Random Matrices and K-Theory for Exact C^*-Algebras

U. Haagerup[1,2] and S. Thorbjørnsen[1,2]

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Abstract. In this paper we find asymptotic upper and lower bounds for the spectrum of random operators of the form

$$S^*S = \Big(\sum_{i=1}^{r} a_i \otimes Y_i^{(n)}\Big)^* \Big(\sum_{i=1}^{r} a_i \otimes Y_i^{(n)}\Big),$$

where $a_1, \dots, a_r$ are elements of an exact C^*-algebra and $Y_1^{(n)}, \dots, Y_r^{(n)}$ are complex Gaussian random $n \times n$ matrices, with independent entries. Our result can be considered as a generalization of results of Geman (1981) and Silverstein (1985) on the asymptotic behavior of the largest and smallest eigenvalue of a random matrix of Wishart type. The result is used to give new proofs of:

(1) Every stably finite exact unital C^*-algebra $\mathcal{A}$ has a tracial state.

(2) If $\mathcal{A}$ is an exact unital C^*-algebra, then every state on $K_0(\mathcal{A})$ is given by a tracial state on $\mathcal{A}$.

The new proofs do not rely on quasitraces or on AW^*-algebra techniques.

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Introduction

Following the terminology in [HT], we let $\mathrm{GRM}(m, n, \sigma^2)$ denote the class of $m \times n$ random matrices $B = (b_{ij})_{1 \le i \le m,\, 1 \le j \le n}$, for which $\big(\mathrm{Re}(b_{ij}), \mathrm{Im}(b_{ij})\big)_{1 \le i \le m,\, 1 \le j \le n}$ form a set of $2mn$ independent Gaussian random variables, all with mean 0 and variance $\frac{1}{2}\sigma^2$. In other words, the

[1] Department of Mathematics and Computer Science, SDU, Odense University, Denmark.

[2] MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from The Danish National Research Foundation.

entries of B are mn independent complex random variables with distribution measure on $\mathbb{C}$ given by

$$\tfrac{1}{\pi\sigma^2}\exp(-\tfrac{|z|^2}{\sigma^2})\ d\mathrm{Re}(z)\ d\mathrm{Im}(z).$$

The theory of exact C^*-algebras has been developed by Kirchberg (see [Ki1], [Ki2], [Ki3], [Was] and references given there). A C^*-algebra $\mathcal{A}$ is exact, if for all pairs $(\mathcal{B},\mathcal{J})$, of a C^*-algebra $\mathcal{B}$ and a closed two-sided ideal $\mathcal{J}$ in $\mathcal{B}$, the sequence

$$0 \longrightarrow \mathcal{A}\underset{\min}{\otimes}\mathcal{J} \longrightarrow \mathcal{A}\underset{\min}{\otimes}\mathcal{B} \longrightarrow \mathcal{A}\underset{\min}{\otimes}\big(\mathcal{B}/\mathcal{J}\big) \longrightarrow 0$$

is exact. Here, for any C^*-algebras $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}\otimes_{\min}\mathcal{D}$ means the completion of the algebraic tensor product $\mathcal{C}\odot\mathcal{D}$ in the minimal (=spatial) tensor norm. Sub-algebras and quotients of exact C^*-algebras are again exact (cf. e.g. [Was, 2.5.2 and Corollary 9.3]), and the class of exact C^*-algebras contains most of the C^*-algebras of current interest, such as all nuclear C^*-algebras, and the non-nuclear reduced group C^*-algebras $C_r^*(\mathbb{F}_n)$, associated with the free group $\mathbb{F}_n$ on n generators $(2\le n\le\infty)$.
For any element T of a unital C^*-algebra, we let $\mathrm{sp}(T)$ denote the spectrum of T. The main result of this paper is

0.1 Main Theorem. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that $\{a_i^*a_j \mid 1\le i,j\le r\}$ is contained in an exact C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$. Let $(\Omega,\mathcal{F},P)$ be a fixed probability space, and let, for each n in $\mathbb{N}$, $Y_1^{(n)},\dots,Y_r^{(n)}$ be independent Gaussian random matrices on Ω in the class $\mathrm{GRM}(n,n,\frac{1}{n})$. Put

$$S_n=\sum_{i=1}^r a_i\otimes Y_i^{(n)},\qquad (n\in\mathbb{N}),$$

and let c,d be positive real numbers. We then have

(i) If $\|\sum_{i=1}^r a_i^*a_i\|\le c$ and $\|\sum_{i=1}^r a_ia_i^*\|\le d$, then for almost all ω in Ω,

$$\limsup_{n\to\infty}\max\big[\mathrm{sp}\big(S_n^*(\omega)S_n(\omega)\big)\big]\le\big(\sqrt{c}+\sqrt{d}\big)^2.$$

(ii) If $\sum_{i=1}^r a_i^*a_i=c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, $\|\sum_{i=1}^r a_ia_i^*\|\le d$, and $d\le c$, then for almost all ω in Ω,

$$\liminf_{n\to\infty}\min\big[\mathrm{sp}\big(S_n^*(\omega)S_n(\omega)\big)\big]\ge\big(\sqrt{c}-\sqrt{d}\big)^2. \qquad\square$$

The Main Theorem can be considered as a generalization of the results of Geman (cf. [Gem]) and Silverstein (cf. [Si]), on the asymptotic behavior of the largest and smallest eigenvalues of a random matrix of Wishart type (see also [BY], [YBK] and [HT]).
The Main Theorem has the following two immediate consequences:

0.2 Corollary. *Let $a_1, \dots, a_r$ be elements of an exact C^*-algebra $\mathcal{A}$, and for each n in $\mathbb{N}$, let $Y_1^{(1)}, \dots, Y_r^{(n)}$ be independent elements of* $\mathrm{GRM}(n, n, \frac{1}{n})$. *Then*

$$\limsup_{n\to\infty} \Big\| \sum_{i=1}^r a_i \otimes Y_i^{(n)}(\omega) \Big\| \le \Big\| \sum_{i=1}^r a_i^* a_i \Big\|^{\frac{1}{2}} + \Big\| \sum_{i=1}^r a_i a_i^* \Big\|^{\frac{1}{2}},$$

for almost all ω in Ω. □

0.3 Corollary. *Let $a_1, \dots, a_r$ and S_n, $n \in \mathbb{N}$, be as in the Main Theorem, and assume that $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\| \sum_{i=1}^r a_i^* a_i \| \le d$, for some positive real numbers c, d, such that $d < c$. Then for almost all ω in Ω,*

$$0 \notin \mathrm{sp}\big(S_n^*(\omega) S_n(\omega)\big), \quad \textit{eventually as } n \to \infty. \qquad \square$$

In a subsequent paper [Th] by the second named author, it is proved, that if $a_1, \dots, a_r$ and S_n, $n \in \mathbb{N}$, are as in the Main Theorem, and if furthermore $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\sum_{i=1}^r a_i a_i^* = d\mathbf{1}_{\mathcal{B}(\mathcal{K})}$, for some positive real numbers c, d, then

$$\lim_{n\to\infty} \max\big[\mathrm{sp}(S_n^* S_n)\big] = \big(\sqrt{c} + \sqrt{d}\big)^2, \qquad \text{almost surely},$$

and if $c \ge d$, then

$$\lim_{n\to\infty} \min\big[\mathrm{sp}(S_n^* S_n)\big] = \big(\sqrt{c} - \sqrt{d}\big)^2, \qquad \text{almost surely}.$$

Hence the asymptotic upper and lower bounds in the Main Theorem cannot, in general, be improved.

Exactness is essential both for the Main Theorem and for the corollaries. An example of violation of the upper bound in the Main Theorem is given in Section 4. The example is based on the non-exact full C^*-algebra $C^*(\mathbb{F}_r)$ associated with the free group on r generators, for $r \ge 6$.

In [Haa], the first named author proved that bounded quasitraces on exact C^*-algebras are traces. Together with results of Handelman (cf. [Han]) and Blackadar and Rørdam (cf. [BR]), this result implies

(1) Every stably-finite exact unital C^*-algebra has a tracial state.

(2) If $\mathcal{A}$ is an exact unital C^*-algebra, then every state on $K_0(\mathcal{A})$ is given by a tracial state on $\mathcal{A}$.

The proof in [Haa] of the above mentioned quasitrace result, relies heavily on ultra product techniques for AW^*-algebras, but the starting point of the proof in [Haa] is the following fairly simple observation: Let $a_1, \dots, a_r$ be r elements in a (not necessarily exact) C^*-algebra $\mathcal{A}$, such that $\sum_{i=1}^r a_i^* a_i = \mathbf{1}_{\mathcal{A}}$ and $\| \sum_{i=1}^r a_i a_i^* \| < 1$. Let further $x_1, \dots, x_r$ be a semi-circular system (in the sense of Voiculescu; cf. [Vo2]) in some C^*-probability space $(\mathcal{B}, \psi)$. Then the operator $s = \sum_{i=1}^r a_i \otimes x_i$ in $\mathcal{A} \otimes C^*(x_1, \dots, x_r, \mathbf{1}_{\mathcal{B}})$, satisfies $0 \notin \mathrm{sp}(s^* s)$ but

$0 \in \mathrm{sp}(ss^*)$, and this implies that $u = s(s^*s)^{-\frac{1}{2}}$ is a non-unitary isometry in the C^*-algebra $\mathcal{A} \otimes C^*(x_1, \dots, x_r, \mathbf{1}_{\mathcal{B}})$.
Corollary 0.3 can be viewed as a random matrix version of the result that $0 \notin \mathrm{sp}(s^*s)$. The corresponding random matrix version of the result that $0 \in \mathrm{sp}(ss^*)$, holds too, i.e., if $a_1, \dots, a_r$ and S_n, $n \in \mathbb{N}$, are as in Corollary 0.3, then with probability 1, $0 \in \mathrm{sp}(S_n S_n^*)$, eventually as $n \to \infty$ (cf. [Th]). In view of Voiculescu's random matrix model for a semi-circular system (cf. [Vo1, Theorem 2.2]), it would have been more natural to substitute $Y_1^{(n)}, \dots, Y_r^{(n)}$ from $\mathrm{GRM}(n, n, \frac{1}{n})$, with a set of independent, *selfadjoint* Gaussian random matrices. However, we found it more tractable to work with the non-selfadjoint random matrices $Y_1^{(n)}, \dots, Y_r^{(n)}$.
In the last section (Section 9), we use Corollary 0.3 to give a new proof of the statements (1) and (2) above. The new proof does not rely on quasitraces or AW^*-algebra techniques. The main step in the new proof of (1) and (2) is to prove, that Corollary 0.3 implies the following

0.4 Proposition. *Let p, q be projections in an exact C^*-algebra $\mathcal{A}$, and assume that there exists an ϵ in $]0,1[$, such that*

$$\tau(q) \le (1-\epsilon)\tau(p),$$

for all lower semi-continuous (possibly unbounded) traces $\tau\colon \mathcal{A}_+ \to [0,\infty]$. Then for some n in $\mathbb{N}$, there exists a partial isometry u in $M_n(\mathcal{A}) = \mathcal{A} \otimes M_n(\mathbb{C})$, such that

$$u^*u = q \otimes \mathbf{1}_{M_n(\mathbb{C})} \qquad \text{and} \qquad uu^* \le p \otimes \mathbf{1}_{M_n(\mathbb{C})}. \qquad \square$$

In the rest of this introduction, we shall briefly discuss the main steps of the proof of the Main Theorem. Observe first, that by a simple scaling argument, it is enough to treat the case $d = 1$. This normalization will be used throughout the paper. The proof of the Main Theorem relies on the following

0.5 Key Estimates. Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, let c be a positive constant, and put $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$, $n \in \mathbb{N}$, as in the Main Theorem. We then have

(a) If $\|\sum_{i=1}^r a_i^* a_i\| \le c$ and $\|\sum_{i=1}^r a_i a_i^*\| \le 1$, then for $0 \le t \le \min\{\frac{n}{2c}, \frac{n}{2}\}$,

$$\mathbb{E}\big[\exp(tS_n^*S_n)\big] \le \exp\big((\sqrt{c}+1)^2 t + (c+1)^2 \tfrac{t^2}{n}\big)\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}. \tag{0.1}$$

(b) If $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, $\sum_{i=1}^r a_i a_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}$ and $c \ge 1$, then for $0 \le t \le \frac{n}{2c}$,

$$\mathbb{E}\big[\exp(-tS_n^*S_n)\big] \le \exp\big(-(\sqrt{c}-1)^2 t + (c+1)^2 \tfrac{t^2}{n}\big)\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}. \tag{0.2}$$

$\square$

We emphasize that the key estimates (0.1) and (0.2) hold without the exactness assumption of the Main Theorem. Once these estimates are proved, a fairly simple application of the Borel-Cantelli Lemma yields, that if $\mathcal{H}$ is *finite dimensional*, and $\lambda_{\max}$ and $\lambda_{\min}$ denote largest and smallest eigenvalues, then one has

$$\limsup_{n\to\infty} \lambda_{\max}(S_n^* S_n) \le \left(\sqrt{c}+1\right)^2, \qquad \text{almost surely,}$$

in the situation of (a) above, and

$$\liminf_{n\to\infty} \lambda_{\min}(S_n^* S_n) \ge \left(\sqrt{c}-1\right)^2, \qquad \text{almost surely,}$$

in the situation of (b) above. (This is completely parallel to the proof of the complex version of the Geman-Silverstein result, given in [HT, Section 7]). To pass from the case $\dim(\mathcal{H}) < \infty$ to the case $\dim(\mathcal{H}) = +\infty$, we need the assumption that the C^*-algebra $C^*(\{a_i^* a_j \mid 1 \le i, j \le r\})$ is exact, as well as the following characterization of exact C^*-algebras, due to Kirchberg (cf. [Ki2] and [Was, Section 7]):
A unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is exact if and only if the inclusion map $\iota\colon \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ has an approximate factorization

$$\mathcal{A} \xrightarrow{\varphi_\lambda} M_{n_\lambda}(\mathbb{C}) \xrightarrow{\psi_\lambda} \mathcal{B}(\mathcal{H}),$$

through a net of full matrix algebras $M_{n_\lambda}(\mathbb{C})$, $\lambda \in \Lambda$. Here, $\varphi_\lambda, \psi_\lambda$ are unital completely positive maps, and

$$\lim_{\lambda} \|\psi_\lambda \circ \varphi_\lambda(x) - x\| = 0, \quad \text{for all } x \text{ in } \mathcal{A}.$$

Finally, we use a dilation argument to pass from the condition $\sum_{i=1}^r a_i a_i^* = \mathbf{1}_{\mathcal{K}}$ of (b) above, to the less restrictive one: $\|\sum_{i=1}^r a_i a_i^*\| \le 1$, which is assumed in (ii) of the Main Theorem (when $d = 1$). The proof of the fact that the key estimates (0.1) and (0.2) imply the Main Theorem, is given in Section 4 for the upper bound, and in Section 8 for the lower bound. Sections 1-3 and 5-7 are used to prove the key estimates (0.1) respectively (0.2).
In Section 1, we associate to any permutation π in the symmetric group S_p, a permutation $\hat{\pi}$ in S_{2p}, for which $\hat{\pi}^2 = \hat{\pi} \circ \hat{\pi} = \mathrm{id}$ and $\hat{\pi}(j) \ne j$ for all j, namely the permutation given by

$$\begin{aligned} \hat{\pi}(2j-1) &= 2\pi^{-1}(j), \qquad (j \in \{1, 2, \dots, p\}) \\ \hat{\pi}(2j) &= 2\pi(j) - 1, \qquad (j \in \{1, 2, \dots, p\}). \end{aligned}$$

Moreover, following [Vo1], we let $\sim_{\hat{\pi}}$ denote the equivalence relation on $\{1, 2, \dots, 2p\}$, generated by the expression:

$$j \sim_{\hat{\pi}} \hat{\pi}(j) + 1, \qquad (\text{addition formed mod. } 2p),$$

and we let $d(\hat{\pi})$ denote the number of equivalence classes for $\sim_{\hat{\pi}}$. We can write $d(\hat{\pi}) = k(\hat{\pi}) + l(\hat{\pi})$, where $k(\hat{\pi})$ (resp. $l(\hat{\pi})$) denotes the number of equivalence classes for $\sim_{\hat{\pi}}$, consisting entirely of even numbers (resp. odd numbers) in $\{1,2,\dots,2p\}$. With this notation we prove, that for any random matrix B from $\mathrm{GRM}(m,n,1)$,

$$\mathbb{E}\circ\mathrm{Tr}_n\big[(B^*B)^p\big] = \sum_{\pi\in S_p} m^{k(\hat{\pi})}n^{l(\hat{\pi})}. \tag{0.3}$$

Consider next the quantity $\sigma(\hat{\pi}) = \frac{1}{2}(p+1-d(\hat{\pi}))$. It turns out, that $\sigma(\hat{\pi})$ is always a non-negative integer, and that $\sigma(\hat{\pi}) = 0$ if and only if $\hat{\pi}$ is non-crossing (cf. Definition 1.14). In Section 2 we show, that if $a_1,\dots,a_r$ are elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $S = \sum_{i=1}^r a_i\otimes Y_i^{(n)}$, where $Y_1^{(n)},\dots,Y_r^{(n)}$ are independent elements of $\mathrm{GRM}(n,n,\frac{1}{n})$, then

$$\mathbb{E}[(S^*S)^p] = \Big(\sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}\cdot\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}}\Big)\otimes\mathbf{1}_{M_n(\mathbb{C})}. \tag{0.4}$$

In [HT, Section 6], we found explicit formulas for the quantities $\mathbb{E}\circ\mathrm{Tr}_n[\exp(tB^*B)]$ and $\mathbb{E}\circ\mathrm{Tr}_n[B^*B\exp(tB^*B)]$, where B is an element of $\mathrm{GRM}(m,n,1)$. In Section 3, a careful comparison of the terms in (0.3) and (0.4), combined with these explicit formulas, allows us to prove, that if $\|\sum_{i=1}^r a_i^*a_i\|\le c$ and $\|\sum_{i=1}^r a_ia_i^*\|\le 1$, then for $0\le t\le\min\{\frac{n}{2c},\frac{n}{2}\}$,

$$\|\mathbb{E}[\exp(tS^*S)]\| \le \exp((c+1)^2\tfrac{t^2}{n})\int_0^\infty \exp(tx)\,d\mu_c(x), \tag{0.5}$$

where μ_c is the free (analog of the) Poisson distribution with parameter c (cf. [VDN] and [HT, Section 6]). The measure μ_c is also called the Marchenko-Pastur distribution (cf. [OP]), and it is given by

$$\mu_c = \max\{1-c,0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\cdot 1_{[a,b]}(x)\,dx,$$

where $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$ and δ_0 is the Dirac measure at 0. Since $\mathrm{supp}(\mu_c)\subseteq[0,b]$, the first key estimate, (0.1), follows immediately from (0.5). To prove the second key estimate, (0.2), we show in Sections 5-6, that under the condition

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}, \qquad \text{and} \qquad \sum_{i=1}^r a_ia_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})},$$

one has, for any q in $\mathbb{N}$, the formula:

$$\mathbb{E}\big[P_q^c(S^*S)\big] = \Big[\sum_{\rho\in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})}\Big(\sum_{1\le i_1,\dots,i_q\le r} a_{i_1}^*a_{i_{\rho(1)}}\cdots a_{i_q}^*a_{i_{\rho(q)}}\Big)\Big]\otimes\mathbf{1}_{M_n(\mathbb{C})}. \tag{0.6}$$

Here $P_0^c(x), P_1^c(x), P_2^c(x), \ldots,$ is the sequence of monic polynomials obtained from $1, x, x^2, \ldots,$ by the Gram-Schmidt orthogonalization process, w.r.t. the inner product

$$\langle f, g\rangle = \int_0^\infty f\overline{g}\; d\mu_c, \qquad (f, g \in L^2(\mathbb{R}, \mu_c)).$$

Moreover, S_q^{irr} denotes the set of permutations ρ in S_q, for which

$$1 \neq \rho(1) \neq 2 \neq \rho(2) \neq \cdots \neq q \neq \rho(q).$$

For fixed t in $\mathbb{R}$, we expand in Section 7 the exponential function $x \mapsto \exp(tx)$, in terms of the polynomials $P_q^c(x)$, $q \in \mathbb{N}_0$:

$$\exp(tx) = \sum_{q=0}^\infty \psi_q^c(t) P_q^c(x), \qquad (x \in [0, \infty[). \tag{0.7}$$

We show that the coefficients $\psi_q^c(t)$ are non-negative for all t in $[0, \infty[$, and that for any q in $\mathbb{N}_0$,

$$|\psi_q^c(-t)| \leq \left(\frac{\int_0^\infty \exp(-tx)\; d\mu_c(x)}{\int_0^\infty \exp(tx)\; d\mu_c(x)}\right) \cdot \psi_q^c(t), \qquad (t \in [0, \infty[). \tag{0.8}$$

By combining (0.6), (0.7) and (0.8) with the proof of (0.5), we obtain that for $c \geq 1$ and $0 \leq t \leq \frac{n}{2c}$,

$$\|\mathbb{E}[\exp(-tS^*S)]\| \leq \exp((c+1)^2 \tfrac{t^2}{n}) \int_0^\infty \exp(-tx)\; d\mu_c(x),$$

and since $\mathrm{supp}(\mu_c) \subseteq [a, \infty[= [(\sqrt{c}-1)^2, \infty[$, when $c \geq 1$, we obtain the second key estimate (0.2).

The rest of the paper is organized in the following way:

1 A Combinatorial Expression for $\mathbb{E}\circ\mathrm{Tr}_n[(B^*B)^p]$, for a Gaussian Random Matrix B in $\mathrm{GRM}(m,n,1)$

For ξ in $\mathbb{R}$ and σ^2 in $]0,\infty[$, we let $N(\xi,\sigma^2)$ denote the Gaussian (or normal) distribution with mean ξ and variance σ^2. In [HT], we introduced the following class of Gaussian random matrices

1.1 Definition. (cf. [HT]) Let $(\Omega,\mathcal{F},P)$ be a classical probability space, let m,n be positive integers, and let

$$B=(b(i,j))_{\substack{1\le i\le m\\ 1\le j\le n}}\colon \Omega\to M_{m,n}(\mathbb{C}),$$

be a complex, random $m\times n$ matrix defined on Ω. We say then that B is a (standard) Gaussian random $m\times n$ matrix with entries of variance σ^2, if the real valued random variables $\mathrm{Re}(b(i,j))$, $\mathrm{Im}(b(i,j))$, $1\le i\le m$, $1\le j\le n$, form a family of $2mn$ independent, identically distributed random variables, with distribution $N(0,\frac{\sigma^2}{2})$. We denote by $\mathrm{GRM}(m,n,\sigma^2)$ the set of such random matrices defined on Ω. Note that σ^2 equals the second absolute moment of the entries of an element from $\mathrm{GRM}(m,n,\sigma^2)$. □

In the following we shall omit mentioning the underlying probability space $(\Omega,\mathcal{F},P)$, and it will be understood that all considered random matrices/variables are defined on this probability space. As a matter of notation, by 1_n we denote the unit of $M_n(\mathbb{C})$, and by tr_n we denote the trace on $M_n(\mathbb{C})$ satisfying that $\mathrm{tr}_n(1_n)=1$. Moreover, we put $\mathrm{Tr}_n=n\cdot\mathrm{tr}_n$.
Let B be an element of $\mathrm{GRM}(m,n,\sigma^2)$. Then for any p in $\mathbb{N}$, $(B^*B)^p$ is a positive definite $n\times n$ random matrix, and $\mathrm{Tr}_n((B^*B)^p)$ is a positive valued, integrable, random variable. The main aim of this section is to derive a combinatorial expression for the moments $\mathbb{E}\circ\mathrm{Tr}_n((B^*B)^p)$ of B^*B w.r.t. $\mathbb{E}\circ\mathrm{Tr}_n$, where $\mathbb{E}$ denotes expectation w.r.t. P.

1.2 Lemma. *Let m,n,r,p be positive integers, let $B_1,B_2,\dots,B_r$ be independent elements of $\mathrm{GRM}(m,n,\sigma^2)$, and for each s in $\{1,2,\dots,r\}$, let $b(u,v,s)$, $1\le u\le m$, $1\le v\le n$, denote the entries of B_s. Then for any $i_1,j_1,i_2,j_2,\dots,i_p,j_p$ in $\{1,2,\dots,r\}$, we have that*

$$\begin{aligned}&\mathbb{E}\circ\mathrm{Tr}_n(B_{i_1}^*B_{j_1}B_{i_2}^*B_{j_2}\cdots B_{i_p}^*B_{j_p})\\ &=\sum_{\substack{1\le u_2,u_4,\dots,u_{2p}\le m\\ 1\le u_1,u_3,\dots,u_{2p-1}\le n}}\mathbb{E}\Big(\overline{b(u_2,u_1,i_1)}b(u_2,u_3,j_1)\cdots\overline{b(u_{2p},u_{2p-1},i_p)}b(u_{2p},u_1,j_p)\Big),\end{aligned}\tag{1.1}$$

and moreover $\mathbb{E}\circ\mathrm{Tr}_n(B_{i_1}^*B_{j_1}B_{i_2}^*B_{j_2}\cdots B_{i_p}^*B_{j_p})=0$, unless there exists a permutation π in the symmetric group S_p, such that $j_h=i_{\pi(h)}$ for all h in $\{1,2,\dots,p\}$.

Proof. Let $f(u,v)$, $1 \le u \le m$, $1 \le v \le n$, denote the usual $m \times n$ matrix units, and let $g(u,v)$, $1 \le u \le n$, $1 \le v \le m$, denote the usual $n \times m$ matrix units. We have then that

$$\begin{aligned}
&\mathbb{E} \circ \mathrm{Tr}_n(B_{i_1}^* B_{j_1} B_{i_2}^* B_{j_2} \cdots B_{i_p}^* B_{j_p}) \\
&= \sum_{\substack{1 \le v_1, u_2, v_3, u_4, \dots, v_{2p-1}, u_{2p} \le m \\ 1 \le u_1, v_2, u_3, v_4, \dots, u_{2p-1}, v_{2p} \le n}} \mathbb{E}\big(b^*(u_1, v_1, i_1) b(u_2, v_2, j_1) \cdots b^*(u_{2p-1}, v_{2p-1}, i_p) b(u_{2p}, v_{2p}, j_p)\big) \\
&\qquad\qquad \cdot \mathrm{Tr}_n\big(g(u_1,v_1) f(u_2,v_2) \cdots g(u_{2p-1}, v_{2p-1}) f(u_{2p}, v_{2p})\big) \\
&= \sum_{\substack{1 \le u_2, u_4, \dots, u_{2p} \le m \\ 1 \le u_1, u_3, \dots, u_{2p-1} \le n}} \mathbb{E}\Big(\overline{b(u_2, u_1, i_1)} b(u_2, u_3, j_1) \cdots \overline{b(u_{2p}, u_{2p-1}, i_p)} b(u_{2p}, u_1, j_p)\Big).
\end{aligned}$$

Note here, that for any $u_2, u_4, \dots, u_{2p}$ in $\{1,2,\dots,m\}$ and $u_1, u_3, \dots, u_{2p-1}$ in $\{1,2,\dots,n\}$, we have because of the independence assumptions,

$$\begin{aligned}
&\mathbb{E}\Big(\overline{b(u_2, u_1, i_1)} b(u_2, u_3, j_1) \cdots \overline{b(u_{2p}, u_{2p-1}, i_p)} b(u_{2p}, u_1, j_p)\Big) \\
&\qquad = \prod_{l=1}^{r} \mathbb{E}\Big(\prod_{h : i_h = l} \overline{b(u_{2h}, u_{2h-1}, l)} \prod_{h : j_h = l} b(u_{2h}, u_{2h+1}, l)\Big),
\end{aligned}$$

where $2h+1$ is calculated mod. $2p$.
Note here, that for any l in $\{1,2,\dots,r\}$, any u in $\{1,2,\dots,m\}$ and any v in $\{1,2,\dots,n\}$, the distribution of $b(u,v,l)$ is invariant under multiplication by complex numbers of norm 1. Hence, for any s,t in $\mathbb{N}_0$, $\mathbb{E}\big[\overline{b(u,v,l)^s} \cdot b(u,v,l)^t\big] = 0$, unless $s = t$. Using this, and the independence assumptions, it follows that for any l in $\{1,2,\dots,r\}$, any $u_2, u_4, \dots, u_{2p}$ in $\{1,2,\dots,m\}$ and any $u_1, u_3, \dots, u_{2p-1}$ in $\{1,2,\dots,n\}$, a necessary condition for the mean

$$\mathbb{E}\Big(\prod_{h : i_h = l} \overline{b(u_{2h}, u_{2h-1}, l)} \cdot \prod_{h : j_h = l} b(u_{2h}, u_{2h+1}, l)\Big)$$

to be distinct from zero is that

$$\mathrm{card}\big(\{h \in \{1,2,\dots,p\} \mid i_h = l\}\big) = \mathrm{card}\big(\{h \in \{1,2,\dots,p\} \mid j_h = l\}\big). \tag{1.2}$$

It follows that $\mathbb{E} \circ \mathrm{Tr}_n(B_{i_1}^* B_{j_1} B_{i_2}^* B_{j_2} \cdots B_{i_p}^* B_{j_p}) = 0$, unless (1.2) holds for all l in $\{1,2,\dots,r\}$, and in this case, it is not hard to construct a permutation π from S_p, with the property described in the lemma. $\square$

1.3 Definition. Let p be a positive integer, and let π be an element of S_p. We associate to π a family $\Lambda(\pi, m, n)$, $m, n \in \mathbb{N}$, of complex numbers, as follows: Let $B_1, B_2, \dots, B_p$ be independent elements of $\mathrm{GRM}(m,n,1)$, and then define

$$\Lambda(\pi, m, n) = \mathbb{E} \circ \mathrm{Tr}_n(B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \cdots B_p^* B_{\pi(p)}). \qquad \square$$

1.4 Remark. Let m, n, r, p be positive integers, and let $B_1, B_2, \dots, B_r$ be arbitrary elements of $\mathrm{GRM}(m, n, \sigma^2)$. Moreover, let $i_1, j_1, \dots, i_p, j_p$ be arbitrary elements of $\{1, 2, \dots, r\}$. We shall need the fact that the quantity $\mathbb{E} \circ \mathrm{Tr}_n(B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p})$ is bounded numerically by some constant $K(m, n, p, \sigma^2)$ depending only on m, n, p, σ^2 and not on r or the distributional relations between $B_1, B_2, \dots, B_r$. For this, adapt the notation from Lemma 1.2, and note then that by (1.1) from that lemma,

$$
\begin{aligned}
&|\mathbb{E} \circ \mathrm{Tr}_n(B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p})| \\
&\quad \le \sum_{\substack{1 \le u_2, u_4, \dots, u_{2p} \le m \\ 1 \le u_1, u_3, \dots, u_{2p-1} \le n}} \Big| \, \mathbb{E}\Big(\overline{b(u_2, u_1, i_1)} b(u_2, u_3, j_1) \cdots \overline{b(u_{2p}, u_{2p-1}, i_p)} b(u_{2p}, u_1, j_p)\Big) \, \Big| .
\end{aligned}
$$

Then let $M(2p, \sigma^2)$ denote the $2p$'th absolute moment of the entries of an element from $\mathrm{GRM}(m, n, \sigma^2)$. A standard computation yields that $M(2p, \sigma^2) = \sigma^{2p} \cdot p!$, but we shall not need this explicit formula. It follows now by the generalized Hölder inequality, that for any $u_2, u_4, \dots, u_{2p}$ in $\{1, 2, \dots, m\}$ and $u_1, u_3, \dots, u_{2p-1}$ in $\{1, 2, \dots, n\}$,

$$
\begin{aligned}
&\Big|\mathbb{E}\Big(\overline{b(u_2, u_1, i_1)} b(u_2, u_3, j_1) \cdots \overline{b(u_{2p}, u_{2p-1}, i_p)} b(u_{2p}, u_1, j_p)\Big) \, \Big| \\
&\quad \le \big\|\overline{b(u_2, u_1, i_1)}\big\|_{2p} \big\|b(u_2, u_3, j_1)\big\|_{2p} \cdots \big\|\overline{b(u_{2p}, u_{2p-1}, i_p)}\big\|_{2p} \big\|b(u_{2p}, u_1, j_p)\big\|_{2p} \\
&\quad = \big(M(2p, \sigma^2)^{\frac{1}{2p}}\big)^{2p} = M(2p, \sigma^2).
\end{aligned}
$$

Thus it follows that we may use $K(m, n, p, \sigma^2) = m^p n^p M(2p, \sigma^2)$. $\square$

1.5 Proposition. *Let B be an element of* $\mathrm{GRM}(m, n, 1)$, *and let p be a positive integer. We then have*

$$
\mathbb{E} \circ \mathrm{Tr}_n[(B^* B)^p] = \sum_{\pi \in S_p} \Lambda(\pi, m, n).
$$

Proof. Let $(B_i)_{i \in \mathbb{N}}$ be a sequence of independent elements of $\mathrm{GRM}(m, n, 1)$. Note then that for any s in $\mathbb{N}$, the matrix $\frac{1}{\sqrt{s}}(B_1 + \cdots + B_s)$ is again an element of $\mathrm{GRM}(m, n, 1)$, and therefore

$$
\begin{aligned}
\mathbb{E} \circ \mathrm{Tr}_n\left[(B^* B)^p\right] &= \mathbb{E} \circ \mathrm{Tr}_n \left[\Big(\big(s^{-\frac{1}{2}}(B_1 + \cdots + B_s)\big)^* \big(s^{-\frac{1}{2}}(B_1 + \cdots + B_s)\big)\Big)^p\right] \\
&= s^{-p} \sum_{1 \le i_1, j_1, \dots, i_p, j_p \le s} \mathbb{E} \circ \mathrm{Tr}_n \left[B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p}\right] .
\end{aligned}
\tag{1.3}
$$

For π in S_p we define

$$
M(\pi, s) = \big\{(i_1, j_1, \dots, i_p, j_p) \in \{1, 2, \dots, s\}^{2p} \;\big|\; j_1 = i_{\pi(1)}, \dots, j_p = i_{\pi(p)}\big\}.
$$

It follows then from Lemma 1.2, that in (1.3), we only have to sum over those $2p$-tuples $(i_1, j_1, \dots, i_p, j_p)$ that belong to $M(\pi, s)$ for some π in S_p, and consequently

$$\mathbb{E} \circ \mathrm{Tr}_n \left[(B^* B)^p \right] = s^{-p} \sum_{(i_1, j_1, \dots, i_p, j_p) \in \cup_{\pi \in S_p} M(\pi, s)} \mathbb{E} \circ \mathrm{Tr}_n \left[B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p} \right].$$

Note though, that the sets $M(\pi, s)$, $\pi \in S_p$, are not disjoint. However, if we put

$$\mathcal{D}(s) = \big\{ (i_1, j_1, \dots, i_p, j_p) \in \{1, 2, \dots, s\}^{2p} \bigm| i_1, i_2, \dots, i_p \text{ are distinct} \big\},$$

then the sets $M(\pi, s) \cap \mathcal{D}(s)$, $\pi \in S_p$, are disjoint. Thus we have

$$\begin{aligned} &\mathbb{E} \circ \mathrm{Tr}_n \left[(B^* B)^p \right] \\ &\qquad = s^{-p} \sum_{\pi \in S_p} \sum_{(i_1, j_1, \dots, i_p, j_p) \in M(\pi, s) \cap \mathcal{D}(s)} \mathbb{E} \circ \mathrm{Tr}_n \left[B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p} \right] \\ &\qquad\quad + s^{-p} \sum_{(i_1, j_1, \dots, i_p, j_p) \in \left(\cup_{\pi \in S_p} M(\pi, s) \right) \setminus \mathcal{D}(s)} \mathbb{E} \circ \mathrm{Tr}_n \left[B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p} \right]. \end{aligned} \tag{1.4}$$

Note here, that if $(i_1, j_1, \dots, i_p, j_p) \in M(\pi, s) \cap \mathcal{D}(s)$, then $B_{i_1}, B_{i_2}, \dots, B_{i_p}$ are independent elements of $\mathrm{GRM}(m, n, 1)$, and hence

$$\mathbb{E} \circ \mathrm{Tr}_n \left[B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p} \right] = \Lambda(\pi, m, n).$$

Thus, the first term on the right hand side of (1.4) equals

$$s^{-p} \sum_{\pi \in S_p} \mathrm{card}(M(\pi, s) \cap \mathcal{D}(s)) \cdot \Lambda(\pi, m, n).$$

Here $\mathrm{card}(M(\pi, s) \cap \mathcal{D}(s)) = s(s-1) \cdots (s-p+1)$, so

$$s^{-p} \cdot \mathrm{card}(M(\pi, s) \cap \mathcal{D}(s)) \to 1 \text{ as } s \to \infty.$$

Hence, the first term on the right hand side of (1.4) tends to $\sum_{\pi \in S_p} \Lambda(\pi, m, n)$ as $s \to \infty$, and since the left hand side of (1.4) does not depend on s, it remains thus to show that the second term on the right hand side of (1.4) tends to 0 as $s \to \infty$. This follows by noting that according to Remark 1.4, for any $(i_1, j_1, \dots, i_p, j_p)$ in $\{1, 2, \dots, s\}^{2p}$, the quantity $|\mathbb{E} \circ \mathrm{Tr}_n [B_{i_1}^* B_{j_1} \cdots B_{i_p}^* B_{j_p}]|$ is bounded by some constant $K(m, n, p)$ depending only on m, n, p; not on s. And moreover,

$$\begin{aligned} s^{-p} \mathrm{card}\big(\left(\cup_{\pi \in S_p} M(\pi, s) \right) \setminus \mathcal{D}(s) \big) &\le \sum_{\pi \in S_p} s^{-p} \mathrm{card}(M(\pi, s) \setminus \mathcal{D}(s)) \\ &= \sum_{\pi \in S_p} \left[s^{-p} \mathrm{card}(M(\pi, s)) - s^{-p} \mathrm{card}(M(\pi, s) \cap \mathcal{D}(s)) \right] \\ &= \sum_{\pi \in S_p} \left[1 - s^{-p} \mathrm{card}(M(\pi, s) \cap \mathcal{D}(s)) \right] \to 0, \end{aligned}$$

as $s \to \infty$. This concludes the proof of the proposition. $\square$

It follows from Proposition 1.5, that in order to obtain a combinatorial expression for the moments $\mathbb{E} \circ \mathrm{Tr}_n((B^*B)^p)$ for a matrix B from $\mathrm{GRM}(m,n,1)$, we need to derive a combinatorial expression for the quantities

$$\Lambda(\pi, m, n) = \mathbb{E} \circ \mathrm{Tr}_n(B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \cdots B_p^* B_{\pi(p)}),$$

where $\pi \in S_p$ and $B_1, \dots, B_p$ are independent elements of $\mathrm{GRM}(m,n,1)$.
As it turns out, it shall be useful to have the relations between the factors in the product $B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \cdots B_p^* B_{\pi(p)}$ determined in terms of a permutation $\hat{\pi}$ in S_{2p}, rather than in terms of the permutation π from S_p.

1.6 Definition. Let p be a positive integer, and let π be a permutation in S_p. Then the permutation $\hat{\pi}$ in S_{2p} is determined by the equations:

$$\begin{aligned} \hat{\pi}(2i-1) &= 2\pi^{-1}(i), \qquad && (i \in \{1,2,\dots,p\}), \\ \hat{\pi}(2i) &= 2\pi(i) - 1, \qquad && (i \in \{1,2,\dots,p\}). \end{aligned}$$ $\square$

1.7 Remark. (a) Let p, π and $\hat{\pi}$ be as in Definition 1.6. Note then that $\hat{\pi}^2 = \hat{\pi} \circ \hat{\pi} = \mathrm{id}$, the identity mapping on $\{1,2,\dots,2p\}$, and that $\hat{\pi}$ maps odd numbers to even numbers, i.e., that $\hat{\pi}(j) - j = 1$ (mod. 2), for all j in $\{1,2,\dots,2p\}$. In particular, $\hat{\pi}$ has no fixed points. It is easy to check that $\{\hat{\pi} \mid \pi \in S_p\}$ is exactly the set of permutations γ in S_{2p}, for which $\gamma^2 = \mathrm{id}$ and $\gamma(j) - j = 1$ (mod. 2), for all j in $\{1,2,\dots,2p\}$. Moreover, the mapping $\pi \mapsto \hat{\pi}$ is injective.
(b) If $B_1, B_2, \dots, B_p$ are independent elements of $\mathrm{GRM}(m,n,1)$, then we may write the product $B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \cdots B_p^* B_{\pi(p)}$ in the form $C_1^* C_2 C_3^* C_4 \cdots C_{2p-1}^* C_{2p}$, where $C_{2i-1} = B_i$ and $C_{2i} = B_{\pi(i)}$ for all i in $\{1,2,\dots,p\}$. Then $\hat{\pi}$ is constructed exactly so that for any j, j' in $\{1,2,\dots,2p\}$, we have

$$C_j = C_{j'} \Leftrightarrow j = j' \text{ or } \hat{\pi}(j) = j'.$$ $\square$

1.8 Definition. We associate to $\hat{\pi}$ an equivalence relation $\sim_{\hat{\pi}}$ on $\mathbb{Z}_{2p}$. This is the equivalence relation (introduced by Voiculescu in [Vo1, Proof of Theorem 2.2]), generated by the expression:

$$j \sim_{\hat{\pi}} \hat{\pi}(j) + 1, \qquad (j \in \{1,2,\dots,2p\}),$$

where addition is formed mod. $2p$. $\square$

1.9 Remark. For a permutation π in S_p, the $\sim_{\hat{\pi}}$-equivalence classes are precisely the orbits in $\{1,2,\dots,2p\}$ for the cyclic subgroup of S_{2p} generated by the permutation $j \mapsto \hat{\pi}(j) + 1$ (addition formed mod. $2p$). Since this subgroup is finite, the equivalence class $[j]_{\hat{\pi}}$ of an element j in $\{1,2,\dots,2p\}$ has the following form:

Let q be the number of elements in $[j]_{\hat{\pi}}$. Then

$$[j]_{\hat{\pi}} = \{j_0, j_1, \dots, j_{q-1}\},$$

where $j_0 = j, j_1 = \hat{\pi}(j_0) + 1, j_2 = \hat{\pi}(j_1) + 1, \dots, j_{q-1} = \hat{\pi}(j_{q-2}) + 1, j_0 = \hat{\pi}(j_{q-1}) + 1$, (addition formed mod. $2p$). $\square$

It follows immediately from the definition of $\hat{\pi}$ and Remark 1.9 that each $\sim_{\hat{\pi}}$-equivalence class consists entirely of even numbers or entirely of odd numbers. This is used in the following definition:

1.10 Definition. Let p be a positive integer, let π be a permutation in S_p, and consider the corresponding permutation $\hat{\pi}$ in S_{2p}. By $k(\hat{\pi})$ and $l(\hat{\pi})$, we denote then the number of $\sim_{\hat{\pi}}$-equivalence classes consisting of even numbers, respectively the number of $\sim_{\hat{\pi}}$-equivalence classes consisting of odd numbers:

$$\begin{aligned} k(\hat{\pi}) &= \operatorname{card}\big(\big\{[j]_{\hat{\pi}} \bigm| j \in \{2,4,\dots,2p\}\big\}\big),\\ l(\hat{\pi}) &= \operatorname{card}\big(\big\{[j]_{\hat{\pi}} \bigm| j \in \{1,3,\dots,2p-1\}\big\}\big). \end{aligned}$$

Moreover, we define the quantities $d(\hat{\pi})$ and $\sigma(\hat{\pi})$ by the equations:

$$\begin{aligned} d(\hat{\pi}) &= k(\hat{\pi}) + l(\hat{\pi}) = \operatorname{card}\big(\big\{[j]_{\hat{\pi}} \bigm| j \in \{1,2,\dots,2p\}\big\}\big),\\ \sigma(\hat{\pi}) &= \tfrac{1}{2}\big(p + 1 - d(\hat{\pi})\big). \end{aligned}$$ $\square$

Regarding the definition of $\sigma(\hat{\pi})$, it will be shown later (cf. Theorem 1.13), that $\sigma(\hat{\pi})$ is always a non-negative integer. The quantity $d(\hat{\pi})$ was introduced by Voiculescu in [Vo1, Proof of Theorem 2.2].

1.11 Theorem. *For any positive integers m, n and any π in S_p, we have that*

$$\Lambda(\pi, m, n) = m^{k(\hat{\pi})} n^{l(\hat{\pi})}.$$

Proof. Consider independent elements $B_1, B_2, \dots, B_p$ of $\mathrm{GRM}(m, n, 1)$, and for each j in $\{1, 2, \dots, p\}$, let $b(u, v, j)$, $1 \le u \le m$, $1 \le v \le n$, denote the entries of B_j. It follows then by (1.1) in Lemma 1.2, that

$$\begin{aligned} &\Lambda(\pi, m, n)\\ &= \mathbb{E} \circ \mathrm{Tr}_n(B_1^* B_{\pi(1)} B_2^* B_{\pi(2)} \cdots B_p^* B_{\pi(p)})\\ &= \sum_{\substack{1 \le u_1, u_3, \dots, u_{2p-1} \le n\\ 1 \le u_2, u_4, \dots, u_{2p} \le m}} \mathbb{E}\Big(\overline{b(u_2, u_1, 1)}\, b(u_2, u_3, \pi(1)) \cdots \overline{b(u_{2p}, u_{2p-1}, p)}\, b(u_{2p}, u_1, \pi(p))\Big). \end{aligned} \tag{1.5}$$

Arguing as in the proof of Lemma 1.2, it follows that the term in the above sum corresponding to $u_1, u_2, \dots, u_{2p}$ is zero, unless the corresponding matrix entries are pairwise conjugate to each other, i.e., unless we have that

$$b(u_{2i}, u_{2i+1}, \pi(i)) = b(u_{2\pi(i)}, u_{2\pi(i)-1}, \pi(i)), \qquad (i \in \{1, 2, \dots, p\}). \tag{1.6}$$

Note also, that if (1.6) is satisfied, then the corresponding term in (1.5) equals 1, and consequently

$$\begin{aligned}&\Lambda(\pi,m,n)\\ &= \operatorname{card}\big(\big\{(u_1,u_2,\dots,u_{2p}) \bigm| 1\le u_{2i-1}\le n,\ 1\le u_{2i}\le m,\ \text{and (1.6) holds}\big\}\big).\end{aligned}$$

To calculate this cardinality, we note first that (1.6) is equivalent to the condition

$$u_{2i} = u_{2\pi(i)} \quad \text{and} \quad u_{2i+1} = u_{2\pi(i)-1}, \qquad (i\in\{1,2,\dots,p\}), \tag{1.7}$$

where addition and subtraction are formed mod. $2p$. Replacing now i by $\pi^{-1}(i)$ in the first equation in (1.7), we get the equivalent condition:

$$u_{2i} = u_{2\pi^{-1}(i)} \quad \text{and} \quad u_{2i+1} = u_{2\pi(i)-1}, \qquad (i\in\{1,2,\dots,p\}).$$

Recall then that by definition of $\hat\pi$, $\hat\pi(2i-1) = 2\pi^{-1}(i)$, and using this formula with i replaced by $\pi(i)$, we get that also $2\pi(i)-1 = \hat\pi\big(\hat\pi(2\pi(i)-1)\big) = \hat\pi(2i)$. Thus (1.6) is equivalent to the condition

$$u_{2i} = u_{\hat\pi(2i-1)}, \quad \text{and} \quad u_{2i+1} = u_{\hat\pi(2i)}, \qquad (i\in\{1,2,\dots,p\}),$$

i.e., the condition

$$u_j = u_{\hat\pi(j-1)}, \qquad (j\in\{1,2,\dots,2p\}).$$

Replacing finally j by $\hat\pi(j)+1$, we conclude that (1.6) is equivalent to the condition

$$u_j = u_{\hat\pi(j)+1}, \qquad (j\in\{1,2,\dots,2p\}),$$

where $\hat\pi(j)+1$ is calculated mod. $2p$. Having realized this, it follows immediately from Remark 1.9 and the definitions of $k(\hat\pi)$ and $l(\hat\pi)$, that the right hand side of (1) equals $m^{k(\hat\pi)}n^{l(\hat\pi)}$, and hence we have the desired formula. ☐

1.12 Corollary. *Let m,n be positive integers and let B be an element of* $\mathrm{GRM}(m,n,1)$. *Then for any positive integer p, we have that*

$$\mathbb{E}\circ\mathrm{Tr}_n\big[(B^*B)^p\big] = \sum_{\pi\in S_p} m^{k(\hat\pi)}n^{l(\hat\pi)}.$$

Proof. This follows immediately by combining Proposition 1.5 and Theorem 1.11. ☐

1.13 Theorem. *Let p be a positive integer, and let π be a permutation in S_p. Then*

(i) $k(\hat\pi)\ge 1$ *and* $l(\hat\pi)\ge 1$.
(ii) $k(\hat\pi)+l(\hat\pi)\le p+1$.
(iii) $\sigma(\hat\pi) = \frac12\big(p+1-k(\hat\pi)-l(\hat\pi)\big)$ *is a non-negative integer.*

Proof. (i) This is clear from Definition 1.10.
(ii) Since $d(\hat{\pi}) = k(\hat{\pi}) + l(\hat{\pi})$ is the number of equivalence classes for $\sim_{\hat{\pi}}$, (ii) follows from [Vo1, Proof of Theorem 2.2].
(iii) The proof of (iii) requires more work. For elements p of $\mathbb{N}$ and k, l of $\mathbb{N}_0$, we define

$$\delta(p,k,l) = \mathrm{card}(\{\pi \in S_p \mid k(\hat{\pi}) = k \text{ and } l(\hat{\pi}) = l\}).$$

By (i) and (ii), $\delta(p,k,l) = 0$ unless $k \geq 1$, $l \geq 1$ and $k + l \leq p + 1$. By Corollary 1.12, we have for an element B of $\mathrm{GRM}(m,n,1)$, that

$$\mathbb{E} \circ \mathrm{Tr}_n\big[(B^*B)^p\big] = \sum_{\substack{k,l\in\mathbb{N} \\ k+l\leq p+1}} \delta(p,k,l) m^k n^l.$$

On the other hand, by the recursion formula for the moments $\mathbb{E} \circ \mathrm{Tr}_n\big[(B^*B)^p\big]$, $(p \in \mathbb{N})$, found in [HT, Theorem 8.2], it follows that for p in $\mathbb{N}$, the moment $\mathbb{E} \circ \mathrm{Tr}_n\big[(B^*B)^p\big]$ can be expressed as a polynomial in m and n of the form:

$$\mathbb{E} \circ \mathrm{Tr}_n\big[(B^*B)^p\big] = \sum_{\substack{k,l\in\mathbb{N} \\ k+l\leq p+1}} \delta'(p,k,l) m^k n^l,$$

for suitable coeffecients $\delta'(p,k,l)$. By the remarks following the proof of [HT, Theorem 8.2], only terms of homogeneous degree $p + 1 - 2j$, $j \in \{0, 1, 2, \dots, [\frac{p-1}{2}]\}$, appear in this polynomial, i.e.,

$$\delta'(p,k,l) = 0, \quad \text{when} \quad k + l = p \ (\text{mod. } 2).$$

If polynomials of two variables coincide on $\mathbb{N}^2$, then they are equal. Therefore, $\delta(p,k,l) = \delta'(p,k,l)$ for all k, l, which proves that

$$\mathrm{card}(\{\pi \in S_p \mid k(\hat{\pi}) = k \text{ and } l(\hat{\pi}) = l\}) = 0, \quad \text{if} \quad k + l = p \ (\text{mod. } 2).$$

Hence, $\sigma(\hat{\pi})$ is an integer for all π in S_p, and by (ii), $\sigma(\hat{\pi}) \geq 0$. This proves (iii). □

In the rest of this section, we shall introduce a method of "reductions of permutations", which will be needed to determine the asymptotic lower bound of the spectrum of $S_n^* S_n$ (cf. Sections 5-8).
Let p be a positive integer, let π be a permutation in S_p, and consider the corresponding permutation $\hat{\pi}$ in S_{2p}, introduced in Definition 1.6. Since $\hat{\pi}^2 = \mathrm{id}$ and $\hat{\pi}$ has no fixed points, the orbits under the action of $\hat{\pi}$ form a partition of $\{1, 2, \dots, 2p\}$ into p sets, each with two elements.

1.14 Definition. Let p be a positive integer, and let π be a permutation in S_p. Following the standard definition of crossings in partitions of $\{1, 2, \dots, 2p\}$ into sets of cardinality 2 (see e.g. [Sp]), we say that (a, b, c, d) is a *crossing* for $\hat{\pi}$, if $a, b, c, d \in \{1, 2, \dots, 2p\}$ such that

$$a < b < c < d, \qquad \text{and} \qquad \hat{\pi}(a) = c, \ \hat{\pi}(b) = d. \tag{1.8}$$

If $\hat{\pi}$ has no such crossings, we say that $\hat{\pi}$ is a non-crossing permutation, and we let $S_p^{\rm nc}$ denote the set of permutations π in S_p for which $\hat{\pi}$ is non-crossing. $\square$

1.15 Definition. Let p be a positive integer, let π be a permutation in S_p, and let e be an element of $\{1, 2, \dots, 2p-1\}$. We say then that $(e, e+1)$ is a *pair of neighbors* for $\hat{\pi}$, if $\hat{\pi}(e) = e+1$. Note, that a pair of neighbors for $\hat{\pi}$ is either of the form

$$(2k-1, 2k), \quad \text{where} \quad k \in \{1, \dots, p\},$$

or of the form

$$(2k, 2k+1), \quad \text{where} \quad k \in \{1, \dots, p-1\}.$$

In the first case $k = \pi(k)$, and in the second case $\pi(k) = k+1$. $\square$

1.16 Definition. Let p be a positive integer, let π be a permutation in S_p, and consider the permutation $\hat{\pi}$ in S_{2p} introduced in Definition 1.6. We say then that $\hat{\pi}$ is *irreducible* if $\hat{\pi}$ has no pair of neighbors (in the sense of Definition 1.15), i.e., if $\hat{\pi}(j) \neq j+1$ for all j in $\{1, 2, \dots, 2p-1\}$. We denote by $S_p^{\rm irr}$ the set of permutations π in S_p for which $\hat{\pi}$ is irreducible. Note that

$$\pi \in S_p^{\rm irr} \iff 1 \neq \pi(1) \neq 2 \neq \pi(2) \neq \cdots \neq p \neq \pi(p).$$

If $\pi \in S_p \setminus S_p^{\rm irr}$, we say that $\hat{\pi}$ is *reducible*. Note, that we do not require that $\hat{\pi}(2p) \neq 1$ in order for $\hat{\pi}$ to be irreducible. Thus, irreducibility of $\hat{\pi}$ is *not* invariant under cyclic permutations of $\{1, 2, \dots, 2p\}$. $\square$

1.17 Lemma. *Let p be a positive integer, and let π be a permutation in $S_p^{\rm nc}$. Then $\hat{\pi}$ has a pair of neighbors, i.e., $\hat{\pi}$ is reducible in the sense of Definition 1.16. In other words, we have the inclusion $S_p^{\rm nc} \subseteq S_p \setminus S_p^{\rm irr}$ or equivalently $S_p^{\rm irr} \subseteq S_p \setminus S_p^{\rm nc}$.*

Proof. We prove the inclusion: $S_p^{\rm irr} \subseteq S_p \setminus S_p^{\rm nc}$. So let π from $S_p^{\rm irr}$ be given, and consider the set $M = \{j \in \{1, 2, \dots, 2p\} \mid \hat{\pi}(j) \geq j\}$. Note that $M \neq \emptyset$, since clearly $1 \in M$. Define now

$$\alpha = \min\{\hat{\pi}(j) - j \mid j \in M\}.$$

Since $\hat{\pi}$ has no fixed points and no pairs of neighbors (since $\pi \in S_p^{\rm irr}$), we must have $\alpha \geq 2$. Choose j in $\{1, 2, \dots, 2p\}$ such that $\hat{\pi}(j) - j = \alpha$. Since $\alpha \geq 2$, $\hat{\pi}(j) \neq j+1$, or equivalently (since $\hat{\pi}^2 = \mathrm{id}$), $\hat{\pi}(j+1) \neq j$. Combining this with the definition of α, and the fact that $\hat{\pi}$ has no fixed points, it follows that

$$\hat{\pi}(j+1) \notin \{j, j+1, \dots, j+\alpha\} = \{j, j+1, \dots, \hat{\pi}(j)\},$$

i.e., either $\hat{\pi}(j+1) < j$ or $\hat{\pi}(j+1) > \hat{\pi}(j)$. In the first case $(\hat{\pi}(j+1), j, j+1, \hat{\pi}(j))$ is a crossing for $\hat{\pi}$, and in the second case $(j, j+1, \hat{\pi}(j), \hat{\pi}(j+1))$ is a crossing for $\hat{\pi}$. In all cases, $\pi \in S_p \setminus S_p^{\rm nc}$, as desired. $\square$

1.18 DEFINITION. Let p be a positive integer, greater than or equal to 2, let π be a permutation in S_p, and assume that the permutation $\hat{\pi}$ in S_{2p} has a pair of neighbors $(e, e+1)$. Let φ be the order preserving bijection of $\{1, 2, \dots, 2p-2\}$ onto $\{1, 2, \dots, 2p\} \setminus \{e, e+1\}$, i.e.,

$$\varphi(i) = \begin{cases} i, & \text{if } 1 \le i \le e-1, \\ i+2, & \text{if } e \le i \le 2p-2. \end{cases} \tag{1.9}$$

By π_0 we denote then the unique permutation in S_{p-1}, satisfying that

$$\hat{\pi}_0 = \varphi^{-1} \circ \hat{\pi} \circ \varphi.$$

We say that $\hat{\pi}_0$ is obtained from $\hat{\pi}$ by *cancellation of the pair* $(e, e+1)$. □

A few words are appropriate about the introduction of π_0 in the definition above. Note first of all that $\varphi^{-1} \circ \hat{\pi} \circ \varphi$ is a well-defined permutation of $\{1, 2, \dots, 2p-2\}$, since $\hat{\pi}^2 = \mathrm{id}$ and $\hat{\pi}(e) = e+1$, so that $\hat{\pi}(\{1, 2, \dots, 2p\} \setminus \{e, e+1\}) = \{1, 2, \dots, 2p\} \setminus \{e, e+1\}$. To see that this permutation is actually of the form $\hat{\pi}_0$ for some (necessarily uniquely determined) permutation π_0 in S_{p-1}, it suffices, by Remark 1.7(a), to check that $(\varphi^{-1} \circ \hat{\pi} \circ \varphi)^2 = \mathrm{id}$, and that $\varphi^{-1} \circ \hat{\pi} \circ \varphi(j) - j = 1 \pmod{2}$, for all j in $\{1, 2, \dots, 2p-2\}$. But these properties follow from the corresponding properties of $\hat{\pi}$, and the fact that $\varphi(j) = j \pmod{2}$, for all j.

1.19 REMARK. Let p be a positive integer, greater than or equal to 2, let π be a permutation in S_p, and assume that the permutation $\hat{\pi}$ in S_{2p} has a pair of neighbors $(e, e+1)$. Let π_0 be the permutation in S_{p-1} obtained from π as in Definition 1.18.

(a) If $(e, e+1) = (2k-1, 2k)$ for some k in $\{1, \dots, p\}$, then $\pi_0 = \psi^{-1} \circ \pi \circ \psi$, where $\psi\colon \{1, \dots, p-1\} \to \{1, \dots, p\} \setminus \{k\}$ is the bijection given by

$$\psi(j) = \begin{cases} j, & \text{if } 1 \le j \le k-1, \\ j+1, & \text{if } k \le j \le p-1. \end{cases} \tag{1.10}$$

(b) If $(e, e+1) = (2k, 2k+1)$ for some k in $\{1, \dots, p-1\}$, then $\pi_0 = \chi^{-1} \circ \pi \circ \psi$, where $\chi\colon \{1, \dots, p-1\} \to \{1, \dots, p\} \setminus \{k+1\}$ is the bijection given by

$$\chi(j) = \begin{cases} j, & \text{if } 1 \le j \le k, \\ j+1, & \text{if } k+1 \le j \le p-1, \end{cases} \tag{1.11}$$

and where ψ is given by (1.10). □

1.20 LEMMA. *Let p be a positive integer, greater than or equal to 2, and let π be a permutation in $S_p \setminus S_p^{\mathrm{irr}}$. Let $(e, e+1)$ be a pair of neighbors for $\hat{\pi}$ and let π_0 be the permutation in S_{p-1}, for which $\hat{\pi}_0$ is the permutation obtained from $\hat{\pi}$ by cancellation of $(e, e+1)$. Then $\hat{\pi}$ is non-crossing if and only if $\hat{\pi}_0$ is non-crossing.*

Proof. Let $\varphi\colon \{1,2,\dots,2p-2\} \to \{1,2,\dots,2p\} \setminus \{e,e+1\}$ be the bijection introduced in (1.9). We show that $\hat{\pi}_0$ has a crossing if and only if $\hat{\pi}$ does.
Assume first that $\hat{\pi}_0$ has a crossing (a,b,c,d). Then since φ is (strictly) monotone and since (by definition of π_0) $\hat{\pi}(\varphi(a)) = \varphi(c)$, $\hat{\pi}(\varphi(b)) = \varphi(d)$, it follows that $(\varphi(a),\varphi(b),\varphi(c),\varphi(d))$ is a crossing for $\hat{\pi}$.
Assume conversely that $\hat{\pi}$ has a crossing (a',b',c',d'). Then clearly

$$\{a',b',c',d'\} \cap \{e,e+1\} = \emptyset,$$

so that the numbers $\varphi^{-1}(a'),\varphi^{-1}(b'),\varphi^{-1}(c'),\varphi^{-1}(d')$ are well-defined. It follows then, as above, that $(\varphi^{-1}(a'),\varphi^{-1}(b'),\varphi^{-1}(c'),\varphi^{-1}(d'))$ is a crossing for $\hat{\pi}_0$. $\square$

1.21 Lemma. *Let m,n be positive integers, and let B be an element of* $\mathrm{GRM}(m,n,1)$. *Then*

$$\mathbb{E}(B^*B) = m\mathbf{1}_n, \qquad \textit{and} \qquad \mathbb{E}(BB^*) = n\mathbf{1}_m. \tag{1.12}$$

Proof. Let $(b_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ be the entries of B. Then

$$\mathbb{E}(\overline{b_{ij}}b_{st}) = \begin{cases} 1, & \text{if } (i,j)=(s,t),\\ 0, & \text{otherwise.}\end{cases} \tag{1.13}$$

Since $(B^*B)_{ij} = \sum_{s=1}^m \overline{b_{si}}b_{sj}$ and $(BB^*)_{ij} = \sum_{s=1}^n b_{is}\overline{b_{js}}$, for all i,j, (1.12) follows readily from (1.13). $\square$

1.22 Proposition. *Let p be a positive integer, greater than or equal to 2, and let π be a permutation in $S_p \setminus S_p^{\mathrm{irr}}$. Let $(e,e+1)$ be a pair of neighbors for $\hat{\pi}$ and let π_0 be the permutation in S_{p-1}, for which $\hat{\pi}_0$ is the permutation obtained from $\hat{\pi}$ by cancellation of $(e,e+1)$. Then with $k(\cdot),l(\cdot),d(\cdot)$ and $\sigma(\cdot)$ as introduced in Definition 1.10, we have that*
(i) *If e is odd, then $k(\hat{\pi}_0) = k(\hat{\pi}) - 1$ and $l(\hat{\pi}_0) = l(\hat{\pi})$.*
(ii) *If e is even, then $k(\hat{\pi}_0) = k(\hat{\pi})$ and $l(\hat{\pi}_0) = l(\hat{\pi}) - 1$.*
In both cases, $d(\hat{\pi}_0) = d(\hat{\pi}) - 1$ and $\sigma(\hat{\pi}_0) = \sigma(\hat{\pi})$.

Proof. Let m,n be positive integers, and let $B_1,\dots,B_p$ be independent random matrices from $\mathrm{GRM}(m,n,1)$. By Theorem 1.11, we have then that

$$\mathbb{E}\circ\mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_p^*B_{\pi(p)}\big] = m^{k(\hat{\pi})}n^{l(\hat{\pi})}. \tag{1.14}$$

(i) Assume that e is odd, i.e., that $(e,e+1) = (2q-1,2q)$ for some q in $\{1,2,\dots,p\}$. Then $\pi(q) = q$, and hence the set of random matrices

$$(B_1^*,B_{\pi(1)},\dots,B_{q-1}^*,B_{\pi(q-1)},B_{q+1}^*,B_{\pi(q+1)},\dots,B_p^*,B_{\pi(p)})$$

is independent from the set $(B_q^*, B_{\pi(q)})$. Therefore,

$$\begin{aligned}
&\mathbb{E}\circ \mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_p^*B_{\pi(p)}\big] \\
&\qquad = \mathbb{E}\circ \mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_{q-1}^*B_{\pi(q-1)}\mathbb{E}(B_q^*B_{\pi(q)})B_{q+1}^*\cdots B_p^*B_{\pi(p)}\big] \\
&\qquad = m\cdot \mathbb{E}\circ \mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_{q-1}^*B_{\pi(q-1)}B_{q+1}^*\cdots B_p^*B_{\pi(p)}\big],
\end{aligned} \tag{1.15}$$

where the last equality follows from Lemma 1.21. Note that only the random matrices $B_1,\dots,B_{q-1},B_{q+1},\dots,B_p$ occur in the last expression in (1.15). Define now for i in $\{1,2,\dots,p-1\}$,

$$B_i' = \begin{cases} B_i, & \text{if } 1\le i\le q-1, \\ B_{i+1}, & \text{if } q\le i\le p-1. \end{cases}$$

Then by Remark 1.19(a), it follows that the last expression in (1.15) is equal to

$$m\cdot \mathbb{E}\circ \mathrm{Tr}_n\big[(B_1')^*B_{\pi_0(1)}'\cdots (B_{p-1}')^*B_{\pi_0(p-1)}'\big],$$

which, by Theorem 1.11, is equal to $m\cdot m^{k(\hat\pi_0)}n^{l(\hat\pi_0)}$. Altogether, we have shown that

$$m^{k(\hat\pi)}n^{l(\hat\pi)} = m\cdot m^{k(\hat\pi_0)}n^{l(\hat\pi_0)},$$

and since this holds for all positive integers m,n, it follows that $k(\hat\pi)=k(\hat\pi_0)+1$ and $l(\hat\pi)=l(\hat\pi_0)$. This proves (i).
(ii) Assume that e is even, i.e., that $(e,e+1)=(2q,2q+1)$, for some q in $\{1,2,\dots,p-1\}$. Then $\pi(q)=q+1$, and arguing now as in the proof of (i), we find that

$$\begin{aligned}
m^{k(\hat\pi)}n^{l(\hat\pi)} &= \mathbb{E}\circ \mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_p^*B_{\pi(p)}\big] \\
&= \mathbb{E}\circ \mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_q^*\mathbb{E}(B_{\pi(q)}B_{q+1}^*)B_{\pi(q+1)}\cdots B_p^*B_{\pi(p)}\big] \\
&= n\cdot \mathbb{E}\circ \mathrm{Tr}_n\big[B_1^*B_{\pi(1)}\cdots B_q^*B_{\pi(q+1)}\cdots B_p^*B_{\pi(p)}\big],
\end{aligned} \tag{1.16}$$

where the last equality follows from Lemma 1.21. Defining, this time, for each i in $\{1,2,\dots,p-1\}$,

$$B_i' = \begin{cases} B_i, & \text{if } 1\le i\le q, \\ B_{i+1}, & \text{if } q+1\le i\le p-1, \end{cases}$$

we get by application of Remark 1.19(b), that the last expression in (1.16) is equal to

$$n\cdot \mathbb{E}\circ \mathrm{Tr}_n\big[(B_1')^*B_{\pi_0(1)}'\cdots (B_{p-1}')^*B_{\pi_0(p-1)}'\big],$$

which, by Theorem 1.11, equals $n\cdot m^{k(\hat\pi_0)}n^{l(\hat\pi_0)}$. Arguing then as in the proof of (i), it follows that $k(\hat\pi)=k(\hat\pi_0)$ and $l(\hat\pi)=l(\hat\pi_0)+1$. This proves (ii).
The last statements of Proposition 1.22 follow immediately from (i), (ii) and Definition 1.10. $\square$

1.23 Proposition. *Let p be a positive integer, and let π be a permutation in S_p. By finitely many (or possibly none) successive cancellations of pairs, $\hat{\pi}$ can be reduced to either*
(i) $\hat{e}_1$, *where e_1 is the trivial permutation in S_1,*
or
(ii) $\hat{\rho}$, *where ρ is a permutation in S_q^{irr} for some q in $\{2,\dots,p\}$.*
Case (i) appears if and only if $\pi \in S_p^{\mathrm{nc}}$.

Proof. It is clear, that by finitely many (or possibly none) successive cancellations of pairs, $\hat{\pi}$ can be reduced to a permutation $\hat{\rho}$, where either $\rho \in S_1$ or $\rho \in S_q^{\mathrm{irr}}$ for some q in $\{2,3,\dots,p\}$. By Lemma 1.20, $\hat{\pi}$ is non-crossing if and only if $\hat{\rho}$ is. Since $S_1 = S_1^{\mathrm{nc}} = \{e_1\}$, and $S_q^{\mathrm{irr}} \cap S_q^{\mathrm{nc}} = \emptyset$ for all q in $\{2,3,\dots,p\}$, by Lemma 1.17, it follows thus, that either case (i) or case (ii) occurs, and that case (i) occurs if and only if $\hat{\pi}$ is non-crossing. $\square$

The following corollary is a special case of [Sh, Lemma 2.3]. For the convenience of the reader, we include a proof based on Propositions 1.22 and 1.23.

1.24 Corollary. *Let p be a positive integer and let π be a permutation in S_p. Then $\hat{\pi}$ is non-crossing if and only if $k(\hat{\pi}) + l(\hat{\pi}) = p+1$, or, equivalently, if and only if $\sigma(\hat{\pi}) = 0$.*

Proof. Assume first that $\hat{\pi}$ is non-crossing. It follows then from Proposition 1.23, that by successive cancellations of pairs, $\hat{\pi}$ may be reduced to $\hat{e}_1$, where e_1 is the unique permutation in S_1. Since $\sigma(\cdot)$ is invariant under cancellations of pairs, (cf. Lemma 1.22), it follows that $\sigma(\hat{\pi}) = \sigma(\hat{e}_1)$, and it is straightforward to check that $\sigma(\hat{e}_1) = 0$.
Assume next that $\hat{\pi}$ has a crossing. Then, by Proposition 1.23, there exist q in $\{2,\dots,p\}$ and a permutation ρ in S_q^{irr}, such that $\hat{\pi}$ may be reduced to $\hat{\rho}$ by finitely many (or possibly none) successive cancellations of pairs. By Proposition 1.22, $\sigma(\hat{\pi}) = \sigma(\hat{\rho})$, and hence it suffices to show that $\sigma(\hat{\rho}) > 0$, i.e., that $d(\hat{\rho}) < q+1$. Note for this, that since $\hat{\rho}$ is irreducible, $\hat{\rho}(j) \neq j+1$, for all j in $\{1,2,\dots,2q-1\}$. Since $\hat{\rho}^2 = \mathrm{id}$, this is equivalent to the condition that $\hat{\rho}(j) \neq j-1$, for all j in $\{2,3,\dots,2q\}$, and by Remark 1.9, this implies that $\mathrm{card}([j]_{\hat{\rho}}) \geq 2$, for all j in $\{2,3,\dots,2q\}$. Letting r denote the number of $\sim_{\hat{\rho}}$-equivalence classes, that are distinct from $[1]_{\hat{\rho}}$, we have thus the inequality

$$2r + \mathrm{card}([1]_{\hat{\rho}}) \leq 2q.$$

Since $r = d(\hat{\rho})-1$, and since $\mathrm{card}([1]_{\hat{\rho}}) \geq 1$, this implies that $2(d(\hat{\rho})-1)+1 \leq 2q$, and hence that $d(\hat{\rho}) \leq q$, as desired. $\square$

2 A Combinatorial Expression for the Moments of S^*S

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let r be a positive integer, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$. Moreover, let n be a fixed positive integer, and let

$Y_1, \dots, Y_r$ be independent elements of $\mathrm{GRM}(n, n, \frac{1}{n})$. We then define

$$S = \sum_{i=1}^{r} a_i \otimes Y_i.$$

Note that S is a random variable taking values in $\mathcal{B}(\mathcal{H}, \mathcal{K}) \otimes M_n(\mathbb{C})$. The aim of this section is to derive combinatorial expressions for the moments

$$(\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes (\mathbb{E} \circ \mathrm{tr}_n))[(S^*S)^p] \quad \text{and} \quad (\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes \mathbb{E})[(S^*S)^p], \quad (p \in \mathbb{N}),$$

where $\mathrm{id}_{\mathcal{B}(\mathcal{H})}$ denotes the identity mapping on $\mathcal{B}(\mathcal{H})$. Moreover, we shall obtain another combinatorial expression, which is an upper estimate for the norm of $(\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes \mathbb{E})[(S^*S)^p]$. For the sake of short notation, in the following we shall just write $\mathbb{E} \circ \mathrm{tr}_n$ and $\mathbb{E}$ instead of $\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes (\mathbb{E} \circ \mathrm{tr}_n)$ and $\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes \mathbb{E}$.
We start with the following generalization of Proposition 1.5.

2.1 PROPOSITION. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, let r be a positive integer, and let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$. Moreover, let m, n be fixed positive integers, and let $B_1, \dots, B_r$ be independent elements of* $\mathrm{GRM}(m, n, 1)$. *Then with $T = \sum_{i=1}^{r} a_i \otimes B_i$, we have for any positive integer p, that*

$$\mathbb{E} \circ \mathrm{Tr}_n[(T^*T)^p] = \sum_{\pi \in S_p} m^{k(\hat{\pi})} n^{l(\hat{\pi})} \cdot \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}.$$

Proof. Let $(B(1,h))_{h \in \mathbb{N}}, \dots, (B(r,h))_{h \in \mathbb{N}}$ be sequences of elements from $\mathrm{GRM}(m, n, 1)$, such that (the entries of) the random matrices $B(i,h)$, $1 \le i \le r$, $h \in \mathbb{N}$, are jointly independent. Then for h in $\mathbb{N}$, we define

$$T_h = \sum_{i=1}^{r} a_i \otimes B(i,h).$$

Note then, that for each s in $\mathbb{N}$,

$$s^{-\frac{1}{2}} \sum_{h=1}^{s} T_h = s^{-\frac{1}{2}} \sum_{h=1}^{s} \sum_{i=1}^{r} a_i \otimes B(i,h) = \sum_{i=1}^{r} a_i \otimes \Big(s^{-\frac{1}{2}} \sum_{h=1}^{s} B(i,h) \Big),$$

where the random matrices $s^{-\frac{1}{2}} \sum_{h=1}^{s} B(1,h), \dots, s^{-\frac{1}{2}} \sum_{h=1}^{s} B(r,h)$ are independent elements of $\mathrm{GRM}(m, n, 1)$. It follows thus, that the moments of $s^{-1}(\sum_{h=1}^{s} T_h)^* \sum_{h=1}^{s} T_h$ w.r.t. $\mathbb{E} \circ \mathrm{Tr}_n$ are equal to those of T^*T. Thus for any p, s in $\mathbb{N}$, we have

$$\begin{aligned} \mathbb{E} \circ \mathrm{Tr}_n[(T^*T)^p] &= \mathbb{E} \circ \mathrm{Tr}_n \left[s^{-p} \Big(\Big(\sum_{h=1}^{s} T_h \Big)^* \sum_{h=1}^{s} T_h \Big)^p \right] \\ &= s^{-p} \cdot \sum_{\substack{1 \le h_1, h_2, \dots, h_p \le s \\ 1 \le g_1, g_2, \dots, g_p \le s}} \mathbb{E} \circ \mathrm{Tr}_n \left[T_{h_1}^* T_{g_1} T_{h_2}^* T_{g_2} \cdots T_{h_p}^* T_{g_p} \right]. \end{aligned} \tag{2.1}$$

Consider here an arbitrary $2p$–tuple $(h_1, g_1, \dots, h_p, g_p)$ in $\{1, 2, \dots, s\}^{2p}$. Recalling then the definition of T_h, we find that

$$\begin{aligned}&\mathbb{E} \circ \mathrm{Tr}_n \Big[T_{h_1}^* T_{g_1} T_{h_2}^* T_{g_2} \cdots T_{h_p}^* T_{g_p}\Big] \\ &= \sum_{\substack{1 \le i_1, \dots, i_p \le r \\ 1 \le j_1, \dots, j_p \le r}} (a_{i_1}^* a_{j_1} \cdots a_{i_p}^* a_{j_p}) \cdot \mathbb{E}\big[\mathrm{Tr}_n(B(i_1, h_1)^* B(j_1, g_1) \cdots B(i_p, h_p)^* B(j_p, g_p))\big].\end{aligned}$$

Since $B(i, h)$ is independent of $B(j, g)$ unless $i = j$ and $h = g$, it follows here from Lemma 1.2 in Section 1, that

$$\begin{aligned}&\mathbb{E} \circ \mathrm{Tr}_n \left[B(i_1, h_1)^* B(j_1, g_1) \cdots B(i_p, h_p)^* B(j_p, g_p)\right] \neq 0 \\ &\Longrightarrow \exists \pi \in S_p : \ (j_1, g_1) = (i_{\pi(1)}, h_{\pi(1)}), \dots, (j_p, g_p) = (i_{\pi(p)}, h_{\pi(p)}).\end{aligned} \tag{2.2}$$

In particular it follows that in (2.1), we only have to sum over $(h_1, g_1, \dots, h_p, g_p)$ in $\cup_{\pi \in S_p} M(\pi, s)$, where, as in the proof of Proposition 1.5 in Section 1 ,

$$M(\pi, s) = \big\{(h_1, g_1, \dots, h_p, g_p) \in \{1, 2, \dots, s\}^{2p} \bigm| g_1 = h_{\pi(1)}, \dots, g_p = h_{\pi(p)}\big\},$$

for any π in S_p. Following still the proof of Proposition 1.5 in Section 1, we define,

$$\mathcal{D}(s) = \big\{(h_1, g_1, \dots, h_p, g_p) \in \{1, 2, \dots, s\}^{2p} \bigm| h_1, \dots, h_p \text{ are distinct } \big\},$$

and then the sets $M(\pi, s) \cap \mathcal{D}(s)$, $\pi \in S_p$, are disjoint and

$$\begin{aligned}&\mathbb{E} \circ \mathrm{Tr}_n[(T^*T)^p] \\ &\qquad = s^{-p} \sum_{\pi \in S_p} \sum_{(h_1, g_1, \dots, h_p, g_p) \in M(\pi, s) \cap \mathcal{D}(s)} \mathbb{E} \circ \mathrm{Tr}_n[T_{h_1}^* T_{g_1} \cdots T_{h_p}^* T_{g_p}] \\ &\qquad\quad + s^{-p} \sum_{(h_1, g_1, \dots, h_p, g_p) \in \left(\cup_{\pi \in S_p} M(\pi, s)\right) \setminus \mathcal{D}(s)} \mathbb{E} \circ \mathrm{Tr}_n[T_{h_1}^* T_{g_1} \cdots T_{h_p}^* T_{g_p}].\end{aligned} \tag{2.3}$$

As was noted in the proof of Proposition 1.5, we have here that

$$s^{-p} \cdot \mathrm{card}(M(\pi, s) \cap \mathcal{D}(s)) \to 1, \text{ as } s \to \infty, \quad (\pi \in S_p), \tag{2.4}$$

and that

$$s^{-p} \cdot \mathrm{card}(\big(\cup_{\pi \in S_p} M(\pi, s)\big) \setminus \mathcal{D}(s)) \to 0, \text{ as } s \to \infty. \tag{2.5}$$

Moreover, for any $h_1, g_1, \dots, h_p, g_p$ in $\mathbb{N}$, we have that

$$\begin{aligned}
&\big\|\mathbb{E}\circ \mathrm{Tr}_n[T_{h_1}^* T_{g_1}\cdots T_{h_p}^* T_{g_p}]\big\| \\
&\le \sum_{\substack{1\le i_1,\dots,i_p\le r\\ 1\le j_1,\dots,j_p\le r}} \|a_{i_1}^* a_{j_1}\cdots a_{i_p}^* a_{j_p}\| \cdot \big|\mathbb{E}\big[\mathrm{Tr}_n(B(i_1,h_1)^* B(j_1,g_1)\cdots B(i_p,h_p)^* B(j_p,g_p))\big]\big| \\
&\le K(m,n,p,1)\cdot \sum_{\substack{1\le i_1,\dots,i_p\le r\\ 1\le j_1,\dots,j_p\le r}} \|a_{i_1}^* a_{j_1}\cdots a_{i_p}^* a_{j_p}\|,
\end{aligned}$$

where $K(m,n,p,1)$ is the constant introduced in Remark 1.4 in Section 1. Since this constant does not depend on s, it follows thus, by (2.5), that the second term on the right hand side of (2.3) tends to 0 as $s\to\infty$.
Regarding the first term on the right hand side of (2.3), for any π in S_p and any $2p$–tuple $(h_1, g_1, \dots, h_p, g_p)$ in $M(\pi, s)\cap \mathcal{D}(s)$, we have that

$$\begin{aligned}
\mathbb{E}\circ \mathrm{Tr}_n[T_{h_1}^* T_{g_1}\cdots T_{h_p}^* T_{g_p}] &= \mathbb{E}\circ \mathrm{Tr}_n[T_{h_1}^* T_{h_{\pi(1)}}\cdots T_{h_p}^* T_{h_{\pi(p)}}] \\
&= \sum_{\substack{1\le i_1,\dots,i_p\le r\\ 1\le j_1,\dots,j_p\le r}} (a_{i_1}^* a_{j_1}\cdots a_{i_p}^* a_{j_p}) \\
&\qquad\cdot \mathbb{E}\big[\mathrm{Tr}_n(B(i_1,h_1)^* B(j_1,h_{\pi(1)})\cdots B(i_p,h_p)^* B(j_p,h_{\pi(p)}))\big].
\end{aligned}$$

Recalling here the statement (2.2) and that $h_1,\dots,h_p$ are distinct, it follows that the term in the above sum corresponding to $(i_1, j_1, \dots, i_p, j_p)$ is 0, unless $j_1 = i_{\pi(1)}, \dots, j_p = i_{\pi(p)}$. Thus we have that

$$\begin{aligned}
&\mathbb{E}\circ \mathrm{Tr}_n[T_{h_1}^* T_{g_1}\cdots T_{h_p}^* T_{g_p}] \\
&\quad= \sum_{1\le i_1,\dots,i_p\le r} (a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}}) \\
&\qquad\qquad\cdot \mathbb{E}\big[\mathrm{Tr}_n(B(i_1,h_1)^* B(i_{\pi(1)},h_{\pi(1)})\cdots B(i_p,h_p)^* B(i_{\pi(p)},h_{\pi(p)}))\big].
\end{aligned}$$

Note here, that since $h_1,\dots,h_p$ are distinct, $B(i_1,h_1),\dots,B(i_p,h_p)$ are independent for any choice of $i_1,\dots,i_p$ in $\{1,2,\dots,r\}$, and consequently

$$\mathbb{E}\big[\mathrm{Tr}_n(B(i_1,h_1)^* B(i_{\pi(1)},h_{\pi(1)})\cdots B(i_p,h_p)^* B(i_{\pi(p)},h_{\pi(p)}))\big] = \Lambda(\pi,m,n),$$

for any $i_1,\dots,i_p$ in $\{1,2,\dots,r\}$. Thus, we may conclude that

$$\mathbb{E}\circ \mathrm{Tr}_n[T_{h_1}^* T_{g_1}\cdots T_{h_p}^* T_{g_p}] = \Lambda(\pi,m,n)\cdot \sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}},$$

and this holds for any $(h_1, g_1, \dots, h_p, g_p)$ in $M(\pi,s)\cap\mathcal{D}(s)$. Therefore the first term on the right hand side of (2.3) equals

$$\sum_{\pi\in S_p} s^{-p}\cdot \mathrm{card}(M(\pi,s)\cap\mathcal{D}(s))\cdot \Lambda(\pi,m,n)\cdot \sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}},$$

and by (2.4), this tends to

$$\sum_{\pi \in S_p} \Lambda(\pi, m, n) \cdot \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}},$$

as $s \to \infty$. Since the left hand side of (2.3) does not depend on s, we get thus by letting $s \to \infty$ in (2.3), that

$$\mathbb{E} \circ \mathrm{Tr}_n[(T^*T)^p] = \sum_{\pi \in S_p} \Lambda(\pi, m, n) \cdot \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}.$$

Combining finally with Theorem 1.11, we obtain the desired formula. □

2.2 Corollary. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, let r be a positive integer, and let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$. Moreover, let n be a fixed positive integer, and let $Y_1, \dots, Y_r$ be independent elements of* $\mathrm{GRM}(n, n, \frac{1}{n})$. *Then with $S = \sum_{i=1}^r a_i \otimes Y_i$, we have for any positive integer p, that*

$$\mathbb{E} \circ \mathrm{tr}_n[(S^*S)^p] = \sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})} \cdot \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}, \tag{2.6}$$

where $\sigma(\hat{\pi})$ is the quantity introduced in Definition 1.10 in Section 1.

Proof. With $B_i = \sqrt{n} \cdot Y_i$, $i \in \{1, 2, \dots, r\}$, we have that $B_1, \dots, B_r$ are independent elements of $\mathrm{GRM}(n, n, 1)$. It follows thus from Proposition 2.1, that for any p in $\mathbb{N}$,

$$n^p \cdot \mathbb{E} \circ \mathrm{Tr}_n[(S^*S)^p] = \sum_{\pi \in S_p} n^{k(\hat{\pi}) + l(\hat{\pi})} \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}},$$

and consequently

$$\mathbb{E} \circ \mathrm{tr}_n[(S^*S)^p] = \sum_{\pi \in S_p} n^{-p-1+k(\hat{\pi})+l(\hat{\pi})} \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}.$$

Formula (2.6) now follows by noting that,

$$p + 1 - k(\hat{\pi}) - l(\hat{\pi}) = p + 1 - d(\hat{\pi}) = 2\sigma(\hat{\pi}),$$

for any π in S_p. □

Our next objective is to derive a matrix version of formula (2.6). In other words, we shall obtain a combinatorial expression for $\mathbb{E}[(S^*S)^p]$.

2.3 Lemma. *Let n, r be positive integers and let $Y_1, \dots, Y_r$ be independent elements of* $\mathrm{GRM}(n, n, \sigma^2)$. *Then for any (non–random) unitary $n \times n$ matrices $u_1, \dots, u_r$, the random matrices $u_1 Y_1 u_1^*, \dots, u_r Y_r u_r^*$ are again independent elements of* $\mathrm{GRM}(n, n, \sigma^2)$.

Proof. Note first that for each i in $\{1, 2, \dots, r\}$, the entries of $u_i Y_i u_i^*$ are all measurable w.r.t. the σ–algebra generated by the entries of Y_i. It follows therefore immediately that $u_1 Y_1 u_1^*, \dots, u_r Y_r u_r^*$ are independent random matrices. We note next, that it follows easily from Definition 1.1, that the joint distribution of the entries of an element from $\mathrm{GRM}(n, n, \sigma^2)$ has the following density w.r.t. Lebesgue measure on $M_n(\mathbb{C}) \simeq \mathbb{R}^{2n^2}$:

$$y \mapsto \big(\tfrac{1}{\pi\sigma^2}\big)^{n^2} \exp\big(-\tfrac{1}{\sigma^2} \cdot \mathrm{Tr}_n(y^* y)\big), \quad (y \in M_n(\mathbb{C})). \tag{2.7}$$

(Here the identification of $M_n(\mathbb{C})$ with $\mathbb{R}^{2n^2}$ is given by

$$y \mapsto (\mathrm{Re}(y_{jk}), \mathrm{Im}(y_{jk}))_{1 \le j,k \le n}.)$$

Now let u be a unitary $n \times n$ matrix, and consider then the linear mapping

$$\mathrm{Ad}u \colon y \mapsto uyu^* \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}).$$

Under the identification of $M_n(\mathbb{C})$ with $\mathbb{R}^{2n^2}$, the Euclidean structure on $\mathbb{R}^{2n^2}$ is given by the inner product:

$$\langle y, z \rangle = \mathrm{Re}(\mathrm{Tr}_n(z^* y)), \quad (y, z \in M_n(\mathbb{C})).$$

Thus $\mathrm{Ad}u \colon \mathbb{R}^{2n^2} \to \mathbb{R}^{2n^2}$ is a linear isometry, and hence the Jacobi determinant of this mapping equals 1. Combining this fact with (2.7) and the usual transformation theorem for Lebesgue measure, it follows that for any Y in $\mathrm{GRM}(n, n, \sigma^2)$, the joint distribution of the entries of uYu^* equals that of the entries of Y. $\square$

2.4 Lemma. *Let $\mathcal{B}$ be a C^*–algebra with unit $\mathbf{1}$, let n be a positive integer, and consider the tensor product $\mathcal{B} \otimes M_n(\mathbb{C})$. If $x \in \mathcal{B} \otimes M_n(\mathbb{C})$, such that $(\mathbf{1} \otimes u)x(\mathbf{1} \otimes u)^* = x$ for any unitary $n \times n$ matrix u, then $x \in \mathcal{B} \otimes \mathbf{1}_n$.*

Proof. Assume that $x \in \mathcal{B} \otimes M_n(\mathbb{C})$, and that $(\mathbf{1} \otimes u)x(\mathbf{1} \otimes u)^* = x$ for any unitary $n \times n$ matrix u. Since $M_n(\mathbb{C})$ is the linear span of its unitaries, it follows that

$$x \in \big\{y \in \mathcal{B} \otimes M_n(\mathbb{C}) \bigm| yT = Ty \quad \text{for all } T \text{ in } \mathbf{1} \otimes M_n(\mathbb{C})\big\} = \mathcal{B} \otimes \mathbf{1}_n,$$

where the last equality follows by standard matrix considerations; thinking of $\mathcal{B} \otimes M_n(\mathbb{C})$ as the set of $n \times n$ matrices with entries from $\mathcal{B}$. $\square$

2.5 Proposition. *Let S be as in Corollary 2.2. Then for any positive integer p, we have that:*

$$\mathbb{E}[(S^* S)^p] = \Big(\sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})} \cdot \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big) \otimes \mathbf{1}_n.$$

Proof. Let u be an arbitrary unitary $n\times n$ matrix, and define: $S_u=\sum_{i=1}^r a_i\otimes(uY_iu^*)$. Note then that $S_u^*S_u=(\mathbf{1}_{\mathcal{H}}\otimes u)S^*S(\mathbf{1}_{\mathcal{H}}\otimes u)^*$, where $\mathbf{1}_{\mathcal{H}}$ denotes the unit of $\mathcal{B}(\mathcal{H})$. It follows now by Lemma 2.3, that

$$\begin{aligned}\mathbb{E}[(S^*S)^p]&=\mathbb{E}[(S_u^*S_u)^p]\\&=\mathbb{E}[(\mathbf{1}_{\mathcal{H}}\otimes u)(S^*S)^p(\mathbf{1}_{\mathcal{H}}\otimes u)^*]=(\mathbf{1}_{\mathcal{H}}\otimes u)\mathbb{E}[(S^*S)^p](\mathbf{1}_{\mathcal{H}}\otimes u)^*.\end{aligned}$$

Since this holds for any unitary u, it follows from Lemma 2.4, that $\mathbb{E}[(S^*S)^p]\in\mathcal{B}(\mathcal{H})\otimes\mathbf{1}_n$, and consequently

$$\mathbb{E}[(S^*S)^p]=\Big(\mathrm{tr}_n(\mathbb{E}[(S^*S)^p])\Big)\otimes\mathbf{1}_n=\Big(\mathbb{E}\circ\mathrm{tr}_n[(S^*S)^p]\Big)\otimes\mathbf{1}_n.$$

The proposition now follows by application of Corollary 2.2. □

In the next section, we shall obtain combinatorial expressions that are upper estimates for the moments $\mathbb{E}[(S^*S)^p]$. It follows from Proposition 2.5, that in order to obtain such combinatorial estimates, we should concentrate on deriving combinatorial estimates for the quantities

$$\Big\|\sum_{1\le i_1,\dots,i_p\le r}a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}}\Big\|,$$

where $\pi\in S_p$, and $a_1,\dots,a_r$ are arbitrary bounded operators from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$.

2.6 Definition. Let p be a positive integer, let π be a permutation in S_p and consider the permutation $\hat\pi$ in S_{2p}. We then put

$$\begin{aligned}\kappa(\hat\pi)&=\mathrm{card}(\{j\in\{1,3,\dots,2p-1\}\mid\hat\pi(j)>j\}),\\\lambda(\hat\pi)&=\mathrm{card}(\{j\in\{1,3,\dots,2p-1\}\mid\hat\pi(j)<j\})+1.\end{aligned}$$ □

We note, that since $\hat\pi$ has no fixed points, it follows that

$$\kappa(\hat\pi)+\lambda(\hat\pi)=p+1,\quad(p\in\mathbb{N},\ \pi\in S_p).\tag{2.8}$$

Recalling that by definition of $\hat\pi$, $\hat\pi(2h-1)=2\pi^{-1}(h)$ for all h in $\{1,2,\dots,p\}$, it follows furthermore that

$$\begin{aligned}\kappa(\hat\pi)&=\mathrm{card}(\{h\in\{1,2,\dots,p\}\mid 2\pi^{-1}(h)>2h-1\})\\&=\mathrm{card}(\{h\in\{1,2,\dots,p\}\mid\pi^{-1}(h)\ge h\})\\&=\mathrm{card}(\{h\in\{1,2,\dots,p\}\mid h\ge\pi(h)\}),\end{aligned}\tag{2.9}$$

where the last equality follows by replacing h by $\pi^{-1}(h)$. Similarly we have that

$$\begin{aligned}\lambda(\hat\pi)&=p+1-\kappa(\hat\pi)\\&=\mathrm{card}(\{h\in\{1,2,\dots,p\}\mid\pi^{-1}(h)<h\})+1\\&=\mathrm{card}(\{h\in\{1,2,\dots,p\}\mid h<\pi(h)\})+1.\end{aligned}\tag{2.10}$$

We note also, that since $\hat{\pi}(j) = j+1$ mod. 2 and $\hat{\pi}(\hat{\pi}(j)) = j$ for all j, we have that

$$\begin{aligned} \kappa(\hat{\pi}) &= \operatorname{card}(\hat{\pi}\big[\{j \in \{1,3,\dots,2p-1\} \mid \hat{\pi}(j) > j\}\big]) \\ &= \operatorname{card}(\{j \in \{2,4,\dots,2p\} \mid \hat{\pi}(j) < j\}), \end{aligned} \tag{2.11}$$

and similarly

$$\lambda(\hat{\pi}) = \operatorname{card}(\{j \in \{2,4,\dots,2p\} \mid \hat{\pi}(j) > j\}) + 1. \tag{2.12}$$

In connection with products of the form $a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}$, note that $\kappa(\hat{\pi})$ denotes the number of h's in $\{1,2,\dots,p\}$ for which the factor $a_{i_h}^*$ appears before the factor a_{i_h} in this product. Similarly $\lambda(\hat{\pi})-1$ denotes the number of h's in $\{1,2,\dots,p\}$ for which the factor a_{i_h} appears before the factor $a_{i_h}^*$.

2.7 Proposition. *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, let r be a positive integer, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$. Let further c and d be positive real numbers, such that*

$$\Big\| \sum_{i=1}^r a_i^* a_i \Big\| \le c \qquad \text{and} \qquad \Big\| \sum_{i=1}^r a_i a_i^* \Big\| \le d. \tag{2.13}$$

Then for any positive integer p and any permutation π in S_p, we have that

$$\Big\| \sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}} \Big\| \le c^{\kappa(\hat{\pi})} d^{\lambda(\hat{\pi})-1}.$$

Proof. Let $\mathcal{V}$ be an infinite dimensional Hilbert space, and choose r isometries $s_1,\dots,s_r$ in $\mathcal{B}(\mathcal{V})$, with orthogonal ranges, i.e.,

$$s_i^* s_j = \delta_{i,j} \mathbf{1}_{\mathcal{B}(\mathcal{V})}, \qquad (i,j \in \{1,2,\dots,r\}). \tag{2.14}$$

Consider then the Hilbert space $\tilde{\mathcal{V}} = \mathcal{V}\otimes\cdots\otimes\mathcal{V}$ (p factors), and for each i in $\{1,2,\dots,r\}$ and h in $\{1,2,\dots,p\}$, define the operator $s(i,h)$ in $\mathcal{B}(\tilde{\mathcal{V}})$ by the equation

$$s(i,h) = \mathbf{1}_{\mathcal{B}(\mathcal{V})} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}(\mathcal{V})} \otimes \underset{\substack{\uparrow \\ h\text{'th position}}}{S_i} \otimes \mathbf{1}_{\mathcal{B}(\mathcal{V})} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}(\mathcal{V})}. \tag{2.15}$$

Next, put

$$t(i,h) = \begin{cases} s(i,h), & \text{if } h \le \pi^{-1}(h), \\ s(i,h)^*, & \text{if } h > \pi^{-1}(h), \end{cases} \qquad (i \in \{1,2,\dots,r\},\ h \in \{1,2,\dots,p\}), \tag{2.16}$$

and

$$A_h = \sum_{i=1}^{r} a_i \otimes t(i,h), \qquad (h \in \{1,2,\dots,p\}). \tag{2.17}$$

We consider A_h as an element of $\mathcal{B}(\mathcal{H}\otimes\tilde{\mathcal{V}}, \mathcal{K}\otimes\tilde{\mathcal{V}})$ in the usual way. We claim then that

$$A_1^* A_{\pi(1)} A_2^* A_{\pi(2)} \cdots A_p^* A_{\pi(p)} = \Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}} \Big) \otimes \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{V}})}. \tag{2.18}$$

To prove (2.18), observe first that

$$\begin{aligned} &A_1^* A_{\pi(1)} \cdots A_p^* A_{\pi(p)} \\ &\quad = \sum_{\substack{1\le i_1,i_2,\dots,i_p\le r\\ 1\le j_1,j_2,\dots,j_p\le r}} (a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p}) \otimes \Pi(i_1,j_1,i_2,j_2,\dots,i_p,j_p), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} &\Pi(i_1,j_1,\dots,i_p,j_p) \\ &\quad = t(i_1,1)^* t(j_1,\pi(1)) t(i_2,2)^* t(j_2,\pi(2)) \cdots t(i_p,p)^* t(j_p,\pi(p)), \end{aligned} \tag{2.20}$$

for all $i_1,j_1,\dots,i_p,j_p$ in $\{1,2,\dots,r\}$. By (2.15) and (2.16), $t(i,h)$ and $t(i,h)^*$ both commute with $t(j,k)$ and $t(j,k)^*$, as long as $h \neq k$. Hence, we can reorder the factors in the product on the right hand side of (2.20), according to the second index in $t(\cdot,\cdot)$ and $t(\cdot,\cdot)^*$, in the following way

$$\Pi(i_1,j_1,\dots,i_p,j_p) = T(1)T(2)\cdots T(p),$$

where

$$T(h) = \begin{cases} t(i_h,h)^* t(j_{\pi^{-1}(h)},h), & \text{if } h \le \pi^{-1}(h), \\ t(j_{\pi^{-1}(h)},h) t(i_h,h)^*, & \text{if } h > \pi^{-1}(h), \end{cases}$$

for each h in $\{1,2,\dots,p\}$. By (2.16), it follows that

$$T(h) = \begin{cases} s(i_h,h)^* s(j_{\pi^{-1}(h)},h), & \text{if } h \le \pi^{-1}(h), \\ s(j_{\pi^{-1}(h)},h)^* s(i_h,h), & \text{if } h > \pi^{-1}(h), \end{cases}$$

and thus by (2.14)-(2.15), we get that for all $i_1,j_1,\dots,i_p,j_p$ in $\{1,2,\dots,r\}$ and all h in $\{1,2,\dots,p\}$,

$$T(h) = \begin{cases} \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{V}})}, & \text{if } i_h = j_{\pi^{-1}(h)}, \\ 0, & \text{if } i_h \neq j_{\pi^{-1}(h)}. \end{cases}$$

Therefore, $\Pi(i_1,j_1,\dots,i_p,j_p) = 0$, unless $i_h = j_{\pi^{-1}(h)}$, for all h in $\{1,2,\dots,p\}$, or equivalently, unless $i_{\pi(h)} = j_h$, for all h in $\{1,2,\dots,p\}$, in which case $\Pi(i_1,j_1,\dots,i_p,j_p) = \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{V}})}$. Combining this with (2.19), we obtain (2.18).

Using again that $s(i,h)^*s(j,h) = \delta_{i,j}\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{V}})}$, for all i,j in $\{1,2,\dots,r\}$, we get that if $h \le \pi^{-1}(h)$,

$$A_h^*A_h = \sum_{i,j=1}^{r} a_i^*a_j \otimes s(i,h)^*s(j,h) = \sum_{i=1}^{r} a_i^*a_i \otimes \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{V}})},$$

and if $h > \pi^{-1}(h)$,

$$A_hA_h^* = \sum_{i=1}^{r} a_ia_i^* \otimes \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{V}})}.$$

By (2.13), it follows thus, that

$$\begin{aligned} \|A_h\|^2 &= \|A_h^*A_h\| \le c, \qquad \text{if } h \le \pi^{-1}(h),\\ \|A_h\|^2 &= \|A_hA_h^*\| \le d, \qquad \text{if } h > \pi^{-1}(h), \end{aligned}$$

so by (2.9) and (2.10),

$$\|A_1^*A_{\pi(1)}\cdots A_p^*A_{\pi(p)}\| \le \prod_{h=1}^{p}\|A_h\|^2 \le c^{\kappa(\hat{\pi})}d^{\lambda(\hat{\pi})-1}.$$

Together with (2.18), this proves the proposition. □

2.8 Corollary. *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, let r be a positive integer, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$. Moreover, let n be a fixed positive integer, and let $Y_1,\dots,Y_r$ be independent elements of $\mathrm{GRM}(n,n,\frac{1}{n})$. Then with $S = \sum_{i=1}^r a_i\otimes Y_i$, $c = \|\sum_{i=1}^r a_i^*a_i\|$ and $d = \|\sum_{i=1}^r a_ia_i^*\|$, we have for any positive integer p, that*

$$\|\mathbb{E}[(S^*S)^p]\| \le \sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}c^{\kappa(\hat{\pi})}d^{\lambda(\hat{\pi})-1}.$$

Proof. This follows immediately by combining Propositions 2.5 and 2.7. □

In Section 3 we shall estimate further the quantity $\|\mathbb{E}[(S^*S)^p]\|$. As preparation for this, we will in Proposition 2.10 below, compare the numbers $\kappa(\hat{\pi})$ and $\lambda(\hat{\pi})$ with the numbers $k(\hat{\pi})$ and $l(\hat{\pi})$, defined in Section 1.

2.9 Lemma. *Let p be a positive integer, let π be a permutation in S_p, and consider the permutation $\hat{\pi}$ in S_{2p} and the corresponding equivalence relation $\sim_{\hat{\pi}}$. Then any equivalence class for $\sim_{\hat{\pi}}$, except possibly $[1]_{\hat{\pi}}$, contains an element j with the property that $\hat{\pi}(j) < j$.*

Proof. Let j' be an element of $\{1,2,\dots,2p\}$, such that $1 \notin [j']_{\hat{\pi}}$. We show that $[j']_{\hat{\pi}}$ contains an element j such that $\hat{\pi}(j) < j$. For this, note first, that we may assume that j' is the smallest element of $[j']_{\hat{\pi}}$. Then, by assumption, $j' \ge 2$. Now write in the usual manner (cf. Remark 1.9)

$$[j']_{\hat{\pi}} = \{j_0, j_1, \dots, j_q\}.$$

In particular, $\hat{\pi}(j_q) + 1 = j_0 = j'$ (addition formed mod. $2p$). Now, since $j' \geq 2$, we have that $j' - 1 < j'$, even when the subtraction is formed mod. $2p$. Therefore, since j' is the smallest element of $[j']_{\hat{\pi}}$, $\hat{\pi}(j_q) = j' - 1 < j' \leq j_q$. Thus we may choose $j = j_q$. $\square$

2.10 Proposition. *Let p be a positive integer, let π be a permutation in S_p, and consider the permutation $\hat{\pi}$ in S_{2p}. We then have*
(i) $\kappa(\hat{\pi}) \geq k(\hat{\pi})$ *and* $\lambda(\hat{\pi}) \geq l(\hat{\pi})$.
(ii) $(\kappa(\hat{\pi}) - k(\hat{\pi})) + (\lambda(\hat{\pi}) - l(\hat{\pi})) = 2\sigma(\hat{\pi})$.
(iii) $\kappa(\hat{\pi}) = k(\hat{\pi})$ *and* $\lambda(\hat{\pi}) = l(\hat{\pi})$ *if and only if $\hat{\pi}$ is non–crossing.*

Proof. (i) By Lemma 2.9 and the definition of $l(\hat{\pi})$, it follows that

$$l(\hat{\pi}) - 1 \leq \mathrm{card}\big(\big\{j \in \{1,3,\dots,2p-1\} \,\big|\, \hat{\pi}(j) < j\big\}\big) = \lambda(\hat{\pi}) - 1.$$

Similarly we find by application of (2.11), that

$$k(\hat{\pi}) \leq \mathrm{card}\big(\big\{j \in \{2,4,\dots,2p\} \,\big|\, \hat{\pi}(j) < j\big\}\big) = \kappa(\hat{\pi}).$$

(ii) We find by application of (2.8), that

$$(\kappa(\hat{\pi}) - k(\hat{\pi})) + (\lambda(\hat{\pi}) - l(\hat{\pi})) = (\kappa(\hat{\pi}) + \lambda(\hat{\pi})) - d(\hat{\pi}) = p + 1 - d(\hat{\pi}) = 2\sigma(\hat{\pi}).$$

(iii) This follows immediately by combining (i), (ii) and Corollary 1.24. $\square$

3 An upper bound for $\mathbb{E}\big[\exp(tS^*S)\big]$, $t \geq 0$

In the previous section, we computed $\mathbb{E}[(S^*S)^p]$, for p in $\mathbb{N}$ and $S = \sum_{i=1}^r a_i \otimes Y_i$, where $a_1, \dots, a_r \in \mathcal{B}(\mathcal{H},\mathcal{K})$, for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and where $Y_1, \dots, Y_r$ are independent random matrices in $\mathrm{GRM}(n,n,\frac{1}{n})$. For fixed p in $\mathbb{N}$, the function $\omega \mapsto \big(S^*(\omega)S(\omega)\big)^p$ only takes values in a finite dimensional subspace of $\mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C})$. This is not the case for the function $\omega \mapsto \exp\big(tS^*(\omega)S(\omega)\big)$, so in order to give precise meaning to the mean $\mathbb{E}\big[\exp(tS^*S)\big]$, we will need the following definition (cf. [Ru, Definition 3.26]).

3.1 Definition. Let $\mathcal{X}$ be a Banach space, let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $f\colon \Omega \to \mathcal{X}$ be a mapping, that satisfies the following two conditions
(a) $\forall \varphi \in \mathcal{X}^*\colon \varphi \circ f \in L^1(\Omega, \mathcal{F}, P)$
(b) $\exists x_0 \in \mathcal{X}\ \forall \varphi \in \mathcal{X}^*\colon\ \int_\Omega \varphi \circ f(\omega)\ dP(\omega) = \varphi(x_0)$.
We say then that f is *integrable* in $\mathcal{X}$, and we call x_0 the integral of f, and write

$$\mathbb{E}(f) = \int_\Omega f\ dP = x_0. \qquad \square$$

Note that in the above definition, x_0 is uniquely determined by (b). Note also, that we do not require that $\int_\Omega \|f\|\ dP < \infty$, in order for f to be integrable. However, if $\mathcal{X}$ is finite dimensional, then this follows automatically from (a).

3.2 PROPOSITION. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let $a_1, \dots, a_r$ be elements of $B(\mathcal{H}, \mathcal{K})$, and let γ be a strictly positive number, such that*

$$\max\left\{\|\textstyle\sum_{i=1}^r a_i^* a_i\|, \|\sum_{i=1}^r a_i a_i^*\|\right\} \le \gamma.$$

Furthermore, let n be a positive integer, let $Y_1, \dots, Y_r$ be independent random matrices in $\mathrm{GRM}(n, n, \frac{1}{n})$, and put $S = \sum_{i=1}^r a_i \otimes Y_i$.
Then for any complex number t, such that $|t| < \frac{n}{\gamma}$, the function

$$\omega \mapsto \exp\big(tS^*(\omega)S(\omega)\big), \qquad (\omega \in \Omega),$$

is integrable in $\mathcal{B}(\mathcal{H}^n)$, in the sense of Definition 3.1, and

$$\mathbb{E}\big[\exp(tS^*S)\big] = \sum_{p=0}^{\infty} \tfrac{t^p}{p!}\mathbb{E}\big[(S^*S)^p\big], \tag{3.1}$$

where the series on the right hand side is absolutely convergent in $\mathcal{B}(\mathcal{H}^n)$.

Proof. By Proposition 2.5, we have for any p in $\mathbb{N}$,

$$\mathbb{E}\big[(S^*S)^p\big] = \Big(\sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})} \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big) \otimes \mathbf{1}_n,$$

and by Proposition 2.7 and formula (2.8), we have here for all π in S_p, that

$$\Big\| \sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}} \Big\| \le \gamma^p. \tag{3.2}$$

Hence the absolute convergence of the right hand side of (3.1) will follow, if we can prove that

$$1 + \sum_{p=1}^{\infty} \tfrac{(\gamma|t|)^p}{p!} \Big(\sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})}\Big) < \infty, \tag{3.3}$$

whenever $|t| < \frac{n}{\gamma}$. For this, consider an element B of $\mathrm{GRM}(n, n, 1)$, and recall then from Corollary 1.12, that

$$\mathbb{E} \circ \mathrm{Tr}_n\big[(B^*B)^p\big] = \sum_{\pi \in S_p} n^{k(\hat{\pi}) + l(\hat{\pi})} = n^{p+1} \sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})}.$$

Hence for positive numbers s, we have

$$\mathbb{E} \circ \mathrm{Tr}_n\big[\exp(sB^*B)\big] = n\Big(1 + \sum_{p=1}^{\infty} \tfrac{(ns)^p}{p!} \sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})}\Big). \tag{3.4}$$

From [HT, Theorem 6.4], we know that

$$\mathbb{E} \circ \mathrm{Tr}_n\big[\exp(sB^*B)\big] < \infty, \quad \text{when} \quad 0 \le s < 1.$$

Hence the sum in (3.4) is finite, whenever $0 \le s < 1$, and this implies that (3.3) holds whenever $|t| < \frac{n}{\gamma}$.

Consider now the state space $\mathcal{S}(\mathcal{B}(\mathcal{H}^n))$ of $\mathcal{B}(\mathcal{H}^n)$ and an element φ of $\mathcal{S}(\mathcal{B}(\mathcal{H}^n))$. For any ω in Ω, we have then that

$$\varphi\big[\exp\big(tS^*(\omega)S(\omega)\big)\big] = \sum_{p=0}^{\infty} \tfrac{t^p}{p!}\varphi\big[(S^*(\omega)S(\omega))^p\big],$$

which is clearly a positive measurable function of ω (since φ is a state). Moreover, by Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned}
\mathbb{E}\big[\varphi\big(\exp(tS^*S)\big)\big] &= \sum_{p=0}^{\infty} \tfrac{t^p}{p!}\mathbb{E}\big[\varphi\big((S^*S)^p\big)\big] \\
&= \sum_{p=0}^{\infty} \tfrac{t^p}{p!}\varphi\big(\mathbb{E}\big[(S^*S)^p\big]\big) \\
&= 1 + \sum_{p=0}^{\infty} \tfrac{t^p}{p!}\varphi\Big(\sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}}\Big)\otimes \mathbf{1}_n\Big) \\
&\le 1 + \sum_{p=0}^{\infty} \tfrac{t^p}{p!}\sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}\Big\|\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}}\Big\|,
\end{aligned} \tag{3.5}$$

and by (3.2) and (3.3), the latter sum is finite, when $|t| < \frac{n}{\gamma}$. Since $\mathcal{B}(\mathcal{H}^n)^* = \operatorname{span}\big(\mathcal{S}(\mathcal{B}(\mathcal{H}^n))\big)$, it follows that the function $\omega \mapsto \exp(tS^*(\omega)S(\omega))$, is integrable, and (by the first two equalities in (3.5)) that $\mathbb{E}\big[\exp(tS^*S)\big]$ is given by (3.1). $\square$

The main result of this section is the following

3.3 Theorem. *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, satisfying that*

$$\sum_{i=1}^{r} a_i^* a_i \le c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \quad \text{and} \quad \sum_{i=1}^{r} a_i a_i^* \le \mathbf{1}_{\mathcal{B}(\mathcal{K})},$$

for some constant c in $]0,\infty[$. Consider furthermore independent elements $Y_1,\dots,Y_r$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and put $S = \sum_{i=1}^{r} a_i \otimes Y_i$. Then for any t in $[0,\frac{n}{2c}]\cap[0,\frac{n}{2}]$, we have that

$$\mathbb{E}\big[\exp(tS^*S)\big] \le \exp\big((\sqrt{c}+1)^2 t + (c+1)^2\cdot \tfrac{t^2}{n}\big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.$$

For the proof of Theorem 3.3, we need three lemmas. Before stating these lemmas, we introduce some notation:

For any p, k, l in $\mathbb{N}$, we put

$$\delta(p,k,l) = \operatorname{card}(\{\pi \in S_p \mid k(\hat{\pi}) = k \text{ and } l(\hat{\pi}) = l\}). \tag{3.6}$$

Note that for any p, k, l in $\mathbb{N}$, $\delta(p,k,l) = 0$, unless $k + l \le p + 1$ (cf. Theorem 1.13).
For any complex number w and any n in $\mathbb{N}_0$, we put

$$(w)_n = \begin{cases} 1, & \text{if } n = 0, \\ w(w+1)(w+2)\cdots(w+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

We recall then, that the hyper-geometric function F, is defined by the formula

$$F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

for a, b, c, x in $\mathbb{C}$, such that $c \notin \mathbb{Z} \setminus \mathbb{N}$, and $|x| < 1$.

3.4 Lemma. *For all positive real numbers α, β, we have that*

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{t^{p-1}}{(p-1)!} & \sum_{\substack{k,l \in \mathbb{N} \\ k+l \le p+1}} \delta(p,k,l) \alpha^{k-1} \beta^{l-1} \\ & = \frac{F(1-\alpha, 1-\beta, 2, t^2)}{(1-t)^{\alpha+\beta}}, \qquad (t \in \mathbb{C},\ |t| < 1). \end{aligned} \tag{3.7}$$

Proof. Assume first that $\alpha = n$ and $\beta = m$, where $m, n \in \mathbb{N}$, and consider an element B of $\mathrm{GRM}(m,n,1)$. Then by [HT, Theorem 6.4],

$$\begin{aligned} \frac{F(1-n, 1-m, 2, t^2)}{(1-t)^{m+n}} &= \frac{1}{mn} \mathbb{E} \circ \mathrm{Tr}_n \big[B^* B \exp(t B^* B) \big] \\ &= \frac{1}{mn} \sum_{p=1}^{\infty} \frac{t^{p-1}}{(p-1)!} \mathbb{E} \circ \mathrm{Tr}_n \big[(B^* B)^p \big]. \end{aligned}$$

But from Section 1 of this paper, we know that for any p in $\mathbb{N}$

$$\mathbb{E} \circ \mathrm{Tr}_n \big[(B^* B)^p \big] = \sum_{\pi \in S_p} m^{k(\hat{\pi})} n^{l(\hat{\pi})} = \sum_{\substack{k,l \in \mathbb{N} \\ k+l \le p+1}} \delta(p,k,l) m^k n^l,$$

and thus (3.7) holds for all α, β in $\mathbb{N}$. In particular, the left hand side (3.7) is finite for all α, β in $\mathbb{N}$. Since the left hand side of (3.7) is an increasing function of both α and β, it is therefore finite for all α, β in $]0, \infty[$.
To prove (3.7) for general positive real numbers, α, β, we get first, as in [HT, Proof of Proposition 8.1], by multiplying the power series

$$F(1-\alpha, 1-\beta, 2; t^2) = \sum_{j=0}^{\infty} \frac{1}{j+1} \binom{\alpha-1}{j} \binom{\beta-1}{j} t^{2j}, \qquad (|t| < 1),$$

and

$$(1-t)^{-(\alpha+\beta)} = \sum_{k=0}^{\infty} \binom{\alpha+\beta+k-1}{k} t^k, \qquad (|t|<1),$$

that the power series expansion for $\frac{F(1-\alpha,1-\beta,2;t^2)}{(1-t)^{\alpha+\beta}}$ is given by

$$\frac{F(1-\alpha,1-\beta,2;t^2)}{(1-t)^{\alpha+\beta}} = \sum_{p=1}^{\infty} \psi(p,\alpha,\beta)t^{p-1}, \qquad (|t|<1), \tag{3.8}$$

where for all p in $\mathbb{N}$,

$$\psi(p,\alpha,\beta) = \sum_{j=0}^{[\frac{p-1}{2}]} \frac{1}{j+1}\binom{\alpha-1}{j}\binom{\beta-1}{j}\binom{\alpha+\beta+p-2j-2}{p-2j-1}. \tag{3.9}$$

Since we know that (3.7) holds for all α,β in $\mathbb{N}$, we have, on the other hand, that

$$\psi(p,\alpha,\beta) = \frac{1}{(p-1)!} \sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}} \delta(p,k,l)\alpha^{k-1}\beta^{l-1}, \tag{3.10}$$

for all α,β in $\mathbb{N}$. Thus, for fixed p, the right hand sides of (3.9) and (3.10) coincide whenever $\alpha,\beta \in \mathbb{N}$, and since these two right hand sides are both polynomials in α and β, they must therefore coincide for all α,β in $]0,\infty[$. In other words, (3.10) holds for all α,β in $]0,\infty[$, and inserting this in (3.8), we get the desired formula. $\square$

3.5 Lemma. *Let α,β be positive numbers, and assume that either α or β is an integer. Then*

$$F(1-\alpha,1-\beta,2;t^2) \le \sum_{j=0}^{\infty} \frac{(\alpha\beta)^j t^{2j}}{j!(j+1)!}, \qquad \text{whenever } 0\le t<1. \tag{3.11}$$

Proof. We recall first, that

$$F(1-\alpha,1-\beta,2;t^2) = \sum_{j=0}^{\infty} \frac{1}{j+1}\binom{\alpha-1}{j}\binom{\beta-1}{j} t^{2j}, \qquad (t\in[0,1[).$$

If both α and β are integers, then

$$0 \le \binom{\alpha-1}{j} \le \frac{\alpha^j}{j!} \qquad \text{and} \qquad 0 \le \binom{\beta-1}{j} \le \frac{\beta^j}{j!},$$

for all j in $\mathbb{N}_0$, and (3.11) follows immediately. By symmetry of (3.11) in α and β, it is therefore sufficient to treat the case where α is an integer and β is not. In this case, we have

$$F(1-\alpha,1-\beta,2;t^2) = \sum_{j=0}^{\alpha-1} \frac{1}{j+1}\binom{\alpha-1}{j}\binom{\beta-1}{j} t^{2j}.$$

If $\beta \geq \alpha$, we have for any j in $\{0, 1, \dots, \alpha - 1\}$, that $0 < \binom{\alpha-1}{j} \leq \frac{\alpha^j}{j!}$ and $0 < \binom{\beta-1}{j} \leq \frac{\beta^j}{j!}$, and again (3.11) follows immediately.
Assume then that $\beta < \alpha$, and let n be the integer for which $n - 1 < \beta < n$. Since α is an integer, and $\alpha > \beta$, we have that $\alpha \geq n$. Forming now Taylor expansion on the function $f(s) = F(1-\alpha, 1-\beta, 2; s)$, $(s > 0)$, it follows that

$$F(1-\alpha, 1-\beta, 2; s) = \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{\alpha-1}{j} \binom{\beta-1}{j} s^j + r_n(s), \qquad (s > 0), \tag{3.12}$$

where $r_n(s) = \frac{f^{(n)}(\xi(s))}{n!} s^n$, for some $\xi(s)$ in $]0, s[$. It suffices thus to show that $f^{(n)}(\xi) \leq 0$, for all ξ in $[0, 1[$, since this will imply that for all s in $[0, 1[$,

$$F(1-\alpha, 1-\beta, 2; s) \leq \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{\alpha-1}{j} \binom{\beta-1}{j} s^j,$$

where, as above, $0 < \binom{\alpha-1}{j} \leq \frac{\alpha^j}{j!}$ and $0 < \binom{\beta-1}{j} \leq \frac{\beta^j}{j!}$, for all j in $\{0, 1, \dots, n-1\}$.
To show that $f^{(n)}(\xi) \leq 0$ for all ξ in $[0, 1[$, we note that by [HTF, Vol. 1, p. 58, formula (7)],

$$\begin{aligned} f^{(n)}(\xi) &= \frac{d^n}{d\xi^n} F(1-\alpha, 1-\beta, 2; \xi) \\ &= \frac{(1-\alpha)_n (1-\beta)_n}{(n+1)!} F(n+1-\alpha, n+1-\beta, n+2; \xi), \end{aligned}$$

for all ξ in $[0, 1[$. Note here that

$$(1-\alpha)_n (1-\beta)_n = (\alpha-1)(\alpha-2)\cdots(\alpha-n)(\beta-1)(\beta-2)\cdots(\beta-n) \leq 0,$$

because $\alpha \geq n$ and $n - 1 < \beta < n$. Moreover, by [HTF, Vol. 1, p. 105, formula (2)], we have for all ξ in $[0, 1[$

$$\begin{aligned} F(n+1-\alpha, n+1-\beta, n+2; \xi) &= (1-\xi)^{\alpha+\beta-n} F(\alpha+1, \beta+1, n+2; \xi) \\ &= (1-\xi)^{\alpha+\beta-n} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j (\beta+1)_j}{j!(n+2)_j} \xi^j, \end{aligned}$$

and therefore $F(n+1-\alpha, n+1-\beta, n+2; \xi) > 0$ for all ξ in $[0, 1[$. Taken together, it follows that $f^{(n)}(\xi) \leq 0$ for all ξ in $[0, 1[$, as desired. □

For any c in $]0, \infty[$, we let μ_c denote the free Poisson distribution with parameter c, i.e., the probability measure on $\mathbb{R}$, given by

$$\mu_c = \max\{1-c, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx, \tag{3.13}$$

where $a=(\sqrt{c}-1)^2$, $b=(\sqrt{c}+1)^2$ and δ_0 is the Dirac measure at 0 (cf. [HT, Definition 6.5]).

3.6 Lemma. *Let α,β be strictly positive real numbers, and assume that either α or β is an integer. Then for any t in $[0,\frac{1}{2}]$,*

$$1+\sum_{p=1}^{\infty}\frac{t^p}{p!}\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)\alpha^k\beta^{l-1}\le\exp((\alpha+\beta)t^2)\int_0^\infty\exp(\beta tx)\ d\mu_{\frac{\alpha}{\beta}}(x),$$

Proof. Using that $-\log(1-t)=\sum_{n=1}^{\infty}\frac{t^n}{n}\le t+t^2$, whenever $0\le t\le\frac{1}{2}$, we note first that

$$(1-t)^{-(\alpha+\beta)}\le\exp((\alpha+\beta)t)\exp((\alpha+\beta)t^2),\qquad(t\in[0,\tfrac{1}{2}]).$$

Hence by Lemma 3.4 and Lemma 3.5,

$$\begin{aligned}\sum_{p=1}^{\infty}\frac{t^{p-1}}{(p-1)!}&\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)\alpha^{k-1}\beta^{l-1}\\&\le\exp((\alpha+\beta)t)\exp((\alpha+\beta)t^2)\sum_{j=0}^{\infty}\frac{(\alpha\beta)^jt^{2j}}{j!(j+1)!}\end{aligned}\tag{3.14}$$

Put $c=\frac{\alpha}{\beta}$ and $s=\beta t$. From [HT, Formula (6.27)], it follows then that

$$\begin{aligned}\int_0^\infty x\exp(sx)\ d\mu_c(x)&=c\exp((c+1)s)\sum_{j=0}^{\infty}\frac{c^js^{2j}}{j!(j+1)!}\\&=c\exp((\alpha+\beta)t)\sum_{j=0}^{\infty}\frac{(\alpha\beta)^jt^{2j}}{j!(j+1)!}.\end{aligned}$$

Hence (3.14) can be rewritten as

$$\begin{aligned}\sum_{p=1}^{\infty}\frac{t^{p-1}}{(p-1)!}&\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)\alpha^{k-1}\beta^{l-1}\\&\le\frac{\beta}{\alpha}\exp((\alpha+\beta)t^2)\int_0^\infty x\exp(\beta tx)\ d\mu_{\frac{\alpha}{\beta}}(x).\end{aligned}\tag{3.15}$$

Using then that $\frac{t^p}{p!}=\int_0^t\frac{u^{p-1}}{(p-1)!}\,du$, for all p in $\mathbb{N}$, and that $\exp((\alpha+\beta)u^2)\le\exp((\alpha+\beta)t^2)$, whenever $0\le u\le t$, we get by termwise integration of (3.15) (after replacing t by u), that

$$\begin{aligned}
\sum_{p=1}^{\infty}\frac{t^p}{p!}&\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)\alpha^{k-1}\beta^{l-1}\\
&\le \tfrac{\beta}{\alpha}\exp((\alpha+\beta)t^2)\int_0^t\Big(\int_0^\infty x\exp(\beta ux)\,d\mu_{\frac{\alpha}{\beta}}(x)\Big)\,du\\
&= \tfrac{\beta}{\alpha}\exp((\alpha+\beta)t^2)\int_0^\infty x\,\frac{\exp(\beta tx)-1}{\beta x}\,d\mu_{\frac{\alpha}{\beta}}(x)\\
&= \tfrac{1}{\alpha}\exp((\alpha+\beta)t^2)\int_0^\infty(\exp(\beta tx)-1)\,d\mu_{\frac{\alpha}{\beta}}(x).
\end{aligned}$$

Hence, using that $\mu_{\frac{\alpha}{\beta}}$ is a probability measure, it follows that

$$\begin{aligned}
1+\sum_{p=1}^{\infty}\frac{t^p}{p!}&\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)\alpha^{k}\beta^{l-1}\\
&\le 1+\exp((\alpha+\beta)t^2)\Big(\int_0^\infty\exp(\beta tx)\,d\mu_{\frac{\alpha}{\beta}}(x)-1\Big)\\
&\le \exp((\alpha+\beta)t^2)\int_0^\infty\exp(\beta tx)\,d\mu_{\frac{\alpha}{\beta}}(x).
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 3.3. Let $a_1,\dots,a_r$, $Y_1,\dots,Y_r$ and S be as set out in Theorem 3.3. By Proposition 2.5 and Proposition 2.7, we have then that

$$\begin{aligned}
\mathbb{E}\big[(S^*S)^p\big]&=\Big(\sum_{\pi\in S_p}n^{-2\sigma(\hat\pi)}\sum_{1\le i_1,\dots,i_p\le r}a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}}\Big)\otimes\mathbf{1}_n\\
&\le\Big(\sum_{\pi\in S_p}n^{-2\sigma(\hat\pi)}c^{\kappa(\hat\pi)}\Big)\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)},
\end{aligned}\tag{3.16}$$

where $\kappa(\hat\pi)$ was introduced in Definition 2.6.

We assume first that $c\ge 1$. By Proposition 2.10(i) and (ii), we have that

$$\kappa(\hat\pi)\le k(\hat\pi)+2\sigma(\hat\pi),\qquad(\pi\in S_p).$$

Hence,

$$\mathbb{E}\big[(S^*S)^p\big]\le\Big(\sum_{\pi\in S_p}\big(\tfrac{n}{c}\big)^{-2\sigma(\hat\pi)}c^{k(\hat\pi)}\Big)\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.$$

Using now that $2\sigma(\hat\pi)=p+1-d(\hat\pi)=p+1-k(\hat\pi)-l(\hat\pi)$, we find that

$$\begin{aligned}
\mathbb{E}\big[(S^*S)^p\big]&\le\Big(\big(\tfrac{c}{n}\big)^{p+1}\sum_{\pi\in S_p}n^{k(\hat\pi)}\big(\tfrac{n}{c}\big)^{l(\hat\pi)}\Big)\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}\\
&=\Big(\big(\tfrac{c}{n}\big)^{p}\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)n^k\big(\tfrac{n}{c}\big)^{l-1}\Big)\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)},
\end{aligned}$$

and therefore, for $0 \le t \le \frac{n}{\max\{c,1\}} = \frac{n}{c}$, it follows by application of Proposition 3.2, that

$$\begin{aligned}\mathbb{E}\big[\exp(tS^*S)\big] &= \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)} + \sum_{p=1}^{\infty} \tfrac{t^p}{p!}\mathbb{E}\big[(S^*S)^p\big] \\ &\le \Big(1 + \sum_{p=1}^{\infty} \tfrac{1}{p!}\big(\tfrac{ct}{n}\big)^p \sum_{\substack{k,l\in\mathbb{N}\\k+l\le p+1}} \delta(p,k,l) n^k \big(\tfrac{n}{c}\big)^{l-1}\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.\end{aligned}$$

Using now Lemma 3.6, we get for $0 \le \frac{ct}{n} \le \frac{1}{2}$, that

$$\begin{aligned}\mathbb{E}\big[\exp(tS^*S)\big] &\le \Big(\exp\big((n+\tfrac{n}{c})(\tfrac{ct}{n})^2\big)\int_0^\infty \exp\big(\tfrac{n}{c}(\tfrac{ct}{n})x\big)\ d\mu_c(x)\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)} \\ &= \Big(\exp\big(c(c+1)\tfrac{t^2}{n}\big)\int_0^\infty \exp(tx)\ d\mu_c(x)\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)} \\ &\le \Big(\exp\big((c+1)^2\cdot\tfrac{t^2}{n}\big)\int_0^\infty \exp(tx)\ d\mu_c(x)\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.\end{aligned}$$

Since $\mathrm{supp}(\mu_c) \subseteq \big[0,(\sqrt{c}+1)^2\big]$, it follows that

$$\mathbb{E}\big[\exp(tS^*S)\big] \le \exp\big((c+1)^2\cdot\tfrac{t^2}{n}\big)\exp\big((\sqrt{c}+1)^2 t\big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)},$$

and this proves the theorem in the case where $c \ge 1$.
Assume then that $c < 1$. In this case we use (3.16) together with the fact that $\kappa(\hat{\pi}) \ge k(\hat{\pi})$ for all π in S_p, (Proposition 2.10(ii)) to obtain

$$\begin{aligned}\mathbb{E}\big[(S^*S)^p\big] &\le \Big(\sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}c^{k(\hat{\pi})}\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)} \\ &\le \Big(\tfrac{1}{n^{p+1}}\sum_{\pi\in S_p}(nc)^{k(\hat{\pi})}n^{l(\hat{\pi})}\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)} \\ &= \Big(\tfrac{1}{n^p}\sum_{\substack{k,l\in\mathbb{N}\\k+l\le p+1}}\delta(p,k,l)(nc)^k n^{l-1}\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.\end{aligned}$$

Hence for $0 \le t < \frac{n}{\max\{c,1\}} = n$, we get by application of Proposition 3.2,

$$\begin{aligned}\mathbb{E}\big[\exp(tS^*S)\big] &\le \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)} + \sum_{p=1}^{\infty} \tfrac{t^p}{p!}\mathbb{E}\big[(S^*S)^p\big] \\ &\le \Big(1 + \sum_{p=1}^{\infty} \tfrac{1}{p!}\big(\tfrac{t}{n}\big)^p \sum_{\substack{k,l\in\mathbb{N}\\k+l\le p+1}} \delta(p,k,l)(nc)^k n^{l-1}\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.\end{aligned}$$

Hence by Lemma 3.6, we have for $0 \le \frac{t}{n} \le \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\big[\exp(tS^*S)\big] &\le \Big(\exp\big((nc+n)(\tfrac{t}{n})^2\big)\int_0^\infty \exp\big(n(\tfrac{t}{n})x\big)\ d\mu_c(x)\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}\\ &= \Big(\exp\big((c+1)\tfrac{t^2}{n}\big)\int_0^\infty \exp(tx)\ d\mu_c(x)\Big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}\\ &\le \exp\big((c+1)^2\cdot \tfrac{t^2}{n})\big)\exp\big((\sqrt{c}+1)^2 t\big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)},\end{aligned}$$

and this completes the proof. □

3.7 Remark. Assume that $a_1,\dots,a_r \in \mathcal{B}(\mathcal{H},\mathcal{K})$, satisfying that $\sum_{i=1}^r a_i^*a_i \le c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\sum_{i=1}^r a_ia_i^* \le d\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, for some positive constants c and d. Consider furthermore independent elements $Y_1,\dots,Y_r$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and put $S = \sum_{i=1}^r a_i\otimes Y_i$. Applying then Theorem 3.3 to $a_i' = \frac{1}{\sqrt{d}}a_i$ and $c' = \frac{c}{d}$, we get the following extension of Theorem 3.3:
For any t in $[0,\frac{n}{2c}]\cap[0,\frac{n}{2d}]$,

$$\mathbb{E}\big[\exp(tS^*S)\big] \le \exp\big((\sqrt{c}+\sqrt{d})^2 t + (c+d)^2\cdot\tfrac{t^2}{n}\big)\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}. \qquad \square$$

4 Asymptotic Upper Bound on the Spectrum of $S_n^*S_n$ in the Exact Case

Throughout this section, we consider elements $a_1,\dots,a_r$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$ (for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$), satisfying that

$$\Big\|\sum_{i=1}^r a_i^*a_i\Big\| \le c, \quad \text{and} \quad \Big\|\sum_{i=1}^r a_ia_i^*\Big\| \le 1, \tag{4.1}$$

for some constant c in $]0,\infty[$. Let $\mathcal{A}$ denote the unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the family $\big\{a_i^*a_j \bigm| i,j\in\{1,\dots,r\}\big\}\cup\{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$. Furthermore, for each n in $\mathbb{N}$, we consider independent elements $Y_1^{(n)},\dots,Y_r^{(n)}$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and we define

$$S_n = \sum_{i=1}^r a_i\otimes Y_i^{(n)}. \tag{4.2}$$

In this section, we shall determine (almost surely) the asymptotic behavior (as $n\to\infty$) of the largest element of the spectrum of $S_n^*S_n$ (i.e., the norm of $S_n^*S_n$), under the assumption that $\mathcal{A}$ is an exact C^*-algebra. We start by studying the corresponding asymptotic behavior for the image of $S_n^*S_n$ under certain matrix valued completely positive mappings. More precisely, let d be a fixed positive integer, and let $\Phi\colon \mathcal{A}\to M_d(\mathbb{C})$ be a unital completely positive mapping. For each n in $\mathbb{N}$, let $\mathrm{id}_n\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ denote the identity mapping on $M_n(\mathbb{C})$. We then define

$$V_n = \big(\Phi \otimes \mathrm{id}_n\big)(S_n^* S_n) = \sum_{i,j=1}^{r} \Phi(a_i^* a_j) \otimes \big(Y_i^{(n)}\big)^* Y_j^{(n)}, \qquad (n \in \mathbb{N}). \tag{4.3}$$

Note that V_n is a random variable taking values in $M_d(\mathbb{C}) \otimes M_n(\mathbb{C}) \simeq M_{dn}(\mathbb{C})$. As indicated above, our first objective is to determine the asymptotic behavior of the largest eigenvalue of V_n. We emphasize, that this step does not require that $\mathcal{A}$ be exact.

The following lemma is a version of Jensen's Inequality, which we shall need significantly in this section and in Section 8. The lemma has been proved in much more general settings by Brown and Kosaki (cf. [BK]) and by Petz (cf. [Pe]). For the reader's convenience, we include a short proof, handling the special case needed here.

4.1 Lemma. (i) *Let $\mathcal{L}$ be a Hilbert space, and let P be a finite dimensional projection in $\mathcal{B}(\mathcal{L})$. Let tr denote the normalized trace on $\mathcal{B}(P(\mathcal{L}))$. Then for any selfadjoint element a of $\mathcal{B}(\mathcal{L})$, and any convex function $g\colon \mathbb{R} \to \mathbb{R}$, we have that*

$$\mathrm{tr}\big[g(PaP)\big] \le \mathrm{tr}\big[Pg(a)P\big]. \tag{4.4}$$

(ii) *Let $\mathcal{B}$ be a C^*-algebra, let m be a positive integer and let $\Psi : \mathcal{B} \to M_m(\mathbb{C})$ be a unital completely positive mapping. Then for any selfadjoint element a of $\mathcal{B}$ and any convex function $g\colon \mathbb{R} \to \mathbb{R}$, we have that*

$$\mathrm{tr}_m\big[g(\Psi(a))\big] \le \mathrm{tr}_m\big[\Psi(g(a))\big].$$

Proof. (i) Note first that g is continuous (being convex on the whole real line). Let m denote the dimension of $P(\mathcal{L})$, and choose an orthonormal basis $(e_1, \dots, e_m)$ for $P(\mathcal{L})$ consisting of eigenvectors for PaP. Let $\lambda_1, \dots, \lambda_m$ be the corresponding eigenvalues for PaP, i.e.,

$$\lambda_i = \langle PaPe_i, e_i\rangle = \langle ae_i, e_i\rangle, \qquad (i \in \{1, 2, \dots, m\}).$$

Then $g(\lambda_1), \dots, g(\lambda_m)$ are the eigenvalues of $g(PaP)$, and hence

$$\mathrm{tr}\big[g(PaP)\big] = \sum_{i=1}^{m} g(\lambda_i) = \sum_{i=1}^{m} g(\langle ae_i, e_i\rangle). \tag{4.5}$$

Since the trace on $\mathcal{B}(P(\mathcal{L}))$ is independent of the choice of orthonormal basis for $P(\mathcal{L})$, we have at the same time, that

$$\mathrm{tr}\big[Pg(a)P\big] = \sum_{i=1}^{m} \langle Pg(a)Pe_i, e_i\rangle = \sum_{i=1}^{m} \langle g(a)e_i, e_i\rangle. \tag{4.6}$$

Comparing (4.5) and (4.6), we see that it suffices to show that $\langle g(a)e_i, e_i\rangle \geq g(\langle ae_i, e_i\rangle)$, for all i in $\{1, 2, \ldots, m\}$. But for each i, this follows from the classical Jensen Inequality, applied to the distribution of a w.r.t. the state $\langle\,\cdot\, e_i, e_i\rangle$, i.e., the probability measure μ_i supported on $\mathrm{sp}(a)$, and satisfying that $\langle f(a)e_i, e_i\rangle = \int_{\mathrm{sp}(a)} f(t)\, d\mu_i(t)$, for all functions f in $C(\mathrm{sp}(a))$. This concludes the proof of (i).
(ii) By Stinespring's Theorem, we may choose a Hilbert space $\mathcal{L}$, a $*$-representation $\pi\colon \mathcal{B} \to \mathcal{B}(\mathcal{L})$ of $\mathcal{B}$ on $\mathcal{L}$, and an embedding $\iota\colon \mathbb{C}^m \to \mathcal{L}$ of $\mathbb{C}^m$ into $\mathcal{L}$, such that

$$\Psi(b) = P_K\pi(b)P_K, \qquad (b \in \mathcal{B}), \tag{4.7}$$

where $K = \iota(\mathbb{C}^m)$, and P_K is the orthogonal projection of $\mathcal{L}$ onto K. Moreover, the equality (4.7) is modulo the natural identifications associated with ι. Let tr_K denote the normalized trace on $\mathcal{B}(K)$. By application of (i), it follows then that

$$\begin{aligned}\mathrm{tr}_m\big[g(\Psi(a))\big] &= \mathrm{tr}_K\big[g(P_K\pi(a)P_K)\big] \leq \mathrm{tr}_K\big[P_K g(\pi(a))P_K\big]\\ &= \mathrm{tr}_K\big[P_K\pi(g(a))P_K\big] = \mathrm{tr}_m\big[\Psi(g(a))\big],\end{aligned}$$

and this proves (ii). □

4.2 Lemma. *Let V_n, $n \in \mathbb{N}$, be as in (4.3), and let $\lambda_{\max}(V_n)$ denote the largest eigenvalue of V_n (considered as an element of $M_{dn}(\mathbb{C})$). Then for any ϵ in $]0, \infty[$, we have that*

$$\sum_{n=1}^{\infty} P\big(\lambda_{\max}(V_n) \geq (\sqrt{c}+1)^2 + \epsilon\big) < \infty. \tag{4.8}$$

Proof. The proof proceeds along the same lines as the proof of [HT, Lemma 7.3]; the main difference being that in the present situation, we have to rely on the estimate obtained in Theorem 3.3. Consider first a fixed n in $\mathbb{N}$. We find then for any t in $]0, \infty[$, that

$$\begin{aligned}P\big(\lambda_{\max}(V_n) \geq (\sqrt{c}+1)^2 + \epsilon\big) &= P\big(\exp\big(t\lambda_{\max}(V_n) - t(\sqrt{c}+1)^2 - t\epsilon\big) \geq 1\big)\\ &\leq \mathbb{E}\big[\exp\big(t\lambda_{\max}(V_n) - t(\sqrt{c}+1)^2 - t\epsilon\big)\big]\\ &= \exp(-t(\sqrt{c}+1)^2 - t\epsilon)\cdot\mathbb{E}\big[\lambda_{\max}\big(\exp(tV_n)\big)\big]\\ &\leq \exp(-t(\sqrt{c}+1)^2 - t\epsilon)\cdot\mathbb{E}\big[\mathrm{Tr}_{dn}\big(\exp(tV_n)\big)\big],\end{aligned} \tag{4.9}$$

where the last inequality follows by noting, that since $\exp(tV_n)$ is a positive $dn \times dn$ matrix, $\lambda_{\max}(\exp(tV_n)) \leq \mathrm{Tr}_{dn}(\exp(tV_n))$. Note now, that since the mapping $\Phi \otimes \mathrm{id}_n$ is unital, completely positive, and since the function $x \mapsto$

$e^{tx}: \mathbb{R} \to \mathbb{R}$ is convex, it follows from Lemma 4.1(ii), that

$$\begin{aligned} \mathrm{tr}_{dn}\big[\exp(tV_n)\big] &= \mathrm{tr}_{dn}\big[\exp\big(t(\Phi\otimes \mathrm{id}_n)(S_n^*S_n)\big)\big] \\ &\le \mathrm{tr}_{dn}\big[\big(\Phi\otimes \mathrm{id}_n\big)(\exp(tS_n^*S_n))\big] \\ &= \mathrm{tr}_d\otimes \mathrm{tr}_n\big[\big(\Phi\otimes \mathrm{id}_n\big)(\exp(tS_n^*S_n))\big] = \phi\otimes \mathrm{tr}_n\big[\exp(tS_n^*S_n)\big], \end{aligned} \tag{4.10}$$

where ϕ is the state $\mathrm{tr}_d \circ \Phi$ on $\mathcal{A}$. Note here, that by Definition 3.1 and Theorem 3.3,

$$\begin{aligned} \mathbb{E}\big[\phi\otimes \mathrm{tr}_n\big(\exp(tS_n^*S_n)\big)\big] &= \phi\otimes \mathrm{tr}_n\big(\mathbb{E}\big[\exp(tS_n^*S_n)\big]\big) \\ &\le \exp\big(t(\sqrt{c}+1)^2 + \tfrac{t^2}{n}(c+1)^2\big), \end{aligned} \tag{4.11}$$

for all t in $]0, \frac{n}{2c}]$.
Combining now (4.9)-(4.11), we get that for all t in $]0, \frac{n}{2c}]$,

$$\begin{aligned} &P\big(\lambda_{\max}(V_n) \ge (\sqrt{c}+1)^2+\epsilon\big) \\ &\qquad \le dn\cdot \exp(-t(\sqrt{c}+1)^2 - t\epsilon)\cdot \exp\big(t(\sqrt{c}+1)^2 + \tfrac{t^2}{n}(c+1)^2\big) \\ &\qquad = dn\cdot \exp\big(t(\tfrac{t}{n}(c+1)^2-\epsilon)\big), \end{aligned}$$

Now choose $t = t_n = \frac{n\epsilon}{2(c+1)^2}$, and note that $t_n \in\,]0, \frac{n}{2c}]$, as long as $\epsilon \le 1$. Clearly it suffices to prove the lemma for such ϵ, so we assume that $\epsilon \le 1$. It follows then that

$$P\big(\lambda_{\max}(V_n) \ge (\sqrt{c}+1)^2+\epsilon\big) \le dn\cdot\exp\big(t_n(\tfrac{t_n}{n}(c+1)^2-\epsilon)\big) = dn\cdot\exp\big(\tfrac{-n\epsilon^2}{4(c+1)^2}\big).$$

Since this estimate holds for all n in $\mathbb{N}$, it follows immediately that (4.8) holds. □

4.3 Proposition. *Let V_n, $n\in\mathbb{N}$, be as in* (4.3). *We then have*

$$\limsup_{n\to\infty} \lambda_{\max}(V_n) \le \big(\sqrt{c}+1\big)^2, \quad \textit{almost surely.}$$

Proof. It suffices to show, that for any ϵ from $]0,\infty[$,

$$P\Big(\limsup_{n\to\infty}\lambda_{\max}(V_n) \le (\sqrt{c}+1)^2+\epsilon\Big) = 1,$$

and this will follow, if we show that

$$P\big(\lambda_{\max}(V_n) \le (\sqrt{c}+1)^2+\epsilon, \ \text{ for all but finitely many } n\big) = 1,$$

for all ϵ in $]0,\infty[$. But this follows from the Borel-Cantelli Lemma (cf. [Bre, Lemma 3.14]) together with Lemma 4.2. □

The next step is to replace V_n in Proposition 4.3 by $S_n^*S_n$ itself. This is where we need to assume that $\mathcal{A}$ is an exact C^*-algebra. The key point in this step is the important result of E. Kirchberg that exactness implies nuclear embeddability (cf. [Ki2, Theorem 4.1] and [Was, Theorem 7.3]).
Let $\mathcal{B}$ be a unital C^*-algebra. Recall then that an operator system in $\mathcal{B}$ is a subspace E of $\mathcal{B}$, such that $\mathbf{1}_{\mathcal{B}} \in E$ and $x^* \in E$ for all x in E.

4.4 Proposition. *Let $\mathcal{B}$ be a unital exact C^*-algebra, and let E be a finite dimensional operator system in $\mathcal{B}$. Then for any ϵ in $]0,\infty[$, there exist d in $\mathbb{N}$ and a unital completely positive mapping $\Phi\colon \mathcal{B} \to M_d(\mathbb{C})$, such that*

$$\big\|\big(\Phi \otimes \mathrm{id}_n\big)(x)\big\| \geq (1-\epsilon)\|x\|,$$

for all n in $\mathbb{N}$ and all x in $M_n(E)$.

Proof. Clearly we may assume that $\mathcal{B}$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{L})$ for some Hilbert space $\mathcal{L}$. Let N denote the dimension of E. Then by Auerbach's Lemma (cf. [LT, Proposition 1.c.3]), we may choose linear bases $e_1, \dots, e_N$ of E and $e_1^*, \dots, e_N^*$ of the dual space E^*, such that

$$\|e_i\| = \|e_i^*\| = 1, \quad \text{and} \quad e_i^*(e_j) = \delta_{i,j}, \qquad (i,j \in \{1,2,\dots,N\}). \tag{4.12}$$

Now since $\mathcal{B}$ is exact, and hence nuclear embeddable, there exist d in $\mathbb{N}$, and unital completely positive mappings $\Phi\colon \mathcal{B} \to M_d(\mathbb{C})$ and $\Psi\colon M_d(\mathbb{C}) \to \mathcal{B}(\mathcal{L})$, such that

$$\|\Psi(\Phi(e_i)) - e_i\| \leq \tfrac{\epsilon}{N}, \qquad (i \in \{1,2,\dots,N\}), \tag{4.13}$$

(cf. [Was, p. 60]). We show that this Φ has the property set out in the proposition. For this, it suffices to show that

$$\big\|(\Psi \circ \Phi - \iota_{\mathcal{B}})_{|E}\big\|_{\mathrm{cb}} \leq \epsilon, \tag{4.14}$$

where $\iota_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}(\mathcal{L})$ is the embedding of $\mathcal{B}$ into $\mathcal{B}(\mathcal{L})$. Indeed, knowing the validity of (4.14), we have for n in $\mathbb{N}$ and x in $M_n(E)$, that

$$\begin{aligned} \|x\| &\leq \big\|\big((\Psi\circ\Phi)\otimes \mathrm{id}_n\big)(x) - x\big\| + \big\|\big((\Psi\circ\Phi)\otimes \mathrm{id}_n\big)(x)\big\| \\ &\leq \epsilon\|x\| + \big\|\big((\Psi\circ\Phi)\otimes \mathrm{id}_n\big)(x)\big\|, \end{aligned}$$

and hence that

$$(1-\epsilon)\|x\| \leq \big\|\big((\Psi\circ\Phi)\otimes \mathrm{id}_n\big)(x)\big\| \leq \|(\Phi\otimes \mathrm{id}_n)(x)\|,$$

where the last inequality is due to the fact that Ψ, being unital completely positive, is a complete contraction.
To verify (4.14) note first, that for x in E, we have by (4.12),

$$x = \sum_{i=1}^N e_i^*(x) e_i,$$

and hence

$$\Psi\circ\Phi(x)-x=\sum_{i=1}^{N}e_i^*(x)\big(\Psi\circ\Phi(e_i)-e_i\big)=\sum_{i=1}^{N}e_i^*(x)f_i,$$

where $f_i = \Psi\circ\Phi(e_i)-e_i$. Note that by (4.13), $\|f_i\|\le\frac{\epsilon}{N}$, for all i in $\{1,2,\dots,N\}$.

Consider now n in $\mathbb{N}$ and $x=(x_{rs})_{1\le r,s\le n}$ in $M_n(E)$. We then have

$$\begin{aligned}\big((\Psi\circ\Phi)\otimes\mathrm{id}_n\big)(x)-x&=\big[(\Psi\circ\Phi)(x_{rs})-x_{rs}\big]_{1\le r,s\le n}\\&=\Big[\textstyle\sum_{i=1}^{N}e_i^*(x_{rs})f_i\Big]_{1\le r,s\le n}\\&=\sum_{i=1}^{N}\Big([e_i^*(x_{rs})]_{1\le r,s\le n}\cdot\mathrm{diag}_n(f_i,\dots,f_i)\Big),\end{aligned}\tag{4.15}$$

where $\mathrm{diag}_n(f_i,\dots,f_i)$ is the $n\times n$ diagonal matrix with f_i in all the diagonal positions. Note here that by (4.12), $\|e_i^*\|_{\mathrm{cb}}=\|e_i^*\|=1$, for all i (cf. [Pa, Proposition 3.7]). Consequently,

$$\big\|[e_i^*(x_{rs})]_{1\le r,s\le n}\big\|\le\|e_i^*\|_{\mathrm{cb}}\cdot\|x\|=\|x\|,\qquad(i\in\{1,2,\dots,N\}),$$

and using this in (4.15), we get that

$$\big\|\big((\Psi\circ\Phi)\otimes\mathrm{id}_n\big)(x)-x\big\|\le\sum_{i=1}^{N}\|x\|\cdot\|f_i\|\le\sum_{i=1}^{N}\|x\|\tfrac{\epsilon}{N}=\epsilon\|x\|,$$

which proves (4.14). □

4.5 Theorem. *Let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that $\|\sum_{i=1}^r a_i^*a_i\|\le c$, and $\|\sum_{i=1}^r a_ia_i^*\|\le 1$, for some constant c in $]0,\infty[$. Assume, in addition, that the C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, generated by $\{a_i^*a_j\mid i,j\in\{1,2,\dots,r\}\}\cup\{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$, is exact. Consider furthermore, for each n in $\mathbb{N}$, independent elements $Y_1^{(n)},\dots,Y_r^{(n)}$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and put $S_n=\sum_{i=1}^r a_i\otimes Y_i^{(n)}$. We then have*

$$\limsup_{n\to\infty}\max\big[\mathrm{sp}(S_n^*S_n)\big]\le\big(\sqrt{c}+1\big)^2,\qquad\textit{almost surely.}$$

Proof. It suffices to show, that for any ϵ from $]0,\infty[$, the set

$$\mathcal{T}_\epsilon=\Big\{\omega\in\Omega\;\Big|\;\limsup_{n\to\infty}\max\big[\mathrm{sp}(S_n(\omega)^*S_n(\omega))\big]\le\tfrac{1}{1-\epsilon}(\sqrt{c}+1)^2\Big\},$$

has probability 1. So let ϵ from $]0,\infty[$ be given, and put

$$E=\mathrm{span}\big(\{\mathbf{1}_{\mathcal{A}}\}\cup\{a_i^*a_j\mid i,j\in\{1,2,\dots,r\}\}\big).$$

Note that $x^* \in E$ for all x in E, and that $\mathbf{1}_{\mathcal{A}} \in E$. Hence E is a finite dimensional operator system in $\mathcal{A}$. Since $\mathcal{A}$ is exact, it follows thus from Proposition 4.4, that we may choose d in $\mathbb{N}$ and a completely positive mapping $\Phi\colon \mathcal{A} \to M_d(\mathbb{C})$, such that

$$\big\|\big(\Phi \otimes \mathrm{id}_n\big)(x)\big\| \geq (1-\epsilon)\|x\|, \qquad (n \in \mathbb{N},\ x \in M_n(E)). \tag{4.16}$$

Now put

$$V_n = \big(\Phi \otimes \mathrm{id}_n\big)(S_n^* S_n), \qquad (n \in \mathbb{N}),$$

and define furthermore

$$\mathcal{V} = \Big\{\omega \in \Omega \;\Big|\; \limsup_{n\to\infty} \|V_n(\omega)\| \leq (\sqrt{c}+1)^2\Big\}.$$

By Proposition 4.3, $P(\mathcal{V}) = 1$, and hence it suffices to show that $\mathcal{T}_\epsilon \supseteq \mathcal{V}$. But if $\omega \in \mathcal{V}$, it follows from (4.16) that

$$\limsup_{n\to\infty} \|S_n(\omega)^* S_n(\omega)\| \leq (1-\epsilon)^{-1} \limsup_{n\to\infty} \|V_n(\omega)\| \leq (1-\epsilon)^{-1}(\sqrt{c}+1)^2,$$

which shows that $\omega \in \mathcal{T}_\epsilon$. This concludes the proof. □

4.6 Corollary. *Let $a_1, \dots, a_r$ be elements of an exact C^*-algebra $\mathcal{A}$, and let, for each n in $\mathbb{N}$, $Y_1^{(n)}, \dots, Y_r^{(n)}$ be independent elements of* $\mathrm{GRM}(n, n, \frac{1}{n})$. *Then*

$$\limsup_{n\to\infty} \Big\| \sum_{i=1}^r a_i \otimes Y_i^{(n)} \Big\| \leq \Big\| \sum_{i=1}^r a_i^* a_i \Big\|^{\frac{1}{2}} + \Big\| \sum_{i=1}^r a_i a_i^* \Big\|^{\frac{1}{2}}, \qquad \textit{almost surely.}$$

Proof. We may assume that not all a_i are zero. Put $\gamma = \| \sum_{i=1}^r a_i^* a_i \| > 0$ and $\delta = \| \sum_{i=1}^r a_i a_i^* \| > 0$. We may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then the unital C^*-algebra $\tilde{\mathcal{A}} = C^*(\mathcal{A}, \mathbf{1}_{\mathcal{B}(\mathcal{H})})$ is also exact, and hence so is every C^*-subalgebra of $\tilde{\mathcal{A}}$ (cf. [Ki1] and [Was, 2.5.2]). Therefore Corollary 4.6 follows by applying Theorem 4.5 to $a_i' = \frac{1}{\sqrt{\delta}} a_i$, $i = 1, \dots, r$, and $c = \frac{\gamma}{\delta}$. □

Regarding the corollary above, consider arbitrary elements $a_1, \dots, a_r$ of an arbitrary C^*-algebra $\mathcal{A}$, and let $\{y_1, \dots, y_r\}$ be a circular (or semi-circular) system in some C^*-probability space $(\mathcal{B}, \psi)$ (cf. [Vo2]), and normalized so that $\psi(y_i^* y_i) = 1$, $i = 1, 2, \dots, r$. In [HP, Proof of Proposition 4.8], G. Pisier and the first named author showed, that in this setting, the following inequality holds:

$$\Big\| \sum_{i=1}^r a_i \otimes y_i \Big\| \leq 2 \max\Big\{ \Big\| \sum_{i=1}^r a_i^* a_i \Big\|^{\frac{1}{2}}, \Big\| \sum_{i=1}^r a_i a_i^* \Big\|^{\frac{1}{2}} \Big\}. \tag{4.17}$$

In [HP, Proof of Proposition 4.8], the factor 2 on the right hand side of (4.17) is missing, but this is due to a different choice of normalization of semi-circular and circular families. By application of [Haa, Section 1], it is not hard to strengthen (4.17) to the inequality

$$\Big\| \sum_{i=1}^{r} a_i \otimes y_i \Big\| \le \Big\| \sum_{i=1}^{r} a_i^* a_i \Big\|^{\frac{1}{2}} + \Big\| \sum_{i=1}^{r} a_i a_i^* \Big\|^{\frac{1}{2}}, \tag{4.18}$$

both for semi-circular and circular systems. Since independent elements $Y_1^{(n)}, \dots, Y_r^{(n)}$ of $\mathrm{GRM}(n, n, \frac{1}{n})$ can be considered as a random matrix model for the circular system $\{y_1, \dots, y_r\}$, in the sense of [Vo1, Theorem 2.2], we should thus consider Corollary 4.6 as a random matrix version of (4.18). However, the random matrix version holds only under the assumption that the C^*-algebra $\mathcal{A}$ be exact. In fact, we shall spend the remaining part of this section, showing that the assumption in Theorem 4.5 that the C^*-algebra $\mathcal{A}$ be exact, can not be omitted. We start with two lemmas, the first of which is a slightly strengthened version of [HT, Theorem 7.4] (which, in turn, is a special case of a theorem of Wachter (cf. [Wac])).

4.7 Lemma. *Let c be a positive number, and let (m_n) be a sequence of positive integers, such that $\frac{m_n}{n} \to c$ as $n \to \infty$. Let furthermore (Y_n) be a sequence of random matrices, such that for each n in $\mathbb{N}$, $Y_n \in \mathrm{GRM}(m_n, n, \frac{1}{n})$. Then for any continuous function $f \colon [0, \infty[\to \mathbb{C}$, we have that*

$$\lim_{n\to\infty} \mathrm{tr}_n\big[f(Y_n^* Y_n)\big] = \int_0^b f(x)\, d\mu_c(x), \qquad \textit{almost surely}, \tag{4.19}$$

where $b = (\sqrt{c}+1)^2$ and μ is the measure introduced in (3.13).

Proof. By splitting f in its real and imaginary parts, it is clear, that we may assume that f is a real valued continuous function on $[0, \infty[$. We note next, that it follows from [HT, Theorem 7.4] and the definition of weak convergence (cf. [HT, Definition 2.2]), that (4.19) holds for all continuous *bounded* functions $f \colon [0, \infty[\to \mathbb{R}$. Thus, our objective is to pass from bounded to unbounded continuous functions, and the key to this, is the fact (cf. [HT, Theorem 7.1]), that

$$\lim_{n\to\infty} \|Y_n^* Y_n\| = \big(\sqrt{c}+1\big)^2, \qquad \text{almost surely}. \tag{4.20}$$

Indeed, it follows from (4.20), that (for example)

$$P\big(\ \|Y_n^* Y_n\| \le (\sqrt{c}+1)^2 + 1, \text{ for all but finitely many } n\big) = 1,$$

and hence, given any ϵ in $]0, \infty[$, we may choose N in $\mathbb{N}$, such that

$$P(F_N) \ge 1 - \epsilon,$$

where

$$F_N = \big\{\omega \in \Omega \ \big|\ \|Y_n(\omega)^* Y_n(\omega)\| \le (\sqrt{c}+1)^2 + 1, \text{ whenever } n \ge N\big\}.$$

Now, given a continuous function $f\colon [0,\infty[\ \to \mathbb{R}$, let $f_1\colon [0,\infty[\ \to \mathbb{R}$ be an arbitrary continuous function, satisfying that $f_1 = f$ on $[0,(\sqrt{c}+1)^2+1]$, and that $\mathrm{supp}(f)$ is compact. Then for any ω in F_N, we have that

$$f_1(Y_n(\omega)^*Y_n(\omega)) = f(Y_n(\omega)^*Y_n(\omega)), \quad \text{whenever } n \geq N,$$

and hence, since f_1 is bounded,

$$\begin{aligned}\lim_{n\to\infty} \mathrm{tr}_n\big[f(Y_n(\omega)^*Y_n(\omega))\big] &= \lim_{n\to\infty} \mathrm{tr}_n\big[f_1(Y_n(\omega)^*Y_n(\omega))\big] = \int_a^b f_1(x)\ d\mu_c(x)\\ &= \int_a^b f(x)\ d\mu_c(x).\end{aligned}$$

It follows thus, that

$$P\Big(\lim_{n\to\infty} \mathrm{tr}_n\big[f(Y_n^*Y_n)\big] = \textstyle\int_a^b f(x)\ d\mu_c(x)\Big) \geq P(F_N) \geq 1-\epsilon,$$

and since this holds for any ϵ in $]0,\infty[$, we obtain the desired conclusion. $\square$

Next, we shall study the polar decomposition of Gaussian random matrices. Let n be a positive integer and let Y be an element of $\mathrm{GRM}(n,n,\frac{1}{n})$, defined on $(\Omega,\mathcal{F},P)$. Furthermore, let $\mathcal{U}_n$ denote the unitary group of $M_n(\mathbb{C})$.
By a measurable unitary sign for Y, we mean a random matrix $U\colon \Omega \to \mathcal{U}_n$, such that for almost all ω in Ω, the polar-decomposition of $Y(\omega)$ is given by:

$$Y(\omega) = U(\omega)|Y(\omega)|,$$

where, as usual, $|Y(\omega)| = [Y(\omega)^*Y(\omega)]^{\frac{1}{2}}$. To see that such measurable unitary signs do exist, we note first that by [HT, Theorem 5.2], $Y(\omega)$ is invertible for almost all ω. Thus, for example the random matrix $U\colon \Omega \to \mathcal{U}_n$ given by

$$U(\omega) = \begin{cases} Y(\omega)\big[Y(\omega)^*Y(\omega)\big]^{-\frac{1}{2}}, & \text{if } Y(\omega) \text{ is invertible},\\ \mathbf{1}_n, & \text{otherwise},\end{cases}$$

is a measurable unitary sign for Y.

4.8 Lemma. *For each n in $\mathbb{N}$, let $Y_1^{(n)},\dots,Y_r^{(n)}$ be (not necessarily independent) random matrices in $\mathrm{GRM}(n,n,\frac{1}{n})$, and let $U_1^{(n)},\dots,U_r^{(n)}$ be measurable unitary signs for $Y_1^{(n)},\dots,Y_r^{(n)}$, respectively. Furthermore, let $\overline{U}_1^{(n)},\dots,\overline{U}_r^{(n)}$, denote the complex conjugated matrices of $U_1^{(n)},\dots,U_r^{(n)}$. We then have*

$$\liminf_{n\to\infty} \Big\| \sum_{i=1}^r \overline{U}_i^{(n)} \otimes Y_i^{(n)} \Big\| \geq \tfrac{8}{3\pi}\cdot r, \qquad \textit{almost surely.}$$

Proof. Let $(e_1, \dots, e_n)$ be the usual orthonormal basis for $\mathbb{C}^n$, and consider then the unit vector $\xi = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$ in $\mathbb{C}^n \otimes \mathbb{C}^n$. Note then that for any $A = (a_{jk})$ and $B = (b_{jk})$ in $M_n(\mathbb{C})$, we have that

$$\begin{aligned} \big\langle (A \otimes B)\xi, \xi \big\rangle &= \tfrac{1}{n} \sum_{j,k=1}^n \big\langle (A \otimes B)(e_j \otimes e_j), e_k \otimes e_k \big\rangle \\ &= \tfrac{1}{n} \sum_{j,k=1}^n \langle A e_j, e_k \rangle \cdot \langle B e_j, e_k \rangle \\ &= \tfrac{1}{n} \sum_{j,k=1}^n a_{kj} b_{kj} = \mathrm{tr}_n(AB^t) = \mathrm{tr}_n(A^t B). \end{aligned}$$

It follows thus, that

$$\begin{aligned} \Big\| \sum_{i=1}^r \overline{U}_i^{(n)} \otimes Y_i^{(n)} \Big\| \geq \Big| \Big\langle \Big(\sum_{i=1}^r \overline{U}_i^{(n)} \otimes Y_i^{(n)} \Big) \xi, \xi \Big\rangle \Big| &= \Big| \sum_{i=1}^r \mathrm{tr}_n \big[\big(U_i^{(n)}\big)^* Y_i^{(n)} \big] \Big| \\ &= \sum_{i=1}^r \mathrm{tr}_n \big(|Y_i^{(n)}| \big), \end{aligned} \tag{4.21}$$

where the last equation holds almost surely. By Lemma 4.7, we have for all i in $\{1, \dots, r\}$, that

$$\lim_{n \to \infty} \mathrm{tr}_n \big(|Y_i^{(n)}| \big) = \int_0^4 \sqrt{x} \; d\mu_1(x), \qquad \text{almost surely,}$$

and combining this with (4.21), it follows that

$$\liminf_{n \to \infty} \Big\| \sum_{i=1}^r \overline{U}_i^{(n)} \otimes Y_i^{(n)} \Big\| \geq r \int_0^4 \sqrt{x} \; d\mu_1(x), \qquad \text{almost surely.}$$

We note finally that

$$\int_0^4 \sqrt{x} \; d\mu_1(x) = \int_0^4 \sqrt{x} \cdot \tfrac{\sqrt{x(4-x)}}{2\pi x} \; dx = \tfrac{1}{2\pi} \int_0^4 \sqrt{4-x} \; dx = \tfrac{8}{3\pi},$$

and this concludes the proof. ☐

We are now ready to give an example where the conclusion of Theorem 4.5 fails, due to lack of exactness of the C^*-algebra $\mathcal{A}$. Consider a fixed positive integer r, greater than or equal to 2, and let $\mathbb{F}_r$ denote the free group on r generators. Let $g_1, \dots, g_r$ denote the generators of $\mathbb{F}_r$, and let $C^*(\mathbb{F}_r)$ denote the *full* C^*-algebra associated to $\mathbb{F}_r$. Recall that there is a canonical unitary representation $u_{\mathbb{F}_r} \colon \mathbb{F}_r \to C^*(\mathbb{F}_r)$, and that the pair $(C^*(\mathbb{F}_r), u_{\mathbb{F}_r})$ is characterized (up to $*$-isomorphism) by the universal property, that given any unital C^*-algebra $\mathcal{B}$

and any unitary representation $u\colon \mathbb{F}_r \to \mathcal{B}$, there exists a unique unital $*$-homomorphism $\Phi_u\colon C^*(\mathbb{F}_r) \to \mathcal{B}$, such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{F}_r \arrow[r, "u_{\mathbb{F}_r}"] \arrow[d, "u"'] & C^*(\mathbb{F}_r) \arrow[dl, "\Phi_u"] \\ \mathcal{B} & \end{tikzcd}$$

It is well-known (cf. [Was, Corollary 3.7]) that $C^*(\mathbb{F}_r)$ is *not* exact. We let $u_1, \dots, u_r$ be the canonical unitaries in $C^*(\mathbb{F}_r)$ associated to $g_1, \dots, g_r$ respectively, i.e., $u_i = u_{\mathbb{F}_r}(g_i)$, $i = 1, \dots, r$. We then define

$$a_i = \tfrac{1}{\sqrt{r}} u_i, \qquad (i \in \{1, \dots, r\}). \tag{4.22}$$

Then clearly,

$$\sum_{i=1}^r a_i^* a_i = \sum_{i=1}^r a_i a_i^* = \mathbf{1}_{C^*(\mathbb{F}_r)}. \tag{4.23}$$

Consider now, in addition, for each n in $\mathbb{N}$, independent elements $Y_1^{(n)}, \dots, Y_r^{(n)}$ of $\mathrm{GRM}(n, n, \frac{1}{n})$, and define

$$S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}, \qquad (n \in \mathbb{N}). \tag{4.24}$$

We then have the following

4.9 Proposition. *With $a_1, \dots, a_r$ and S_n, $n \in \mathbb{N}$, as introduced in* (4.22) *and* (4.24)*, we have that*

(i) $\liminf_{n\to\infty} \|S_n^* S_n\| \geq \left(\frac{8}{3\pi}\right)^2 \cdot r$, *almost surely.*

(ii) *The conclusion of Theorem 4.5 does not hold for these $a_1, \dots, a_r$, whenever $r \geq 6$.*

In particular, the assumption in Theorem 4.5, that $\mathcal{A}$ be exact, can not, in general, be omitted.

Proof. (i) For each positive integer n, choose measurable unitary signs $U_1^{(n)}, \dots, U_r^{(n)}$ for $Y_1^{(n)}, \dots, Y_r^{(n)}$ respectively, and let $\overline{U}_1^{(n)}, \dots, \overline{U}_r^{(n)}$ denote the complex conjugated matrices of $U_1^{(n)}, \dots, U_r^{(n)}$. Since $\mathbb{F}_r$ is the group free product of r copies of $\mathbb{Z}$, it follows that for each ω in Ω and each n in $\mathbb{N}$, there exists a unitary representation $u_\omega^{(n)}\colon \mathbb{F}_r \to M_n(\mathbb{C})$, such that

$$u_\omega^{(n)}(g_i) = \overline{U}_i^{(n)}(\omega), \qquad (i \in \{1, \dots, r\}).$$

By the universal property of $C^*(\mathbb{F}_r)$ it follows then, that for each ω in Ω and each n in $\mathbb{N}$, we may choose a $*$-homomorphism $\Phi_\omega^{(n)}\colon C^*(\mathbb{F}_r) \to M_n(\mathbb{C})$, such that

$$\Phi_\omega^{(n)}(u_i) = \overline{U}_i^{(n)}(\omega), \qquad (i \in \{1, \dots, r\}).$$

For each ω in Ω and each n in $\mathbb{N}$, note now that

$$\begin{aligned}\Big\|\sum_{i=1}^r u_i\otimes Y_i^{(n)}(\omega)\Big\| &\ge \Big\|\big(\Phi_\omega^{(n)}\otimes\mathrm{id}_n\big)\Big(\sum_{i=1}^r u_i\otimes Y_i^{(n)}(\omega)\Big)\Big\| \\ &= \Big\|\sum_{i=1}^r \overline{U}_i^{(n)}(\omega)\otimes Y_i^{(n)}(\omega)\Big\|.\end{aligned}$$

Applying then Lemma 4.8, it follows that

$$\liminf_{n\to\infty}\Big\|\sum_{i=1}^r u_i\otimes Y_i^{(n)}\Big\| \ge \tfrac{8}{3\pi}\cdot r,\qquad \text{almost surely,}$$

and hence that

$$\liminf_{n\to\infty}\Big\|\sum_{i=1}^r a_i\otimes Y_i^{(n)}\Big\| \ge \tfrac{8}{3\pi}\cdot \sqrt{r},\qquad \text{almost surely.}$$

Since $\|S_n^*S_n\| = \|S_n\|^2$, we get the desired formula.
(ii) By (4.23), $a_1,\dots,a_r$ introduced in (4.22) satisfy condition (4.1) in the case $c=1$. Thus, if the conclusion of Theorem 4.5 were to hold for these $a_1,\dots,a_r$, it would mean that

$$\limsup_{n\to\infty}\Big\|\sum_{i=1}^r a_i\otimes Y_i^{(n)}\Big\| \le 2,\qquad \text{almost surely.}$$

However, Proposition 4.9 shows that

$$\liminf_{n\to\infty}\Big\|\sum_{i=1}^r a_i\otimes Y_i^{(n)}\Big\| \ge \big(\tfrac{8}{3\pi}\big)\cdot \sqrt{r},\qquad \text{almost surely,}$$

and thus the conclusion of Theorem 4.5 breaks down, for $c=1$, whenever $r > (\frac{3\pi}{4})^2 \approx 5.55$, i.e., for $r\ge 6$. $\square$

5 A New Combinatorial Expression for $\mathbb{E}\big[(S^*S)^p\big]$

Throughout this section, we consider elements $a_1,\dots,a_r$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. In Section 2 we proved that if $Y_1,\dots,Y_r$ are independent random matrices in $\mathrm{GRM}(n,n,\frac{1}{n})$, and we put $S=\sum_{i=1}^r a_i\otimes Y_i$, then

$$\mathbb{E}\big[(S^*S)^p\big] = \Big(\sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}\cdot\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}}\Big)\otimes \mathbf{1}_n. \tag{5.1}$$

In this section, we shall assume that $a_1,\dots,a_r$ satisfy the condition

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})},\quad\text{and}\quad \sum_{i=1}^r a_ia_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \tag{5.2}$$

for some number c in $]0,\infty[$. Under this assumption, and by application of the method of "reductions of permutations", introduced in Section 1, we show that $\mathbb{E}\big[(S^*S)^p\big]$ can be expressed as a constant plus a linear combination of the sums:

$$\sum_{\rho\in S_q^{\rm irr}} n^{-2\sigma(\hat{\rho})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^*a_{i_{\rho(1)}}\cdots a_{i_q}^*a_{i_{\rho(q)}}\Big), \qquad (q=2,\dots,p),$$

where $S_q^{\rm irr}$, as in Section 1, denotes the set of permutations ρ in S_q for which $\hat{\rho}$ is irreducible in the sense of Definition 1.16.

5.1 Lemma. *Let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, and assume that (5.2) holds. Let p be a positive integer, greater than or equal to 2, let π be a permutation in $S_p\setminus S_p^{\rm irr}$, and let π_0 be the permutation in S_{p-1} obtained by cancellation of a pair $(e,e+1)$ for $\hat{\pi}$ (cf. Definition 1.18). We then have*
(i) *If e is odd, then $k(\hat{\pi}_0)=k(\hat{\pi})-1$, and*

$$\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}} = c\cdot\Big(\sum_{1\le i_1,\dots,i_{p-1}\le r} a_{i_1}^*a_{i_{\pi_0(1)}}\cdots a_{i_{p-1}}^*a_{i_{\pi_0(p-1)}}\Big). \tag{5.3}$$

(ii) *If e is even, then $k(\hat{\pi}_0)=k(\hat{\pi})$, and*

$$\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}} = \sum_{1\le i_1,\dots,i_{p-1}\le r} a_{i_1}^*a_{i_{\pi_0(1)}}\cdots a_{i_{p-1}}^*a_{i_{\pi_0(p-1)}}. \tag{5.4}$$

Proof. (i) Assume that e is odd. Then $k(\hat{\pi}_0)=k(\hat{\pi})-1$ by Proposition 1.22. Moreover, $(e,e+1)$ is of the form $(2j-1,2j)$ for some j in $\{1,2,\dots,p\}$, and therefore $\pi(j)=j$ (cf. Definition 1.15). Hence, the index i_j occur only at the $2j-1$'th and the $2j$'th factor in the product $a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_p}^*a_{i_{\pi(p)}}$, and therefore the sum on the left hand side of (5.3) is equal to

$$\sum_{1\le i_1,\dots,i_{j-1},i_{j+1},\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_{\pi(j-1)}}\Big(\sum_{i_j=1}^r a_{i_j}^*a_{i_j}\Big)a_{i_{j+1}}^*\cdots a_{i_p}^*a_{i_{\pi(p)}},$$

which by (5.2) is equal to

$$c\cdot\Big(\sum_{1\le i_1,\dots,i_{j-1},i_{j+1},\dots,i_p\le r} a_{i_1}^*a_{i_{\pi(1)}}\cdots a_{i_{\pi(j-1)}}a_{i_{j+1}}^*\cdots a_{i_p}^*a_{i_{\pi(p)}}\Big). \tag{5.5}$$

Note here, that if we relabel the indices $i_{j+1},\dots,i_p$ by $i_j,\dots,i_{p-1}$, then it follows from Remark 1.19(a), that (5.5) is equal to

$$c\cdot\Big(\sum_{1\le i_1,\dots,i_{p-1}\le r} a_{i_1}^*a_{i_{\pi_0(1)}}\cdots a_{i_{p-1}}^*a_{i_{\pi_0(p-1)}}\Big),$$

and this proves (5.3).
(ii) Assume that e is even. Then $k(\hat{\pi}_0) = k(\hat{\pi})$ by Proposition 1.22, and $(e, e+1) = (2j, 2j+1)$, for some j in $\{1, 2, \dots, p-1\}$, so that $\pi(j) = j+1$ (c.f. Definition 1.15). Hence, the left hand side of (5.4) is equal to

$$\sum_{1 \le i_1, \dots, i_j, i_{j+2}, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_j}^* \Big(\sum_{i_{j+1}=1}^{r} a_{i_{j+1}} a_{i_{j+1}}^* \Big) a_{i_{\pi(j+1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}. \tag{5.6}$$

Here, $\sum_{i_{j+1}=1}^r a_{i_{j+1}} a_{i_{j+1}}^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}$, by (5.2), and proceeding then as in the proof of (i), we obtain by Remark 1.19(b) (after relabeling $i_{j+2}, \dots, i_p$ by $i_{j+1}, \dots, i_{p-1}$), that (5.6) is equal to

$$\sum_{1 \le i_1, \dots, i_{p-1} \le r} a_{i_1}^* a_{i_{\pi_0(1)}} \cdots a_{i_{p-1}}^* a_{i_{\pi_0(p-1)}}.$$

This proves (5.4) $\square$

Recall that for p in $\mathbb{N}$, S_p^{nc} denotes the set of permutations π in S_p, for which the permutation $\hat{\pi}$ is non-crossing in the sense of Definition 1.14.

5.2 Lemma. *Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, such that* (5.2) *holds, let p be a positive integer, and let π be a permutation in S_p^{nc}. Then*

$$\sum_{1 \le i_1, \dots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}} = c^{k(\hat{\pi})} \mathbf{1}_{\mathcal{B}(\mathcal{H})}, \tag{5.7}$$

and

$$\sum_{1 \le i_1, \dots, i_p \le r} a_{i_1} a_{i_{\pi(1)}}^* \cdots a_{i_p} a_{i_{\pi(p)}}^* = c^{l(\hat{\pi})-1} \mathbf{1}_{\mathcal{B}(\mathcal{K})}. \tag{5.8}$$

Proof. We start by proving (5.7); proceeding by induction on p. The case $p = 1$ is clear from (5.2). Assume now that $p \ge 2$, and that (5.7) holds for $p-1$ instead of p, and all permutations in S_{p-1}^{nc}. Consider then a permutation π from S_p^{nc}, and recall from Lemma 1.17 that $\hat{\pi}$ has a pair of neighbors $(e, e+1)$. Let π_0 be the permutation in S_{p-1} obtained by cancellation of this pair. Then by Lemma 1.20, $\pi_0 \in S_{p-1}^{\mathrm{nc}}$, and hence by the induction hypothesis,

$$\sum_{1 \le i_1, \dots, i_{p-1} \le r} a_{i_1}^* a_{i_{\pi_0(1)}} \cdots a_{i_{p-1}}^* a_{i_{\pi_0(p-1)}} = c^{k(\hat{\pi}_0)} \mathbf{1}_{\mathcal{B}(\mathcal{H})}. \tag{5.9}$$

But by Lemma 5.1, (5.9) implies (5.7), both when e is odd, and when e is even. This completes the proof of (5.7).
To prove (5.8), we put $b_i = \frac{1}{\sqrt{c}} a_i^*$, $i = 1, 2, \cdots, r$. Then

$$\sum_{i=1}^r b_i^* b_i = c^{-1} \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \quad \text{and} \quad \sum_{i=1}^r b_i b_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{H})}.$$

Applying then (5.7), with c replaced by c^{-1}, it follows that

$$\sum_{1\le i_1,\dots,i_p\le r} b_{i_1}^* b_{i_{\pi(1)}} \cdots b_{i_p}^* b_{i_{\pi(p)}} = c^{-k(\hat{\pi})}\mathbf{1}_{\mathcal{B}(\mathcal{K})},$$

i.e., that

$$\sum_{1\le i_1,\dots,i_p\le r} a_{i_1} a_{i_{\pi(1)}}^* \cdots a_{i_p} a_{i_{\pi(p)}}^* = c^{p-k(\hat{\pi})}\mathbf{1}_{\mathcal{B}(\mathcal{K})}.$$

Recall finally, that since $\hat{\pi}$ is non-crossing, $k(\hat{\pi}) + l(\hat{\pi}) = p + 1$ (cf. Corollary 1.24), and hence it follows that (5.8) holds. $\square$

As in Section 3, for any c in $]0,\infty[$, μ_c denotes the probability measure on $[0,\infty[$, given by

$$\mu_c = \max\{1-c, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx,$$

where $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$ and δ_0 is the Dirac measure at 0. Recall from [OP] or [HT, Remark 6.8], that the moments of μ_c are given by

$$\int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{p}\sum_{j=1}^p \tbinom{p}{j}\tbinom{p}{j-1}c^j, \qquad (p \in \mathbb{N}). \tag{5.10}$$

5.3 LEMMA. *For any positive integer p, we have*

$$\sum_{\pi\in S_p^{\mathrm{nc}}} c^{k(\hat{\pi})} = \tfrac{1}{p}\sum_{j=1}^p \tbinom{p}{j}\tbinom{p}{j-1}c^j, \tag{5.11}$$

and

$$\sum_{\pi\in S_p^{\mathrm{nc}}} c^{l(\hat{\pi})-1} = \tfrac{1}{p}\sum_{j=1}^p \tbinom{p}{j}\tbinom{p}{j-1}c^{j-1}. \tag{5.12}$$

Proof. To prove (5.11), recall from Corollary 1.12, that for B in $\mathrm{GRM}(m,n,1)$, we have that

$$\mathbb{E}\circ \mathrm{Tr}_n\big[(B^*B)^p\big] = \sum_{\pi\in S_p} m^{k(\hat{\pi})} n^{l(\hat{\pi})}.$$

Hence, for Y in $\mathrm{GRM}(m,n,\frac{1}{n})$,

$$\mathbb{E}\circ \mathrm{tr}_n\big[(Y^*Y)^p\big] = n^{-p-1}\sum_{\pi\in S_p} m^{k(\hat{\pi})} n^{l(\hat{\pi})} = \sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}\big(\tfrac{m}{n}\big)^{k(\hat{\pi})}, \tag{5.13}$$

where we have used that $\sigma(\hat{\pi}) = \frac{1}{2}(p+1-k(\hat{\pi})-l(\hat{\pi}))$. Consider now a sequence (m_n) of positive integers, such that $\frac{m_n}{n} \to c$ as $n\to\infty$, and for each n in $\mathbb{N}$, let Y_n be an element of $\mathrm{GRM}(m_n, n, \frac{1}{n})$. It follows then from (5.13), that

$$\lim_{n\to\infty} \mathbb{E}\circ \mathrm{tr}_n\big[(Y^*Y)^p\big] = \sum_{\substack{\pi\in S_p\\ \sigma(\hat{\pi})=0}} c^{k(\hat{\pi})} = \sum_{\pi\in S_p^{\mathrm{nc}}} c^{k(\hat{\pi})}, \tag{5.14}$$

where the last equality follows from Corollary 1.24. On the other hand, it follows from [HT, Theorem 6.7(ii)] and (5.10), that

$$\lim_{n\to\infty} \mathbb{E}\circ \mathrm{tr}_n\big[(Y^*Y)^p\big] = \int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{p}\sum_{j=1}^p \binom{p}{j}\binom{p}{j-1}c^j. \tag{5.15}$$

Combining (5.14) and (5.15), we obtain (5.11).
To prove (5.12), we use, again, that $k(\hat{\pi}) + l(\hat{\pi}) = p+1$ for all π in S_p^{nc}. It follows thus, that

$$\sum_{\pi\in S_p^{\mathrm{nc}}} c^{l(\hat{\pi})-1} = c^p \sum_{\pi\in S_p^{\mathrm{nc}}} c^{-k(\hat{\pi})}. \tag{5.16}$$

But by (5.11) (with c replaced by c^{-1}), the right hand side of (5.16) is equal to

$$\tfrac{1}{p}\sum_{j=1}^p \binom{p}{j}\binom{p}{j-1}c^{p-j}. \tag{5.17}$$

Substituting finally j with $p+1-j$ in (5.17), we obtain (5.12). □

5.4 COROLLARY. *Let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that* (5.2) *holds. Then for any p in $\mathbb{N}$, we have that*

(i) $$\sum_{\pi\in S_p^{\mathrm{nc}}}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}}\Big) = \Big[\tfrac{1}{p}\sum_{j=1}^p \binom{p}{j}\binom{p}{j-1}c^j\Big]\cdot \mathbf{1}_{\mathcal{B}(\mathcal{H})},$$

and

(ii) $$\sum_{\pi\in S_p^{\mathrm{nc}}}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1} a_{i_{\pi(1)}}^*\cdots a_{i_p} a_{i_{\pi(p)}}^*\Big) = \Big[\tfrac{1}{p}\sum_{j=1}^p \binom{p}{j}\binom{p}{j-1}c^{j-1}\Big]\cdot \mathbf{1}_{\mathcal{B}(\mathcal{K})}.$$

Proof. Combine Lemma 5.2 and Lemma 5.3. □

5.5 DEFINITION. (a) A subset I of $\mathbb{Z}$ is called an *interval of integers*, if it is the form
$$I = \{\alpha, \alpha+1,\dots,\beta\},$$
for some α,β in $\mathbb{Z}$, such that $\alpha\le\beta$.
(b) Let p be a positive integer, let π be a permutation in S_p, and let I be an interval of integers, such that $I\subseteq\{1,2,\dots,2p\}$. We say then that the restriction $\hat{\pi}_{|I}$ of $\hat{\pi}$ to I is non-crossing, if $\hat{\pi}(I) = I$, and $\hat{\pi}$ has no crossing (a,b,c,d) where $a,b,c,d\in I$. In this case, we refer to I as a *non-crossing interval of integers* for $\hat{\pi}$. □

5.6 Remark. Let p be a positive integer, let π be a permutation in S_p and let I be an interval of integers, such that $I \subseteq \{1, 2, \dots, 2p\}$ and $\hat{\pi}(I) = I$. Since $\hat{\pi}^2 = \mathrm{id}$ and $\hat{\pi}$ has no fixed points, it follows then, that $\mathrm{card}(I)$ is an even number. Put $t = \frac{1}{2}\mathrm{card}(I)$, and consider the unique order preserving bijection $\varphi\colon \{1, 2, \dots, 2t\} \to I$ of $\{1, 2, \dots, 2t\}$ onto I (i.e., $\varphi(j) = \min(I) - 1 + j$, for all j in $\{1, 2, \dots, 2t\}$). It is clear then, that the mapping $\varphi^{-1} \circ (\hat{\pi}_{|I}) \circ \varphi$ is a permutation of $\{1, 2, \dots, 2t\}$, and that we may choose a (unique) permutation π_1 in S_t, such that

$$\hat{\pi}_1 = \varphi^{-1} \circ (\hat{\pi}_{|I}) \circ \varphi, \tag{5.18}$$

(cf. Remark 1.7(a)). It is clear too, that the restriction $\hat{\pi}_{|I}$ of $\hat{\pi}$ to I is non-crossing in the sense of Definition 5.5, if and only if $\hat{\pi}_1$ is a non-crossing permutation in the usual sense (cf. Definition 1.14). □

5.7 Lemma. *Let p be a positive integer, and let π be a permutation in S_p.*
(i) *If I is an interval of integers such that $I \subseteq \{1, 2, \dots, 2p\}$ and $\hat{\pi}_{|I}$ is non-crossing, then there exists e in I, such that $e + 1 \in I$ and $\hat{\pi}(e) = e + 1$.*
(ii) *If $\pi \in S_p^{\mathrm{irr}}$, then $\hat{\pi}$ has no non-crossing interval of integers.*

Proof. (i) Assume that $I \subseteq \{1, 2, \dots, 2p\}$ and that $\hat{\pi}_{|I}$ is non-crossing. Put $t = \frac{1}{2}\mathrm{card}(I)$, let φ be the order preserving bijection of $\{1, 2, \dots, 2t\}$ onto I, and let π_1 be the permutation in S_t given by (5.18). Then $\pi_1 \in S_t^{\mathrm{nc}}$, and hence $\hat{\pi}_1$ has a pair of neighbors $(e', e' + 1)$ by Lemma 1.17. Putting $e = \varphi(e')$, it follows that $e + 1 = \hat{\pi}(e) \in I$, and this proves (i).
(ii) This follows immediately from (i). □

5.8 Lemma. *Let p be a positive integer, and let π be a permutation in S_p, such that $\hat{\pi}$ is reducible. Consider furthermore a family $(I_\lambda)_{\lambda \in \Lambda}$ of intervals of integers, such that $I_\lambda \subseteq \{1, 2, \dots, 2p\}$ for all λ, and such that the union $I = \cup_{\lambda \in \Lambda} I_\lambda$ is again an interval of integers. If each I_λ is a non-crossing interval of integers for $\hat{\pi}$, then so is I.*

Proof. Assume that each I_λ is a non-crossing interval of integers for $\hat{\pi}$. Then $\hat{\pi}(I_\lambda) = I_\lambda$ for all λ, and hence also $\hat{\pi}(I) = I$. Assume then that I contains a crossing for $\hat{\pi}$, i.e., that there exist a, b, c, d in I, such that $a < b < c < d$ and $\hat{\pi}(a) = c$, $\hat{\pi}(b) = d$. Choose λ in Λ such that $a \in I_\lambda$. Then $c = \hat{\pi}(a) \in I_\lambda$, and since I_λ is an interval of integers, also $b \in I_\lambda$. But then $d = \hat{\pi}(b) \in I_\lambda$ too, and hence (a, b, c, d) is a crossing for $\hat{\pi}$ contained in I_λ; a contradiction. Therefore I too is a non-crossing interval of integers for $\hat{\pi}$. □

5.9 Definition. Let p be a positive integer and let π be a permutation in S_p. By $\mathcal{J}(\hat{\pi})$ we denote then the family of all non-crossing intervals of integers for

$\hat{\pi}$. Moreover, we put

$$\mathrm{NC}(\hat{\pi}) = \bigcup_{I\in\mathcal{J}(\hat{\pi})} I, \tag{5.19}$$

$$\mathrm{IRR}(\hat{\pi}) = \{1,2,\dots,2p\}\setminus \mathrm{NC}(\hat{\pi}). \tag{5.20}$$

We refer to $\mathrm{NC}(\hat{\pi})$ (respectively $\mathrm{IRR}(\hat{\pi})$) as the non-crossing set (respectively irreducible set) for $\hat{\pi}$. □

5.10 Lemma. *Let p be a positive integer and let π be a permutation in S_p. We then have*
(i) $\mathrm{NC}(\hat{\pi}) = \{1,2,\dots,2p\}$ *if and only if $\hat{\pi}$ is non-crossing.*
(ii) $\mathrm{NC}(\hat{\pi}) = \emptyset$ *if and only if $\hat{\pi}$ is irreducible.*

Proof. (i) If $\mathrm{NC}(\hat{\pi}) = \{1,2,\dots,2p\}$, then is follows from Lemma 5.8, that $\hat{\pi}$ is non-crossing. If, conversely, $\hat{\pi}$ is non-crossing, then $\{1,2,\dots,2p\}\in\mathcal{J}(\hat{\pi})$, and hence $\mathrm{NC}(\hat{\pi}) = \{1,2,\dots,2p\}$.
(ii) If $\mathrm{NC}(\hat{\pi}) = \emptyset$, then for any j in $\{1,2,\dots,2p-1\}$, $\{j,j+1\}$ can not be a non-crossing interval of integers for $\hat{\pi}$. Hence $\hat{\pi}(j)\neq j+1$ for all j in $\{1,2,\dots,2p-1\}$, which means that $\hat{\pi}$ is irreducible. If, conversely, $\hat{\pi}$ is irreducible, then $\mathcal{J}(\hat{\pi}) = \emptyset$ by Lemma 5.7(ii), and hence also $\mathrm{NC}(\hat{\pi}) = \emptyset$. □

5.11 Proposition. *Let p be a positive integer, let π be a permutation in S_p, and assume that $\hat{\pi}$ has a crossing. Then the set $\mathrm{IRR}(\hat{\pi})$ is of the form*

$$\mathrm{IRR}(\hat{\pi}) = \{s_1,s_2,\dots,s_{2q}\},$$

where $q\in\{1,\dots,p\}$, and $1\le s_1<s_2<\cdots<s_{2q}\le 2p$. Moreover, $s_1,s_2,\dots,s_{2q}$ have the following properties:
(i) *The set $\{s_1,s_2,\dots,s_{2q}\}$ is $\hat{\pi}$-invariant and $\hat{\pi}(s_i)\neq s_{i+1}$, for all i in $\{1,2,\dots,2q-1\}$.*
(ii) *If we put $s_0=0$ and $s_{2q+1}=2p+1$, then for each i in $\{0,1,\dots,2q\}$, the set*

$$I_i =]s_i,s_{i+1}[\,\cap\,\mathbb{Z}$$

is either the empty set or a non-crossing interval of integers for $\hat{\pi}$.

Proof. By Definition 5.5(b), each I in $\mathcal{J}(\hat{\pi})$ is $\hat{\pi}$-invariant. Therefore $\mathrm{NC}(\hat{\pi})$ is $\hat{\pi}$-invariant too, and hence so is $\mathrm{IRR}(\hat{\pi})$. Since $\hat{\pi}^2=\mathrm{id}$ and $\hat{\pi}$ has no fixed points, it follows that $\mathrm{card}(\mathrm{IRR}(\hat{\pi}))=2q$ for some q in $\{0,1,\dots,p\}$, and since $\hat{\pi}$ has a crossing, Lemma 5.10(i) shows that $q\ge 1$. Thus, we may write $\mathrm{IRR}(\hat{\pi})$ in the form $\{s_1,s_2,\dots,s_{2q}\}$, where $s_1<s_2<\cdots<s_{2q}$, and it remains to show that these $s_1,s_2,\dots,s_{2q}$ satisfy (i) and (ii).
We start by proving (ii). For all I from $\mathcal{J}(\hat{\pi})$, $I\cap\{s_1,s_2,\dots,s_{2q}\}=\emptyset$, and hence each such I is contained in one of the sets $I_i =]s_i,s_{i+1}[\,\cap\,\mathbb{Z}$,

$i = 0, 1, \dots, 2q$. Therefore

$$\mathcal{J}(\hat{\pi}) = \bigcup_{i=0}^{2q} \mathcal{J}_i(\hat{\pi}), \tag{5.21}$$

where $\mathcal{J}_i(\hat{\pi}) = \{I \in \mathcal{J}(\hat{\pi}) \mid I \subseteq I_i\}$, for all i in $\{0, 1, \dots, 2q\}$. Note here that

$$\bigcup_{I \in \mathcal{J}_i(\hat{\pi})} I \subseteq I_i, \qquad (i \in \{0, 1, \dots, 2q\}), \tag{5.22}$$

and that

$$\bigcup_{I \in \mathcal{J}(\hat{\pi})} I = \mathrm{NC}(\hat{\pi}) = \{1, 2, \dots, 2p\} \setminus \mathrm{IRR}(\hat{\pi}) = \bigcup_{i=0}^{2q} I_i. \tag{5.23}$$

Combining (5.21)-(5.23), it follows that we actually have equality in (5.22), i.e.,

$$\bigcup_{I \in \mathcal{J}_i(\hat{\pi})} I = I_i, \qquad (i \in \{0, 1, \dots, 2q\}). \tag{5.24}$$

Since each I_i is either empty or an interval of integers, (ii) follows now by combining (5.24) with Lemma 5.8.

It remains to prove (i). We already noted (and used) that $\mathrm{IRR}(\hat{\pi})$ is $\hat{\pi}$-invariant. Assume then that $\hat{\pi}(s_i) = s_{i+1}$ for some i in $\{1, \dots, 2q-1\}$. Then, by (ii), the set

$$\tilde{I}_i = \{s_i\} \cup I_i \cup \{s_{i+1}\},$$

is a non-crossing interval of integers for $\hat{\pi}$. But this contradicts that $s_i \notin \mathrm{NC}(\hat{\pi})$, and hence we have proved (i). $\square$

We prove next the following converse of Proposition 5.11.

5.12 Proposition. *Let p be a positive integer, let π be a permutation in S_p, and assume that there exist q in $\{1, \dots, p\}$ and $s_1 < s_2 < \cdots < s_{2q}$ in $\{1, 2, \dots, 2p\}$, such that*

(i) *The set $\{s_1, s_2, \dots, s_{2q}\}$ is $\hat{\pi}$-invariant and $\hat{\pi}(s_i) \neq s_{i+1}$, for all i in $\{1, 2, \dots, 2q-1\}$.*

(ii) *If we put $s_0 = 0$ and $s_{2q+1} = 2p+1$, then for each i in $\{0, 1, \dots, 2q\}$, the set $I_i = \,]s_i, s_{i+1}[\, \cap \mathbb{Z}$ is either the empty set or a non-crossing interval of integers for $\hat{\pi}$.*

Then $\{s_1, s_2, \dots, s_{2q}\} = \mathrm{IRR}(\hat{\pi})$.

Proof. It follows from (i), that there exists a (unique) permutation γ in S_{2q}, such that

$$\hat{\pi}(s_i) = s_{\gamma(i)}, \qquad (i \in \{1, 2, \dots, 2q\}),$$

and moreover

$$\gamma(i) \neq i+1, \qquad (i \in \{1,2,\dots,2q-1\}). \tag{5.25}$$

Our first objective is to prove that γ is of the form $\hat{\rho}$ for some (unique) permutation ρ in S_q^{irr}. For this, note first that by (ii), $\mathrm{card}(I_i)$ is an even number for all i in $\{0,1,\dots,2q\}$. Hence $s_{i+1}-s_i$ is odd for all i in $\{0,1,\dots,2q\}$, and this implies that

$$\begin{aligned} s_1, s_3, \dots, s_{2q-1} &\quad \text{are odd numbers} \\ s_2, s_4, \dots, s_{2q} &\quad \text{are even numbers} \end{aligned}$$

Since $\hat{\pi}^2 = \mathrm{id}$ and $\hat{\pi}(j)-j$ is odd for all j in $\{1,2,\dots,2p\}$, it follows now that $\gamma^2 = \mathrm{id}$ and that $\gamma(i)-i$ is odd for all i in $\{1,2,\dots,2q\}$. Therefore, by Remark 1.7(a), $\gamma = \hat{\rho}$ for some (unique) ρ in S_q, and (5.25) shows that in fact $\rho \in S_q^{\mathrm{irr}}$.

Returning now to the proof of the equation $\{s_1,s_2,\dots,s_{2q}\} = \mathrm{IRR}(\hat{\pi})$, note first that $\cup_{i=0}^{2q} I_i \subseteq \mathrm{NC}(\hat{\pi})$, and therefore

$$\{s_1,s_2,\dots,s_{2q}\} = \{1,2,\dots,2p\} \setminus \cup_{i=0}^{2q} I_i \supseteq \mathrm{IRR}(\hat{\pi}).$$

Suppose then that $\mathrm{IRR}(\hat{\pi})$ is a proper subset of $\{s_1,s_2,\dots,s_{2q}\}$. Then there exists j_0 in $\{1,2,\dots,2q\}$, such that $s_{j_0} \in \mathrm{NC}(\hat{\pi})$, i.e., $s_{j_0} \in I$, for some non-crossing interval of integers for $\hat{\pi}$. For this I, define

$$J = \big\{ j \in \{1,2,\dots,2q\} \bigm| s_j \in I \big\}.$$

Then $J \neq \emptyset$, and since $s_1 < s_2 < \cdots < s_{2q}$, J is an interval of integers. Consider now the permutation ρ in S_q^{irr}, introduced above. Then, since $\hat{\pi}(I) = I$, we have also that $\hat{\rho}(J) = J$. Moreover, J is a non-crossing interval of integers for $\hat{\rho}$. Indeed, if (a,b,c,d) were a crossing for $\hat{\rho}$ contained in J, then clearly (s_a,s_b,s_c,s_d) would be a crossing for $\hat{\pi}$ contained in I, which is impossible. Altogether, ρ is both irreducible and has a non-crossing interval of integers, and by Lemma 5.10(ii), this is impossible. Thus, we have reached a contradiction, which means that we must also have the inclusion $\{s_1,s_2,\dots,s_{2q}\} \subseteq \mathrm{IRR}(\hat{\pi})$. $\square$

5.13 Lemma. *Let p be a positive integer, and let π be a permutation in $S_p \setminus S_p^{\mathrm{nc}}$. Write then, as in Proposition 5.11, $\mathrm{IRR}(\hat{\pi})$ in the form*

$$\mathrm{IRR}(\hat{\pi}) = \{s_1,s_2,\dots,s_{2q}\},$$

where $q \in \{1,\dots,p\}$ and $1 \le s_1 < s_2 < \cdots < s_{2q} \le 2p$. Then $s_1,s_2,\dots,s_{2q}$ satisfy, in addition, that

(i) *$s_1, s_3, \dots, s_{2q-1}$ are odd numbers.*

(ii) *$s_2, s_4, \dots, s_{2q}$ are even numbers.*

(iii) *There is one and only one permutation ρ in S_q^{irr}, such that $\hat{\pi}(s_j) = s_{\hat{\rho}(j)}$ for all j in $\{1,2,\dots,2q\}$.*

Proof. This follows immediately from Proposition 5.11 and the first part of the proof of Proposition 5.12. $\square$

5.14 Definition. Let p be a positive integer, let π be a permutation in $S_p \backslash S_p^{\mathrm{nc}}$, and let q, $s_1, s_2, \ldots, s_{2q}$ and $I_0, I_1, \ldots, I_{2q}$, be as in Proposition 5.11. Then put

$$t_i = \tfrac{1}{2}\mathrm{card}(I_i), \qquad (i \in \{0, 1, \ldots, 2q\}),$$

and note that since I_i is either empty or a non-crossing interval of integers for $\hat{\pi}$, $t_i \in \mathbb{N}_0$ for all i. If $t_i > 0$, then as in Remark 5.6, we consider the order-preserving bijection φ_i of $\{1, 2, \ldots, 2t_i\}$ onto I_i, and we let π_i denote the (unique) permutation in S_{t_i}, satisfying that $\hat{\pi}_i = \varphi_i^{-1} \circ (\hat{\pi}_{|I_i}) \circ \varphi$. Clearly $\pi_i \in S_p^{\mathrm{nc}}$.
It is convenient to consider the permutation group S_0 of the empty set, as a group with one element $\pi_\emptyset$. Then, in the setting considered above, we put $\pi_i = \pi_\emptyset$, for all i in $\{0, 1, \ldots, 2q\}$, for which $t_i = 0$. By convention, we put

$$k(\hat{\pi}_\emptyset) = 0, \qquad \text{and} \qquad l(\hat{\pi}_\emptyset) = 1. \tag{5.26}$$

$\square$

5.15 Lemma. *Let p be a positive integer, let π be a permutation in $S_p \setminus S_p^{\mathrm{nc}}$, and let ρ be the irreducible permutation introduced in Lemma 5.13(iii). Then $\sigma(\hat{\rho}) = \sigma(\hat{\pi})$.*

Proof. Let q, $s_1, s_2, \ldots, s_{2q}$ and $I_0, I_1, \ldots, I_{2q}$, be as in Proposition 5.11, and for each i in $\{0, 1, \ldots, 2q\}$, let t_i and π_i be as in Definition 5.14. If $t_i > 0$, then $\hat{\pi}_i$ is non-crossing, and hence, by Proposition 1.23, $\hat{\pi}_i$ may be reduced to $\hat{e}_1$ (where e_1 is the permutation in S_1), by a series of successive cancellations of pairs. Here $\hat{e}_1$ consists exactly of one pair of neighbors, so, formally speaking, $\hat{e}_1$ can be reduced $\hat{\pi}_\emptyset$, by cancellation of this pair. Thus, $\hat{\pi}_i$ can be reduced to $\hat{\pi}_\emptyset$, by a series of successive cancellations of pairs, and forming the corresponding series of cancellations of pairs to $\hat{\pi}_{|I_i}$, it follows that $\hat{\pi}$ can be reduced to a permutation, which is, loosely speaking, obtained by "cutting out" $\hat{\pi}_{|I_i}$ from $\hat{\pi}$. Forming these reductions for each i in $\{0, 1, \ldots, 2q\}$, for which $t_i > 0$, it follows that $\hat{\pi}$ can be reduced to $\hat{\rho}$ by a series of successive cancellations of pairs. By Proposition 1.22, this implies that $\sigma(\hat{\pi}) = \sigma(\hat{\rho})$. $\square$

5.16 Proposition. *Let p be a positive integer, let π be a permutation in $S_p \setminus S_p^{\mathrm{nc}}$, and let q, $s_1, s_2, \ldots, s_{2q}$ be as in Proposition 5.11. Let further ρ be the permutation in S_q^{irr} introduced in Lemma 5.13(iii), and let $\pi_0, \pi_1, \ldots, \pi_{2q}$ be as in Definition 5.14. Then for any elements $a_1, \ldots, a_r$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ for which* (5.2) *holds, we have*

$$\sum_{1 \le i_1, \ldots, i_p \le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}} = c^{h(\hat{\pi})} \sum_{1 \le i_1, \ldots, i_q \le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}}, \tag{5.27}$$

where

$$h(\hat{\pi}) = k(\hat{\pi}_0) + (l(\hat{\pi}_1) - 1) + k(\hat{\pi}_2) + \cdots + (l(\hat{\pi}_{2q-1}) - 1) + k(\hat{\pi}_{2q}). \tag{5.28}$$

Proof. We start by introducing some notation. Let t be a positive integer, and let η be a permutation in S_t. We then put

$$\Gamma(\hat{\eta}) = \sum_{1 \le i_1, \dots, i_t \le r} a_{i_1}^* a_{i_{\eta(1)}} \cdots a_{i_t}^* a_{i_{\eta(t)}}, \tag{5.29}$$

and moreover, we put

$$\Gamma(\hat{\pi}_\emptyset) = \mathbf{1}_{\mathcal{B}(\mathcal{H})}. \tag{5.30}$$

Note that $\Gamma(\hat{\eta})$ can be expressed in terms of $\hat{\eta}$ only, namely as

$$\Gamma(\hat{\eta}) = \sum_{(i_1, i_2, i_3, i_4, \dots, i_{2t}) \in N(\hat{\eta})} a_{i_1}^* a_{i_2} a_{i_3}^* a_{i_4} \cdots a_{i_{2t-1}}^* a_{i_{2t}}, \tag{5.31}$$

where

$$N(\hat{\eta}) = \big\{(i_1, i_2, \dots, i_{2t}) \in \{1, 2, \dots, r\}^{2t} \bigm| i_j = i_{\hat{\eta}(j)}, \text{ for all } j \text{ in } \{1, 2, \dots, 2t\}\big\}, \tag{5.32}$$

(cf. Remark 1.7(b)). Consider next an interval of integers I, such that $I \subseteq \{1, 2, \dots, 2t\}$ and $\hat{\eta}(I) = I$. Write I in the form $\{\alpha, \alpha+1, \dots, \beta\}$, and note that $\beta - \alpha + 1 = \mathrm{card}(I)$ is an even number. We then put

$$N(\hat{\eta}, I) = \big\{(i_\alpha, \dots, i_\beta) \in \{1, 2, \dots, r\}^{\beta-\alpha+1} \bigm| i_j = i_{\hat{\eta}(j)},\ j = \alpha, \alpha+1, \dots, \beta\big\} \tag{5.33}$$

and

$$\Gamma(\hat{\eta}, I) = \begin{cases} \displaystyle\sum_{(i_\alpha, \dots, i_\beta) \in N(\hat{\eta}, I)} a_{i_\alpha}^* a_{i_{\alpha+1}} \cdots a_{i_{\beta-1}}^* a_{i_\beta}, & \text{if } \alpha \text{ is odd,} \\ \displaystyle\sum_{(i_\alpha, \dots, i_\beta) \in N(\hat{\eta}, I)} a_{i_\alpha} a_{i_{\alpha+1}}^* \cdots a_{i_{\beta-1}} a_{i_\beta}^*, & \text{if } \alpha \text{ is even.} \end{cases} \tag{5.34}$$

Now, to prove (5.27), consider p in $\mathbb{N}$ and π in $S_p \setminus S_p^{\mathrm{nc}}$, and let q, $s_1, s_2, \dots, s_{2q}$ and $I_0, I_1, \dots, I_{2q}$, $t_0, t_1, \dots, t_{2q}$ be as in Proposition 5.11. Note then, that we may write $N(\hat{\pi})$ as

$$N(\hat{\pi}) = \bigcup_{(i_{s_1}, \dots, i_{s_{2q}}) \in N_1(\hat{\pi})} N(\hat{\pi}, I_0) \times \{i_{s_1}\} \times N(\hat{\pi}, I_1) \times \{i_{s_2}\} \times \cdots \times \{i_{s_{2q}}\} \times N(\hat{\pi}, I_{2p}), \tag{5.35}$$

with the convention that $N(\hat{\pi}, I_i)$ is omitted in the product sets when $2t_i = \mathrm{card}(I_i) = 0$, and where

$$N_1(\hat{\pi}) = \big\{(i_{s_1}, \dots, i_{s_{2q}}) \in \{1,2,\dots,r\}^{2q} \;\big|\; i_{s_j} = i_{\hat{\pi}(s_j)},\ j = 1,2,\dots,2q\big\}. \tag{5.36}$$

It follows thus, by (5.31), that

$$\Gamma(\hat{\pi}) = \sum_{(i_{s_1},\dots,i_{s_{2q}}) \in N_1(\hat{\pi})} \Gamma(\hat{\pi}, I_0) a^*_{i_{s_1}} \Gamma(\hat{\pi}, I_1) a_{i_{s_2}} \cdots a_{i_{s_{2q}}} \Gamma(\hat{\pi}, I_{2q}), \tag{5.37}$$

with the convention that if $\mathrm{card}(I_i) = 0$,

$$\Gamma(\hat{\pi}, I_i) = \begin{cases} \mathbf{1}_{\mathcal{B}(\mathcal{H})}, & \text{if } s_i \text{ is even,} \\ \mathbf{1}_{\mathcal{B}(\mathcal{K})}, & \text{if } s_i \text{ is odd.} \end{cases} \tag{5.38}$$

To calculate $\Gamma(\hat{\pi}, I_0), \dots, \Gamma(\hat{\pi}, I_{2q})$, consider the non-crossing permutations $\pi_0, \pi_1, \dots, \pi_{2q}$ introduced in Definition 5.14. Note then, that for each v in $\{0,1,\dots,2q\}$, such that $t_v > 0$, we have by a suitable relabeling of indices,

$$\begin{aligned} N(\hat{\pi}, I_v) &= \big\{(i_1, i_2, \dots, i_{2t_v}) \in \{1,2,\dots,r\}^{2t_v} \;\big|\; i_j = i_{\hat{\pi}_v(j)},\ j = 1,2,\dots,2t_v\big\} \\ &= N(\hat{\pi}_v). \end{aligned}$$

It follows thus, that if $t_v > 0$,

$$\Gamma(\hat{\pi}, I_v) = \begin{cases} \displaystyle\sum_{1 \le i_1,\dots,i_{t_v} \le r} a^*_{i_1} a_{i_{\pi_v(1)}} \cdots a^*_{i_{t_v}} a_{i_{\pi_v(t_v)}}, & \text{if } v \text{ is even,} \\ \displaystyle\sum_{1 \le i_1,\dots,i_{t_v} \le r} a_{i_1} a^*_{i_{\pi_v(1)}} \cdots a_{i_{t_v}} a^*_{i_{\pi_v(t_v)}}, & \text{if } v \text{ is odd,} \end{cases}$$

and hence by Lemma 5.2 (since $\hat{\pi}_v$ is non-crossing),

$$\Gamma(\hat{\pi}, I_v) = \begin{cases} c^{k(\hat{\pi}_v)} \mathbf{1}_{\mathcal{B}(\mathcal{H})}, & \text{if } v \text{ is even,} \\ c^{l(\hat{\pi}_v)-1} \mathbf{1}_{\mathcal{B}(\mathcal{K})}, & \text{if } v \text{ is odd.} \end{cases} \tag{5.39}$$

If $t_v = 0$, then by definition,

$$\Gamma(\hat{\pi}, I_v) = \begin{cases} \mathbf{1}_{\mathcal{B}(\mathcal{H})}, & \text{if } v \text{ is even,} \\ \mathbf{1}_{\mathcal{B}(\mathcal{K})}, & \text{if } v \text{ is odd,} \end{cases} = \begin{cases} c^{k(\hat{\pi}_v)} \mathbf{1}_{\mathcal{B}(\mathcal{H})}, & \text{if } v \text{ is even,} \\ c^{l(\hat{\pi}_v)-1} \mathbf{1}_{\mathcal{B}(\mathcal{K})}, & \text{if } v \text{ is odd,} \end{cases} \tag{5.40}$$

with $k(\hat{\pi}_\emptyset), l(\hat{\pi}_\emptyset)$ as defined in (5.26). Combining (5.37),(5.39) and (5.40), it follows that with $h(\hat{\pi})$ given in (5.28), we have

$$\Gamma(\hat{\pi}) = c^{h(\hat{\pi})} \sum_{(i_{s_1},\dots,i_{s_{2q}}) \in N_1(\hat{\pi})} a^*_{i_{s_1}} a_{i_{s_2}} \cdots a^*_{i_{2q-1}} a_{i_{s_{2q}}}. \tag{5.41}$$

Note finally, that with ρ the permutation introduced in Lemma 5.13(iii), we have that

$$\begin{aligned} N_1(\hat{\pi}) &= \big\{(i_1, i_2, \dots, i_{2q}) \in \{1, 2, \dots, r\}^{2q} \;\big|\; i_j = i_{\hat{\rho}(j)},\ j = 1, 2, \dots, 2q\big\} \\ &= N(\hat{\rho}), \end{aligned}$$

and therefore

$$\sum_{(i_{s_1}, \dots, i_{s_{2q}}) \in N_1(\hat{\pi})} a_{i_{s_1}}^* a_{i_{s_2}} \cdots a_{i_{2q-1}}^* a_{i_{s_{2q}}} = \sum_{1 \le i_1, \dots, i_q \le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}}.$$

Inserting this in (5.41), we obtain (5.27). $\square$

5.17 Definition. Let c be a positive number. Then for any p in $\mathbb{N}_0$, we define

$$g_c(p) = \begin{cases} \frac{1}{p} \sum_{j=1}^{p} \binom{p}{j} \binom{p}{j-1} c^j, & \text{if } p \in \mathbb{N}, \\ 1, & \text{if } p = 0, \end{cases} \tag{5.42}$$

and

$$h_c(p) = \begin{cases} \frac{1}{p} \sum_{j=1}^{p} \binom{p}{j} \binom{p}{j-1} c^{j-1}, & \text{if } p \in \mathbb{N}, \\ 1, & \text{if } p = 0. \end{cases} \tag{5.43}$$

Moreover, for p, q in $\mathbb{N}_0$, such that $p \ge q$, we put

$$\nu'(c, p, q) = \sum_{\substack{r_0, r_1, \dots, r_{2q} \ge 0 \\ r_0 + r_1 + \cdots + r_{2q} = p - q}} g_c(r_0) h_c(r_1) g_c(r_2) h_c(r_3) \cdots g_c(r_{2q}). \tag{5.44}$$

$\square$

We are now ready to prove the main result of this section.

5.18 Theorem. *Let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, let c be a positive number, and assume that $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, and $\sum_{i=1}^r a_i a_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}$. Consider furthermore independent elements $Y_1, \dots, Y_r$ of $\mathrm{GRM}(n, n, \frac{1}{n})$, and put $S = \sum_{i=1}^r a_i \otimes Y_i$. Then for any positive integer p,*

$$\begin{aligned} &\mathbb{E}\big[(S^*S)^p\big] \\ &= \Big[\nu'(c, p, 0)\mathbf{1}_{\mathcal{B}(\mathcal{H})} \\ &+ \sum_{q=1}^{p} \nu'(c, p, q) \Big(\sum_{\rho \in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})} \sum_{1 \le i_1, \dots, i_q \le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}} \Big) \Big] \otimes \mathbf{1}_n. \end{aligned} \tag{5.45}$$

Proof. Let p from $\mathbb{N}$ be given. Then for each q in $\{1,2,\dots,p\}$, we define

$$S_{p,q} = \big\{\pi \in S_p \;\big|\; \mathrm{card}(\mathrm{IRR}(\hat{\pi})) = 2q\big\}, \tag{5.46}$$

and

$$M_q = \sum_{\pi \in S_{p,q}} n^{-2\sigma(\hat{\pi})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big) \tag{5.47}$$

It follows then by (5.1), that

$$\begin{aligned}\mathbb{E}\big[(S^*S)^p\big] &= \Big[\sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big)\Big] \otimes \mathbf{1}_n \\ &= \sum_{q=0}^{p} M_q \otimes \mathbf{1}_n.\end{aligned} \tag{5.48}$$

By Lemma 5.10, $S_{p,0} = S_p^{\mathrm{nc}}$ and $S_{p,p} = S_p^{\mathrm{irr}}$. Hence

$$M_p = \sum_{\pi \in S_p^{\mathrm{irr}}} n^{-2\sigma(\hat{\pi})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big), \tag{5.49}$$

and by Corollary 5.4(i) and Corollary 1.24,

$$M_0 = g_c(p)\mathbf{1}_{\mathcal{B}(\mathcal{H})} = \nu'(c,p,0)\mathbf{1}_{\mathcal{B}(\mathcal{H})}. \tag{5.50}$$

To calculate $M_1, M_2 \dots, M_{p-1}$, we let, for each π in S_p, $\rho(\pi)$ denote the irreducible permutation ρ associated to π in Lemma 5.13(iii). Then for any q in $\{1,2,\dots,p-1\}$ and any ρ in S_q^{irr}, we define

$$R(p,\rho) = \{\pi \in S_{p,q} \mid \rho(\pi) = \rho\}.$$

Then we have the following disjoint union

$$S_{p,q} = \dot{\bigcup_{\rho \in S_q^{\mathrm{irr}}}} R(p,\rho),$$

and therefore

$$M_q = \sum_{\rho \in S_q^{\mathrm{irr}}} \sum_{\pi \in R(p,\rho)} n^{-2\sigma(\hat{\pi})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big). \tag{5.51}$$

Note here, that for any ρ in S_q^{irr}, we have by Proposition 5.16 and Lemma 5.15,

$$\begin{aligned}&\sum_{\pi \in R(p,\rho)} n^{-2\sigma(\hat{\pi})}\Big(\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}}\Big) \\ &\qquad = \Big(\sum_{\pi \in R(p,\rho)} c^{h(\hat{\pi})}\Big) n^{-2\sigma(\hat{\rho})}\Big(\sum_{1\le i_1,\dots,i_q\le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}}\Big),\end{aligned} \tag{5.52}$$

where for each π in $R(p,\rho)$,

$$h(\hat{\pi}) = k(\hat{\pi}_0) + (l(\hat{\pi}_1) - 1) + k(\hat{\pi}_2) + \cdots + (l(\hat{\pi}_{2q-1}) - 1) + k(\hat{\pi}_{2q}),$$

and where $\pi_0, \pi_1, \dots, \pi_{2q}$ are the permutations introduced in Definition 5.14. For any ρ in S_q^{irr} and any π in $R(p,\rho)$, it follows from Proposition 5.11 and Lemma 5.13, that $\hat{\pi}$ can be obtained from $\hat{\rho}$ in a unique way, by "stuffing in" the intervals (or empty sets) $I_0, I_1, \dots, I_{2q}$, and the corresponding non-crossing permutations $\hat{\pi}_0, \hat{\pi}_1, \dots, \hat{\pi}_{2q}$. Conversely, if $\pi \in S_p$ such that $\hat{\pi}$ can be obtained from $\hat{\rho}$ by "stuffing in" intervals (or empty sets) $J_0, J_1, \dots, J_{2q}$ and corresponding non-crossing permutations $\hat{\eta}_0, \hat{\eta}_1, \dots, \hat{\eta}_{2q}$, then, by Proposition 5.12, $\pi \in R(p,\rho)$ and $J_j = I_j$, $\eta_j = \pi_j$, for all j in $\{0, 1, \dots, 2q\}$. It follows thus, that the mapping

$$\pi \mapsto (\pi_0, \pi_1, \dots, \pi_{2q})$$

is a bijection of $R(p,\rho)$ onto the set of $(2q+1)$-tuples $(\pi_0, \pi_1, \dots, \pi_{2q})$ of permutations for which there exist $t_0, t_1, \dots, t_{2q}$ in $\mathbb{N}_0$, such that $\pi_i \in S_{t_i}^{\text{nc}}$ for all i, and $\sum_{i=0}^{2q} t_i = p - q$ (here we have used the convention that $S_0^{\text{nc}} = S_0 = \{\pi_\emptyset\}$). Using this description of $R(p,\rho)$, it follows that

$$\sum_{\pi \in R(p,\rho)} c^{h(\hat{\pi})} = \sum_{\substack{t_0, \dots, t_{2q} \geq 0 \\ t_0 + \cdots + t_{2q} = p-q}} \; \sum_{\pi_0 \in S_{t_0}^{\text{nc}}, \dots, \pi_{2q} \in S_{t_{2q}}^{\text{nc}}} c^{k(\hat{\pi}_0)} c^{(l(\hat{\pi}_1)-1)} c^{k(\hat{\pi}_2)} \cdots c^{k(\hat{\pi}_{2q})}. \tag{5.53}$$

Recall here from Definition 5.17 and Lemma 5.3, that for any t in $\mathbb{N}$,

$$\sum_{\eta \in S_t^{\text{nc}}} c^{k(\hat{\eta})} = g_c(t), \qquad \text{and} \qquad \sum_{\eta \in S_t^{\text{nc}}} c^{l(\hat{\eta})-1} = h_c(t),$$

and by (5.26) this formula holds for $t = 0$ too. Using this in (5.53), it follows that

$$\begin{aligned} \sum_{\pi \in R(p,\rho)} c^{h(\hat{\pi})} &= \sum_{\substack{t_0, t_1, \dots, t_{2q} \geq 0 \\ t_0 + t_1 + \cdots + t_{2q} = p-q}} g_c(t_0) h_c(t_1) g_c(t_2) h_c(t_3) \cdots g_c(t_{2q}) \\ &= \nu'(c,p,q). \end{aligned} \tag{5.54}$$

Note, in particular, that the right hand side depends only on p and q, and not on ρ itself. Combining (5.51),(5.52) and (5.54), it follows that for any q in $\{1, 2, \dots, p-1\}$,

$$M_q = \nu'(c,p,q) \sum_{\rho \in S_q^{\text{irr}}} n^{-2\sigma(\hat{\rho})} \Big(\sum_{1 \leq i_1, \dots, i_q \leq r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}} \Big). \tag{5.55}$$

Since $\nu'(c,p,p) = 1$, (5.55) holds for $q = p$ too, by (5.49), and combining this with (5.48) and (5.50), we obtain, finally, (5.45). $\square$

5.19 Proposition. *Let* $a_1, \dots, a_r$ *in* $\mathcal{B}(\mathcal{H},\mathcal{K})$, c *in* $]0,\infty[$ *and* $S = \sum_{i=1}^r a_i \otimes Y_i$, *be as in Theorem 5.18. Then for any* p *in* $\mathbb{N}$, *we have that*

$$\sum_{\pi \in S_p} n^{-2\sigma(\hat{\pi})} \Big\| \sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}} \cdots a_{i_p}^* a_{i_{\pi(p)}} \Big\|$$
$$= \nu'(c,p,0) + \sum_{q=1}^{p} \nu'(c,p,q) \sum_{\rho\in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})} \Big\| \sum_{1\le i_1,\dots,i_q\le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}} \Big\|.$$

Proof. This follows by exactly the same proof as for Theorem 5.18. □

5.20 Example. Let $a_1, \dots, a_r$ in $\mathcal{B}(\mathcal{H},\mathcal{K})$ and c from $]0,\infty[$ be as in Theorem 5.18.
(a) For $p=1$ or $p=2$, we have $S_p = S_p^{\mathrm{nc}}$. Hence by (5.1), Corollary 1.24 and Corollary 5.4(i), we get that

$$\mathbb{E}\big[S^*S\big] = c\mathbf{1}_{\mathcal{B}(\mathcal{H})\otimes M_n(\mathbb{C})}, \qquad \text{and} \qquad \mathbb{E}\big[(S^*S)^2\big] = (c^2+c)\mathbf{1}_{\mathcal{B}(\mathcal{H})\otimes M_n(\mathbb{C})}.$$

This can also easily be obtained directly from (5.1) and (5.2).
(b) For $p=3$, $\mathrm{card}(S_3)=6$ and $\mathrm{card}(S_3^{\mathrm{nc}})=5$. The only element of $S_3 \setminus S_3^{\mathrm{nc}}$ is the irreducible permutation π given by

$$\pi(1)=3,\ \pi(2)=1,\ \pi(3)=2.$$

For this π, $\sigma(\hat{\pi})=1$, and it follows then by (5.1) and Corollary 5.4(i), that

$$\mathbb{E}\big[(S^*S)^3\big] = (c^3+3c^2+c)\mathbf{1}_{\mathcal{B}(\mathcal{H})\otimes M_n(\mathbb{C})} + \Big(n^{-2}\sum_{i,j,k=1}^{r} a_i^* a_k a_j^* a_i a_k^* a_j\Big)\otimes \mathbf{1}_n.$$

This follows also from Theorem 5.18, because $S_1^{\mathrm{irr}} = S_2^{\mathrm{irr}} = \emptyset$ and $S_3^{\mathrm{irr}} = \{\pi\}$.
□

6 The Sequence of Orthogonal Polynomials for the Measure μ_c

Throughout this section we consider a fixed positive constant c, and elements $a_1,\dots,a_r$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$, satisfying that

$$\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \sum_{i=1}^r a_i a_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{H})}.$$

Moreover, we put

$$S = \sum_{i=1}^r a_i \otimes Y_i,$$

where $Y_1,\dots,Y_r$ are independent elements of $\mathrm{GRM}(n,n,\frac{1}{n})$.

As in Section 3, we let μ_c denote the probability measure on $\mathbb{R}$, given by

$$\mu_c = \max\{1-c, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx,$$

where $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$.
The asymptotic upper bound for the spectrum of S^*S obtained in Section 4 (in the exact case), was obtained by making careful estimates of the moments $\mathbb{E}\big[(S^*S)^p\big]$, $p \in \mathbb{N}$. However, these estimates cannot be used to give good asymptotic lower bounds for the spectrum of S^*S in the case $c > 1$. To obtain such lower bounds, we shall instead consider the operators $\mathbb{E}\big[P_q^c(S^*S)\big]$, where $(P_q^c)_{q\in\mathbb{N}_0}$ is the sequence of *monic* polynomials, obtained by Gram-Schmidt orthogonalization of the polynomials $1, x, x^2, \ldots$, w.r.t. the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)}\; d\mu_c(x), \qquad (f, g \in L^2(\mathbb{R}, \mu_c)).$$

The main result of this section is the equation

$$\mathbb{E}\big[P_q^c(S^*S)\big] = \Big[\sum_{\rho\in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})}\Big(\sum_{1\le i_1,\ldots,i_q\le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}}\Big)\Big]\otimes \mathbf{1}_n, \ (q \in \mathbb{N}),$$

where S_q^{irr} is the set of permutations ρ in S_q, satisfying that

$$1 \neq \rho(1) \neq 2 \neq \rho(2) \neq \cdots \neq \rho(q)$$

(cf. Definition 1.16).

6.1 Proposition. *Let $(P_q^c)_{q\in\mathbb{N}_0}$ be the sequence of polynomials on $\mathbb{R}$, defined by the recursion formulas:*

$$\begin{aligned} P_0^c(x) &= 1, && (6.1)\\ P_1^c(x) &= x - c, && (6.2)\\ P_{q+1}^c(x) &= (x-c-1)P_q^c(x) - cP_{q-1}^c(x), \qquad (q \ge 1). && (6.3) \end{aligned}$$

We then have
(i) *For each q in $\mathbb{N}_0$, $P_q^c(x)$ is a monic polynomial of degree q, and $P_q^c(x) \in \mathbb{R}$ for all real numbers x.*

(ii) $$P_q^c(c+1+2\sqrt{c}\cos\theta) = \frac{c^{\frac{q}{2}}\sin((q+1)\theta) + c^{\frac{q-1}{2}}\sin(q\theta)}{\sin\theta}, \quad (\theta \in \,]0,\pi[\,).$$

(iii) $$\int_a^b P_q^c(x)P_{q'}^c(x)\; d\mu_c(x) = \begin{cases} c^q, & \textit{if} \quad q = q',\\ 0, & \textit{if} \quad q \neq q',\end{cases} \qquad (q, q' \in \mathbb{N}_0).$$

In particular, $(P_q^c)_{q\in\mathbb{N}_0}$ is the sequence of monic orthogonal polynomials obtained by Gram-Schmidt orthogonalization of $1, x, x^2, \ldots$, in the Hilbert space $L^2(\mathbb{R}, \mu_c)$.

Proof. (i) This is clear from (6.1)-(6.3).
(ii) Consider the sequences $(R_q^c)_{q\in\mathbb{N}_0}$ and $(T_q^c)_{q\in\mathbb{N}_0}$ of polynomials, given by the recursion formulas

$$\begin{aligned} R_0^c(x) &= 1, && (6.4)\\ R_1^c(x) &= x-c-1, && (6.5)\\ R_{q+1}^c(x) &= (x-c-1)R_q^c(x) - cR_{q-1}^c(x), \qquad (q\geq 1), && (6.6)\end{aligned}$$

respectively

$$\begin{aligned} T_0^c(x) &= 0, && (6.7)\\ T_1^c(x) &= 1, && (6.8)\\ T_{q+1}^c(x) &= (x-c-1)T_q^c(x) - cT_{q-1}^c(x), \qquad (q\geq 1). && (6.9)\end{aligned}$$

Note here, that the conditions (6.6) and (6.9) are the same, and therefore, the sequence $(R_q+T_q)_{q\in\mathbb{N}_0}$ satisfies this condition too. Moreover, the sequence $(R_q+T_q)_{q\in\mathbb{N}_0}$ also satisfies (6.1) and (6.2), and it follows thus, that

$$P_q^c(x) = R_q^c(x) + T_q^c(x), \qquad (q\in\mathbb{N}_0).$$

Note also, that $T_2^c(x) = x-c-1$, so that the sequence $(T_{q+1}^c)_{q\in\mathbb{N}_0}$ satisfies (6.4)-(6.6), and hence

$$T_q^c(x) = R_{q-1}^c(x), \qquad (q\in\mathbb{N}).$$

Altogether, it follows that

$$\begin{aligned} P_q^c(x) &= R_q^c(x) + R_{q-1}^c(x), \qquad (q\geq 1), && (6.10)\\ P_0^c(x) &= R_0^c(x). && (6.11)\end{aligned}$$

To prove (ii), it suffices therefore to show, that with $x = c+1+2\sqrt{c}\cos\theta$, $\theta\in\,]0,\pi[$, one has

$$R_q^c(x) = \frac{c^{\frac{q}{2}}\sin((q+1)\theta)}{\sin\theta}, \qquad (q\in\mathbb{N}_0). \tag{6.12}$$

For $q=0$, this is clear from (6.4), and for $q=1$, it follows easily from (6.5), using that $\sin 2\theta = 2\sin\theta\cos\theta$. Proceeding by induction, assume now that $p\geq 1$ and that (6.12) has been proved for all q in $\{0,1,\dots,p\}$. Then by (6.6),

$$R_{p+1}^c(x) = \frac{2\sqrt{c}\cos\theta\cdot c^{\frac{p}{2}}\sin((p+1)\theta)}{\sin\theta} - \frac{c^{\frac{p+1}{2}}\sin(p\theta)}{\sin\theta},$$

when $x = c+1+2\sqrt{c}\cos\theta$, $\theta\in\,]0,\pi[$. But $2\cos\theta\sin((p+1)\theta) = \sin((p+2)\theta) + \sin(p\theta)$, and therefore

$$R_{p+1}^c(x) = \frac{c^{\frac{p+1}{2}}\sin((p+2)\theta)}{\sin\theta},$$

which means that (6.12) holds for $q = p + 1$. Thus, by induction, (6.12) holds for all q in $\mathbb{N}_0$, and this concludes the proof of (ii).
(iii) We show first, that for any m, n in $\mathbb{N}_0$,

$$\int_0^\infty x R_m^c(x) R_n^c(x)\ d\mu_c(x) = \begin{cases} 0, & \text{if } n \neq m, \\ c^{m+1}, & \text{if } n = m, \end{cases} \tag{6.13}$$

where $R_0^c, R_1^c, R_2^c, \dots$, are the polynomials determined by (6.4)-(6.6). Note for this, that if $c < 1$, then the atom for μ_c at 0, does not contribute to the integral on the left hand side of (6.13). Hence, for all values of c in $]0, \infty[$, we have

$$\int_0^\infty x R_m^c(x) R_n^c(x)\ d\mu_c(x) = \tfrac{1}{2\pi} \int_a^b R_m^c(x) R_n^c(x) \sqrt{(x-a)(b-x)}\ dx. \tag{6.14}$$

By the substitution $x = c + 1 + 2\sqrt{c}\cos\theta$, $\theta \in]0, \pi[$, and by (6.12), the integral on the right hand side of (6.14) can be reduced to

$$\tfrac{2c}{\pi} \int_0^\pi c^{\frac{m+n}{2}} \sin((m+1)\theta) \sin((n+1)\theta)\ d\theta,$$

which is equal to $c^{m+1}\delta_{m,n}$. This proves (6.13).
We show next that

$$x R_m^c(x) = P_{m+1}^c(x) + c P_m^c(x), \qquad (m \in \mathbb{N}_0). \tag{6.15}$$

For $m = 0$, this is clear from (6.1),(6.2) and (6.4), and for $m \geq 1$, we get from (6.6) and (6.10), that

$$x R_m^c(x) = R_{m+1}^c(x) + (c+1) R_m^c(x) + c R_{m-1}^c(x) = P_{m+1}^c(x) + c P_m^c(x).$$

This proves (6.15). Define now

$$\gamma_{m,n} = \int_0^\infty P_m^c(x) P_n^c(x)\ d\mu_c(x), \qquad (m, n \in \mathbb{N}_0).$$

It follows then from (6.15), that

$$\gamma_{m+1,n} + c\gamma_{m,n} = \int_0^\infty x R_m^c(x) P_n^c(x)\ d\mu_c(x), \qquad (m, n \in \mathbb{N}_0),$$

and applying then (6.10),(6.11) and (6.13), we get that

$$\gamma_{m+1,n} + c\gamma_{m,n} = c^{m+1}(\delta_{m,n} + \delta_{m,n-1}), \qquad (m \in \mathbb{N}_0, n \in \mathbb{N}), \tag{6.16}$$

and

$$\gamma_{m+1,0} + c\gamma_{m,0} = c^{m+1}\delta_{m,0}, \qquad (m \in \mathbb{N}_0). \tag{6.17}$$

Since μ_c is a probability measure, $\gamma_{0,0} = 1$, and using this and induction on (6.17), it follows that $\gamma_{m,0} = 0$ for all m in $\mathbb{N}$. Thus

$$\gamma_{0,n} = \gamma_{n,0} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases} \tag{6.18}$$

Consider now a fixed n in $\mathbb{N}$. By (6.16), we have then that

$$\gamma_{m+1,n} + c\gamma_{m,n} = \begin{cases} 0, & \text{if } m \in \{0,1,\dots,n-2\}, \\ c^n, & \text{if } m = n-1. \end{cases}$$

By induction in m $(0 \leq m \leq n)$, we get then, by application of (6.18), that

$$\gamma_{m,n} = \begin{cases} 0, & \text{if } m < n, \\ c^n, & \text{if } m = n, \end{cases}$$

and this completes the proof of (iii). $\square$

6.2 Lemma. *For any non-negative integers p, q, put*

$$\nu(c,p,q) = c^{-q} \int_a^b x^p P_q^c(x)\, d\mu_c(x). \tag{6.19}$$

We then have
(i) *For any p in $\mathbb{N}_0$, $x^p = \sum_{q=0}^p \nu(c,p,q) P_q^c(x)$.*
(ii) *For any p, q in $\mathbb{N}_0$,*

$$\begin{aligned} \nu(c,p,q) &\geq 0, \quad \text{if } q \leq p, & (6.20)\\ \nu(c,p,p) &= 1, & (6.21)\\ \nu(c,p,q) &= 0, \quad \text{if } q > p. & (6.22) \end{aligned}$$

Proof. (i) Consider a fixed p from $\mathbb{N}_0$. By Proposition 6.1, $\operatorname{span}\{P_0^c, P_1^c, \dots, P_p^c\}$ is equal to the set of all polynomials of degree less than or equal to p. In particular we have that $x^p = \sum_{q=0}^p \gamma_q P_q^c(x)$, for suitable complex numbers $\gamma_0, \dots, \gamma_p$ (depending on c and p). Applying then the orthogonality relation in Proposition 6.1(iii), it follows that $\gamma_q = \nu(c,p,q)$ for all q in $\{0,1,\dots,p\}$, and this proves (i).
(ii) By (6.1)-(6.3), it follows that

$$\begin{aligned} xP_0^c(x) &= P_1^c(x) + cP_0^c(x), & (6.23)\\ xP_q^c(x) &= P_{q+1}^c(x) + (c+1)P_q^c(x) + cP_{q-1}^c(x), \qquad (q \geq 1), & (6.24) \end{aligned}$$

so by induction in p, we get that $x^p (= x^p P_0^c(x))$, can be expressed as a linear combination of $P_0^c(x), P_1^c(x), \dots, P_p^c(x)$, in which all coefficients are non-negative. By (i) and the linear independence of $P_0^c(x), P_1^c(x), \dots, P_p^c(x)$, these

coefficients are exactly $\nu(c,p,0),\nu(c,p,1),\dots,\nu(c,p,p)$, and hence (6.20) follows.
Note next that (6.21) follows from (i) and the facts that $P_p^c(x)$ is a monic polynomial of degree p, whereas $P_0^c(x),\dots,P_{p-1}^c(x)$ are all of degree at most $p-1$.
Finally, (6.22) follows from (i) and the orthogonality relation in Proposition 6.1(iii). ☐

6.3 Lemma. *Let $\nu(c,p,q)$, $p,q\in\mathbb{N}_0$, be as in Lemma 6.2. Then for any fixed q in $\mathbb{N}_0$, the power series*

$$\sum_{p=0}^{\infty}\nu(c,p,q)t^p, \tag{6.25}$$

converges for all t in the open complex ball $B(0,\frac{1}{b})$, where $b=(\sqrt{c}+1)^2$. Moreover, the function

$$J_q^c(t)=\sum_{p=0}^{\infty}\nu(c,p,q)t^p,\qquad (t\in B(0,\tfrac{1}{b})),$$

is for all t in $B(0,\frac{1}{b})\setminus\{0\}$, given by

$$J_q^c(t)=\frac{1-(c-1)t-\sqrt{(1-at)(1-bt)}}{2t}\left(\frac{1-(c+1)t-\sqrt{(1-at)(1-bt)}}{2ct}\right)^q, \tag{6.26}$$

where $\sqrt{\cdot}$ is the principal branch of the complex square-root.

Proof. Consider the Hilbert space $L^2(\mathbb{R},\mu_c)$, and let A be the bounded operator on $L^2(\mathbb{R},\mu_c)$, given by

$$[A(f)](x)=xf(x),\qquad (f\in L^2(\mathbb{R},\mu_c),\ x\in\mathbb{R}).$$

Note that $A^*=A$ and that $\mathrm{sp}(A)=\mathrm{supp}(\mu_c)\subseteq[0,b]$. Thus, letting $\mathbf{1}$ denote the identity operator on $L^2(\mathbb{R},\mu_c)$, $\mathbf{1}-tA$ is invertible for all complex numbers t such that $|t|<\frac{1}{b}$, and moreover, for such t,

$$(\mathbf{1}-tA)^{-1}=\sum_{p=0}^{\infty}t^pA^p,\qquad \text{(norm convergence)}.$$

For any t in $B(0,\frac{1}{b})$, we have thus that

$$\begin{aligned}\sum_{p=0}^{\infty}\nu(c,p,q)t^p=c^{-q}\sum_{p=0}^{\infty}\langle x^p,P_q^c\rangle t^p&=c^{-q}\sum_{p=0}^{\infty}\langle A^pP_0^c,P_q^c\rangle t^p\\&=c^{-q}\langle(1-tA)^{-1}P_0^c,P_q^c\rangle.\end{aligned}$$

This shows that the series in (6.25) converges for all t in $B(0,\frac{1}{b})$, and moreover, that

$$J_q^c(t) = c^{-q}\langle (1-tA)^{-1}P_0^c, P_q^c\rangle, \qquad (t\in B(0,\tfrac{1}{b})). \tag{6.27}$$

To prove (6.26), we shall calculate the right hand side of (6.27). For this, consider for each z in $B(0,\frac{1}{\sqrt{c}})$ the series $\sum_{q=0}^{\infty} z^q P_q^c$, and note that by Lemma 6.1(iii), this series converges in $\|\cdot\|_2$-norm in $L^2(\mathbb{R},\mu_c)$. We may thus define

$$\omega_z = \sum_{q=0}^{\infty} z^q P_q^c \in L^2(\mathbb{R},\mu_c), \qquad (z\in B(0,\tfrac{1}{\sqrt{c}})). \tag{6.28}$$

With A as above, it follows now by (6.23) and (6.24), that for any z in $B(0,\frac{1}{\sqrt{c}})\setminus\{0\}$,

$$\begin{aligned}
A\omega_z &= \sum_{n=0}^{\infty} z^n A P_n^c = cP_0^c + P_1^c + \sum_{n=1}^{\infty} z^n(cP_{n-1}^c + (c+1)P_n^c + P_{n+1}^c)\\
&= (c+cz)P_0^c + \sum_{n=1}^{\infty}(z^{n-1} + (c+1)z^n + cz^{n+1})P_n^c\\
&= (c+cz)P_0^c + z^{-1}(1+(c+1)z+cz^2)\sum_{n=1}^{\infty} z^n P_n^c\\
&= \big(c+cz - z^{-1}(1+(c+1)z+cz^2)\big)P_0^c + z^{-1}(1+(c+1)z+cz^2)\omega_z\\
&= -z^{-1}(1+z)P_0^c + z^{-1}(1+z)(1+cz)\omega_z,
\end{aligned}$$

where the infinite sums converge in $\|\cdot\|_2$-norm. From this it follows that

$$(z^{-1}(1+z)(1+cz)\mathbf{1} - A)\omega_z = z^{-1}(1+z)P_0^c, \qquad (z\in B(0,\tfrac{1}{\sqrt{c}})\setminus\{0\}),$$

and hence that

$$\big(\mathbf{1} - \tfrac{z}{(1+z)(1+cz)}A\big)\omega_z = \tfrac{1}{1+cz}P_0^c, \qquad (z\in B(0,\tfrac{1}{\sqrt{c}})\setminus\{-1,-\tfrac{1}{c}\}). \tag{6.29}$$

Define now

$$\varphi(z) = \frac{z}{(1+z)(1+cz)}, \qquad (z\in\mathbb{C}\setminus\{-1,\tfrac{1}{c}\}).$$

Since $\mathrm{sp}(A)\subseteq[0,b]$, it follows that $(\mathbf{1}-\varphi(z)A)$ is invertible whenever $\varphi(z)\notin[\frac{1}{b},\infty[$, and in particular, as long as $|\varphi(z)|<\frac{1}{b}$. Note then, that φ is analytic on $\mathbb{C}\setminus\{-1,-\frac{1}{c}\}$, and that $\varphi(0)=0$, $\varphi'(0)=1$. It follows thus, that we may choose neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of 0 in $\mathbb{C}$, such that φ is a bijection of $\mathcal{U}$ onto $\mathcal{V}$. We may assume, in addition, that

$$\mathcal{U}\subseteq B(0,\tfrac{1}{\sqrt{c}})\setminus\{-1,-\tfrac{1}{c}\}, \qquad \text{and} \qquad \mathcal{V}\subseteq B(0,\tfrac{1}{b}).$$

For z in $\mathcal{U}$, it follows now from (6.29), that

$$\omega_z = \tfrac{1}{1+cz}\big(1-\varphi(z)A\big)^{-1}P_0^c,$$

and hence, by (6.27) and Lemma 6.1(iii),

$$J_q^c\big(\varphi(z)\big) = (1+cz)\cdot c^{-q}\langle\omega_z, P_q^c\rangle = (1+cz)z^q, \qquad (z\in\mathcal{U}). \tag{6.30}$$

It remains to invert φ. By solving the equation

$$t = \frac{z}{(1+z)(1+cz)},$$

w.r.t. z, we find that

$$\varphi^{-1}(t) = \frac{1-(c+1)t\pm\sqrt{(1-at)(1-bt)}}{2ct}, \qquad (t\in\mathcal{V}\setminus\{0\}),$$

where, as usual, $a=(\sqrt{c}-1)^2$ and $b=(\sqrt{c}+1)^2$. Since $\varphi^{-1}(t)\to 0$ as $t\to 0$, it follows that for some neighbourhood $\mathcal{V}_0$ of 0, such that $\mathcal{V}_0\subseteq\mathcal{V}$, we must have

$$\varphi^{-1}(t) = \frac{1-(c+1)t-\sqrt{(1-at)(1-bt)}}{2ct}, \qquad (t\in\mathcal{V}_0\setminus\{0\}), \tag{6.31}$$

where $\sqrt{\cdot}$ is the principal part of the square root. Hence, we have also that

$$1+c\varphi^{-1}(t) = \frac{1-(c-1)t-\sqrt{(1-at)(1-bt)}}{2t}, \qquad (t\in\mathcal{V}_0\setminus\{0\}). \tag{6.32}$$

Inserting (6.31) and (6.32) in (6.30), we obtain that (6.26) holds for all t in $\mathcal{V}_0\setminus\{0\}$.

To show that (6.26) actually holds for all t in $B(0,\frac{1}{b})\setminus\{0\}$, note that for all such t, $\mathrm{Re}(1-at)>0$ and $\mathrm{Re}(1-bt)>0$, so that $(1-at)(1-bt)\in\mathbb{C}\setminus\,]-\infty,0]$. Hence, with $\sqrt{\cdot}$ the principal branch of the square root, $t\mapsto\sqrt{(1-at)(1-bt)}$ is an analytic function of $t\in B(0,\frac{1}{b})$. By uniqueness of analytic continuation, it follows thus, that (6.26) holds for all t in $B(0,\frac{1}{b})\setminus\{0\}$. $\square$

6.4 Lemma. *Let $g_c(p)$ and $h_c(p)$, $p\in\mathbb{N}_0$, be as in Definition 5.17. Then the power series*

$$G_c(t) = \sum_{p=0}^{\infty} g_c(p)t^p, \tag{6.33}$$

and

$$H_c(t) = \sum_{p=0}^{\infty} h_c(p)t^p, \tag{6.34}$$

are convergent for all t in $B(0,\frac{1}{b})$, and

$$J_q^c(t) = t^q G_c(t)^{q+1}H_c(t)^q, \qquad (t\in B(0,\tfrac{1}{b})). \tag{6.35}$$

Proof. By (5.10), we have

$$g_c(p) = \int_0^\infty x^p \, d\mu_c(x), \qquad (p \in \mathbb{N}),$$

and since $g_c(0) = 1$, the same formula holds for $p = 0$. Hence $g_c(p) = \nu(c, p, 0)$, for all p in $\mathbb{N}_0$, so by Lemma 6.3, the series in (6.33) converges for all t in $B(0, \frac{1}{b})$, and

$$G_c(t) = J_0^c(t) = \frac{1 - (c-1)t - \sqrt{(1-at)(1-bt)}}{2t}, \qquad (t \in B(0, \tfrac{1}{b}) \setminus \{0\}). \tag{6.36}$$

Since $h_c(0) = 1$ and since $h_c(p) = \frac{1}{c} g_c(p)$, for all p in $\mathbb{N}$, the series in (6.34) is also convergent for all t in $B(0, \frac{1}{b})$, and

$$H_c(t) = 1 + \tfrac{1}{c}(G_c(t) - 1), \qquad (t \in B(0, \tfrac{1}{c})).$$

Hence by (6.34)

$$H_c(t) = \frac{1 + (c-1)t - \sqrt{(1-at)(1-bt)}}{2ct}, \qquad (t \in B(0, \tfrac{1}{b}) \setminus \{0\}). \tag{6.37}$$

By (6.36) and (6.37), we get now for all t in $B(0, \frac{1}{b}) \setminus \{0\}$,

$$\begin{aligned} & G_c(t) H_c(t) \\ &= \frac{\big(1 - \sqrt{(1-at)(1-bt)}\big)^2 - (c-1)^2 t^2}{4ct^2} \\ &= \frac{1 + (1-at)(1-bt) - 2\sqrt{(1-at)(1-bt)} - (c-1)^2 t^2}{4ct^2} \\ &= \frac{1 + (1 - 2(c+1)t + (c-1)^2 t^2) - 2\sqrt{(1-at)(1-bt)} - (c-1)^2 t^2}{4ct^2} \\ &= \frac{1 - (c+1)t - \sqrt{(1-at)(1-bt)}}{2ct^2}. \end{aligned}$$

Combining this with (6.36) and (6.26), it follows that

$$J_c^q(t) = G_c(t) \big(t G_c(t) H_c(t)\big)^q, \qquad (t \in B(0, \tfrac{1}{b})),$$

and the same formula holds trivially for $t = 0$, by (6.22). This proves (6.35). □

6.5 Lemma. *For all p, q in $\mathbb{N}_0$ such that $p \geq q$, let $\nu(c, p, q)$ be as introduced in Definition 5.17. Then*

$$\nu'(c, p, q) = \nu(c, p, q), \qquad (p, q \in \mathbb{N}_0, \ q \leq p).$$

Proof. Recall from Definition 5.17, that for p, q in $\mathbb{N}_0$, such that $p \geq q$, we have

$$\nu'(c,p,q) = \sum_{\substack{r_0,r_1,\dots,r_{2q} \geq 0 \\ r_0+r_1+\cdots+r_{2q}=p-q}} g_c(r_0)h_c(r_1)g_c(r_2)h_c(r_3)\cdots g_c(r_{2q}).$$

Hence $\nu'(c,p,q)$ is the coefficient to t^{p-q} in the power series for

$$G_c(t)H_c(t)G_c(t)H_c(t)\cdots G_c(t), \qquad (2q+1 \text{ factors}),$$

and therefore $\nu'(c,p,q)$ is the coefficient to t^p in the power series for $t^q G_c(t)^{q+1} H_c(t)^q$. Thus, by Lemma 6.3 and Lemma 6.4, it follows that

$$\nu'(c,p,q) = \nu(c,p,q), \quad \text{for all } p,q \text{ in } \mathbb{N}_0, \text{ such that } p \geq q. \qquad \square$$

6.6 Theorem. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, and let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, satisfying that $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\sum_{i=1}^r a_i a_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}$, for some positive real number c. Furthermore, let $Y_1, \dots, Y_r$ be independent elements of* GRM$(n, n, \frac{1}{n})$*, and put $S = \sum_{i=1}^r a_i \otimes Y_i$. Then for any q in $\mathbb{N}$,*

$$\mathbb{E}\big[P_q^c(S^*S)\big] = \Big[\sum_{\rho \in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})} \Big(\sum_{1 \leq i_1,\dots,i_q \leq r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}}\Big)\Big] \otimes \mathbf{1}_n.$$

Proof. For each q in $\mathbb{N}$, put

$$T_q = \sum_{\rho \in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})} \Big(\sum_{1 \leq i_1,\dots,i_q \leq r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}}\Big),$$

and put $T_0 = \mathbf{1}_{\mathcal{B}(\mathcal{H})}$. By Theorem 5.18 and Lemma 6.5, it follows then that

$$\mathbb{E}\big[(S^*S)^p\big] = \sum_{q=0}^p \nu(c,p,q) \cdot T_q \otimes \mathbf{1}_n, \qquad (p \in \mathbb{N}_0). \tag{6.38}$$

On the other hand, it follows from Lemma 6.2(i), that

$$\mathbb{E}\big[(S^*S)^p\big] = \sum_{q=0}^p \nu(c,p,q)\mathbb{E}\big[P_q^c(S^*S)\big], \qquad (p \in \mathbb{N}_0). \tag{6.39}$$

We prove that

$$\mathbb{E}\big[P_q^c(S^*S)\big] = T_q \otimes \mathbf{1}_n, \qquad (q \in \mathbb{N}_0), \tag{6.40}$$

by induction in q. Note that (6.40) is trivial for $q = 0$. Consider then p from $\mathbb{N}$, and assume that (6.40) has been proved for $q = 0, 1, \dots, p-1$. Since

$\nu(c,p,p) = 1$, by Lemma 6.2(ii), it follows then from (6.39) and (6.38), that

$$\begin{aligned}\mathbb{E}\big[P_p^c(S^*S)\big] &= \mathbb{E}\big[(S^*S)^p\big] - \sum_{q=0}^{p-1} \nu(c,p,q)\mathbb{E}\big[P_q^c(S^*S)\big] \\ &= \mathbb{E}\big[(S^*S)^p\big] - \sum_{q=0}^{p-1} \nu(c,p,q) \cdot T_q \otimes \mathbf{1}_n \\ &= T_p \otimes \mathbf{1}_n.\end{aligned}$$

Thus, (6.40) holds for $q = p$, and this completes the proof. □

6.7 Example. By (6.1)-(6.3), it follows that

$$\begin{aligned}P_1^c(x) &= x - c, &(6.41)\\ P_2^c(x) &= x^2 - (2c+1)x + c^2, &(6.42)\\ P_3^c(x) &= x^3 - (3c+2)x^2 + (3c^2+2c+1)x - c^3. &(6.43)\end{aligned}$$

By Example 5.20, $S_p^{\mathrm{irr}} = \emptyset$ if $p \in \{1,2\}$, and $S_3^{\mathrm{irr}} = \{\pi\}$, where π is the permutation given by $\pi(1) = 3, \pi(2) = 1, \pi(3) = 2$, so that $\sigma(\hat{\pi}) = 1$. It follows thus by Theorem 6.6, that

$$\begin{aligned}\mathbb{E}\big[P_1^c(S^*S)\big] &= 0,\\ \mathbb{E}\big[P_2^c(S^*S)\big] &= 0,\\ \mathbb{E}\big[P_3^c(S^*S)\big] &= n^{-2}\sum_{i,j,k=1}^{r} a_i^* a_k a_j^* a_i a_k^* a_j.\end{aligned}$$

These three formulas can also easily be derived directly from Example 5.20, using the formulas (6.41)-(6.43). □

7 An Upper Bound for $\mathbb{E}\big[\exp(-tS^*S)\big],\ t \geq 0$

Throughout this section, we consider elements $a_1, \dots, a_r$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$ (for given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$), satisfying that

$$\sum_{i=1}^{r} a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \quad \text{and} \quad \sum_{i=1}^{r} a_i a_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})},$$

for some constant c in $[1,\infty[$. Moreover, we consider independent elements $Y_1, \dots, Y_r$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and put

$$S = \sum_{i=1}^{r} a_i \otimes Y_i.$$

As in Section 3, we let μ_c denote the probability measure on $\mathbb{R}$, given by

$$\mu_c = \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx,$$

where $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. Furthermore, we let $(P_q^c)_{q\in\mathbb{N}_0}$ be the sequence of monic orthogonal polynomials w.r.t. μ_c as defined in Section 6. In particular $P_0^c \equiv 1$.

7.1 Lemma. *Let, as above, $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. Then for any q in $\mathbb{N}_0$,*
(i) $P_q^c(x) \geq P_q^c(b) > 0$, *for all x in $]b,\infty[$.*
(ii) $|P_q^c(x)| \leq P_q^c(b)$, *for all x in $[a,b]$.*
(iii) $|P_q^c(x)| \leq P_q^c(2c+2-x)$, *for all x in $]-\infty,a[$.*

Proof. We start by proving (ii). If $x \in [a,b]$, then $x = c+1+2\sqrt{c}\cos\theta$, for some θ in $[0,\pi]$. For θ in $]0,\pi[$, we have from Proposition 6.1(ii), that

$$P_q^c(c+1+2\sqrt{c}\cos\theta) = \frac{c^{\frac{q}{2}}\sin((q+1)\theta) + c^{\frac{q-1}{2}}\sin(q\theta)}{\sin\theta}. \tag{7.1}$$

Note here that for any k in $\mathbb{N}_0$,

$$\frac{\sin((k+1)\theta)}{\sin\theta} = e^{-k\theta}\big(1+e^{2i\theta}+e^{4i\theta}+\cdots+e^{2ki\theta}\big), \tag{7.2}$$

so that $\left|\frac{\sin((k+1)\theta)}{\sin\theta}\right| \leq k+1$. It follows thus that

$$|P_q^c(x)| \leq c^{\frac{q}{2}}(q+1) + c^{\frac{q-1}{2}}q, \qquad (x \in]a,b[), \tag{7.3}$$

and by continuity, (7.3) holds also for $x = a$ and $x = b$. By (7.2), $\lim_{\theta\to 0}\frac{\sin((k+1)\theta)}{\sin\theta} = k+1$, for any k in $\mathbb{N}_0$, and hence the right hand side of (7.3) is equal to $P_q^c(b)$. This proves (ii).
To prove (i), we note first, that by uniqueness of analytic continuation, (7.1) actually holds for all θ in $\mathbb{C}\setminus\pi\mathbb{Z}$. If we put $\theta = i\rho$, $\rho > 0$, we get the equation:

$$P_q^c(c+1+2\sqrt{c}\cosh\rho) = \frac{c^{\frac{q}{2}}\sinh((q+1)\rho) + c^{\frac{q-1}{2}}\sinh(q\rho)}{\sinh\rho}, \qquad (\rho \in]0,\infty[), \tag{7.4}$$

which covers the values of $P_q(x)$ for all x in $]b,\infty[$. Note here that for any k in $\mathbb{N}_0$,

$$\frac{\sinh((k+1)\rho)}{\sinh\rho} = e^{-k\rho}\big(1+e^{2\rho}+e^{4\rho}+\cdots+e^{2k\rho}\big),$$

and hence, if k is even,

$$\frac{\sinh((k+1)\rho)}{\sinh\rho} = 1+2\cosh(2\rho)+2\cosh(4\rho)+\cdots+2\cosh(k\rho),$$

whereas, if k is odd,

$$\frac{\sinh((k+1)\rho)}{\sinh\rho} = 2\cosh(\rho) + 2\cosh(3\rho) + \cdots + 2\cosh(k\rho),$$

so in both cases $\frac{\sinh((k+1)\rho)}{\sin\rho}$ is an increasing function of $\rho > 0$. It follows thus from (7.4), that $P_q^c(x) \geq P_q^c(b)$ for all x in $]b, \infty[$. Moreover, as we saw in the proof of (ii), $P_q^c(b) > 0$. This concludes the proof of (i).
Finally, to prove (iii), we put $\theta = \pi + i\rho$ in (7.1), and get for ρ in $]0, \infty[$, that

$$\begin{aligned}\left|P_q^c(c+1-2\sqrt{c}\cosh\rho)\right| &= \left|\frac{(-1)^q c^{\frac{q}{2}}\sinh((q+1)\rho) + (-1)^{q-1}c^{\frac{q-1}{2}}\sinh(q\rho)}{\sinh\rho}\right| \\ &\leq \frac{c^{\frac{q}{2}}\sinh((q+1)\rho) + c^{\frac{q-1}{2}}\sinh(q\rho)}{\sinh\rho} \\ &= P_q^c(c+1+2\sqrt{c}\cosh\rho).\end{aligned}$$

This proves (iii). □

7.2 Definition. For each q in $\mathbb{N}_0$, we define the function $\psi_q^c \colon \mathbb{R} \to \mathbb{R}$, by the equation

$$\psi_q^c(t) = c^{-q}\int_a^b \exp(tx)P_q^c(x)\ d\mu_c(x), \qquad (t \in \mathbb{R}).$$ □

7.3 Lemma. *Consider the sequence $(\psi_q^c)_{q\in\mathbb{N}_0}$ of functions, introduced in Definition 7.2, and for each p in $\mathbb{N}_0$, let, as in Section 6,*

$$\nu(c,p,q) = c^{-q}\int_a^b x^p P_q^c(x)\ d\mu_c(x), \qquad (p, q \in \mathbb{N}_0).$$

We then have

(i) $\psi_q^c(t) = \sum_{p=q}^{\infty}\frac{t^p}{p!}\nu(c,p,q)$, *for all t in $\mathbb{R}$.*

(ii) $\sum_{q=0}^{\infty}|\psi_q^c(t)|\cdot|P_q^c(x)| \leq \exp(|t|x) + \exp(|t|(2c+2))$, *for all t in $\mathbb{R}$ and all x in $[0, \infty[$.*

(iii) $\exp(tx) = \sum_{q=0}^{\infty}\psi_q^c(t)\cdot P_q^c(x)$, *for all t in $\mathbb{R}$ and x in $[0, \infty[$, and for fixed t in $\mathbb{R}$, the series converges uniformly in x on compact subsets of $[0, \infty[$.*

Proof. (i) By Lemma 6.2(ii), $\nu(c,p,q) = 0$ whenever $q > p$. Hence (i) follows from the power series expansion of $\exp(tx)$.
(ii) Let $\beta \colon \mathbb{R} \to [b, \infty[$ be the *continuous* function defined by:

$$\beta(x) = \begin{cases} x, & \text{if } x > b, \\ b, & \text{if } a \leq x \leq b, \\ 2c+2-x, & \text{if } x < a, \end{cases}$$

It follows then from Lemma 7.1, that

$$|P_q^c(x)| \le P_q^c(\beta(x)), \qquad (x \in \mathbb{R},\ q \in \mathbb{N}_0). \tag{7.5}$$

Recall that $x^p = \sum_{q=0}^{p} \nu(c,p,q) P_q^c(x)$, for all p in $\mathbb{N}$ (c.f. Lemma 6.2(i)). Hence, for x, t in $\mathbb{R}$, we have that

$$\exp(tx) = \sum_{p=0}^{\infty} \tfrac{t^p}{p!} x^p = \sum_{p=0}^{\infty} \tfrac{t^p}{p!} \Big(\sum_{q=0}^{p} \nu(c,p,q) P_q^c(x) \Big). \tag{7.6}$$

Substituting x with $\beta(x)$ and t with $|t|$ in this formula, and recalling from Lemma 6.2(ii), that $\nu(c,p,q) \ge 0$, for $0 \le q \le p$, we get by application of (7.5),

$$\begin{aligned} \sum_{p=0}^{\infty} \tfrac{|t|^p}{p!} \Big(\sum_{q=0}^{p} \nu(c,p,q) |P_q^c(x)| \Big) &\le \sum_{p=0}^{\infty} \tfrac{|t|^p}{p!} \Big(\sum_{q=0}^{p} \nu(c,p,q) P_q^c(\beta(x)) \Big) \\ &= \exp(|t|\beta(x)) < \infty. \end{aligned}$$

Hence, we can apply Fubini's theorem to the double sum in (7.6), and obtain that

$$\exp(tx) = \sum_{q=0}^{\infty} \Big(\sum_{p=q}^{\infty} \tfrac{t^p}{p!} \nu(c,p,q) \Big) P_q^c(x), \qquad (x, t \in \mathbb{R}). \tag{7.7}$$

Similarly we have that

$$\exp(|t|\beta(x)) = \sum_{q=0}^{\infty} \Big(\sum_{p=q}^{\infty} \tfrac{|t|^p}{p!} \nu(c,p,q) \Big) P_q^c(\beta(x)), \qquad (x, t \in \mathbb{R}). \tag{7.8}$$

Note here that by (i) proved above, we have that,

$$|\psi_q^c(t)| \le \sum_{p=q}^{\infty} \tfrac{|t|^p}{p!} \nu(c,p,q). \tag{7.9}$$

Since $\beta(x) \le \max\{2c+2, x\}$ for all x in $[0,\infty[$, (7.5) and (7.7)-(7.9) imply that for all t in $\mathbb{R}$ and x in $[0,\infty[$,

$$\sum_{q=0}^{\infty} |\psi_q^c(t)| \cdot |P_q^c(x)| \le \exp(|t|\beta(x)) \le \exp(|t|(2c+2)) + \exp(|t|x),$$

and this proves (ii).
(iii) The summation formula in (iii) follows from (i) and (7.7). To prove that the convergence is uniform in x on compact subsets, we observe that for any

Q in $\mathbb{N}$,

$$\begin{aligned}
\Big|\exp(tx) - \sum_{q=0}^{Q} \psi_q^c(t) P_q^c(x)\Big| &\le \sum_{q=Q+1}^{\infty} |\psi_q^c(t)| \cdot |P_q^c(x)| \\
&\le \sum_{q=Q+1}^{\infty} \Big(\sum_{p=q}^{\infty} \tfrac{|t|^p}{p!} \nu(c,p,q) P_q^c(\beta(x)) \Big) \\
&\le \sum_{p=Q+1}^{\infty} \tfrac{|t|^p}{p!} \Big(\sum_{q=0}^{p} \nu(c,p,q) P_q^c(\beta(x)) \Big) \\
&= \sum_{p=Q+1}^{\infty} \tfrac{(|t|\beta(x))^p}{p!}.
\end{aligned} \tag{7.10}$$

Since β is continuous, and hence bounded on compact sets, it follows readily from (7.10) that for fixed t in $\mathbb{R}$, the series in (iii) converges uniformly in x on compact subsets of $[0,\infty[$. □

7.4 PROPOSITION. *Consider the sequence $(\psi_q^c)_{q\in\mathbb{N}_0}$ of functions, introduced in Definition 7.2. Then for any t in $\mathbb{R}$ such that $|t| < \frac{n}{c}$, the function $\omega \mapsto \exp(tS^*(\omega)S(\omega))$ is integrable in the sense of Definition 3.1, and*

$$\mathbb{E}\big[\exp(tS^*S)\big] = \sum_{q=0}^{\infty} \psi_q^c(t) \mathbb{E}\big[P_q^c(S^*S)\big], \tag{7.11}$$

where the sum on the right hand side is absolutely convergent in $\mathcal{B}(\mathcal{H}^n)$.

Proof. We start by proving that the right hand side of (7.11) is absolutely convergent in $\mathcal{B}(\mathcal{H}^n)$. Since $|\psi_q^c(t)| \le \psi_q^c(|t|)$ by Lemma 7.3(i) and (7.9), it suffices to consider the case where $t \ge 0$.
By Lemma 7.3(i), we have for any t in $[0,\infty[$,

$$\sum_{q=0}^{\infty} \psi_q^c(t) \big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| = \sum_{p=0}^{\infty} \frac{t^p}{p!} \Big(\sum_{q=0}^{p} \nu(c,p,q) \big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| \Big). \tag{7.12}$$

Note here, that by Theorem 6.6,

$$\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| \le \sum_{\rho\in S_q^{\mathrm{irr}}} n^{-2\sigma(\hat{\rho})} \Big\| \sum_{1\le i_1,\dots,i_q\le r} a_{i_1}^* a_{i_{\rho(1)}} \cdots a_{i_q}^* a_{i_{\rho(q)}} \Big\|,$$

for any q in $\mathbb{N}$, whereas

$$\big\|\mathbb{E}\big[P_0^c(S^*S)\big]\big\| = \|\mathbb{E}(\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)})\| = 1.$$

Hence, by Proposition 5.19, Lemma 6.5 and Proposition 2.7, we have for any p in $\mathbb{N}$,

$$\begin{aligned}\sum_{q=0}^{p}\nu(c,p,q)\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| &\le \sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}\Big\|\sum_{1\le i_1,\dots,i_p\le r} a_{i_1}^* a_{i_{\pi(1)}}\cdots a_{i_p}^* a_{i_{\pi(p)}}\Big\| \\ &\le \sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}c^{\kappa(\hat{\pi})}.\end{aligned} \tag{7.13}$$

Using now that $c\ge 1$, and that $\kappa(\hat{\pi})\le k(\hat{\pi})+2\sigma(\hat{\pi})$ (c.f. Proposition 2.10), it follows that for any p in $\mathbb{N}$,

$$\sum_{\pi\in S_p} n^{-2\sigma(\hat{\pi})}c^{\kappa(\hat{\pi})} \le \sum_{\pi\in S_p}\big(\tfrac{n}{c}\big)^{-2\sigma(\hat{\pi})}c^{k(\hat{\pi})}. \tag{7.14}$$

For $p=0$, we note that

$$\sum_{q=0}^{p}\nu(c,p,q)\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| = 1. \tag{7.15}$$

Combining now (7.12)-(7.15), we get that

$$\sum_{q=0}^{\infty}\psi_q^c(t)\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| \le 1+\sum_{p=1}^{\infty}\tfrac{t^p}{p!}\Big(\sum_{\pi\in S_p}\big(\tfrac{n}{c}\big)^{-2\sigma(\hat{\pi})}c^{k(\hat{\pi})}\Big). \tag{7.16}$$

Using then that $-2\sigma(\hat{\pi}) = k(\hat{\pi})+l(\hat{\pi})-p-1$, it follows that

$$\begin{aligned}\sum_{q=0}^{\infty}\psi_q^c(t)\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| &\le 1+\sum_{p=1}^{\infty}\tfrac{1}{p!}\big(\tfrac{ct}{n}\big)^p\sum_{\pi\in S_p} n^{k(\hat{\pi})}\big(\tfrac{n}{c}\big)^{l(\hat{\pi})-1} \\ &\le 1+ct\sum_{p=1}^{\infty}\tfrac{1}{(p-1)!}\big(\tfrac{ct}{n}\big)^{p-1}\sum_{\pi\in S_p} n^{k(\hat{\pi})-1}\big(\tfrac{n}{c}\big)^{l(\hat{\pi})-1},\end{aligned} \tag{7.17}$$

where the last equality follows by noting that $\frac{1}{p!}\le\frac{1}{(p-1)!}$ for all p in $\mathbb{N}$. By Lemma 3.4, the last quantity in (7.17) is finite whenever $0\le\frac{ct}{n}<1$, and this shows that the right hand side of (7.11) is absolutely convergent for all t in $]-\frac{n}{c},\frac{n}{c}[$, as desired.

It remains now (cf. Definition 3.1) to show, that for any state φ on $\mathcal{B}(\mathcal{H}^n)$,

$$\mathbb{E}\big[\varphi(\exp(tS^*S))\big] = \sum_{q=0}^{\infty}\psi_q^c(t)\varphi\big(\mathbb{E}\big[P_q^c(S^*S)\big]\big), \qquad (t\in]-\tfrac{n}{c},\tfrac{n}{c}[). \tag{7.18}$$

So consider a fixed t from $]-\frac{n}{c},\frac{n}{c}[$ and a fixed state φ on $\mathcal{B}(\mathcal{H}^n)$. Since the spectrum of $S^*(\omega)S(\omega)$ is compact for each ω in Ω, it follows then by Lemma 7.3, that

$$\varphi\big[\exp(tS^*(\omega)S(\omega))\big]=\sum_{p=0}^{\infty}\psi_q^c(t)\varphi\big[P_q^c(S_n(\omega)^*S_n(\omega))\big], \tag{7.19}$$

so we need to show that we can integrate termwise in the sum on the right hand side. Note for this, that by Lemma 7.3(ii), and the function calculus for selfadjoint operators on Hilbert spaces,

$$\sum_{p=0}^{\infty}|\psi_q^c(t)|\cdot|P_q^c(S(\omega)^*S(\omega))|\leq\exp(2(c+1)|t|)\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}+\exp(|t|S(\omega)^*S(\omega)), \tag{7.20}$$

where $|T|=(T^2)^{\frac{1}{2}}$, for any selfadjoint T in $\mathcal{B}(\mathcal{H}^n)$. For such T, we have also that $|\varphi(T)|\leq\varphi(|T|)$, and hence it follows from (7.20), that

$$\sum_{p=0}^{\infty}|\psi_q^c(t)|\cdot\big|\varphi\big[P_q^c(S(\omega)^*S(\omega))\big]\big|\leq\exp(2(c+1)|t|)+\varphi\big[\exp(|t|S(\omega)^*S(\omega))\big]. \tag{7.21}$$

Since $\mathbb{E}\big[\varphi(\exp(|t|S^*S))\big]<\infty$, by Proposition 3.2, it follows from (7.21) and Lebesgue's theorem on dominated convergence, that we may integrate termwise in (7.19), and hence obtain (7.18). This concludes the proof. $\square$

In order to obtain the upper bound for $\mathbb{E}\big[\exp(-tS^*S)\big]$ in Theorem 7.8 below, we need more precise information about the behavior of the function $\psi_q^c(t)$ for $t<0$.

7.5 Proposition. *Consider the sequence $(\psi_q^c)_{q\in\mathbb{N}_0}$ of functions, defined in Definition 7.2. Then for any q in $\mathbb{N}_0$, and any t in $]0,\infty[$, we have that*
(i) $\psi_q^c(t)>0$.
(ii) $(-1)^q\psi_q^c(-t)>0$.
(iii) $|\psi_q^c(-t)|\leq\frac{\psi_0^c(-t)}{\psi_0^c(t)}\psi_q^c(t)$.

Proof. (i) This follows from Lemma 7.3(i), but for completeness we include a different proof, which will also be needed in the proof of (ii) and (iii). For each q in $\mathbb{N}_0$, we put

$$\rho_q^c(x)=c^{-\frac{q}{2}}P_q^c(x),\qquad (x\in\mathbb{R}).$$

Then by Proposition 6.1, $(\rho_q^c)_{q\in\mathbb{N}_0}$ is an orthonormal basis for $L_2([a,b],\mu_c)$. Let A be the (bounded) operator for multiplication by x in $L_2([a,b],\mu_c)$. Then by

(6.23) and (6.24), the matrix $M(A)$ of A w.r.t. $(\rho_q^c)_{q\in\mathbb{N}_0}$, is given by

$$M(A) = \begin{pmatrix} c & \sqrt{c} & & & 0 \\ \sqrt{c} & c+1 & \sqrt{c} & & \\ & \sqrt{c} & c+1 & \sqrt{c} & \\ & & \ddots & \ddots & \ddots \\ & & & & \\ 0 & & & & \end{pmatrix} \tag{7.22}$$

From this, it follows, that for any p in $\mathbb{N}$,

$$\begin{aligned} M(A^p)_{jk} &> 0, \quad \text{when } |j-k| \le p, \\ M(A^p)_{jk} &= 0, \quad \text{when } |j-k| > p. \end{aligned}$$

Hence, for any t in $[0,\infty[$,

$$M(\exp(tA))_{jk} = \delta_{j,k} + \sum_{p=1}^{\infty} \tfrac{t^p}{p!} M(A^p)_{jk} > 0, \qquad (j,k \in \mathbb{N}_0).$$

Since $\exp(tA)$ is the operator for multiplication by $\exp(tx)$ in $L_2([a,b],\mu_c)$, and since $P_0^c(x) \equiv 1$, we get that

$$\begin{aligned} \psi_q^c(t) &= c^{-q} \int_a^b \exp(tx) P_q^c(x) P_0^c(x)\, d\mu_c(x) = c^{-\frac{q}{2}} \langle \exp(tA)\rho_q^c, \rho_0^c \rangle \\ &= c^{-\frac{q}{2}} M(\exp(tA))_{0,q} > 0, \end{aligned} \tag{7.23}$$

and this proves (i).

(ii) To prove (ii), we consider the operator

$$B = A + 2P_0,$$

where P_0 is the projection onto $\mathbb{C}\rho_0^c$ in $\mathcal{B}\big(L_2([a,b],\mu_c)\big)$. Then

$$M(B) = \begin{pmatrix} c+2 & \sqrt{c} & & & 0 \\ \sqrt{c} & c+1 & \sqrt{c} & & \\ & \sqrt{c} & c+1 & \sqrt{c} & \\ & & \ddots & \ddots & \ddots \\ & & & & \\ 0 & & & & \end{pmatrix}, \tag{7.24}$$

so as above, we get that

$$M(\exp(tB))_{jk} > 0, \quad \text{for all } j,k \text{ in } \mathbb{N}_0. \tag{7.25}$$

Let U be the unitary operator on $L_2([a,b],\mu_c)$, defined by the equation:

$$U\rho_q^c = (-1)^q \rho_q^c, \qquad (q \in \mathbb{N}_0).$$

Then

$$M(UBU^*) = \begin{pmatrix} c+2 & -\sqrt{c} & & & 0 \\ -\sqrt{c} & c+1 & -\sqrt{c} & & \\ & -\sqrt{c} & c+1 & -\sqrt{c} & \\ & & \ddots & \ddots & \ddots \\ & & & & \\ 0 & & & & \end{pmatrix} = M(2(c+1)\mathbf{1} - A).$$

Hence $A = 2(c+1)\mathbf{1} - UBU^*$, and for t in $[0,\infty[$, we have thus that

$$\exp(-tA) = \exp(-2(c+1)t)\exp(tUBU^*) = \exp(-2(c+1)t)U\exp(tB)U^*.$$

Therefore,

$$M(\exp(-tA))_{jk} = (-1)^{j+k}\exp(-2(c+1)t)M(\exp(tB))_{jk}, \qquad (j,k \in \mathbb{N}_0), \tag{7.26}$$

so in particular, by (7.25),

$$(-1)^{j+k}M(\exp(-tA))_{jk} > 0, \qquad (j,k \in \mathbb{N}_0).$$

For t in $[0,\infty[$, we note here that

$$\psi_q^c(-t) = c^{-q}\int_a^b \exp(-tx)P_q^c(x)P_0^c(x)\, d\mu_c(x) = c^{-\frac{q}{2}}M(\exp(-tA))_{q0}, \tag{7.27}$$

and hence it follows that $(-1)^q\psi_q(-t) > 0$, which proves (ii).
To prove (iii), we need the following technical lemma:

7.6 Lemma. *Let C and D be bounded positive selfadjoint operators on $\ell_2(\mathbb{N}_0)$, and assume that the corresponding matrices $(c_{jk})_{j,k\in\mathbb{N}_0}$ and $(d_{jk})_{j,k\in\mathbb{N}_0}$ satisfy the following conditions:*

(a) $c_{jk} \geq 0$ *for all* j,k *in* $\mathbb{N}_0$.
(b) $c_{jk} = 0$ *when* $|j-k| \geq 2$.
(c) $d_{jk} = c_{jk}$, *when* $(j,k) \neq (0,0)$.
(d) $d_{00} \geq c_{00}$.

For φ,ψ in $\ell_2(\mathbb{N}_0)$, we define

$$[\varphi,\psi]_{j,k} = \varphi(j)\psi(k) - \varphi(k)\psi(j), \qquad (j,k \in \mathbb{N}_0).$$

Consider then furthermore f,g from $\ell_2(\mathbb{N}_0)$, satisfying that

(e) $f(k) \geq 0$ *and* $g(k) \geq 0$ *for all* k *in* $\mathbb{N}_0$.
(f) $[f,g]_{j,k} \geq 0$, *for all* k,j *in* $\mathbb{N}_0$ *such that* $k > j$.

Then for all j,k in $\mathbb{N}_0$, such that $k > j$, we have that

(i) $[Cf,Cg]_{j,k} \geq 0$.
(ii) $[Df,Cg]_{j,k} \geq 0$.
(iii) $[D^nf,C^ng]_{j,k} \geq 0$, *for all* n *in* $\mathbb{N}$.
(iv) $[\exp(tD)f,\exp(tC)g]_{j,k} \geq 0$, *for all* t *in* $[0,\infty[$.

7.7 Remark. If φ, ψ are strictly positive functions in $\ell_2(\mathbb{N}_0)$, then the statement

$$[\varphi, \psi]_{j,k} \geq 0, \quad \text{for all } j, k \text{ in } \mathbb{N}_0, \text{ such that } k > j,$$

is equivalent to the condition that

$$\frac{\varphi(0)}{\psi(0)} \geq \frac{\varphi(1)}{\psi(1)} \geq \frac{\varphi(2)}{\psi(2)} \geq \cdots . \qquad \square$$

Proof of Lemma 7.6. Note first that for any φ, ψ in $\ell_2(\mathbb{N}_0)$ and j, k in $\mathbb{N}_0$, we have that $[\varphi, \psi]_{j,k} = -[\varphi, \psi]_{k,j}$. In particular,

$$[\varphi, \psi]_{j,j} = 0, \qquad (\varphi, \psi \in \ell_2(\mathbb{N}_0),\ j \in \mathbb{N}_0). \tag{7.28}$$

Note also that the positivity of C implies that

$$\det \begin{pmatrix} c_{jj} & c_{jk} \\ c_{kj} & c_{kk} \end{pmatrix} \geq 0, \qquad \text{for all } j, k \text{ in } \mathbb{N}_0, \text{ such that } j \neq k. \tag{7.29}$$

To prove (i), consider k, j in $\mathbb{N}_0$, such that $k > j \geq 0$. We then have

$$(Cf)(j) = \begin{cases} c_{j,j-1} f(j-1) + c_{j,j} f(j) + c_{j,j+1} f(j+1), & \text{if } j \geq 1, \\ c_{0,0} f(0) + c_{0,1} f(1), & \text{if } j = 0, \end{cases}$$

and since $k \neq 0$,

$$(Cg)(k) = c_{k,k-1} g(k-1) + c_{k,k} g(k) + c_{k,k+1} g(k+1).$$

Thus,

$$[Cf, Cg]_{j,k} = \begin{cases} \sum_{l=j-1}^{j+1} \sum_{m=k-1}^{k+1} c_{jl} c_{km} [f, g]_{l,m}, & \text{if } j \geq 1, \\ \sum_{l=0}^{1} \sum_{m=k-1}^{k+1} c_{0l} c_{km} [f, g]_{l,m}, & \text{if } j = 0. \end{cases}$$

Assume first that $k \geq j+2$. In this case, $l \leq j+1 \leq k-1 \leq m$, for all terms in the above sums, and thus, by (f) and (7.28), $[f, g]_{l,m} \geq 0$. Since $c_{lm} \geq 0$ for all l, m in $\mathbb{N}_0$ (by (a)), it follows thus that $[Cf, Cg]_{j,k} \geq 0$.
Assume next that $k = j+1$, and consider first the case $j \geq 1$. Then

$$[Cf, Cg]_{j,k} = \sum_{l=j-1}^{j+1} \sum_{m=j}^{j+2} c_{jl} c_{j+1,m} [f, g]_{l,m}. \tag{7.30}$$

In 8 of the 9 terms in the sum above, $l \leq m$, and hence $[f, g]_{l,m} \geq 0$. Only in the case $(l, m) = (j+1, j)$, do we have $l > m$. However, the sum of the two terms corresponding to $(l, m) = (j, j+1)$ and $(l, m) = (j+1, j)$ is non-negative, since

$$\begin{aligned} c_{jj} c_{j+1,j+1} [f, g]_{j,j+1} &+ c_{j,j+1} c_{j+1,j} [f, g]_{j+1,j} \\ &= (c_{jj} c_{j+1,j+1} - c_{j,j+1} c_{j+1,j}) [f, g]_{j,j+1}, \end{aligned}$$

which is non-negative by (7.29). Since the remaining 7 terms in the sum on the right hand side of (7.30) are also non-negative, it follows that $[Cf, Cg]_{j,k} \geq 0$. If $j = 0$, and $k = j + 1 = 1$, the same argument can be used to show that

$$[Cf, Cg]_{0,1} = \sum_{l=0}^{1} \sum_{m=0}^{2} c_{0l} c_{1m} [f, g]_{l,m} \geq 0.$$

This proves (i).
To prove (ii), note first that by (a) and (c), we have

$$\big(Df\big)(j) = \big(Cf\big)(j), \quad \text{if } j \geq 1,$$

and

$$\big(Df\big)(0) = \big(Cf\big)(0) + (d_{00} - c_{00}) f(0).$$

Hence, if $k > j \geq 1$, we get from (i), that

$$[Df, Cg]_{j,k} = [Cf, Cg]_{j,k} \geq 0.$$

If $k > j = 0$, then

$$\begin{aligned} [Df, Cg]_{0,k} &= \big(Df\big)(0)\big(Cg\big)(k) - \big(Df\big)(k)\big(Cg\big)(0) \\ &= [Cf, Cg]_{0,k} + (d_{00} - c_{00}) f(0) (Cg)(k). \end{aligned}$$

But $(d_{00} - c_{00}) f(0) \geq 0$ by (d) and (e), and since also $(Cg)(k) = \sum_{l=0}^{\infty} c_{kl} g(l) \geq 0$, by (a) and (e), it follows by (i), that also $[Df, Cg]_{0,k} \geq 0$. This proves (ii).
Next, (iii) follows from (ii) and induction on n, and from noting (by induction), that $(D^n f)(j), (C^n g)(j) \geq 0$ for all n in $\mathbb{N}$ and j in $\mathbb{N}_0$.
To prove (iv), we let t be a fixed number in $[0, \infty[$, and put

$$C_n = \mathbf{1} + \tfrac{t}{n} C, \quad \text{and} \quad D_n = \mathbf{1} + \tfrac{t}{n} D, \qquad (n \in \mathbb{N}_0).$$

Then, for all n, C_n and D_n are positive selfadjoint operators on $\ell_2(\mathbb{N}_0)$, which also satisfy the requirements (a)-(d). Hence, if $f, g \in \ell_2(\mathbb{N}_0)$ which satisfy (e) and (f), we conclude from (iii), that

$$\big[\big(\mathbf{1} + \tfrac{t}{n} D\big)^n f, \big(\mathbf{1} + \tfrac{t}{n} C\big)^n g\big]_{j,k} \geq 0, \qquad \text{when } j > k,$$

and hence, letting $n \to \infty$, we get that

$$\big[\exp(tD) f, \exp(tC) g\big]_{j,k} \geq 0, \qquad \text{when } j > k,$$

as desired. □

End of Proof of Proposition 7.5. Only (iii) in Proposition 7.5 remains to be proved. Let A, B from $\mathcal{B}\big(L_2([a, b], \mu_c)\big)$ be as in the first part of the proof of Proposition 7.5. Since A is the multiplication operator associated to a positive function on $[a, b]$, and since $B \geq A$, both A and B are positive selfadjoint operators on $L_2([a,b],\mu_c)$. Let C and D be the operators in $\mathcal{B}\big(\ell_2(\mathbb{N}_0)\big)$ corresponding to A and B respectively, via the natural Hilbert space isomorphism between $L_2([a,b],\mu_c)$ and $\ell_2(\mathbb{N}_0)$, given by the orthonormal basis $(\rho_q^c)_{q\in\mathbb{N}_0}$ for $L_2([a,b],\mu_c)$. Then C and D are positive selfadjoint operators and by (7.22) and (7.24), they satisfy the conditions (a)-(d) of Lemma 7.6. Now, let both f and g be the first basis vector in the natural basis for $\ell_2(\mathbb{N}_0)$ (i.e., $f(k)=g(k)=\delta_{k,0}$ for all k in $\mathbb{N}_0$). Then (e),(f) of Lemma 7.6 are also satisfied, and hence we obtain from (iv) of that lemma, that for all j,k in $\mathbb{N}_0$ such that $k>j$,

$$\big(\exp(tD)f\big)(j)\big(\exp(tC)f\big)(k)-\big(\exp(tD)f\big)(k)\big(\exp(tC)f\big)(j)\geq 0,$$

i.e.,

$$\big\langle \exp(tB)\rho_0^c,\rho_j^c\big\rangle\cdot\big\langle\exp(tA)\rho_0^c,\rho_k^c\big\rangle\geq\big\langle\exp(tB)\rho_0^c,\rho_k^c\big\rangle\cdot\big\langle\exp(tA)\rho_0^c,\rho_j^c\big\rangle.$$

For $j=0$, we get in particular,

$$\frac{M(\exp(tB))_{k,0}}{M(\exp(tA))_{k,0}}\leq\frac{M(\exp(tB))_{0,0}}{M(\exp(tA))_{0,0}},\qquad (k\in\mathbb{N}_0).\tag{7.31}$$

Note here, that by (7.26),

$$(-1)^kM(\exp(-tA))_{k,0}=\exp(-2(c+1)t)M(\exp(tB))_{k,0}>0,\qquad (k\in\mathbb{N}_0).$$

Inserting this in (7.31), it follows that

$$\frac{(-1)^kM(\exp(-tA))_{k,0}}{M(\exp(tA))_{k,0}}\leq\frac{M(\exp(-tA))_{0,0}}{M(\exp(tA))_{0,0}},\qquad (k\in\mathbb{N}_0).\tag{7.32}$$

By (7.23) and (7.27),

$$M(\exp(\pm tA))_{k,0}=c^{-\frac{k}{2}}\int_a^b\exp(\pm tx)P_k^c(x)\,d\mu_c(x)=c^{\frac{k}{2}}\psi_k^c(\pm t),\qquad (k\in\mathbb{N}_0).$$

Hence, (iii) in Proposition 7.5 follows from (7.32). □

7.8 Theorem. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\sum_{i=1}^r a_i^*a_i=c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, and $\sum_{i=1}^r a_ia_i^*=\mathbf{1}_{\mathcal{B}(\mathcal{K})}$, for some constant c in $[1,\infty[$. Consider furthermore independent elements $Y_1,\dots,Y_r$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and put $S=\sum_{i=1}^r a_i\otimes Y_i$. Then for any t in $[0,\frac{n}{2c}]$,*

$$\mathbb{E}\big[\exp(-tS^*S)\big]\leq\exp\big(-(\sqrt{c}-1)^2t+(c+1)^2\cdot\tfrac{t^2}{n}\big)\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.\tag{7.33}$$

Proof. Consider a fixed t in $[0,\frac{n}{2c}]$. By Proposition 7.4 and Proposition 7.5 we then have

$$\begin{aligned}\big\|\mathbb{E}\big[\exp(-tS^*S)\big]\big\|&\leq\sum_{q=0}^{\infty}|\psi_q^c(-t)|\cdot\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\|\\&\leq\frac{\psi_0^c(-t)}{\psi_0^c(t)}\sum_{q=0}^{\infty}\psi_q^c(t)\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\|.\end{aligned}\tag{7.34}$$

From (7.16) in the proof of Proposition 7.4, we have here that

$$\begin{aligned}\sum_{q=0}^{\infty}\psi_q^c(t)\cdot\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| &\le \sum_{p=0}^{\infty}\tfrac{1}{p!}\big(\tfrac{ct}{n}\big)^p\sum_{\pi\in S_p}n^{k(\hat{\pi})}\big(\tfrac{n}{c}\big)^{l(\hat{\pi})-1}\\ &\le \sum_{p=0}^{\infty}\tfrac{1}{p!}\big(\tfrac{ct}{n}\big)^p\sum_{\substack{k,l\in\mathbb{N}\\ k+l\le p+1}}\delta(p,k,l)n^k\big(\tfrac{n}{c}\big)^{l-1},\end{aligned}$$

where $\delta(p,k,l)$ was introduced in (3.6). Applying now Lemma 3.6, we get for t in $[0,\frac{n}{2c}]$, that

$$\begin{aligned}\sum_{q=0}^{\infty}\psi_q^c(t)\cdot\big\|\mathbb{E}\big[P_q^c(S^*S)\big]\big\| &\le \exp\big((n+\tfrac{n}{c})(\tfrac{ct}{n})^2\big)\int_a^b\exp\big(\tfrac{n}{c}(\tfrac{ct}{n}x)\big)\,d\mu_c(x)\\ &\le \exp\big((c+1)^2\cdot\tfrac{t^2}{n}\big)\int_a^b\exp(tx)\,d\mu_c(x).\end{aligned}$$

Note here, that $\psi_0^c(t)=\int_a^b\exp(tx)\,d\mu_c(x)$, and hence we get by (7.34), that

$$\begin{aligned}\big\|\mathbb{E}\big[\exp(-tS^*S)\big]\big\| &\le \exp\big((c+1)^2\cdot\tfrac{t^2}{n}\big)\psi_0^c(-t)\\ &= \exp\big((c+1)^2\cdot\tfrac{t^2}{n}\big)\int_a^b\exp(-tx)\,d\mu_c(x).\end{aligned}$$

But $\exp(-tx)\le\exp(-ta)=\exp(-t(\sqrt{c}+1)^2)$ for all x in $[a,b]$, and hence it follows that

$$\big\|\mathbb{E}\big[\exp(-tS^*S)\big]\big\|\le\exp\big((c+1)^2\cdot\tfrac{t^2}{n}\big)\exp(-(\sqrt{c}-1)^2t),\qquad(t\in[0,\tfrac{n}{2c}]).$$

This proves (7.33). □

7.9 Remark. By application of the method of Remark 3.7, it is easy to extend Theorem 7.8, to the case where

$$\sum_{i=1}^{r}a_i^*a_i=c\mathbf{1}_{\mathcal{B}(\mathcal{H})},\qquad\text{and}\qquad\sum_{i=1}^{r}a_ia_i^*=d\mathbf{1}_{\mathcal{B}(\mathcal{K})},$$

for constants c,d such that $c\ge d>0$. In this case, one obtains that for t in $[0,\frac{n}{2c}]$,

$$\mathbb{E}\big[\exp(-tS^*S)\big]\le\exp\big(-(\sqrt{c}-\sqrt{d})^2t+(c+d)^2\cdot\tfrac{t^2}{n}\big)\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H}^n)}.\qquad\square$$

8 Asymptotic Lower Bound on the Spectrum of $S_n^*S_n$ in the Exact Case

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and consider elements $a_1, \dots, a_r$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$. Let $\mathcal{A}$ denote the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, generated by the family $\{a_i^*a_j \mid i,j \in \{1,2,\dots,r\}\}$. Consider furthermore, for each n in $\mathbb{N}$, independent elements $Y_1^{(n)}, \dots, Y_r^{(n)}$ of $\mathrm{GRM}(n, n, \frac{1}{n})$, and define

$$S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}, \qquad (n \in \mathbb{N}). \tag{8.1}$$

In this section, we shall determine (almost surely), the asymptotic behavior of the *smallest* element of the spectrum of $S_n^*S_n$, under the assumptions that $\mathcal{A}$ is an exact C^*-algebra and that $a_1, \dots, a_r$ satisfy the condition

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \sum_{i=1}^r a_ia_i^* \le \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \tag{8.2}$$

for some constant c in $[1, \infty[$. We start, however, by considering the simpler case, where, instead of (8.2), $a_1, \dots, a_r$, satisfy the stronger condition

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \qquad \text{and} \qquad \sum_{i=1}^r a_ia_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})}, \tag{8.3}$$

for some constant c in $[1, \infty[$. Once this simpler case has been handled, we obtain the more general case by virtue of a dilation result.
As in Section 4, we determine first the asymptotic behavior of the smallest eigenvalue of V_n, where

$$V_n = \big(\Phi \otimes \mathrm{id}_n\big)(S_n^*S_n), \qquad (n \in \mathbb{N}), \tag{8.4}$$

and $\Phi\colon \mathcal{A} \to M_d(\mathbb{C})$ is a completely positive mapping, for some d in $\mathbb{N}$.

8.1 Lemma. *Let S_n, $n \in \mathbb{N}$, and V_n, $n \in \mathbb{N}$, be as in* (8.1) *and* (8.4), *and assume that $a_1, \dots, a_r$ satisfy the condition* (8.3). *Let $\lambda_{\min}(V_n)$ denote the smallest eigenvalue of V_n (considered as an element of $M_{dn}(\mathbb{C})$). Then for any ϵ in $]0, \infty[$, we have that*

$$\sum_{n=1}^\infty P\big(\lambda_{\min}(V_n) \le (\sqrt{c} - 1)^2 - \epsilon\big) < \infty.$$

Proof. The proof is basically the same as the proof of Lemma 4.2; the main difference being that in this proof we apply Theorem 7.8 instead of Theorem 3.3. Consequently, we shall not repeat all details in this proof.

For fixed n in $\mathbb{N}$, and arbitrary t in $]0,\infty[$, we find that

$$\begin{aligned} P\big(\lambda_{\min}(V_n) &\le (\sqrt{c}-1)^2-\epsilon\big) \\ &= P\big(\exp\big(-t\lambda_{\min}(V_n)+t(\sqrt{c}-1)^2-t\epsilon\big)\ge 1\big) \\ &\le \exp(t(\sqrt{c}-1)^2-t\epsilon)\cdot\mathbb{E}\big[\exp\big(-t\lambda_{\min}(V_n)\big)\big] \\ &\le \exp(t(\sqrt{c}-1)^2-t\epsilon)\cdot\mathbb{E}\circ\mathrm{Tr}_{dn}\big[\exp(-tV_n)\big]. \end{aligned} \tag{8.5}$$

By application of Lemma 4.1(ii), we have here, that

$$\begin{aligned} \mathrm{tr}_{dn}\big[\exp(-tV_n)\big] &= \mathrm{tr}_{dn}\big[\exp\big(-t(\Phi\otimes\mathrm{id}_n)(S_n^*S_n)\big)\big] \\ &\le \mathrm{tr}_{dn}\big[\big(\Phi\otimes\mathrm{id}_n\big)(\exp(-tS_n^*S_n))\big] \\ &= \mathrm{tr}_d\otimes\mathrm{tr}_n\big[\big(\Phi\otimes\mathrm{id}_n\big)(\exp(-tS_n^*S_n))\big] \\ &= \phi\otimes\mathrm{tr}_n\big[\exp(-tS_n^*S_n)\big], \end{aligned} \tag{8.6}$$

where ϕ is the state $\mathrm{tr}_d\circ\Phi$ on $\mathcal{A}$. It follows here from Definition 3.1 and Theorem 7.8, that

$$\begin{aligned} \mathbb{E}\big[\phi\otimes\mathrm{tr}_n\big(\exp(-tS_n^*S_n)\big)\big] &= \phi\otimes\mathrm{tr}_n\big(\mathbb{E}\big[\exp(-tS_n^*S_n)\big]\big) \\ &\le \exp\big(-t(\sqrt{c}-1)^2+\tfrac{t^2}{n}(c+1)^2\big), \end{aligned} \tag{8.7}$$

for all t in $]0,\frac{n}{2c}]$. Combining now (8.5)-(8.7), it follows that for all t in $]0,\frac{n}{2c}]$,

$$\begin{aligned} P\big(\lambda_{\min}(V_n) \le&(\sqrt{c}-1)^2-\epsilon\big) \\ &\le dn\cdot\exp(t(\sqrt{c}-1)^2-t\epsilon)\cdot\exp\big(-t(\sqrt{c}-1)^2+\tfrac{t^2}{n}(c+1)^2\big) \\ &= dn\cdot\exp\big(t(\tfrac{t}{n}(c+1)^2-\epsilon)\big). \end{aligned}$$

From here, the proof is concluded exactly as the proof of Theorem 4.2. □

8.2 Proposition. *Let S_n, $n\in\mathbb{N}$, and V_n, $n\in\mathbb{N}$, be as in* (8.1) *and* (8.4), *and assume that $a_1,\dots,a_r$ satisfy the condition* (8.3). *We then have*

$$\liminf_{n\to\infty}\lambda_{\min}(V_n)\ge\big(\sqrt{c}-1\big)^2, \quad \textit{almost surely.}$$

Proof. By Lemma 4.2 and the Borel-Cantelli Lemma (cf. [Bre, Lemma 3.14]), we have for any ϵ from $]0,\infty[$, that

$$P\big(\lambda_{\min}(V_n)\ge(\sqrt{c}-1)^2-\epsilon, \text{ for all but finitely many } n\big)=1,$$

and from this the proposition follows readily. □

The next two lemmas enable us to pass from the situation considered in Proposition 8.2 to the more general situation, where it is only assumed that $a_1,\dots,a_r$ satisfy (8.2).

8.3 Lemma. *Let c be a number in $[1,\infty[$, and put $q = 2+[c]$, where $[c]$ denotes the integer part of c. Then there exist elements $x_1,\dots,x_q$ in the Cuntz algebra O_2, such that*

$$\sum_{i=1}^{q} x_i^* x_i = c\mathbf{1}_{O_2}, \qquad \text{and} \qquad \sum_{i=1}^{q} x_i x_i^* = \mathbf{1}_{O_2}.$$

Proof. Recall that O_2 is the unital C^*-algebra $C^*(s_1,s_2)$ generated by two operators s_1, s_2 satisfying that $s_i^* s_j = \delta_{i,j}\mathbf{1}_{O_2}$, $i,j \in \{1,2\}$, and that $s_1 s_1^* + s_2 s_2^* = \mathbf{1}_{O_2}$. Define then $t_1,\dots,t_{q-1}$ in O_2, by the expression

$$t_j = \begin{cases} s_2^{j-1}s_1, & \text{if } j \in \{1,2,\dots,q-2\}, \\ s_2^{q-2}, & \text{if } j = q-1. \end{cases}$$

Then $t_i^* t_j = \delta_{i,j}\mathbf{1}_{O_2}$, for all i,j in $\{1,2,\dots,q-1\}$, and

$$\sum_{j=1}^{q-1} t_j t_j^* = \sum_{i=0}^{q-3} s_2^i(\mathbf{1}_{O_2} - s_2 s_2^*)(s_2^i)^* + s_2^{q-2}(s_2^{q-2})^* = \mathbf{1}_{O_2}, \tag{8.8}$$

(i.e., $t_1,\dots,t_{q-1}$ generates a copy of O_{q-1} inside O_2). Define now $x_1,\dots,x_q$ in O_2, by

$$x_i = \begin{cases} \Big(\dfrac{c-1}{q-2}\Big)^{\frac12} t_i, & \text{if } i \in \{1,2,\dots,q-1\} \\ \Big(\dfrac{q-1-c}{q-2}\Big)^{\frac12}\mathbf{1}_{O_2}, & \text{if } i = q. \end{cases}$$

Then

$$\sum_{i=1}^{q} x_i^* x_i = (q-1)\cdot\frac{c-1}{q-2}\cdot\mathbf{1}_{O_2} + \frac{q-1-c}{q-2}\cdot\mathbf{1}_{O_2} = c\mathbf{1}_{O_2},$$

and by (8.8),

$$\sum_{i=1}^{q} x_i x_i^* = \frac{c-1}{q-2}\cdot\mathbf{1}_{O_2} + \frac{q-1-c}{q-2}\cdot\mathbf{1}_{O_2} = \mathbf{1}_{O_2}.$$

Thus, $x_1,\dots,x_q$ have the desired properties. $\square$

8.4 Lemma. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, and $\sum_{i=1}^r a_i a_i^* \le \mathbf{1}_{\mathcal{B}(\mathcal{K})}$.*
Then there exist Hilbert spaces $\tilde{\mathcal{H}},\tilde{\mathcal{K}}$, s in $\{r, r+1, r+2,\dots\}$ and elements $\tilde a_1,\dots,\tilde a_s$ of $\mathcal{B}(\tilde{\mathcal{H}},\tilde{\mathcal{K}})$, such that the following conditions hold:
(i) $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ *and* $\tilde{\mathcal{K}} \supseteq \mathcal{K}$.
(ii) $\tilde a_{i|\mathcal{H}} = \begin{cases} a_i, & \text{if } 1 \le i \le r, \\ 0, & \text{if } r+1 \le i \le s. \end{cases}$
(iii) $\sum_{i=1}^s \tilde a_i^* \tilde a_i = c\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})}$ *and* $\sum_{i=1}^s \tilde a_i \tilde a_i^* = \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{K}})}$.

Proof. By Lemma 8.3, we may choose finitely many elements $x_1, \dots, x_q$ of the Cuntz algebra O_2, such that $\sum_{i=1}^q x_i^* x_i = c\mathbf{1}_{O_2}$ and $\sum_{i=1}^q x_i x_i^* = \mathbf{1}_{O_2}$. We assume that O_2 is represented on some Hilbert space $\mathcal{L}$, so that $x_1, \dots, x_r \in \mathcal{B}(\mathcal{L})$. Define then

$$\tilde{\mathcal{H}} = (\mathcal{H} \otimes \mathcal{L}) \oplus (\mathcal{K} \otimes \mathcal{L}) \quad \text{and} \quad \tilde{\mathcal{K}} = (\mathcal{K} \otimes \mathcal{L}) \oplus (\mathcal{H} \otimes \mathcal{L}).$$

For Hilbert spaces $\mathcal{V}, \mathcal{W}$, an element v of $\mathcal{B}(\mathcal{V}, \mathcal{W})$, and an element y of $\mathcal{B}(\mathcal{L})$, we consider $v \otimes y$ as an element of $\mathcal{B}(\mathcal{V} \otimes \mathcal{L}, \mathcal{W} \otimes \mathcal{L})$ in the natural manner. Moreover, given v_{11} in $\mathcal{B}(\mathcal{H} \otimes \mathcal{L}, \mathcal{K} \otimes \mathcal{L})$, v_{12} in $\mathcal{B}(\mathcal{K} \otimes \mathcal{L})$, v_{21} in $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ and v_{22} in $\mathcal{B}(\mathcal{K} \otimes \mathcal{L}, \mathcal{H} \otimes \mathcal{L})$, we shall consider the matrix $(v_{ij})_{1 \le i,j \le 1}$ as an element of $\mathcal{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ in the usual way. With these conventions, consider now the following elements of $\mathcal{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$,

$$\begin{aligned}
\tilde{a}_i &= \begin{pmatrix} a_i \otimes \mathbf{1}_{\mathcal{B}(\mathcal{L})} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (i \in \{1, 2, \dots, r\}), \\
b_j &= \begin{pmatrix} 0 & (\mathbf{1}_{\mathcal{B}(\mathcal{K})} - \sum_{i=1}^r a_i a_i^*)^{\frac{1}{2}} \otimes x_j \\ 0 & 0 \end{pmatrix}, \qquad (j \in \{1, 2, \dots, q\}), \\
c_{i,j,k} &= \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{c}} \cdot a_i^* \otimes (x_j x_k) \end{pmatrix}, \qquad (i \in \{1, 2, \dots, r\},\ j, k \in \{1, 2, \dots, q\}).
\end{aligned}$$

It follows then by direct calculation, that

$$\begin{aligned}
&\sum_{i=1}^r \tilde{a}_i^* \tilde{a}_i + \sum_{j=1}^q b_j^* b_j + \sum_{i=1}^r \sum_{j,k=1}^q c_{i,j,k}^* c_{i,j,k} \\
&\quad = \begin{pmatrix} \big[\sum_{i=1}^r a_i^* a_i\big] \otimes \mathbf{1}_{\mathcal{B}(\mathcal{L})} & 0 \\ 0 & \big[c(\mathbf{1}_{\mathcal{B}(\mathcal{K})} - \sum_{i=1}^r a_i a_i^*) + c \sum_{i=1}^r a_i a_i^*\big] \otimes \mathbf{1}_{\mathcal{B}(\mathcal{L})} \end{pmatrix} \\
&\quad = c\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})},
\end{aligned}$$

and that

$$\begin{aligned}
&\sum_{i=1}^r \tilde{a}_i \tilde{a}_i^* + \sum_{j=1}^q b_j b_j^* + \sum_{i=1}^r \sum_{j,k=1}^q c_{i,j,k} c_{i,j,k}^* \\
&\quad = \begin{pmatrix} \big[\sum_{i=1}^r a_i a_i^* + (\mathbf{1}_{\mathcal{B}(\mathcal{K})} - \sum_{i=1}^r a_i a_i^*)\big] \otimes \mathbf{1}_{\mathcal{B}(\mathcal{L})} & 0 \\ 0 & \big[\frac{1}{c} \sum_{i=1}^r a_i^* a_i\big] \otimes \mathbf{1}_{\mathcal{B}(\mathcal{L})} \end{pmatrix} \\
&\quad = \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{K}})}.
\end{aligned}$$

Thus, if we put $s = r + q + rq^2$, and let $\tilde{a}_{r+1}, \tilde{a}_{r+2}, \dots, \tilde{a}_s$, be new names for the elements in the set $\{b_j \mid j \in \{1, \dots, q\}\} \cup \{c_{i,j,k} \mid i \in \{1, \dots, r\},\ j, k \in \{1, \dots, q\}\}$, then it follows that $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_s$ satisfy condition (iii).

Choosing a fixed unit vector ξ in $\mathcal{L}$, we have natural embeddings $\iota_{\mathcal{H}}\colon \mathcal{H} \to \tilde{\mathcal{H}}$ and $\iota_{\mathcal{K}}\colon \mathcal{K} \to \tilde{\mathcal{K}}$ given by the equations

$$\iota_{\mathcal{H}}(h) = (h \otimes \xi) \oplus 0, \qquad (h \in \mathcal{H}),$$
$$\iota_{\mathcal{K}}(k) = (k \otimes \xi) \oplus 0, \qquad (k \in \mathcal{K}).$$

This justifies (i), and moreover, it is straightforward to check, that under the identifications of $\mathcal{H}$ with $\iota_{\mathcal{H}}(\mathcal{H})$ and $\mathcal{K}$ with $\iota_{\mathcal{K}}(\mathcal{K})$, condition (ii) is satisfied. This concludes the proof. $\square$

8.5 Proposition. *Let S_n, $n \in \mathbb{N}$, and V_n, $n \in \mathbb{N}$, be as in* (8.1) *and* (8.4), *and assume now that $a_1, \dots, a_r$ satisfy the condition* (8.2). *Then*

$$\liminf_{n\to\infty} \lambda_{\min}(V_n) \geq \big(\sqrt{c} - 1\big)^2, \qquad \textit{almost surely.}$$

Proof. By Lemma 8.4, we may choose Hilbert spaces $\tilde{\mathcal{H}}, \tilde{\mathcal{K}}$, s in $\{r, r+1, \dots, \}$ and elements $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_s$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, such that conditions (i)-(iii) of Lemma 8.4 are satisfied. If $r < s$, then for each n in $\mathbb{N}$ we choose additional elements $Y_{r+1}^{(n)}, \dots, Y_s^{(n)}$ of $\mathrm{GRM}(n, n, \frac{1}{n})$, such that $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_s^{(n)}$ are independent. We then define

$$\tilde{S}_n = \sum_{i=1}^{s} \tilde{a}_i \otimes Y_i^{(n)}, \qquad (n \in \mathbb{N}).$$

Recall from (8.4), that

$$V_n = \big(\Phi \otimes \mathrm{id}_n\big)(S_n^* S_n), \qquad (n \in \mathbb{N}),$$

where $\Phi\colon \mathcal{A} \to M_d(\mathbb{C})$ is a completely positive mapping from the C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ generated by $\{a_i^* a_j \mid i, j \in \{1, 2, \dots, r\}\}$, into the matrix algebra $M_d(\mathbb{C})$. By [Pa, Theorem 5.2], there exists a completely positive mapping $\Phi_1\colon \mathcal{B}(\mathcal{H}) \to M_d(\mathbb{C})$ extending Φ. Note that since Φ is unital, so is Φ_1.
Consider next the orthogonal projection $P_{\mathcal{H}}$ of $\tilde{\mathcal{H}}$ onto $\mathcal{H}$. Then the mapping

$$C_{P_{\mathcal{H}}}\colon b \mapsto P_{\mathcal{H}} b P_{\mathcal{H}}\colon \mathcal{B}(\tilde{\mathcal{H}}) \to P_{\mathcal{H}} \mathcal{B}(\tilde{\mathcal{H}}) P_{\mathcal{H}} \simeq \mathcal{B}(\mathcal{H}),$$

is unital completely positive. Hence, so is the mapping $\Phi_2\colon \mathcal{B}(\tilde{\mathcal{H}}) \to M_d(\mathbb{C})$, given by

$$\Phi_2(b) = \Phi_1(P_{\mathcal{H}} b P_{\mathcal{H}}) = \Phi_1 \circ C_{P_{\mathcal{H}}}(b), \qquad (b \in \mathcal{B}(\tilde{\mathcal{H}})).$$

Thus, if we define

$$\tilde{V}_n = \big(\Phi_2 \circ \mathrm{id}_n\big)(\tilde{S}_n^* \tilde{S}_n), \qquad (n \in \mathbb{N}),$$

then it follows from Lemma 8.4(iii) and Proposition 8.2, that

$$\liminf_{n\to\infty} \lambda_{\min}(\tilde{V}_n) \geq \big(\sqrt{c} - 1\big)^2, \qquad \text{almost surely.} \tag{8.9}$$

However, by Lemma 8.4(ii), we have here that

$$\begin{aligned}\tilde{V}_n &= \big(\Phi_2\otimes \mathrm{id}_n\big)\Big[\sum_{i,j=1}^{s} \tilde{a}_i^*\tilde{a}_j\otimes \big(Y_i^{(n)}\big)^*Y_j^{(n)}\Big] = \sum_{i,j=1}^{s}\Phi_2\big(\tilde{a}_i^*\tilde{a}_j\big)\otimes\big(Y_i^{(n)}\big)^*Y_j^{(n)}\\ &= \sum_{i,j=1}^{s}\Phi_1\big(P_{\mathcal{H}}\tilde{a}_i^*\tilde{a}_jP_{\mathcal{H}}\big)\otimes\big(Y_i^{(n)}\big)^*Y_j^{(n)} = \sum_{i,j=1}^{r}\Phi_1(a_i^*a_j)\otimes\big(Y_i^{(n)}\big)^*Y_j^{(n)}\\ &= \sum_{i,j=1}^{r}\Phi(a_i^*a_j)\otimes\big(Y_i^{(n)}\big)^*Y_j^{(n)} = V_n.\end{aligned}$$

Therefore (8.9) yields the desired conclusion. □

It remains now to show that we can replace V_n in Proposition 8.5 by $S_n^*S_n$ itself. Before proceeding with this task, we draw attention to the following simple observation:

8.6 Lemma. *For each n in $\mathbb{N}$, let $\mathcal{B}_n$ be a unital C^*-algebra, and let b_n be an element of $\mathcal{B}_n$. Then for any R in $[0,\infty[$, the following two conditions are equivalent:*

(i) $\limsup_{n\to\infty}\|b_n\|\le R$.

(ii) $\limsup_{n\to\infty}\max(\mathrm{sp}(b_n))\le R$, *and* $\liminf_{n\to\infty}\min(\mathrm{sp}(b_n))\ge -R$.

Proof. This is clear, since, for each n, $\|b_n\|$ is the largest of the two numbers $\max(\mathrm{sp}(b_n))$ and $-\min(\mathrm{sp}(b_n))$. □

8.7 Theorem. *Let $a_1,\dots,a_r$ be elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, such that $\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\sum_{i=1}^r a_ia_i^*\le \mathbf{1}_{\mathcal{B}(\mathcal{K})}$, for some constant c in $[1,\infty[$. Assume, in addition, that the unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, generated by the set $\{a_i^*a_j \mid i,j,\in\{1,2,\dots,r\}\}$, is exact. Consider furthermore, for each n in $\mathbb{N}$, independent elements $Y_1^{(n)},\dots,Y_r^{(n)}$ of $\mathrm{GRM}(n,n,\frac{1}{n})$, and put $S_n=\sum_{i=1}^r a_i\otimes Y_i^{(n)}$, $n\in\mathbb{N}$. We then have*

$$\liminf_{n\to\infty}\min\big[\mathrm{sp}(S_n^*S_n)\big]\ge(\sqrt{c}-1)^2,\qquad \textit{almost surely.}\tag{8.10}$$

Proof. Put $E=\mathrm{span}\{a_i^*a_j\mid i,j\in\{1,2,\dots,r\}\}$, and note that $x^*\in E$ for all x in E, and that $\mathbf{1}_{\mathcal{A}}=c^{-1}\sum_{i=1}^r a_i^*a_i\in E$. Thus, E is a finite dimensional operator system, and since $\mathcal{A}$ is exact, it follows thus from Proposition 4.4, that for any ϵ from $]0,\infty[$, there exist d in $\mathbb{N}$ and a unital completely positive mapping $\Phi\colon\mathcal{A}\to M_d(\mathbb{C})$, such that

$$\big\|\big(\Phi\otimes\mathrm{id}_n\big)(x)\big\|\ge(1-\epsilon)\|x\|,\qquad (n\in\mathbb{N},\ x\in M_n(E)).\tag{8.11}$$

Consider now a fixed ϵ from $]0,\infty[$, let d, Φ be as described above, and define

$$V_n=\big(\Phi\otimes\mathrm{id}_n\big)(S_n^*S_n),\qquad(n\in\mathbb{N}).$$

Recall then from Proposition 4.3 and Proposition 8.5, that

$$\begin{aligned}\limsup_{n\to\infty} \max\big[\mathrm{sp}(V_n)\big] &\le c+1+2\sqrt{c}, &&\text{almost surely,}\\ \liminf_{n\to\infty} \min\big[\mathrm{sp}(V_n)\big] &\ge c+1-2\sqrt{c}, &&\text{almost surely,}\end{aligned}$$

and hence that

$$\begin{aligned}\limsup_{n\to\infty} \max\big[\mathrm{sp}\big(V_n-(c+1)\mathbf{1}_{dn}\big)\big] &\le 2\sqrt{c}, &&\text{almost surely,}\\ \liminf_{n\to\infty} \min\big[\mathrm{sp}\big(V_n-(c+1)\mathbf{1}_{dn}\big)\big] &\ge -2\sqrt{c}, &&\text{almost surely.}\end{aligned}$$

By Lemma 8.6, this means that

$$\limsup_{n\to\infty} \big\|V_n-(c+1)\mathbf{1}_{dn}\big\| \le 2\sqrt{c}, \qquad \text{almost surely.} \tag{8.12}$$

Note here, that since $S_n^*S_n-(c+1)\mathbf{1}_{\mathcal{A}\otimes M_n(\mathbb{C})} \in M_n(E)$, for all n, it follows from (8.11), that

$$\begin{aligned}\big\|S_n^*S_n-(c+1)\mathbf{1}_{\mathcal{A}\otimes M_n(\mathbb{C})}\big\| &\le (1-\epsilon)^{-1}\big\|\big(\Phi\otimes \mathrm{id}_n\big)\big[S_n^*S_n-(c+1)\mathbf{1}_{\mathcal{A}\otimes M_n(\mathbb{C})}\big]\big\|\\ &= (1-\epsilon)^{-1}\big\|V_n-(c+1)\mathbf{1}_{dn}\big\|,\end{aligned}$$

for all n in $\mathbb{N}$. Hence (8.12) implies that

$$\limsup_{n\to\infty} \big\|S_n^*S_n-(c+1)\mathbf{1}_{\mathcal{A}\otimes M_n(\mathbb{C})}\big\| \le (1-\epsilon)^{-1}\cdot 2\sqrt{c}, \qquad \text{almost surely.}$$

Since this holds for arbitrary ϵ from $]0,\infty[$, it follows that actually

$$\limsup_{n\to\infty} \big\|S_n^*S_n-(c+1)\mathbf{1}_{\mathcal{A}\otimes M_n(\mathbb{C})}\big\| \le 2\sqrt{c}, \qquad \text{almost surely.}$$

By Lemma 8.6, this implies, in particular, that

$$\liminf_{n\to\infty} \min\big[\mathrm{sp}(S_n^*S_n)-(c+1)\big] \ge -2\sqrt{c}, \qquad \text{almost surely,}$$

and this proves (8.10). □

8.8 Remark. As for the upper bound (cf. Section 4), Theorem 8.7 does not, in general, hold without the condition, that the C^*-algebra generated by $\{a_i^*a_j \mid 1\le i,j\le r\}$ be exact. In fact, for any c in $]1,\infty[$, it is possible to choose a finite set of elements $a_1,\dots,a_r$ of $\mathcal{B}(\mathcal{H})$, for an infinite dimensional Hilbert space $\mathcal{H}$, such that

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \quad\text{and}\quad \sum_{i=1}^r a_ia_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{H})},$$

but at the same time

$$P\Big(\ 0 \in \mathrm{sp}(S_n^* S_n),\ \text{for all but finitely many } n\Big) = 1,$$

where $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$, as in (8.1). The proof of this is, however, much more complicated than the corresponding proof of the possible violation for the upper bound (cf. Proposition 4.9(ii)), and it will be presented elsewhere. $\square$

9 Comparison of Projections in Exact C^*-algebras and states on the K_0-group

In [Haa], the first named author proved that quasitraces on exact, unital C^*-algebras are traces. This result implies the following two theorems

9.1 Theorem. (cf. [Han], [Haa]) *If $\mathcal{A}$ is an exact, unital, stably finite C^*-algebra, then $\mathcal{A}$ has a tracial state.*

9.2 Theorem. (cf. [BR, Corollary 3.4]) *If $\mathcal{A}$ is an exact, unital C^*-algebra, then every state on $K_0(\mathcal{A})$ comes from a tracial state on $\mathcal{A}$.*

The proof given in [Haa] of the fact that quasitraces in exact unital C^*-algebras are traces, is based on an ultra-product argument, involving ultra products of finite AW^*-algebras. The aim of this section is to show that Theorem 9.1 and Theorem 9.2 can be obtained from the random matrix results of the previous sections, without appealing to results on quasitraces and AW^*-algebras.
We start by recapturing some of the standard notions and notation in connection with comparison theory for projections in C^*-algebras (see e.g. [Bl1],[Bl2],[Cu] and [Go2]). For a C^*-algebra $\mathcal{A}$, we put

$$M_\infty(\mathcal{A}) = \bigcup_{n\in\mathbb{N}} M_n(\mathcal{A}),$$

where elements are identified via the (non-unital) embeddings $M_n(\mathcal{A}) \hookrightarrow M_{n+1}(\mathcal{A})$, given by addition of a row and a column of zeroes. Given two projections p, q in $M_\infty(\mathcal{A})$, we say, as usual, that p and q are (Murray-von Neumann) equivalent, and write $p \sim q$, if there exists a u in $M_\infty(\mathcal{A})$, such that $u^*u = p$ and $uu^* = q$. We let $V(\mathcal{A})$ denote the set of equivalence classes $\langle p\rangle$ of projections p in $M_\infty(\mathcal{A})$, w.r.t. Murray-von Neumann equivalence, and we equip $V(\mathcal{A})$ with an order structure and an addition, as follows: For projections p, q in $M_\infty(\mathcal{A})$, we write $\langle q\rangle \le \langle p\rangle$ if $q \prec p$, i.e., if q is equivalent to a sub-projection of p. Moreover, we define $\langle p\rangle + \langle q\rangle$ to be $\langle p' + q'\rangle$, where p', q' are projections in $M_\infty(\mathcal{A})$, satisfying that $p' \sim p$, $q' \sim q$ and $p' \perp q'$. Finally, for k in $\mathbb{N}$, we let $k\langle p\rangle$ denote the equivalence class $\langle p\rangle + \cdots + \langle p\rangle$ (k summands).
Recall that for a unital C^*-algebra $\mathcal{A}$, $K_0(\mathcal{A})$ is the additive group obtained from the semi group $V(\mathcal{A})$, via the Grothendieck construction (cf. [Bl1]), and

that $K_0(\mathcal{A})_+$ denotes the range of $V(\mathcal{A})$ under the natural map

$$\rho\colon V(\mathcal{A}) \to K_0(\mathcal{A}).$$

In particular, we have that $K_0(\mathcal{A}) = K_0(\mathcal{A})_+ - K_0(\mathcal{A})_+$.
For a projection p in $M_\infty(\mathcal{A})$, we put

$$[p] = \rho(\langle p \rangle).$$

Note then, that for projections p, q in $M_\infty(\mathcal{A})$, $[p] = [q]$ if and only if there exists a projection r in $M_\infty(\mathcal{A})$, such that $\langle p \rangle + \langle r \rangle = \langle q \rangle + \langle r \rangle$.
The four lemmas 9.3-9.6 below are well known and easy, but since we have not been able to find precise references in the literature, we have included proofs of these lemmas.

9.3 Lemma. *Let $\mathcal{A}$ be a C^*-algebra, and let p, q be projections in $\mathcal{A}$. Then with $I(p)$ the ideal in $\mathcal{A}$ generated by p, the following three conditions are equivalent:*
(i) $\langle q \rangle \le k\langle p \rangle$, *for some k in $\mathbb{N}$.*
(ii) $q \in I(p)$.
(iii) $q \in \overline{I(p)}$.

Proof. (i) $\Rightarrow$ (ii) : Assume that (i) holds, i.e., that there exists k in $\mathbb{N}$ and u in $M_k(\mathcal{A})$, such that

$$u^*u = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad uu^* \le \begin{pmatrix} p & & 0 \\ & \ddots & \\ 0 & & p \end{pmatrix}.$$

This implies that u is of the form

$$u = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{k1} & 0 & \cdots & 0 \end{pmatrix},$$

where $u_{11}, u_{21}, \dots, u_{k1} \in p\mathcal{A}q$. It follows thus, that

$$q = \sum_{j=1}^k u_{j1}^* u_{j1} = \sum_{j=1}^k u_{j1}^* p u_{j1} \in I(p),$$

as desired.
(ii) $\Rightarrow$ (iii) : This is trivial.
(iii) $\Rightarrow$ (i) : Assume that (iii) holds. Then there exist k in $\mathbb{N}$ and $a_1, \dots, a_k$, $b_1, \dots, b_k$ in $\mathcal{A}$, such that

$$\Big\| \sum_{j=1}^k a_j p b_j - q \Big\| < 1. \tag{9.1}$$

Thus, by [Go2, 10.7],

$$\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \prec \begin{pmatrix} p & & 0 \\ & \ddots & \\ 0 & & p \end{pmatrix} \quad \text{in} \quad M_k(\mathcal{A}),$$

i.e., $\langle q \rangle \leq k\langle p \rangle$. $\square$

9.4 Lemma. *Let $\mathcal{M}$ be a von Neumann algebra, and let p be a projection in $\mathcal{M}$. Then any σ-weakly lower semi-continuous trace*

$$\tau\colon \big(p\mathcal{M}p\big)_+ \to [0,\infty],$$

has an extension to a σ-weakly lower semi-continuous trace $\tilde{\tau}$ on $\mathcal{M}_+$.

Proof. We can assume that $p \neq 0$. Choose then a maximal family $(p_i)_{i\in I}$ of pairwise orthogonal projections in $\mathcal{M}$, such that $p_i \prec p$ for all i in I. Then, by standard comparison theory, it follows that

$$\sum_{i\in I} p_i = c(p),$$

where $c(p)$ denotes the central support of p in $\mathcal{M}$. Choose next, for each i in I, a partial isometry v_i in $\mathcal{M}$, such that

$$v_i^* v_i = p_i \quad \text{and} \quad v_i v_i^* \leq p, \qquad (i \in I).$$

Define then $\tilde{\tau}\colon \mathcal{M}_+ \to [0,\infty]$, by the equation

$$\tilde{\tau}(a) = \sum_{i\in I} \tau(v_i a v_i^*), \qquad (a \in \mathcal{M}_+).$$

Clearly $\tilde{\tau}$ is additive, homogeneous and σ-weakly lower semi-continuous. To show that $\tilde{\tau}$ has the trace property, note first that since $pv_i = v_i$ for all i, we have also that $c(p)v_i = v_i$ for all i. Since $c(p)$ is in the center of $\mathcal{M}$, it follows thus, that for any x in $\mathcal{M}$,

$$\begin{aligned} \tilde{\tau}(xx^*) &= \sum_{i\in I} \tau\big(v_i x x^* v_i^*\big) = \sum_{i\in I} \tau\big(c(p) v_i x x^* v_i^*\big) \\ &= \sum_{i\in I} \tau\big(v_i x c(p) x^* v_i^*\big) = \sum_{i\in I}\sum_{j\in I} \tau\big((v_i x v_j^*)(v_j x^* v_i^*)\big), \end{aligned}$$

and similarly

$$\tilde{\tau}(x^*x) = \sum_{j\in I}\sum_{i\in I} \tau\big((v_j x^* v_i^*)(v_i x v_j^*)\big).$$

But by the trace property of τ on $p\mathcal{M}p$, we have that

$$\tau\big((v_i x v_j^*)(v_j x^* v_i^*)\big) = \tau\big((v_j x^* v_i^*)(v_i x v_j^*)\big),$$

for all i, j, and since all the terms in the above sums are positive, we can permute their order without changing the sums, and thus obtain

$$\tilde{\tau}(xx^*) = \tilde{\tau}(x^*x).$$

Taken together, we have verified that $\tilde{\tau}$ is a σ-weakly lower semi-continuous trace on $\mathcal{M}_+$, and it remains thus to show that $\tilde{\tau}$ coincides with τ on $\big(p\mathcal{M}p\big)_+$. Given a from $\big(p\mathcal{M}p\big)_+$, we have that $v_i a^{\frac{1}{2}} \in p\mathcal{M}p$, for all i, and therefore

$$\tilde{\tau}(a) = \sum_{i\in I} \tau\big((v_i a^{\frac{1}{2}})(a^{\frac{1}{2}} v_i^*)\big) = \sum_{i\in I} \tau\big(a^{\frac{1}{2}} v_i^* v_i a^{\frac{1}{2}}\big) = \tau\big(a^{\frac{1}{2}} c(p) a^{\frac{1}{2}}\big) = \tau(a),$$

as desired. $\square$

9.5 Lemma. *Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathbf{1}$ denote the unit of $\mathcal{M}$. Let furthermore p, q be projections in $\mathcal{M}$, that satisfy the following two conditions:*
(i) $\mathbf{1} \in I(p)$.
(ii) $\tau(q) \le \tau(p)$, for any normal, tracial state τ on $\mathcal{M}$.
Then $q \prec p$.

Proof. Let $\mathcal{M} = e\mathcal{M} \oplus (\mathbf{1}-e)\mathcal{M}$, be the decomposition of $\mathcal{M}$ into a finite part $e\mathcal{M}$ and a properly infinite part $(\mathbf{1}-e)\mathcal{M}$, by a central projection e. Since any normal, tracial state on $\mathcal{M}$ must vanish on $(\mathbf{1}-e)\mathcal{M}$, condition (ii) is equivalent to the condition

$$\tau(eq) \le \tau(ep), \quad \text{for any normal tracial state } \tau \text{ on } e\mathcal{M}.$$

By comparison theory for finite von Neumann algebras (cf. e.g. [KR, Theorem 8.4.3(vii)]), this condition implies that

$$eq \prec ep \quad \text{in} \quad e\mathcal{M}, \tag{9.2}$$

By Lemma 9.3, condition (i) implies that there exists a k in $\mathbb{N}$, such that

$$\mathbf{1} \otimes e_{11} \prec p \otimes \mathbf{1}_k \quad \text{in} \quad M_k(\mathcal{M}),$$

where $(e_{ij})_{1\le i,j\le k}$ are the usual matrix units in $M_k(\mathbb{C})$. Therefore, we have also that

$$(\mathbf{1}-e) \otimes e_{11} \prec (\mathbf{1}-e)p \otimes \mathbf{1}_k \quad \text{in} \quad M_k((\mathbf{1}-e)\mathcal{M}).$$

At the same time, since $\mathbf{1}-e$ is a properly infinite projection in $\mathcal{M}$, we have that

$$(\mathbf{1}-e) \otimes e_{11} \sim (\mathbf{1}-e) \otimes \mathbf{1}_k \quad \text{in} \quad M_k((\mathbf{1}-e)\mathcal{M}).$$

It follows thus, that

$$(\mathbf{1}-e)q \otimes \mathbf{1}_k \le (\mathbf{1}-e) \otimes \mathbf{1}_k \sim (\mathbf{1}-e) \otimes e_{11} \prec (\mathbf{1}-e)p \otimes \mathbf{1}_k \quad \text{in} \quad M_k((\mathbf{1}-e)\mathcal{M}),$$

and by [KR, Exercise 6.9.14], this implies that

$$(\mathbf{1}-e)q \prec (\mathbf{1}-e)p \quad \text{in} \quad (\mathbf{1}-e)\mathcal{M}. \tag{9.3}$$

Combining (9.2) and (9.3), it follows that $q \prec p$, as desired. □

9.6 Lemma. *Let $\mathcal{M}$ be a von Neumann algebra, and let p, q be projections in $\mathcal{M}$. Then the following two conditions are equivalent*
(i) $q \prec p$.
(ii) $q \in I(p)$, and $\tau(q) \le \tau(p)$ for every σ-weakly lower semi-continuous trace τ on $\mathcal{M}_+$.

Proof. Clearly (i) implies (ii). To show that (ii) implies (i), assume that (ii) holds. By Lemma 9.3 there exists then a k in $\mathbb{N}$, such that $\langle q\rangle \le k\langle p\rangle$, i.e., such that

$$q \otimes e_{11} \sim q' \le p \otimes \mathbf{1}_k,$$

for some projection q' in $M_k(\mathcal{M})$. Consider now the von Neumann algebra

$$\mathcal{N} = M_k(p\mathcal{M}p),$$

with unit $\mathbf{1}_{\mathcal{N}} = p \otimes \mathbf{1}_k$. Set $p' = p \otimes e_{11}$. Then p', q' are both projections in $\mathcal{N}$, and

$$\mathbf{1}_{\mathcal{N}} \in I_{\mathcal{N}}(p'), \tag{9.4}$$

where $I_{\mathcal{N}}(p')$ is the ideal in $\mathcal{N}$ generated by p'.
We show next, that

$$\tau(q') \le \tau(p'), \quad \text{for any normal, tracial state } \tau \text{ on } \mathcal{N}. \tag{9.5}$$

Indeed, if τ is a normal, tracial state on $\mathcal{N}$, then by Lemma 9.4, the restriction $\tau_{|\mathcal{N}_+}$ of τ to $\mathcal{N}_+$ can be extended to a σ-weakly lower semi-continuous trace $\tilde{\tau}$ on $M_k(\mathcal{M})_+$. Then the mapping

$$a \mapsto \tilde{\tau}(a \otimes e_{11}), \qquad (a \in \mathcal{M}_+),$$

is a σ-weakly lower semi-continuous trace on $\mathcal{M}_+$, and hence the assumption (ii) yields that

$$\tilde{\tau}(q \otimes e_{11}) \le \tilde{\tau}(p \otimes e_{11}).$$

Since $q' \sim q \otimes e_{11}$, $p' = p \otimes e_{11}$ and $p', q' \in \mathcal{N}$, it follows thus that

$$\tau(q') = \tilde{\tau}(q') = \tilde{\tau}(q \otimes e_{11}) \le \tilde{\tau}(p \otimes e_{11}) = \tilde{\tau}(p') = \tau(p'),$$

which proves (9.5).
Applying now Lemma 9.5, it follows from (9.4) and (9.5), that $q' \prec p'$ in $\mathcal{N}$, and hence that

$$q \otimes e_{11} \sim q' \prec p' = p \otimes e_{11} \quad \text{in} \quad M_k(\mathcal{M}),$$

which implies that $q \prec p$ in $\mathcal{M}$. □

9.7 Proposition. *Let $\mathcal{A}$ be a C^*-algebra, and let p, q be projections in $\mathcal{A}$. Then the following two conditions are equivalent:*
(i) $q \prec p$ in $\mathcal{A}^{**}$.
(ii) $\tau(q) \le \tau(p)$, for every (norm) lower semi-continuous trace τ on $\mathcal{A}_+$.

Proof. (i) $\Rightarrow$ (ii) : Assume that $q \prec p$ in $\mathcal{A}^{**}$, and choose u in $\mathcal{A}^{**}$, such that $u^*u = q$ and $uu^* \le p$. Then $\|u\| \le 1$, and hence by the Kaplansky Density Theorem, we may choose a net $(u_\beta)_{\beta \in B}$ from $\mathcal{A}$, such that $\|u_\beta\| \le 1$, for all β in B, and $u_\beta \to u$ in the strong (operator) topology.
Define now: $v_\beta = pu_\beta q$, $(\beta \in B)$, and note that $v_\beta \to puq = u$ in the strong (operator) topology, so that $v_\beta^* v_\beta \to u^*u = q$ in the weak (operator) topology. Since $\|v_\beta\| \le 1$ for all β, this implies that actually

$$v_\beta^* v_\beta \to q \ \text{ in the } \sigma\text{-weak topology.}$$

Note also, that since $\|u_\beta\| \le 1$ for all β,

$$v_\beta v_\beta^* \le p, \qquad (\beta \in B). \tag{9.6}$$

Recall now that the σ-weak topology on $\mathcal{A}^{**}$ is the weak* topology i.e., the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology, and hence its restriction to $\mathcal{A}$ is the weak topology, i.e., the $\sigma(\mathcal{A}, \mathcal{A}^*)$-topology. Since $v_\beta \in \mathcal{A}$ for all β, we have thus, that

$$v_\beta^* v_\beta \to q \ \text{ in the } \sigma(\mathcal{A}, \mathcal{A}^*)\text{-topology.}$$

Consider then the convex hull K of $\{v_\beta^* v_\beta \mid \beta \in B\}$. Then $q \in K^{-\sigma(\mathcal{A},\mathcal{A}^*)}$, but since convex sets in a Banach space have the same closure in weak and norm topology (cf. [KR, Theorem 1.3.4]), it follows that actually $q \in K^{-\mathrm{norm}}$. Hence, we may choose a sequence $(w_n)_{n\in\mathbb{N}}$ from K, which converges to q in norm. Then, for any (norm) lower semi-continuous trace $\tau \colon \mathcal{A}_+ \to [0, \infty]$,

$$\tau(q) \le \liminf_{n\to\infty} \tau(w_n) \le \sup_{\beta \in B} \tau(v_\beta^* v_\beta) = \sup_{\beta\in B} \tau(v_\beta v_\beta^*) \le \tau(p), \tag{9.7}$$

and this proves (i).
(ii) $\Rightarrow$ (i) : Assume (ii) holds. We set out to show that condition (ii) in Lemma 9.6 is satisfied, in the case $\mathcal{M} = \mathcal{A}^{**}$. Consider first the function $\tau_0 \colon \mathcal{A}_+ \to [0, \infty]$, defined by

$$\tau_0(a) = \begin{cases} 0, & \text{if } a \in \overline{I_{\mathcal{A}}(p)_+}, \\ \infty, & \text{if } a \in \mathcal{A}_+ \setminus \overline{I_{\mathcal{A}}(p)_+}. \end{cases}$$

Then τ_0 is a (norm) lower semi-continuous trace on $\mathcal{A}_+$, and hence the assumption (ii) yields that $\tau_0(q) \le \tau_0(p) = 0$, which means that $q \in \overline{I_{\mathcal{A}}(p)_+}$. According to Lemma 9.3, this implies that actually $q \in I_{\mathcal{A}}(p) \subseteq I_{\mathcal{A}^{**}}(p)$.
Note next, that for any σ-weakly lower semi-continuous trace τ on $(\mathcal{A}^{**})_+$, the restriction $\tau_{|\mathcal{A}_+}$ is a (norm) lower semi-continuous trace on $\mathcal{A}$, and hence, by the assumption (ii), $\tau(q) \le \tau(p)$.

Taken together, we have verified that the projections p, q satisfy the condition (ii) in Lemma 9.6, in the case $\mathcal{M} = \mathcal{A}^{**}$, and hence this lemma yields that $q \prec p$ in $\mathcal{A}^{**}$, as desired. $\square$

9.8 Corollary. *Let $\mathcal{A}$ be a C^*-algebra, and let p, q be projections in $\mathcal{A}$. Then the following two conditions are equivalent:*
(i) $\exists k \in \mathbb{N}$: $k\langle q\rangle \leq (k-1)\langle p\rangle$ in $V(A^{**})$.
(ii) $\exists \epsilon > 0$: $\tau(q) \leq (1-\epsilon)\tau(p)$, for any (norm) lower semi-continuous trace τ on $\mathcal{A}_+$.

Proof. (i) $\Rightarrow$ (ii) : Assume that (i) holds, and define, for the existing k, $q' = q \otimes \mathbf{1}_k$ and $p' = p \otimes \big(\sum_{i=1}^{k-1} e_{ii}\big)$. Then q', p' are projections in $M_k(\mathcal{A})$, and the assumption (i) implies that

$$q' \prec p' \quad \text{in} \quad M_k(\mathcal{A}^{**}). \tag{9.8}$$

Given now any (norm) lower semi-continuous trace τ on $\mathcal{A}_+$, note that the expression

$$\tau_k(a) = \sum_{i=1}^{k} \tau(a_{ii}), \qquad (a = (a_{ij}) \in M_k(\mathcal{A})_+),$$

defines a (norm) lower semi-continuous trace τ on $\mathcal{M}_k(\mathcal{A})_+$. Thus, by Proposition 9.7, (9.8) implies that $\tau_k(q') \leq \tau_k(p')$, i.e., that $k\tau(q) \leq (k-1)\tau(p)$. This shows that (ii) holds for any ϵ in $]0, \frac{1}{k}]$.
(ii) $\Rightarrow$ (i) : Assume that (ii) holds, and choose, for the existing ϵ, a k in $\mathbb{N}$ such that $\frac{1}{k} \leq \epsilon$. Define then, for this k, q' and p' as above.
Now, for any (norm) lower semi-continuous trace τ on $M_k(\mathcal{A})_+$, the mapping

$$a \mapsto \tau(a \otimes e_{11}), \qquad (a \in \mathcal{A}_+),$$

is a (norm) lower semi-continuous trace on $\mathcal{A}_+$, and thus the assumption (ii) yields that

$$\tau(q \otimes e_{11}) \leq (1-\epsilon)\tau(p \otimes e_{11}) \leq \tfrac{k-1}{k} \cdot \tau(p \otimes e_{11}),$$

and hence that

$$\tau(q') = k \cdot \tau(q \otimes e_{11}) \leq (k-1) \cdot \tau(p \otimes e_{11}) = \tau(p').$$

According to Proposition 9.7, this means that $q' \prec p'$ in $M_k(\mathcal{A}^{**})(= M_k(\mathcal{A})^{**})$, which shows that (i) holds. $\square$

9.9 Lemma. *Let $\mathcal{A}$ be a C^*-algebra, and let p, q be projections in $\mathcal{A}$. Then the following two conditions are equivalent:*
(i) There exists an ϵ in $]0, \infty[$, such that

$$\tau(q) \leq (1-\epsilon)\tau(p), \quad \text{for any (norm) lower semi-continuous trace } \tau \text{ on } \mathcal{A}_+.$$

(ii) There exist ϵ in $]0,\infty[$, r in $\mathbb{N}$ and $a_1,\dots,a_r$ in $\mathcal{A}$, such that

$$\sum_{i=1}^r a_i^* a_i = q, \qquad \text{and} \qquad \sum_{i=1}^r a_i a_i^* \le (1-\epsilon)p.$$

Proof. The proof follows the ideas of the first section of [Haa].
Note first that (ii) clearly implies (i). To show the converse implication, assume that (i) holds. Then, by Corollary 9.8, there exists a k in $\mathbb{N}$, such that

$$q \otimes \mathbf{1}_k \prec p \otimes \left(\textstyle\sum_{i=1}^{k-1} e_{ii}\right) \quad \text{in} \quad M_k(\mathcal{A}^{**}),$$

i.e., such that

$$u^*u = q \otimes \mathbf{1}_k, \qquad \text{and} \qquad uu^* \le p \otimes \left(\textstyle\sum_{i=1}^{k-1} e_{ii}\right), \tag{9.9}$$

for some $u = (u_{ij})_{1\le i,j\le k}$ in $M_k(\mathcal{A}^{**})$. For this u, we have then that

$$\sum_{j=1}^k \sum_{i=1}^k u_{ij}^* u_{ij} = \sum_{j=1}^k (u^*u)_{jj} = kq,$$

and that

$$\sum_{i=1}^k \sum_{j=1}^k u_{ij} u_{ij}^* = \sum_{i=1}^k (uu^*)_{ii} \le (k-1)p.$$

Thus, if $b_1,\dots,b_{k^2} \in \mathcal{A}^{**}$ denote the elements $\frac{1}{\sqrt{k}} u_{ij}$, $i,j \in \{1,2,\dots,k\}$, listed in any fixed order, then we have that

$$\sum_{i=1}^{k^2} b_i^* b_i = q, \qquad \text{and} \qquad \sum_{i=1}^{k^2} b_i b_i^* \le \tfrac{k-1}{k} p.$$

Note also, that (9.9) implies that $b_i \in p\mathcal{A}^{**}q$ for all i. Consider then the subset K of $\mathcal{A} \oplus \mathcal{A}$, defined by

$$K = \left\{ \left(\textstyle\sum_{i=1}^r c_i^* c_i,\, g + \sum_{i=1}^r c_i c_i^*\right) \,\middle|\, r \in \mathbb{N},\ c_1,\dots,c_r \in p\mathcal{A}q,\ g \in (p\mathcal{A}p)_+ \right\}.$$

Then K is clearly closed under addition and multiplication by a non-negative scalar, and thus K is a convex cone in $\mathcal{A} \oplus \mathcal{A}$.
Recall next, that the σ-strong* topology on a von Neumann algebra $\mathcal{M}$, is generated by the semi-norms

$$x \mapsto \varphi(x^*x + xx^*)^{\frac{1}{2}}, \qquad (\varphi \in (\mathcal{M}_*)_+).$$

Since the σ-strong* continuous functionals on $\mathcal{M}$ are also σ-weakly continuous (i.e., belong to $\mathcal{M}_*$; cf. [Ta, Lemma II.2.4]), any convex set in $\mathcal{M}$ has the same closure in σ-strong* and σ-weak topology. In particular it follows that

$p\mathcal{A}q$ is σ-strong* dense in $p\mathcal{A}^{**}q$, and $(p\mathcal{A}p)_+$ is σ-strong* dense in $(p\mathcal{A}^{**}p)_+$.

Thus, we may choose a net $(c_1^\alpha, \dots, c_{k^2}^\alpha, g^\alpha)_{\alpha \in A}$ in $\big[\oplus_{j=1}^{k^2} p\mathcal{A}q\big] \oplus (p\mathcal{A}p)$, such that

- $c_i^\alpha \to b_i,$ in the σ-strong* topology, for all i in $\{1, 2, \dots, k^2\}$,
- $g^\alpha \geq 0,$ for all α,
- $g^\alpha \to \frac{k-1}{k}p - \sum_{i=1}^{k^2} b_i b_i^*,$ in the σ-strong* topology.

It follows then that

$$\lim_\alpha \Big(\sum_{i=1}^{k^2} (c_i^\alpha)^* c_i^\alpha \Big) = q, \quad \sigma\text{-weakly},$$

and that

$$\lim_\alpha \Big(g^\alpha + \sum_{i=1}^{k^2} c_i^\alpha (c_i^\alpha)^* \Big) = \tfrac{k-1}{k}p, \quad \sigma\text{-weakly}.$$

But since the σ-weak topology on $\mathcal{A}^{**}$ is just the weak*-topology (i.e., the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology), its restriction to $\mathcal{A}$ is the weak topology (i.e., the $\sigma(\mathcal{A}, \mathcal{A}^*)$-topology) on $\mathcal{A}$. It follows thus that

$$\big(q, \tfrac{k-1}{k}p\big) \in K^{-\sigma(\mathcal{A}\oplus\mathcal{A}, \mathcal{A}^*\oplus\mathcal{A}^*)}.$$

But convex sets in a Banach space have the same closure in weak and norm topology (cf. [KR, Theorem 1.3.4]), so it follows that in fact

$$\big(q, \tfrac{k-1}{k}p\big) \in K^{-\mathrm{norm}}. \tag{9.10}$$

Since $(1-\delta)^{-1}\big(\frac{k-1}{k} + \delta\big) \to \frac{k-1}{k} < 1$, as $\delta \to 0$, we may choose δ, ϵ in $]0,1[$, such that

$$(1-\delta)^{-1}\big(\tfrac{k-1}{k} + \delta\big) = 1 - \epsilon.$$

By (9.10), there exist then r in $\mathbb{N}$, $c_1, \dots, c_r$ in $p\mathcal{A}q$ and g in $(p\mathcal{A}p)_+$, such that

$$\Big\| q - \textstyle\sum_{i=1}^r c_i^* c_i \Big\| < \delta \qquad \text{and} \qquad \Big\| \tfrac{k-1}{k}p - g - \big(\textstyle\sum_{i=1}^r c_i c_i^*\big) \Big\| < \delta. \tag{9.11}$$

The first inequality in (9.11) implies that $\sum_{i=1}^r c_i^* c_i$ is invertible in the C^*-algebra $q\mathcal{A}q$. Let $h \in (q\mathcal{A}q)_+$ denote the inverse of $\sum_{i=1}^r c_i^* c_i$ in $q\mathcal{A}q$. Since

$$(1-\delta)q \leq \sum_{i=1}^r c_i^* c_i \leq (1+\delta)q,$$

it follows then that

$$(1+\delta)^{-1} q \leq h \leq (1-\delta)^{-1} q. \tag{9.12}$$

Define now: $a_i = c_i h^{\frac{1}{2}}$, $i \in \{1, 2, \dots, r\}$. Then $\sum_{i=1}^r a_i^* a_i = q$, and moreover, by (9.12) and the second inequality in (9.11),

$$\begin{aligned}\sum_{i=1}^r a_i a_i^* = \sum_{i=1}^r c_i h c_i^* &\leq (1-\delta)^{-1} \sum_{i=1}^r c_i c_i^* \leq (1-\delta)^{-1} \Big(g + \sum_{i=1}^r c_i c_i^* \Big) \\ &\leq (1-\delta)^{-1}\big(\tfrac{k-1}{k} + \delta\big)p = (1-\epsilon)p.\end{aligned}$$

Thus, it follows that (ii) holds. ☐

9.10 Theorem. *Let $\mathcal{A}$ be an exact C^*-algebra, and let p, q be projections in $\mathcal{A}$. Assume that there exists ϵ in $]0, \infty[$, such that*

$$\tau(q) \leq (1-\epsilon)\tau(p)$$

for any (norm) lower semi-continuous trace $\tau\colon \mathcal{A}_+ \to [0, \infty]$.
Then there exists n in $\mathbb{N}$, such that

$$q \otimes \mathbf{1}_n \prec p \otimes \mathbf{1}_n \quad \textit{in} \quad M_n(\mathcal{A}).$$

Proof. By Lemma 9.9, we get (after multiplying the a_i's from Lemma 9.9(ii) by $(1-\epsilon)^{-\frac{1}{2}}$), that there exist c in $]1, \infty[$, r in $\mathbb{N}$ and $a_1, \dots, a_r$ in $\mathcal{A}$, such that

$$\sum_{i=1}^{r} a_i^* a_i = cq, \qquad \text{and} \qquad \sum_{i=1}^{r} a_i a_i^* \leq p. \tag{9.13}$$

We may assume that $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then (9.13) implies that we may consider $a_1, \dots, a_r$ as elements of $\mathcal{B}(q(\mathcal{H}), p(\mathcal{H}))$, and that

$$\sum_{i=1}^{r} a_i^* a_i = c\mathbf{1}_{q(\mathcal{H})}, \qquad \text{and} \qquad \sum_{i=1}^{r} a_i a_i^* \leq \mathbf{1}_{p(\mathcal{H})}.$$

Moreover, the set $\{a_i^* a_j \mid i, j \in \{1, 2, \dots, r\}\}$ is contained in the exact, unital C^*-algebra $q\mathcal{A}q$. Choosing now, for each n in $\mathbb{N}$, independent elements $Y_1^{(n)}, \dots, Y_r^{(n)}$ of $\mathrm{GRM}(n, n, \frac{1}{n})$, it follows from Theorem 8.7, that with

$$S_n = \sum_{i=1}^{r} a_i \otimes Y_i^{(n)}, \qquad (n \in \mathbb{N}),$$

we have that

$$\liminf_{n\to\infty} \big[\min\{\mathrm{sp}(S_n(\omega)^* S_n(\omega))\}\big] \geq \big(\sqrt{c}-1\big)^2, \quad \text{for almost all } \omega \text{ in } \Omega.$$

In particular, there exists one(!) ω in Ω, and an n in $\mathbb{N}$, such that $S_n(\omega)^* S_n(\omega)$ is invertible in the C^*-algebra $M_n(q\mathcal{A}q)$. For this pair (ω, n), we define

$$u = S_n(\omega)\big[S_n(\omega)^* S_n(\omega)\big]^{-\frac{1}{2}},$$

where the inverse is formed w.r.t. $M_n(q\mathcal{A}q)$. Then $u \in M_n(p\mathcal{A}q)$, and

$$u^* u = \mathbf{1}_{q(\mathcal{H})} \otimes \mathbf{1}_n = q \otimes \mathbf{1}_n. \tag{9.14}$$

Moreover, $uu^* \in M_n\big(\mathcal{B}(p(\mathcal{H}))\big)$, and since u^*u is a projection in $M_n\big(\mathcal{B}(q(\mathcal{H}))\big)$, uu^* is a projection in $M_n\big(\mathcal{B}(p(\mathcal{H}))\big)$, so that

$$uu^* \leq \mathbf{1}_{p(\mathcal{H})} \otimes \mathbf{1}_n = p \otimes \mathbf{1}_n. \tag{9.15}$$

Combining (9.14) and (9.15), we obtain the desired conclusion. ☐

9.11 Corollary. *If $\mathcal{A}$ is an exact, unital and simple C^*-algebra, and p, q are projections in $\mathcal{A}$, such that $p \neq 0$ and $\tau(q) < \tau(p)$ for all tracial states τ on $\mathcal{A}$, then for some n in $\mathbb{N}$*

$$q \otimes \mathbf{1}_n \prec p \otimes \mathbf{1}_n \quad \textit{in} \quad M_n(\mathcal{A}). \tag{9.16}$$

Proof. By simplicity of $\mathcal{A}$, $\tau(p) > 0$ for all tracial states τ on $\mathcal{A}$, and hence by weak* compactness of the set of tracial states on $\mathcal{A}$, there exists ϵ in $]0, \infty[$, such that

$$\tau(q) \leq (1-\epsilon)\tau(p),$$

for all tracial states τ on $\mathcal{A}$. By the assumptions on $\mathcal{A}$, $\mathcal{A}$ is algebraically simple. Hence, every non-zero trace $\tau \colon \mathcal{A}_+ \to [0, \infty]$ is either equal to $+\infty$ on all of $\mathcal{A}_+ \setminus \{0\}$, or proportional to a tracial state. Hence we can apply Theorem 9.10. $\square$

9.12 Remark. In the "inequality" (9.16) in Corollary 9.11, the tensoring with $\mathbf{1}_n$ can in general not be avoided. This follows from Villadsen's result in [Vi] that there exist nuclear (and hence exact) unital simple C^*-algebras with weak perforation. Recall that a unital C^*-algebra $\mathcal{A}$ has weak perforation, if there exists x in $K_0(\mathcal{A})$, such that $x \notin K_0(\mathcal{A})_+$, but $nx \in K_0(\mathcal{A})_+ \setminus \{0\}$, for some n in $\mathbb{N}$. To see how Villadsen's result implies, that we cannot, in general, avoid tensoring with $\mathbf{1}_n$ in (9.16), let $\mathcal{A}$ be a unital exact simple C^*-algebra, and assume that $x \in K_0(\mathcal{A})$, such that $x \notin K_0(\mathcal{A})_+$ and $nx \in K_0(\mathcal{A})_+ \setminus \{0\}$, for some positive integer n. Write then x in the form $x = [p] - [q]$, where p, q are projections in $M_k(\mathcal{A})$ for some k in $\mathbb{N}$. By the assumption that $nx \in K_0(\mathcal{A})_+ \setminus \{0\}$, and the simplicity of $\mathcal{A}$, it is not hard to deduce that

$$(\tau \otimes \mathrm{tr}_k)(p) > (\tau \otimes \mathrm{tr}_k)(q),$$

for all tracial states τ on $\mathcal{A}$, and hence $\tilde{\tau}(p) > \tilde{\tau}(q)$ for all tracial states $\tilde{\tau}$ on $M_k(\mathcal{A})$. However, since $x \notin K_0(\mathcal{A})_+$, q cannot be equivalent to a sub-projection of p. $\square$

9.13 Theorem. *Let $\mathcal{A}$ be a unital, exact C^*-algebra. Then the following two conditions are equivalent:*
(i) *$\mathcal{A}$ has no tracial states.*
(ii) *For some n in $\mathbb{N}$ there exist projections p, q in $M_n(\mathcal{A})$, such that*

$$p \perp q \qquad \textit{and} \qquad p \sim q \sim \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n.$$

Proof. Clearly, (ii) implies (i). To show the converse implication, assume that (i) holds, and consider then the two projections p', q' in $M_2(\mathcal{A})$ given by

$$p' = \begin{pmatrix} \mathbf{1}_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad q' = \begin{pmatrix} \mathbf{1}_{\mathcal{A}} & 0 \\ 0 & \mathbf{1}_{\mathcal{A}} \end{pmatrix}.$$

Since $\mathcal{A}$ has no tracial states, $\mathcal{A}^{**}$ has no normal tracial states, and hence $\mathcal{A}^{**}$ is a properly infinite von Neumann algebra. Therefore,

$$\langle \mathbf{1}_{\mathcal{A}} \rangle = 4\langle \mathbf{1}_{\mathcal{A}} \rangle \quad \text{in} \quad V(\mathcal{A}^{**}),$$

which implies that

$$\langle p' \rangle = 2\langle q' \rangle \quad \text{in} \quad V\big(M_2(\mathcal{A}^{**})\big).$$

Hence by Corollary 9.8 and Theorem 9.10, there exists an n in $\mathbb{N}$, such that

$$q' \otimes \mathbf{1}_n \prec p' \otimes \mathbf{1}_n \quad \text{in} \quad M_{2n}(\mathcal{A}).$$

Here, $p' \otimes \mathbf{1}_n \sim \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}$, and thus there exists u in $M_{2n}(\mathcal{A})$, such that

$$u^*u = \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n \end{pmatrix}, \qquad \text{and} \qquad uu^* \leq \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.17}$$

The inequality in (9.17) implies that u has the form

$$u = \begin{pmatrix} u_{11} & u_{12} \\ 0 & 0 \end{pmatrix},$$

for suitable u_{11}, u_{12} from $M_n(\mathcal{A})$. The equality in (9.17) yields then subsequently that

$$u_{11}^* u_{11} = u_{12}^* u_{12} = \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n, \qquad \text{and} \qquad u_{11}^* u_{12} = 0.$$

Defining now

$$p = u_{11}u_{11}^* \qquad \text{and} \qquad q = u_{12}u_{12}^*,$$

it follows that p, q are orthogonal projections in $M_n(\mathcal{A})$, such that $p \sim q \sim \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n$. This shows that (ii) holds. □

In particular, Theorem 9.13 implies the validity of Theorem 9.1:

9.14 Corollary. *If $\mathcal{A}$ is an exact, unital, stably finite C^*-algebra, then $\mathcal{A}$ has a tracial state.*

Proof. This is an obvious consequence of Theorem 9.13. □

Consider next an arbitrary unital C^*-algebra $\mathcal{A}$. A function $\varphi\colon V(\mathcal{A}) \to \mathbb{R}$ is said to be a state on $V(\mathcal{A})$, if it satisfies the following three conditions:

- $\varphi(x) \geq 0$, for all x in $V(\mathcal{A})$.
- $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all x, y in $V(\mathcal{A})$.
- $\varphi\big(\langle \mathbf{1}_{\mathcal{A}} \rangle\big) = 1$.

Similarly, a function $\psi\colon K_0(\mathcal{A}) \to \mathbb{R}$ is said to be a state on $K_0(\mathcal{A})$, if it satisfies the conditions:

- $\psi(z) \geq 0$, for all z in $K_0(\mathcal{A})_+$.
- $\psi(z + w) = \psi(z) + \psi(w)$, for all z, w in $K_0(\mathcal{A})$.

- $\psi([\mathbf{1}_{\mathcal{A}}]) = 1$.

The set of states on $V(\mathcal{A})$ (resp. $K_0(\mathcal{A})$) is denoted by $\mathcal{S}(V(\mathcal{A}))$ (resp. $\mathcal{S}(K_0(\mathcal{A}))$). Note that $\mathcal{S}(V(\mathcal{A}))$ and $\mathcal{S}(K_0(\mathcal{A}))$ are both convex compact sets in "the topology of pointwise convergence". Let $\rho\colon V(\mathcal{A}) \to K_0(\mathcal{A})$ be the natural map introduced in the beginning of this section. Then it is clear, that the map $\psi \mapsto \psi \circ \rho$, $\psi \in \mathcal{S}(K_0(\mathcal{A}))$, gives a one-to-one correspondence between the states on $K_0(\mathcal{A})$ and the states on $V(\mathcal{A})$. Moreover, this map is an affine homeomorphism of $\mathcal{S}(K_0(\mathcal{A}))$ onto $\mathcal{S}(V(\mathcal{A}))$.

9.15 Lemma. *Let $\mathcal{A}$ be a unital, exact C^*-algebra, and let p, q be projections in $\mathcal{A}$, such that*

$$\tau(q) \le \tau(p), \quad \text{for any tracial state } \tau \text{ on } \mathcal{A}. \tag{9.18}$$

Then for any k in $\mathbb{N}$, there exists n in $\mathbb{N}$, such that

$$nk\langle q\rangle \le nk\langle p\rangle + n\langle \mathbf{1}_{\mathcal{A}}\rangle.$$

Proof. Let k from $\mathbb{N}$ be given, and consider then the projections p', q' in $M_{k+1}(\mathcal{A})$ defined by:

$$p' = p \otimes \big(\textstyle\sum_{i=1}^k e_{ii}\big) + \mathbf{1}_{\mathcal{A}} \otimes e_{k+1,k+1}, \qquad \text{and} \qquad q' = q \otimes \big(\textstyle\sum_{i=1}^k e_{ii}\big).$$

Given now an arbitrary non-zero, bounded trace τ on $M_{k+1}(\mathcal{A})$, note that the mapping

$$a \mapsto \tau(a \otimes e_{11}), \qquad (a \in \mathcal{A}),$$

is proportional to a tracial state on $\mathcal{A}$. It follows thus from the assumption (9.18), that $\tau(q \otimes e_{11}) \le \tau(p \otimes e_{11})$, and hence

$$\tau(q') = k \cdot \tau(q \otimes e_{11}) \le k \cdot \tau(p \otimes e_{11}) = \tfrac{k}{k+1} \cdot \tau(p \otimes \mathbf{1}_{k+1}) \le \tfrac{k}{k+1} \cdot \tau(p').$$

Since $\mathbf{1}_{\mathcal{A}} \otimes e_{11} \prec p'$, any unbounded (lower semi-continuous) trace τ on $M_{k+1}(\mathcal{A})$ must take the value $+\infty$ at p', and hence we have also in this case, that

$$\tau(q') \le \tfrac{k}{k+1} \cdot \tau(p').$$

Applying now Theorem 9.10, it follows that there exists an n in $\mathbb{N}$, such that $n\langle q'\rangle \le n\langle p'\rangle$, and hence such that $nk\langle q\rangle \le nk\langle p\rangle + n\langle \mathbf{1}_{\mathcal{A}}\rangle$, as desired. ☐

Next, we need the following version of the Goodearl-Handelman theorem (see [Bl2, 3.4.7], [Go1, 7.11] and [BR, Lemma 2.9]).

9.16 Lemma. *Let $\mathcal{A}$ be a unital C^*-algebra, and consider a convex subset K of $\mathcal{S}(V(\mathcal{A}))$, which is closed in "the topology of pointwise convergence". Assume furthermore that the following implication holds*

$$\forall x, y \in V(\mathcal{A})\colon \quad \big[\forall \varphi \in K\colon \varphi(x) \le \varphi(y)\big] \Longrightarrow \big[\forall \varphi \in \mathcal{S}(V(\mathcal{A}))\colon \varphi(x) \le \varphi(y)\big]. \tag{9.19}$$

Then $K = \mathcal{S}(V(\mathcal{A}))$.

Proof. By the one-to-one correspondence between states on $K_0(\mathcal{A})$ and states on $V(\mathcal{A})$, we can find a convex compact subset L of $\mathcal{S}(K_0(\mathcal{A}))$, such that

$$K = \{\psi \circ \rho \mid \psi \in L\}.$$

Since $K_0(\mathcal{A}) = \rho(V(\mathcal{A})) - \rho(V(\mathcal{A}))$, condition (9.19) is equivalent to the condition:

$$\forall z \in K_0(\mathcal{A})\colon \quad \big[\forall \psi \in L\colon \psi(z) \geq 0\big] \Longrightarrow \big[\forall \psi \in \mathcal{S}(K_0(\mathcal{A}))\colon \psi(z) \geq 0\big].$$

Thus by [Go1, Corollary 7.11], all the extreme points of $\mathcal{S}(K_0(\mathcal{A}))$ are contained in $\overline{L} = L$. Hence by Krein-Milman's theorem,

$$\mathcal{S}(K_0(\mathcal{A})) \subseteq \overline{\mathrm{conv}(L)} = L,$$

and therefore $L = \mathcal{S}(K_0(\mathcal{A}))$, which is equivalent to the equation: $K = \mathcal{S}(V(\mathcal{A}))$. □

9.17 Theorem. *Let $\mathcal{A}$ be a unital, exact C^*-algebra. Then for any state φ on $V(\mathcal{A})$, there exists a tracial state τ on $\mathcal{A}$, such that*

$$\varphi(\langle p\rangle) = (\tau \otimes \mathrm{Tr}_m)(p), \quad \textit{for all projections } p \textit{ in } M_m(\mathcal{A}), \textit{ and } m \textit{ in } \mathbb{N}. \tag{9.20}$$

Proof. Let K denote the subset of $\mathcal{S}\big(V(\mathcal{A})\big)$ consisting of those states on $V(\mathcal{A})$, that are given by (9.20) for some tracial state τ on $\mathcal{A}$. Then K is clearly a convex, compact subset of $\mathcal{S}\big(V(\mathcal{A})\big)$, and hence, by Lemma 9.16, it suffices to verify that K satisfies condition (9.19). So consider projections p, q in $M_\infty(\mathcal{A})$. We may assume that $p, q \in M_m(\mathcal{A})$, for some sufficiently large positive integer m. Suppose then that

$$(\tau \otimes \mathrm{Tr}_m)(q) \leq (\tau \otimes \mathrm{Tr}_m)(p), \quad \text{for all tracial states } \tau \text{ on } \mathcal{A}.$$

Since any tracial state on $M_m(\mathcal{A})$ has the form $\frac{1}{m} \cdot \tau \otimes \mathrm{Tr}_m$, for some tracial state τ on $\mathcal{A}$, it follows then from Lemma 9.15, that for any k in $\mathbb{N}$, there exists an n in $\mathbb{N}$, such that

$$nk\langle q\rangle \leq nk\langle p\rangle + n\langle \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_m\rangle.$$

Hence for any φ in $\mathcal{S}\big(V(\mathcal{A})\big)$, and any k in $\mathbb{N}$, we have that

$$\varphi(\langle q\rangle) \leq \varphi(\langle p\rangle) + \tfrac{m}{k},$$

and this shows that K satisfies condition (9.19). □

Using the one-to-one correspondence between states on $K_0(\mathcal{A})$ and states on $V(\mathcal{A})$, Theorem 9.17 gives a new proof, not relying on quasitraces, for the following

9.18 Corollary. *Let $\mathcal{A}$ be a unital, exact C^*-algebra. Then any state on $K_0(\mathcal{A})$ comes from a tracial state on $\mathcal{A}$, i.e., for every state ψ on $K_0(\mathcal{A})$, there exists a tracial state τ on $\mathcal{A}$, such that*

$$\psi\big([p]\big) = (\tau \otimes \mathrm{Tr}_m)(p), \textit{ for all projections } p \textit{ in } M_m(\mathcal{A}), \textit{ and all } m \textit{ in } \mathbb{N}. \quad \square$$

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U. Haagerup
Department of Mathematics
and Computer Science
SDU, Odense University
Campusvej 55, 5230 Odense
Denmark
haagerup@imada.ou.dk

S. Thorbjørnsen
Department of Mathematics
and Computer Science
SDU, Odense University
Campusvej 55, 5230 Odense
Denmark
steenth@imada.ou.dk