On the Cyclic Homology of Ringed Spaces and Schemes

Bernhard Keller

Received: March 22, 1998
Revised: September 21, 1998

Communicated by Peter Schneider

Abstract. We prove that the cyclic homology of a scheme with an ample line bundle coincides with the cyclic homology of its category of algebraic vector bundles. As a byproduct of the proof, we obtain a new construction of the Chern character of a perfect complex on a ringed space.

1991 Mathematics Subject Classification: Primary: 16E40; Secondary: 18E30, 14F05.

Keywords: Cyclic homology, Chern character, Ringed space, Scheme, Perfect complex, Derived category.

1. Introduction

1.1. The main theorem. Let k be a field and X a scheme over k which admits an ample line bundle (e.g. a quasi-projective variety). Let $\mathrm{vec}(X)$ denote the category of algebraic vector bundles on X. We view $\mathrm{vec}(X)$ as an exact category in the sense of Quillen [27]: By definition, a short sequence of vector bundles is admissible exact iff it is exact in the category of sheaves on X. Moreover, the category $\mathrm{vec}(X)$ is k-linear, i.e. it is additive and its morphism sets are k-vector spaces such that the composition is bilinear. In [18], we have defined, for each k-linear exact category $\mathcal{A}$, a cyclic homology theory $HC_*^{\mathrm{der}}(\mathcal{A})$. The superscript *der* indicates that the definition is modeled on that of the derived category of $\mathcal{A}$. In [loc. cit.] it was denoted by $HC_*(\mathcal{A})$. As announced in [loc. cit.], in this article, we will show that the cyclic homology of the scheme X coincides with the cyclic homology of the k-linear exact category $\mathrm{vec}(X)$: There is a canonical isomorphism (cf. Corollary 5.2)

$$HC_*(X) \xrightarrow{\sim} HC_*^{\mathrm{der}}(\mathrm{vec}(X)). \tag{1.1.1}$$

The definition of the cyclic homology of a scheme is an important technical point which will be discussed below in 1.4. Note that by definition [27, Par. 7], there is an analogous isomorphism in K-theory.

1.2. Motivation. Our motivation for proving the isomorphism 1.1.1 is twofold: Firstly, it allows the computation of $HC_*(X)$ for some non-trivial examples. Indeed, suppose that k is algebraically closed and that X is a smooth projective algebraic variety. Suppose moreover that X admits a *tilting bundle*, i.e. a vector bundle without higher selfextensions whose direct summands generate the bounded derived category of the category of coherent sheaves on X. Examples of varieties satisfying these hypotheses are projective spaces, Grassmannians, and smooth quadrics [3], [12], [13], [14]. In 5.3, we deduce from 1.1.1 that for such a variety, the Chern character induces an isomorphism

$$K_0X \otimes_{\mathbf{Z}} HC_*k \xrightarrow{\sim} HC_*X.$$

Here the left hand side is explicitly known since the group K_0X is free and admits a basis consisting of the classes of the pairwise non-isomorphic indecomposable direct summands of the tilting bundle. Cyclic homology of projective spaces was first computed by Beckmann [2] using a different method.

Our second motivation for proving the isomorphism 1.1.1 is that it provides further justification for the definition of HC_*^{der}. Indeed, there is a 'competing' (and previous) definition of cyclic homology for k-linear exact categories due to R. McCarthy [22]. Let us denote by $HC_*^{\mathrm{McC}}(\mathcal{A})$ the graded k-module which he associates with $\mathcal{A}$. McCarthy proved in [loc. cit.] a number of good properties for HC_*^{McC}. The most fundamental of these is the existence of an agreement isomorphism

$$HC_*(A) \xrightarrow{\sim} HC_*^{\mathrm{McC}}(\mathrm{proj}(A)),$$

where A is a k-algebra and $\mathrm{proj}(A)$, the category of finitely generated projective A-modules endowed with the split exact sequences. In particular, if we take A to be commutative, we obtain the isomorphism

$$HC_*(X) \xrightarrow{\sim} HC_*^{\mathrm{McC}}(\mathrm{vec}(X))$$

for all *affine schemes* $X = \mathrm{Spec}(A)$ (to identify the left hand side, we use Weibel's isomorphism [32] between the cyclic homology of an affine scheme and the cyclic homology of its coordinate algebra). Whereas for HC_*^{der}, this ismorphism extends to more general schemes, this cannot be the case for HC_*^{McC}. Indeed, for $n \geq 0$, the group $H^n(X, \mathcal{O}_X)$ occurs as a direct factor of $HC_{-n}(X)$. However, the group HC_{-n}^{McC} vanishes for $n > 0$ by its very definition.

1.3. Generalization, Chern character. Our proof of the isomorphism 1.1.1 actually yields a more general statement: Let X be a quasi-compact separated scheme over k. Denote by $\mathrm{per}\,X$ the pair formed by the category of perfect sheaves (4.1) on X and its full subcategory of acyclic perfect sheaves. The pair $\mathrm{per}\,X$ is a localization pair in the sense of [18, 2.4] and its cyclic homology $HC_*(\mathrm{per}\,X)$ has been defined in [loc. cit.]. We will show (5.2) that there is a canonical isomorphism

$$HC_*(X) \xrightarrow{\sim} HC_*(\mathrm{per}\,X). \tag{1.3.1}$$

If X admits an ample line bundle, we have an isomorphism

$$HC_*^{\mathrm{der}}(\mathrm{vec}(X)) \xrightarrow{\sim} HC_*(\mathrm{per}\,X)$$

so that the isomorphism 1.1.1 results as a special case.

The first step in the proof of 1.3.1 will be to construct a map

$$HC_*(\operatorname{per} X) \to HC_*(X).$$

This construction will be carried out in 4.2 for an arbitrary topological space X endowed with a sheaf of (possibly non-commutative) k-algebras. As a byproduct, we therefore obtain a new construction of the Chern character of a perfect complex P. Indeed, the complex P yields a functor between localization pairs

$$? \otimes_k P : \operatorname{per} \mathrm{pt} \to \operatorname{per} X$$

and hence a map

$$HC_*(\operatorname{per} \mathrm{pt}) \to HC_*(\operatorname{per} X) \to HC_*(X).$$

The image of the class

$$ch([k]) \in HC_*(\operatorname{per} \mathrm{pt}) = HC_*(k)$$

under this map is the value of the Chern character at the class of P. An analogous construction works for the other variants of cyclic homology, in particular for negative cyclic homology. The first construction of a Chern character for perfect complexes is due to Bressler–Nest–Tsygan, who needed it in their proof [5] of Schapira-Schneiders' conjecture [28]. They even construct a generalized Chern character defined on all higher K-groups. Several other constructions of a classical Chern character are due to B. Tsygan (unpublished).

1.4. CYCLIC HOMOLOGY OF SCHEMES. Let k be a commutative ring and X a scheme over k. The cyclic homology of X was first defined by Loday [20]: He sheafified the classical bicomplex to obtain a complex of sheaves $CC(\mathcal{O}_X)$. He then defined the cyclic homology of X to be the hypercohomology of the (total complex of) $CC(\mathcal{O}_X)$. Similarly for the different variants of cyclic homology. There arise three problems:

(1) The complex $CC(\mathcal{O}_X)$ is unbounded to the left. So there are (at least) two non-equivalent possibilities to define its hypercohomology: should one take Cartan-Eilenberg hypercohomology (cf. [32]) or derived functor cohomology in the sense of Spaltenstein [29] ?
(2) Is the cyclic homology of an affine scheme isomorphic to the cyclic homology of its coordinate ring ?
(3) If a morphism of schemes induces an isomorphism in Hochschild homology, does it always induce an isomorphism in cyclic homology ?

Problem (1) is related to the fact that in a category of sheaves, products are not exact in general. We refer to [32] for a discussion of this issue.

In the case of a noetherian scheme of finite dimension, Beckmann [2] and Weibel-Geller [34] gave a positive answer to (2) using Cartan-Eilenberg hypercohomology. By proving the existence of an SBI-sequence linking cyclic homology and Hochschild homology they also settled (3) for this class of schemes, whose Hochschild homology vanishes in all sufficiently negative degrees. Again using Cartan-Eilenberg hypercohomology, Weibel gave a positive answer to (2) in the general case in [32]. There, he also showed that cyclic homology is a homology theory on the category of quasi-compact quasi-separated schemes. Problem (3) remained open.

We will show in A.2 that Cartan-Eilenberg hypercohomology agrees with Spaltenstein's derived functor hypercohomology on all complexes with quasi-coherent homology if X is quasi-compact and separated. Since $CC(\mathcal{O}_X)$ has quasi-coherent homology [34], this shows that problem (1) does not matter for such schemes. As a byproduct of A.2, we deduce in B.1 a (partially) new proof of Boekstedt-Neeman's theorem [4] which states that for a quasi-compact separated scheme X, the unbounded derived category of quasi-coherent sheaves on X is equivalent to the full subcategory of the unbounded derived category of all $\mathcal{O}_X$-modules whose objects are the complexes with quasi-coherent homology. A different proof of this was given by Alonso–Jeremías–Lipman in [30, Prop. 1.3].

In order to get rid of problem (3), we will slightly modify Loday's definition: Using sheaves of mixed complexes as introduced by Weibel [33] we will show that the image of the Hochschild complex $C(\mathcal{O}_X)$ under the derived global section functor is canonically a mixed complex $M(X)$. The *mixed cyclic homology* of X will then be defined as the cyclic homology of $M(X)$. For the mixed cyclic homology groups, the answer to (2) is positive thanks to the corresponding theorem in Hochschild homology due to Weibel–Geller [34]; the answer to (3) is positive thanks to the definition. The mixed cyclic homology groups coincide with Loday's groups if the derived global section functor commutes with infinite sums. This is the case for quasi-compact separated schemes as we show in 5.10.

1.5. ORGANIZATION OF THE ARTICLE. In section 2, we recall the mixed complex of an algebra and define the mixed complex $M(X, \mathcal{A})$ of a ringed space $(X, \mathcal{A})$. In section 3, we recall the definition of the mixed complex associated with a localization pair and give a 'sheafifiable' description of the Chern character of a perfect complex over an algebra. In section 4, we construct a morphism from the mixed complex associated with the category of perfect complexes on $(X, \mathcal{A})$ to the mixed complex $M(X, \mathcal{A})$. We use it to construct the Chern character of a perfect complex on $(X, \mathcal{A})$. In section 5, we state and prove the main theorem and apply it to the computation of the cyclic homology of smooth projective varieties admitting a tilting bundle. In appendix A, we prove that Cartan-Eilenberg hypercohomology coincides with derived functor cohomology for (unbounded) complexes with quasi-coherent homology on quasi-compact separated schemes. In appendix B, we apply this to give a (partially) new proof of a theorem of Boekstedt-Neeman [4].

1.6. ACKNOWLEDGMENT. The author thanks the referee for his suggestions, which helped to make this article more readable.

2. HOMOLOGY THEORIES FOR RINGED SPACES

Let k be a field, X a topological space, and $\mathcal{A}$ a sheaf of k-algebras on X. In this section, we consider the possible definitions of the cyclic homology of $(X, \mathcal{A})$. In 2.1 we recall the definition suggested by Loday [20]. In 2.2, we point out that with this definition, it is not clear that a morphism inducing isomorphisms in Hochschild homology also does so in cyclic homology and its variants. This is our main reason for introducing the 'mixed homologies'. These also have the advantage of allowing a unified and simultaneous treatment of all the different homology theories. For the sequel, the two fundamental invariants are the mixed complex of sheaves $M(\mathcal{A})$ and its image $M(X, \mathcal{A}) = \mathbf{R}\Gamma(X, M(\mathcal{A}))$ under the derived global section functor. Both are canonical up to quasi-isomorphism and are therefore viewed as objects of the corresponding mixed derived categories. In the case of a point and a sheaf given by an algebra A, these complexes specialize to the mixed complex $M(A)$ associated with the algebra. The mixed cyclic homology $HC_{mix,*}(\mathcal{A})$ is defined to be the cyclic homology of the mixed complex $M(X, \mathcal{A})$.

2.1. Hochschild and cyclic homologies. Following a suggestion by Loday [20], the *Hochschild complex* $C(\mathcal{A})$, and the bicomplexes $CC(\mathcal{A})$, $CC^{-}(\mathcal{A})$, and $CC^{per}(\mathcal{A})$ are defined in [5, 4.1] by composing the classical constructions (cf. [21], for example) with sheafification. The *Hochschild homology, cyclic homology* ... of $\mathcal{A}$ are then obtained as the homologies of the complexes

$$\mathbf{R}\Gamma(X, C(\mathcal{A}))\ ,\ \mathbf{R}\Gamma(X, CC(\mathcal{A}))\ ,\ \ldots$$

where $\mathbf{R}\Gamma(X, ?)$ is the total right derived functor in the sense of Spaltenstein [29] of the global section functor.

2.2. Mixed cyclic homologies. Suppose that $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a morphism of spaces with sheaves of k-algebras inducing isomorphisms in Hochschild homology. With the above definitions, it does not seem to follow that f also induces isomorphisms in cyclic homology, negative cyclic homology, and periodic cyclic homology. This is one of the reasons why we need to replace the above definitions by slightly different variants defined in terms of the mixed complex associated with $\mathcal{A}$. This complex was introduced by C. Weibel in [33]. However, the 'mixed homologies' we consider do not always coincide with the ones of [33] (cf. the end of this section).

Let us first recall the case of ordinary algebras: For an algebra A, we denote by $M(A)$ the mapping cone over the differential $1 - t$ linking the first two columns of the bicomplex $CC(A)$. We endow $M(A)$ with the operator $B : M(A) \to M(A)[-1]$ induced by the norm map N from the first to the second column of the bicomplex. Then endowed with its differential d and with the operator B the complex $M(A)$ becomes a mixed complex in the sense of Kassel [15], i.e. we have

$$d^2 = 0\ ,\ B^2 = 0\ ,\ dB + Bd = 0.$$

The mixed complex $M(A)$ completely determines the homology theories of A. Indeed, we have a canonical quasi-isomorphism

$$C(A) \to M(A),$$

which shows that Hochschild homology is determined by $M(A)$. We also have canonical quasi-isomorphisms

$$CC(A) \xrightarrow{\sim} M(A) \otimes^{\mathbf{L}}_{\Lambda} k \quad , \quad CC^{-}(A) \xrightarrow{\sim} \mathbf{R}\operatorname{Hom}_{\Lambda}(k, M(A))$$

where the right hand sides are defined by viewing mixed complexes as objects of the mixed derived category, i.e. differential graded (=dg) modules over the dg algebra Λ generated by an indeterminate ε of chain degree 1 with $\varepsilon^2 = 0$ and $d\varepsilon = 0$ (cf. [15], [16]). Finally, we have a quasi-isomorphism

$$CC^{per}(A) \to (\mathbf{R}\varprojlim) P_k[-2n] \otimes_{\Lambda} M(A)$$

where P_k is a cofibrant resolution (= 'closed' resolution in the sense of [17, 7.4] ='semi-free' resolution in the sense of [1]) of the dg Λ-module k and the transition map $P_k[-2(n+1)] \to P_k[-2n]$ comes from a chosen morphism of mixed complexes

$P_k \to P_k[2]$ which represents the canonical morphism $k \to k[2]$ in the mixed derived category. For example, one can take

$$P_k = \bigoplus_{i \in \mathbf{N}} \Lambda[2i]$$

as a Λ-module endowed with the differential mapping the generator 1_i of $\Lambda[2i]$ to $\varepsilon 1_{i-1}$. The periodicity morphism then takes 1_i to 1_{i-1} and 1_0 to 0. Note that the functor $\varprojlim P_k[-2n] \otimes_\Lambda ?$ is actually exact so that $\mathbf{R}\varprojlim$ may be replaced by $\varprojlim$ in the above formula.

Following Weibel [33, Section 2] we sheafify this construction to obtain a mixed complex of sheaves $M(\mathcal{A})$. We view it as an object of the *mixed derived category* $\mathcal{DM}ix\,(X)$ *of sheaves on* X, i.e. the derived category of dg sheaves over the constant sheaf of dg algebras with value Λ. The global section functor induces a functor from mixed complexes of sheaves to mixed complexes of k-modules. By abuse of notation, the total right derived functor of the induced functor will still be denoted by $\mathbf{R}\Gamma(X, ?)$. The *mixed complex of the ringed space* $(X, \mathcal{A})$ is defined as

$$M(X, \mathcal{A}) = \mathbf{R}\Gamma(X, M(\mathcal{A})).$$

The fact that the functor $\mathbf{R}\Gamma(X, ?)$ (and the mixed derived category of sheaves) is well defined is proved by adapting Spaltenstein's argument of section 4 of [29]. Since the underlying complex of k-modules of $M(\mathcal{A})$ is quasi-isomorphic to $C(\mathcal{A})$, we have a canonical isomorphism

$$HH_*(\mathcal{A}) \xrightarrow{\sim} H_*\mathbf{R}\Gamma(X, M(\mathcal{A})).$$

We define the 'mixed variants'

$$HC_{mix,*}(\mathcal{A})\ ,\ HC^-_{mix,*}(\mathcal{A})\ ,\ HC^{per}_{mix,*}(\mathcal{A})$$

of the homologies associated with $\mathcal{A}$ by applying the functors

$$? \otimes^L_\Lambda k\ ,\ \mathbf{R}\operatorname{Hom}_\Lambda(k, ?) \quad \text{resp.} \quad \mathbf{R}\varprojlim P_k[-2n] \otimes_\Lambda ?$$

to $M(X, \mathcal{A})$ and taking homology.

These homology theories are slightly different from those of Bressler–Nest–Tsygan [5], Weibel [32], [33], and Beckmann [2]. We prove in 5.10 that mixed cyclic homology coincides with the cyclic homology defined by Weibel if the global section functor $\mathbf{R}\Gamma(X, ?)$ commutes with countable coproducts and that this is the case if $(X, \mathcal{A})$ is a quasi-compact separated scheme.

For a closed subset $Z \subset X$, we obtain versions with support in Z by applying the corresponding functors to $\mathbf{R}\Gamma_Z(X, M(\mathcal{A}))$.

Now suppose that a morphism $(X, \mathcal{A}) \to (Y, \mathcal{B})$ induces an isomorphism in HH_*. Then by definition, it induces an isomorphism in the mixed derived category

$$\mathbf{R}\Gamma(X, M(\mathcal{A})) \leftarrow \mathbf{R}\Gamma(Y, M(\mathcal{B}))$$

and thus in $HC_{mix,*}$, $HC^-_{mix,*}$, and $HC^{per}_{mix,*}$.

3. Homology theories for categories

In this section, we recall the definition of the cyclic homology (or rather: the mixed complex) of a localization pair from [18]. We apply this to give a description of the Chern character of a perfect complex over an algebra A (=sheaf of algebras over a point). This description will later be generalized to sheaves of algebras over a general topological space.

A localization pair is a pair consisting of a (small) differential graded k-category and a full subcategory satisfying certain additional assumptions. To define its mixed complex, we proceed in three steps: In 3.1, the classical definition for algebras is generalized to small k-categories following an idea of Mitchell's [24]; then, in 3.2, we enrich our small k-categories over the category of differential complexes, i.e. we define the mixed complex of a differential graded small k-category; by making this definition relative we arrive, in 3.3, at the definition of the mixed complex of a localization pair. For simplicity, we work only with the Hochschild complex at first.

We illustrate each of the three stages by considering the respective categories associated with a k-algebra A : the k-category $\mathrm{proj}(A)$ of finitely generated projective A-modules, the differential graded k-category $C^b(\mathrm{proj}(A))$ of bounded complexes over $\mathrm{proj}(A)$, and finally the localization pair formed by the category of all perfect complexes over A together with its full subcategory of all acyclic perfect complexes. The three respective mixed complexes are canonically quasi-isomorphic. Thanks to this fact the mixed complex of an algebra is seen to be functorial with respect to exact functors between categories of perfect complexes. This is the basis for our description of the Chern character in 4.2.

3.1. k-CATEGORIES. Let $\mathcal{C}$ be a small k-category, i.e. a small category whose morphism spaces carry structures of k-modules such that the composition maps are bilinear. Following Mitchell [24] one defines the Hochschild complex $C(\mathcal{C})$ to be the complex whose nth component is

$$\coprod \mathcal{C}(X_n, X_0) \otimes \mathcal{C}(X_{n-1}, X_n) \otimes \mathcal{C}(X_{n-2}, X_{n-1}) \otimes \ldots \otimes \mathcal{C}(X_0, X_1) \tag{3.1.1}$$

where the sum runs over all sequences $X_0, \ldots, X_n$ of objects of $\mathcal{C}$. The differential is given by the alternating sum of the face maps

$$d_i(f_n, \ldots, f_i, f_{i-1}, \ldots, f_0) = \begin{cases} (f_n, \ldots, f_i f_{i-1}, \ldots f_0) & \text{if } i > 0 \\ (-1)^n (f_0 f_n, \ldots, f_1) & \text{if } i = 0 \end{cases}$$

For example, suppose that A is a k-algebra. If we view A as a category $\mathcal{C}$ with one object, the Hochschild complex $C(\mathcal{C})$ coincides with $C(A)$. We have a canonical functor

$$A \to \mathrm{proj}\, A\ ,$$

where $\mathrm{proj}\, A$ denotes the category of finitely generated projective A-modules. By a theorem of McCarthy [22, 2.4.3], this functor induces a quasi-isomorphism

$$C(A) \to C(\mathrm{proj}\, A).$$

3.2. DIFFERENTIAL GRADED CATEGORIES. Now suppose that the category $\mathcal{C}$ is a differential graded k-category. This means that $\mathcal{C}$ is enriched over the category of differential $\mathbf{Z}$-graded k-modules (=dg k-modules), i.e. each space $\mathcal{C}\,(X, Y)$ is a dg k-module and the composition maps

$$\mathcal{C}\,(Y, Z) \otimes_k \mathcal{C}\,(X, Y) \to \mathcal{C}\,(X, Z)$$

are morphisms of dg k-modules. Then we obtain a double complex whose columns are the direct sums of (3.1.1) and whose horizontal differential is the alternating sum of the face maps

$$d_i(f_n, \dots, f_i, f_{i-1}, \dots, f_0) = \begin{cases} (f_n, \dots, f_i f_{i-1}, \dots f_0) & \text{if } i > 0 \\ (-1)^{(n+\sigma)}(f_0 f_n, \dots, f_1) & \text{if } i = 0 \end{cases}$$

where $\sigma = (\deg f_0)(\deg f_1 + \cdots + \deg f_{n-1})$. The Hochschild complex $C(\mathcal{C})$ of the dg category $\mathcal{C}$ is by definition the (sum) total complex of this double complex. The dg categories we will encounter are all obtained as subcategories of a category $\mathbf{C}(\mathcal{X})$ of differential complexes over a k-linear category $\mathcal{X}$ (a k-linear category is a k-category which admits all finite direct sums). In this case, the dg structure is given by the complex $\mathcal{H}om^\bullet_{\mathcal{X}}(X, Y)$ associated with two differential complexes X and Y.

Hence if A is a k-algebra, the category $\mathbf{C}^b(\operatorname{proj} A)$ of bounded complexes of finitely generated projective A-modules is a dg category and the functor

$$\operatorname{proj} A \to \mathbf{C}^b(\operatorname{proj} A)$$

mapping a module P to the complex concentrated in degree 0 whose zero component is P becomes a dg functor if we consider $\operatorname{proj} A$ as a dg category whose morphism spaces are concentrated in degree 0. By [17, lemma 1.2], the functor $\operatorname{proj} A \to \mathbf{C}^b(\operatorname{proj} A)$ induces a quasi-isomorphism

$$C(\operatorname{proj} A) \to C(\mathbf{C}^b(\operatorname{proj} A)).$$

3.3. Pairs of dg categories. Now suppose that $\mathcal{C}_0 \subset \mathcal{C}_1$ are full subcategories of a category of complexes $\mathbf{C}(\mathcal{X})$ over a small k-linear category $\mathcal{X}$. We define the Hochschild complex $C(\mathcal{C})$ of the pair $\mathcal{C} : \mathcal{C}_0 \subset \mathcal{C}_1$ to be the cone over the morphism

$$C(\mathcal{C}_0) \to C(\mathcal{C}_1)$$

induced by the inclusion (here both $\mathcal{C}_0$ and $\mathcal{C}_1$ are viewed as dg categories). For example, let A be a k-algebra. Recall that a perfect complex over A is a complex of A-modules which is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. Let $\operatorname{per} A$ denote the pair of subcategories of the category of complexes of A-modules formed by the category $\operatorname{per}_1 A$ of perfect A-modules and its full subcategory $\operatorname{per}_0 A$ of acyclic perfect A-modules. Clearly we have a functor $\operatorname{proj} A \to \operatorname{per} A$, i.e. a commutative diagram of dg categories

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \operatorname{per}_0 A \arrow[d] \\ \operatorname{proj} A \arrow[r] & \operatorname{per}_1 A \end{tikzcd}$$

This functor induces a quasi-isomorphism

$$C(\operatorname{proj} A) \to C(\operatorname{per} A)$$

by theorem 2.4 b) of [18].

3.4. Mixed complexes and characteristic classes. In the preceding paragraph, we have worked with the Hochschild complex, but it is easy to check that everything we said carries over to the mixed complex (2.2). The conclusion is then that if A is a k-algebra, we have the following isomorphisms in the mixed derived category

$$M(A) \xrightarrow{\sim} M(\operatorname{proj} A) \xrightarrow{\sim} M(\operatorname{per} A).$$

This shows that $M(A)$ is functorial with respect to morphisms of pairs $\operatorname{per} A \to \operatorname{per} B$, i.e. functors from perfect complexes over A to perfect complexes over B which respect the dg structure and preserve acyclicity. For example, if P is a perfect complex over A, we have the functor

$$? \otimes_k P : \operatorname{per} k \to \operatorname{per} A$$

which induces a morphism

$$M(? \otimes_k P) : M(\operatorname{per} k) \to M(\operatorname{per} A)$$

and hence a morphism

$$M(P) : M(k) \to M(A).$$

If we apply the functors H_0 resp. $H^* \mathbf{R} \operatorname{Hom}_\Lambda(k, ?)$ to this morphism we obtain morphisms

$$HH_0(k) \to HH_0(A) \quad \text{and} \quad HC^-_{mix,*}(k) \to HC^-_{mix,*}(A)$$

which map the canonical classes in $HH_0(k)$ resp. $HC^-_{mix,*}(k) = HC^-_*(k)$ to the Euler class resp. the Chern character of the perfect complex P.

4. Characteristic Classes for Ringed spaces

Let k be a field, X a topological space, and $\mathcal{A}$ a sheaf of k-algebras on X. In this section, we consider, for each open subset U of X, the localization pair of perfect complexes on U denoted by $\operatorname{per} \mathcal{A}|_U$. The mixed complexes $M(\operatorname{per} \mathcal{A}|_U)$ associated with these localization pairs are assembled into a sheaf of mixed complexes $M(\wp er\, \mathcal{A})$. In 4.1, we show that this sheaf is quasi-isomorphic to the sheaf $M(\mathcal{A})$ of mixed complexes associated with $\mathcal{A}$. In 4.2, this isomorphism is used to construct the trace morphism

$$\tau : M(\operatorname{per} \mathcal{A}) \to \mathbf{R}\Gamma(X, M(\mathcal{A})).$$

The construction of the characteristic classes of a perfect complex is then achieved using the functoriality of the mixed complex $M(\operatorname{per} \mathcal{A})$ with respect to exact functors between localization pairs.

The main theorem (5.2) will state that τ is invertible if $(X, \mathcal{A})$ is a quasi-compact separated scheme.

4.1. The presheaf of categories of perfect complexes. Recall that a *strictly perfect complex* is a complex P of $\mathcal{A}$-modules such that each point $x \in X$ admits an open neighbourhood U such that $P|_U$ is isomorphic to a bounded complex of direct summands of finitely generated free $\mathcal{A}|_U$-modules (note that such modules have no reason to be projective objects in the category of $\mathcal{A}|_U$-modules). A *perfect complex* is a complex P of $\mathcal{A}$-modules such that each point $x \in X$ admits an open neighbourhood U such that $P|_U$ is quasi-isomorphic to a strictly perfect complex.

We denote by $\operatorname{per}\mathcal{A}$ the pair formed by the category of perfect complexes and its full subcategory of acyclic perfect complexes. For each open $U \subset X$, we denote by $\operatorname{per}\mathcal{A}|_U$ the corresponding pair of categories of perfect $\mathcal{A}|_U$-modules. Via the restriction functors, the assignment $U \mapsto M(\operatorname{per}(\mathcal{A}|_U))$ becomes a presheaf of mixed complexes on X. We denote by $M(\wp er\,\mathcal{A})$ the corresponding sheaf of mixed complexes.

For each open $U \subset X$, we have a canonical functor

$$\operatorname{proj}\mathcal{A}(U) \to \operatorname{per}\mathcal{A}|_U\ ,$$

whence morphisms

$$M(\mathcal{A}(U)) \to M(\operatorname{proj}\mathcal{A}(U)) \to M(\operatorname{per}\mathcal{A}|_U)$$

and a morphism of sheaves

$$M(\mathcal{A}) \to M(\wp er\,\mathcal{A}).$$

Key Lemma. *The above morphism is a quasi-isomorphism*

Remark 4.1. This is the analog in cyclic homology of lemma 4.7.1 of [5] (with the same proof, as P. Bressler has kindly informed me).

Proof. We will show that the morphism induces quasi-isomorphisms in the stalks. Let $x \in X$. Clearly we have an isomorphism

$$M(\wp er\,\mathcal{A})_x \xrightarrow{\sim} M(\varinjlim \operatorname{per}\mathcal{A}|_U)\ ,$$

where U runs through the system of open neighbourhoods of x. We will show that the canonical functor

$$\varinjlim \operatorname{per}\mathcal{A}|_U \to \operatorname{per}\mathcal{A}_x$$

induces a quasi-isomorphism in the mixed complexes. For this, it is enough to show that it induces equivalences in the associated triangulated categories, by [18, 2.4 b)]. Now we have a commutative square

$$\begin{tikzcd} \varinjlim \operatorname{per}\mathcal{A}|_U \arrow[r] & \operatorname{per}\mathcal{A}_x \\ \varinjlim \operatorname{strper}\mathcal{A}|_U \arrow[r] \arrow[u] & \operatorname{strper}\mathcal{A}_x \arrow[u] \end{tikzcd}$$

Here, we denote by strper the pair formed by the category of strictly perfect complexes and its subcategory of acyclic complexes. For an algebra A, we have $\operatorname{strper} A = \mathbf{C}^b(\operatorname{proj} A)$ by definition. It is easy to see that the two vertical arrows induce equivalences in the triangulated categories, and the bottom arrow is actually itself an equivalence of categories. Indeed, we have the commutative square

$$\begin{tikzcd} \varinjlim \operatorname{strper}\mathcal{A}|_U \arrow[r] & \operatorname{strper}\mathcal{A}_x \\ \varinjlim \operatorname{strper}\mathcal{A}(U) \arrow[r] \arrow[u] & \operatorname{strper}\mathcal{A}_x \arrow[u] \end{tikzcd}$$

Here the right vertical arrow is the identity and the left vertical arrow and the bottom arrow are clearly equivalences.

The claim follows since the composition of the morphism

$$M(\mathcal{A}_x) \to M(\varinjlim \operatorname{per} \mathcal{A}|_U)$$

with the quasi-isomorphism $M(\varinjlim \operatorname{per} \mathcal{A}|_U) \to M(\operatorname{per} \mathcal{A}_x)$ is the canonical quasi-isomorphism $M(\mathcal{A}_x) \to M(\operatorname{per} \mathcal{A}_x)$.

$\surd$

4.2. Characteristic classes. By definition of $M(\wp er\, \mathcal{A})$ we have a morphism of mixed complexes $M(\operatorname{per} \mathcal{A}) \to \Gamma(X, M(\wp er\, \mathcal{A}))$. By the key lemma (4.1), the canonical morphism $M(\mathcal{A}) \to M(\wp er\, \mathcal{A})$ is invertible in the mixed derived category. Thus we can define the trace morphism

$$\tau : M(\operatorname{per} \mathcal{A}) \to \mathbf{R}\Gamma(X, M(\mathcal{A}))$$

by the following commutative diagram

$$\begin{tikzcd} M(\operatorname{per} \mathcal{A}) \arrow[r] \arrow[d, "\tau"'] & \Gamma(X, M(\wp er\, \mathcal{A})) \arrow[d] \\ \mathbf{R}\Gamma(X, M(\mathcal{A})) \arrow[r] & \mathbf{R}\Gamma(X, M(\wp er\, \mathcal{A})) \end{tikzcd}$$

Now let P be a perfect complex. It yields a functor

$$? \otimes_k P : \operatorname{per} k \to \operatorname{per} \mathcal{A}$$

and hence a morphism in the mixed derived category

$$M(k) \xrightarrow{\sim} M(\operatorname{per} k) \xrightarrow{M(P)} M(\operatorname{per} \mathcal{A}) \xrightarrow{\tau} \mathbf{R}\Gamma(X, M(\mathcal{A})) = M(X, \mathcal{A}).$$

If we apply the functor H_0 resp. $\mathbf{R}\operatorname{Hom}_\Lambda(k, ?)$ to this morphism, we obtain morphisms

$$HH_0(k) \to HH_0(\mathcal{A}) \quad \text{resp.} \quad HC_*^-(k) = HC_{mix,*}^-(k) \to HC_{mix,*}^-(\mathcal{A})$$

mapping the canonical classes to the Euler class respectively to the Chern character of the perfect complex P.

Remark 4.2. The trace morphism $\tau : M(\operatorname{per} \mathcal{A}) \to M(X, \mathcal{A})$ is a quasi-isomorphism if X is a point (by 3.3) or if $(X, \mathcal{A})$ is a quasi-compact separated scheme (by 5.2 below).

Remark 4.3. (B. Tsygan) Let P be a perfect complex and $A = \mathcal{H}om^\bullet_X(P, P)$ the dg algebra of endomorphisms of P. So if P is fibrant (cf. A.1), then the ith homology of A identifies with $\operatorname{Hom}_{\mathcal{D}X}(P, P[i])$. The dg category with one object whose endomorphism algebra is A naturally embeds into $\operatorname{per}_1 \mathcal{A}$ and we thus obtain a morphism

$$M(A) \to M(\operatorname{per}_1 \mathcal{A}) \to M(\operatorname{per} \mathcal{A}) \xrightarrow{\tau} \mathbf{R}\Gamma(X, M(\mathcal{A}))$$

whose composition with the canonical map $M(k) \to M(A)$ coincides with the morphism constructed above.

4.3. VARIANT WITH SUPPORTS. Let $Z \subset X$ be a closed subset. Let $\mathrm{per}(\mathcal{A} \text{ on } X)$ be the pair formed by the category of perfect complexes acyclic off Z and its full subcategory of acyclic complexes. For each open $U \subset X$ denote by $\mathrm{per}(\mathcal{A}|_U \text{ on } Z)$ the corresponding pair of categories of perfect $\mathcal{A}|_U$-modules. Via the restriction functors, the assignment $U \mapsto M(\mathrm{per}(\mathcal{A}|_U \text{ on } Z))$ becomes a presheaf of mixed complexes on X. We denote by $M(\wp er\,(\mathcal{A} \text{ on } Z))$ the corresponding sheaf of mixed complexes. We claim that $M(\wp er\,(\mathcal{A} \text{ on } Z))_x$ is acyclic for $x \notin Z$. Indeed, if $U \subset X \setminus Z$ is an open neighbourhood of x, then by definition, the inclusion

$$\mathrm{per}_0(\mathcal{A}|_U \text{ on } Z) \to \mathrm{per}_1(\mathcal{A}|_U \text{ on } Z)$$

is the identity so that $M(\mathrm{per}(\mathcal{A}|_U \text{ on } Z))$ is nullhomotopic. It follows that the canonical morphism $M(\wp er\,(\mathcal{A} \text{ on } Z)) \to M(\wp er\,\mathcal{A})$ uniquely factors through

$$\mathbf{R}\Gamma_Z M(\wp er\,\mathcal{A}) \to M(\wp er\,\mathcal{A})$$

in $\mathcal{DM}ix\,(X)$. Using the quasi-isomorphism $M(\mathcal{A}) \to M(\wp er\,\mathcal{A})$ we thus obtain a canonical morphism $M(\wp er\,(\mathcal{A} \text{ on } Z)) \to \mathbf{R}\Gamma_Z M(\mathcal{A})$ making the following diagram commutative

$$\begin{tikzcd}
M(\wp er\,(\mathcal{A} \text{ on } Z)) \arrow[r] \arrow[dr] & \mathbf{R}\Gamma_Z M(\wp er\,\mathcal{A}) \arrow[r] & M(\wp er\,\mathcal{A}) \\
& \mathbf{R}\Gamma_Z M(\mathcal{A}) \arrow[u] \arrow[r] & M(\mathcal{A}) \arrow[u]
\end{tikzcd}$$

We now define the trace morphism $\tau_Z : M(\mathrm{per}(\mathcal{A} \text{ on } Z)) \to \mathbf{R}\Gamma_Z(X, M(\mathcal{A}))$ as the composition

$$M(\mathrm{per}(\mathcal{A} \text{ on } Z)) \to \Gamma(X, M(\wp er\,(\mathcal{A} \text{ on } Z))) \to \mathbf{R}\Gamma_Z(X, M(\mathcal{A})).$$

We then have a commutative diagram

$$\begin{tikzcd}
M(\mathrm{per}(\mathcal{A} \text{ on } Z)) \arrow[r] \arrow[d] & M(\mathrm{per}\,\mathcal{A}) \arrow[d] \\
\mathbf{R}\Gamma_Z(X, M(\mathcal{A})) \arrow[r] & \mathbf{R}\Gamma(X, M(\mathcal{A}))
\end{tikzcd}$$

This yields a canonical lift of the classes constructed in section 4.2 to the theories supported in Z. The trace morphism τ_Z is invertible if X and $U = X \setminus Z$ are quasi-compact separated schemes (by 5.2 below).

5. THE MAIN THEOREM, EXAMPLES, PROOF

This section is devoted to the main theorem 5.2. Let k be a field and X a quasi-compact separated scheme over k. The mixed complex associated with X is defined as $M(X) = \mathbf{R}\Gamma(X, M(\mathcal{O}_X))$. The main theorem states that the trace map $\tau : M(\mathrm{per}\,X) \to M(X)$ of 4.2 is invertible in the mixed derived category.

In 5.1, we define $M(\mathrm{per}\,X)$ and examine its functoriality with respect to morphisms of schemes following [31]. In 5.2, we state the theorem and, as a corollary, the case of quasi-projective varieties. As an application, we compute, in 5.3, the cyclic homology of smooth projective varieties admitting a tilting bundle as described in the introduction.

The proof of the main theorem occupies subsections 5.4 to 5.9. It proceeds by induction on the number of open affines needed to cover X. The case of affine X is treated in section 5.4. The induction step uses a Mayer-Vietoris theorem (5.8) which is based on the description of the fiber of the morphism of mixed complexes induced by the localization at a quasi-compact open subscheme. This description is achieved in 5.7. It is based on Thomason-Trobaugh's localization theorem, which we recall in section 5.5 in a suitable form, and on the localization theorem for cyclic homology of localization pairs [18, 2.4 c)], which we adapt to our needs in 5.6.

5.1. Definition and functoriality. We adapt ideas of Thomason-Trobaugh [31]: Let X be a quasi-compact separated scheme over a field k. We put $\operatorname{per} X = \operatorname{per} \mathcal{O}_X$ (cf. 4.1). We claim that the assignment $X \mapsto M(\operatorname{per} X)$ is a functor of X. Indeed, let $\operatorname{flatper} X$ be the pair formed by the category of right bounded perfect complexes with flat components and its subcategory of acyclic complexes. Then the inclusion

$$\operatorname{flatper} X \to \operatorname{per} X$$

induces an equivalence in the associated triangulated categories (by [31, 3.5]) and hence an isomorphism

$$M(\operatorname{flatper} X) \to M(\operatorname{per} X)$$

by [18, 2.4 b)]. Now if $f : X \to Y$ is a morphism of schemes, then f^* clearly induces a a functor $\operatorname{flatper} Y \to \operatorname{flatper} X$ and hence a morphism $M(\operatorname{per} Y) \to M(\operatorname{per} X)$. Notice that this morphism is compatible with the map $M(\operatorname{per} X) \to \mathbf{R}\Gamma(X, M(\wp er\, X))$ of section 4.2.

Now suppose that X admits an ample family of line bundles. Then the inclusion

$$\operatorname{strper} X \to \operatorname{per} X$$

induces an equivalence in the associated triangulated categories [31, 3.8.3] and hence an isomorphism $M(\operatorname{strper} X) \to M(\operatorname{per} X)$. Note that $\operatorname{strper} X$ is simply the category of bounded complexes over the category $\operatorname{vec} X$ of algebraic vector bundles on X (together with its subcategory of acyclic complexes). Hence we have the equality $M(\operatorname{strper} X) = M(\operatorname{vec} X)$ where $M(\operatorname{vec} X)$ denotes the mixed complex associated with the *exact category* $\operatorname{vec} X$ as defined in [18]. In particular, if $X = \operatorname{Spec} A$ is affine, we have canonical isomorphisms

$$M(A) \tilde{\to} M(\operatorname{proj} A) \tilde{\to} M(\operatorname{vec} X) \tilde{\to} M(\operatorname{per} X).$$

5.2. The main theorem. Let X be a quasi-compact separated scheme over a field k. The mixed complex associated with X is defined as $M(X) = \mathbf{R}\Gamma(X, M(\mathcal{O}_X))$. Note that by definition, we have

$$HC_{mix,*}(X) = HC_* M(X)\,, \quad HC^-_{mix,*}(X) = HC^-_{mix,*} M(X)\,, \; \ldots .$$

Theorem. *The trace morphism (4.2)*

$$\tau : M(\operatorname{per} X) \to M(X)$$

is invertible. More generally, if Z is a closed subset of X such that $U = X \setminus Z$ is quasi-compact, then the trace morphism

$$\tau_Z : M(\operatorname{per}(X \text{ on } Z)) \to \mathbf{R}\Gamma_Z(X, M(\mathcal{O}_X))$$

is invertible.

COROLLARY. *Let X be a quasi-compact separated scheme over a field k. Then there is a canonical isomorphism*

$$HC_*(\operatorname{per} X) \xrightarrow{\sim} HC_*(X).$$

In particular, if X admits an ample line bundle (e.g. if X is a quasi-projective variety), there is a canonical isomorphism

$$HC_*^{\mathrm{der}}(\operatorname{vec} X) \xrightarrow{\sim} HC_*(X).$$

The corollary was announced in [18, 1.10], where we wrote $HC_*(\operatorname{vec} X)$ instead of $HC_*^{\mathrm{der}}(\operatorname{vec} X)$. It is immediate from the theorem once we prove that for quasi-compact separated schemes, there is an isomorphism

$$HC_*(X) \xrightarrow{\sim} HC_{mix,*}(X).$$

This will be done in 5.10.

The theorem will be proved in 5.9. The plan of the proof is described in the introduction to this section.

5.3. THE EXAMPLE OF VARIETIES WITH TILTING BUNDLES. Suppose that k is an algebraically closed field and that X is a smooth projective algebraic variety. Suppose moreover that X admits a *tilting bundle*, i.e. a vector bundle T without higher selfextensions whose direct summands generate the bounded derived category of the category of coherent sheaves on X as a triangulated category. Examples of varieties satisfying these hypotheses are projective spaces, Grassmannians, and smooth quadrics [3], [12], [13], [14].

PROPOSITION. *The Chern character induces an isomorphism*

$$K_0(X) \otimes_{\mathbf{Z}} HC_*(k) \to HC_*(X).$$

Here the left hand side is explicitly known since the group $K_0(X)$ is free and admits a basis consisting of the classes of the pairwise non-isomorphic indecomposable direct summands of the tilting bundle. For example, if X is the Grassmannian of k-dimensional subspaces of an n-dimensional space, the indecomposables are indexed by all Young diagrams with at most k rows and at most $n-k$ columns. Cyclic homology of projective spaces was first computed by Beckmann [2] using a different method.

The proposition shows that if X is a smooth projective variety such that $H^n(X, \mathcal{O}_X) \neq 0$ for some $n > 0$, then X cannot admit a tilting bundle. Indeed, the group $H^n(X, \mathcal{O}_X)$ occurs as a direct factor of $HC_{-n}(X)$ and therefore has to vanish if the assumptions of the proposition are satisfied.

Proof. Let A be the endomorphism algebra of the tilting bundle T and r the Jacobson radical of A. We assume without restriction of generality that T is a direct sum of pairwise non-isomorphic indecomposable bundles. Then A/r is a product of copies of k (since k is algebraically closed). We will show that the mixed complex $M(X)$ is canonically isomorphic to $M(A/r)$. For this, consider the exact functor

$$?\otimes_A : \operatorname{proj}(A) \to \operatorname{vec}(X).$$

It induces an equivalence in the bounded derived categories

$$\mathcal{D}^b(\operatorname{proj}(A)) \to \mathcal{D}^b(\operatorname{vec}(X)).$$

Indeed, we have a commutative square

$$\begin{tikzcd} \mathcal{D}^b(\mathrm{proj}(A)) \arrow[r, "?\otimes_A T"] \arrow[d] & \mathcal{D}^b(\mathrm{vec}(X)) \arrow[d] \\ \mathcal{D}^b(\mathrm{mod}(A)) \arrow[r, "\mathbf{L}(?\otimes_A T)"] & \mathcal{D}^b(\mathrm{coh}(X)) \end{tikzcd}$$

where $\mathrm{mod}(A)$ denotes the abelian category of all finitely generated right A-modules and $\mathrm{coh}(X)$ the abelian category of all coherent sheaves on X. Since T is a tilting bundle, the bottom arrow is an equivalence. Since X is smooth projective, it follows that A is of finite global dimension. Hence the left vertical arrow is an equivalence. Again because X is smooth projective, the right vertical arrow is an equivalence. Hence the top arrow is an equivalence. So the functor

$$? \otimes_A T : \mathrm{per}(A) \to \mathrm{per}(X)$$

induces an equivalence in the associated triangulated categories and hence an isomorphism

$$M(\mathrm{per}(A)) \xrightarrow{\sim} M(\mathrm{per}(X))$$

by [18, 2.4 b)]. Of course, it also induces an isomorphism $K_0(\mathrm{proj}(A)) \xrightarrow{\sim} K_0(\mathrm{vec}(X))$ and the Chern character is compatible with these isomorphisms by its description in 4.2. So we are reduced to proving that the Chern character induces an isomorphism

$$K_0(A) \otimes_Z HC_*(k) \xrightarrow{\sim} HC_*(A).$$

For this, let $E \subset A$ be a semi-simple subalgebra such that E identifies with the quotient A/r. The algebra E is a product of copies of k and of course, the inclusion $E \subset A$ induces an isomorphism in K_0. It also induces an isomorphism in HC_* by [17, 2.5] since A is finite-dimensional and of finite global dimension. These isomorphisms are clearly compatible with the Chern character and we are reduced to the corresponding assertion for $HC_*(E)$. This is clear since E is a product of copies of k. $\surd$

5.4. Proof of the main theorem in the affine case. Suppose that $X = \mathrm{Spec}\, A$. Then we know by section 5.2 that the canonical morphism $M(A) \to M(\mathrm{per}\, X)$ is invertible. Now Weibel-Geller have shown in [34, 4.1] that the canonical morphism

$$M(A) \to \mathbf{R}_{ce}\Gamma(X, M(\mathcal{O}_X))$$

is invertible where $M(\mathcal{O}_X)$ is viewed as a complex of sheaves on X and $\mathbf{R}_{ce}\Gamma(X, ?)$ denotes Cartan-Eilenberg hypercohomology (cf. section A.2). Moreover, Weibel-Geller have shown in [34, 0.4] that the complex $M(\mathcal{O}_X)$ has quasi-coherent homology. By section A.2, it follows that the canonical morphism

$$\mathbf{R}\Gamma(X, M(\mathcal{O}_X)) \to \mathbf{R}_{ce}\Gamma(X, M(\mathcal{O}_X))$$

is invertible. Using the commutative diagram

$$\begin{array}{ccc} M(A) & \longrightarrow & \mathbf{R}_{ce}\Gamma(X, M(\mathcal{O}_X)) \\ \downarrow & & \uparrow \\ M(\operatorname{per} X) & \longrightarrow & \mathbf{R}\Gamma(X, M(\mathcal{O}_X)) \end{array}$$

we conclude that $M(\operatorname{per} X) \to \mathbf{R}\Gamma(X, M(\mathcal{O}_X))$ is invertible for affine X.

5.5. THOMASON-TROBAUGH'S LOCALIZATION THEOREM. Let X be a quasi-compact quasi-separated scheme. We denote by $\mathcal{T}\operatorname{per} X$ the full subcategory of the (unbounded) derived category of the category of $\mathcal{O}_X$-modules whose objects are the perfect complexes. This category identifies with the triangulated category associated with the localization pair $\operatorname{per} X$ as defined in [18, 2.4]. Recall that a triangle functor $\mathcal{S} \to \mathcal{T}$ is an *equivalence up to factors* if it is an equivalence onto a full subcategory whose closure under forming direct summands is all of $\mathcal{T}$. A sequence of triangulated categories

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{T} \to 0$$

is *exact up to factors* if the first functor is an equivalence up to factors onto the kernel of the second functor and the induced functor $\mathcal{S}/\mathcal{R} \to \mathcal{T}$ is an equivalence up to factors.

THEOREM. [31]

a) *Let $U \subset X$ be a quasi-compact open subscheme and let $Z = X \setminus U$. Then the sequence*

$$0 \to \mathcal{T}\operatorname{per}(X \textit{ on } Z) \to \mathcal{T}\operatorname{per} X \to \mathcal{T}\operatorname{per} U \to 0$$

is exact up to factors.

b) *Suppose that $X = V \cup W$, where V and W are quasi-compact open subschemes and put $Z = X \setminus W$. Then the lines of the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}\operatorname{per}(X \textit{ on } Z) & \longrightarrow & \mathcal{T}\operatorname{per} X & \longrightarrow & \mathcal{T}\operatorname{per} W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j^*} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}\operatorname{per}(V \textit{ on } Z) & \longrightarrow & \mathcal{T}\operatorname{per} V & \longrightarrow & \mathcal{T}\operatorname{per}(V \cap W) & \longrightarrow & 0 \end{array}$$

are exact up to factors and the functor j^* is an equivalence up to factors.

The theorem was proved in section 5 of [31]. Note that the first assertion of part b) follows from a). The second assertion of b) is a special case of the main assertion in [31, 5.2] (take $U = V$, $Z = X \setminus W$ in [loc.cit.]). A new proof of the theorem is due to A. Neeman [25], [26].

5.6. LOCALIZATION IN CYCLIC HOMOLOGY OF DG CATEGORIES. In this section, we adapt the localization theorem [18, 4.9] to our needs. Let

$$0 \to \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \to 0$$

be a sequence of small flat exact DG categories such that F is fully faithful, $GF = 0$, and the induced sequence of stable categories

$$0 \to \underline{\mathcal{A}} \to \underline{\mathcal{B}} \to \underline{\mathcal{C}} \to 0$$

is exact up to factors (5.5).

Theorem. *The morphism*

$$\mathrm{Cone}(M(\mathcal{A}) \xrightarrow{M(F)} M(\mathcal{B})) \to M(\mathcal{C})$$

induced by $M(G)$ is a quasi-isomorphism.

Proof. The proof consists in extracting the relevant information from [18] : Indeed, since F is fully faithful, we may consider $\mathcal{A} \xrightarrow{F} \mathcal{B}$ as a localization pair and since $GF = 0$, the square

$$\begin{CD} \mathcal{A} @>{F}>> \mathcal{B} \\ @VVV @VV{G}V \\ 0 @>>> \mathcal{C} \end{CD}$$

as a morphism of localization pairs, i.e. a morphism of the category $\mathcal{L}^b_{str}$ of [18, 4.3]. By applying the completion functor $?^+$ of [loc. cit.] we obtain a morphism

(5.6.1)
$$\begin{CD} (\mathcal{A} \xrightarrow{F} \mathcal{B})^+ \\ @VVV \\ (0 \to \mathcal{C})^+ \end{CD}$$

of the category $\mathcal{L}$. Applying the functor Cm to this morphism yields the morphism

$$\begin{CD} (M(\mathcal{A}) \to M(\mathcal{B})) \\ @VV{(0,M(G))}V \\ (0 \to M(\mathcal{C})) \end{CD}$$

of $\mathcal{DM}or\mathcal{M}ix$ by the remarks following proposition 4.3 of [18]. On the other hand, applying the functor I_λ of [18, 4.8] to the morphism (5.6.1) yields the identity of $\mathcal{C}^+$ in $\mathcal{M}$ and applying M (denoted by C in [18]) yields the identity of $M(\mathcal{C})$ in $\mathcal{DM}ix$. By the naturality of the isomorphism of functors in [18, 4.9 a)], call it ψ, we obtain a commutative square in $\mathcal{DM}ix$

$$\begin{CD} \mathrm{Cone}(M(\mathcal{A}) \to M(\mathcal{B})) @>{\psi}>> M(\mathcal{C}) \\ @V{(0,M(G))}VV @VV{\mathbf{1}}V \\ \mathrm{Cone}(0 \to M(\mathcal{C})) @>{\psi}>> M(\mathcal{C}) \end{CD}$$

So the left vertical arrow of the square is invertible in $\mathcal{DM}ix$, which is what we had to prove. $\surd$

5.7. PERFECT COMPLEXES WITH SUPPORT AND LOCAL COHOMOLOGY. Let X be a quasi-compact quasi-separated scheme, $U \subset X$ a quasi-compact open subscheme, and $Z = X \setminus U$. Let $j : U \to X$ be the inclusion.

PROPOSITION. *The sequence*

$$M(\wp er\,(X \;\; on\; Z)) \to M(\wp er\, X) \to j^* M(\wp er\, U)$$

embeds into a triangle of $\mathcal{DM}ix\,(X)$. *This triangle is canonically isomorphic to the Z-local cohomology triangle associated with* $M(\wp er\, X)$. *In particular, there is a canonical isomorphism*

$$M(\wp er\,(X \;\; on\; Z)) \xrightarrow{\sim} \mathbf{R}\Gamma_Z(X, M(\wp er\, X)).$$

Moreover, the canonical morphisms fit into a morphism of triangles

$$\begin{tikzcd}
M(\mathrm{per}(X \;\; on\; Z)) \arrow[r] \arrow[d] & M(\mathrm{per}\, X) \arrow[r] \arrow[d] & M(\mathrm{per}\, U) \arrow[r] \arrow[d] & M(\mathrm{per}(X \;\; on\; Z))[1] \arrow[d] \\
\Gamma_Z M(\wp er\, X)) \arrow[r] & \Gamma M(\wp er\, X)) \arrow[r] & \Gamma M(\wp er\, U)) \arrow[r] & \Gamma_Z M(\wp er\, X))[1]
\end{tikzcd}$$

in the mixed derived category, where Γ *and* Γ_Z *are short for* $\mathbf{R}\Gamma(X, ?)$ *and* $\mathbf{R}\Gamma_Z(X, ?)$.

Proof. Let $V \subset X$ be open. Consider the sequence

$$M(\mathrm{per}(V \text{ on } Z)) \to M(\mathrm{per}\, V) \to M(\mathrm{per}(V \cap U)). \tag{5.7.1}$$

If we let V vary, it becomes a sequence of presheaves on X. We will show that there is a sequence of mixed complexes of presheaves

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{5.7.2}$$

such that

- we have $gf = 0$ in the category of mixed complexes of presheaves
- in the derived category of mixed complexes of presheaves, the sequence 5.7.2 becomes isomorphic to the sequence 5.7.1.
- for each quasi-compact open subscheme $V \subset X$, the canonical morphism from the cone over the morphism $A(V) \to B(V)$ to $C(V)$ induced by g is a quasi-isomorphism.

This implies that firstly, the sequence of sheaves associated with the sequence 5.7.2 embeds canonically into a triangle

$$\tilde{A} \to \tilde{B} \to \tilde{C} \to \tilde{A}[1]\,,$$

where the tilde denotes sheafification and the connecting morphism is constructed as the composition

$$\tilde{C} \xleftarrow{\sim} \mathrm{Cone}(\tilde{A} \to \tilde{B}) \to \tilde{A}[1]\,,$$

and secondly we have a morphism of triangles

$$\begin{tikzcd}
A(V) \arrow[r] \arrow[d] & B(V) \arrow[r] \arrow[d] & C(V) \arrow[r] \arrow[d] & A(V)[1] \arrow[d] \\
\mathbf{R}\Gamma(V,\tilde{A}) \arrow[r] & \mathbf{R}\Gamma(V,\tilde{B}) \arrow[r] & \mathbf{R}\Gamma(V,\tilde{C}) \arrow[r] & \mathbf{R}\Gamma(V,\tilde{A}[1])
\end{tikzcd}$$

for each quasi-compact open subscheme $V \subset X$ (to prove this last assertion, we use that $\mathbf{R}\Gamma(V,?)$ lifts to a derived functor defined on the category of all sequences

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C'$$

with $g'f' = 0$).

To construct the sequence 5.7.2, we have to (pre-) sheafify a part of the proof of [18, 2.4]. For this, let $\operatorname{iper} X$ denote the category of all fibrant (A.1) perfect complexes. Then the inclusion $\operatorname{iper} X \to \operatorname{per} X$ induces an equivalence in the associated triangulated categories and thus we have an isomorphism $M(\operatorname{iper} X) \xrightarrow{\sim} M(\operatorname{per} X)$ in $\mathcal{DM}ix$. Note that this even holds if X is an arbitrary ringed space. In particular, it holds for each open subscheme $V \subset X$ instead of X. Hence the presheaf $V \mapsto M(\operatorname{per} V)$ is isomorphic in the derived category of presheaves to $V \mapsto M(\operatorname{iper} V)$. Similarly for the other terms of the sequence, so that we are reduced to proving the assertion for the sequence of presheaves whose value at V is

$$M(\operatorname{iper}(V \text{ on } Z)) \to M(\operatorname{iper} V) \to M(\operatorname{iper}(U \cap V)).$$

For this, let $\mathcal{I}(V)$ be the exact dg category [18, 2.1] of fibrant (A.1) complexes on V and let $\tilde{\mathcal{I}}(V)$ be the category whose objects are the exact sequences

$$0 \to K \xrightarrow{i} L \xrightarrow{p} M \to 0$$

of $\mathcal{I}(V)$ such that i has split monomorphic components, K is acyclic off Z and i_x is a quasi-isomorphism for each $x \in Z$. Then $\tilde{\mathcal{I}}(V)$ is equivalent to a full exact dg subcategory of the category of filtered objects of $\mathcal{I}(V)$ (cf. example 2.2 d) of [18]). Let $\tilde{\mathcal{I}}(V \text{ on } Z)$ be the full subcategory of $\tilde{\mathcal{I}}(X)$ whose objects are the sequences

$$0 \to K \xrightarrow{\sim} L \to 0 \to 0$$

and $\tilde{\mathcal{I}}(U \cap V)$ the full subcategory whose objects are the sequences

$$0 \to 0 \to M \xrightarrow{\sim} L \to 0.$$

Let $G : \tilde{\mathcal{I}}(V) \to \tilde{\mathcal{I}}(V \cap U)$ be the functor

$$(0 \to K \to L \to M \to 0) \mapsto (0 \to 0 \to M \xrightarrow{\mathbf{1}} M \to 0)$$

and $F : \tilde{\mathcal{I}}(V \text{ on } Z) \to \tilde{\mathcal{I}}(V)$ the inclusion. Then the sequence

$$0 \to \tilde{\mathcal{I}}(V \text{ on } Z) \xrightarrow{F} \tilde{\mathcal{I}}(V) \xrightarrow{G} \tilde{\mathcal{I}}(V \cap U) \to 0 \tag{5.7.3}$$

is an exact sequence of the category $\mathcal{M}_{str}$ of [18, 4.4] and in particular we have $GF = 0$. We take the subsequence of perfect objects : Let $\widetilde{\operatorname{iper}}(V \text{ on } Z)$ be the full subcategory of $\tilde{\mathcal{I}}(V \text{ on } Z)$ whose objects are the $K \xrightarrow{\sim} L \to 0$ with $K \in \operatorname{iper}(V \text{ on } Z)$, let $\widetilde{\operatorname{iper}}(V)$ be the full subcategory of the $K \to L \to M$ with $M \in \operatorname{per} V$, and let

$\widetilde{\mathrm{iper}}(V \cap U)$ be the full subcategory of the $0 \to L \to M$ with $M|_U \in \mathrm{per}(V \cap U)$. Consider the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{iper}}(V \text{ on } Z) \arrow[r, "F"] \arrow[d] & \widetilde{\mathrm{iper}}(V) \arrow[r, "G"] \arrow[d] & \widetilde{\mathrm{iper}}(V \cap U) \arrow[d] \\
\mathrm{iper}(V \text{ on } Z) \arrow[r] & \mathrm{iper}(V) \arrow[r] & \mathrm{iper}(V \cap U),
\end{tikzcd}$$

where the three vertical functors are given by

$$\begin{array}{rcl}
K \xrightarrow{\sim} L \to 0 & \mapsto & K \\
K \to L \to M & \mapsto & M \\
0 \to L \to M & \mapsto & M|_U .
\end{array}$$

Its left hand square is commutative up to isomorphism and its right hand square is commutative up to the homotopy [18, 3.3]

$$L|_U \xrightarrow{p|_U} M|_U .$$

The vertical arrows clearly induce equivalences in the associated triangulated categories. By applying the functor M to the diagram and letting V vary we obtain a commutative diagram in the derived category of presheaves of mixed complexes on X. The vertical arrows become invertible and the top row becomes

$$M(\widetilde{\mathrm{iper}}(V \text{ on } Z)) \to M(\widetilde{\mathrm{iper}}(V)) \to M(\widetilde{\mathrm{iper}}(V \cap U))$$

where V runs through the open subsets of X. This is the sequence of presheaves $A \to B \to C$ announced at the beginning of the proof. Using theorem 5.5 a) and theorem 5.6 one sees that it has the required properties. $\surd$

5.8. Mayer-Vietoris sequences. Let X be a quasi-compact quasi-separated scheme and $V, W \subset X$ quasi-compact open subschemes such that $X = V \cup W$.

Proposition. *There is a canonical morphism of triangles in the mixed derived category*

$$\begin{tikzcd}
M(\mathrm{per}\, X) \arrow[r] \arrow[d] & M(\mathrm{per}\, V) \oplus M(\mathrm{per}\, W) \arrow[r] \arrow[d] & M(\mathrm{per}(V \cap W)) \arrow[r] \arrow[d] & {} \\
\Gamma M(\wp er\, X) \arrow[r] & \Gamma M(\wp er\, V) \oplus \Gamma M(\wp er\, W)) \arrow[r] & \Gamma M(\wp er\, (V \cap W)) \arrow[r] & {}
\end{tikzcd}$$

where Γ *is short for* $\mathbf{R}\Gamma(X, ?)$.

Proof. Put $Z = X \setminus W$. The first line of the diagram is deduced from theorem 5.5 b) using [18, 2.7]. Clearly the two squares appearing in the diagram are commutative.

We have to show that the square involving the arrows of degree 1

$$\begin{tikzcd} M(\mathrm{per}(V\cap W)) \arrow[r] \arrow[d] & M(\mathrm{per}\,X)[1] \arrow[d] \\ \Gamma M(\wp er\,(V\cap W)) \arrow[r] & \Gamma M(\wp er\,X))[1] \end{tikzcd}$$

is commutative as well. By [loc.cit.], the connecting morphism is the composition

$$\begin{tikzcd} & M(\mathrm{per}(X \text{ on } Z)[1] \arrow[r] \arrow[d] & M(\mathrm{per}\,X)[1] \\ M(\mathrm{per}(V\cap W)) \arrow[r] & M(\mathrm{per}(V \text{ on } Z))[1] & \end{tikzcd}$$

Here the vertical morphism is invertible by theorem 5.5 b) and [18, 2.4 b)]. The second line of the diagram is the Mayer-Vietoris triangle for hypercohomology. So the connecting morphism of the second line is obtained as the composition

$$\Gamma M(\wp er\,(V\cap W)) \to \Gamma_Z M(\wp er\,V)[1] \xleftarrow{\sim} \Gamma_Z M(\wp er\,X)[1] \to \Gamma M(\wp er\,X)[1]\,,$$

where Γ and Γ_Z are short for $\mathbf{R}\Gamma(X,?)$ and $\mathbf{R}\Gamma_Z(X,?)$. Now it follows from proposition 5.7 that the rightmost square of the diagram of the assertion is commutative as well. $\surd$

5.9. Proof of theorem 5.1. Let $V_1,\dots,V_n$ be open affines covering X. If $n=1$, theorem 5.2 holds by section 5.4. If $n>1$, we cover X by $V=V_1$ and $W=\bigcup_{i=2\dots n}V_i$. The intersection $V\cap W$ is then covered by the $n-1$ sets $V\cap V_i$, $2\le i\le n$. These are affine, since X is separated. So theorem 5.2 holds for V, W, and $V\cap W$ by the induction hypothesis. Thus it holds for $X=V\cup W$ by proposition 5.8. The assertion for τ_Z now follows by proposition 5.7.

5.10. Proof of corollary 5.1. In [32] (cf. also [33]), C. Weibel defined $HC_*(X)$ as the homology of the complex of k-modules

$$\mathbf{R}\Gamma_{ce}(X, CC(\mathcal{O}_X))$$

where $\mathbf{R}\Gamma_{ce}$ denotes Cartan-Eilenberg hypercohomology (cf. section A.2) and $CC(\mathcal{O}_X)$ is the sheafification of the classical bicomplex. Now Weibel-Geller have shown in [34] that the Hochschild complex $C(\mathcal{O}_X)$ has quasi-coherent homology. Thus each column of $CC(\mathcal{O}_X)$ has quasi-coherent homology and hence (the sum total complex of) $CC(\mathcal{O}_X)$ has itself quasi-coherent homology. Hence by theorem A.2, the above complex is isomorphic to

$$\mathbf{R}\Gamma(X, CC(\mathcal{O}_X)).$$

Now, as in the case of an algebra (cf. [21, 2.5.13]), $CC(\mathcal{O}_X)$ may also be viewed as the (sum total complex of the) bicomplex $\mathcal{BC}(M(\mathcal{O}_X))$ associated with the mixed complex of sheaves $M(\mathcal{O}_X)$ (cf. [33, Section 2]). What remains to be proved then is that the canonical map

$$\mathcal{BC}(\mathbf{R}\Gamma(X, M(\mathcal{O}_X)) \to \mathbf{R}\Gamma(X, \mathcal{BC}(M(\mathcal{O}_X))$$

is invertible in the derived category of k-vector spaces. Now indeed, more generally, we claim that we have

$$\mathcal{B}C(\mathbf{R}\Gamma(X, M)) \xrightarrow{\sim} \mathbf{R}\Gamma(X, \mathcal{B}C(M))$$

for any mixed complex of sheaves M with quasi-coherent homology. As the reader will easily check, this is immediate once we know that the functor $\mathbf{R}\Gamma(X, ?)$ commutes with countable direct sums when restricted to the category of complexes with quasi-coherent homology. This follows from Corollary 3.9.3.2 in [19]. It may also be proved by the argument of [26, 1.4]. For completeness, we include a proof : Let K_i, $i \in I$, be a family of complexes with quasi-coherent homology. It is enough to prove that $H^0(X, ?)$ takes $K = \bigoplus K_i$ to the sum of the $H^0(X, K_i)$. Now $\Gamma(X, ?)$ is of finite cohomological dimension on the category of quasi-coherent modules. Indeed, for an affine X, this follows from Serre's theorem [9, III, 1.3.1], and for arbitrary X it is proved by induction on the size of an affine cover of X (here we use that X is quasi-compact and separated). It therefore follows from by theorem A.2 b), lemma A.3, and Serre's theorem [9, III, 1.3.1]. that we have an isomorphism $H^0(X, K_i) \xrightarrow{\sim} H^0(X, \tau^{\geq n} K_i)$ and similarly for K for some fixed $n < 0$ (cf. the proof of theorem A.2 for the definition of the truncation functor $\tau^{\geq n}$). So we may assume that the K_i and K are uniformly bounded below. But then, we may compute the $H^0(X, K_i)$ using resolutions $K_i \to F_i$ by uniformly bounded below complexes of flasque sheaves. The sum of the F_i is again bounded below with flasque components and is clearly quasi-isomorphic to K. Now $\Gamma(X, ?)$ commutes with infinite sums since X is quasi-compact, so the claim follows.

APPENDIX A. ON CARTAN-EILENBERG RESOLUTIONS

We prove that Cartan-Eilenberg hypercohomology coincides with derived functor hypercohomology on all (unbounded) complexes of sheaves with quasi-coherent homology on a quasi-compact separated scheme. More precisely, we prove that in this situation, Cartan-Eilenberg resolutions are actually K-injective resolutions in the sense of [29].

A.1. TERMINOLOGY. Let $\mathcal{A}$ be a Grothendieck category. Spaltenstein [29] defined a complex I over $\mathcal{A}$ to be *K-injective* if, in the homotopy category, there are no non zero morphisms from an acyclic complex to I. This is the case iff each morphism $M \to I$ in the derived category is represented by a unique homotopy class of morphisms of complexes.

In [33, A.2], C. Weibel proposed the use of the term *fibrant* for K-injective. Indeed, one can show that a complex is K-injective iff it is homotopy equivalent to a complex which is fibrant for the 'global' closed model structure on the category of complexes in which cofibrations are the componentwise monomorphisms. This structure is an additive analogue of the global closed model structure on the category of simplicial sheaves on a Grothendieck site. The existence of the global structure in the case of simplicial sheaves was proved by Joyal [11] (cf. [10, 2.7]). We have not been able to find a published proof of the fact that the category of complexes over a Grothendieck category admits the global structure (an unpublished proof is due to F. Morel). However, the key step may be found in [8, Prop. 1].

Whereas in the homotopy category, the notions of 'fibrant for the global structure' and 'K-injective' become essentially equivalent, there is a slight difference at the level of complexes: fibrant objects for the global structure are exactly the K-injective complexes with injective components.

We will adopt the terminology proposed by Weibel: We *call a complex fibrant* iff it is K-injective in the sense of Spaltenstein. This will not lead to ambiguities since we will not use the global closed model structure.

A.2. Sheaves with quasi-coherent cohomology. Let X be a scheme and K a complex of $\mathcal{O}_X$-modules (unbounded to the right and to the left). Let I be a Cartan-Eilenberg resolution of K, i.e.

a) I is a $\mathbf{Z} \times \mathbf{Z}$-graded $\mathcal{O}_X$-module endowed with differentials d_I of bidegree $(1,0)$ and d_{II} of bidegree $(0,1)$ such that $(d_I + d_{II})^2 = 0$,
b) I^{pq} vanishes for $q < 0$ and
c) I is endowed with an augmentation $\varepsilon : K \to I$, i.e. a morphism of differential $\mathbf{Z} \times \mathbf{Z}$-graded $\mathcal{O}_X$-modules, where K is viewed as concentrated on the p-axis, such that for each p, the induced morphisms $K^p \to I^{p,\bullet}$ and $H^pK \to H^p_I I$ are injective resolutions.

It follows that for each p, the induced morphisms $B^pK \to B^p_I K$ and $Z^pK \to Z^p_I I$ are injective resolutions and that the rows of I are products of complexes of the form

$$\ldots 0 \to M \to 0 \ldots \quad \text{or} \quad \ldots 0 \to M \xrightarrow{\mathbf{1}} M \to 0 \ldots ,$$

where M is injective.

Let $J = \widehat{\operatorname{Tot}}\, I$ denote the product total complex of I and $\eta : K \to J$ the morphism of complexes induced by ε. The morphism η is called a *total Cartan-Eilenberg resolution* of K. The *Cartan-Eilenberg hypercohomology* of K is the cohomology of the complex

$$\mathbf{R}\Gamma_{ce}(X,K) = \Gamma(X,J).$$

The morphism η is usually *not* a quasi-isomorphism.

Theorem. a) *The complex J is fibrant (A.1).*
b) *If K has quasi-coherent homology, the morphism $\eta : K \to J$ is a quasi-isomorphism. Hence, Cartan-Eilenberg hypercohomology of K coincides with derived functor hypercohomology of K in the sense of Spaltenstein* [29].

Part a) holds more generally whenever K is a complex of objects over an abelian category having enough injectives and admitting all countable products. This was proved by C. Weibel in [32, A.3]. For completeness, we include a proof of a) below. Part b) was proved by C. Weibel in [loc. cit.] for the case of complete abelian categories with enough injectives and *exact products*, for example module categories. The case we consider here is implicit in [29, 3.13]. Nevertheless, we thought it useful to include the explicit statement and a complete proof.

In preparation of the proof, let us recall the notion of a homotopy limit (cf. [4] for example) : If $\mathcal{T}$ is a triangulated category admitting all countable products and

$$\ldots \to X_{p+1} \xrightarrow{f_p} X_p \to \ldots \to X_0 \, , \; p \in \mathbf{N} \, ,$$

is a sequence in $\mathcal{T}$, its *homotopy limit* $\operatorname{holim} X_p$ is defined by the Milnor triangle [23]

$$\operatorname{holim} X_p \to \prod X_p \xrightarrow{\Phi} \prod X_q \to (\operatorname{holim} X_p)[1] \, ,$$

where the morphism Φ has the components

$$\prod_p X_p \xrightarrow{\text{can}} X_{q+1} \oplus X_q \xrightarrow{[-f_q, \mathbf{1}]} X_q.$$

Note that the homotopy limit is unique only up to non unique isomorphism. We will encounter the following situation : Consider a sequence of complexes

$$\ldots \to K_p \xrightarrow{f_p} K_{p-1} \to \ldots \to K_0$$

over an additive category admitting all countable products such that the f_p are componentwise split epi (or, more generally, for each n and p, the morphism $X^n_{p+k} \to X^n_p$ is split epi for some $k \gg 0$). Then we have a componentwise split short exact sequence of complexes

$$0 \to \varprojlim K_p \to \prod_p K_p \xrightarrow{\Phi} \prod_q K_q \to 0$$

and hence the inverse limit $\varprojlim K_p$ is then isomorphic to $\operatorname{holim} K_p$ in the homotopy category.

Proof of the theorem. a) Note that the bicomplex I is the inverse limit of its quotient complexes $I^{\bullet,q]}$ obtained by killing all rows of index greater than q. Let J_q be the product total complex of $I^{\bullet,q]}$. Then the sequence of the J_q has inverse limit $\widehat{\mathrm{Tot}}\, I$ and its structure maps are split epi in each component. Hence I is isomorphic to the homotopy limit of the sequence of the J_q. Since the class of fibrant complexes is stable under extensions and products, it is stable under homotopy limits. Therefore it is enough to show that the J_q are fibrant. Clearly the J_q are iterated extensions of rows of I (suitably shifted). So it is enough to show that the rows of I are fibrant. But each row of I is homotopy equivalent to a complex with vanishing differential and injective components. Such a complex is the product of its components placed in their respective degrees and is thus fibrant.

b) For $p \in \mathbf{Z}$, define $\tau^{\geq p} K$ to be the quotient complex of K given by

$$\ldots \to 0 \to K^p/B^pK \to K^{p+1} \to K^{p+2} \to \ldots$$

and $\tau^{<p}K$ to be the subcomplex of K given by

$$\ldots \to K^{p-2} \to K^{p-1} \to B^pK \to 0 \to \ldots .$$

Define $\tau^{\geq p} J$ and $\tau^{<p} J$ by applying the respective functor to each row of J. Then the morphism $\tau^{\geq p}K \to \tau^{\geq p}J$ is a Cartan-Eilenberg resolution for each $p \in \mathbf{Z}$. Since $\tau^{\geq p}K$ is left bounded, it follows that the induced morphism $\tau^{\geq p}K \to \widehat{\mathrm{Tot}}\, \tau^{\geq p}J$ is a quasi-isomorphism for each $p \in \mathbf{Z}$. Now fix $n \in \mathbf{Z}$ and consider the diagram

$$\begin{array}{ccc} H^nK & \longrightarrow & H^n\tau^{\geq p}K \\ \downarrow & & \downarrow \\ H^n\widehat{\mathrm{Tot}}\, J & \longrightarrow & H^n\widehat{\mathrm{Tot}}\, \tau^{\geq p}J. \end{array}$$

For $p < n$, the top morphism is invertible. It now suffices to show that for $p \ll 0$, the bottom morphism is invertible. Equivalently, it is enough to show that $H^n\widehat{\mathrm{Tot}}\, \tau^{<p}J$

vanishes for $p \ll 0$. For this let $x \in X$. We have to show that $(H^n \widehat{\mathrm{Tot}}\, \tau^{<p} J)_x$ vanishes. Since taking the stalk is an exact functor, this reduces to showing that the complex $(\widehat{\mathrm{Tot}}\, \tau^{<p} J)_x$ is acyclic in degree n. For this, it is enough to show that $(\widehat{\mathrm{Tot}}\, \tau^{<p} J)(U)$ is acyclic in degree n for each affine neighbourhood of x. Now $\tau^{<p} J$ is a Cartan-Eilenberg resolution of $\tau^{<p} K$. Therefore, if we apply proposition A.3 below to the functor $F = \Gamma(U, ?)$, we see that $(\widehat{\mathrm{Tot}}\, \tau^{<p} J)(U)$ is acyclic in all degrees $n \geq p$. Indeed, we have $(\mathbf{R}^i F)(H^p K) = 0$ for all p and all $i > 0$ by Serre's theorem [9, III, 1.3.1], since $H^p K$ is quasi-coherent.

$\surd$

A.3. Unbounded complexes with uniformly bounded cohomology. Let $\mathcal{A}$ be an abelian category with enough injectives which admits all countable products and let $F : \mathcal{A} \to \mathcal{A}b$ be an additive functor commuting with all countable products.

Let K be a complex over $\mathcal{A}$ and let $K \to J$ a Cartan-Eilenberg resolution. Suppose that $K^p = 0$ for all $p > 0$ and that there is an integer n with

$$(\mathbf{R}^i F)(H^p K) = 0$$

for all $i \geq n$ and all $p \in \mathbf{Z}$.

Lemma. *We have $H^p F \widehat{\mathrm{Tot}}\, J = 0$ for all $p \geq n$.*

Note that this assertion is clear if K is (homologically) left bounded. The point is that it remains true without this hypothesis.

Proof. Define $\tau^{\geq p} K$ and $\tau^{\geq p} J$ as in the proof of proposition A.2. The canonical morphisms $\tau^{\geq p} J \to \tau^{\geq p+1} J$ are split epi in each bidegree and J identifies with the inverse limit of the $\tau^{\geq p} J$. Hence we have $\widehat{\mathrm{Tot}}\, J = \varprojlim \widehat{\mathrm{Tot}}\, \tau^{\geq p} J$ and the morphisms

$$\widehat{\mathrm{Tot}}\, \tau^{\geq p} J \to \widehat{\mathrm{Tot}}\, \tau^{\geq p+1} J$$

are componentwise split epi. Since F commutes with countable products, we therefore have $F(\widehat{\mathrm{Tot}}\, J) = \varprojlim F \widehat{\mathrm{Tot}}\, \tau^{\geq p} J$. By lemma A.4 below, it is therefore enough to show that the groups $H^i F(\widehat{\mathrm{Tot}}\, L_p)$ vanish for all $i \geq n$ and all p where L_p is the kernel of the canonical morphism $\tau^{\geq p} J \to \tau^{\geq p+1} J$. Now L_p is in fact a Cartan-Eilenberg resolution of the kernel of the morphism $\tau^{\geq p} K \to \tau^{\geq p+1} K$, which is isomorphic to the complex

$$\ldots 0 \to K^{p-1}/B^{p-1} K \to Z^p K \to 0 \to \ldots$$

This complex is quasi-isomorphic to $H^p K$ placed in degree p. So $\widehat{\mathrm{Tot}}\, L_p$ is homotopy equivalent to an injective resolution of $H^p K$ shifted by p degrees. Hence

$$H^i F \widehat{\mathrm{Tot}}\, L_p = H^i \mathbf{R} F(H^p K[-p]) = (\mathbf{R}^{i-p} F)(H^p K).$$

By assumption, this vanishes for $i - p \geq n$. $\surd$

A.4. A Mittag-Leffler lemma. Let n be an integer and let

$$\ldots \to K_{p+1} \xrightarrow{\pi_{p+1}} K_p \to \ldots \to K_0 \xrightarrow{\pi_0} K_{-1} = 0\,,\ p \in \mathbf{N}\,,$$

be an inverse system of complexes of abelian groups such that the π_p are surjective in each component and $H^i K'_p = 0$ for all $i \geq n$ and all p, where K'_p is the kernel of π_p.

Lemma. *We have $H^i \varprojlim K_p = 0$ for all $i \geq n$.*

Proof. By induction, we find that $H^i K_p = 0$ for all $i \geq n$. Now we have exact sequences

$$0 \to Z^i K_p \to K_p^i \to Z^{i+1} K_p \to 0 \,,$$

for all $i \geq n-1$. Since $B^i K_p \xrightarrow{\sim} Z^i K_p$, the maps $Z^i K_{p+1} \to Z^i K_p$ are surjective for $i \geq n$. The fact that $H^n K'_{p+1} = 0$ implies that the maps $Z^{n-1} K_{p+1} \to Z^{n-1} K_p$ are surjective as well. By the Mittag-Leffler lemma [9, 0_{III}, 13.1], the sequence

$$0 \to \varprojlim Z^i K_p \to \varprojlim K_p^i \to \varprojlim Z^{i+1} K_p \to 0$$

is still exact for $i \geq n-1$. Since $\varprojlim Z^i K_p \xleftarrow{\sim} Z^i \varprojlim K_p$, this means that $H^i \varprojlim K_p = 0$ for $i \geq n$. $\surd$

Appendix B. A comparison of derived categories

B.1. Boekstedt-Neeman's theorem. Let X be a quasi-compact separated scheme, $\mathcal{D}\,\mathrm{Qcoh}\,X$ the derived category of the category $\mathrm{Qcoh}\,X$ of quasi-coherent sheaves on X, $\mathcal{D}X$ the derived category of all sheaves of $\mathcal{O}_X$-modules on X, and $\mathcal{D}_{qc}X$ its full subcategory whose objects are the complexes with quasi-coherent homology.

As an application of theorem A.2, we give a partially new proof of the following result of Boekstedt-Neeman. We refer to [30, Prop. 1.3] for yet another proof.

Theorem. [4, 5.5] *The canonical functor $\mathcal{D}\,\mathrm{Qcoh}\,X \to \mathcal{D}_{qc}X$ is an equivalence of categories.*

The proof proceeds by induction on the size of an affine cover of X. The crucial step is the case where X is affine. Our proof for this case is new. For completeness, we have included the full induction argument.

Proof. In a first step, suppose that X is affine : $X = \mathrm{Spec}\,A$. We identify $\mathrm{Qcoh}\,X$ with $\mathrm{Mod}\,A$ and then have to show that the sheafification functor $F : \mathcal{D}\,\mathrm{Mod}\,A \to \mathcal{D}X$ induces an equivalence $\mathcal{D}\,\mathrm{Mod}\,A \to \mathcal{D}_{qc}X$. Clearly, the image of A (viewed as a complex of A-modules concentrated in degree 0) is $\mathcal{O}_X$. By the lemma below, it suffices therefore to show that

a) We have $A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}X}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathrm{Hom}_{\mathcal{D}X}(\mathcal{O}_X, \mathcal{O}_X[n]) = 0$ for each $n \neq 0$,

b) The object $\mathcal{O}_X$ is compact in $\mathcal{D}_{qc}X$ i.e. the associated functor

$$\mathrm{Hom}_{\mathcal{D}_{qc}X}(\mathcal{O}_X, ?)$$

commutes with infinite direct sums.

c) An object $K \in \mathcal{D}_{qc}X$ vanishes if $\mathrm{Hom}_{\mathcal{D}X}(\mathcal{O}_X, K[n])$ vanishes for all $n \in \mathbf{Z}$.

The three assertions a), b), and c) all follow easily from the fact that we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}_{qc}X}(\mathcal{O}_X, ?) \xrightarrow{\sim} \Gamma(X, H^0(?)) \,,$$

which we will now prove : Indeed, let $K \in \mathcal{D}_{qc}X$. By definition, we have

$$\mathrm{Hom}_{\mathcal{D}X}(\mathcal{O}_X, K) = H^0 \mathbf{R}\Gamma(X, K).$$

Now we have morphisms

$$H^0 \mathbf{R}\Gamma(X, K) \xleftarrow{\alpha} H^0 \mathbf{R}\Gamma(X, \tau_{\leq 0} K) \xrightarrow{\beta} H^0 \mathbf{R}\Gamma(X, H^0 K) = \Gamma(X, H^0 K).$$

The morphism α is invertible because $\mathbf{R}\Gamma(X, ?)$ is a right derived functor. The morphism β is invertible by theorem A.2 b), lemma A.3, and Serre's theorem [9, III, 1.3.1].

Now suppose that X is the union of n open affine sets $U_1, \dots, U_n$. By induction on n and the affine case, we may assume that the claim is proved for $U = U_1$ and $V = \bigcup_{i=2\dots n} U_i$. Let $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusions. Let $Y = X \setminus U$ and let $i : Y \to X$ be the inclusion. For any object $K \in \mathcal{D}_{qc}X$, we have a triangle

$$\mathbf{R}\Gamma_Y K \to K \to j_{1*}j_1^*K \to \mathbf{R}\Gamma_Y K[1].$$

Here the second morphism is the adjunction morphism and $\mathbf{R}\Gamma_Y K$ is defined (up to unique isomorphism) by the triangle. The object j_1^*K is a complex of sheaves on U and $H^n j_1^* K = j_1^* H^n K$ is quasi-coherent. So j_1^*K is in the faithful image of $\mathcal{D}\,\mathrm{Qcoh}\,U$. Because X is separated, j_{1*} preserves quasi-coherence (cf. [19, 3.9.2]). So the triangle lies in $\mathcal{D}_{qc}X$. The subset $Y \subset X$ is a closed subset of V and $i = j_1\, i_2$, where i_2 is the inclusion of Y into V. This implies that $\mathbf{R}\Gamma_Y K = j_{2*}(\mathbf{R}\Gamma_{Y\subset V} K)$. The above triangle thus shows that $\mathcal{D}_{qc}X$ is generated by the $j_{1*}K'$ and the $j_{2*}K''$, where K' belongs to $\mathcal{D}\,\mathrm{Qcoh}\,U$ and K'' to $\mathcal{D}\,\mathrm{Qcoh}\,V$. It remains to be checked that morphisms between $j_{1*}K'$ and $j_{2*}K''$ in $\mathcal{D}\,\mathrm{Mod}\,\mathcal{O}_X$ are in bijection with those in $\mathcal{D}\,\mathrm{Qcoh}\,X$. Indeed, we have

$$\mathrm{Hom}_{\mathcal{D}X}(j_{1*}K', j_{2*}K'') = \mathrm{Hom}_{\mathcal{D}V}(j_2^* j_{1*}K', K'').$$

By the induction hypothesis, the latter group identifies with

$$\mathrm{Hom}_{\mathcal{D}\,\mathrm{Qcoh}\,V}(j_2^* j_{1*}K', K'') = \mathrm{Hom}_{\mathcal{D}\,\mathrm{Qcoh}\,X}(j_{1*}K', j_{2*}K'').$$

The same argument applies to morphisms from $j_{2*}K''$ to $j_{1*}K'$. This ends the proof. $\surd$

B.2. Derived categories of modules. Let A be a ring and $\mathcal{T}$ a triangulated category admitting all (infinite) direct sums. Suppose that $F : \mathcal{D}\,\mathrm{Mod}\,A \to \mathcal{T}$ is a triangle functor commuting with all direct sums. For the convenience of the reader, we include a proof of the following more and more well-known

Lemma. *The functor F is an equivalence if and only if*

a) *We have $A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(FA, FA)$ and $\mathrm{Hom}_{\mathcal{T}}(FA, FA[n]) = 0$ for all $n \neq 0$.*
b) *The object FA is compact in $\mathcal{T}$, i.e. $\mathrm{Hom}_{\mathcal{T}}(FA, ?)$ commutes with infinite direct sums.*
c) *An object X of $\mathcal{T}$ vanishes iff $\mathrm{Hom}_{\mathcal{T}}(FA, X[n]) = 0$ for all $n \in \mathbf{Z}$.*

Proof. Let $\mathcal{S} \subset \mathcal{T}$ be the smallest triangulated subcategory of $\mathcal{T}$ containing FA and stable under forming infinite direct sums. Then, since FA is compact, the inclusion $\mathcal{S} \to \mathcal{T}$ admits a right adjoint R by Brown's representability theorem [6] (cf. also [16, 5.2], [26], [8]). Now if $X \in \mathcal{T}$ and $RX \to X \to X' \to RX[1]$ is a triangle over the adjunction morphism, then $\mathrm{Hom}_{\mathcal{T}}(FA, X'[n])$ vanishes for all $n \in \mathbf{Z}$ by the long exact sequence associated with the triangle. So X' vanishes by assumption c) and $\mathcal{S}$ coincides with $\mathcal{T}$. So FA is a compact generator for $\mathcal{T}$. Now the claim follows from [16, 4.2]. $\surd$

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Bernhard Keller
UFR de Mathématiques
Université Paris 7
Institut Mathématique de Jussieu
UMR 9994 du CNRS
Case 7012
2, place Jussieu
75251 Paris Cedex 05
France
keller@math.jussieu.fr
www.math.jussieu.fr/ ˜ keller