polystable iff E is a direct sum $E = E_1 \oplus E_2 \oplus \ldots \oplus E_k$ of d''-invariant and g-stable subbundles E_i with $\mu_g(d''|_{E_i} = \mu_g(d'')$ for $i = 1,2,\ldots,k$.*

Definition 4.8 *An integrable Higgs operator d'' in E is called simple iff for every $f \in A^0(\mathrm{End}E)$ with $d'' \circ f = f \circ d''$ it holds $f = a \cdot \mathrm{id}_E$ with $a \in \mathbb{C}$.*

As in the case of stable vector bundles or flat connections, (semi)-stability can equivalently be defined using quotients of E; again it follows

Lemma 4.9 *i) A g-stable integrable Higgs operator in E is simple.*
ii) Let d_1'', d_2'' be g-stable integrable Higgs operators in bundles E_1, E_2 on X such that $\mu_g(d_1'') = \mu_g(d_2'')$. If $f \in A^0(\mathrm{Hom}(E_1,E_2))$ satisfies $d_2'' \circ f = f \circ d_1''$, then either $f = 0$ or f is an isomorphism.

Furthermore, using arguments similar to those in the proof of Proposition 3.13, we get the following consequence of Proposition 4.5.

Proposition 4.10 *Let d'' be a simple integrable Higgs operator in E. If a g-Einstein metric in (E,d'') exists, then it is unique up to a positive scalar.*

The proof of the next result is a straightforward generalization of that in the Kähler case [S2] (just as for the proof of the corresponding statement for Hermite-Einstein metrics in vector bundles, see [LT]).

Proposition 4.11 *Let d'' be an integrable Higgs operator in E, and h a g-Einstein metric in (E,d'') with Einstein constant c; then d'' is g-semistable. If d'' is not g-stable, then d'' is g-polystable; more precisely, $E = E_1 \oplus E_2 \oplus \ldots \oplus E_k$ is an h-orthogonal direct sum of d''-invariant and g-stable subbundles such that $\mu_g(d''|_{E_i}) = \mu_g(d'')$ for $i = 1,2,\ldots,k$. Furthermore, $h|_{E_i}$ is a g-Einstein metric in $(E_i, d''|_{E_i})$ with Einstein constant c for all i, and the direct sum is invariant with respect to the connection $D_h = I_h^{-1}(d'')$.*

Let d'', $\tilde{d}''$ be integrable Higgs operators in bundles E, $\tilde{E}$ with g-Einstein metrics h, $\tilde{h}$. Let $E = \bigoplus_{i=1}^{k} E_i$ and $\tilde{E} = \bigoplus_{i=1}^{l} \tilde{E}_i$ be the orthogonal, invariant splittings given by Proposition 4.11. We write $d_i'' := d''|_{E_i}$, $\tilde{d}_i'' := \tilde{d}''|_{\tilde{E}_i}$, $h_i := h|_{E_i}$, $\tilde{h}_i := \tilde{h}|_{\tilde{E}_i}$.
As in the previous section (but now using Lemma 4.9 and Proposition 4.10) we deduce

Corollary 4.12 *Suppose that there exists an isomorphism* $f \in A^0(\mathrm{Hom}(E, \tilde{E}))$ *satisfying* $f \circ d'' = \tilde{d}'' \circ f$. *Then it holds* $k = l$, *and, after renumbering of the summands if necessary, there are isomorphisms* $f_i \in A^0(\mathrm{Hom}(E_i, \tilde{E}_i))$ *such that* $f_i \circ d_i'' = \tilde{d}_i'' \circ f$ *and* $f_*(h_i) = \tilde{h}_i$.

Remark 4.13 *We expect that the existence of a g-Einstein metric for a g-stable Higgs operator d'' can be proved by solving (again using the continuity method as in [LT]) the differential equation*

$$K_h + i\Lambda_g d''(f^{-1} \circ d'(f)) = c \cdot \mathrm{id}_E$$

for a positive definite and h-selfadjoint endomorphism f of E, where h is a suitable fixed metric in E.

5 Surfaces.

In this section we consider the special case $n = 2$, i.e. where X is a compact complex surface; again we fix a Hermitian metric g in X. In this case, the real Chern numbers $c_1^2(E), c_2(E) \in H^4(X, \mathbb{R}) \cong \mathbb{R}$ can be calculated by integrating the corresponding Chern forms of any connection in E, independently of the chosen metric g. In particular, if E admits a flat connection, then these Chern numbers vanish.

Proposition 5.1 *Suppose that $D \in \mathcal{A}_f(E)$ is a flat connection of g-degree 0, and that h is a g-Einstein metric in (E, D). Then it holds $G_h = 0$. In particular, the Higgs operator d_h'' associated to D and h is integrable with $\deg_g(d_h'') = 0$, and h is a g-Einstein metric for (E, d_h'').*

Proof: (see [S2]) For $\epsilon > 0$ we define a new connection $B_\epsilon := d + \frac{1}{\epsilon}\theta + \epsilon\theta^*$, and $F_\epsilon := B_\epsilon^2$. Observe that $n = 2$ implies $F_\epsilon^2 = \frac{1}{\epsilon^2}\nabla_\epsilon^4$, where $\nabla_\epsilon = d_h'' + \epsilon d'$. The vanishing of the Chern numbers of E implies $\int_X \mathrm{tr} F_\epsilon^2 = 0$, and hence $\int_X \mathrm{tr}\nabla_\epsilon^4 = 0$ for all $\epsilon > 0$. Taking the limit $\epsilon \to 0$ it follows

$$\int_X \mathrm{tr} G_h^2 = 0. \tag{9}$$

Write $G_h = G_{1,1} + G_2$, where $G_{1,1}$ is the component of the 2-form G_h of type $(1,1)$. Then it holds

$$*_g G_{1,1} = -G_{1,1}, *_g G_2 = G_2; \tag{10}$$

the first equation is a consequence of $\Lambda_g G_h = 0$, which follows from the assumption and Lemma 3.4. On the other hand, it holds $G_h = \bar\partial^2 + \bar\partial(\theta) + \theta \wedge \theta$, so Lemma 2.4 implies

$$G_{1,1} = \bar\partial(\theta) = \partial(\theta^*)^* = -\bar\partial(\theta)^* = -G_{1,1}^*, \tag{11}$$

and

$$G_2 = \bar\partial^2 + \theta \wedge \theta = -\theta^* \wedge \theta^* - \theta \wedge \theta = (\theta \wedge \theta + \theta^* \wedge \theta^*)^* = G_2^*. \tag{12}$$

(11) and (12) combined with (10) give $*_g G_h^* = G_h$, so from (9) it follows

$$0 = \int\limits_X \mathrm{tr} G_h^2 = \int\limits_X \mathrm{tr}(G_h \wedge *_g G_h^*) = \int\limits_X |G_h|^2 \mathrm{vol}_g,$$

implying $(d_h'')^2 = G_h = 0$. Hence d_h'' is integrable, $\deg_g(d_h'')$ vanishes (Remark 3.5), and h is g-Einstein for $(E, d'')h)$ because the curvature of d_h'' with respect to h equals $D^2 = 0$. □

Proposition 5.2 *Suppose that $c_1^2(E) = c_2(E) = 0$, that d'' is an integrable Higgs operator of g-degree 0, and that h is a g-Einstein metric in (E, d''). Then it holds $F_h = 0$. In particular, the connection D_h associated to d'' and h is flat with $\deg_g(D_h) = 0$, and h is a g-Einstein metric for (E, D_h).*

Proof: Define $F_{1,1} := d^2 + [\theta, \theta^*]$, $F_2 := \partial(\theta) + \bar\partial(\theta^*)$; then $F_h = F_{1,1} + F_2$. Observe that $F_{1,1}$ is of type (1,1) because d is a unitary connection in the holomorphic bundle $(E, \bar\partial)$. Since $\deg_g(d'') = 0$, Lemma 4.3 implies $0 = \Lambda_g F_h = \Lambda_g F_{1,1}$, hence it holds $*_g F_{1,1} = -F_{1,1}$ and $*_g F_2 = F_2$. On the other hand, it is easy to see that $F_{1,1}^* = -F_{1,1}$ and $F_2^* = F_2$. Combining these relations we get $*_g F_h^* = F_h$. Since $c_1^2(E)$ resp. $c_2(E)$ are obtained by integrating $-\frac{1}{4\pi^2}(\mathrm{tr} F_h)^2$ resp. $-\frac{1}{8\pi^2}((\mathrm{tr} F_h)^2 - \mathrm{tr}(F_h^2))$, we get

$$0 = \int\limits_X \mathrm{tr}(F_h^2) = \int\limits_X \mathrm{tr}(F_h \wedge *_g F_h^*) = \|F_h\|^2,$$

implying $D_h^2 = F_h = 0$. Hence D_h is flat, $\deg_g(D_h)$ vanishes (Remark 4.4), and h is g-Einstein for (E, D_h) because the pseudocurvature of D_h with respect to h equals $(d'')^2 = 0$. □

Remark 5.3 *The above proposition implies in particular the following: Suppose that $c_1^2(E) = c_2(E) = 0$; if there exists an integrable Higgs operator d'' in E with g-degree 0 admitting a g-Einstein metric, then the real Chern class $c_1(E)_{\mathbb{R}} \in H^2(X, \mathbb{R})$ vanishes, because there is a flat connection in E.*

We define $\mathcal{A}_f(E)^0_g$ to be the space of $D \in \mathcal{A}_f(E)$ of g-degree 0 such that there exists a g-Einstein metric in (E, D), and $\mathcal{A}''_i(E)^0_g$ to be the space of $d'' \in \mathcal{A}''_i(E)$ of g-degree 0 such that there exists a g-Einstein metric in (E, d''). By Remark 3.5 and Remark 4.4, the two moduli sets

$$\mathcal{M}_f(E)^0_g := {}^{\mathcal{A}_f(E)^0_g}\!\big/_{\text{isomorphy of connections}}$$

and

$$\mathcal{M}''(E)^0_g := {}^{\mathcal{A}''_i(E)^0_g}\!\big/_{\text{isomorphy of Higgs operators}}$$

are well defined. The main result of this section is

Theorem 5.4 *There is a natural bijection*

$$I : \mathcal{M}_f(E)^0_g \longrightarrow \mathcal{M}''(E)^0_g.$$

Proof: First observe that we may assume that the real Chern classes of E vanish, since otherwise both spaces are empty (see Remark 5.3).

Let D be a flat connection in E with g-degree 0, and h a g-Einstein metric in (E, D). By Proposition 5.1, the associated Higgs operator $d''_h = I_h(D)$ is integrable with g-degree 0, and h is a g-Einstein metric in (E, d''_h). We will show that the map I defined by $I([D]) := [d''_h]$ is well defined and bijective.

Suppose that $D, \tilde{D} \in \mathcal{A}_f(E)^0_g$ are isomorphic via the automorphism f of E; then f_*h is g-Einstein in $(E, \tilde{D})$ (Remark 3.5), the Higgs-operator $\tilde{d}''$ associated to $\tilde{D}$ and f_*h is isomorphic to d'' via f (Corollary 3.7), and f_*h is a g-Einstein metric in $(E, \tilde{d}'')$ (Remark 4.4). To prove that I is well defined it thus suffices to show that two different g-Einstein metrics $h, \tilde{h}$ for a fixed $D \in \mathcal{A}_f(E)^0_g$ produce isomorphic Higgs operators $d''_h, d''_{\tilde{h}}$. For this consider the D-invariant and h- resp. $\tilde{h}$-orthogonal splittings $E = \bigoplus_{i=1}^{k} E_i$ resp. $E = \bigoplus_{i=1}^{l} \tilde{E}_i$ associated to h resp. $\tilde{h}$ by Proposition 3.12. According to Corollary 3.14 (with $E = \tilde{E}$, $D = \tilde{D}$, $f = \mathrm{id}_E$) it holds $k = l$, and we may assume that there are isomorphisms $f_i : (E_i, D_i, h_i) \longrightarrow (\tilde{E}_i, \tilde{D}_i, \tilde{h}_i)$ of flat bundles of g-degree 0 with g-Einstein metrics, where $D_i := D|_{E_i}$, $\tilde{D}_i := D|_{\tilde{E}_i}$, $h_i := h|_{E_i}$, $\tilde{h}_i := \tilde{h}|_{\tilde{E}_i}$. This means in particular that the Higgs operator d''_i in E_i associated to D_i and h_i is isomorphic via f_i to the Higgs operator $\tilde{d}''_i$ in $\tilde{E}_i$ associated to $\tilde{D}_i$ and $\tilde{h}_i$. Hence $d''_h = d''_1 \oplus \ldots d''_k$ is isomorphic to $d''_{\tilde{h}} = \tilde{d}''_1 \oplus \ldots \oplus \tilde{d}''_k$ via the isomorphism $f := f_1 \oplus \ldots \oplus f_k$.

In the same way, but using Proposition 5.2 and the results of section 4, one shows that there is a well defined map from $\mathcal{M}''(E)^0_g$ to $\mathcal{M}_f(E)^0_g$, associating to the class of an integrable Higgs operator d'' with g-Einstein metric h the class of the connection $D_h = I_h^{-1}(d'')$; this obviously is an inverse of I. □

6 Line bundles on non-Kähler surfaces.

Isomorphism classes of flat complex *line* bundles (L, D) on a manifold X are parametrized by $H^1(X, \mathbb{C}^*)$. On the other hand, an integrable Higgs operator $d'' = \bar\partial + \theta$ in a complex line bundle L consists of a holomorphic structure $\bar\partial$ in L and a holomorphic 1-form θ on X (the condition $\theta \wedge \theta = 0$ now is trivial). Furthermore, two integrable Higgs operators d''_1 and d''_2 in L are isomorphic if and only if the two holomorphic line bundles $(L, \bar\partial_1)$ and $(L, \bar\partial_2)$ are isomorphic and $\theta_1 = \theta_2$. Hence, the space parametrizing isomorphism classes of integrable Higgs operators is $H^1(X, \mathcal{O}^*) \oplus H^0(X, \Omega^1(X)) = \mathrm{Pic}(X) \oplus H^{1,0}(X)$. In particular, the moduli sets $\mathcal{M}_f(L)^0_g$ and $\mathcal{M}''(L)^0_g$ defined in the previous section are subsets of $H^1(X, \mathbb{C}^*)$ resp. $\mathrm{Pic}(X) \oplus H^{1,0}(X)$.

Lemma 6.1 *Let L be a complex line bundle on X, and g a Hermitian metric in X. Then every flat connection in L and every integrable Higgs operator in L admits a g-Einstein metric.*

Proof: Let h_0 be fixed metric in L, then every metric is of the form $h_f = e^f \cdot h_0$ with $f \in \mathcal{C}^\infty(X, \mathbb{R})$. Let D be a flat connection in L; then h_f is a g-Einstein metric for D if and only if it is a solution of the equation $i\Lambda_g G_{h_0} - \frac{i}{2}\Lambda_g \bar\partial\partial(f) = c$ with a real constant c. Such a solution exists by [LT] Corollary 7.2.9. A similar argument works for integrable Higgs operators. □

From now on let X be a *surface*, and g a fixed Hermitian metric in X. Then the map $\deg_g : \mathrm{Pic}(X) \longrightarrow \mathbb{R}$ is a morphism of Lie groups ([LT] Proposition 1.3.7; recall that $\deg_g = \deg_{\tilde g}$ for some Gauduchon metric $\tilde g$). We define

$$H^1(X, \mathbb{C}^*)^f := \{\ [(L, D)] \in H^1(X, \mathbb{C}^*) \mid \deg_g(D) = 0\ \},$$

$$\mathrm{Pic}(X)^T := \{\ [(L, \bar\partial)] \in \mathrm{Pic}(X) \mid c_1(L)_{\mathbb{R}} = 0\ \},$$

and

$$\mathrm{Pic}(X)^f := \ker(\deg_g |_{\mathrm{Pic}(X)^T}).$$

Observe that $\mathrm{Pic}(X)^f$ can be identified with the set of isomorphism classes of line bundles admitting a flat *unitary* connection ([LT] Proposition 1.3.13). Theorem 5.4 and Lemma 6.1 imply

Proposition 6.2 *There is a natural bijection*

$$I_1 : H^1(X, \mathbb{C}^*)^f \longrightarrow \mathrm{Pic}(X)^f \times H^{1,0}(X).$$

If X admits a Kähler metric, i.e. if the first Betti number of $b_1(X)$ is even, then $\deg_g$ is a topological invariant for every metric g ([LT] Corollary 1.3.12 i)). Hence in this case it holds $H^1(X, \mathbb{C}^*)^f = H^1(X, \mathbb{C}^*)$ and $\mathrm{Pic}(X)^f = \mathrm{Pic}(X)^T$, and I_1 is the natural bijection from the moduli space of isomorphism classes of flat line bundles to the moduli space of integrable Higgs operators in line bundles with vanishing first real Chern class, which (e.g. by the work of Simpson) already is known to exist for a Kähler metric g.

So let us assume that $b_1(X)$ is odd. Then $\deg_g|_{\mathrm{Pic}^0(X)} : \mathrm{Pic}^0(X) \longrightarrow \mathbb{R}$ is surjective, and it holds

$$\mathrm{Pic}(X)^T\big/_{\mathrm{Pic}(X)^f} \cong \mathrm{Pic}^0(X)\big/_{\mathrm{Pic}^0(X)^f} \cong \mathbb{R}$$

([LT] Corollary 1.3.12 and Proposition 1.3.13). We will show that I_1 extends to a (non-natural) bijection from $H^1(X, \mathbb{C}^*)$ to $\mathrm{Pic}(X)^T \times H^{1,0}(X)$ in this case, too.

Lemma 6.3 *There is a bijection* $i : \mathrm{Pic}(X)^T \longrightarrow \mathrm{Pic}(X)^f \times \mathbb{R}$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{Pic}(X)^T & \xrightarrow{\deg_g} & \mathbb{R} \\ i\downarrow & & \| \\ \mathrm{Pic}(X)^f \times \mathbb{R} & \xrightarrow{proj.} & \mathbb{R} \end{array}$$

commutes.

Proof: $\deg_g|_{\mathrm{Pic}^0(X)}$ is surjective, so we can choose $\mathcal{L}_1 := [(L_1, \bar\partial_1)] \in \mathrm{Pic}^0(X)$ with $\deg_g(\mathcal{L}_1) = \deg_g(\bar\partial_1) = 1$, and a class $\alpha \in H^1(X, \mathcal{O})$ such that $\mathcal{L}_1 = \pi(\alpha)$ where $\pi : H^1(X, \mathcal{O}) \longrightarrow \mathrm{Pic}^0(X)$ is the natural surjection. For $\lambda \in \mathbb{R}$ define

$$\mathcal{L}_\lambda := \pi(\lambda \cdot \alpha);$$

then $\deg_g(\mathcal{L}_\lambda) = \lambda$ since $\deg_g \circ\pi : H^1(X, \mathcal{O}) \longrightarrow \mathbb{R}$ is linear. Now define i by

$$i(\mathcal{L}) := (\mathcal{L} \otimes \mathcal{L}_{-\deg_g(\mathcal{L})}, \deg_g(\mathcal{L}));$$

then it is obvious that the inverse of i is given by $(\mathcal{L}, \lambda) \mapsto \mathcal{L} \otimes \mathcal{L}_\lambda$, and that the diagram above commutes. □

In the proof of a similar statement for $H^1(X, \mathbb{C}^*)$ we will use

Lemma 6.4 *The natural map*

$$l^1 : H^1(X, \mathbb{C}^*) \longrightarrow \mathrm{Pic}(X)^T, l^1([(L, D)]) := [(L, D'')].$$

is surjective, i.e. a holomorphic structure $\bar\partial$ *in a differentiable line bundle* L *on* X *is the (0,1)-part of a flat connection if and only if the real first Chern class* $c_1(L)_{\mathbb{R}}$ *vanishes.*

Proof: $\mathrm{Pic}(X)^f$ is naturally identified with $H^1(X, U(1))$, such that the inclusion $\mathrm{Pic}(X)^f \hookrightarrow \mathrm{Pic}(X)$ becomes the injection $k^1 : H^1(X, U(1)) \hookrightarrow H^1(X, \mathcal{O}^*)$ ([LT] p. 38). Observe that k^1 is the composition of the natural map $H^1(X, U(1)) \longrightarrow H^1(X, \mathbb{C}^*)$ and l^1, so it holds

$$\mathrm{Pic}(X)^f = \mathrm{im}(k^1) \subset \mathrm{im}(l^1).$$

Each component of $\mathrm{Pic}(X)^T$ contains a component of $\mathrm{Pic}(X)^f$ ([LT] Remark 1.3.10), hence for each component

$$\mathrm{Pic}^c(X) := \{ \, [(L,\bar\partial)] \in \mathrm{Pic}(X) \mid c_1(L)_{\mathbb{Z}} = c \, \} \subset \mathrm{Pic}(X)^T$$

there exists a class $[(L_c, D_c)] \in H^1(X, \mathbb{C}^*)$ such that $l^1([(L_c, D_c)]) \in \mathrm{Pic}^c(X)$. Define $H^1(X,\mathbb{C}^*)^0 := \{ \, [(L,D)] \in H^1(X,\mathbb{C}^*) \mid c_1(L)_{\mathbb{Z}} = 0 \, \}$. The commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{exp} & \mathbb{C}^* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{exp} & \mathcal{O}^* & \longrightarrow & 0 \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} H^1(X,\mathbb{C}) & \longrightarrow & H^1(X,\mathbb{C}^*)^0 \\ h^1 \downarrow & & \downarrow l^1 \\ H^1(X,\mathcal{O}) & \longrightarrow & \mathrm{Pic}^0(X) \end{array}$$

with surjective horizontal arrows. Since X is a surface, the left vertical arrow h^1 is also surjective ([BPV] p. 117), hence l^1 maps $H^1(X,\mathbb{C}^*)^0$ surjectively onto $\mathrm{Pic}^0(X)$. Now it is easy to see that every element of $\mathrm{Pic}^c(X) \subset \mathrm{Pic}(X)^T$ is of the form $l^1([(L_c \otimes L, D_c \otimes D)])$ for some $[(L,D)] \in H^1(X,\mathbb{C}^*)^0$. $\square$

Lemma 6.5 *There is a bijection $j : H^1(X,\mathbb{C}^*) \longrightarrow H^1(X,\mathbb{C}^*)^f \times \mathbb{R}$ such that the diagram*

$$\begin{array}{ccc} H^1(X,\mathbb{C}^*) & \xrightarrow{\deg'_g} & \mathbb{R} \\ j \downarrow & & \| \\ H^1(X,\mathbb{C}^*)^f \times \mathbb{R} & \xrightarrow{proj.} & \mathbb{R} \end{array}$$

commutes, where $\deg'_g := \deg_g \circ l^1$ is the map associated to the g-degree of flat connections.

Proof: Choose $\mathcal{L}_1 \in \mathrm{Pic}^0(X)$, $\alpha \in H^1(X,\mathcal{O})$ as in the proof of Lemma 6.3, and a class $\beta \in H^1(X,\mathbb{C})$ with $h^1(\beta) = \alpha$. Let $\pi' : H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathbb{C}^*)$ be the map induced by $exp : \mathbb{C} \longrightarrow \mathbb{C}^*$, and define $\mathcal{L}'_1 := \pi'(\beta) \in H^1(X,\mathbb{C}^*)$. Since the diagram

$$\begin{array}{ccc} H^1(X,\mathbb{C}) & \xrightarrow{\pi'} & H^1(X,\mathbb{C}^*) \\ h^1 \downarrow & & \downarrow l^1 \\ H^1(X,\mathcal{O}) & \longrightarrow & \mathrm{Pic}(X)^T \end{array}$$

commutes, it holds $\deg'_g(\mathcal{L}'_1) = 1$. The rest of the proof is as for Lemma 6.3. □

We conclude

Theorem 6.6 *The composition*

$$\bar{I} : H^1(X, \mathbb{C}^*) \xrightarrow{i} H^1(X, \mathbb{C}^*)^f \times \mathbb{R} \xrightarrow{I_1 \times \mathrm{id}_{\mathbb{R}}} H^{1,0}(X) \times \mathrm{Pic}(X)^f \times \mathbb{R}$$
$$\xrightarrow{\mathrm{id}_{H^{1,0}(X)} \times i^{-1}} H^{1,0}(X) \times \mathrm{Pic}(X)^T$$

is a bijective extension of the map I_1, and preserves the g-degree.

We finish with the obvious remark that the map $l^1 : H^1(X, \mathbb{C}^*) \longrightarrow \mathrm{Pic}(X)^T$ in general does not coincide with the composition of $\bar{I}$ and projection onto $\mathrm{Pic}(X)^T$.

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