Local Heights on Abelian Varieties and Rigid Analytic Uniformization

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Abstract. We express classical and p-adic local height pairings on an abelian variety with split semistable reduction in terms of the corresponding pairings on the abelian part of the Raynaud extension (which has good reduction). Here we use an approach to height pairings via splittings of biextensions which is due to Mazur and Tate. We conclude with a formula comparing Schneider's p-adic height pairing to the p-adic height pairing in the semistable ordinary reduction case defined by Mazur and Tate.

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1 Introduction

In this paper we express classical and p-adic local height pairings on an abelian variety A_K with split semistable reduction in terms of the corresponding pairings on the abelian part B_K of the Raynaud extension. Since B_K is an abelian variety with good reduction, this result provides a rather explicit step from the class of local height pairings on all abelian varieties with good reduction to the class of local height pairings on arbitrary abelian varieties. As an application of this principle we show a formula comparing two local p-adic height pairings on A_K, namely the canonical Mazur-Tate pairing in the ordinary reduction case and Schneider's norm-adapted pairing.

Besides these two p-adic height pairings, we study Néron's classical real-valued pairing. We use an approach to height pairings developped in [Ma-Ta]. Let K be a non-archimedean local ground field. For any homomorphism $\rho : K^\times \to Y$ to some abelian group Y, we can define a local height pairing on A_K with values in Y whenever we can continue ρ to a "bihomomorphic" map, a so-called ρ-splitting $\sigma : P_{A_K \times A'_K}(K) \to Y$ on the K-rational points of the Poincaré biextension associated to A_K and its dual abelian variety A'_K. For our three types of height pairings the corresponding ρ-splittings can be uniquely characterized by certain properties. (We recall these facts in section 2.)

We assume that A_K has semistable reduction with split torus part, which can always be achieved after a finite base change. In section 3, we recall that A_K and A'_K are rigid analytic quotients of semiabelian varieties E_K respectively E'_K after certain lattices M_K and M'_K. Here the abelian quotients B_K respectively B'_K of E_K respectively E'_K have good reduction and are dual to each other. Let $P_{B_K \times B'_K}$ be the Poincaré biextension expressing the duality. We show that the biextension $P^{an}_{A_K \times A'_K}$ is a quotient of the pullback of the biextension $P^{an}_{B_K \times B'_K}$ to $E^{an}_K \times E'^{an}_K$.

Then, in section 4, we define (under a certain condition) for a given ρ-splitting σ on $P_{B_K \times B'_K}(K)$ a ρ-splitting τ on $P_{A_K \times A'_K}(K)$, and we describe the relation between the corresponding height pairings on B_K respectively on A_K.

In section 5 we show that if we start with the ρ-splitting σ corresponding to Néron's local height pairing, the height pairing on A_K defined by our ρ-splitting τ is also Néron's local height pairing. From this we can deduce a formula relating the Néron pairings on A_K and B_K. A similar formula was already proved in [Hi].

Then we investigate Schneider's p-adic height pairing in section 6. First we show that if B_K has good ordinary reduction, our existence condition for τ is equivalent to the existence condition for Schneider's height pairing on A_K, namely that the group of universal norms with respect to a certain $\mathbb{Z}_p$-extension associated to ρ has finite index in $A_K(K)$. Afterwards, we prove that if we use the ρ-splitting σ defining Schneider's p-adic height pairing on B_K to construct τ, then the height pairing on A_K defined by τ is also Schneider's p-adic pairing.

In the last section we compare the canonical Mazur-Tate ρ-splittings in the ordinary case on B_K and on A_K. Then we calculate the difference between Schneider's p-adic height pairing and the p-adic Mazur-Tate pairing on A_K, using the fact that they coincide on abelian varieties with good ordinary reduction. Thereby we correct an error in the comparison formula for Tate curves given in [MTT], p.34.

This paper generalizes [We], where we showed formulas for Néron's and Schneider's height pairings on abelian varieties with split multiplicative reduction.

We adopt the following terminological conventions:

For a group scheme G over a base T we denote the unit section by $e_{G/T}$. When we refer to extensions or biextensions of T-group schemes, we work in the $fppf$-site over T. But note that we will often consider extensions and biextensions by $\mathbb{G}_m$, so that many sequences will also be exact in the big Zariski site. For a biextension Q of T-group schemes X and X' by G over T we refer to the element $e_{Q/X}(e_{X/T}) \in Q(T)$ as the unit section of Q. By [SGA7, I], VII, 2.2, this is a symmetrical notion, i.e. $e_{Q/X}(e_{X/T}) = e_{Q/X'}(e_{X'/T})$.

We will often work with rigid analytic varieties over a complete non-archimedean field K, endowed with their rigid analytic Grothendieck topology. (See [BGR], 9.3, Def. 4.) There is a rigid analytic GAGA functor, associating to a K-scheme X locally of finite type a rigid analytic variety X^{an}, see [BGR], 9.3, Ex. 2. The analogies of Serre's complex analytic GAGA theorems hold, see [Kö]. Extensions or biextensions of rigid analytic group varieties are always to be understood in the "big Zariski site", i.e. the category of rigid analytic varieties endowed with their Grothendieck topology.

Throughout this paper, K will be a non-archimedean field, locally compact with respect to a non-trivial absolute value, R will be its ring of integers, and k the residue class field. By $v_K : K^\times \to \mathbb{Z}$ we denote the valuation map, mapping a prime element to 1.

2 Height pairings

We fix an abelian variety A_K over K and a dual abelian variety $(A'_K, P_{A_K\times A'_K})$, where $P_{A_K\times A'_K}$ is the Poincaré biextension expressing the duality. (See [SGA7, I], VII, 2.9). We write $P = P_{A_K\times A'_K}$, when confusion seems unlikely. Note that $P(K)$ is a biextension of $A(K) \times A'(K)$ by $K^\times$ in the category of sets. Let $\rho : K^\times \to Y$ be a homomorphism to some abelian group Y. We call a map $\sigma : P(K) \to Y$ a ρ-splitting if it is compatible with the biextension structure on $P(K)$, i.e. if the following conditions hold:

i) $\sigma(\alpha x) = \rho(\alpha) + \sigma(x)$ for all $\alpha \in K^\times$ and $x \in P(K)$.

ii) For all $a \in A_K(K)$ (respectively $a' \in A'_K(K)$) the restriction of σ to $P(K) \times_{(A_K(K)\times A'_K(K))} \{a\} \times A'_K(K)$ (respectively $P(K) \times_{(A_K(K)\times A'_K(K))} A_K(K) \times \{a'\}$) is a group homomorphism. (See [Ma-Ta], 1.4.)

Let $\mathrm{Div}^0 A_K$ denote the group of divisors on A_K which are algebraically equivalent to zero, and let $Z^0(A_K/K)$ denote the group of zero cycles on A_K with degree zero and K-rational support. By $(\mathrm{Div}^0 A_K \times Z^0(A_K/K))'$ we denote the set of all pairs (D, z) with disjoint supports.

Whenever we have a homomorphism $\rho : K^\times \to Y$ and a ρ-splitting $\sigma : P(K) \to Y$, we can define a bilinear Mazur-Tate (height) pairing with values in Y:

$$\begin{array}{rccl} (\ ,\ )_{MT,\sigma} : & (\mathrm{Div}^0 A_K \times Z^0(A_K/K))' & \longrightarrow & Y \\ & (D, z) & \longmapsto & \sigma(s_D(z)), \end{array}$$

where s_D is a rational section of $P|_{A_K\times\{d\}} \to A_K$ with divisor D, and where d is the point in $A'_K(K)$ corresponding to D. The rational section s_D is defined only up to a constant in $K^\times$ which vanishes when we continue s_D linearly to $Z^0(A_K/K)$.

Let us denote by A respectively A' the Néron models of A_K respectively A'_K over R, and by A^0 respectively ${A'}^0$ their identity components.

In this paper, we will deal with three situations in which good ρ-splittings can be singled out:

I) The Mazur-Tate splitting in the unramified case

Assume that ρ is unramified, i.e. that ρ vanishes on $R^\times$, and that Y is uniquely divisible by m_A, the exponent of the group $A_k(k)/A_k^0(k)$. There exists a biextension $P_{A^0\times A'}$ of A^0 and A' by $\mathbb{G}_{m,R}$ with generic fibre P, see [SGA7, I, exp. VIII], 7.1 b). The canonical ρ-splitting σ_ρ is defined as the unique ρ-splitting vanishing on $P_{A^0\times A'}(R) \subset P(K)$ ([Ma-Ta], 1.5.2). If $\rho = \log|\ |_K : K^\times \to \mathbb{R}$, then the Mazur-Tate height pairing corresponding to the canonical ρ-splitting is just Néron's local height pairing, see [Ma-Ta], 2.3.1.

II) Schneider's p-adic height pairing

Here we take $Y = \mathbb{Q}_p$. Let K be a finite extension of $\mathbb{Q}_l$, and let $\rho : K^\times \to \mathbb{Q}_p$ be a non-trivial continuous homomorphism. Then ρ is continuous for the profinite topology on $K^\times$ and extends therefore uniquely to a homomorphism $\rho^\wedge$ on the profinite completion $K^{\times\wedge}$ of $K^\times$. By local class field theory, $K^{\times\wedge}$ is topologically isomorphic to $\mathrm{Gal}(K^{ab}/K)$. Then $\rho^\wedge$ determines a $\mathbb{Z}_p$-extension K_∞/K with intermediate fields K_ν which are the uniquely determined cyclic extensions of degree p^ν of K such that $\rho(N_{K_\nu/K}K_\nu^\times) = p^\nu\rho(K^\times) \subset \mathbb{Q}_p$ (see [Ma-Ta], 1.11.1). For any commutative group scheme G over K we denote by $NG(K) \subset G(K)$ the group of universal norms with respect to K_∞/K. Furthermore, let $P(K_\nu, K)$ be the set of points in $P(K_\nu)$ which project to $A_K(K_\nu) \times A'_K(K)$. We define $NP(K) \subset P(K)$ as the intersection of all $N_{K_\nu/K}P(K_\nu, K)$, where we use the group structure of P over A'_K to define norms.

If ρ is not unramified, assume that $NA_K(K)$ has finite index in $A_K(K)$. Then there exists a unique ρ-splitting $\sigma_\rho : P(K) \to \mathbb{Q}_p$ vanishing on $NP(K)$, see [Sch1], and [Ma-Ta], 1.11.5. If ρ is unramified (which e.g. is the case if the residue characteristic l is not equal to p), let σ_ρ the canonical ρ-splitting in case I).

We call $(\ ,\ )_{MT,\sigma_\rho}$ Schneider's local p-adic height pairing with respect to ρ. It was originally defined in [Sch1].

The following result characterizes the existence condition for Schneider's height in the good reduction case.

THEOREM 2.1 *Assume that A_K has good reduction.*

i)(Mazur) If A_K has good ordinary reduction, then for any non-trivial continuous homomorphism ρ the universal norm group $NA_K(K)$ has finite index in $A_K(K)$, i.e. Schneider's local p-adic height pairing exists.

ii)(Schneider) Conversely, if ρ is not unramified, and if $NA_K(K)$ has finite index in $A_K(K)$, then A_K has good ordinary reduction.

PROOF: See [Sch2], Theorem 2, and [Ma], 4.39. □

In section 6, we will investigate the existence condition for Schneider's local height in the case of semistable ordinary reduction.

III) The canonical Mazur-Tate splitting in the ordinary case

Let $\rho : K^\times \to Y$ be a homomorphism. Assume that A has ordinary reduction, i.e. that the formal completion of A_k at the origin is isomorphic to a product of copies of $\mathbb{G}_m^f$ over the algebraic closure of k. This is equivalent to the fact that A_k^0 is an extension of an ordinary abelian variety B_k by a torus T_k, see [Ma-Ta], 1.1. Note that in particular A has semistable reduction.

Now let T_k respectively T'_k be the maximal tori in A_k respectively A'_k, and denote by n_A respectively $n_{A'}$ the exponents of $A_k^0(k)/T_k(k)$ respectively $A_k^{'0}(k)/T'_k(k)$. Assume that Y is uniquely divisible by $m_A m_{A'} n_A n_{A'}$. Moreover, denote by A^t and $A^{'t}$ the formal completions of A and A' along T_k and T'_k, and let $P^t_{A^0\times A'}$ be the formal completion of $P_{A^0\times A'}$ along the inverse image of $T_k \times T'_k$ in $P_{A^0\times A'}$. Then $P^t_{A^0\times A'}$ is a formal biextension of A^t and $A^{'t}$ by $\mathbb{G}^\wedge_{mR}$ (the formal completion of $\mathbb{G}_{m,R}$ along its special fibre). By [Ma-Ta], 5.11.1, $P^t_{A^0\times A'}$ admits a unique splitting $\sigma_0 : P^t_{A^0\times A'} \to \mathbb{G}^\wedge_{mR}$. Hence there exists a unique ρ-splitting $\tilde{\sigma} : P(K) \to Y$ such that for all $x \in P^t_{A^0\times A'}(R)$ we have $\tilde{\sigma}(x) = \rho \circ \sigma_0(x)$. This defines a local p-adic height pairing $(\ ,\ )_{MT,\tilde{\sigma}}$.

If additionally ρ is unramified, then $\tilde{\sigma}$ coincides with the canonical splitting in case I).

On the other hand, if we take $Y = \mathbb{Q}_p$ and ρ is non-trivial and continuous but not unramified, case III) gives us a p-adic height pairing $(\ ,\ )_{MT,\tilde{\sigma}} : (\mathrm{Div}^0 A_K \times Z^0(A_K/K))' \to \mathbb{Q}_p$, if A has ordinary reduction. If A has *good* ordinary reduction, then $(\ ,\ )_{MT,\tilde{\sigma}}$ coincides with Schneider's p-adic height pairing from case II), see [Ma-Ta], 1.11.6. But in general both pairings may differ. We will compare these two p-adic height pairings in section 7.

Let us conclude this section with a general remark. If we start with a local field K and a homomorphism $\rho : K^\times \to Y$, we can extend ρ to any finite field extension L of K such that Y is uniquely divisible by $[L : K]$ by the formula $\rho_L(x) = [L :$

$K]^{-1}\rho(N_{L/K}(x))$. Note that in all three cases discussed above the restriction of the canonical ρ_L-splitting to $P(K)$ coincides with the canonical ρ-splitting. Hence we can investigate the corresponding local height pairings after finite base changes.

Finally, if we start with an abelian variety A_F over a global field F, we can define global height pairings by summing over all local ones, see [Ma-Ta], section 3.

3 RIGID ANALYTIC UNIFORMIZATION

We still fix A_K, A'_K and $P_{A_K \times A'_K}$. We say that an abelian variety over K has split semistable reduction, if the special fibre of the identity component of its Néron model over R is an extension of an abelian variety by a split torus.

From now on we assume that A_K (and hence A'_K) has split semistable reduction. Note that by Grothendieck's semistable reduction theorem, we are always in this situation after a finite base change. Even for an abelian variety A_F over a *global* field F we can find a finite extension E of F such that the Néron model of $A_F \otimes E$ has split semistable reduction at all finite places. Since our local height pairings are compatible with finite base changes, we can always place ourselves in the situation of the assumption if we want to deal with the local height pairings on A_F at the finite places of F.

Let us now recall some facts about the rigid analytic uniformization of A_K and A'_K. We can associate the following data to A_K, A'_K and $P_{A_K \times A'_K}$:

i) Since A_K has split semistable reduction, there is an extension, the so-called Raynaud extension,

$$0 \longrightarrow T_K \longrightarrow E_K \xrightarrow{p} B_K \longrightarrow 0,$$

such that T_K is a split torus of dimension t over K, and B_K is an abelian variety over K with good reduction. Let M' be the character group of T_K. Then M' is a free $\mathbb{Z}$-module of rank t. We denote the corresponding constant K-group scheme by M'_K. Fix once and for all a dual abelian variety $(B'_K, P_{B_K \times B'_K})$ of B_K, where $P_{B_K \times B'_K}$ is the Poincaré biextension expressing the duality. We will always identify B'_K with $\underline{\mathrm{Ext}}^1(B_K, \mathbb{G}_{m,K})$ via $b' \mapsto P_{B_K \times B'_K}|_{B_K \times \{b'\}}$ for functorial points b' of B'_K. Then E_K corresponds to a homomorphism $\phi' : M' \to B'_K$ (see e.g. [SGA7, I], VIII, 3.7.)

Besides, there is a rigid analytic homomorphism $\pi : E_K^{an} \to A_K^{an}$ inducing a short exact sequence

$$0 \longrightarrow M_K^{an} \xrightarrow{i} E_K^{an} \xrightarrow{\pi} A_K^{an} \longrightarrow 0,$$

where M_K is the constant group scheme corresponding to a free $\mathbb{Z}$-module M of rank t. (See [Bo-Lü1], section 1, and [Ray].)

ii) We can construct a "dual" uniformization of A'_K: The embedding $i : M_K \to E_K$ induces a homomorphism $\phi : M_K \xrightarrow{i} E_K \xrightarrow{p} B_K$, which gives us an extension E'_K (again by [SGA7, I], VIII, 3.7)

$$0 \longrightarrow T'_K \longrightarrow E'_K \xrightarrow{p'} B'_K \longrightarrow 0,$$

where T'_K is the split torus of dimension t over K with character group M. Besides, i induces a trivialization of the pullback $P_{B_K \times B'_K}|_{M_K \times M'_K}$ of $P_{B_K \times B'_K}$ via the homorphism $\phi \times \phi' : M_K \times M'_K \to B_K \times B'_K$ in the following way: Fix $m' \in M'$.

Then, tautologically, $\phi'(m') \in B'_K(K) = \mathrm{Ext}^1_K(B_K, G_m)$ corresponds to the extension $P_{B_K \times \{\phi'(m')\}}$, and by the definition of ϕ', this is the extension we get by pushout:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_K & \longrightarrow & E_K & \longrightarrow & B_K & \longrightarrow & 0 \\
 & & {\scriptstyle m'}\big\downarrow & & {\scriptstyle h_{m'}}\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & P_{B_K \times \{\phi'(m')\}} & \longrightarrow & B_K & \longrightarrow & 0
\end{array}$$

We define a bilinear map $<,>$: $E_K \times M'_K \to P_{B_K \times B'_K}$ by $< e, m' > := h_{m'}(e)$. Then the restriction of $<,>$ to $M_K \times M'_K \overset{i \times id}{\to} E_K \times M'_K$ induces a trivialization of $P_{B_K \times B'_K}|_{M_K \times M'_K}$. On the other hand, this trivialization defines an embedding $i' : M'_K \to E'_K$ such that $p' \circ i' = \phi'$, see [Bo-Lü1], 3.2. As above, by definition of ϕ we have a pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T'_K & \longrightarrow & E'_K & \longrightarrow & B'_K & \longrightarrow & 0 \\
 & & {\scriptstyle m}\big\downarrow & & {\scriptstyle h_m}\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & P_{\{\phi(m)\} \times B'_K} & \longrightarrow & B'_K & \longrightarrow & 0.
\end{array}$$

We get a bilinear map $<,>$: $M_K \times E'_K \to P_{B_K \times B'_K}$ defined by $<m, e'> := h_m(e')$, which coincides with the previous pairing on $M_K \times M'_K \overset{id \times i'}{\to} M_K \times E'_K$.

iii) The extension E'_K is a rigid analytic uniformization of A'_K: There is a rigid analytic homomorphism $\pi' : E'^{an}_K \to A'^{an}_K$ such that the sequence

$$0 \longrightarrow M'^{an}_K \xrightarrow{i'} E'^{an}_K \xrightarrow{\pi'} A'^{an}_K \longrightarrow 0$$

is exact, and such that we have the following description of $P_{A_K \times A'_K}$:

The $\mathbb{G}^{an}_{m,K}$-torsor $P^{an}_{A_K \times A'_K}$ is the quotient of $(p^{an} \times p'^{an})^* P^{an}_{B_K \times B'_K}$ after the $M \times M'$-linearization given by

$$u_{(m,m')} : P^{an}_{B_K \times B'_K} \times_{B^{an}_K \times B'^{an}_K} E^{an}_K \times E'^{an}_K \longrightarrow P^{an}_{B_K \times B'_K} \times_{B^{an}_K \times B'^{an}_K} E^{an}_K \times E'^{an}_K,$$

mapping a (functorial) point (ω, e, e'), such that ω in $P^{an}_{B_K \times B'_K}$ projects to $p(e) \times p'(e')$ in $B^{an}_K \times B'^{an}_K$, to

$$(([<e, m'> \bullet <m, m'>] \odot [<m, e'> \bullet \omega]), me, m'e').$$

Here $\odot$ is the group law on $P^{an}_{B_K \times B'_K}$ as a B_K-group, and $\bullet$ is the group law on $P^{an}_{B_K \times B'_K}$ as a B'_K-group. (See [Bo-Lü1], Theorem 6.8.)

Hence, in particular, we have a quotient morphism of analytic $\mathbb{G}^{an}_{m,K}$-torsors:

$$\theta : P^{an}_{B_K \times B'_K} \times_{B^{an}_K \times B'^{an}_K} E^{an}_K \times E'^{an}_K \to P^{an}_{A_K \times A'_K}.$$

Note that both sides also carry biextension structures. After multiplying θ by an element of $\mathbb{G}_{m,K}(K)$, we may assume that θ maps the unit section in $P^{an}_{B_K \times B'_K} \times_{B^{an}_K \times B'^{an}_K} E^{an}_K \times E'^{an}_K(K)$ to the unit section in $P^{an}_{A_K \times A'_K}(K)$.

PROPOSITION 3.1 *θ is a morphism of biextensions.*

PROOF: For better readability, we put $P = P_{A_K \times A'_K}$ and $Q = (p \times p')^* P_{B_K \times B'_K}$. We investigate the map

$$\alpha : Q^{an} \times_{E_K^{'an}} Q^{an} \longrightarrow P^{an}$$

defined by $\alpha(x, y) = \theta(xy)\theta(x)^{-1}\theta(y)^{-1}$, where we multiply and take inverses with respect to the group structures over $E_K^{'an}$ respectively $A_K^{'an}$. Since θ is a torsor homomorphism, the composition of α with the projection $P^{an} \to A_K^{an} \times A_K^{'an}$ factorizes through the unit section of the $A_K^{'an}$-group $A_K^{an} \times A_K^{'an}$, hence there is a $A_K^{'an}$-morphism $\alpha' : Q^{an} \times_{E_K^{'an}} Q^{an} \to \mathbb{G}_{m,K}^{an} \times A_K^{'an}$ which yields α when composed with the natural embedding $\mathbb{G}_{m,K}^{an} \times A_K^{'an} \to P^{an}$. Besides, α (and hence α') is equivariant with respect to the operation by $\mathbb{G}_{m,K}^{an} \times \mathbb{G}_{m,K}^{an}$ we get from the torsor structure of Q^{an}. Hence α' is derived from a map

$$\beta : E_K^{an} \times E_K^{an} \times E_K^{'an} \to \mathbb{G}_{m,K}^{an}$$

by composition with the natural projection $Q^{an} \times_{E_K^{'an}} Q^{an} \to E_K^{an} \times E_K^{an} \times E_K^{'an}$. Now let ω_1 and ω_2 be (functorial) points in $P_{B_K \times B'_K}^{an}$ with the same projection to $B_K^{'an}$, and let e_1, e_2 be points in E_K^{an} and e' a point in $E_K^{'an}$ such that $x_1 = (\omega_1, e_1, e')$ and $x_2 = (\omega_2, e_2, e')$ are in Q^{an}. Besides, fix m_1, m_2 in M and m' in M'. Then

$$u_{(m_1, m')}(x_1) \bullet u_{(m_2, m')}(x_2) = u_{(m_1 m_2, m')}(\omega_1 \bullet \omega_2, e_1 e_2, e'),$$

where on the left hand side we use the symbol $\bullet$ also for the group law on Q^{an} as an $E_K^{'an}$-group. This implies $\alpha(u_{(m_1,m')}(x_1), u_{(m_2,m')}(x_2)) = \alpha(x_1, x_2)$. From that we can deduce that β is invariant under the action of $M \times M \times M'$ on $E_K^{an} \times E_K^{an} \times E_K^{'an}$, which implies that there is a morphism $\beta_1 : A_K^{an} \times A_K^{an} \times A_K^{'an} \to \mathbb{G}_{m,K}^{an}$ such that $\beta = \beta_1 \circ (\pi \times \pi \times \pi')$.

But since A_K and A'_K are projective, β_1 must be constant. Since θ respects the unit sections, it follows that β_1 is equal to $1 \in \mathbb{G}_{m,K}^{an}(K)$, hence $\beta = 1$ and α factorizes through $e_{P^{an}/A_K^{'an}}$. This means that θ respects the group structures over $E_K^{'an}$ respectively $A_K^{'an}$. A parallel argument now shows that θ is also a group homomorphism with respect to the group structures over E_K^{an} respectively A_K^{an}. □

4 DEFINITION OF A LOCAL HEIGHT PAIRING VIA THE RAYNAUD EXTENSIONS

For the rest of this paper, we fix an abelian variety A_K with split semistable reduction and its dual abelian variety $(A'_K, P_{A_K \times A'_K})$. Let $\rho : K^\times \to Y$ be a homomorphism to some commutative ring Y, and let $\sigma : P_{B_K \times B'_K}(K) \to Y$ be a ρ-splitting on $P_{B_K \times B'_K}$. We will show how to construct in certain cases from σ a ρ-splitting τ on $P_{A_K \times A'_K}(K)$.

We fix once and for all bases $m_1, \dots, m_t$ for M and $m'_1, \dots, m'_t$ for M'.

DEFINITION 4.1 *We call σ M-invertible, if the $(t \times t)$-matrix $(\sigma(<m_i, m'_j>)_{i,j})$ with entries in Y is invertible over Y.*

(This definition differs slightly from Definition 4.4 in [We].) For M-invertible σ we will now define a ρ-splitting τ^* on $(P_{B_K \times B'_K} \times_{B_K \times B'_K} E_K \times E'_K)(K)$, which descends to a ρ-splitting τ on $P(K)$.

PROPOSITION 4.2 *Assume that σ is M-invertible, and let Σ be the inverse matrix of $(\sigma(<m_i, m'_j>)_{i,j})$.*

Define the ρ-splitting $\tau^* : (P_{B_K \times B'_K} \times_{B_K \times B'_K} E_K \times E'_K)(K) \to Y$ by

$$(\omega, e, e') \longmapsto \sigma(\omega) - (\sigma <e, m'_1>, \dots, \sigma <e, m'_t>) \, \Sigma \, {}^t(\sigma <m_1, e'>, \dots, \sigma <m_t, e'>)$$

for $\omega \in P_{B_K \times B'_K}(K)$ and $(e, e') \in (E_K \times E'_K)(K)$ with the same projection to $(B_K \times B'_K)(K)$.

Then there is a uniquely determined ρ-splitting $\tau : P(K) \to Y$ such that $\tau^* = \tau \circ \theta$.

PROOF: First of all, note that τ^* is indeed a ρ-splitting. Besides, we claim that for all $m \in M$, $m' \in M'$, $e \in E_K(K)$ and $e' \in E'_K(K)$ we have

i) $\tau^*(<m, e'>, m, e') = 0$ and

ii) $\tau^*(<e, m'>, e, m') = 0$.

We will only show i), since the argument for ii) is completely parallel. Note that it suffices to prove i) for our basis $m_1, \dots, m_t$, since the left hand side is additive in m. By definition of Σ, we find that for all $i = 1, \dots, t$ the vector $(\sigma <m_i, m'_1>, \dots, \sigma <m_i, m'_t>)\Sigma$ is the i-th unit vector, hence

$$\begin{aligned} &\tau^*(<m_i, e'>, m_i, e') \\ &= \sigma <m_i, e'> -(\sigma <m_i, m'_1>, \dots, \sigma <m_i, m'_t>) \, \Sigma \, {}^t(\sigma <m_1, e'>, \dots, \sigma <m_t, e'>) \\ &= \sigma <m_i, e'> -\sigma <m_i, e'> = 0, \end{aligned}$$

as desired. Now we can calculate

$$\begin{aligned} &\tau^*(u_{(m,m')}(\omega, e, e')) \\ &= \tau^*(<e, m'>, e, m') + \tau^*(<m, m'>, m, m') + \tau^*(<m, e'>, m, e') + \tau^*(\omega, e, e') \\ &= \tau^*(\omega, e, e'). \end{aligned}$$

Hence there is a uniquely determined map $\tau : P_{A_K \times A'_K}(K) \to Y$ such that $\tau^* = \tau \circ \theta$, and since θ is a homomorphism of biextensions by 3.1, τ is also a ρ-splitting. $\square$

Hence we can define a local height pairing on A_K

$$(\ ,\ )_{MT,\tau} : (\mathrm{Div}^0 A_K \times Z^0(A_K/K))' \to Y$$

for any M-invertible ρ-splitting σ on $P_{B_K \times B'_K}(K)$.

In the next three sections, we will investigate the connection to the canonical height pairings in our three cases.

But first we will describe our local height pairing $(\ ,\ )_{MT,\tau}$ a bit more explicitely.

Obviously, the description of $P^{an}_{A_K \times A'_K}$ as a quotient via θ implies that θ induces an isomorphism $P^{an}_{B_K \times B'_K} \times_{B^{an}_K \times B'^{an}_K} E^{an}_K \times E'^{an}_K \simeq (\pi \times \pi')^* P^{an}_{A_K \times A'_K}$. Restricting this isomorphism we find for any $a' \in A'_K(K)$ and any preimage $e' \in E'_K(K)$ of a' an isomorphism $\nu : \pi^* P^{an}_{A_K \times \{a'\}} \to p^{an*} P^{an}_{B_K \times \{p'(e')\}}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \pi^* P^{an}_{A_K \times \{a'\}} & \xrightarrow{\nu} & p^{an*} P^{an}_{B_K \times \{p'(e')\}} \\ \downarrow & & \downarrow{\scriptstyle \theta(-,-,e')} \\ P^{an}_{A_K \times \{a'\}} & \xrightarrow{=} & P^{an}_{A_K \times \{a'\}}. \end{array}$$

Now consider a divisor $D \in \mathrm{Div}^0(A_K)$ with divisor class $a' \in A'_K(K)$. We choose a preimage $e' \in E'_K(K)$ of a' and we denote by b' the point $p'(e') \in B'_K(K)$. Let $D^\sim$ be a divisor in $\mathrm{Div}^0(B_K)$ whose class corresponds to b'. Then there is a meromorphic function h on E_K^{an} such that $\pi^* D^{an} = p^{an*} D^{\sim an} + \mathrm{div}(h)$.

Let s_D and $s_{D^\sim}$ be rational sections corresponding to D respectively $D^\sim$ (both are uniquely defined up to a constant). s_D induces a meromorphic section s_D^{an} of $P^{an}_{A_K \times \{a'\}}$, which we can pull back to a meromorphic section $\pi^* s_D^{an}$ of $\pi^* P^{an}_{A_K \times \{a'\}}$ with divisor $\pi^* D^{an}$. Via the isomorphism ν, this induces a meromorphic section $\nu(\pi^* s_D^{an})$ of $p^{an*} P^{an}_{B_K \times \{b'\}}$ with the same divisor. Besides, the rational section $s_{D^\sim}$ gives a meromorphic section $p^{an*} s^{an}_{D^\sim}$ of $p^{an*} P^{an}_{B_K \times \{b'\}}$ with divisor $p^{an*} D^{\sim an}$.

Hence the meromorphic sections $\nu(\pi^* s_D^{an})$ and $h \cdot p^{an*} s^{an}_{D^\sim}$ of $p^{an*} P_{B_K \times \{b'\}}$ differ by a function $g \in \Gamma(E_K^{an}, \mathcal{O}^\times)$. We put $h^\diamond = hg$. Then we have $\pi^* D^{an} = p^{an*} D^{\sim an} + \mathrm{div}(h^\diamond)$ and $\nu(\pi^* s_D^{an}) = h^\diamond \cdot (p^{an*} s^{an}_{D^\sim})$.

Now let $a \in A_K(K)$ be a point not lying in the support of D. For any preimage $e \in E_K(K)$ of a we can calculate

$$\begin{aligned}
\tau(s_D(a)) &= \tau(s_D(\pi e)) \\
&= \tau(\theta(-,-,e') \circ \nu \circ \pi^* s_D^{an}(e)) \\
&= \tau(\theta(h^\diamond(e) \cdot s_{D^\sim}(pe), e, e')) \\
&= \tau^*(h^\diamond(e) \cdot s_{D^\sim}(pe), e, e') \\
&= \rho(h^\diamond(e)) + \sigma(s_{D^\sim}(pe)) \\
&\quad - (\sigma <e, m_1'>, \ldots, \sigma <e, m_t'>)\, \Sigma\, {}^t(\sigma <m_1, e'>, \ldots, \sigma <m_t, e'>).
\end{aligned}$$

This proves the following

Theorem 4.3 *For $(D, \sum_i n_i a_i) \in (Div^0 A_K \times Z^0(A_K/K))'$ let $a' \in A'_K(K)$ be the point corresponding to D, and choose a preimage $e' \in E'_K(K)$ of a'. Then there exists a divisor $D^\sim \in Div^0(B_K)$ and a meromorphic function $h^\diamond$ on E_K^{an} such that $\pi^* D^{an} = p^{an*} D^{\sim an} + div(h^\diamond)$ and such that for all rational sections s_D and $s_{D^\sim}$ corresponding to D respectively $D^\sim$ the meromorphic sections $\nu(\pi^* s_D^{an})$ and $h^\diamond \cdot (p^{an*} s^{an}_{D^\sim})$ of $p^{an*} P^{an}_{B_K \times \{p'(e')\}}$ differ by a constant.*

For any choice of preimages $e_i \in E_K(K)$ of the a_i we have the following formula for the canonical Mazur-Tate pairing associated to τ:

$$\begin{aligned}
(D, \sum n_i a_i)_{MT,\tau} &= (D^\sim, \sum n_i p(e_i))_{MT,\sigma} + \sum n_i \rho(h^\diamond(e_i)) \\
&\quad -(\sigma <\sum n_i e_i, m_1'>, \ldots, \sigma <\sum n_i e_i, m_t'>)\, \Sigma\, {}^t(\sigma <m_1, e'>, \ldots, \sigma <m_t, e'>).
\end{aligned}$$

5 Néron's local height pairing

In this section we show that our ρ-splitting τ coincides with the canonical Mazur-Tate splitting in the unramified case if σ is the canonical Mazur-Tate splitting on B_K. Hence we can use τ to "calculate" Néron's local height pairing on A_K in terms of Néron's local height pairing on B_K.

We need some notation first. Put $S = \mathrm{Spec} R$. We denote by A respectively A' the Néron models of A_K respectively A'_K, and by B and B' the Néron models of B_K and B'_K. Note that the split torus T_K and the semiabelian variety E_K have

Néron models T and E over R by [BLR], 10.1, Proposition 7, and that the identity component T^0 of T is isomorphic to $\mathbb{G}_{m,R}^t$ by [BLR], 10.1, Example 5. Similarly, let T' and E' be the Néron models of T'_K and E'_K.

By [SGA7, I], VIII, 7.1, we can (up to canonical isomorphism) uniquely extend $P_{B_K \times B'_K}$ to a biextension $P_{B\times B'}$ of B and B' by $\mathbb{G}_{m,R}$, and $P_{A_K\times A'_K}$ to a biextension $P_{A^0\times A'}$ of A^0 and A' by $\mathbb{G}_{m,R}$. Now the sequences of identity components

$$0 \longrightarrow T^0 \longrightarrow E^0 \longrightarrow B \longrightarrow 0$$

and

$$0 \longrightarrow T'^0 \longrightarrow E'^0 \longrightarrow B' \longrightarrow 0$$

are exact ([BLR], 10.1, proof of Proposition 7). Denote by $D(T^0)$ and $D(T'^0)$ the Cartier duals of T^0 and T'^0. Then these sequences induce homomorphims $\phi : M_S \simeq D(T'^0) \to B$ and $\phi' : M'_S \simeq D(T^0) \to B'$, which extend our previous maps $\phi : M_K \to B_K$ respectively $\phi' : M'_K \to B'_K$. Here M_S and M'_S of course denote the constant S-group schemes corresponding to M and M'.

Hence we have pushout homomorphisms $h_{m'} : E^0 \to P_{B\times\{\phi'(m')\}}$ and $h_m : E'^0 \to P_{\{\phi(m)\}\times B'}$ for m in M and m' in M'. The pairings $<,>_S$: $E^0 \times M'_S \to P_{B\times B'}$ and $<,>_S$: $M_S \times E'^0 \to P_{B\times B'}$ defined by $<e,m'>_S = h_{m'}(e)$ and $<m,e'>_S = h_m(e')$ extend the pairings from section 3, ii).

We will also use some results from formal and rigid geometry. Recall that there is a canonical functor associating to a formal S-scheme X, flat and locally of topologically finite type over S, its rigid analytic generic fibre X^{rig}, see [Bo-Lü2]. It is defined locally by associating to a formal affine scheme Spf$A$ the affinoid variety $\mathrm{Sp}(A\otimes_R K)$. Note that $X(R) = \mathrm{Mor}_{\mathrm{formal/R}}(\mathrm{Spf}R, X) = \mathrm{Mor}_{\mathrm{rigid/K}}(\mathrm{Sp}K, X^{rig}) = X^{rig}(K)$ by a standard argument: It suffices to check this for formal affine $X = \mathrm{Spf}A$. Then one uses the fact that the supremum semi-norm is contractive ([BGR], Prop. 1, p. 238) to show that every K-homomorphism $A\otimes_R K \to K$ restricts to an R-homomorphism $A \to R$.

Moreover, we use the theory of formal Néron models of rigid analytic groups as developped in [Bo-Sch]. A formal Néron model of a smooth rigid analytic K-variety Y is a smooth formal R-scheme Z such that its generic fibre Z^{rig} is an open rigid subspace of Y and such that for any smooth formal R-scheme Z' all rigid K-morphisms $Z'^{rig} \to Y$ extend uniquely to formal R-morphisms $Z' \to Z$. For all S-schemes X, we denote by $X^\wedge$ the completion along the special fibre. If X is separated and of finite type over S, there is a canonical open immersion $X^{\wedge rig} \to X_K^{an}$. We will often use the fact that for a commutative smooth K-group scheme X_K of finite type, the formal completion $X^\wedge$ of its ordinary Néron model X is a formal Néron model of X_K^{an}, see [Bo-Sch], Theorem 6.2.

Lemma 5.1 *Let Y be a commutative ring, and let $v : K^\times \to Y$ be the homomorphism $x \mapsto v_K(x)1_Y$ given by the valuation map $v_K : K^\times \to \mathbb{Z}$. We denote by σ_v the canonical v-splitting in the unramified case on $P_{B_K\times B'_K}(K)$. Then σ_v is M-invertible iff Y is uniquely divisible by m_A, the exponent of $A_k(k)/A_k^0(k)$.*

Proof: First of all, note that σ_v exists since B_K has good reduction (which implies $m_B = 1$). Our claim could be proven along the same lines as Lemma 4.9 in [We], but we prefer to give a different argument here.

Now M_K is a split lattice in E_K by [Bo-Lü1], Thm. 1.2, which means that the map $M \to \mathbb{R}^t$ given by $m \mapsto (\sigma_{v_K} <m, m'_1>, \ldots, \sigma_{v_K} <m, m'_t>)$ is a bijection onto a lattice in $\mathbb{R}^t$. This implies that the pairing $M \times M' \to \mathbb{Z}$, mapping (m, m') to $\sigma_{v_K}(<m, m'>)$, induces an injection $j : M \to \mathrm{Hom}(M', \mathbb{Z})$. (Note that this pairing coincides with the monodromy pairing. This is claimed in [SGA7, I], IX, 14.2.5 and proven in [Co].) From the rigid analytic uniformization of A_K one can deduce that the component group $\phi_A = A_k / A_k^0$ is constant, and that there exists an exact sequence

$$0 \to M \xrightarrow{j} \mathrm{Hom}(M', \mathbb{Z}) \to \phi_A(k) \to 0.$$

(See e.g. [Bo-Xa], 5.2.) Hence the number of elements in $\phi_A(k)$ is equal to $|\det(\sigma_{v_K}(<m_i, m'_j>))|$. Moreover, we have $H^1(k_{ét}, A_k^0) = 0$ by [La]. Hence we find that the natural inclusion $A_k(k)/A_k^0(k) \to \phi_A(k)$ is actually an isomorphism.

This implies that $\det(\sigma_v(<m_i, m'_j>))$ is a unit in Y iff m_A is a unit in Y. Hence our claim follows. □

Now we can compare our splitting τ to the canonical Mazur-Tate splitting:

THEOREM 5.2 *Let $\rho : K^\times \to Y$ be an unramified homomorphism to the commutative ring Y, and let σ be the canonical ρ-splitting on $P_{B_K \times B'_K}(K)$ in the unramified case.*

i) If σ is M-invertible, then Y is uniquely divisible by m_A. Conversely, if Y is uniquely divisible by m_A, and $\rho(r)$ is a unit in Y for one (and hence for any) prime element r in R, then σ is M-invertible.

ii) Assume that σ is M-invertible. Then our ρ-splitting τ from 4.2 is the canonical ρ-splitting in the unramified case.

PROOF: i) Since ρ is unramified, we find $\rho(x) = v_K(x)\rho(r)$, where r is a prime element in $K^\times$. Let σ_v denote as above the canonical v-splitting of $P_{B_K \times B'_K}(K)$. Then for all $z \in P_{B_K \times B'_K}(K)$ we have $\sigma(z) = \sigma_v(z)\rho(r)$. Hence $\det(\sigma(<m_i, m'_j>)_{i,j}) = \rho(r)^t \det(\sigma_v(<m_i, m'_j>)_{i,j})$, which, together with 5.1, implies our claim.

ii) In order to show that τ coincides with the canonical Mazur-Tate-splitting, we have to show that τ vanishes on $P_{A^0 \times A'}(R) \subset P_{A_K \times A'_K}(K)$. We fix a point $a' \in A'_K(K) = A'(R)$ and a preimage e' of a' in $E'_K(K) = E'(R)$. Let $b' \in B'_K(K) = B'(R)$ be the projection of e'. Since $P_{A_K \times \{a'\}}$ is semiabelian, it has a Néron model Q over S, which is an extension of A by G_S, the Néron model of $\mathbb{G}_{m,K}$. Its formal completion $Q^\wedge$ is a formal Néron model of $P^{an}_{A_K \times \{a'\}}$.

Let us write ϑ for the map $\theta(-,-,e') : P^{an}_{B_K \times \{b'\}} \times_{B_K^{an}} E_K^{an} \to P^{an}_{A_K \times \{a'\}}$. By the universal property of formal Néron models, the homomorphism of rigid analytic K-groups

$$(P^\wedge_{B \times \{b'\}} \times_{B^\wedge} E^{0\wedge})^{rig} \hookrightarrow P^{an}_{B_K \times \{b'\}} \times_{B_K^{an}} E_K^{an} \xrightarrow{\vartheta} P^{an}_{A_K \times \{a'\}}$$

is induced from a unique formal morphism

$$P^\wedge_{B \times \{b'\}} \times_{B^\wedge} E^{0\wedge} \xrightarrow{f} Q^\wedge,$$

which means that it coincides with

$$(P^\wedge_{B \times \{b'\}} \times_{B^\wedge} E^{0\wedge})^{rig} \xrightarrow{f^{rig}} (Q^\wedge)^{rig} \hookrightarrow P^{an}_{A_K \times \{a'\}}.$$

Moreover, f is a homomorphism of formal S-group schemes. Now $P^{\wedge}_{B\times\{b'\}}\times_{B^\wedge} E^{0\wedge}$ is connected (use e.g. [EGA IV] 4.5.7), hence its image via f is contained in $Q^{0\wedge}$, the identity component of $Q^\wedge$. Hence we get the following commutative diagram:

$$\begin{array}{ccc}
(P^{\wedge}_{B\times\{b'\}}\times_{B^\wedge} E^{0\wedge})^{rig} & \longrightarrow & P^{an}_{B_K\times\{b'\}}\times_{B_K^{an}} E_K^{an} \\
\big\downarrow & & \big\downarrow \vartheta \\
(Q^{0\wedge})^{rig} & \longrightarrow & P^{an}_{A_K\times\{a'\}}
\end{array}$$

which induces on K-rational points the commutative diagram

$$\begin{array}{ccc}
(P_{B\times\{b'\}}\times_B E^0)(R) & \xrightarrow{\subset} & (P^{an}_{B_K\times\{b'\}}\times_{B_K^{an}} E_K^{an})(K) \\
\big\downarrow & & \big\downarrow \vartheta \\
Q^0(R) & \xrightarrow{\subset} & P^{an}_{A_K\times\{a'\}}(K)
\end{array}$$

According to [SGA7, I], VIII, 7.1, taking the generic fibre induces a fully faithful functor from the category of extensions of A^0 by $\mathbb{G}_{m,R}$ to the category of extensions of A_K by $\mathbb{G}_{m,K}$. Hence there is an isomorphism $Q^0 \xrightarrow{\sim} P_{A^0\times\{a'\}}$ inducing the identity on the generic fibre. This implies that θ maps $(P_{B\times\{b'\}}\times_B E^0)(R)$ to $P_{A^0\times\{a'\}}(R)$.

Now take a point x in $P_{A^0\times\{a'\}}(R)$ projecting to $a\in A^0(R)$. The homomorphism $E^{0\wedge}\to A^{0\wedge}$ induced by $\pi: E_K^{an}\to A_K^{an}$ is an isomorphism (see [Bo-Xa], Thm. 2.3), hence we find a point $y\in(P_{B\times\{b'\}}\times_B E^0)(R)$ projecting to $a\in A^0(R)$. Since $\vartheta(y)$ lies in $P_{A^0\times\{a'\}}(R)$, it follows that $x=\alpha\vartheta(y)=\vartheta(\alpha y)=\theta(\alpha y,e')$ for some $\alpha\in\mathbb{G}_{m,R}(R)$. So $\tau(x)=\tau^*(z)$ for some $z\in(P_{B\times B'}\times_{B\times B'}(E^0\times E'))(R)$. Since σ vanishes on $P_{B\times B'}(R)$, and since $<e,m'>=<e,m'>_R\in P_{B\times B'}(R)$ for all $e\in E^0(R)$ and $m'\in M'$, it follows from the definition of τ^*, that $\tau(x)=\tau^*(z)=0$. Hence τ coincides with the canonical Mazur-Tate splitting in the unramified case. □

From this theorem we immediately get the following

Corollary 5.3 *If $\rho=\log|\ |_K: K^\times\to\mathbb{R}$, and $\sigma: P_{B_K\times B'_K}(K)\to\mathbb{R}$ is the canonical ρ-splitting, then σ is M-invertible. Define τ as in 4.2. Then $(\ ,\ )_{MT,\tau}$ coincides with Néron's local height pairing on A_K.*

According to this corollary, Theorem 4.3 expresses Néron's local height pairing on A_K with the one on B_K. There is a similar result by Hindry who even relates the Néron functions for *arbitrary* divisors on A_K to certain Néron functions on B_K. Let us denote by $(\ ,\ )_{N,A_K}$ the local Néron pairing on A_K. For $D\in\mathrm{Div}^0(A_K)$ completely antisymmetric, i.e. of the shape $D=(-1)^*D'-D'$ for some divisor D' on A_K, Hindry's result is the following: There exists a completely antisymmetric divisor $D^\sim\in\mathrm{Div}^0(B_K)$ and a meromorphic function h with $h(e^{-1})=h(e)^{-1}$ on E_K^{an} such that $\pi^*D^{an}=p^{an*}D^{\sim an}+\mathrm{div}(h)$ and such that for $\sum_i n_ia_i\in Z^0(A_K/K)$ disjoint from the support of D and all preimages e_i of a_i in $E_K(K)$ the following formula holds

$$(D,\sum_i n_ia_i)_{N,A_K}=\sum_i n_i\log|h(e_i)|_K+(D^\sim,\sum_i n_ip(e_i))_{N,B_K}+\sum_i n_iJ(e_i),$$

where $J : E_K(K) \to \mathbb{R}$ is a linear function determined by its values on M which are given by

$$J(m) = \log \left| \frac{h(e)}{h(me)} \right|_K + (D^{\sim}, p(e) - p(me))_{N,B_K}$$

for arbitrary e. (See [Hi], Lemme 3.4 and Théorème D, but note that our height pairing differs from his by a sign, since we started with $\rho = \log|\ |_K$.)

Our result in 4.3 can be used to deduce an expression for Hindry's linear term $J(e)$ for general $e \in E_K(K)$.

6 Schneider's local p-adic height pairing

Let K be a finite extension of $\mathbb{Q}_l$, and let $\rho : K^\times \to \mathbb{Q}_p$ be a non-trivial continuous homomorphism with corresponding $\mathbb{Z}_p$-extension K_∞/K. Since we already dealt with the unramified case in section 5, we will in this section assume that ρ is not unramified. Recall that this implies that $l = p$. Since $R^\times \subset K^\times$ is mapped to a non-trivial compact subgroup of $\mathbb{Q}_p$, there is an integer s so that $\rho(K^\times) = p^s\mathbb{Z}_p$. The goal of this section is to prove the following two theorems:

Theorem 6.1 *Assume that ρ is not unramified, and that B_K has ordinary reduction. Let σ_ρ be the canonical Schneider ρ-splitting on $P_{B_K \times B'_K}(K)$. Then the universal norm group $NA_K(K)$ with respect to K_∞/K has finite index in $A_K(K)$ iff σ_ρ is M-invertible, i.e. iff the matrix $(\sigma_\rho <m_i, m'_j>_{i,j})$ is invertible over $\mathbb{Q}_p$.*

With other words, this theorem says that in the semistable ordinary reduction case Schneider's local p-adic height exists iff our ρ-splitting τ from 4.2 exists.

Theorem 6.2 *Assume that B_K has ordinary reduction, and let σ be the canonical Schneider ρ-splitting on $P_{B_K \times B'_K}(K)$. If σ is M-invertible, our ρ-splitting τ from 4.2 is equal to Schneider's ρ-splitting σ_ρ on $P_{A_K \times A'_K}(K)$.*

Hence $(\ ,\ )_{MT,\tau}$ coincides with Schneider's p-adic height pairing on A_K.

Let us prove two lemmas first. We will use the notation from the beginning of section 5.

Lemma 6.3 *The map*

$$\begin{array}{llll} \mu : & E^0 & \to P_{B\times\{\phi'(m'_1)\}} \times_B \ldots \times_B P_{B\times\{\phi'(m'_t)\}} =: \times_B P_{B\times\{\phi'(m'_j)\}} \\ & x & \mapsto (<x, m'_1>_S, \ldots, <x, m'_t>_S) \end{array}$$

is an isomorphism.

Proof: Look at the following commutative diagram in the category of abelian sheaves on the big flat site over S:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^0 & \longrightarrow & E^0 & \longrightarrow & B & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle (m'_1,\ldots,m'_t)} & & \big\downarrow{\scriptstyle \mu} & & \| & & | \\ 0 & \longrightarrow & \mathbb{G}^t_{m,S} & \longrightarrow & \times_B P_{B\times\{\phi'(m'_j)\}} & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Both horizontal sequences are exact. This is clear for the upper one. For the lower one, it follows from the exactness of the sequences

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow P_{B\times\{\phi'(m'_j)\}} \longrightarrow B \longrightarrow 0$$

for all j. Hence, since $(m'_1, \dots m'_t) : T^0 \to \mathbb{G}^t_{m,S}$ is an isomorphism, μ is also an isomorphism. □

LEMMA 6.4 *Assume that B_K has ordinary reduction. Then taking universal norms with respect to K_∞/K in the short exact sequence $0 \to T_K \to E_K \to B_K \to 0$ yields a short exact sequence*

$$0 \longrightarrow NT_K(K) \longrightarrow NE_K(K) \longrightarrow NB_K(K) \longrightarrow 0.$$

PROOF: Recall Theorem 2.1,i) and imitate the proof of Lemma 3 in section 2 of [Sch1], substituting $\mathbb{G}_{m,K}$ by T_K. □

Now we are ready to prove Theorem 6.1:

PROOF OF THEOREM 6.1: We write N_ν for the norm map $N_{K_\nu/K}$ where K_ν/K is the intermediate layer of degree p^ν in the $\mathbb{Z}_p$-extension K_∞ belonging to ρ.

(1) π induces an isomorphism

$$E_K(K)/\cap_\nu\,(MN_\nu E_K(K_\nu)) \xrightarrow{\sim} A_K(K)/NA_K(K).$$

(2) Since B_K has good ordinary reduction, the quotient $B_K(K)/NB_K(K)$ is finite by 2.1. Let d be the number of elements of this group. Since B'_K is isogeneous to B_K, the quotient $B'_K(K)/NB'_K(K)$ is also finite, and we denote its cardinality by d'.

By construction, Schneider's canonical ρ-splitting σ maps a point $x \in P_{B_K\times B'_K}(K)$ to $\sigma(x) = d^{-1}\rho(\alpha)$, where α is an element in $K^\times$ such that $x^d \in \alpha NP(K)$. Hence the image of σ is contained in $d^{-1}\rho(K^\times) = d^{-1}p^s\mathbb{Z}_p$. Besides, if we assume that $x \in N_\nu P_{B_K\times\{b'\}}(K_\nu)$ for some $b' \in B'_K(K)$ and some index ν, we find that $\sigma(x) = d^{-1}\rho(\alpha)$ for some $\alpha \in K^\times \cap N_\nu P_{B_K\times\{b'\}}(K_\nu)$. The proof of Lemma 3 in section 2 of [Sch1], applied to $X = P_{B_K\times\{b'\}}$, shows that $\alpha^{d'}$ is contained in $N_\nu(K_\nu^\times)$. Hence $\sigma(x)$ lies in $d^{-1}d'^{-1}\rho(N_\nu K_\nu^\times) = d^{-1}d'^{-1}p^{\nu+s}\mathbb{Z}_p$.

Since σ induces homomorphisms $P_{B_K\times\{\phi'(m'_j)\}}(K) \to d^{-1}p^s\mathbb{Z}_p$ for each j, we can define a homomorphism

$$\omega : E_K(K) \overset{\mu_K}{\to} \times_{B_K} P_{B_K\times\{\phi'(m'_j)\}}(K) \to \prod_{j=1}^{t} P_{B_K\times\{\phi'(m'_j)\}}(K) \overset{\prod\sigma}{\to} (d^{-1}p^s\mathbb{Z}_p)^t,$$

where μ_K is the generic fibre of the isomorphism μ from Lemma 6.3. Let us first show that the cokernel of ω is annihilated by d. Namely, consider $(\alpha_1, \dots, \alpha_t) \in p^s\mathbb{Z}_p^t$ with $\alpha_j = \rho(x_j)$ for $x_j \in K^\times$. Then $(x_1, \dots, x_t)$ is an element of $\mathbb{G}_m(K)^t$ which embeds naturally into $\times_B P_{B\times\{\phi'(m'_j)\}}(K)$. Hence there exists some $z \in E_K(K)$ such that $\mu_K(z) = (x_1, \dots x_t)$. Then $\omega(z) = (\rho(x_1), \dots, \rho(x_t)) = (\alpha_1, \dots, \alpha_t)$. This proves our claim.

Let us now show that $\omega(\cap_\nu(MN_\nu E_K(K_\nu)))$ is contained in the p-adic closure $\overline{\omega(M)}$ of $\omega(M)$. Take an element z in $\cap_\nu(MN_\nu E_K(K_\nu))$. Then $\omega(z)$

lies in $\omega(M) + \omega(N_\nu E_K(K_\nu))$ for all ν. Now $\omega(N_\nu E_K(K_\nu))$ is contained in $\prod_j \sigma(N_\nu P_{B\times\{\phi'(m'_j)\}}(K_\nu))$, which is contained in $\prod_j (d^{-1}d'^{-1}p^{\nu+s}\mathbb{Z}_p)$, as we showed above. This implies that $\omega(z)$ lies indeed in $\overline{\omega(M)}$. Hence ω induces a homomorphism

$$\overline{\omega} : E_K(K)/\cap_\nu\,(MN_\nu E_K(K_\nu)) \longrightarrow (d^{-1}p^s\mathbb{Z}_p)^t/\overline{\omega(M)}.$$

Since the cokernel of ω is annihilated by d, the same holds for the cokernel of $\overline{\omega}$. Let us now study the kernels of ω and $\overline{\omega}$. For all $z \in E_K(K)$, z^d projects to an element b in $NB_K(K) \subset B_K(K)$. According to Lemma 6.4, b has a preimage z' in $NE_K(K)$. Hence $z^d = z'\alpha$ for some $\alpha \in T_K(K)$. Since σ vanishes on $NP(K)$, we find $\omega(z') = 0$. Hence if we assume that $\omega(z) = 0$, it follows that $\omega(\alpha) = (\rho(m'_1\alpha), \ldots, \rho(m'_t\alpha)) = 0$. Therefore all $m'_j\alpha$ lie in the kernel of ρ, which is equal to $N\mathbb{G}_m(K)$, so that $\alpha \in NT_K(K)$. Hence z^d is contained in $NE_K(K)$, which implies that $\mathrm{Ker}\omega/NE_K(K)$ is annihilated by d.

Let now z be an element in $E_K(K)$ such that $\omega(z)$ lies in $\overline{\omega(M)}$. Hence for all ν we find some $m_\nu \in M$ such that $\omega(z) - \omega(m_\nu) \in p^\nu((p^s\mathbb{Z}_p)^t) = (\rho(N_\nu K_\nu^\times))^t$. So we find $(\alpha_1, \ldots, \alpha_t) \in (N_\nu K_\nu^\times)^t$ such that $\omega(z) - \omega(m_\nu) = (\rho(\alpha_1), \ldots, \rho(\alpha_t))$, which is equal to $\omega(t_\nu)$ for the element $t_\nu \in N_\nu T_K(K_\nu)$ satisfying $m'_j(t_\nu) = \alpha_j$ for all j. Therefore $zm_\nu^{-1}t_\nu^{-1}$ is contained in the kernel of ω, which implies that z^d lies in $MN_\nu E_K(K_\nu)$ for all ν. Hence we find that the kernel of $\overline{\omega}$ is annihilated by d.

(3) We deduce from (1) and (2) that if B_K has ordinary reduction, then $A_K(K)/NA_K(K)$ is finite if and only if $d^{-1}\rho(K^\times)^t/\overline{\omega(M)}$ is finite. (Note that the torsion parts of both groups are finite. For the first group, this follows from Mattuck's Theorem, and the second group is a finitely generated $\mathbb{Z}_p$-module.) So it remains to show that $d^{-1}\rho(K^\times)^t/\overline{\omega(M)}$ is finite iff M is σ-invertible.

Note that $\omega(M)$ is generated by $(\sigma <m_i, m'_1>, \ldots, \sigma <m_i, m'_t>)$ for $i = 1, \ldots, t$. If $d^{-1}\rho(K^\times)^t/\overline{\omega(M)}$ is finite, then $\overline{\omega(M)}$ contains a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p^t$. Hence the same holds for $\omega(M)$, which implies that M is σ-invertible. Conversely, assume that M is σ-invertible. Then $\omega(M)$, and hence $\overline{\omega(M)}$ contains a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p^t$. Thus $d^{-1}\rho(K^\times)^t/\overline{\omega(M)}$ is a finitely generated torsion $\mathbb{Z}_p$-module, hence finite. □

Actually, the proof of 6.1 shows a more general result. By Mattuck's Theorem (or the existence of a logarithm), $A_K(K)/NA_K(K)$ contains a subgroup U of finite index, which is isomorphic to a free $\mathbb{Z}_p$-module. We define $rk_{\mathbb{Z}_p}A_K(K)/NA_K(K)$ to be the rank of this module. The properties of our map $\overline{\omega}$ show that if B_K has ordinary reduction, then

$$rk_{\mathbb{Z}_p}A_K(K)/NA_K(K) = rk_{\mathbb{Z}_p}(d^{-1}p^s\mathbb{Z}_p)^t/\overline{\omega(M)} = t - rk((\sigma <m_i, m'_j>)_{i,j}).$$

Question 6.5 *Is there a formula for the $\mathbb{Z}_p$-rank of $A_K(K)/NA_K(K)$ in terms of data given by the rigid analytic uniformization if B_K has arbitrary (good) reduction?*

Certainly, in such a formula the $\mathbb{Z}_p$-rank of $B_K(K)/NB_K(K)$ should appear. Note that Schneider's result [Sch2], Theorem 2 gives a formula for $rk_{\mathbb{Z}_p}B_K(K)/NB_K(K)$, which does not depend on the choice of a ramified $\mathbb{Z}_p$-extension.

Proof of Theorem 6.2: Since Schneider's ρ-splitting on $P_{A_K\times A'_K}(K)$ is uniquely determined by the fact that it vanishes on $NP_{A_K\times A'_K}(K)$, it suffices to show that our ρ-splitting τ vanishes on this universal norm group. Let a' be a point in $A'_K(K)$, and let x be an element of $NP_{A_K\times\{a'\}}(K)$. Fix a preimage e' of a' in

$E'_K(K)$, and put $b' = p'(e')$. Note that for any intermediate extension K_ν/K the map

$$P_{B_K \times \{b'\}}(K_\nu) \times_{B_K(K_\nu)\times\{b'\}} (E_K(K_\nu) \times \{e'\}) \xrightarrow{\theta} P_{A_K\times\{a'\}}(K_\nu)$$

is surjective. We abbreviate again $N_\nu = N_{K_\nu/K}$. For any ν, there exists some $x_\nu \in P_{A_K\times\{a'\}}(K_\nu)$ such that $x = N_\nu x_\nu$. Let $(\omega_\nu, e_\nu, e') \in P_{B_K\times\{b'\}}(K_\nu) \times_{B_K(K_\nu)\times\{b'\}} (E_K(K_\nu)\times\{e'\})$ be a preimage of x_ν. Then $N_\nu(\omega_\nu, e_\nu, e') \in P_{B_K\times\{b'\}}(K)\times_{B_K(K)\times\{b'\}} (E_K(K) \times \{e'\})$ projects to x under θ. Hence

$$\begin{aligned}\tau(x) = \tau^*(N_\nu(\omega_\nu, e_\nu, e')) = \\ \sigma(N_\nu\omega_\nu) - (\sigma <N_\nu e_\nu, m'_1>, \ldots, \sigma <N_\nu e_\nu, m'_t>)\Sigma\ {}^t(\sigma <m_1, e'>, \ldots, \sigma <m_t, e'>).\end{aligned}$$

We denote again by d respectively d' the number of elements of $B_K(K)/NB_K(K)$ respectively $B'_K(K)/NB'_K(K)$. Now recall from the proof of 6.1 that σ maps $N_\nu P_{B_K\times\{\tilde{b}\}}(K_\nu)$ to $d^{-1}d'^{-1}p^{\nu+s}\mathbb{Z}_p$ for all $\tilde{b} \in B'_K(K)$. If we denote by $u \le 0$ an integer such that the vector $\Sigma\ {}^t(\sigma <m_1, e'>, \ldots, \sigma <m_t, e'>)$ (which does not depend on ν) is contained in $(p^u\mathbb{Z}_p)^t$, we find that $\tau(x)$ is contained in $d^{-1}d'^{-1}p^{\nu+s+u}\mathbb{Z}_p$ for all ν. Hence $\tau(x) = 0$. □

7 The canonical Mazur-Tate height in the ordinary reduction case

Let us put $S_n = \mathrm{Spec} R/\mathcal{M}^{n+1}$, where $\mathcal{M}$ is the maximal ideal in R, and let us indicate base changes over $S = \mathrm{Spec} R$ with S_n by subscripts n. We continue to assume that A_K has split semistable reduction and that B_K has ordinary reduction, and we use the notation of section 5. In particular, we write $Z^\wedge$ for the completion of a S-scheme along the special fibre.

The rigid analytic uniformization map $\pi : E_K^{an} \to A_K^{an}$ induces a homomorphism of formal Néron models $E^\wedge \to A^\wedge$, which is an isomorphism on the identity components by [Bo-Xa], 2.3. This induces compatible isomorphisms $E_n^0 \xrightarrow{\sim} A_n^0$ for all n. In particular, the abelian part of A_k^0 is isomorphic to B_k. Hence A_K has semistable ordinary reduction and $n_A = n_B$. The same reasoning applies to A'_K.

Let $\rho : K^\times \to Y$ be a homomorphism to an abelian group Y which is uniquely divisible by $m_A m_{A'} n_A n_{A'}$. Then the canonical Mazur-Tate splittings in the ordinary case $\tilde{\sigma}_A : P_{A_K\times A'_K}(K) \to Y$ respectively $\tilde{\sigma}_B : P_{B_K\times B'_K}(K) \to Y$ exist.

Let $\nu : P_{A^0\times A'^0} \to A^0 \times A'^0$ and $\mu : P_{B\times B'} \to B \times B'$ be the natural projections. We will ususally write X^Z for the formal completion of a scheme X along a closed subscheme Z, with some exceptions: We denote by A^{0t}, A'^{0t} respectively $P^t_{A^0\times A'^0}$ the completions along T_k, T'_k respectively $\nu^{-1}(T_k \times T'_k)$, and by B^e, B'^e, respectively $P^e_{B\times B'}$ the completion along the unit sections of the special fibre respectively along the preimage of the unit section of $B_k \times B'_k$ under μ. Similar conventions hold for $(B_n)^e$, $(B'_n)^e$ and $(P_{B\times B'})^e_n$.

The isomorphisms $E_n^0 \xrightarrow{\sim} A_n^0$ provide all A_n^0 with the structure of an extension of B_n by T_n^0 in a compatible way. T_n^0 is (up to canonical isomorphism) the uniquely determined torus lifting T_k^0 and $T_n^0 \hookrightarrow A_n^0$ is the unique lift of $T_k^0 \hookrightarrow A_k^0$. Let $p_n : A_n^0 \to B_n$ be the projection map. We can deduce from [SGA7, I], IX, 7.5 that there exists a compatible system of isomorphisms

$$(p_n \times p'_n)^*(P_{B\times B'})_n \xrightarrow{\sim} (P_{A^0\times A'^0})_n.$$

We have a natural commutative diagram of formal biextensions

$$\begin{array}{ccc} (P_{B\times B'})^e_n \times_{B^e_n \times_{S_n} B'^e_n} (A^{0T^0_n} \times_{S_n} A'^{0T'^0_n}) & \longrightarrow & (P_{A^0\times A'^0})_n^{\nu_n^{-1}(T^0_n \times T'^0_n)} \\ \downarrow & & \downarrow \\ (P_{B\times B'})_n \times_{B_n \times_{S_n} B'_n} (A^0_n \times_{S_n} A'^0_n) & \longrightarrow & (P_{A^0\times A'^0})_n \end{array}$$

Passing to the limit, we find a commutative diagram of formal biextensions

$$\begin{array}{ccc} (P_{B\times B'})^e \times_{B^e \times_{S^\wedge} B'^e} (A^{0t} \times_{S^\wedge} A'^{0t}) & \longrightarrow & (P_{A^0\times A'^0})^t \\ \downarrow & & \downarrow \\ (P_{B\times B'})^\wedge \times_{B^\wedge \times_{S^\wedge} B'^\wedge} (A^{0\wedge} \times_{S^\wedge} A'^{0\wedge}) & \underset{\xi}{\xrightarrow{\sim}} & (P_{A^0\times A'^0})^\wedge \end{array}$$

Hence the (uniquely determined) splitting $(P_{B\times B'})^e \to \mathbb{G}^\wedge_{m,R}$ induces the uniquely determined splitting of the biextension $(P_{A^0\times A'^0})^t$. This implies that the relation between $\tilde{\sigma}_A$ and $\tilde{\sigma}_B$ is the following:

LEMMA 7.1 *For $x \in P_{A_K\times A'_K}(K)$ we denote by $x^{(m_A,m_{A'})}$ the point we get by applying to x the m_A-th power map with respect to the group structure over A'_K and the $m_{A'}$-th power map with respect to the group structure over A_K. Let $\alpha \in K^\times$ and $y \in P_{A^0\times A'^0}(R)$ be such that $x^{(m_A,m_{A'})} = \alpha y$. Let ω be the projection to $P_{B\times B'}(R)$ of $\xi^{-1}(y)$. Then*

$$\tilde{\sigma}_A(x) = \frac{1}{m_A m_{A'}}(\rho(\alpha) + \tilde{\sigma}_B(\omega)).$$

PROOF: Both sides are ρ-splittings which are equal to $\rho\circ\sigma_0$ on $(P_{A^0\times A'^0})^t(R)$, where $\sigma_0 : P^t_{A^0\times A'^0} \to \mathbb{G}^\wedge_{m,R}$ is the unique splitting. □

Note that ξ induces a homomorphism of biextensions

$$\xi^{rig} : (P_{B\times B'})^{\wedge rig} \times_{B^{\wedge rig}\times B'^{\wedge rig}} (E^{0\wedge rig} \times E'^{0\wedge rig}) \to (P_{A^0\times A'^0})^{\wedge rig} \hookrightarrow P^{an}_{A_K\times A'_K}.$$

Hence ξ^{rig} differs from the restriction of θ to $(P_{B\times B'})^{\wedge rig} \times_{B^{\wedge rig}\times B'^{\wedge rig}} (E^{0\wedge rig} \times E'^{0\wedge rig}) \hookrightarrow P^{an}_{B_K\times B'_K} \times_{B^{an}_K\times B'^{an}_K} E^{an}_K \times E'^{an}_K$ by a bilinear map $E^{0\wedge rig} \times E'^{0\wedge rig} \to \mathbb{G}^{an}_{m,K}$, which must be equal to one. We find that ξ^{rig} is equal to the restriction of θ. Hence for $y \in P_{A^0\times A'^0}(R)$ the point $\xi^{-1}(y)$ is just the unique preimage of y under θ which lies in $P_{B\times B'}(R) \times_{B(R)\times B'(R)} (E^0(R) \times E'^0(R))$ (after identifying $E^0(R)$ respectively $E'^0(R)$ with $A^0(R)$ respectively $A'^0(R)$).

If $\tilde{\sigma}_B$ is M-invertible, and we use it to construct our ρ-splitting τ, then we can calculate the difference between $\tilde{\sigma}_A$ and τ using 7.1. We apply this to compare Schneider's p-adic height pairing to the one defined by Mazur and Tate in the semistable ordinary reduction case.

THEOREM 7.2 *Let $\rho : K^\times \to \mathbb{Q}_p$ be a non-trivial, continuous homomorphism, and assume that B_K has ordinary reduction and that $NA_K(K)$ has finite index in $A_K(K)$. Let $\sigma_{\rho,A}$ respectively $\sigma_{\rho,B}$ denote the canonical Schneider ρ-splittings on A_K respectively B_K. For $x \in P_{A_K\times A'_K}(K)$ projecting to $(a,a') \in A_K(K)\times A'_K(K)$ let e and e'*

be the uniquely determined preimages of a^{m_A} *respectively* $a'^{m_{A'}}$ *in* $E^0(R)$ *respectively* $E'^0(R)$. *Then*

$$\tilde{\sigma}_A(x) = \sigma_{\rho,A}(x) + \frac{1}{m_A m_{A'}}(\sigma_{\rho,B}<e, m'_1>, \ldots, \sigma_{\rho,B}<e, m'_t>)\ \Sigma\ {}^t(\sigma_{\rho,B}<m_1, e'>, \ldots, \sigma_{\rho,B}<m_t, e'>).$$

PROOF: Recall that $\tilde{\sigma}_B$ is equal to $\sigma_{\rho,B}$ since B_K has good ordinary reduction, and that $\sigma_{\rho,A}$ is equal to τ by 6.2. Then our claim follows from 7.1 and the definition of τ. □

Note that in [MTT], p.34, a comparison formula between these two p-adic height pairings is stated for Tate curves. Let us apply 7.2 to a Tate curve A_K over K with Tate parameter $q \in K^\times$, i.e. $E_K = \mathbb{G}_{m,K}$ and $M = <q> \subset K^\times$. We identify E'_K with $\mathbb{G}_{m,K}$ via the character q. Then Theorem 7.2 boils down to

COROLLARY 7.3 *Assume that* $\rho(q) \neq 0$. *For* $x \in P_{A_K \times A'_K}(K)$ *projecting to* (a, a') *in* $A_K(K) \times A'_K(K)$ *let* (e, e') *be the uniquely defined preimage of* $(a^{m_A}, a'^{m_{A'}})$ *in* $R^\times \times R^\times \subset E_K(K) \times E'_K(K)$. *Then*

$$\tilde{\sigma}_A(x) = \sigma_{\rho,A}(x) + \frac{1}{m_A m_{A'}}\left(\frac{\rho(e)\rho(e')}{\rho(q)}\right).$$

Hence our formula differs from the one in [MTT] by the factors $\mathrm{ord}_v(q_v)$ appearing in the denominators of their correction term. The author consulted the authors of [MTT] about this discrepancy who agreed that the result in [MTT] needs a correction.

Note that a similar formula (without factors $\mathrm{ord}_v(q_v)$) describes the relation between Schneider's height and Nekovar's canonical height on Tate curves, see [Ne], 7.14. It seems very probable that Nekovar's height coincides with the canonical Mazur-Tate height for abelian varieties with semistable ordinary reduction, cf. [Ne], 8.2.

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