On the Automorphism Group of a Complex Sphere

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Abstract. Let X be a compact complex threefold with the integral homology of $\mathbf{S}^6$ and let $Aut(X)$ be its holomorphic automorphism group. By [HKP] and [CDP] the dimension of $Aut(X)$ is at most 2. We prove that $Aut(X)$ cannot be isomorphic to the complex affine group.

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A classical problem in the theory of complex manifolds concerns the existence of complex structures on the six-dimensional sphere $\mathbf{S}^6$. Using octonions one can construct almost-complex structures on $\mathbf{S}^6$, but they are not integrable, and in fact no integrable almost-complex structure is known; it is generally believed that they do not exist, and therefore that $\mathbf{S}^6$ provides an example of almost-complex but non-complex manifold. Examples of this kind are abundant in (real) dimension 4 (as a consequence of our rather good understanding of complex surfaces) but are still lacking in higher dimension (as a manifestation of our rather poor understanding of higher dimensional complex manifolds, except, of course, algebraic or Kähler ones). The case of $\mathbf{S}^6$ is perhaps of particular interest because a complex structure on $\mathbf{S}^6$ would give, by blowing up a point, an exotic complex structure on the familiar $\mathbf{C}P^3$. Moreover, it was proved by Borel and Serre in the fifties that $\mathbf{S}^2$ and $\mathbf{S}^6$ are the only spheres which admit an almost-complex structure.

Recently, two papers add new insights into this problem. Campana, Demailly and Peternell prove in [CDP] that a complex threefold X diffeomorphic to $\mathbf{S}^6$ has no nonconstant meromorphic function. Huckleberry, Kebekus and Peternell prove in [HKP] that a complex threefold X diffeomorphic to $\mathbf{S}^6$ is not almost-homogeneous. Due to [CDP], this last result can be reformulated as: the automorphism group $Aut(X)$ of X has dimension less than or equal to 2 (recall that the automorphism group of a compact complex manifold is a finite dimensional complex Lie group [Huc]).

Our aim is to pursue the study of $Aut(X)$. Let $Aut_0(X)$ be the connected component of the identity: it is a connected complex Lie group of dimension ≤ 2, and if it is not abelian then it is isomorphic to $Aff^k(\mathbf{C})$ for some $k \in \mathbf{N}^+ \cup \{\infty\}$, where $Aff^k(\mathbf{C})$ denotes the k-fold covering of the complex affine group $Aff(\mathbf{C})$. The Lie algebra of $Aff^k(\mathbf{C})$ is generated by two vectors ξ, η satisfying $[\xi, \eta] = \eta$, and if $k \neq \infty$ then ξ is the generator of a subgroup isomorphic to $\mathbf{C}^*$ (more precisely, $\mathbf{C}/2\pi ik\mathbf{Z}$). We shall prove that $Aut_0(X)$ cannot be isomorphic to $Aff^k(\mathbf{C})$, $k \in \mathbf{N}^+$; equivalently, if $Aut_0(X)$ contains a $\mathbf{C}^*$-action then $Aut_0(X)$ is abelian.

More generally, we shall work under the hypothesis that X is a compact complex threefold with the $\mathbf{Z}$-homology of $\mathbf{S}^6$; we shall call such an X a *complex homology sphere*. The results of [CDP] and [HKP] are still valid for any complex homology sphere: this is explicit in [CDP] and can be easily checked in [HKP].

Theorem. *Let X be a complex homology sphere. Then the groups $Aff^k(\mathbf{C})$, $k \in \mathbf{N}^+$, do not act faithfully on X.*

The main step of the proof is a "reduction" of the fixed point set of a $\mathbf{C}^*$-action on a complex sphere (incidentally, this furnishes also some simplifications of sections 7-8 of [HKP]). It has been observed in [HKP] that such a fixed point set is either a pair of points or a smooth rational curve. We shall prove that, if the former case occurs, one can find a bimeromorphic transformation $\phi : X - - \to Y$, where Y is still a complex homology sphere, which maps $Aut_0(X)$ isomorphically onto $Aut_0(Y)$ and moreover maps the $\mathbf{C}^*$-action on X to a $\mathbf{C}^*$-action on Y whose fixed point set is a rational curve. The argument is the following. Using index type considerations [Bot] we find smooth rational curves joining the two fixed points, invariant by the $\mathbf{C}^*$-action, and whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We perform a bimeromorphic transformation (a flop [Kol]) centered on one of these curves, giving a new complex homology sphere and a new $\mathbf{C}^*$-action. A combinatorial argument shows that after a finite number of steps we arrive at a $\mathbf{C}^*$-action whose fixed point set is a rational curve, as desired.

Once that reduction of fixed points has been done, the commutativity of $Aut_0(X)$ in presence of $\mathbf{C}^*$-subgroups will be proved by a somewhat algebraic argument. Assuming (by contradiction) that $Aut_0(X)$ is not commutative, we show that X contains a rational irreducible (singular) surface, invariant by the $\mathbf{C}^*$-action and containing the stable and unstable manifolds of the fixed point set. This turns out to be impossible. We notice that one can prove the existence of such a rational surface even if $Aut_0(X)$ is commutative and bidimensional; however we are not able, in that case, to produce a contradiction and therefore we do not present here that partial result.

1. Reduction of fixed points of $\mathbf{C}^*$-actions

Let $\rho : \mathbf{C}^* \times X \to X$ be a $\mathbf{C}^*$-action on a complex homology sphere. We will assume, without loss of generality, that ρ is faithful: $\rho_t \neq id$ for $t \neq 1$. We denote by $v = \frac{d}{dt}|_{t=1}\rho_t$ its infinitesimal generator. It is a holomorphic vector field, whose flow is $2\pi i$-periodic, and its zero set coincides with the fixed point set $Fix(\rho)$ of ρ. It is a classical fact [Huc] that v is linearizable near each point of $Fix(\rho)$; in particular $Fix(\rho)$ is a smooth complex submanifold of X.

Lemma 1 [HKP]. *$Fix(\rho)$ is either a pair of points or a smooth rational curve.*

Proof. The set $Fix(\rho)$ coincides with the fixed point set of the $\mathbf{S}^1$-action contained in the $\mathbf{C}^*$-action. Therefore, and because X is a $\mathbf{Z}$-homology sphere, we have that the $\mathbf{Z}$-homology of $Fix(\rho)$ is that of $\mathbf{S}^0$ or $\mathbf{S}^2$ or $\mathbf{S}^4$ [Bor,IV.5.9]. The first case gives $Fix(\rho) = \{a, b\}$, the second one $Fix(\rho) = \mathbf{C}P^1$, and the third one is excluded because no compact complex surface has the $\mathbf{Z}$-homology of $\mathbf{S}^4$ (by the signature formula, for instance). An alternative way to exclude the third case is the following [HKP]: the adjunction formula and $b_2(X) = b_4(X) = 0$ imply that the Euler characteristic of a smooth compact complex surface $S \subset X$ is zero:

$$c_2(S) = c_2(X) \cdot S - c_1(S) \cdot c_1(\mathcal{O}_X(S)|_S) = 0$$

and so S cannot have the $\mathbf{Z}$-homology of $\mathbf{S}^4$. *q.e.d.*

Let $p \in Fix(\rho)$ and let p_1, p_2, p_3 be the eigenvalues of (the linear part of) v at p. They are integers, and the faithfulness of ρ implies that

$$GCD(p_1, p_2, p_3) = 1.$$

Lemma 2. *For every $i, j \in \{1, 2, 3\}$, $i \neq j$, we have*

$$GCD(p_i, p_j) = 1.$$

Proof. Suppose by contradiction that $GCD(p_1, p_2) = n \geq 2$ and let ω be a primitive n-root of 1. Then the periodic biholomorphism ρ_ω is not the identity but its fixed point set contains a smooth compact complex surface S with $p \in S$ and $T_pS = E_{p_1} \oplus E_{p_2}$, where E_{p_j} is the eigenspace corresponding to p_j. As observed in the proof of lemma 1, the Euler characteristic of S is zero. The action ρ restricts to S to a nontrivial (and nonfaithful) action whose fixed point set is $Fix(\rho) \cap S$. This set is nonempty (it contains p) and it is either discrete or a rational curve. In both cases the Poincaré - Hopf formula gives $c_2(S) > 0$, contradiction. *q.e.d.*

In particular, if $Fix(\rho) = \mathbf{C}P^1$ (that is, one of the eigenvalues is zero) then there are only two possibilities for (p_1, p_2, p_3) (up to renumbering and up to reversing the action): $(0, 1, 1)$ and $(0, 1, -1)$. Let us exclude the first one.

Lemma 3. *If $Fix(\rho) = \mathbf{C}P^1$ then the two nonvanishing eigenvalues of v along $Fix(\rho)$ have opposite sign.*

Proof. This can be proved using the Bott formula [Bot]. However, that formula is rather complicated in the case of nonisolated fixed points, and so we prefer to give the following elementary proof. Suppose, by contradiction, that the eigenvalues of v along $Fix(\rho)$ are $(0,1,1)$. Take the quotient of X by the $\mathbf{S}^1$-action contained in the $\mathbf{C}^*$-action. It is easy to see that it is a topological compact manifold M of dimension 5, and $Fix(\rho)$ projects on M to an embedded 2-sphere N: near a point of $Fix(\rho)$ the $\mathbf{S}^1$-action is the product of the trivial action on $\mathbf{C}$ and the action on $\mathbf{C}^2$ tangent to each 3-sphere and inducing there the Hopf fibration, so that the quotient of each 3-sphere is $\mathbf{S}^2$ and the quotient of the $\mathbf{C}^2$ factor is a cone over $\mathbf{S}^2$, that is $\mathbf{R}^3$. More explicitely, near a point of $Fix(\rho)$ we can choose local holomorphic coordinates (x,y,z) so that v is expressed by $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$, and then the quotient map is $\mathbf{C}^3\to\mathbf{R}^3\times\mathbf{C}$, $(x,y,z)\mapsto(\sqrt{|x|^2+|y|^2},\frac{x}{y},z)$, where $\mathbf{R}^3$ is coordinatized by polar coordinates $(r,\theta)\in\mathbf{R}^+\times\mathbf{C}P^1$. The 2-sphere N is locally given by $\{r=0\}$. The $\mathbf{R}^*$-action contained in the $\mathbf{C}^*$-action projects on M to an action generated by a vector field V vanishing on N, and only there. Up to changing V to $-V$, the sphere N is an attractor: locally, in the same coordinates (r,θ,z) as before, we have $V=-r\frac{\partial}{\partial r}$. We see that the Poincaré - Hopf index of v at N is equal to 2, hence the Euler characteristic of M is also equal to 2. Since M is odd-dimensional, this is an absurd. *q.e.d.*

We shall prove the following result.

Proposition 1. *Let X be a complex homology sphere and let $\rho:\mathbf{C}^*\times X\to X$ be a $\mathbf{C}^*$-action whose fixed point set $Fix(\rho)$ is a pair of points $\{a,b\}$. Then there exists a complex homology sphere Y and a bimeromorphism $\phi:X--\to Y$ such that:*

i) ϕ conjugates $Aut_0(X)$ to $Aut_0(Y)$;

ii) ϕ conjugates ρ to a $\mathbf{C}^*$-action τ on Y whose fixed point set is a smooth rational curve.

The bimeromorphism ϕ will be a composition of elementary bimeromorphisms that we now describe.

Suppose that X contains a smooth rational curve R whose normal bundle $N_{R,X}$ is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Let $\tilde{X}\xrightarrow{\pi}X$ be the blow-up of X with center R. The exceptional divisor $D\subset\tilde{X}$ is a rational ruled surface over R, more precisely $D=P(N_{R,X})=\mathbf{C}P^1\times\mathbf{C}P^1$. Hence there are *two* rulings on D: the ruling over R, given by $\pi|_D$, and a second ruling $D\xrightarrow{\sigma}\mathbf{C}P^1$ whose fibres are transverse to the fibres of $\pi|_D$. The normal bundle of D in $\tilde{X}$ has degree -1 on the fibres of $\pi|_D$ and also on the fibres of σ. Hence [Moi] we can contract each fibre of σ to a point: the result is a smooth complex threefold Y and a morphism $\pi':\tilde{X}\to Y$. The image of D by π' is a smooth rational curve S with normal bundle $N_{S,Y}=\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, and π' is nothing else than the blow-up of Y with center S. The bimeromorphism $\pi'\circ\pi^{-1}:X--\to Y$ will be called a *flop with center R*. It is in fact the simplest example of a flop [Kol].

Lemma 4. *Y is a complex homology sphere.*

Proof. It follows from

$$H_k(Y,\mathbf{Z}) = H_k(\tilde{X},\mathbf{Z}) = H_k(X,\mathbf{Z}) \qquad \text{if } k \neq 2,4$$

$$H_k(Y,\mathbf{Z}) \oplus \mathbf{Z} = H_k(\tilde{X},\mathbf{Z}) = H_k(X,\mathbf{Z}) \oplus \mathbf{Z} \qquad \text{if } k = 2,4.$$

q.e.d.

It should be possible to prove also that Y is diffeomorphic to X. In fact, there should exist a smooth (non holomorphic!) diffeomorphism of $\tilde{X}$, whose support is localized on a neighbourhood of D, which exchanges the two rulings $\pi|_D$ and σ, proving the diffeomorphicity of X and Y. Remark that the fundamental groups of X and Y are isomorphic. If X is diffeomorphic to $\mathbf{S}^6$ then it is easy to see that Y also is diffeomorphic to $\mathbf{S}^6$, by classical results in differential topology (Smale, Kervaire - Milnor,...).

Lemma 5. *The flop $\pi' \circ \pi^{-1} : X - - \to Y$ realizes an isomorphism between $Aut_0(X)$ and $Aut_0(Y)$.*

Proof. We simply have to check that for every holomorphic vector field on X (resp. on Y) its transform on Y (resp. on X) is still holomorphic. This follows from the negativity of $N_{R,X}$ and $N_{S,Y}$: every holomorphic vector field on X (resp. on Y) is tangent to R (resp. to S). *q.e.d.*

In order to do flops, we have to find rational curves with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. This will be based on the following remarks.

Let $Fix(\rho) = \{a,b\}$ and let a_1,a_2,a_3 be the eigenvalues of v at a, b_1,b_2,b_3 those at b. Suppose that $|a_j| \geq 2$, for some j: then by the same argument of the proof of lemma 2 there is a ρ-invariant smooth complex curve $R \subset X$, with $a \in R$ and $T_aR = E_{a_j}$ (observe that, by lemma 2, this eigenspace is one-dimensional). Clearly R is rational and contains a second fixed point, that is $b \in R$. Moreover, for some i we have $T_bR = E_{b_i}$, and $b_i = -a_j$. To fix ideas, suppose $j = i = 1$. The normal bundle of R will be computed by the following formula.

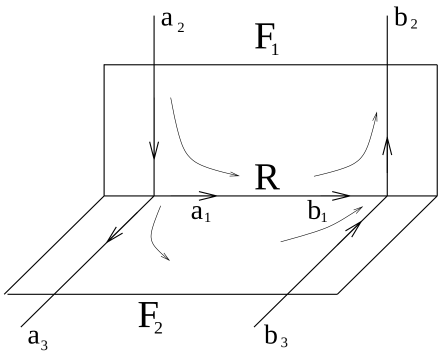
Lemma 6.

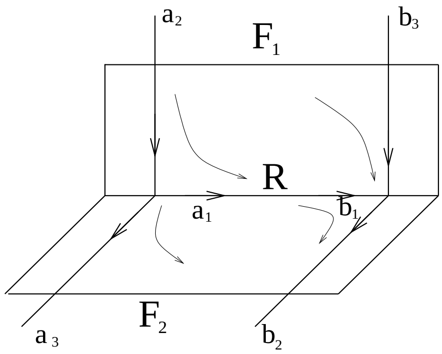
$$N_{R,X} = \mathcal{O}(n) \oplus \mathcal{O}(m)$$

where $n = \frac{a_2-b_2}{a_1}, m = \frac{a_3-b_3}{a_1}$ *or* $n = \frac{a_2-b_3}{a_1}, m = \frac{a_3-b_2}{a_1}$.

Proof. We consider the restriction of ρ to R and its natural extension to $N_{R,X}$, via the differential. We therefore are in the situation of [Bot]: a holomorphic vector field (on R) which acts on a vector bundle. Hence the characteristic numbers of that bundle are localized at zeroes of the vector field, that is at a and b.

The bundle $N_{R,X}$ has a splitting $F_1 \oplus F_2$ by line bundles which are invariant by the action: this corresponds to the fact that a $\mathbf{C}^*$-action on a rational ruled surface (in our case $P(N_{R,X})$) has always two disjoint invariant sections. The fibres $(F_i)_a$, $(F_i)_b$ are invariant and their eigenvalues are a_2,a_3,b_2,b_3. Hence there are two possibilities:





From Bott formula [Bot] we deduce in the first case

$$c_1(F_1) = \frac{a_2}{a_1} + \frac{b_2}{b_1} = \frac{a_2 - b_2}{a_1}, \qquad c_1(F_2) = \frac{a_3 - b_3}{a_1}$$

and in the second case

$$c_1(F_1) = \frac{a_2 - b_3}{a_1}, \qquad c_1(F_2) = \frac{a_3 - b_2}{a_1}.$$

q.e.d.

Observe that by adjunction formula and $c_1(X) = 0$ we have $c_1(N_{R,X}) = -c_1(R) = -2$ and consequently $n + m = -2$, i.e.

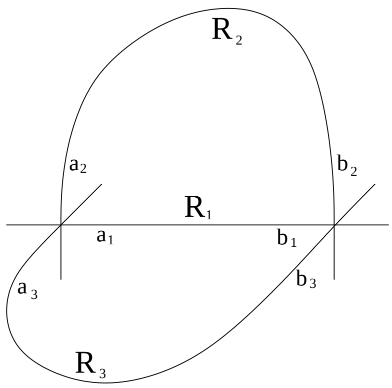
$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

2. PROOF OF PROPOSITION 1

Let a_1, a_2, a_3 (resp. b_1, b_2, b_3) be the eigenvalues of v at a (resp. at b), with $|a_1| \leq |a_2| \leq |a_3|$.

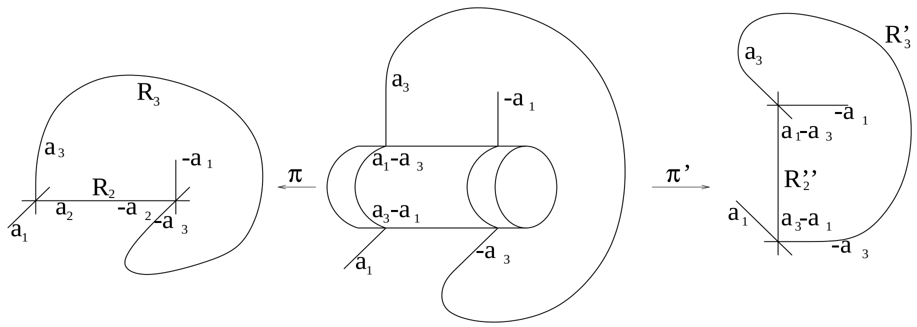
FIRST STEP: from $|a_1| \geq 2$ to $|a_1| = 1$.

If $|a_1| \geq 2$ then, as explained before lemma 6, we have three smooth ρ-invariant rational curves through a and b, each one connecting a and b:



We have $b_j = -a_j$ for every $j = 1, 2, 3$ (up to renumbering the eigenvalues at b). Remark that $|a_1| \geq 2$ implies $|a_2|, |a_3| \geq 3$ (lemma 2). By lemma 6, the normal bundle $N_{R_2,X}$ is either $\mathcal{O}(\frac{2a_1}{a_2}) \oplus \mathcal{O}(\frac{2a_3}{a_2})$ or $\mathcal{O}(\frac{a_1+a_3}{a_2}) \oplus \mathcal{O}(\frac{a_1+a_3}{a_2})$. The former case is excluded because $\frac{2a_1}{a_2}$ and $\frac{2a_3}{a_2}$ are not integers. The latter case is in fact $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, because $c_1(N_{R_2,X}) = -2$. Similarly, $N_{R_3,X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Hence we can perform flops with center R_2 or R_3. Let us see how a flop with center R_2 transforms the eigenvalues of the $\mathbf{C}^*$-action. After a blow-up π with center R_2 we obtain a $\mathbf{C}^*$-action with four fixed points on the exceptional divisor, two over a and two over b. The rational curves $\pi^{-1}(a)$ and $\pi^{-1}(b)$ are invariant by the action and their eigenvalues are $\pm(a_1 - a_3)$. The blow-down π' maps these two curves onto a rational curve R_2'', with eigenvalues $\pm(a_1 - a_3)$:

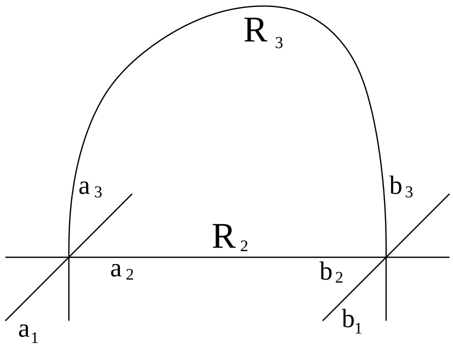


On the new complex homology sphere we therefore have a new (faithful) $\mathbf{C}^*$-action with a fixed point whose eigenvalues are $(a_1, a_3 - a_1, -a_3)$. The strict transform R_3' of R_3 has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, again by lemma 6. Therefore we can perform a second flop with center R_3': we obtain a new complex homology sphere and a $\mathbf{C}^*$-action with a fixed point whose eigenvalues are $(a_1, a_1 - a_3, a_3 - 2a_1) = (a_1, a_2 + 2a_1, a_3 - 2a_1)$ (recall that $\frac{a_1+a_3}{a_2} = -1$, i.e. $a_1 + a_2 + a_3 = 0$). Of course, we can reverse the order: a flop with center R_3 followed by a flop with center R_2' produces a $\mathbf{C}^*$-action with a fixed point with eigenvalues $(a_1, a_2 - 2a_1, a_3 + 2a_1)$. Remark that these new collections of eigenvalues still satisfy the GCD condition of lemma 2.

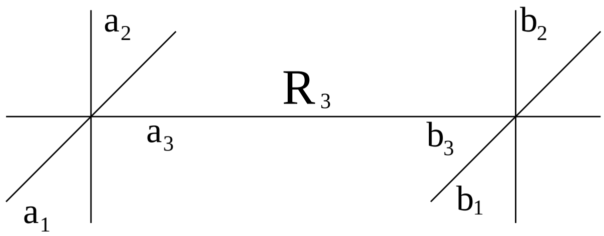
Iterating this process we arrive at a fixed point with eigenvalues (a_1, α, β) and $|\alpha| \leq |a_1|$ ($\alpha = a_2 + 2na_1$ or $a_3 + 2na_1$ for a suitable integer n). Because $GCD(a_1, \alpha) = 1$ and $|a_1| \geq 2$, we have the strict inequalities $0 < |\alpha| < |a_1|$, that is the eigenvalue with smallest modulus has modulus strictly smaller that $|a_1|$. Iterating again we finally arrive at an eigenvalue with modulus equal to 1.

Second step: from $|a_1| = 1$, $|a_2| \geq 2$ to $|a_1| = |a_2| = 1$.

Now we can guarantee only two ρ-invariant rational curves:



We have $b_j = -a_j$ for $j = 2, 3$. We also have $|b_1| = 1$: otherwise $|b_1| \geq 2$ and there would be a third ρ-invariant rational curve tangent to E_{a_1} at a and to E_{b_1} at b, giving $b_1 = -a_1$, thus $|b_1| = |a_1| = 1$, a contradiction.
From $|a_2| \geq 2$ it follows $|a_3| \geq 3$ and (lemma 6) the normal bundle $N_{R_3,X}$ is either $\mathcal{O}(\frac{a_1-b_1}{a_3}) \oplus \mathcal{O}(\frac{2a_2}{a_3})$ or $\mathcal{O}(\frac{a_1+a_2}{a_3}) \oplus \mathcal{O}(\frac{a_2-b_1}{a_3})$. As before, the first possibility is excluded because $\frac{2a_2}{a_3}$ is not an integer. Hence $N_{R_3,X} = \mathcal{O}(\frac{a_1+a_2}{a_3}) \oplus \mathcal{O}(\frac{a_2-b_1}{a_3})$. From $c_1(N_{R_3,X}) = -2$ it follows that $a_1 + 2a_2 - b_1 = -2a_3$. Because $|a_2| \neq |a_3|$, we cannot have $b_1 = a_1$ and so we have $b_1 = -a_1$. This in turn implies $a_1 + a_2 + a_3 = 0$ and $N_{R_3,X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.
If $|a_2| \geq 3$ the same argument applies to R_2, and we obtain $N_{R_2,X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then we proceed as in the first step: a sequence of flops produces a $\mathbf{C}^*$-action with a fixed point with eigenvalues (a_1, α, β), $|\alpha| \leq |a_1|$; that is $|\alpha| = |a_1| = 1$.
If $|a_2| = 2$, from $a_1 + a_2 + a_3 = 0$, $|a_1| = 1$ and $|a_3| \geq 3$ we find $a_2 = 2a_1, a_3 = -3a_1$. It is readily checked that a single flop along R_3 produces a $\mathbf{C}^*$-action with a fixed point with eigenvalues $(a_1, a_1, -2a_1)$.
Third step: the case $|a_1| = |a_2| = 1$, $|a_3| \geq 2$.



We have $b_3 = -a_3$. As before, $|b_1| = |b_2| = 1$. Up to exchanging b_1 and b_2 we obtain $N_{R_3,X} = \mathcal{O}(\frac{a_1-b_1}{a_3}) \oplus \mathcal{O}(\frac{a_2-b_2}{a_3})$ and $a_1 - b_1 + a_2 - b_2 = -2a_3$. From $|a_3| \geq 2$ it follows $a_1 = a_2 = -b_1 = -b_2$ and $a_3 = -2a_1$, therefore $N_{R_3,X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. A flop along R_3 gives a $\mathbf{C}^*$-action with a rational curve of fixed points (and eigenvalues $(0, 1, -1)$).
Last step. To complete the proof of proposition 1 we need to show that the case $|a_1| = |a_2| = |a_3| = 1$ never happens. By the usual argument, if $|a_j| = 1$ for every j then also $|b_j| = 1$ for every j. We now take the Bott formula [Bot]

for $c_1^3(X)$:

$$\frac{(a_1+a_2+a_3)^3}{a_1a_2a_3} + \frac{(b_1+b_2+b_3)^3}{b_1b_2b_3} = c_1^3(X) = 0.$$

If $|a_j| = 1$ for every j then the residue $\frac{(a_1+a_2+a_3)^3}{a_1a_2a_3}$ can take only two values: 27 and -1. The same for $\frac{(b_1+b_2+b_3)^3}{b_1b_2b_3}$. Hence their sum cannot vanish. *q.e.d.*

Remark: we could use the Bott formula since the beginning of the proof and not only in the last step, but it turns out that this would give only minor simplifications (for instance, in the second step we can use the Bott formula to deduce $b_1 = -a_1$ from $|b_1| = |a_1|$). It also turns out that the analogous formula for $c_1(X) \cdot c_2(X)$ yields no further information. The formula for $c_3(X)$ is equivalent to the Poincaré - Hopf formula and was already used, more or less, in lemmata 1, 2 and 3. And, of course, all these formulae do not contradict the existence of a $\mathbf{C}^*$-action with a rational curve of fixed points, with eigenvalues $(0, 1, -1)$.

The automorphism group is abelian

From now on $\rho : \mathbf{C}^* \times X \to X$ will denote a faithful $\mathbf{C}^*$-action on a complex homology sphere with $Fix(\rho) = Z_0$ a smooth rational curve. Around each point of Z_0 we can choose local coordinates (x, y, z) such that the infinitesimal generator v is expressed by $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ (and $Z_0 = \{x = y = 0\}$). If we take a sufficiently small tubular neighbourhood V of Z_0 then the sets

$$W_V^s = \{p \in V | \rho_t(p) \in V \ \forall t, |t| \geq 1, \text{ and } \lim_{t\to\infty} \rho_t(p) = \theta^+(p) \in Z_0\}$$

and

$$W_V^u = \{p \in V | \rho_t(p) \in V \ \forall t, |t| \leq 1, \text{ and } \lim_{t\to 0} \rho_t(p) = \theta^-(p) \in Z_0\}$$

are smooth complex open surfaces, containing Z_0 and intersecting transversely along Z_0. In the above local coordinates, $W_V^s = \{x = 0\}$ and $W_V^u = \{y = 0\}$. Suppose now that

$$dim\ Aut_0(X) = 2.$$

This means that there exists a second holomorphic vector field w on X, linearly independent of v. In fact, w cannot be collinear to v at a generic point of X, because X has no nonconstant meromorphic function [CDP]. The commutator $[v, w]$ is a linear combination $av + bw$, $a, b \in \mathbf{C}$, since the Lie algebra of holomorphic vector fields on X is two-dimensional, spanned by v and w. Because the flow of v is $2\pi i$-periodic, one easily sees that if $b = 0$ then also $a = 0$: when $[v, w] = av$, the flows ϕ_t $(= \rho_{\exp t})$ of v and ψ_t of w are related by $\psi_s \circ \phi_t = \phi_{t\exp(as)} \circ \psi_s$ for every $t, s \in \mathbf{C}$, in particular

$\phi_{2\pi i \exp(as)} = \psi_s \circ \phi_{2\pi i} \circ \psi_{-s} = id$ for every $s \in \mathbf{C}$, so that $\exp(as)$ is an integer for every s and therefore $a = 0$. Hence, up to replacing w by $w + \frac{a}{b}v$ (if $b \neq 0$), we have

$$[v, w] = bw$$

where $b \in \mathbf{Z}$, again for the $2\pi i$-periodicity of the flow of v. Up to changing v to $-v$, we may suppose that $b \in \mathbf{N}$.

In this section we shall prove the commutativity of $Aut_0(X)$, concluding the proof of the theorem stated in the introduction.

Proposition 2. *Let v, w be holomorphic vector fields on a complex homology sphere, where v generates a $\mathbf{C}^*$-action. Then v and w commute: $[v, w] = 0$.*

Let us consider the wedge product $v \wedge w \in H^0(X, TX \wedge TX)$, whose zero set $E \subset X$ is the analytic subset of X where v and w are collinear. Define $O(v \wedge w, W_V^s)$, resp. $O(v \wedge w, W_V^u)$, as the vanishing order of $v \wedge w$ along W_V^s, resp. W_V^u. Of course, if W_V^s is not contained in E (for instance, if W_V^s is not a piece of a compact analytic subset of X) then $O(v \wedge w, W_V^s) = 0$.

Lemma 7.

$$O(v \wedge w, W_V^s) = O(v \wedge w, W_V^u) + b.$$

Proof. We shall conclude by a local computation. Take $p \in Z_0$ and local coordinates (x, y, z) so that $v = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, $w = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}$, $W_V^s = \{x = 0\}$, $W_V^u = \{y = 0\}$. Hence

$$v \wedge w = (xB + yA)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + xC\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - yC\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

and we see that

$$O(v \wedge w, W_V^s) = \min\{O(A, W_V^s), O(B, W_V^s) + 1, O(C, W_V^s)\}$$

$$O(v \wedge w, W_V^u) = \min\{O(A, W_V^u) + 1, O(B, W_V^u), O(C, W_V^u)\}.$$

From $[v, w] = bw$ we obtain the following system of equations:

$$\begin{cases} xA_x - yA_y = (b+1)A \\ xB_x - yB_y = (b-1)B \\ xC_x - yC_y = bC \end{cases}$$

Write $A(x, y, z) = x^h y^k a(x, y, z)$, $h = O(A, W_V^s)$, $k = O(A, W_V^u)$ (i.e., the functions $a(0, y, z)$ and $a(x, 0, z)$ are not identically zero). From the first equation we obtain:

$$xa_x - ya_y = (b + 1 - h + k)a$$

and restricting to $\{y = 0\}$:

$$xa_x(x, 0, z) = (b + 1 - h + k)a(x, 0, z)$$

that is

$$\frac{a_x(x,0,z)}{a(x,0,z)} = \frac{b+1-h+k}{x}.$$

We deduce that $b+1-h+k \geq 0$, because $a(x,0,z)$ is holomorphic and not identically zero. Restricting to $\{x=0\}$ we find the opposite inequality: $b+1-h+k \leq 0$. Hence $b+1-h+k=0$, or more explicitely

$$O(A, W_V^s) = O(A, W_V^u) + b + 1.$$

In a similar way, from the second and the third equations we find

$$O(B, W_V^s) = O(B, W_V^u) + b - 1$$

and

$$O(C, W_V^s) = O(C, W_V^u) + b$$

from which it follows that

$$O(v \wedge w, W_V^s) = O(v \wedge w, W_V^u) + b.$$

q.e.d.

In order to prove proposition 2, suppose now by contradiction that b is strictly positive. In particular $O(v \wedge w, W_V^s) > 0$, so that $v \wedge w$ does vanish on W_V^s. In other words, there exists an irreducible component $N \subset E$, *dim* $N=2$, which contains W_V^s. Take the restriction of the $\mathbf{C}^*$-action ρ to N, and take an equivariant resolution of singularities $\tilde{N} \to N$, over which ρ can be lifted. On $\tilde{N}$ we therefore have a $\mathbf{C}^*$-action with a rational curve of fixed points (arising from Z_0). It follows from the classification of $\mathbf{C}^*$-actions on compact complex surfaces [Hau] that $\tilde{N}$ is algebraic (and even rational) and that the closure of each orbit of the $\mathbf{C}^*$-action is a (possibly singular) rational curve. Returning to X, we therefore see that for each $p \in W_V^s$ not only $\lim_{t\to\infty} \rho_t(p)$ is a single point on Z_0 (as the definition of W_V^s claims) but also $\lim_{t\to 0} \rho_t(p)$ is a single point, necessarily on Z_0, and so the ρ-orbit through p cuts W_V^u. Varying p on W_V^s we also see that the full W_V^u belongs to N. But this contradicts lemma 7: because $O(v \wedge w, W_V^s) \neq O(v \wedge w, W_V^u)$, the sets W_V^s and W_V^u cannot belong to the same irreducible component of E. This contradiction proves proposition 2.

References

[Bor] A. Borel, *Seminar on transformation groups*, Princeton Univ. Press, Ann. of Math. Studies 46 (1960)

[Bot] R. Bott, *A residue formula for holomorphic vector fields*, Jour. Diff. Geom. 1 (1967), 311-330

[CDP] F. Campana, J.-P. Demailly, T. Peternell, *The algebraic dimension of compact complex threefolds with vanishing second Betti number*, Comp. Math. 112 (1998), 77-91
[Hau] J. Hausen, *Zur Klassifikation glatter kompakter* $\mathbf{C}^*$*-flächen*, Math. Ann. 301 (1995), 763-769
[Huc] A. Huckleberry, *Actions of groups of holomorphic transformations*, Enc. Math. Sci. vol. 69 (Several complex variables VI), Springer Verlag (1990)
[HKP] A. Huckleberry, S. Kebekus, T. Peternell, *Group actions on* $\mathbf{S}^6$ *and complex structure on* $\mathbf{P}^3$, preprint math. AG/9812076 (1998)
[Kol] J. Kollár, *Flip and flop*, Proc. ICM Kyoto 1990, Springer Verlag (1991)
[Moi] B. Moishezon, *On* n*-dimensional compact varieties with* n *algebraically independent meromorphic functions*, AMS Translations 63 (1967), 51-177

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