On 14-dimensional quadratic forms in I^3, 8-dimensional forms in I^2, and the common value property

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Abstract. Let F be a field of characteristic $\neq 2$. We define certain properties $D(n)$, $n \in \{2,4,8,14\}$, of F as follows: F has property $D(14)$ if each quadratic form $\varphi \in I^3F$ of dimension 14 is similar to the difference of the pure parts of two 3-fold Pfister forms; F has property $D(8)$ if each form $\varphi \in I^2F$ of dimension 8 whose Clifford invariant can be represented by a biquaternion algebra is isometric to the orthogonal sum of two forms similar to 2-fold Pfister forms; F has property $D(4)$ if any two 4-dimensional forms over F of the same determinant which become isometric over some quadratic extension always have (up to similarity) a common binary subform; F has property $D(2)$ if for any two binary forms over F and for any quadratic extension E/F we have that if the two binary forms represent over E a common nonzero element, then they represent over E a common nonzero element in F. Property $D(2)$ has been studied earlier by Leep, Shapiro, Wadsworth and the second author. In particular, fields where $D(2)$ does not hold have been known to exist.

In this article, we investigate how these properties $D(n)$ relate to each other and we show how one can construct fields which fail to have property $D(n)$, $n > 2$, by starting with a field which fails to have property $D(2)$ and then passing to transcendental field extensions. Particular emphasis is devoted to the situation where K is a field with a discrete valuation with residue field k of characteristic $\neq 2$. Here, we study how the properties $D(n)$ behave when one passes from K to k or vice versa. We conclude with some applications and an explicit and detailed example involving rational function fields of transcendence degree at most four over the rationals.

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1 Introduction

After Pfister [P] proved his structure results on quadratic forms of even dimension ≤ 12 and of trivial signed discriminant and Clifford invariant (cf. Theorem 2.1(i)–(iv) in this paper) over a field F of characteristic $\neq 2$, there have been various attempts to extend and generalize his results. Merkurjev's theorem [Me 1] implies that even-dimensional forms of trivial signed discriminant and Clifford invariant are exactly the forms whose Witt classes lie in I^3F, the third power of the fundamental ideal IF of even-dimensional forms in the Witt ring WF of F. But there have been no further results concerning the explicit characterization of such forms of a given dimension ≥ 14 until Rost [R] gave a description of 14-dimensional forms with trivial invariants as being transfers of scalar multiples of pure parts of 3-fold Pfister forms defined over a quadratic extension of the base field (cf. Theorem 2.1(v) in this paper). It remained open whether such 14-dimensional forms can always be written up to similarity as the difference of the pure parts of two 3-fold Pfister forms over F. It turns out that this question is related to the question whether 8-dimensional forms in I^2F whose Clifford invariant is given by the class of a biquaternion algebra are always isometric to a sum of scalar multiples of two 2-fold Pfister forms.

Izhboldin suggested a method to construct counterexamples to the second question which then leads to counterexamples to the first one (after a ground field extension). One crucial step to make his approach work depended on the construction of examples of two quaternion algebras over a suitable field F such that there exists a quadratic extension E/F over which these two quaternion algebras have a common slot, but no such common slot over E can be chosen to be an element in F. In this paper, we reduce this existence problem to the existence of quadratic field extensions which do not have a certain property $CV(2,2)$ defined by Leep [Le] (see also [SL]). This property has been studied in [STW], where it is shown that generally quadratic extensions do not have this property $CV(2,2)$. As a consequence, both questions above concerning 14-dimensional forms in I^3F and 8-dimensional forms in I^2F have negative answers in general.

It should be noted that the examples in [STW] of quadratic extensions not having $CV(2,2)$ are all in characteristic 0. Independently, Izhboldin and Karpenko [IK 2] found a method to construct counterexamples to the common slot problem above which is of a very general nature and works in all characteristics, thus also leading to counterexamples to the above questions on quadratic forms and incidentally also providing counterexamples to $CV(2,2)$ for quadratic extensions. Needless to say that they employ machinery quite different from what is used in [STW].

In the next section, we will recall the known results on forms in I^3F and prove certain others which are crucial in the understanding of 14-dimensional forms in I^3F. In section 3 we will then investigate the relations between the questions raised above. We will state these results in terms of certain properties $D(n)$ of the ground field F which describe the behaviour of certain forms of dimension $n \in \{2,4,8,14\}$ over F. In section 4, we consider the situation of a discrete valuation ring R with residue field k of characteristic not 2 and quotient field K. The purpose is to determine how the properties $D(n)$ for k and K relate to each other. These results can then be used to show that starting with a field F which does not have property $D(2)$, one obtains fields which do not have property $D(n)$, $n \in \{4,8,14\}$, by passing to rational field extensions. In section 5, we exhibit the properties $D(n)$ for fields with finite Hasse number and for their power series extensions. Finally, in section 6, we derive some further consequences and exhibit in all detail an example, starting over $\mathbf{Q}(x)$, which will then lead (after going up to rational field extensions over $\mathbf{Q}(x)$) to the explicit construction of counterexamples to all the problems touched upon in this article.

The standard references for those results in the theory of quadratic forms and division algebras which we will need in this paper are Lam's book [L 1] and Scharlau's book [S]. Most of the notations we will use are also borrowed from these two sources.

Fields are always assumed to be of characteristic $\neq 2$, and we only consider nondegenerate finite dimensional quadratic forms. Let φ and ψ be two quadratic forms over a field F. We write $\varphi \simeq \psi$ (resp. $\varphi \sim \psi$) to denote that the two forms are isometric (resp. equivalent in the Witt ring WF). The forms φ and ψ are said to be similar if there exists some $a \in F^\times$ such that $\varphi \simeq a\psi$. We call ψ a subform of φ, and write $\psi \subset \varphi$, if ψ is isometric to an orthogonal summand of φ. The hyperbolic plane $\langle 1, -1\rangle$ is denoted by $\mathbf{H}$. We write $d_\pm(\varphi)$ for the signed discriminant of a form φ, and $c(\varphi)$ for its Clifford invariant. For a field extension E/F, we write $D_E(\varphi)$ to denote the set of elements in $E^\times$ represented by φ_E, the form obtained from φ by scalar extension to E.

We use the convention $\langle\langle a_1, \cdots, a_n\rangle\rangle$ to denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ over F. By P_nF (resp. GP_nF) we denote the set of all forms over F which are isometric (resp. similar) to n-fold Pfister forms.

Forms of dimension 6 with trivial signed discriminant are called *Albert forms*, in reference to the following theorem of Albert:

> The biquaternion algebra $(a_1, a_2)_F \otimes (a_3, a_4)_F$ is a division algebra if and only if the quadratic form $\langle -a_1, -a_2, a_1a_2, a_3, a_4, -a_3a_4\rangle$ is anisotropic.

For a proof, see [A, Th. 3] or [P, p. 123].

2 Pfister's and Rost's results and some consequences

We begin by stating the results of Pfister and Rost on even-dimensional forms with trivial signed discriminant and Clifford invariant. Pfister proved the results on forms of dimension ≤ 12 in [P, Satz 14, Zusatz] (our statement of the 12-dimensional case is a little different but can easily be deduced from Pfister's original proof). The 14-dimensional case is due to Rost [R].

Theorem 2.1 *Let φ be an even-dimensional form over F with $d_\pm\varphi = 1$ and $c(\varphi) = 1$.*

(i) *If $\dim\varphi < 8$ then φ is hyperbolic.*

(ii) *If $\dim\varphi = 8$ then $\varphi \in GP_3F$.*

(iii) *If $\dim\varphi = 10$ then $\varphi \simeq \pi \perp \mathbf{H}$ with $\pi \in GP_3F$.*

(iv) *If $\dim\varphi = 12$ then $\varphi \simeq \alpha \otimes \beta$ for some Albert form α and some binary form β or, equivalently, there exist $r, s, t, u, v, w \in F^\times$ such that $\varphi \sim r(\langle\langle s, t, u\rangle\rangle - \langle\langle s, v, w\rangle\rangle)$ in WF.*

(v) *If $\dim\varphi = 14$ and φ is anisotropic, then there exists a quadratic extension $L = F(\sqrt{d})$ and some $\pi \in P_3L$ such that φ is the trace of $\sqrt{d}\pi'$, where π'*

denotes the pure part of π. *(Here, "trace" means the transfer defined via the trace map.)*

Part (i) of the following corollary can also easily be deduced from the classifications given in [H 2, Th. 4.1, Th. 5.1]. We will give a self-contained proof. Part (ii) is an observation due to Karpenko [K, Cor. 1.3].

Corollary 2.2 *Let* φ *be a form over* F.

(i) *If* $\dim\varphi = 10$ *and there exists* $\sigma \in P_2F$ *such that* $\varphi \equiv \sigma \pmod{I^3F}$, *then there exist* $r \in F^\times$ *and* $\pi \in GP_3F$ *such that* $\varphi \sim \pi + r\sigma$.

(ii) *If* $\dim\varphi = 14$ *and* $\varphi \in I^3F$ *then there exists an Albert form* α *such that* $\alpha \subset \varphi$.

Proof. (i) Let $s \in F^\times$ such that $\varphi \simeq \langle s\rangle \perp \varphi'$, and let σ' be the pure part of σ. Let $\psi := (\varphi' \perp -s\sigma')_{\mathrm{an}}$. Note that $\dim\psi \le 12$. We have

$$\psi \equiv \varphi \perp -s\sigma \equiv \sigma \perp -s\sigma \equiv 0 \pmod{I^3F}.$$

If $\dim\psi \le 10$ then by Th. 2.1 there exists $\pi \in GP_3F$ (possibly hyperbolic) such that $\psi \sim \pi$ in WF. Thus, $\varphi \sim \psi + s\sigma \sim \pi + s\sigma$ in WF and we put $r = s$.

So suppose that $\dim\psi = 12$. Then, by Th. 2.1(iv), there exists a quadratic extension $E = F(\sqrt{d})$ such that ψ_E is hyperbolic, i.e. $\varphi'_E \sim s\sigma'_E$, and comparing dimensions yields that $i_W(\varphi'_E) \ge 3$. In particular, there exist $x, y, z \in F^\times$ such that $\varphi' \simeq \langle 1, -d\rangle \otimes \langle x, y, z\rangle \perp \varphi''$ with $\dim\varphi'' = 3$ (cf. [S, Ch. 2, Lemma 5.1]). Consider $\pi := \langle 1, -d\rangle \otimes \langle x, y, z, xyz\rangle \in GP_3F$ and $\alpha := -xyz\langle 1, -d\rangle \perp \varphi'' \perp \langle s\rangle$. Then $\varphi - \pi \sim \alpha$ in WF and thus $\alpha \equiv \sigma \pmod{I^3F}$. Note that α is an Albert form with $c(\alpha) = c(\sigma)$. It follows from Jacobson's theorem (see, e.g., [MaS]) that there exists $r \in F^\times$ such that $\alpha \sim r\sigma$ and therefore $\varphi \sim \pi + r\sigma$ in WF.

(ii) Any isotropic form of dimension ≥ 7 contains some Albert form as a subform as can readily be verified. Thus, if φ is isotropic, it contains some Albert form (which also follows from Th. 2.1(iv)). So assume that φ is anisotropic. By Th. 2.1(v), there exists a quadratic extension $E = F(\sqrt{d})$ and some form $\langle\langle u, v, w\rangle\rangle \in P_3E$ such that $\varphi \simeq \mathrm{tr}(\sqrt{d}\langle\langle u, v, w\rangle\rangle')$. Let $\alpha := \mathrm{tr}(\sqrt{d}\langle -u, -v, uv\rangle)$. Clearly, $\langle -u, -v, uv\rangle \subset \langle\langle u, v, w\rangle\rangle'$ and thus $\alpha \subset \varphi$. Furthermore, $\dim\alpha = 6$, and we have by [S, Ch. 2, Th. 5.12] that, in $F^\times/F^{\times 2}$, $\det\alpha = d^3 N_{E/F}(\det(\sqrt{d}\langle -u, -v, uv\rangle)) = d^3 N_{E/F}(\sqrt{d}) = -d^4 = -1$. Therefore $\alpha \in I^2F$. Hence, α is an Albert subform of φ. □

Proposition 2.3 *Let* φ *be a form over* F *with* $\dim\varphi = 14$ *and* $\varphi \in I^3F$. *Then there exist forms* $\pi_i \in GP_3F$, $i = 1, 2, 3$, *such that* $\varphi \sim \pi_1 + \pi_2 + \pi_3$ *in* WF. *Furthermore, the following statements are equivalent:*

(i) *There exist* $\tau_1, \tau_2 \in P_3F$ *and* $s_1, s_2 \in F^\times$ *such that* $\varphi \sim s_1\tau_1 + s_2\tau_2$ *in* WF.

(ii) *There exist* $\tau_1, \tau_2 \in P_3F$ *and* $s \in F^\times$ *such that* $\varphi \simeq s(\tau'_1 \perp -\tau'_2)$, *where* τ'_1 *and* τ'_2 *are the pure parts of* τ_1 *resp.* τ_2.

(iii) *There exists* $\sigma \in GP_2F$ *such that* $\sigma \subset \varphi$.

Proof. Let φ be a 14-dimensional form if I^3F. By Cor. 2.2(ii), we can write $\varphi \simeq \alpha \perp \psi$ with an Albert form α and some $\psi \in I^2F$, $\dim\psi = 8$. After scaling, we may assume that $\alpha \sim \sigma_1 - \sigma_2$ in WF with $\sigma_1, \sigma_2 \in P_2F$. Let $x \in F^\times$ such that $\psi \simeq \langle -x \rangle \perp \psi'$ and consider the 10-dimensional form $\psi' \perp x\sigma_1'$. We then have

$$\psi' \perp x\sigma_1' \equiv \psi + x\sigma_1 \equiv \varphi - \alpha + x\sigma_1 \equiv \sigma_2 - \sigma_1 + x\sigma_1 \equiv \sigma_2 \pmod{I^3F}.$$

By Cor. 2.2(i), there exists $y \in F^\times$ and $\pi_3 \in GP_3F$ such that $\psi' \perp x\sigma_1' \sim \psi + x\sigma_1 \sim \pi_3 + y\sigma_2$ in WF. Let now $\pi_1 := \langle\langle x \rangle\rangle \otimes \sigma_1 \in P_3F$ and $\pi_2 := \langle\langle y \rangle\rangle \otimes \sigma_2 \in P_3F$. One checks readily that we have $\varphi \sim \pi_1 - \pi_2 + \pi_3$ in WF.

As for the equivalences, (ii) trivially implies (i), and the converse follows readily after comparing dimensions of φ and $s_1\tau_1 \perp s_2\tau_2$, implying that the latter form is isotropic, and then using the multiplicativity of the Pfister forms τ_1, τ_2.

(ii) implies (iii) since τ_1' as well as τ_2' clearly contain subforms in GP_2F.

Finally, let $\varphi \in I^3F$ with $\dim \varphi = 14$ and suppose there exists $\sigma \in GP_2F$ with $\varphi \simeq \sigma \perp \psi$. Then $\dim \psi = 10$ and $\psi \equiv -\sigma \pmod{I^3F}$. By Cor. 2.2, there exist $\pi_1 \in GP_3F$ and $x \in F^\times$ such that $\psi \sim \pi_1 - x\sigma$ in WF. Let $\pi_2 := \langle\langle x \rangle\rangle \otimes \sigma \in GP_3F$. We then have $\varphi \sim \psi + \sigma = \pi_1 + \pi_2$ in WF, which implies (i). ◻

The fact that each 14-dimensional form in I^3F is Witt equivalent to the sum of three forms in GP_3F has been noticed independently by Izhboldin. A somewhat different proof of the equivalence of the three statements above is given in [IK 2, Prop. 17.2].

Let us now turn our attention to 8-dimensional I^2-forms over a field F. It is well-known that if φ is such a form, then the Clifford invariant $c(\varphi)$ can be represented as the class of $Q_1 \otimes Q_2 \otimes Q_3$ for suitable quaternion algebras Q_i. In particular, its index is 1, 2, 4, or 8. Which of these cases occurs can be determined in terms of the splitting behaviour of φ over (multi)quadratic extensions of F. To this end, we will need results on the Scharlau transfer of certain quadratic forms.

LEMMA 2.4 (i) (See also [S, Ch. 2, Lemma 14.8].) *Let $E = F(\sqrt{d})$ and $\tau \in GP_2E$. Then there exist $a_1, a_2 \in F^\times$, $b_1, b_2, c \in E^\times$, such that in WE, one has $c\tau \sim \langle\langle a_1, b_1 \rangle\rangle - \langle\langle a_2, b_2 \rangle\rangle$.*

(ii) *Let $\varphi \in I^2F$ be anisotropic,* $\dim \varphi = 8$, *and suppose that* $\operatorname{ind} c(\varphi) = 4$. *Then there exists a quadratic extension $E = F(\sqrt{d})$ and some $\tau \in GP_2E$ such that $\varphi \simeq \operatorname{tr}(\tau)$, where "tr" denotes the transfer defined via the trace map (cf. also Theorem 2.1(iv)).*

Proof. (i) After scaling, we may assume that $\tau \simeq \langle\langle x_1, x_2 \rangle\rangle$ with x_1, $x_2 \in E^\times$. If x_1 or x_2 lies in F, then obviously we are done. So let us assume that $x_1, x_2 \notin F$. Since E is 2-dimensional over F, the elements 1, x_1, x_2 are not linearly independent over F, hence we may find a_1, $a_2 \in F^\times$ such that $a_1x_1 + a_2x_2 = 0$ or 1. The form $\langle\langle a_1x_1, a_2x_2 \rangle\rangle$ is then hyperbolic. Multiplying by $\langle a_1, -a_1a_2x_2 \rangle$ both sides of

$$\langle 1, -a_1x_1 \rangle \sim \langle a_1, -a_1x_1 \rangle + \langle 1, -a_1 \rangle$$

we get

$$\langle\langle x_1, a_2x_2 \rangle\rangle \simeq \langle\langle a_1, a_2x_2 \rangle\rangle.$$

Substituting $\langle 1, -a_2x_2 \rangle \sim \langle a_2, -a_2x_2 \rangle + \langle 1, -a_2 \rangle$ in the left side, we obtain

$$a_2\langle\langle x_1, x_2 \rangle\rangle \sim \langle\langle a_1, a_2x_2 \rangle\rangle - \langle\langle a_2, x_1 \rangle\rangle.$$

We may thus choose $b_1 = a_2x_2$ and $b_2 = x_1$.

Part (ii) is due to Izhboldin and Karpenko [IK 2, Th. 16.10], and its proof (which we will omit) is based on Rost's result on 14-dimensional I^3-forms. □

Proposition 2.5 *Let φ be an 8-dimensional form in I^2F. Then* $\operatorname{ind} c(\varphi) \in \{1,2,4,8\}$ *and there exists a multiquadratic extension L/F of degree 1, 2, 4 or 8 such that $\varphi_L \sim 0$. Moreover, for $i = 0$, 1, 2, 3, we have* $\operatorname{ind} c(\varphi) \leq 2^i$ *if and only if there exists a multiquadratic extension L/F of degree $\leq 2^i$ such that $\varphi_L \in GP_3L$. For $i = 1$, 2, 3, this condition is also equivalent to the existence of a multiquadratic extension L'/F of degree $\leq 2^i$ such that $\varphi_{L'} \sim 0$.*

Proof. Write $\varphi \simeq \beta_1 \perp \beta_2 \perp \beta_3 \perp \beta_4$, where the β_i are binary forms with $d_\pm\beta_i = d_i \in F^\times/F^{\times 2}$. Then $d_4 = d_1d_2d_3$ as $\varphi \in I^2F$, and for $L = F(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$, we obviously have $(\beta_i)_L \sim 0$ and thus $\varphi_L \sim 0$. Hence, we also have that $c(\varphi_L) = 0$ in $\mathrm{Br}L$. Thus, $c(\varphi)_L$ is split and it follows readily that $\operatorname{ind} c(\varphi) \in \{1,2,4,8\}$. (Of course, this also follows from the fact mentioned above that $c(\varphi)$ can be represented as the class of some triquaternion algebra.)

As for the remaining statements, the case $i = 0$ follows from Theorem 2.1(ii).

If $\varphi_L \in GP_3L$ for some quadratic extension L/F, then $c(\varphi_L) = 0$ in $\mathrm{Br}L$. We then have $\operatorname{ind} c(\varphi) \leq 2$, hence $c(\varphi) = [Q]$ for some quaternion algebra Q over F. It is well-known that in this case φ is divisible by some binary form β (see for example [H 2, Th. 4.1]). With $d = d_\pm\beta$ and $L' = F(\sqrt{d})$, we get $\varphi_{L'} \sim 0$. Finally, if $\varphi_{L'} \sim 0$ for some quadratic extension L'/F, then $\varphi_{L'} \in GP_3L'$, as it is isometric to the hyperbolic 3-fold Pfister form over L'.

Similarly as above, the existence of a biquadratic extension L'/F such that $\varphi_{L'} \sim 0$ trivially implies the existence of a biquadratic extension L/F with $\varphi_L \in GP_3L$, which in turn implies that $\operatorname{ind} c(\varphi) \leq 4$. It remains to show that $\operatorname{ind} c(\varphi) \leq 4$ implies the existence of L' as above. We may assume by (ii) that $\operatorname{ind} c(\varphi) = 4$. By Lemma 2.4(ii), there exists a quadratic extension $E = F(\sqrt{d})$ and a form $\tau \in GP_2E$ such that $\varphi \simeq \mathrm{tr}(\tau)$. By Lemma 2.4(i), there exist $a_1, a_2 \in F^\times$ and binary forms β_1, β_2 over E such that $\tau \sim \langle\langle a_1\rangle\rangle \otimes \beta_1 + \langle\langle a_2\rangle\rangle \otimes \beta_2$ in WE. By [S, Ch. 2, Th. 5.6], we get

$$\varphi \sim \mathrm{tr}(\tau) \sim \langle\langle a_1\rangle\rangle \otimes \mathrm{tr}(\beta_1) + \langle\langle a_2\rangle\rangle \otimes \mathrm{tr}(\beta_2) \ .$$

Let $L' = F(\sqrt{a_1}, \sqrt{a_2})$. Then $\langle\langle a_i\rangle\rangle_{L'} \sim 0$ and hence $\varphi_{L'} \sim 0$. □

Remark 2.6 Using Rost's description of 14-dimensional I^3-forms as certain transfers, one can prove, similarly as in part (iii) of the previous proposition, that every 14-dimensional I^3-form becomes hyperbolic over some multiquadratic extension of degree ≤ 4. Another way of proving this is as follows. Let $\varphi \in I^3F$, $\dim\varphi = 14$. By Cor. 2.2, we can write $\varphi \simeq \psi \perp \alpha$ for some Albert form α. Let $a \in F^\times$ such that $\psi \perp a\alpha$ is isotropic. Note that the anisotropic part of $\psi \perp a\alpha$ has dimension ≤ 12, and it is again in I^3F. By Theorem 2.1, there exists $b \in F^\times$ such that this anisotropic part is divisible by $\langle\langle b\rangle\rangle$. Thus, for $E = F(\sqrt{a}, \sqrt{b})$ we get

$$\varphi_E \sim (\psi \perp \alpha)_E \sim (\psi \perp a\alpha)_E \sim 0 \ .$$

3 Forms of dimension 14 in I^3, of dimension 8 in I^2, and the property $CV(2,2)$

Let E/F be a field extension. Then E/F is said to have the common value property for pairs of forms of dimension n and m, property $CV(n,m)$ for short, if for any pair of forms φ and ψ over F with $\dim\varphi = n$ and $\dim\psi = m$ we have that if φ_E and ψ_E represent a common element over E, then they already represent a common element of $F^\times$ over E, i.e., if $D_E(\varphi)\cap D_E(\psi) \neq \emptyset$, then $D_E(\varphi)\cap D_E(\psi)\cap F^\times \neq \emptyset$. This definition is originally due to Leep [Le]. Trivially, the property $CV(1,n)$ holds for all n and all extensions E/F. We are interested in the case where E/F is a quadratic extension. The following was shown in [STW, Lemma 2.7].

Lemma 3.1 *Let E/F be a quadratic extension. Then E/F has property $CV(2,2)$ iff E/F has property $CV(n,m)$ for all pairs of positive integers n, m.*

We now define certain properties of a field F pertaining to quadratic forms and quaternion algebras and we will investigate the relationships among them.

Property $D(14)$: Every 14-dimensional form in I^3F is similar to the difference of two forms in P_3F or, equivalently by Prop. 2.3, contains a subform in GP_2F.

Property $D(8)$: Every 8-dimensional form $\varphi\in I^2F$ whose Clifford invariant $c(\varphi)$ can be represented by a biquaternion algebra contains a subform in GP_2F.

Property $D(4)$: Suppose φ_1 and φ_2 are 4-dimensional forms over F with $d_\pm\varphi_1 = d_\pm\varphi_2$. If there is a quadratic extension E/F such that $(\varphi_1)_E\simeq(\varphi_2)_E$, then there is a binary form β over F which is similar to a subform of both φ_1 and φ_2.

Property CS: Suppose Q_1 and Q_2 are quaternion algebras over F and E/F is a quadratic extension. If $(Q_1)_E$ and $(Q_2)_E$ have a common slot over E, then such a slot can be chosen in F, i.e., if there exist $u,v,w\in E^\times$ such that $(Q_1)_E\simeq (u,v)_E$ and $(Q_2)_E\simeq(u,w)_E$, then there exists $u'\in F^\times$, $v',w'\in E^\times$ such that $(Q_1)_E\simeq(u',v')_E$ and $(Q_2)_E\simeq(u',w')_E$.

Property $D(2)$: Every quadratic extension E/F has property $CV(2,2)$.

(The notation $D(n)$ alludes to the fact that the thus-labelled property describes a certain behaviour of certain forms of dimension n over the field in question.)

Remark 3.2 (i) As for property $D(8)$, if there exist a biquaternion algebra B over F and an 8-dimensional form $\varphi\in I^2F$ such that $c(\varphi)=[B]$ in $\mathrm{Br}F$ and such that φ does not contain a subform in GP_2, then B is necessarily a division algebra and φ is anisotropic.

For if φ were isotropic, one could readily find 4-dimensional subforms of determinant 1 as φ would contain the universal form $\mathbf{H}$ as a subform. Furthermore, if B were not a division algebra, then there would exist a quaternion algebra Q such that $c(\varphi)=[B]=[Q]$. By Prop. 2.5, φ would become hyperbolic over some quadratic extension $F(\sqrt{d})$ and would therefore be divisible by $\langle\langle d\rangle\rangle$. The existence of a subform in GP_2F would follow immediately.

(ii) As for property $D(4)$, if there exist forms φ_1 and φ_2 over F with $\dim\varphi_1 = \dim\varphi_2 = 4$ and $d_\pm\varphi_1 = d_\pm\varphi_2 = d$ and a quadratic extension E/F such that $(\varphi_1)_E\simeq$

$(\varphi_2)_E$, but there does not exist a binary form β over F such that β is similar to a subform of both φ_1 and φ_2, then the quadratic extension cannot be given by $F(\sqrt{d})$.

In fact, Wadsworth [W] showed that if two 4-dimensional forms over F of the same determinant d become similar over the extension $F(\sqrt{d})$, then they are already similar over F. In view of this result, it is even more remarkable that there are fields where property $D(4)$ fails.

Furthermore, if the two forms φ_1 and φ_2 are as above, then necessarily $d \notin F^{\times 2}$, i.e. $\varphi_1, \varphi_2 \notin GP_2F$. In fact, suppose that $\varphi_1 \simeq r\langle\langle a,b\rangle\rangle$ and $\varphi_2 \simeq s\langle\langle u,v\rangle\rangle$, and let $\alpha \simeq \langle -a,-b,ab,u,v,-uv\rangle$. If there exists a quadratic extension $E = F(\sqrt{e})/F$, $e \in F^\times \backslash F^{\times 2}$, such that $(\varphi_1)_E \simeq (\varphi_2)_E$, then it follows readily that $\langle\langle a,b\rangle\rangle_E \simeq \langle\langle u,v\rangle\rangle_E$ and hence that α_E is hyperbolic. Suppose that α is anisotropic over F. Then there exists a 3-dimensional form γ over F such that $\alpha \simeq \langle\langle e\rangle\rangle \otimes \gamma$ and therefore $d_\pm \alpha = e$, a contradiction. Hence, α is isotropic and there exists $x \in F^\times$ such that $-x$ is represented by $\langle -a,-b,ab\rangle$ and $\langle -u,-v,uv\rangle$. In particular, there exist $y,z \in F^\times$ such that $\langle\langle a,b\rangle\rangle \simeq \langle\langle x,y\rangle\rangle$ and $\langle\langle u,v\rangle\rangle \simeq \langle\langle x,z\rangle\rangle$. It follows that $\beta := \langle\langle x\rangle\rangle$ is similar to a subform of both φ_1 and φ_2.

The following observation provides a useful criterion as for when an 8-dimensional I^2-form whose Clifford invariant can be represented by a biquaternion algebra contains a subform in GP_2F. We will use it in various proofs involving property $D(8)$ (see also [IK 2, Prop. 16.4]).

Lemma 3.3 *Let φ be an 8-dimensional form in I^2F such that $c(\varphi) = [A]$ for some biquaternion algebra A over F with associated Albert form α. The following are equivalent:*

(i) *φ contains a subform in GP_2F.*

(ii) *There exists a quadratic extension $L = F(\sqrt{d})$ such that φ_L is isotropic and A_L is not a division algebra.*

(iii) *There exists a quadratic extension $L = F(\sqrt{d})$ such that φ_L and α_L are both isotropic.*

(iv) *There exists a binary form over F which is similar to a subform of both φ and α.*

Proof. The equivalence of (ii) and (iii) is clear by Albert's theorem, and the equivalence of (iii) and (iv) is also rather obvious. In view of Remark 3.2(i), we may assume that φ is anisotropic and that A is a division algebra, i.e. α is anisotropic. It remains to show (i) $\Longleftrightarrow$ (ii).

Suppose that (i) holds. Then $\varphi \simeq \psi_1 \perp \psi_2$ with $\psi_i \in GP_2F$. Let $L = F(\sqrt{d})$ be any quadratic extension such that ψ_2 becomes isotropic and hence hyperbolic over L. Then we have $c(\varphi_L) = c((\psi_1)_L) = [A_L]$. Since $\psi_1 \in GP_2F$, there exists a quaternion algebra Q over F such that $c(\psi_1) = [Q]$. Hence, $[Q_L] = [A_L]$, which implies that A_L cannot be a division algebra.

Conversely, suppose that there exists a quadratic extension $L = F(\sqrt{d})$ with φ_L isotropic and A_L not division. Since φ_L is isotropic and in I^2L, there exists a 6-dimensional form $\psi \in I^2L$ with $\varphi_L \sim \psi$, in particular, $c(\psi) = c(\varphi_L) = [A_L]$. By Albert's theorem, ψ must be isotropic, hence the Witt index of φ over L is ≥ 2. Thus, there exists a binary form β over F such that $\langle\langle d\rangle\rangle \otimes \beta \subset \varphi$ (cf. [S, Ch. 2, Lemma 5.1]). (i) now follows as $\langle\langle d\rangle\rangle \otimes \beta \in GP_2F$. □

THEOREM 3.4

$$D(2) \Rightarrow CS \iff D(4) \qquad \textit{and} \qquad D(8) \Rightarrow D(14).$$

Proof. $D(2) \Rightarrow CS$: It is well-known that $(a,b)_F \simeq (a',b')_F$ iff $\langle -a,-b,ab\rangle \simeq \langle -a',-b',a'b'\rangle$. Suppose that F does not have property CS, and let $(a,b)_F$ and $(u,v)_F$ be quaternion algebras over F and let E/F be a quadratic extension such that the quaternion algebras have a common slot over E but such that no common slot over E can be given by an element in F. By the remark above, the fact that they have a common slot over E translates into $D_E(\langle -a,-b,ab\rangle)\cap D_E(\langle -u,-v,uv\rangle)\neq\emptyset$, and the fact that such a common slot cannot be chosen in F translates into $D_E(\langle -a,-b,ab\rangle)\cap D_E(\langle -u,-v,uv\rangle)\cap F^\times = \emptyset$. We conclude that E/F does not have property $CV(3,3)$, which, by Lemma 3.1, yields that F does not have property $D(2)$.

$CS \iff D(4)$: Suppose F does not have property CS and let $(a,b)_F$ and $(u,v)_F$ be quaternion algebras over F such that they have a common slot over $L=F(\sqrt{d})$, but no such common slot can be chosen in F. Let

$$\psi_1 := \langle d,-a,-b,ab\rangle \quad\text{and}\quad \psi_2 := \langle d,-u,-v,uv\rangle\ .$$

We first show that there does not exist a binary form β such that β is similar to a subform of ψ_1 and ψ_2. Then we show that there exists a quadratic extension $E=F(\sqrt{e})$ and some $x\in F^\times$ such that $(\psi_1)_E\simeq(x\psi_2)_E$. This then implies that property $D(4)$ fails.

Suppose there exists a binary form β with, say, $d_\pm\beta = s$ such that β is similar to a subform of ψ_1 and ψ_2. Then the forms $(\psi_1)_L\simeq\langle\langle a,b\rangle\rangle_L$ and $(\psi_2)_L\simeq\langle\langle u,v\rangle\rangle_L$ are, over $L(\sqrt{s})$, isotropic and hence hyperbolic, or, equivalently, the quaternion algebras $(a,b)_L$ and $(u,v)_L$ are split over $L(\sqrt{s})$. Hence, there exist $t,w\in L^\times$ such that $(a,b)_L\simeq(s,t)_L$ and $(u,v)_L\simeq(s,w)_L$, which yields the common slot $s\in F^\times$, a contradiction.

Let now $r\in F^\times$ and consider $\psi_1\perp -r\psi_2\in I^2F$. We then have in WF

$$\begin{aligned}\psi_1\perp -r\psi_2 &\sim \langle d,-rd\rangle + \langle -a,-b,ab\rangle - r\langle -u,-v,uv\rangle\\ &\sim \langle -1,r,d,-rd\rangle + \langle 1,-a,-b,ab\rangle - r\langle 1-u,-v,uv\rangle\\ &\sim \langle\langle a,b\rangle\rangle - r\langle\langle u,v\rangle\rangle - \langle\langle d,r\rangle\rangle\ ,\end{aligned}$$

which yields $c(\psi_1\perp -r\psi_2) = [(a,b)_F(u,v)_F(d,r)_F]$. Now $(a,b)_F$ and $(u,v)_F$ have a common slot over $L=F(\sqrt{d})$, i.e. $(a,b)_F(u,v)_F$ is not a division algebra over L and thus there exist $x,y,z\in F^\times$ such that $(a,b)_F(u,v)_F\simeq(d,x)_F(y,z)_F$, by [LLT, Prop. 5.2]. The above computation then shows that $c(\psi_1\perp -x\psi_2)=[(y,z)_F]$. Hence, $\psi_1\perp -x\psi_2$ is an 8-dimensional form in I^2F whose Clifford invariant is given by the class of a quaternion algebra, thus there exists a quadratic extension $E=F(\sqrt{e})/F$ such that $(\psi_1\perp -x\psi_2)_E$ is hyperbolic (cf. also Rem. 3.2(i)), i.e. $(\psi_1)_E\simeq(x\psi_2)_E$.

As for the converse, suppose that F does not have property $D(4)$ and let φ_1 and φ_2 be two 4-dimensional forms such that $d_\pm\varphi_1 = d_\pm\varphi_2 = d$ and that there exists a quadratic extension E/F such that $(\varphi_1)_E\simeq(\varphi_2)_E$, but there does not exist $\beta\in P_1F$ similar to a subform of both φ_1 and φ_2. After scaling, we may assume that there exist $a,b,u,v,x\in F^\times$ such that

$$\varphi_1\simeq\langle d,-a,-b,ab\rangle \quad\text{and}\quad \varphi_2\simeq x\langle d,-u,-v,uv\rangle\ .$$

Similar to above, we have that $\varphi_1 \perp -\varphi_2 \in I^2F$ and that $c(\varphi_1 \perp -\varphi_2) = [(a,b)_F(u,v)_F(d,x)_F]$. On the other hand, $\varphi_1 \perp -\varphi_2$ is hyperbolic over the quadratic extension E of F. Hence, the index of the Clifford algebra of $\varphi_1 \perp -\varphi_2$ can be at most 2, which implies that the Clifford invariant can be represented by a quaternion algebra, say, $c(\varphi_1 \perp -\varphi_2) = [(y,z)_F]$, $y,z \in F^\times$. In particular, $(a,b)_F(u,v)_F \simeq (d,x)_F(y,z)_F$, and it follows that $(a,b)_F(u,v)_F$ is not a division algebra over $L = F(\sqrt{d})$, i.e. $(a,b)_L$ and $(u,v)_L$ have a common slot. To show that property CS fails, it suffices to show that this common slot cannot be in F.

Suppose there exist $r \in F^\times$ and $s,t \in L^\times$ such that $(a,b)_L \simeq (r,s)_L$ and $(u,v)_L \simeq (r,t)_L$. Let $K = F(\sqrt{r})$. Since $(r,s)_L$ and $(r,t)_L$ split over $L(\sqrt{r}) = K(\sqrt{d})$, one sees easily that $(\varphi_1)_{K(\sqrt{d})}$ and $(\varphi_2)_{K(\sqrt{d})}$ are hyperbolic. On the other hand, $d_\pm\varphi_1 = d_\pm\varphi_2 = d$, and it is well-known and easy to show that an anisotropic 4-dimensional form stays anisotropic over the field obtained by adjoining the square root of the determinant of the form. Hence, $(\varphi_1)_K$ and $(\varphi_2)_K$ are both isotropic, which yields that both φ_1 and φ_2 contain subforms similar to $\langle 1, -r\rangle$, a contradiction.

$D(8) \Rightarrow D(14)$: If F does not have property $D(14)$, there exists a form $\varphi \in I^3F$ with $\dim\varphi = 14$ such that φ does not contain a subform in GP_2F. By Cor. 2.2, we can write $\varphi \simeq \alpha \perp \psi$ with an Albert form α and some 8-dimensional form $\psi \in I^2F$. Clearly $\psi \equiv \alpha \pmod{I^3F}$ and therefore $c(\psi) = c(\alpha)$. Since α is an Albert form, there exists a biquaternion algebra B over F such that $c(\alpha) = c(\psi) = [B]$ in $\mathrm{Br}F$. Furthermore, ψ does not contain a subform in GP_2F as φ does not contain such a subform, hence F does not have property $D(8)$. □

We do not know whether $D(4)$ implies $D(8)$ or not.

4 The properties $D(n)$ over fields with a discrete valuation

Let R be a discrete valuation ring with residue class field k and quotient field K. Suppose that $\operatorname{char} k \neq 2$, and let π be a uniformizing element of R. For each form φ over K, there exist forms φ_1 and φ_2 which have diagonalizations containing only units in $R^\times$ such that $\varphi \simeq \varphi_1 \perp \pi\varphi_2$. The residue forms $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are called the *first* and *second* residue forms respectively; they are uniquely determined by φ (see [S, Ch. 6, Def. 2.5]). If $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are both anisotropic, then φ is anisotropic. The converse holds if R is 2-henselian, by Springer's theorem [S, Ch. 6, Cor. 2.6]. A typical example of such a discrete valuation ring in the equal characteristic case is $R = k[[t]]$, the power series ring in one variable t.

Our aim is to investigate how the properties $D(n)$, $n \in \{2,4,8,14\}$, behave after going down from K to k or going up from k to K (under the extra hypothesis that R is 2-henselian).

We first go down from K to k, assuming that the residue map $R \to k$ has a section, hence that k can be viewed as a subfield of K. (For instance, K may be an intermediate field between the field of rational fractions $k(t)$ and the power series field $k((t))$, and R the t-adic valuation ring.)

Theorem 4.1 *Suppose the residue map $R \to k$ has a section, and view k as a subfield of R.*

(i) *If K has property $D(4)$, then k has property $D(2)$ (hence also $D(4)$).*

(ii) *If K has property $D(8)$, then k has properties $D(4)$ and $D(8)$.*

(iii) *If K has property $D(14)$, then k has property $D(8)$ (hence also $D(14)$).*

Proof. (i) Suppose that k does not have property $D(2)$. It will suffice to show that K does not have property CS, since Theorem 3.4 shows that CS and $D(4)$ are equivalent. Let $a,b,c\in k^\times$ and let $E=k(\sqrt{e})/k$ be a quadratic extension such that $D_E(\langle 1,-a\rangle)\cap D_E(\langle b,-bc\rangle)\neq\emptyset$ but $D_E(\langle 1,-a\rangle)\cap D_E(\langle b,-bc\rangle)\cap k^\times=\emptyset$. Let $L=K(\sqrt{e})$. Then $D_L(\langle -a,-\pi,a\pi\rangle)\cap D_L(\langle -c,-b\pi,bc\pi\rangle)\neq\emptyset$ as these 3-dimensional subforms contain $-\pi\langle 1,-a\rangle_L$ and $-\pi\langle b,-bc\rangle_L$, respectively. We will show that $D_L(\langle -a,-\pi,a\pi\rangle)\cap D_L(\langle -c,-b\pi,bc\pi\rangle)\cap K^\times=\emptyset$, which, by the remark at the beginning of the proof of $D(2)\Rightarrow CS$ in Theorem 3.4, implies that $(a,\pi)_K$ and $(c,b\pi)_K$ have a common slot over L, but no such common slot can be chosen in K, which then shows that property CS fails for K.

In order to do this, we may replace K by its 2-henselization (or by its completion) for the discrete valuation. Then L is 2-henselian with residue field E, and it follows from Springer's theorem (cf. [S, Ch. 6, Cor. 2.6]) that if $D_L(\langle -a,-\pi,a\pi\rangle)\cap D_L(\langle -c,-b\pi,bc\pi\rangle)\cap K^\times\neq\emptyset$, then $D_E(\langle -a\rangle)\cap D_E(\langle -c\rangle)\cap k^\times\neq\emptyset$, which actually implies that $ac\in E^{\times 2}$, or $D_E(\langle 1,-a\rangle)\cap D_E(\langle b,-bc\rangle)\cap k^\times\neq\emptyset$. The latter can be ruled out by our choice of $a,b,c\in k^\times$. Suppose that $ac\in E^{\times 2}$. Then $\langle 1,-a\rangle_E\simeq\langle 1,-c\rangle_E$. Since $D_E(\langle 1,-a\rangle)\cap D_E(\langle b,-bc\rangle)\neq\emptyset$, there exists $r\in E^\times$ such that $\langle 1,-a\rangle_E\simeq r\langle 1,-a\rangle_E$ and $\langle b,-bc\rangle_E\simeq r\langle 1,-c\rangle_E$. These facts together yield

$$\langle b,-bc\rangle_E\simeq r\langle 1,-c\rangle_E\simeq r\langle 1,-a\rangle_E\simeq\langle 1,-a\rangle_E\ .$$

In particular, $1\in D_E(\langle 1,-a\rangle)\cap D_E(\langle b,-bc\rangle)\cap k^\times$, a contradiction.

(ii) Suppose k does not have property $D(4)$. Let φ_1 and φ_2 be 4-dimensional forms over k such that there exists a quadratic extension $E=k(\sqrt{e})/k$ with $(\varphi_1)_E\simeq(\varphi_2)_E$ but such that there does not exist a binary form β over k which is similar to a subform of both φ_1 and φ_2. Let $\varphi:=\varphi_1\perp -\pi\varphi_2\in I^2K$. Then φ becomes hyperbolic over the biquadratic extension $K(\sqrt{e},\sqrt{\pi})$. This shows that the index of the Clifford algebra of φ can be at most 4 and hence there exists a biquaternion algebra B such that $c(\varphi)=[B]$.

In order to prove that K does not have property $D(8)$, it remains to show that φ does not contain a subform in GP_2K. For this, we may replace K by its 2-henselization for the discrete valuation. Suppose $\sigma\in GP_2K$ is such that $\sigma\subset\varphi$. We may decompose $\sigma\simeq\sigma_1\perp -\pi\sigma_2$, where σ_1 and σ_2 are even-dimensional forms which have a diagonalization containing only units in $R^\times$. By Springer's theorem, the residue forms $\overline{\sigma_1}$ and $\overline{\sigma_2}$ satisfy $\overline{\sigma_1}\subset\varphi_1$ and $\overline{\sigma_2}\subset\varphi_2$. If $\dim\sigma_1=0$ or $\dim\sigma_2=0$, then φ_2 or φ_1 lies in GP_2F, which is not possible (cf. Rem. 3.2). Therefore, $\dim\sigma_1=\dim\sigma_2=2$. Since $d_\pm\sigma=1$, there exists $s\in k^\times$ such that $\overline{\sigma_2}\simeq s\overline{\sigma_1}$, in which case $\overline{\sigma_1}\subset\varphi_1$ and $s\overline{\sigma_1}\subset\varphi_2$, a contradiction to the choice of φ_1 and φ_2. We conclude that φ does not contain a subform in GP_2K.

If k does not have property $D(8)$, there exists an 8-dimensional form $\psi\in I^2k$ such that $\operatorname{ind}c(\psi)\leq 4$ which does not contain any subform in GP_2k. As in the preceding argument, we may use residues and Springer's theorem to show that, viewed over K, the form ψ does not contain any subform in GP_2K. Therefore, K does not have property $D(8)$.

(iii) Suppose k does not have property $D(8)$, i.e. there exist an 8-dimensional form $\psi\in I^2k$ and a biquaternion algebra B over k such that $c(\psi)=[B]$, and such that

ψ does not contain a subform in GP_2k. Let α be an Albert form with $c(\alpha) = [B]$. By Remark 3.2, ψ and α are both anisotropic (in the case of α this follows after invoking Albert's theorem because B is a division algebra). In particular, α also does not contain a subform in GP_2k. Consider the form $\varphi := \alpha \perp \pi\psi$ over K. Obviously, $c(\varphi) = c(\alpha)c(\psi) = 1$ in $\mathrm{Br}K$ and thus $\varphi \in I^3K$ and $\dim\varphi = 14$. We will show that φ does not contain a subform in GP_2K which then implies that property $D(14)$ fails for K. For this, we may replace K by its 2-henselization for the discrete valuation.

Suppose there exists $\sigma \in GP_2K$ such that $\sigma \subset \varphi$. As in the proof of (ii) above, we decompose $\sigma \simeq \sigma_1 \perp \pi\sigma_2$ and obtain by Springer's theorem $\overline{\sigma_1} \subset \alpha$ and $\overline{\sigma_2} \subset \psi$. If $\dim\sigma_1 = 0$ or $\dim\sigma_2 = 0$, it follows that ψ or α contains a subform in GP_2k, a contradiction. Therefore, $\dim\sigma_1 = \dim\sigma_2 = 2$ and, since $d_\pm\sigma = 1$, we have $d_\pm\overline{\sigma_1} = d_\pm\overline{\sigma_2}$. Let $d \in k^\times$ be a representative of $d_\pm\overline{\sigma_1}$ and $E = k(\sqrt{d})$. Then α_E and ψ_E are isotropic and it follows from Lemma 3.3 that ψ contains a subform in GP_2k, a contradiction. □

Corollary 4.2 *Let k be a field and let K_i, $1 \le i \le 3$, be any field with $k(t_1,\cdots,t_i) \subset K_i \subset k((t_1))\cdots((t_i))$, where t_1,t_2,t_3 are independent variables over k. If k does not have property $D(2)$, then K_1 does not have property $D(4)$, K_2 does not have property $D(8)$, and K_3 does not have property $D(14)$.*

A more precise statement is in Corollary 6.2 below.

Remark 4.3 The hypothesis that the residue map has a section is used in the proof of Theorem 4.1 to find suitable lifts for quadratic forms over k. If the valuation is 2-henselian, this hypothesis is not needed. Indeed, in the proof of part (i) we may choose any lifts a', b', c', $e' \in R$ of a, b, c, e, and set $L = K(\sqrt{e'})$. Since $D_E(\langle 1,-a\rangle) \cap D_E(\langle b,-bc\rangle) \neq \emptyset$, the 2-henselian hypothesis ensures that $D_L(\langle 1,-a'\rangle) \cap D_L(\langle b',-b'c'\rangle) \neq \emptyset$, hence $D_L(\langle -a',-\pi,a'\pi\rangle) \cap D_L(\langle -c',-b'\pi,b'c'\pi\rangle) \neq \emptyset$. The rest of the proof holds without change.

Similarly, in the proof of part (ii), we may choose for φ the quadratic form over K whose first and second residues are φ_1 and φ_2 respectively, and use the henselian hypothesis to see that φ becomes hyperbolic over the biquadratic extension $L(\sqrt{\pi})$, where L is the quadratic extension of K with residue field E.

For the proof of (iii), choose for φ the quadratic form over K whose first and second residues are α and ψ respectively, and use Witt's theorem on the structure of $\mathrm{Br}K$ (which is a Brauer-group analogue of Springer's theorem) (see [Se, Ch. XII, §3]) to see that $c(\varphi) = 1$.

Our next goal is to lift properties $D(n)$ from k to K, assuming that the valuation is 2-henselian.

Theorem 4.4 *Suppose the valuation ring R is 2-henselian.*

(i) *If k has property $D(2)$, then K has property $D(2)$ (hence also $D(4)$).*

(ii) *If k has properties $D(4)$ and $D(8)$, then K has property $D(8)$.*

(iii) *If k has property $D(8)$, then K has property $D(14)$.*

Proof. (i) If k has property $D(2)$, then property $D(2)$ for K follows from [STW, Th. 3.10].

(ii) Assume that k has properties $D(4)$ and $D(8)$. Let $\varphi \in I^2K$, $\dim\varphi = 8$, such that $c(\varphi)$ can be represented by a biquaternion algebra. We want to show that φ contains a subform in GP_2K. By Remark 3.2(i), we may assume that φ is anisotropic. There exists an Albert form α over K such that $\varphi \equiv \alpha \pmod{I^3K}$. (Note that scaling φ resp. α does not affect this congruence.) With decompositions $\varphi \simeq \varphi_1 \perp \pi\varphi_2$ and $\alpha \simeq \alpha_1 \perp \pi\alpha_2$ as above, and using the fact that $\varphi, \alpha \in I^2K$, we obtain for the first and second residue forms, respectively, that $\overline{\varphi_i}, \overline{\alpha_i} \in Ik$, $i = 1, 2$, and that $d_\pm\overline{\varphi_1} = d_\pm\overline{\varphi_2}$ and $d_\pm\overline{\alpha_1} = d_\pm\overline{\alpha_2}$ in $k^\times/k^{\times 2}$. Furthermore, $(\varphi_1 \perp -\alpha_1) \perp \pi(\varphi_2 \perp -\alpha_2) \in I^3K$, hence $\overline{\varphi_i} \perp -\overline{\alpha_i} \in I^2k$, $i = 1, 2$, and thus in fact $d_\pm\overline{\varphi_1} = d_\pm\overline{\varphi_2} = d_\pm\overline{\alpha_1} = d_\pm\overline{\alpha_2}$.

If $\dim\varphi_1 = 0$ then $\overline{\varphi_2}$ is an 8-dimensional form in I^2k whose Clifford invariant can obviously be represented by some biquaternion algebra over k. Since k has property $D(8)$, $\overline{\varphi_2}$ contains some form in GP_2k as a subform. This subform can be lifted to a form in GP_2K which will be a subform of φ_2 and thus similar to a subform of φ. The case $\dim\varphi_2 = 0$ is treated in an analogous way. Thus, we may assume after scaling φ that $(\dim\varphi_1, \dim\varphi_2) \in \{(2, 6), (4, 4)\}$.

If $\dim\alpha_1 = 0$ or $\dim\alpha_2 = 0$, then $\overline{\alpha_i} \in I^2k$ which, by the above discriminant comparison, yields that $\overline{\varphi_1}, \overline{\varphi_2} \in I^2k$. In the case $\dim\varphi_1 = 2$, this forces $\overline{\varphi_1} \simeq \mathbf{H}$ which in turn implies that φ is isotropic, contrary to our assumption. If $\dim\varphi_1 = 4$, we have $\overline{\varphi_1} \in GP_2k$, and thus we even have $\varphi_1 \in GP_2K$. Hence, we may assume after scaling α that $\dim\alpha_1 = 2$, $\dim\alpha_2 = 4$, and that $\alpha_1 \perp -\varphi_1$ is isotropic.

If $\dim\varphi_1 = 2$, then the isotropy of $\alpha_1 \perp -\varphi_1$ together with $\underline{d_\pm\overline{\varphi_1}} = d_\pm\overline{\alpha_1} = \overline{d}$ for some $d \in R^\times$ implies that $\overline{\varphi_1} \simeq \overline{\alpha_1}$ which in turn is similar to $\overline{\langle 1, -d\rangle}$. Thus, over $\ell = k(\sqrt{\overline{d}})$, we get $(\overline{\alpha_2})_\ell \equiv (\overline{\varphi_2})_\ell \pmod{I^3\ell}$ and $(\overline{\alpha_2})_\ell, (\overline{\varphi_2})_\ell \in I^2\ell$. In particular, $(\overline{\varphi_2})_\ell$ is an Albert form, $(\overline{\alpha_2})_\ell \in GP_2\ell$, and $c((\overline{\varphi_2})_\ell) = c((\overline{\alpha_2})_\ell)$. Since $c((\overline{\alpha_2})_\ell)$ can be represented by a single quaternion algebra, this implies that the Albert form $(\overline{\varphi_2})_\ell$ is isotropic, and $\overline{\varphi_2}$ contains therefore a subform similar to $\overline{\langle 1, -d\rangle}$ over k. After lifting, we see that there exist $x, y \in R^\times$ such that $\varphi_1 \simeq x\langle 1, -d\rangle$ and $y\langle 1, -d\rangle \subset \varphi_2$. Hence, φ contains $\langle x, y\pi\rangle \otimes \langle 1, -d\rangle \in GP_2K$ as a subform.

Finally, suppose that $\dim\varphi_1 = 4$. The fact that φ_1 is anisotropic of dimension 4, $\dim\alpha_1 = 2$ and $\alpha_1 \perp -\varphi_1$ is isotropic imply that $\overline{\psi_1} = (\overline{\alpha_1} \perp -\overline{\varphi_1})_{\mathrm{an}}$ is not hyperbolic and of dimension ≤ 4. Since $d_\pm\overline{\varphi_1} = d_\pm\overline{\alpha_1}$, we also have $\overline{\psi_1} \in I^2k$. All this together yields $\overline{\psi_1} \in GP_2k$. Lifting $\overline{\psi_1}$ to a form $\psi_1 \in GP_2K$, we get by Springer's theorem

$$-\psi_1 + \pi(\varphi_2 \perp -\alpha_2) \sim (\varphi_1 \perp -\alpha_1) + \pi(\varphi_2 \perp -\alpha_2) \in I^3K \ ,$$

thus

$$\psi_1 \equiv \pi(\varphi_2 \perp -\alpha_2) \equiv \varphi_2 \perp -\alpha_2 \pmod{I^3K} \ ,$$

which obviously implies $\overline{\psi_1} \equiv \overline{\varphi_2} \perp -\overline{\alpha_2} \pmod{I^3k}$. Since $\overline{\varphi_2} \perp -\overline{\alpha_2}$ is an 8-dimensional I^2k-form whose Clifford invariant is the same as that of $\overline{\psi_1} \in GP_2k$, i.e., it can be represented by a single quaternion algebra, there exists $e \in R^\times$ such that $\overline{\varphi_2} \perp -\overline{\alpha_2}$ becomes hyperbolic over $k(\sqrt{\overline{e}})$ (see also Remark 3.2(i)), i.e., $\overline{\varphi_2}$ and $\overline{\alpha_2}$ are 4-dimensional forms which become isometric over the quadratic extension $k(\sqrt{\overline{e}})$. Since k has property $D(4)$, there exists $b \in R^\times$ such that $\overline{\langle 1, -b\rangle}$ is similar to a subform of both $\overline{\varphi_2}$ and $\overline{\alpha_2}$. After lifting, this shows that $\langle 1, -b\rangle$ is similar to a subform of both φ and α. It follows from Lemma 3.3 that φ contains a subform in GP_2K.

(iii) Suppose that k has property $D(8)$ and let φ be a 14-dimensional I^3-form over K, which we write as $\varphi \simeq \varphi_1 \perp \pi\varphi_2$ with first resp. second residue form $\overline{\varphi_1}$

resp. $\overline{\varphi_2}$ over k. To establish property $D(14)$, it suffices by Prop. 2.3 to show that φ contains a subform in GP_2K. This is obvious if φ is isotropic, so that we may assume that φ and hence $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are anisotropic. We have that $\overline{\varphi_1}, \overline{\varphi_2} \in I^2k$ as $\varphi \in I^3K$, and after scaling we may assume that $\dim \overline{\varphi_2} \in \{0, 2, 4, 6\}$.

If $\dim \overline{\varphi_2} = 0$, then $\varphi \simeq \varphi_1$ and we have in fact $\overline{\varphi_1} \in I^3k$. Since k has property $D(8)$, it has property $D(14)$ by Theorem 3.4, and by Prop. 2.3, $\overline{\varphi_1}$ contains a subform in GP_2k which can be lifted to a subform of φ in GP_2K.

If $\dim \overline{\varphi_2} = 2$, then $\overline{\varphi_2} \in I^2k$ implies that $\overline{\varphi_2}$ is isotropic, contrary to our assumption.

If $\dim \overline{\varphi_2} = 4$, then $\overline{\varphi_2} \in I^2k$ implies that $\overline{\varphi_2} \in GP_2k$, and after lifting we find again a subform of φ which is in GP_2K.

Finally, if $\dim \overline{\varphi_2} = 6$, then $\overline{\varphi_2}$ is an Albert form over k with associated biquaternion algebra A over k. Furthermore, $\overline{\varphi_1}$ is an 8-dimensional I^2-form over k and one has that $\overline{\varphi_1} \equiv \overline{\varphi_2} \pmod{I^3k}$, so that $c(\overline{\varphi_1}) = [A]$. Since k has property $D(8)$, it follows from Lemma 3.3 that there is a binary form $\overline{\beta}$ over k which is similar to both a subform of $\overline{\varphi_1}$ and of $\overline{\varphi_2}$. Lifting $\overline{\beta}$ to a binary form β over K, we see that φ_1 and φ_2 each contain a subform similar to β, say, $u\beta \subset \varphi_1$ and $v\beta \subset \pi\varphi_2$, $u, v \in K^\times$. Hence, φ contains $\langle u, v\rangle \otimes \beta \in GP_2K$ as a subform. □

Combining Remark 4.3 and Theorem 4.4, we obtain:

Corollary 4.5 (i) *k has property $D(2)$ iff K has property $D(2)$ iff K has property $D(4)$.*

(ii) *k has properties $D(4)$ and $D(8)$ iff K has property $D(8)$.*

(iii) *k has property $D(8)$ iff K has property $D(14)$.*

Note that for $n \in \{4, 8, 14\}$ it is generally *not* true that if $D(n)$ holds over k then $D(n)$ also holds over K, cf. Ex. 5.4 below.

Recall that a field F is called linked if the quaternion algebras over F form a subgroup in $\mathrm{Br}F$, in particular, any two quaternion algebras over F have a common slot and there are therefore no biquaternion division algebras. This readily implies that a linked field F always has properties $D(n)$, $n \in \{4, 8, 14\}$. We will encounter typical examples, like finite, local or global fields, etc., also in Cor. 5.1 below. But first, let us state the following immediate consequences of Theorem 4.4.

Corollary 4.6 *Let $K_0, K_1, K_2, \cdots$ be fields of characteristic $\neq 2$ such that K_{i+1} is the quotient field of a 2-henselian discrete valuation ring R_{i+1} with residue field K_i, $i \geq 0$. If K_0 has property $D(2)$, then K_i has property $D(2)$ for all $i \geq 0$.*

(i) *If K_0 has property $D(2)$ and $D(8)$, then K_i has property $D(n)$ for all $i \geq 0$ and all $n \in \{2, 4, 8, 14\}$.*

(ii) *If K_0 is linked, then K_0 has property $D(n)$ for $n \in \{4, 8, 14\}$, K_1 has properties $D(8)$ and $D(14)$, and K_2 has property $D(14)$.*

Proof. (i) follows by induction from Theorems 3.4 and 4.4, and (ii) is a consequence of the preceding remarks together with Theorem 4.4. □

5 Fields with finite Hasse number

For a field F, the Hasse number $\tilde{u}(F)$ is defined to be the supremum of the dimensions of anisotropic totally indefinite quadratic forms over F, where totally indefinite means indefinite with respect to each ordering on F. If F is not formally real, i.e., if F does not possess any orderings, then $\tilde{u}(F)$ is nothing but the supremum of the dimensions of anisotropic forms over F and coincides with the u-invariant $u(F)$, the supremum of the dimensions of anisotropic torsion forms. In the sequel, we investigate the properties $D(n)$, $n \in \{2,4,8,14\}$, over fields with finite Hasse number and of power series extensions of such fields.

For basic properties of fields with finite Hasse number, we refer the reader to [ELP]. Let us just mention that one always has $\tilde{u}(F) \neq 3,5,7$, and that F is a so-called SAP field if $\tilde{u}(F) < \infty$. Furthermore, using Merkurjev's index reduction formulas [Me 2], one can construct fields F with $\tilde{u}(F) = 2n$ for any integer $n \geq 0$, see for example [L 2], [Ho]. It is also well-known that fields of transcendence degree ≤ 1 over a real closed field have $\tilde{u} \leq 2$ (cf. [ELP, Th. I]), finite fields have $\tilde{u} = 2$, and local and global fields have $\tilde{u} = 4$ (for global fields, this is Meyer's theorem). Furthermore, if $\tilde{u}(F) \leq 4$, then F is linked. Conversely, if F is linked, then $\tilde{u}(F) \in \{0,1,2,4,8\}$ (cf. [EL], [E, Th. 4.7]).

Corollary 5.1 *Let F_0 be a field with $\tilde{u}(F_0) \leq 2$, or let F_0 be a local or global field. Let $F_i = F((t_1))\cdots((t_i))$ be the iterated power series field in i variables over F_0. Then F_i has property $D(n)$ for all $i \geq 0$ and all $n \in \{2,4,8,14\}$.*

Proof. By Cor. 4.6, it suffices to verify that F_0 has properties $D(2)$ and $D(8)$. For property $D(2)$, this follows from [STW, Ths. 3.6, 3.7]. Property $D(8)$ is a consequence of the fact that in each case, F_0 is a linked field (cf. [EL, § 1]). □

In the sequel, X_F denotes the space of orderings on F, and $\mathrm{sgn}_P(\varphi)$ denotes the signature of the form φ at the ordering $P \in X_F$.

Lemma 5.2 (i) *Let φ be an anisotropic form over F. Then*

$$\dim \varphi \leq \sup\{\tilde{u}(F), |\mathrm{sgn}_P(\varphi)|; P \in X_F\} .$$

(ii) *Let $\tilde{u}(F) \leq r$ and let φ_1, φ_2 be forms over F of dimension ≥ 3 such that $\dim \varphi_1 + \dim \varphi_2 \geq r+3$. Then there exists a binary form β which is similar to a subform of both φ_1 and φ_2.*

Proof. (i) If $\dim \varphi > \sup\{|\mathrm{sgn}_P(\varphi)|; P \in X_F\}$, then φ is totally indefinite, hence $\dim \varphi \leq \tilde{u}(F)$.

(ii) Since F is SAP, there exist $a_1, a_2 \in F^\times$ such that $\mathrm{sgn}_P(a_1\varphi_1), \mathrm{sgn}_P(a_2\varphi_2) \geq 0$ for all $P \in X_F$. Hence, $|\mathrm{sgn}_P(a_1\varphi_1 \perp -a_2\varphi_2)| \leq \dim \varphi_1 + \dim \varphi_2 - 3$ for all $P \in X_F$, and since $\dim \varphi_1 + \dim \varphi_2 - 3 \geq \tilde{u}(F)$, it follows from (i) that $\dim(a_1\varphi_1 \perp -a_2\varphi_2)_{\mathrm{an}} \leq \dim \varphi_1 + \dim \varphi_2 - 3$, which in turn yields for the Witt index that $i_W(a_1\varphi_1 \perp -a_2\varphi_2) \geq 2$. This shows that $a_1\varphi_1$ and $a_2\varphi_2$ have a common binary subform. □

We have seen above that iterated power series fields over fields with $\tilde{u} \leq 2$ always have the properties $D(n)$, $n \in \{2,4,8,14\}$. We now ask what happens if the base field has $\tilde{u} \geq 4$. Note that if $\tilde{u} \leq 4$, then F is linked as already mentioned above. (One can see this also by applying Lemma 5.2(ii), which shows that two 4-dimensional forms over F have always up to similarity a common binary subform, which, applied to 2-fold Pfister forms, implies linkage.) Of particular interest is the case $\tilde{u} = 4$ as will be illustrated by Ex. 5.4 below. For this reason, we state explicitly the following special case of Cor. 4.6(ii).

Corollary 5.3 *Let $F_i = F((t_1)) \cdots ((t_i))$ be the iterated power series field in i variables over a field F_0 with $\tilde{u}(F_0) = 4$.*

(i) *F_0 has property $D(n)$ for $n \in \{4, 8, 14\}$;*

(ii) *F_1 has property $D(n)$ for $n \in \{8, 14\}$;*

(iii) *F_2 has property $D(14)$.*

Example 5.4 Let $F = \mathbf{C}(x, y)$, the rational function field in two variables x, y over the complex numbers $\mathbf{C}$. It is well-known that $u(F) = \tilde{u}(F) = 4$. F does *not* have property $D(2)$ (cf. [STW, Remarks 4.18, 5.10]). But it has property $D(n)$, $n \in \{4, 8, 14\}$ by Cor. 5.3. It also shows that linked fields generally do not have property $D(2)$.

By Theorem 4.1, $F_1 = F((t_1))$ does not have property $D(4)$, but it has property $D(n)$ for $n \in \{8, 14\}$ by Cor. 5.3. Similarly, we see that $F_2 = F((t_1))((t_2))$ does not have property $D(8)$, but that it does have property $D(14)$.

All this shows that generally, the statements regarding the properties $D(n)$ in Cor. 5.3 cannot be strengthened. It shows furthermore for $n, m \in \{2, 4, 8, 14\}$, $m > n$, that generally $D(m) \not\Rightarrow D(n)$, so that the implications in Theorem 3.4 cannot be reversed without any further assumptions on the field in question.

For values of $\tilde{u}$ possibly bigger than 4, let us note the following.

Corollary 5.5 (i) *If $\tilde{u}(F) < 12$, then F has properties $D(8)$, $D(14)$, and $F((t))$ has property $D(14)$.*

(ii) *If $\tilde{u}(F) < 14$, then F has property $D(14)$.*

Proof. (i) Let φ be an 8-dimensional I^2-form over F such that $c(\varphi)$ can be represented by a biquaternion algebra A with associated Albert form α. To establish property $D(8)$, it suffices by Lemma 3.3 to show that φ and α have a common binary subform. Since $\tilde{u}(F) < 12$, this is an easy consequence of Lemma 5.2(ii). Property $D(14)$ for $F((t))$ follows from Theorem 4.4.

(ii) Let $\varphi \in I^3F$, $\dim \varphi = 14$. If F is not formally real, then $\tilde{u}(F) < 14$ implies that φ is isotropic and $D(14)$ follows easily. If F is formally real, then we first note that for each $P \in X_F$ we have $\mathrm{sgn}_P(\varphi) \equiv 0 \pmod 8$ because $\varphi \in I^3F$. Hence, $\mathrm{sgn}_P(\varphi) \in \{0, \pm 8\}$ as $\dim \varphi = 14$. By Lemma 5.2(i), $\dim \varphi_{\mathrm{an}} < 14$. Thus, again we have that φ is isotropic and we are done. □

Example 5.6 It is again interesting in this context to consider the example from above based on $\mathbf{C}(x, y)$. As was shown there, the field $F_1 = \mathbf{C}(x, y)((t_1))$ has property $D(8)$, but not $D(4)$, and $F_2 = F_1((t_2))$ has property $D(14)$, but not $D(8)$. $F_3 = F_2((t_3))$ does not even have property $D(14)$. One has $\tilde{u}(F_1) = u(F_1) = 8$, which shows that in part (i) of the above corollary, one cannot always expect that property $D(8)$ carries over to a power series extension. Also, F_2 is a field for which $D(8)$ fails, and we have $\tilde{u}(F_2) = u(F_2) = 16$, which is still a little higher than the bound given in part (i) above which assures that $D(8)$ holds. This naturally raises the question whether the bound given there is the best possible.

We note furthermore that $\tilde{u}(F_3) = u(F_3) = 32$. For F_3, we know that $D(14)$ fails, but its Hasse number is considerably higher than the bound in part (ii) of the above corollary, and therefore this example does not give an indication on how good this bound really is.

Knowing that $D(4)$ always holds if $\tilde{u}(F) \leq 4$ (see Corollaries 5.1 and 5.3) and that it can fail if $\tilde{u}(F) \geq 8$ (see Examples 5.4 and 5.6), it would be interesting to know if there exist fields F with $\tilde{u}(F) = 6$ for which $D(4)$ fails. We do know by Corollary 5.5 that $D(8)$ holds whenever $\tilde{u}(F) \leq 12$, so it holds in particular for all fields with $\tilde{u}(F) \leq 6$. In the following proposition, we establish property $D(8)$ for another class of fields which also contains all fields F with $\tilde{u}(F) \leq 6$.

In the sequel, $I_t^n F = I^n \cap W_t F$, where $W_t F$ denotes the torsion part of the Witt ring. If F is not formally real, then $WF = W_t F$, otherwise $W_t F$ consists of the classes of forms which have total signature zero (Pfister's local-global principle).

PROPOSITION 5.7 *Suppose that $I_t^3 F = 0$ and that F is SAP. Then F has property $D(8)$ (and hence also $D(14)$), and $F((t))$ has property $D(14)$.*

Proof. In view of Theorems 3.4 and 4.4, it suffices to establish property $D(8)$ for F.

Let $\varphi \in I^2 F$, $\dim \varphi = 8$ and $c(\varphi) = c(\alpha)$ with α an Albert form. We have to show that φ contains a subform in $GP_2 F$.

Suppose first that F is not formally real. By Merkurjev's theorem, we have $\varphi - \alpha \in I^3 F = I_t^3 F = 0$, hence $\varphi \sim \alpha$, and comparing dimensions yields that φ is isotropic and therefore contains a subform in $GP_2 F$ (see Remark 3.2(i)).

Hence, we may assume that F is formally real. Since φ, $\alpha \in I^2 F$, we have for all orderings $P \in X_F$ that $\mathrm{sgn}_P(\varphi)$, $\mathrm{sgn}_P(\alpha) \equiv 0 \pmod 4$. Since $\dim \alpha = 6$ and $\dim \varphi = 8$, and since F is SAP, we may assume after scaling that $\mathrm{sgn}_P(\varphi) \in \{0, 4, 8\}$ and $\mathrm{sgn}_P(\alpha) \in \{0, 4\}$. On the other hand, we have $\varphi - \alpha \in I^3 F$ and thus $\mathrm{sgn}_P(\varphi \perp -\alpha) \equiv 0 \pmod 8$. Thus, we always have $\mathrm{sgn}_P(\varphi \perp -\alpha) \in \{0, 8\}$. Now if $\pi \in P_3 F$, then $\mathrm{sgn}_P(\pi) \in \{0, 8\}$, and since F is SAP, there exists $\pi \in P_3 F$ such that $\mathrm{sgn}_P(\pi) = \mathrm{sgn}_P(\varphi \perp -\alpha)$ for all $P \in X_F$. Hence, $\mathrm{sgn}_P(\varphi \perp -\alpha \perp -\pi) = 0$ for all $P \in X_F$, i.e. $\varphi \perp -\alpha \perp -\pi \in I^3 F \cap W_t F = I_t^3 F = 0$. Thus, $\varphi \perp -\pi \sim \alpha$, and comparing dimensions yields that the Witt index of $\varphi \perp -\pi$ is ≥ 5. In particular, φ contains a 5-dimensional Pfister neighbor of π as a subform. It is well-known that 5-dimensional Pfister neighbors always contain a subform in $GP_2 F$. Hence, φ contains a subform in $GP_2 F$. □

REMARK 5.8 (i) Note that the two classes of fields for which we established property $D(8)$, fields with $\tilde{u} < 12$ and SAP-fields with $I_t^3 F = 0$, respectively, are such that one class is not contained in the other. Indeed, using constructions similar to those in [L 2], [Ho], it is not difficult to construct fields F with $\tilde{u}(F) = 8$ or 10 and $I_t^3 F \neq 0$. On the other hand, to any positive integer n, there exist fields with $\tilde{u}(F) = 2n$ and $I_t^3 F = 0$ (cf. [Ho]). Since their Hasse number is finite, they are SAP-fields. Thus, there are SAP-fields with $I_t^3 F = 0$ for which $\tilde{u} \geq 12$.

(ii) We do not know whether $I_t^3F = 0$ alone already suffices for property $D(8)$ (or maybe even $D(4)$) to hold, or whether we can replace SAP by some weaker property which together with $I_t^3F = 0$ would imply property $D(8)$. Consider, for example, the field $F = \mathbf{R}((t_1))\cdots((t_i))$ with $i \geq 2$. We have $I_t^3F = 0$ (in fact, we even have $W_tF = 0$), and it is well-known that F is not SAP. However, F does have property $D(n)$, $n \in \{2, 4, 8, 14\}$ by Corollary 5.1. Note also that $I_t^3F = 0$ alone does not imply property $D(2)$ in general, as exemplified by the field $\mathbf{C}(x, y)$ (see Example 5.4).

(iii) It is well-known that a field F satisfies $I_t^2F = 0$ and SAP if and only if $\tilde{u}(F) \leq 2$ (cf. [ELP, Theorems E, F]). In this case, F and its iterated power series extensions have property $D(n)$, $n \in \{2, 4, 8, 14\}$ by Corollary 5.1.

6 Some further consequences and examples

A field extension K/F is said to be *excellent* if for every quadratic form φ over F there exists a form ψ over F such that $(\varphi_K)_{\mathrm{an}} \simeq \psi_K$, i.e. the anisotropic part of φ over K is defined over F. Izhboldin and Karpenko [IK 1, Part II] considered the question of excellence of extensions K/F where K is the function field of a Severi-Brauer variety $SB(A)$ of a central simple algebra A over F. One of the crucial cases in their investigations was the case where A was an algebra of exponent 2. In this situation, if the algebra is of index ≤ 2, then K/F is excellent as was shown by Arason in [ELW, App. II]. If the index is 8, then K/F is never excellent as was shown in [IK 1, Part II, Th. 3.10]. If the index is equal to 4, i.e. A is a biquaternion division algebra, examples are given in [IK 1] which show that both excellence and nonexcellence are possible for such an extension. Izhboldin himself noticed that if a field F does not have property $D(8)$, then one can readily find examples of biquaternion algebras A over F such that $F(SB(A))/F$ is nonexcellent.

In [Ma], Mammone gave counterexamples to a question raised by Knus concerning the product of a biquaternion algebra B and a quaternion algebra Q over F, both assumed to be division algebras: If $B \otimes_F Q$ is not a division algebra, does it follow that there exists a quadratic extension L/F over which both Q and B are not division (i.e. Q and B have a quadratic extension of F as a common subfield)? Again, if F does not have property $D(8)$ then a pair B, Q can be readily found which provides a counterexample.

The previous two implications for a field where property $D(8)$ fails are summarized in the following proposition.

Proposition 6.1 *Let F be a field where property $D(8)$ fails. Then the following holds:*

(i) (Izhboldin) *There exists a biquaternion division algebra A over F such that $F(SB(A))/F$ is nonexcellent.*

(ii) *There exist a biquaternion division algebra B over F and a quaternion division algebra Q over F which have the following properties:*

  (a) *$B \otimes_F Q$ is not a division algebra, and yet*

  (b) *there does not exist a quadratic extension L/F which is a common subfield of B and Q.*

Proof. Since F does not have property $D(8)$, there exist a biquaternion division algebra A over F and a form $\varphi \in I^2F$, $\dim \varphi = 8$ such that $c(\varphi) = [A]$ and such that φ does not contain a subform in GP_2F. After scaling, we may assume that $1 \in D(\varphi)$.

(i) Let $K = F(SB(A))$. By Rem. 3.2(i), φ is anisotropic and thus φ_K is also anisotropic (cf. [La, Th. 4]). In particular, φ_K is an anisotropic form in I^3K representing 1. Hence, $\varphi_K \in P_3K$. Let $\varphi \simeq \langle 1, -a, -b, \cdots \rangle$, $a, b \in F^\times$. It follows readily that there exists $c \in K^\times$ such that $\varphi_K \simeq \langle\langle a, b, c\rangle\rangle_K$. Suppose that K/F is excellent. Then, by [ELW, Prop. 2.11], we may assume that $c \in F^\times$ and we put $\pi := \langle\langle a, b, c\rangle\rangle \in P_3F$.

Let $\psi := (\varphi \perp -\pi)_{\mathrm{an}}$. We have $\psi \in I^2F$, $c(\psi) = c(\varphi) = [A]$, and $\dim \psi \leq 10$. If $\dim \psi \leq 6$ then φ and π have at least a 5-dimensional subform in common, i.e., φ contains a Pfister neighbor of π. Now each 5-dimensional Pfister neighbor contains a subform in GP_2F, thus φ contains a subform in GP_2F, a contradiction.

If $\dim \psi = 8$, then it follows again from [La, Th. 4] that ψ_K is anisotropic, a contradiction because we have by construction that ψ_K is hyperbolic.

Finally, suppose that $\dim \psi = 10$. Let $E = F(\psi)$. Then $\dim(\psi_E)_{\mathrm{an}} = 8$ or 6 (cf. [H 1, Cor. 1]). If $\dim(\psi_E)_{\mathrm{an}} = 8$, then, since $c(\psi_E) = [A_E]$ in $\mathrm{Br}E$, we have again that $(\psi_E)_{\mathrm{an}}$ stays anisotropic over $E(SB(A_E))$, obviously a contradiction to ψ becoming hyperbolic over $K = F(SB(A))$. Hence, $\dim(\psi_E)_{\mathrm{an}} = 6$, and by [H 2, Lemma 3.3] it follows that there exist a 6-dimensional form β and an anisotropic $\tau \in GP_4F$ such that $\psi \perp \beta \simeq \tau$. On the other hand, ψ and thus τ contain a 5-dimensional subform of $-\pi \in GP_3F$. Hence, τ becomes hyperbolic over $F(\pi)$. Using the multiplicativity of Pfister forms and the fact that $\tau \in W(F(\pi)/F)$ is anisotropic, we conclude readily that there exists $x \in F^\times$ such that $\tau \simeq -\pi \perp x\pi$. In the Witt ring, we thus get

$$\psi + \beta \sim \varphi - \pi + \beta \sim -\pi + x\pi$$

and hence $x\pi - \varphi \sim \beta$. Comparing dimensions yields that φ and $x\pi$ have a 5-dimensional subform in common, i.e., φ contains a Pfister neighbor of π and we get a contradiction as before.

(ii) After scaling, we may assume that $\varphi \simeq \langle -x, -y, xy\rangle \perp \varphi'$ for suitable $x, y \in F^\times$ and some form φ' over F with $\dim \varphi' = 5$ and $\det \varphi' = 1$. Now φ' does not represent $1 = \det \varphi'$ as φ' does not contain a subform in GP_2F. In particular, the Albert form $\beta := \varphi' \perp \langle -1\rangle$ is anisotropic, and therefore the biquaternion algebra B with $c(\beta) = [B]$ is a division algebra by Albert's theorem. Since $\langle -x, -y, xy\rangle$ is anisotropic, we also have that the quaternion algebra $Q = (x, y)_F$ is a division algebra. Furthermore, $\varphi \sim \langle\langle x, y\rangle\rangle + \beta$ in WF and therefore

$$[A] = c(\varphi) = c(\langle\langle x, y\rangle\rangle \perp \beta) = c(\langle\langle x, y\rangle\rangle)c(\beta) = [Q][B]$$

and it follows that $Q \otimes_F B$ is not a division algebra.

Suppose there exists a quadratic extension $L = F(\sqrt{d})/F$ such that Q_L and B_L are both not division. Then $\langle\langle x, y\rangle\rangle_L$ is hyperbolic and β_L is isotropic. It follows that φ_L is isotropic and A_L is not division. By Lemma 3.3, this implies that φ contains a subform in GP_2F, a contradiction. ◻

For an element $a \in F^\times$, let $N_F(a)$ denote the norm group $D_F(\langle 1, -a\rangle)$. Let now $a, b, c \in F^\times$ and let $E = F(\sqrt{c})$. Consider the following factor group:

$$N_1(a, b, c) = \frac{F^\times \cap N_E(a)N_E(b)}{(F^\times \cap N_E(a))(F^\times \cap N_E(b))}\ .$$

Corollary 6.2 *Let F be a field such that there exist $a,b,c\in F^\times$ with $N_1(a,b,c)\neq 1$. Let $E=F(\sqrt{c})$ and let $d\in F^\times\cap N_E(a)N_E(b)\setminus(F^\times\cap N_E(a))(F^\times\cap N_E(b))$. Let t_1,t_2,t_3 be independent variables over F and $F_i=F(t_1,\cdots,t_i)$ (or $F_i=F((t_1))\cdots((t_i))$), $i=1,2,3$, and let $E_i=F_i(\sqrt{c})$.*

(i) *$\langle 1,-a\rangle$ and $d\langle 1,-b\rangle$ represent a common element over $E=F(\sqrt{c})$, but there does not exist an element in $F^\times$ which is represented by $\langle 1,-a\rangle$ and $d\langle 1,-b\rangle$ over $E=F(\sqrt{c})$.*

(ii) *The two quaternion algebras $(a,t_1)_{F_1}$ and $(b,t_1d)_{F_1}$ have a common slot over E_1, but such a common slot cannot be chosen in F_1.*

(iii) *Let $\psi_1:=\langle c,-a,-t_1,t_1a\rangle$ and $\psi_2:=\langle c,-b,-t_1d,t_1db\rangle$. Then there exist $u,v\in F_1^\times$ such that for $L=F_1(\sqrt{u})$ one has $(\psi_1)_L\simeq v(\psi_2)_L$, but there does not exist a binary form over F_1 which is similar to a subform of both ψ_1 and ψ_2.*

(iv) *The Clifford invariant of the form $\psi:=\psi_1\perp -t_2\psi_2\in I^2F_2$ can be represented by a biquaternion algebra A over F_2, but ψ does not contain any subform in GP_2F_2.*

(v) *Let α be the Albert form over F_2 associated to A, and let $\varphi:=\alpha\perp t_3\psi$. Then $\varphi\in I^3F_3$, $\dim\varphi=14$, but φ is not similar to the difference of the pure parts of two forms in P_3F_3.*

Proof. Let $d=rs$, where $r\in N_E(a)$ and $s\in N_E(b)$. By multiplicativity of the norm form, we have $s^{-1}\in N_E(b)$, and the equality $r=ds^{-1}$ shows that $r\in D_E(\langle 1,-a\rangle)$ is represented by $d\langle 1,-b\rangle$. Suppose $D_E(\langle 1,-a\rangle)\cap D_E(d\langle 1,-b\rangle)$ contains an element $x\in F^\times$; then $x\in F^\times\cap N_E(a)$ and $x=dy$ for some $y\in N_E(b)$. Since $y=d^{-1}x\in F^\times$, we have $y\in F^\times\cap N_E(b)$. It follows that $d\in(F^\times\cap N_E(a))(F^\times\cap N_E(b))$ since $d=xy^{-1}$. This proves (i) (see also [STW, p. 69]). The remaining statements follow from Theorem 4.1 and its proof. □

Part (i) shows that property $D(2)$ fails for F if there exist $a,b,c\in F^\times$ with $N_1(a,b,c)\neq 1$. Actually, tracing back through the proof, it is easily seen that property $D(2)$ is equivalent to the vanishing of the group $N_1(a,b,c)$ for all $a,b,c\in F^\times$ (see [STW, Cor. 2.14]).

The group $N_1(a,b,c)$ occurs in [STW] as the homology group of a certain complex associated with the multiquadratic extension $M=F(\sqrt{a},\sqrt{b},\sqrt{c})$. A more symmetric description of this group is given in [G, Prop. 3]:

$$N_1(a,b,c)\simeq\frac{N_F(a)\cap N_F(b)\cap N_F(c)}{F^{\times 2}N_{M/F}(M^\times)}.$$

As mentioned in the introduction, there exist fields F such that $D(2)$ fails, i.e., there exist $a,b,c\in F^\times$ with $N_1(a,b,c)\neq 1$. In [STW, Cor. 5.6 and 5.7], it is for example shown that $D(2)$ fails for finitely generated extensions of transcendence degree ≥ 2 (resp. ≥ 1) over any field of characteristic 0 (resp. over $\mathbf{Q}$).

Examples where $N_1(a,b,c)\neq 1$ arise in various contexts: in [LW], they are related to transfer ideals: for an arbitrary finite extension K/F, let $\mathcal{T}_{K/F}$ denote the image of the Witt ring WK in WF under the Scharlau transfer map associated with any nonzero linear form $s\colon K \to F$. Leep and Wadsworth show in [LW, Prop. 2.4] that if $N_1(a,b,c) \neq 1$, then for $M = F(\sqrt{a}, \sqrt{b}, \sqrt{c})$ we have

$$\mathcal{T}_{M/F} \neq \mathcal{T}_{F(\sqrt{a})/F} \cap \mathcal{T}_{F(\sqrt{b})/F} \cap \mathcal{T}_{F(\sqrt{c})/F}.$$

The group $N_1(a,b,c)$ is also related to problems in Galois cohomology and to the rationality problem for group varieties: over the field $L = F((t_1))((t_2))((t_3))$, consider the division algebra $D = (a,t_1)_L \otimes (b,t_2)_L \otimes (c,t_3)_L$ and the 8-dimensional quadratic form $q \in I^2L$ such that

$$q \sim \langle\langle a,t_1\rangle\rangle - \langle\langle b,t_2\rangle\rangle - a\langle\langle c,t_3\rangle\rangle.$$

Using the alternative description of $N_1(a,b,c)$ above, it is shown in [KLST, p. 283] and [Me 3, p. 329] that if $N_1(a,b,c) \neq 1$, then

$$L^{\times 2}\mathrm{Nrd}(D^\times) \neq \{x \in L^\times \mid (x) \cup (D) = 0 \text{ in } H^3(L,\mu_2)\},$$

where Nrd is the reduced norm, $(D) \in H^2(L,\mu_2)$ is the Galois cohomology class corresponding to D under the canonical isomorphism mapping $H^2(L,\mu_2)$ to the 2-torsion part of the Brauer group of L, and $(x) \in H^1(L,\mu_2)$ corresponds to $x \in L^\times$ under the canonical isomorphism $H^1(L,\mu_2) \simeq L^\times/L^{\times 2}$. On the other hand, under the same hypothesis, Gille shows in [G] that the adjoint group $\mathrm{PSO}(q)$ over L is not R-trivial, hence not stably L-rational.

To conclude, we illustrate Corollary 6.2 by an explicit example over $\mathbf{Q}(x)$ which is derived from the example given in [STW, Remark 5.4].

EXAMPLE 6.3 Let $F = \mathbf{Q}(x)$ be the rational function field in one variable over the rationals. Then it follows from [STW, Remark 5.4] that $N_1(x+4, x+1, x) \neq 1$ and that the two binary forms $\langle 1, -(x+4)\rangle$ and $2\langle 1, -(x+1)\rangle$ represent a common element over $E = F(\sqrt{x})$, but no element in $F^\times$ is represented by both these forms over E.

In fact, we have

$$\begin{array}{rcl}\langle 1,-(x+4)\rangle \perp -2\langle 1,-(x+1)\rangle & \simeq & \langle 2,-1,-(x+4),2(x+1)\rangle \\ & \simeq & \langle -1,x,2(x+2)(x+4),-2x(x+1)(x+2)\rangle\ ,\end{array}$$

which shows that the difference of these two binary forms becomes isotropic over $E = F(\sqrt{x})$, i.e., the two forms represent a common element over E. Indeed, we can compute such an element directly. We have that

$$\begin{array}{rcl}(\sqrt{x}+2)^2-(x+4) & = & 4\sqrt{x} \in D_E(\langle 1,-(x+4)\rangle), \\ 2(\sqrt{x}+1)^2-2(x+1) & = & 4\sqrt{x} \in D_E(2\langle 1,-(x+1)\rangle),\end{array}$$

and therefore $\sqrt{x} \in D_E(\langle 1,-(x+4)\rangle) \cap D_E(2\langle 1,-(x+1)\rangle)$.

Over $F_1 = F(t_1) = \mathbf{Q}(x,t_1)$, we now define the two 4-dimensional forms

$$\begin{array}{rcl}\psi_1 & = & \langle x,-(x+4)\rangle \perp -t_1\langle 1,-(x+4)\rangle \\ \psi_2 & = & \langle x,-(x+1)\rangle \perp -2t_1\langle 1,-(x+1)\rangle\end{array}$$

and the two quaternion algebras

$$\begin{array}{rcl} Q_1 & = & (x+4, t_1)_{F_1} \\ Q_2 & = & (x+1, 2t_1)_{F_1} \end{array}$$

over F_1. By our construction, we know that Q_1 and Q_2 have a common slot over $E_1 = F_1(\sqrt{x})$, but that no such common slot can be chosen in F_1. A common slot over E_1 is given by $\sqrt{x}t_1$.

Consider now the biquaternion algebra $B = Q_1 \otimes Q_2$ with associated Albert form

$$\beta \simeq \langle x+1, -(x+4)\rangle \perp t_1\langle 1, -x, -2(x+2)(x+4), 2x(x+1)(x+2)\rangle \sim \psi_1 \perp -\psi_2 \ .$$

We then get

$$\begin{array}{rcl} x(x+4)\beta & \simeq & \langle -x, -t_1(x+4), t_1x(x+4)\rangle \\ & & \perp \langle x(x+1)(x+4), -2t_1x(x+2), 2t_1(x+1)(x+2)(x+4)\rangle \end{array}$$

from which we conclude that

$$B = (x, t_1(x+4))_{F_1} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_1} \ .$$

As in the proof of $CS \iff D(4)$ in Theorem 3.4, we get for $u \in F_1^\times$ that $c(\psi_1 \perp -u\psi_2) = [B \otimes (u,x)_{F_1}]$, and by putting $u = t_1(x+4)$, we obtain

$$c(\psi_1 \perp -t_1(x+4)\psi_2) = [(x(x+1)(x+4), -2t_1x(x+2))_{F_1}] \ .$$

Now with $\langle x, -(x+1)\rangle \simeq \langle -1, x(x+1)\rangle$, we obtain

$$\begin{array}{rcl} \psi_1 \perp -t_1(x+4)\psi_2 & \simeq & \langle x, -(x+4), 2(x+4), -2(x+1)(x+4)\rangle \\ & & \perp t_1\langle -1, (x+4), (x+4), -x(x+1)(x+4)\rangle \ . \end{array}$$

Also, $\langle -1, x+4, x+4\rangle \simeq \langle x, -x(x+4), x+4\rangle$ represents $xx^2+(x+4)x^2 = 2x^2(x+2)$. Hence,

$$\begin{array}{rcl} \langle x, 2t_1x^2(x+2)\rangle & \simeq & x\langle 1, 2t_1x(x+2)\rangle \\ & \subset & \psi_1 \perp -t_1(x+4)\psi_2 \ . \end{array}$$

Let $L = F_1(\sqrt{-2t_1x(x+2)})$. The above shows that $\psi_1 \perp -t_1(x+4)\psi_2$ becomes isotropic over L. On the other hand, $[(x(x+1)(x+4), -2t_1x(x+2))_L] = 0$, and it follows that $(\psi_1 \perp -t_1(x+4)\psi_2)_L$ is an isotropic 8-dimensional form in I^3L and hence hyperbolic. Thus, $(\psi_1)_L \simeq (t_1(x+4)\psi_2)_L$. However, by construction there does not exist a binary form over F_1 which is similar to a subform of both ψ_1 and ψ_2.

Let us now consider $\psi := \psi_1 \perp -t_2\psi_2$ over $F_2 = \mathbf{Q}(x, t_1, t_2)$. Then $\psi \in I^2F_2$ is of dimension 8, by construction it does not contain a subform in GP_2F_2, and for its Clifford invariant we get

$$c(\psi) = [B \otimes (t_2, x)_{F_2}] = [(x, t_1t_2(x+4))_{F_2} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_2}] \ .$$

Consider the biquaternion algebra

$$A = (x, t_1t_2(x+4))_{F_2} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_2} \ ,$$

which by our construction is necessarily a division algebra, and an associated Albert form

$$\begin{aligned}\alpha &\simeq \langle -x, -t_1t_2(x+4), t_1t_2x(x+4)\rangle \\ &\quad \perp \langle x(x+1)(x+4), 2t_1(x+1)(x+2)(x+4), -2t_1x(x+2)\rangle\ .\end{aligned}$$

Then, over $F_3 = \mathbf{Q}(x, t_1, t_2, t_3)$, the form $\varphi := \alpha \perp t_3\psi$ is a 14-dimensional form in I^3F_3 which is not similar to the difference of the pure parts of two forms in P_3F_3.

We summarize the above results.

- The two forms $\langle 1, -(x+4)\rangle$ and $2\langle 1, -(x+1)\rangle$ over $\mathbf{Q}(x)$ both represent $\sqrt{x}$ over $\mathbf{Q}(x)(\sqrt{x})$, but there is no element in $\mathbf{Q}(x)^\times$ which is represented by both forms over $\mathbf{Q}(x)(\sqrt{x})$. In particular, $\mathbf{Q}(x)$ does not have property $D(2)$.
- The two quaternion algebras $(x+4, t_1)_{F_1}$ and $(x+1, 2t_1)_{F_1}$ over $F_1 = \mathbf{Q}(x, t_1)$ have a common slot over $\mathbf{Q}(x, t_1)(\sqrt{x})$, for example $t_1\sqrt{x}$, but no such common slot can be chosen in $\mathbf{Q}(x, t_1)$. In particular, $\mathbf{Q}(x, t_1)$ does not have property CS.
- The two forms $\psi_1 = \langle x, -(x+4)\rangle \perp -t_1\langle 1, -(x+4)\rangle$ and $\psi_2 = \langle x, -(x+1)\rangle \perp -2t_1\langle 1, -(x+1)\rangle$ over $\mathbf{Q}(x, t_1)$ do not simultaneously become isotropic over any quadratic extension of $\mathbf{Q}(x, t_1)$, i.e., there is no binary form over $\mathbf{Q}(x, t_1)$ which is similar to a subform of both ψ_1 and ψ_2. However, the forms ψ_1 and $t_1(x+4)\psi_2$ become isometric over $\mathbf{Q}(x, t_1)(\sqrt{-2t_1x(x+2)})$. In particular, $\mathbf{Q}(x, t_1)$ does not have property $D(4)$.
- The Clifford invariant of the 8-dimensional form $\psi = \psi_1 \perp -t_2\psi_2 \in I^2F_2$, where $F_2 = \mathbf{Q}(x, t_1, t_2)$, is represented by the biquaternion algebra

  $$A = (x, t_1t_2(x+4))_{F_2} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_2}\ .$$

  However, ψ does not contain a subform in GP_2F_2. In particular, $\mathbf{Q}(x, t_1, t_2)$ does not have property $D(8)$.
- The extension $F_2(SB(A))/F_2$ is not excellent (cf. Prop. 6.1(i)).
- With

  $$\begin{aligned}\psi &\sim \langle -(x+4), -t_1, t_1(x+4), x, -t_2x, t_2\rangle \\ &\quad - t_2\langle 1, -(x+1), -2t_1, 2t_1(x+1)\rangle\end{aligned}$$

  as above, and with

  $$\begin{aligned}c(\langle -(x+4), -t_1, t_1(x+4), x, -t_2x, t_2\rangle) &= [(x+4, t_1)_{F_2} \otimes (x, t_2)_{F_2}] \\ c(\langle 1, -(x+1), -2t_1, 2t_1(x+1)\rangle) &= [(x+1, 2t_1)_{F_2}]\ ,\end{aligned}$$

  we have that $(x+4, t_1)_{F_2} \otimes (x, t_2)_{F_2} \otimes (x+1, 2t_1)_{F_2}$ is not a division algebra, but $(x+4, t_1)_{F_2} \otimes (x, t_2)_{F_2}$ and $(x+1, 2t_1)_{F_2}$ have no proper common quadratic subextension of $F_2 = \mathbf{Q}(x, t_1, t_2)$ (cf. Prop. 6.1(ii)).
- With α an Albert form associated to A, the form $\alpha \perp t_3\psi$ of dimension 14 over $F_3 = \mathbf{Q}(x, t_1, t_2, t_3)$ is in I^3F_3, but it is not similar to the difference of the pure parts of two forms in P_3F_3. In particular, $\mathbf{Q}(x, t_1, t_2, t_3)$ does not have property $D(14)$.

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