Partition Regular Systems of Linear Inequalities

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Received: Nov. 5, 1997
Revised: July 24, 1998

Communicated by Günter M. Ziegler

Introduction

In 1930 Ramsey published his paper *On a problem in formal logic* [12]. He established a result, nowadays known as Ramsey's Theorem:

> Let k and r be positive integers. Then for every r-coloring of the k-element subsets of ω there exists an infinite subset $S \subseteq \omega$ such that all k-element subsets of S are colored the same.

Already in 1927 van der Waerden published his theorem on arithmetic progressions [15]. He proved that for every coloring of the natural numbers with finitely many colors there exists a monochromatic arithmetic progression of given length. Van der Waerden's result can be seen in the context of Schur's investigations [14] on the distribution of quadratic residues and nonresidues. Schur knew about the existence of monochromatic solutions of $x + y = z$. He worked on such problems in order to resolve Fermat's conjecture, which was proved by Wiles in 1994.

The above mentioned work of Ramsey [12] and van der Waerden [15] gave rise to the part of discrete mathematics, known as Ramsey Theory or Partition Theory. An important contribution was made by Rado [10] in 1933. Working on his dissertation, supervised by Schur, he was able to prove a common generalization of Schur's and van der Waerden's results by introducing the concept of *regularity*: A system of linear equations $A\vec{x} = \vec{0}$ is called regular over a ring R if it has monochromatic solutions for every coloring of R with finitely many colors. In his *Studien zur Kombinatorik* (1933) [10] Rado gave a complete characterization of all regular systems of linear equations over the rational numbers. The property Rado used in order to describe regular systems of linear equations is an syntactical property of the matrix. It is

[1]For this work, the author has been awarded with the Richard-Rado-Preis 1998, which is granted every two years for outstanding dissertations in discrete mathematics by the Fachgruppe Diskrete Mathematik of the Deutsche Mathematiker-Vereinigung.

characterized by certain linear dependences of the columns of the matrix A and is called *column property.*

It is possible to generalize the concept of regularity to systems of linear inequalities. We call a system of linear inequalities $A\vec{x} \leq \vec{0}$ partition regular if for every coloring of the natural numbers with finitely many colors there exists a monochromatic solution of $A\vec{x} \leq \vec{0}$. Rado considered systems of linear inequalities only incidentally. He stated the following proposition which is easy to prove:

> Let the system $\sum_{j=1}^{n} a_{ij}x_j = 0, \quad 1 \leq i \leq m$ be partition regular and assume that the following system of inequalities has a solution in the natural numbers:
>
> $$(*) \quad \sum_{j=1}^{n} a_{ij}x_j \begin{cases} = 0 & \text{for} \quad 1 \leq i \leq m_1, \\ > 0 & \text{for} \quad m_1 < i \leq m. \end{cases}$$
>
> Then also $(*)$ is partition regular.

Of course this observation is far away from being a characterization of partition regular systems of inequalities but it can be taken as a starting point for our investigations.

The characterization of all partition regular systems of linear inequalities is a central goal of this paper. In the first chapter we define a generalized column property called *cpi*, which can be used to characterize partition regular systems of linear inequalities. It is an interesting feature of Rado's proof that the linear system $A\vec{x} = \vec{0}$ is already regular if there exists a monochromatic solution with respect to one (number theoretic) type of coloring. Systems of inequalities let things tend to be more difficult.

Several years after finishing his *Studien zur Kombinatorik*, Rado [11] considered systems of linear equations with coefficients in $\mathbb{R}$ and he also extended the set of partitioned numbers to the field of real numbers. It turned out that it is possible to carry over the previous results from the natural numbers to the reals. We will show in chapter 1 that our arguments can also be used if we consider real systems of inequalities partitioning the set of reals.

As well as for homogeneous systems the column property can be used to describe partition regularity of inhomogeneous systems of inequalities. We will give a complete characterization of those systems which are partition regular, over the natural numbers, over the set of integers and over the rationals.

The column property for systems of inequalities as well as the column property in the sense of Rado is a syntactical property of the matrix and does not explicitly refer to the set of solutions of the system. In 1973 Deuber [1] gave a semantical characterization of partition regular systems of equations. The approach is by a description of the arithmetic structure of the sets of solutions of regular linear systems $A\vec{x} = \vec{0}$. The central notion is the one of $(m,p,c)-$sets. He proved the following theorem:

> A system $A\vec{x} = \vec{0}$ is partition regular if and only if there exist positive integers m,p,c such that every $(m,p,c)-$set contains a solution of $A\vec{x} = \vec{0}$.

In chapter two we will show that $(m, p, c)-$sets can also be used to characterize solution spaces of partition regular systems of linear inequalities.

Starting with results of Erdös and Rado [4] another part of partition theory was developed, which is nowadays known as *Canonical Ramsey Theory*. In Canonical Ramsey Theory one considers colorings with no restriction on the number of colors. The first result is a canonical version of Ramsey's theorem. Later Erdös and Graham [3] proved a generalization of van der Waerden's theorem:

> For every coloring Δ of the natural numbers with arbitrary many colors there exists an arithmetic progression, which is colored monochromatic or injective with respect to Δ.

A canonical analogue of the Rado-Deuber-Theorem on regular systems of equations and (m, p, c)-sets was proved by Lefman [7]. His result states:

> Let $A\vec{x} = \vec{0}$ be a partition regular system of linear equations. For every coloring Δ of the natural numbers with arbitrary many colors there exists a solution of the system $A\vec{x} = \vec{0}$ such that Δ restricted to this solution is either monochromatic, injective or a *block-coloring*.

The third case is related to the partitioning of the columns of A into blocks, corresponding to the column property and to the rows of the (m, p, c)-sets. In chapter 3. we prove a canonical partition theorem for systems of inequalities.

Acknowledgment: I would like to thank Prof. Dr. Walter Deuber for his encouragement and guidance and Dr. Wolfgang Thumser for helpful discussions.

1. Systems of Homogeneous, Linear Inequalities

Notations By $\mathbb{N} = \{1, 2, 3, \ldots\}$ we denote the set of positive integers; $[n] = \{1, 2, \ldots, n\}$ is the set of the natural numbers less or equal than n. A matrix A with m rows and n columns is denoted by $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$, where a_{ij} is the entry of A which belongs to the ith row and jth column. For $i, j \le n$ the jth column of a matrix A is denoted by $a^{(j)}$ the ith row by $a_{(i)}$. For a matrix $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ the system

$$\sum_{j=1}^{n} a_{ij} x_j \le 0, \quad 1 \le i \le m$$

is abbreviated as $A\vec{x} \le \vec{0}$. For a given matrix $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$, $k \le n$ and $\epsilon > 0$ by $A^k(\epsilon) = (a^k_{ij}(\epsilon))_{1 \le i \le m, 1 \le j \le n}$ we denote the following matrix:

$$\begin{pmatrix} a_{11} & \ldots & a_{1k-1} & a_{1k} - \epsilon & a_{1k+1} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mk-1} & a_{mk} - \epsilon & a_{mk+1} & \ldots & a_{mn} \end{pmatrix},$$

obtained from A by subtracting ϵ in column k.

For $k, l \in [n]$, $k < l$ and $\epsilon > 0$ the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1k-1} & a_{1k}-\epsilon & a_{1k+1} & \dots & a_{1l-1} & a_{1l+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mk-1} & a_{mk}-\epsilon & a_{mk+1} & \dots & a_{ml-1} & a_{ml+1} & \dots & a_{mn} \end{pmatrix}$$

obtained by deleting column l in $A^k(\epsilon)$, is denoted by $A^k_l(\epsilon)$ and the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1k-1} & a_{1k}+a_{1l}-\epsilon & a_{1k+1} & \dots & a_{1l-1} & a_{1l+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mk-1} & a_{mk}+a_{ml}-\epsilon & a_{mk+1} & \dots & a_{ml-1} & a_{ml+1} & \dots & a_{mn} \end{pmatrix},$$

obtained from $A^k(\epsilon)$ by adding the kth and the lth column, is denoted by $A^{(k)+(l)}(\epsilon)$.

Rado considered systems of linear equations over $\mathbb{Q}$. In his paper, published in 1933 [10], Rado gives a characterization of all systems of linear homogeneous equations which have for every coloring of the natural numbers with finitely many colors a solution in one color class. Rado called those systems regular. The central definition in this context is the following:

DEFINITION 1.1. *Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ be a matrix with m rows an n columns and with entries $a_{ij} \in \mathbb{Z}$. A has the column property if there exists $l \in \mathbb{N}$ and a partition $[n] = I_0 \cup I_1 \cup \dots I_l$ of the column indices such that*

1. *for all $1 \le i \le m$ we have $\sum_{j\in I_0} a_{ij} = 0$ and*
2. *for all $k < l, j \in \cup_{s\le k} I_s$ there exist $c_k, c_{kj} \in \mathbb{N}$ such that for all $1 \le i \le m$ we have*
$$\sum_{j\in\cup_{s\le k}I_s} c_{jk}a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij} = 0.$$

Rado proved the following theorem:

THEOREM 1.1. (RADO 1933) *A system of homogeneous linear equations $A\vec{x} = \vec{0}$ is regular if and only if A has the column property.*

In the following we will consider systems of linear inequalities rather than systems of linear equations. First we define partition regularity for systems of inequalities.

DEFINITION 1.2. *Let $A = (a_{ij})_{1\le i\le m 1\le j\le n}$ be a rational matrix and let $\vec{b} = (b_1, \dots, b_m) \in \mathbb{Q}^m$. The system*

$$(*) \qquad \sum_{j=1}^{n} a_{ij}x_j \le b_i, \quad 1 \le i \le m$$

is called partition regular over $\mathbb{N}$ *if for every* $c \in \mathbb{N}$ *and every c-coloring of the natural numbers* $\Delta : \mathbb{N} \to [c]$ *there exists a solution* $\vec{x} = (x_1, \dots x_n) \in \mathbb{N}^n$ *of* $(*)$ *such that* $\Delta \big|_{\{x_1,\dots,x_n\}} = const.$

In the following section we will give a characterization of all systems of homogeneous linear inequalities which are partition regular over $\mathbb{N}$. It turns out that a natural generalization of Rado's column property can be used to describe these systems.

Definition 1.3. *Let* $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ *be a rational matrix. A has the column property for systems of inequalities (abbreviated as cpi) over* $\mathbb{N}$ *if there exists* $l \in \mathbb{N}$ *and a partition* $[n] = I_0 \cup I_1 \dots \cup I_l$ *such that*

1. *for all* $1 \le i \le m$ *we have* $\sum_{j\in I_0} a_{ij} \le 0$ *and*
2. *for all* $k < l, j \in \cup_{s\le k} I_s$ *there exist* $c_k, c_{jk} \in \mathbb{N}$ *such that for all* $1 \le i \le m$ *we have*
$$\sum_{j\in \cup_{s\le k} I_s} c_{kj} a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij} \le 0.$$

If a matrix A has the column property (in the sense of Rado) [10] the system $A\vec{x} \le \vec{0}$ obviously is partition regular. But there are many other systems of inequalities which are partition regular without A having Rado's column property. For example the matrix
$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
has *cpi* but not the column property.

Theorem 1.2. *Let* $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ *be a rational matrix. The system of inequalities* $(*)$ $A\vec{x} \le \vec{0}$ *is partition regular over* $\mathbb{N}$ *if and only if A has cpi over* $\mathbb{N}$.

Both implications stated in theorem 1.5. are not completely trivial to prove. We start by showing that *cpi* implies partition regularity. This part of the proof proceeds along the general lines of the corresponding proof for systems of equations [10]. The following lemma combines arithmetic progressions and partition regular systems of linear inequalities:

Lemma 1.1. *Let* $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ *be a rational matrix,* $A\vec{x} \le \vec{0}$ *a partition regular system of inequalities and let* $p \in \mathbb{N}$. *Then for every* $c \in \mathbb{N}$ *and every c-coloring* $\Delta : \mathbb{N} \to [c]$ *there exists* $\vec{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$ *and* $d \in \mathbb{N}$ *such that*

1. $A\vec{x} \le \vec{0}$ *and*
2. *for all* $i, j \le n$, *for all* $k, l \le p$ *we have* $\Delta(x_i + ld) = \Delta(x_j + kd)$.

Proof of lemma 1.1.: $A\vec{x} \le \vec{0}$ is partition regular. Thus by compactness [6] for every $c \in \mathbb{N}$ there exists $N^* = N^*(c) \in \mathbb{N}$ such that for every c-coloring $\Delta : [N^*] \to [c]$ there exists a monochromatic solution $\vec{x} = (x_1 \dots x_n)$ of $A\vec{x} \le \vec{0}$ such that for all $1 \le i \le n$ we have $x_i \le N^*$.

Let $\Delta : \mathbb{N} \to [c]$ be an arbitrary c-coloring. Define the following coloring $\Delta^* : \mathbb{N} \to [r^{N^*}]$ by

$$\Delta^*(x) = (\Delta(ix))_{1 \leq i \leq N^*}.$$

By van der Waerden's theorem [15] there exists a "long" arithmetic progression which is monochromatic with respect to Δ^*, i. e. there exist $a', d' \in \mathbb{N}$ such that for all $l \leq pN^{*^{n-1}}$ we have $\Delta^*(a' + ld')$ =const.
Define $\Delta^{**} : \mathbb{N} \to [c]$ by

$$\Delta^{**}(x) = \Delta(a'x).$$

By the choice of N^* there exists a solution $\vec{x'} = (x'_1, \ldots, x'_n) \in [N^*]^n$ of $A\vec{x} \leq \vec{0}$ which is monochromatic for Δ^*. For all $i \leq n$ let $x_i = x'_i a'$. By homogeneity $\vec{x} = (x_1, \ldots x_n)$ is a solution of $A\vec{x} \leq \vec{0}$ and because of the definition of Δ^{**} for all $i, j \leq n$ we have $\Delta(x_i a') = \Delta(x_j a')$.
Let $d = d' x'_1 \ldots x'_n$. Then for $i \leq n$ and $l \leq p$ we have:

$$x'_i a' + ld = x'_i(a' + ld' x'_1 \ldots x'_{i-1} x'_{i+1} \ldots x'_n).$$

Hence by the definition of a', d' and Δ^* for all $l \leq p$ we have $\Delta(x'_i a' + ld)$ =const. $\square_{lemma\ \ 1.6.}$

Proof of theorem 1.2. (first part): First we show that if A has *cpi* over $\mathbb{N}$ then $(*)$ is partition regular. We know by assumption that there is some $l \in \mathbb{N}$ and a partition $[n] = I_0 \cup I_1 \cup \ldots \cup I_l$ such that

1. for all $1 \leq i \leq m$ we have $\sum_{j \in I_0} a_{ij} \leq 0$ and
2. for all $k < l$, for all $j \in \cup_{s \leq k} I_s$ there exist $c_{kj}, c_k \in \mathbb{N}$, such that for all $1 \leq i \leq m$ we have
$$\sum_{j \in \cup_{s \leq k} I_s} c_{kj} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij} \leq 0.$$

To prove that $(*)$ is partition regular we will use a double induction. We proceed by main induction on the number of colors c and by subsidiary induction on l, the number of column classes.

Let $A_k = (a_{ij})_{1 \leq i \leq m, j \in \cup_{s \leq k} I_s}$ be the submatrix of A which only consists of the columns belonging to block 1 up to k. We will show by induction that for all $k \leq l$ A_k is partition regular.

For $k = 0$ there is nothing to show because every singleton forms a solution of the system $A_0\vec{x} \leq \vec{0}$. Assume that $A_k\vec{x} \leq \vec{0}$ is partition regular for some $k \geq 0$ (which will be kept fix by now), i. e. (by compactness) for every $c \in \mathbb{N}$ there exists $R(c, A_k) \in \mathbb{N}$ such that for every c-coloring $\Delta : [R(c, A_k)] \to [c]$ there exists a monochromatic solution $(x_j)_{j \in \cup_{s \leq k} I_s}$, such that $A_k\vec{x} \leq \vec{0}$ and for all $j \in \cup_{s \leq k} I_s$ we have $x_j \leq R(c, A_k)$. We will show that A_{k+1} is partition regular, i. e. for all $c \in \mathbb{N}$ there exists $R(c, A_{k+1}) \in \mathbb{N}$

First we observe that $x_j = c_{jk}$ for $j \in \cup_{s \le k} I_s$ and $x_j = c_k$ for $j \in I_{k+1}$ form a solution of the system $A_{k+1}\vec{x} \le \vec{0}$. So we are done if only one color is used for the coloring, i. e. there exists $R(1, A_{k+1})$. Now assume that $R(c, A_{k+1})$ exists for some (fixed) $c \ge 1$. We will show that $R(c+1, A_{k+1})$ exists.

Let $\Delta : \mathbb{N} \to [c+1]$ be an arbitrary (c+1)-coloring. Use lemma 1.6. for the (by assumption) partition regular system $A_k\vec{x} \le \vec{0}$ with
$p = R(c, A_{k+1}) \cdot (max_{j \in \cup_{s \le k} I_s}\{c_{kj}\})$. Hence there exists $(y_j)_{j \in \cup_{s \le k} I_s}$, such that for all $1 \le i \le m$ we have

$$\sum_{j \in \cup_{s \le k} I_s} a_{ij} y_j \le 0$$

and there exists $d \in \mathbb{N}$ such that for all $j \in \cup_{s \le k} I_s$ and $t \le p$ we have

$$\Delta(y_j + td) = const.$$

for all $1 \le i \le m$ and $t \in [R(c, A_{k+1})]$ it follows

$$\begin{aligned} & \sum_{j \in \cup_{s \le k} I_s} (y_j + c_{kj}td)a_{ij} + \sum_{j \in I_{k+1}} c_k t d a_{ij} \\ = & \sum_{j \in \cup_{s \le k} I_s} y_j a_{ij} + td(\sum_{j \in \cup_{s \le k} I_s} c_{kj} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij}) \quad \le 0. \end{aligned}$$

Further for all $j \in \cup_{s \le k} I_s$ and $t \le p$ we have

$$\Delta(y_j + c_{kj}td) = const.$$

Say $\Delta(y_j + c_{kj}td) = c+1$.

We distinguish the following cases:

1. There exist $t \in [R(c, A_{k+1})]$ such that $\Delta(c_k td) = c+1$. Then we are done.

2. For all $t \in [R(c, A_{k+1})]$ the relation $\Delta(c_k td) \in [c]$ holds. Then consider the c-coloring: $\Delta' : [R(c, A_{k+1})] \to [c]$ which is defined by

   $$\Delta'(x) = \Delta(c_k x d).$$

   By definition of $R(c, A_{k+1})$ there exists a solution $(t_j)_{j \in \cup_{s \le k} I_s}$ of the system $A_{k+1}\vec{x} \le \vec{0}$ which is monochromatic for Δ'. Hence $(c_k d t_j)_{j \in \cup_{s \le k+1} I_s}$ forms a solution of $A_{k+1}\vec{x} \le \vec{0}$ which is monochromatic with respect to Δ.

$\square_{theorem \quad 1.2.(first \quad part)}$

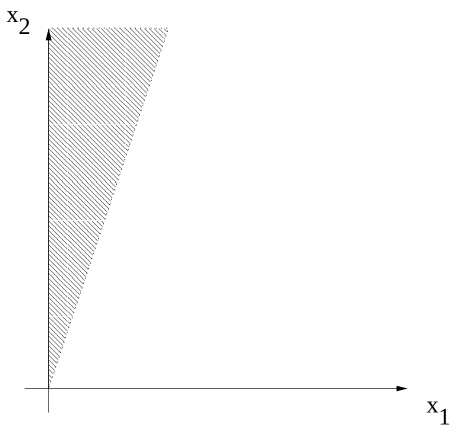
In order to demonstrate the structure of the proof of the second part of theorem 1.5. we will give a short overview. For his characterization of regular systems of linear equations Rado [10] had to prove that for each systems $A\vec{x} = \vec{0}$, which is regular, A has the column property. It is an interesting feature of Rado's proof that a system $A\vec{x} = \vec{0}$ is regular if there exists a monochromatic solution with respect to one type of coloring. For systems of linear inequalities $A\vec{x} \le \vec{0}$ with A having only two columns there also exists a certain type of coloring such that $A\vec{x} \le \vec{0}$ is partition regular if it has a monochromatic solution with respect to this type of coloring. In lemma 1.12. we will show, that a system $(*) \quad a \le \frac{x_1}{x_2} \le b$, where $a, b \in \mathbb{Q}$ and $1 < a \le b$, is not partition regular. It is easy to see that essentially each system $A\vec{x} \le \vec{0}$ with A having only two columns can be transformed into a system $(*)$ for suitable a and b. If such a system is partition regular this means that one of the following cases holds:

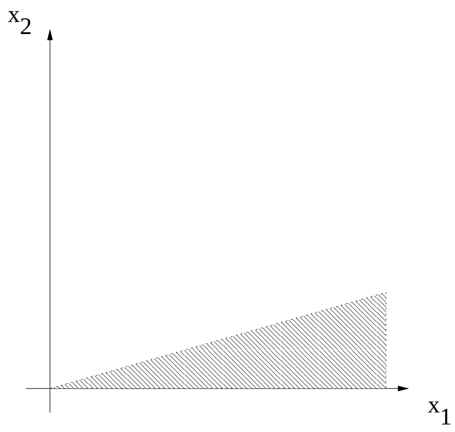
1. $a \le 0$ and $b > 0$ or
2. $a \le 1$ and $b \ge 1$.

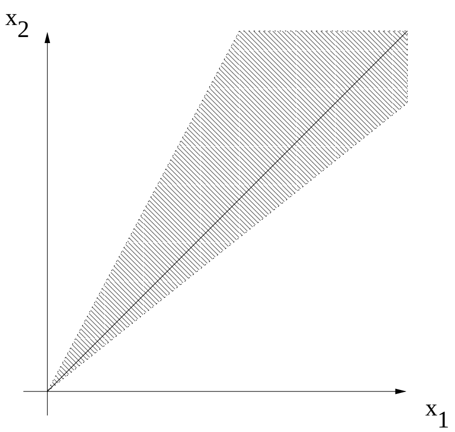
It is not difficult to see that these conditions exactly lead to *cpi*. If we visualize a partition regular system

$$(**) \begin{cases} a_{11}x_1 + a_{12}x_2 \le 0 \\ a_{21}x_1 + a_{22}x_2 \le 0 \end{cases}$$

geometrically then obviously the solutions are bounded by two straight lines. Three typical cases occur, i.e. one of the axes is a limiting line or the diagonal is contained in the solution space:







We will prove theorem 1.5. by induction on the number of columns of A. In order to start the induction we described the situation for $n = 2$. Let us consider a rational matrix A with n columns. Assume that the system

$$A\vec{x} \leq \vec{0} \quad (***)$$

is partition regular. Under certain assumptions we can transform the system $A\vec{x} \leq \vec{0}$ for each choice of k, l with $1 \leq k < l \leq n$ into the following system:

$$-\frac{a_{sl}}{a_{sk}} - \sum_{j=1, j\neq l,k}^{n} \frac{a_{sj}}{a_{sk}} \frac{x_j}{x_l} \leq \frac{x_k}{x_l} \leq -\frac{a_{tl}}{a_{tk}} - \sum_{j=1, j\neq l,k}^{n} \frac{a_{tj}}{a_{tk}} \frac{x_j}{x_l}$$

for all s with $a_{sk} < 0$ and for all t with $a_{tk} > 0$. Thus we have a similar situation as in $(*)$ except that the fraction $\frac{x_k}{x_l}$ is not bounded by constant terms a and b but by terms which depend on $x_1 \ldots x_{k-1}, x_{k-2}, \ldots x_n$. Thus we cannot directly apply lemma 1.12. Consider this situation for fixed k and l. Assume that there are colorings of the natural numbers with finitely many colors such that for each monochromatic solution $x_1, \ldots x_n$ of the system $(***)$ either

1. there exists $\epsilon_1 > 0$ and $r \in \mathbb{N}$ such that $1 + \epsilon_1 \leq \frac{x_k}{x_l} \leq r$ or
2. there exists $\epsilon_2 > 0$ and $\epsilon_3 > 0$ such that $\epsilon_2 \leq \frac{x_k}{x_l} \leq 1 - \epsilon_3$.

Then again by lemma 1.12. $(***)$ cannot be partition regular. To avoid such situations the terms $-\frac{a_{sl}}{a_{sk}} - \sum_{j=1,j\neq l,k}^{n} \frac{a_{sj}}{a_{sk}} \frac{x_j}{x_l}$ and $-\frac{a_{tl}}{a_{tk}} - \sum_{j=1,j\neq l,k}^{n} \frac{a_{tj}}{a_{tk}} \frac{x_j}{x_l}$ have to fulfill certain conditions for every coloring. This is what is shown in lemma 1.13. With this kind of arguments it is possible to show that for every choice of k and l with $1 \leq k < l \leq n$ either for all $\epsilon > 0$ the system $A_l^k(\epsilon)$ is partition regular or for all $\epsilon > 0$ the system $A_k^l(\epsilon)$ is partition regular, if the system $A\vec{x} \leq \vec{0}$ is partition regular. By induction we can conclude that either for all $\epsilon > 0$ the matrix $A_l^k(\epsilon)$ has *cpi* or for all $\epsilon > 0$ the matrix $A_k^l(\epsilon)$ has *cpi*. Therefore we define:

Definition 1.4. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix. A has the ϵ-property if the following conditions are satisfied:*

1. *The system $A\vec{x} \leq \vec{0}$ has a solution in the natural numbers and*
2. *For all $1 \leq k < l \leq n$ one of the following conditions is satisfied:*
   (a) *For all $\epsilon > 0$ the matrix $A^k(\epsilon)$ has cpi over $\mathbb{N}$,*
   (b) *for all $\epsilon > 0$ the matrix $A^l(\epsilon)$ has cpi over $\mathbb{N}$,*

   i.e. for at most one r with $1 \leq r \leq n$ there is an $\epsilon_0 > 0$ such that $A^r(\epsilon_0)$ has not cpi.

Note that if the matrix $A^k(\epsilon_0)$ has *cpi* for some $\epsilon_0 > 0$ then for all $\epsilon \geq \epsilon_0$ $A^k(\epsilon)$ has *cpi*.

Remark 1.1. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix, such that $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Let $1 \leq k < l \leq n$.*

1. *If the matrix $A_l^k(\epsilon)$ has cpi then $A^k(\epsilon)$ has cpi.*

   A_l^k has cpi. Let $I_0, \ldots, I_r$ be the corresponding partition of the column indices. Define $I_{r+1} = \{l\}$. Then $I_0, \ldots, I_{r+1}$ is a partition of $[n]$ which proves cpi for $A^k(\epsilon)$.

2. *If the matrix $A^{(k)+(l)}(\epsilon)$ has cpi then the matrices $A^k(\epsilon)$ and $A^l(\epsilon)$ have cpi.*

   Let the blocks for $A^{(k)+(l)}(\epsilon)$ be $I'_0, \ldots, I'_q$ and assume that the column

$$a^{(k')}(\epsilon) = \begin{pmatrix} a_{1k} + a_{1l} - \epsilon \\ a_{2k} + a_{2l} - \epsilon \\ \vdots \\ a_{mk} + a_{ml} - \epsilon \end{pmatrix}$$

   belongs to the block I'_p. Then $A^k(\epsilon)$ and $A^l(\epsilon)$ have cpi with the corresponding blocks being $I_r = I'_r$ for $r \neq p$ and $I_p = I'_p - \{k'\} \cup \{k, l\}$.

Up to now we did not succeed in proving that A has *cpi*, but we know that if we transform A only a little then the transformed matrix has *cpi* and it is possible to do this transformations in nearly each column. What we will show in lemma 1.9. is that the property *cpi* is continuous in a certain manner.

LEMMA 1.2. *If $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ is a rational matrix, which satisfies the ϵ-property, then A has cpi.*

In order to prove lemma 1.9. we need the following lemma:
LEMMA 1.3. *Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ be a rational matrix such that for all $1 \le i \le m$ the entries of row i sum up to zero, i.e. $\sum_{j=1}^n a_{ij} = 0$. Let $s_1, \ldots, s_m \in \mathbb{Q}$. For all $\epsilon > 0$ let $A'(\epsilon) = (a'_{ij}(\epsilon))_{1\le i\le m, 1\le j\le n+1}$, be the matrix with entries $a'_{ij}(\epsilon) = a_{ij}$ for $1 \le i \le m, 1 \le j \le n$ and $a_{in+1} = s_i - \epsilon$ for $1 \le i \le m$. Further let $A' = A'(0)$.*
If for all $\epsilon > 0$ the system $A'(\epsilon)\vec{x} \le \vec{0}$ has a solution in $\mathbb{N}$, then the system $A'\vec{x} \le \vec{0}$ has a solution in $\mathbb{N}$.

PROOF OF LEMMA 1.3.: Let A, $A'(\epsilon)$ and A' be as in the assumptions of lemma 1.10. Assume that for all $1 \le i \le m$ we have $\sum_{j=1}^n a_{ij} = 0$. Thus the system $A\vec{x} \le \vec{0}$ can be transformed into the following system

$$(*) \qquad \sum_{j=1}^{n-1} a_{ij}(x_j - x_n) \le 0, \quad 1 \le i \le m,$$

which will be abbreviated in the following as $A^*\vec{y} \le \vec{0}$, where $A^* = (a_{ij})_{1\le i\le m, 1\le j\le n-1}$, and $y_j = x_j - x_n$ for $1 \le j \le n-1$.
The system $A\vec{x} \le \vec{0}$ (resp. $A\vec{x} < \vec{0}$) has a solution in $\mathbb{N}$ if and only if $(*)$ (resp. $A^*\vec{y} < 0$) has a solution in $\mathbb{Z}$.

In the following we will consider A^* instead of A. (The entries of A^* will be denoted without $*$.) Assume that the set of rows of A^* is linear independent over $\mathbb{Q}$. Then there exists $\vec{y} = (y_1, \ldots y_{n-1}) \in \mathbb{Q}^{n-1}$ such that $A^*\vec{y} < \vec{0}$. Multiplication with the least common multiple of the denominators of y_j yields a solution $\vec{y}' = (y'_1 \ldots y'_{n-1}) \in \mathbb{Z}^{n-1}$ of the system $A^*\vec{y} < 0$. Thus the system $A\vec{x} < \vec{0}$ has a solution in $\mathbb{N}$ and therefore $A'\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Hence we are done in this case.

Next we consider the case where the set of rows of A^* is not linear independent. Assume that A^* consists of the rows $a_{(1)}, \ldots a_{(k)}, b_{(k+1)}, \ldots, b_{(m)}$ for some $k \geq 0$, where $a_{(1)}, \ldots, a_{(k)}$ are linear independent and for all $k+1 \leq i \leq m$ we have $b_{(i)} = \sum_{s=1}^{k} c_s^i a_{(s)}$ for suitable $c_s^i \in \mathbb{Q}$.

We will prove the lemma by induction on k. If $k = 0$ then A^* is the zero-matrix. Hence A is the zero-matrix and therefore the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$ if and only if for all $1 \leq i \leq m$ we have $s_i - \epsilon \leq 0$. This is true for all $\epsilon > 0$ by assumption and therefore for all $1 \leq i \leq m$ we have $s_i \leq 0$.
If $k = 1$ for all $2 \leq i \leq m$ we have $b_{(i)} = c_1^i a_{(i)}$ for suitable $c_1^i \in \mathbb{Q}$. We distinguish the following cases:

1. for all $2 \leq i \leq m$ we have $c_1^i > 0$.

   If $a_{(1)}\vec{y} < 0$ holds then for all $2 \leq i \leq m$ we have $b_{(i)}\vec{y} < 0$. Because $a_{(1)}$ is not the zero-vector there exists a solution $\vec{y} \in \mathbb{Z}^n$ such that $A^*\vec{y} < \vec{0}$ and hence we are done in this case.

2. There exists i such that $c_1^i = 0$.

   In this case we have $b_{(i)} = \vec{0}$ and the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution only if $s_i - \epsilon \leq 0$. Because this is true for every $\epsilon > 0$, we have $s_i \leq 0$. Hence $(b_{(i)} s_i)\vec{x} \leq \vec{0}$ is true for every choice of $\vec{x}$ where $x_{n+1} \geq 0$. Therefore the matrix keeps its properties if we omit the row $b_{(i)}$.

3. There exists i such that $c_1^i < 0$.

   Let i be arbitrary with $c_1^i < 0$. By assumption we know that for every $\epsilon > 0$ the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution. Let $\vec{x}(\epsilon) = (x_1(\epsilon), \ldots, x_n(\epsilon)), x(\epsilon)$ be one specific solution of the system $A'(\epsilon)\vec{x} \leq \vec{0}$, i. e.

   $$a_{(1)}\vec{x}(\epsilon) + (s_1 - \epsilon)x(\epsilon) \leq 0,$$

   which is equivalent to

   $$\sum_{j=1}^{n} a_{1j}x_j(\epsilon) \leq -(s_1 - \epsilon)x(\epsilon)$$

   and correspondingly we have

   $$b_{(i)}\vec{x}(\epsilon) + (s_i - \epsilon)x(\epsilon) \leq 0,$$

   which is equivalent to

   $$c_1^i(\sum_{j=1}^{n} a_{1j}x_j(\epsilon)) \leq -(s_i - \epsilon)x(\epsilon)$$

Dividing by $c_1^i > 0$ we obtain

$$\sum_{j=1}^{n} a_{1j} x_j(\epsilon) \geq -\frac{s_i - \epsilon}{c_1^i} x(\epsilon).$$

Hence a solution $x_1(\epsilon), \ldots, x_n(\epsilon)$ exists if and only if

$$-\frac{s_i - \epsilon}{c_1^i} \leq s_1 - \epsilon,$$

which means

$$s_i \leq -c_1^i s_i + (c_1^i - 1)\epsilon.$$

This is true for all $\epsilon > 0$ and hence

$$s_i \leq -c_1^i s_1$$

holds.

Thus the statement is true for $k = 1$.

Assume that our statement is true for some (fixed) $k \geq 1$. Let A^* consist of the rows $a_{(1)}, \ldots, a_{(k+1)}, b_{(k+2)}, \ldots, b_{(m)}$, where $a_{(i)}$ are linear independent and for $k+1 \leq i \leq m$ let

$$b_{(i)} = \sum_{s=1}^{k+1} c_s^i a_{(s)}$$

for suitable $c_s^i \in \mathbb{Q}$. Further assume that for every $\epsilon > 0$ the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. We distinguish the following cases:

1. There exists $1 \leq s \leq k+1$ such that for all $k+2 \leq i \leq m$ we have $c_s^i > 0$.

   Let $c = max_{k+1<i<m, 1\leq l\leq k, l\neq s}|c_l^i|$. $a_{(1)}, \ldots, a_{(k+1)}$ are linearly independent by assumption. Hence there exists $\vec{y} = (y_1, \ldots, y_n)$ such that for all $1 \leq i \leq k$ we have $a_{(i)}\vec{y} < 0$ and

   $$min_{k+1\leq i\leq m}|c_s^i(a_{(s)}\vec{y})| > c \cdot (max_{1\leq l\leq k, l\neq s}|a_{(l)}\vec{y}|)(k-1).$$

   Then $y_1, \ldots y_{n-1}$ form a solution for the whole system $A^*\vec{y} < \vec{0}$ and hence $A'\vec{x} \leq \vec{0}$ has a solution.

2. There exists s such that for all $k+1 \leq i \leq m$ we have $c_s^i \geq 0$ and $c_s^i = 0$ for at least one i.

   Without loss of generality let $s = 1$ and $c_1^i > 0$ for $k+1 \leq i \leq l$ and $c_1^i = 0$ for $l < i \leq m$. Then the matrix which consists of the rows $a_{(1)}, \ldots, a_{(k+1)}, b_{(k+2)}, \ldots, b_{(l)}$ is dealt within case 1. But the rows $b_{(l+1)}$ up to $b_{(m)}$ only depend on the $k-1$ generators $a_{(2)}$ up to $a_{(k+1)}$. Hence by induction we obtain a solution $y_1, \ldots, y_n$ for the rows $a_{(2)}, \ldots, a_{(k+1)}, b_{(k+2)}, \ldots, b_{(m)}$ which are independent of $a_{(1)}$. Thus we also obtain a solution for the whole system.

3. For $a \leq i \leq k+1$ we define
$$c_j^i = \begin{cases} 1 & \text{for} \quad j = i \\ 0 & \text{for} \quad j \neq i. \end{cases}$$

Then it remains to consider the case where there exist $1 \leq i_1, i_2 \leq m$ and there exists $1 \leq s \leq k+1$ such that $c_s^{i_1} > 0$ and $c_s^{i_2} < 0$.

Without loss of generality let $s = 1$. Further we can divide the entries of each row i by $|c_1^i|$, if $|c_1^i| \neq 0$, such that we may assume that $|c_1^i| = 1$ for each i, where $|c_1^i| \neq 0$.

For every $\epsilon > 0$ the system $A^*(\epsilon)'\vec{x} \leq \vec{0}$ has a solution. Let $\vec{y}^\epsilon = (y_1^\epsilon, \ldots, y_{n-1}^\epsilon), x^\epsilon$ be such a solution, i. e.

$$\sum_{s=1}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon) + (s_i - \epsilon)x^\epsilon \leq 0 \quad \text{for } k+2 \leq i \leq m$$

and
$$a_{(i)}\vec{y}^\epsilon \leq -(s_i - \epsilon)x^\epsilon \quad \text{for } 1 \leq i \leq k+1.$$

Thus we have
$$\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y})^\epsilon + (s_i - \epsilon)x^\epsilon \leq -c_1^i a_{(1)}\vec{y}^\epsilon.$$

Dividing by $-c_1^i$ leads to

$$\sum_{s=2}^{k+1} c_s^r(a_{(s)}\vec{y}^\epsilon) + (s_r - \epsilon)x^\epsilon \leq a_{(1)}\vec{y}^\epsilon \leq -\sum_{s=2}^{k+1} c_s^j(a_{(s)}\vec{y}^\epsilon) - (s_j - \epsilon)x^\epsilon$$

for all r with $c_1^r = -1$ and j with $c_1^j = 1$. Further we know that $a_{(1)}\vec{y} \leq -(s_1 - \epsilon)x^\epsilon$. Hence we additionally obtain:

$$\sum_{s=2}^{k+1} c_s^r(a_{(s)}\vec{y}^\epsilon) \leq -(s_1 - \epsilon)x^\epsilon$$

for all i satisfying $c_1^i = -1$ and

$$\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon) \leq -(s_i - \epsilon)x^\epsilon$$

for all i satisfying $c_1^i = 0$. Transforming these inequalities we get the following system of inequalities:

$$(***) \begin{cases} a_{(i)}\vec{y}^\epsilon + (s_i - \epsilon)x^\epsilon \leq 0 & 2 \leq i \leq k+1 \\ (\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon)) + (s_i - \epsilon)x^\epsilon \leq 0 & \text{for all } i \text{ with} \\ & c_1^i = 0 \\ (\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon)) + (s_i + s_1 - 2\epsilon)x^\epsilon \leq 0 & \text{for all } i \text{ with} \\ & c_1^i = -1 \\ (\sum_{s=2}^{k+1}(c_s^i + c_s^j)(a_{(s)}\vec{y}^\epsilon)) + (s_i + s_j - 2\epsilon)x^\epsilon \leq 0 & \text{for all } i,j \text{ with} \\ & c_1^i = -1, c_1^j = 1 \end{cases}$$

By assumption we know that for all $\epsilon > 0$ the system $A^*(\epsilon)'\vec{x} \leq \vec{0}$ has a solution. Hence the system $(***)$ has a solution for every $\epsilon > 0$. In system $(***)$ only k row vectors are linear independent, namely $a_{(2)}, \ldots, a_{(k+1)}$. Thus we can use induction to show that the system $(***)$ has a solution for $\epsilon = 0$. Thus the system $A'\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$.

$\Box_{lemma \quad 1.2.}$

CLAIM 1.1. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix which has cpi with the first block being $I_0^A = \{1, \ldots k\}$ and $\sum_{j=1}^k a_{ij} = 0$. Let*

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{l1} & b_{l2} & \ldots & b_{ln} \\ & & A & \end{pmatrix}$$

such that for all $1 \leq i \leq l$ the relation $\sum_{j=1}^k b_{ij} < 0$ holds. Then B has cpi.

PROOF OF CLAIM 1.1.: Obviously $I_0^B = \{1, \ldots, k\}$ satisfies the first condition of *cpi*. Let $I_0^A, \ldots, I_v^A$ be the partition of columns of A and for $1 \leq r < v, j \in \cup_{s\leq r} I_s$ let $c_{rj}^A, c_r^A \in \mathbb{N}$ be the corresponding coefficients . Let the parameters $b(r), \delta, B(r), c(r) \quad 1 \leq r \leq v$ be "big enough", in particular we define:

$$\begin{aligned} b(r) &= max_{1\leq i\leq l}\{\sum_{j\in I_{r+1}^A} b_{ij},\} \\ \delta &= max_{1\leq i\leq l}\{\sum_{j=1}^k b_{ij}\} \quad (< 0), \\ B(r) &= max_{1\leq i\leq l}\{\sum_{j\in\cup_{w\leq r} I_w^A} |b_{ij}|\}, \\ c(r) &= max_{j\in\cup_{w\leq r} I_w^A}\{c_{rj}^A, c_r^A\}. \end{aligned}$$

and let $a(r) \in \mathbb{N}$ be minimal such that

$$a(r)\delta \leq -(c(r)B(r) + c_r b(r)).$$

Such an a=a(r) exists because δ is negative. Let $c_{rj}^B = c_{rj}^A + a$ if $j \leq k$ and $c_{rj}^B = c_{rj}$ otherwise. For $1 \leq r \leq v$ let $c_r^B = c_r^A$ and $I_r^B = I_r^A$. Then for all $1 \leq i \leq l$ we have:

$$\sum_{j=1}^k (a + c_{rj})b_{ij} + \sum_{j\in\cup_{w\leq r} I_w^A, j>k} c_{rj}b_{ij} + c_r \sum_{j\in I_{r+1}^A} b_{ij}$$

$$= a\sum_{j=1}^k b_{ij} + \sum_{j=1}^k c_{rj}b_{ij} + \sum_{j\in\cup_{w\leq r} I_w^A, j>k} c_{rj}b_{ij} + c_r \sum_{j\in I_{r+1}^A} b_{ij}$$

$$\leq a\delta + c(r)B(r) + c_r b(r) \leq 0.$$

Further for all $1 \leq i \leq l$ we have:

$$\begin{aligned} \sum_{j=1}^{k}(a + c_{ij})a_{ij} &= (\sum_{j=1}^{k} c_{rj}a_{ij}) + a(\sum_{j=1}^{k} a_{ij}) \\ &= \sum_{j=1}^{k} c_{rj}a_{ij}. \end{aligned}$$

Hence B has *cpi*. $\Box_{claim}$ 1.1.

PROOF OF LEMMA 1.2.: Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix which has the ϵ-property, i. e. for all $1 \leq k < l \leq n$ either $A^k(\epsilon)$ or $A^l(\epsilon)$ has *cpi* for every $\epsilon > 0$. We will prove that A has *cpi*. If the matrix $A^k(\epsilon)$ has *cpi* for some $k \leq n$, let $I_0^k, \ldots, I_{l_k}^k$ be a partition of columns of $A^k(\epsilon)$, which certifies *cpi*. We can assume that the partition of $[n]$ into blocks does not depend on ϵ because there are only finitely many possibilities of partitioning $[n]$ into blocks. By the pigeonhole principle at least one partition has to occur for arbitrary small $\epsilon > 0$. But if a matrix $A^k(\epsilon_0)$ has *cpi* with blocks $I_0^k(\epsilon_0), \ldots I_{l_k}^k(\epsilon_0)$ then for all $\epsilon > \epsilon_0$ the matrix $A^k(\epsilon)$ has *cpi* with the same blocks.

We will prove lemma 1.3. by a downward induction on the size of the block I_0^k which is maximal for $k \leq n$, for which the matrix $A^k(\epsilon)$ has *cpi* for all $\epsilon > 0$. To illustrate the main idea of the proof we first show the theorem for matrices with one and two columns.

$n = 1$:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}.$$

The system $A\vec{x} \leq \vec{0}$ has a solution $x \in \mathbb{N}$. Therefore we have $a_{i1} \leq 0$ and thus A has *cpi* with $I_0 = \{1\}$.

$n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix}$$

There are only three (finitely many) possibilities to arrange the columns of A into blocks. Hence we can assume that there is an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ the partition of the columns of $A^i(\epsilon)$ into blocks is the same.

1. $I_0^1 = \{1\}$ or $I_0^2 = \{2\}$ resp.

   For all $1 \le i \le m$ and all $\epsilon > 0$ we have $a_{i1} - \epsilon \le 0$. Hence for all $1 \le i \le m$ the relation $a_{i1} \le 0$ holds. So the first condition of *cpi* is satisfied with $I_0 = \{1\}$.

   Further by the definition of the ϵ-property the system $A\vec{x} \le \vec{0}$ has a solution in $\mathbb{N}$, Let x_1^*, x_2^* be such a solution. Then the second condition is fulfilled with $c_{11} = x_1^*$ and $c_1 = x_2^*$, i. e. for all $1 \le i \le m$ we have $c_{11}a_{i1} + c_1 a_{i2} \le 0$. Hence A has *cpi*.

2. $I_0^1 = \{1, 2\}$

   In this case for all $1 \le i \le m$ and for all $\epsilon > 0$ we have $a_{i1} + a_{i2} - \epsilon \le 0$. Hence for all $1 \le i \le m$ we have $a_{i1} + a_{i2} \le 0$. Therefore A has *cpi* with $I_0 = \{1, 2\}$.

Now we will prove the lemma for matrices of arbitrary size.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Let $1 \le k < l \le n$. We know by assumption that for all $\epsilon > 0$ either $A^k(\epsilon)$ or $A^l(\epsilon)$ has *cpi*. As mentioned above we can assume that the partition of $[n]$ into blocks does not depend on ϵ. In order to start the induction we consider the case where we can find some $1 \le k \le n$ such that $A^k(\epsilon)$ has *cpi* for every $\epsilon > 0$ and $|I_0^k|$=n, i. e. the sum over all columns of $A^k(\epsilon)$ is less of equal to zero. In this case for all $1 \le i \le m$ and every $\epsilon > 0$ we have

$$a_{i1} + a_{i2} + \ldots + a_{in} - \epsilon \le 0.$$

Hence for all $1 \le i \le m$ we have

$$a_{i1} + a_{i2} + \ldots + a_{in} \le 0$$

and therefore A has *cpi* with $I_0 = [n]$.
Next we consider the case where we can find some k, $1 \le k \le n$ such that $A^k(\epsilon)$ has *cpi* for every $\epsilon > 0$ and $|I_0^k| = n-1$. First assume that $k \in I_0^k$. Then for all $1 \le i \le m$ and all $\epsilon > 0$ we have:

$$(\sum_{j \in I_0^k} a_{ij}) - \epsilon \le 0.$$

In this case for all $1 \le i \le m$ we obtain

$$\sum_{j \in I_0^k} a_{ij} \le 0.$$

If $k \notin I_0^k$ for all $1 \le i \le m$ we also have

$$\sum_{j \in I_0^k} a_{ij} \le 0.$$

Thus in both cases the first condition of *cpi* is satisfied choosing $I_0 = I_0^k$.
Let $I_1 = [n] - I_0$. Note that $|I_1| = 1$ and assume $p \in I_1$. We know that the system $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Let $x_1^*, \dots x_n^*$ be such a solution. Then for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0} c_{1j} a_{ij} + c_1 a_{ip} \leq 0,$$

if we choose $c_{1j} = x_j^*$ for $j \in I_0$ and $c_1 = x_p^*$.

Assume inductively that the following is true for some (fixed) $k \leq n-1$: Let A be a rational matrix with m rows and n columns which has the ϵ- property. If there exists a column s, such that for all $\epsilon > 0$ $A^s(\epsilon)$ has *cpi* and $|I_0^s| \geq k$, then A has *cpi*.

In the following we will show that if A is a rational matrix which has the ϵ-property and there exists a column s, such that for all $\epsilon > 0$ $A^s(\epsilon)$ has *cpi* and $|I_0^s| = k-1$, then A has *cpi*. Without loss of generality we can assume that $I_0^s = \{1, \dots k-1\}$ for some (fixed) s. For $k - 1 \leq n-2$, we have $|[n] - I_0^s| \geq 2$. A has the ϵ- property, therefore either $A^k(\epsilon)$ or $A^{k+1}(\epsilon)$ has *cpi* for all $\epsilon > 0$. Without loss of generality we can assume that $A^k(\epsilon)$ has *cpi*. We will consider several cases:

1. $I_0^k \not\subseteq I_0^s$.

   In this case for all $\epsilon > 0$ and all $1 \leq i \leq m$ we have

$$(\sum_{j=1}^{k-1} a_{ij}) - \epsilon \leq 0$$

   and therefore

$$\sum_{j=1}^{k-1} a_{ij} \leq 0.$$

   Further for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0^k} a_{ij} \leq 0.$$

   We distinguish the following cases:

   (a) $I_0^k \cap I_0^s = \emptyset$

   Then we have

$$\sum_{j \in I_0^k \cup I_0^s} a_{ij} \leq 0.$$

   Let $I_0 = I_0^k \cup I_0^s$ and $I_l = I_l^k - I_0^s$. Because of the definition of I_l^k for all $\epsilon > 0$ and for all $j \in \cup_{s \leq l} I_s^k$ there exists $c_{lj}^k(\epsilon)$ and $c_l^k(\epsilon)$ such that for all $1 \leq i \leq m$ we have

$$\sum_{j \in \cup_{s \leq l} I_s^k} c_{lj}^k(\epsilon) a_{ij}^k(\epsilon) + c_l^k(\epsilon) \sum_{j \in I_{l+1}^k} a_{ij}^k(\epsilon) \leq 0$$

and therefore

$$\sum_{j\in\cup_{s\leq l}I_s^k} c_{lj}^k(\epsilon)a_{ij}^k(\epsilon) + \sum_{j\in(I_0^s-I_{l+1}^k)} a_{ij}^k(\epsilon) + \sum_{j\in I_0^s\cap I_{l+1}^k} (1+c_l^k(\epsilon))a_{ij}^k(\epsilon) +$$

$$\sum_{j\in(I_{l+1}^k-I_0^s)} c_l(\epsilon)a_{ij}^k(\epsilon) \leq 0.$$

Hence we conclude that we can choose $I_0 = I_0^k \cup I_0^s$ to prove *cpi* and $|I_0| > |I_0^s| = k-1$. So we are done by induction.

(b) $I_0^k \cap I_0^s \neq \emptyset$

Without loss of generality we can assume that $I_0^k \cap I_0^s = \{1,\ldots,l\}$. Consider the matrix

$$B = \begin{pmatrix} 2a_{11} & 2a_{12} & \ldots & 2a_{1l} & a_{1l+1} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2a_{m1} & 2a_{m2} & \ldots & 2a_{ml} & a_{ml+1} & \ldots & a_{mn} \end{pmatrix} = (b_{ij})_{1\leq i\leq m, 1\leq j\leq n}.$$

We claim that B has the ϵ-property. This is true because

(i) the system $A'\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$ for if $x_1,\ldots,x_n$ is a solution of $A\vec{x} \leq \vec{0}$, then $x_1,\ldots,x_l, 2x_{l+1},\ldots,2x_n$ forms a solution of $B\vec{x} \leq \vec{0}$.

(ii) Let $1 \leq p \leq n$ such that $A^p(\epsilon)$ has *cpi* for every $\epsilon > 0$ with blocks $I_0^p, I_1^p, \ldots$. Let $I_0'^p = I_0^k \cup I_0^s$. Then for all $1 \leq i \leq m$ the following is true:

$$0 \geq \sum_{j\in I_0^k} a_{ij}^k(\epsilon) + \sum_{j\in I_0^s} a_{ij}^s(\epsilon) = \sum_{j\in I_0'^p} b_{ij}^p(\epsilon).$$

Let $I_r'^p = I_{r-1}^p - (I_0^k \cup I_0^s)$. $A^p(\epsilon)$ has *cpi* for every $\epsilon > 0$. Hence there exist $c_{r-1j}^p = c_{r-1j}^p(\epsilon), c_{r-1}^p = c_{r-1}^p(\epsilon)$ such that for all $1 \leq i \leq m$ we have

$$\sum_{j\in\cup_{q\leq r-1}I_q^p} c_{r-1j}a_{ij}^p(\epsilon) + c_{r-1}\sum_{j\in I_r^p} a_{ij}^p(\epsilon) \leq 0.$$

Hence we have

$$\begin{aligned}&\sum_{j=1}^{l} b_{ij}^p(\epsilon) + \sum_{j\in I_0'^p-\{1,\ldots,l\}} 2b_{ij}^p(\epsilon) + \sum_{j\in\cup_{q\leq r-1}I_q^p\cap(I_0^k\cup I_0^s)} c_{r-1j}2b_{ij}^p(\epsilon) + \\ &\sum_{j\in(\cup_{q\leq r-1}I_q^p)-(I_0^k\cup I_0^s)} c_{r-1j}2b_{ij}^p(\epsilon) + \sum_{j\in I_r^p\cap(I_0^k\cup I_0^s)} c_{r-1}2b_{ij}^p(\epsilon) + \sum_{j\in I_r'^p} c_{r-1}2b_{ij}^p(\epsilon) \\ &\leq 0.\end{aligned}$$

Hence $B^p(\epsilon)$ has *cpi* if $A^p(\epsilon)$ has *cpi*. Therefore B has the ϵ-property and $|I_0'^p| \geq k$. Hence B has *cpi* by induction.

We claim that if B has *cpi* then A has *cpi*.

Let the partition into blocks for B be $I_0^B, I_1^B, \ldots, I_v^B$. Let $I_0 = \{1, \ldots, k-1\}$. We know that

$$\sum_{j=1}^{k-1} a_{ij} \leq 0.$$

Let $I_1 = (I_0^k \cup I_0^s) - \{1, \ldots, k-1\}$, let $c_{01} = \ldots = c_{0l} = 2, c_{0l+1} = \ldots = c_{0k-1} = 1$ and $c_0 = 1$. Then for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0} c_{j0} a_{ij} + c_0 \sum_{j \in I_1} a_{ij} \leq 0.$$

Let $I_r = I_{r-2}^B - (I_0^k \cup I_0^s)$. We know that there exist c_{r-2j}^B, c_{r-2}^B such that we have

$$\sum_{j \in \cup_{w \leq r-3} I_w^B} c_{r-2j}^B b_{ij} + c_{r-2}^B \sum_{j \in I_{r-2}^B} b_{ij} \leq 0$$

and thus

$$\begin{aligned} &\sum_{j=1}^{l} 2a_{ij} + \sum_{j \in (I_0^k \cup I_0^s) - \{1,\ldots,l\}} a_{ij} + \sum_{j \in (\cup_{w \leq r-3} I_w^B) - (I_0^k \cup I_0^s)} c_{r-2j}^B a_{ij} + \\ &\sum_{j \in (\cup_{w \leq r-3} I_w^B) \cap (I_0^k \cup I_0^s)} c_{r-2j}^B a_{ij} + c_{r-2}^B \sum_{j \in I_{r-2}^B \cap (I_0^k \cup I_0^s)} a_{ij} + c_{r-2}^B \sum_{j \in I_r} a_{ij} \\ &\leq 0. \end{aligned}$$

Hence A has *cpi*.

2. $I_0^k \subseteq I_0^s = \{a^{(1)}, \ldots a^{(k-1)}\}$.
   (If $I_0^s \subset I_0^k$, we would have $|I_0^k| \geq k$ and we were done by induction.)
   Without loss of generality we can assume that $I_0^k = I_0^s$, because otherwise it is possible to choose I_0^k as the first block for the matrix $A^s(\epsilon)$. We distinguish the following cases:
   (a) $k \notin I_1^k$
   In this case there exist $c_{1j} \in \mathbb{N}, c_1 \in \mathbb{N}$ such that for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0^k} c_{1j} a_{ij} + c_1 \sum_{j \in I_1^k} a_{ij} \leq 0.$$

   Consider the following matrix $B = (b_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, where for all $1 \leq i \leq m$ b_{ij} is defined by

$$b_{ij} = \begin{cases} c_{1j} a_{ij} & \text{for} \quad 1 \leq j \leq k-1 \\ c_1 a_{ij} & \text{for} \quad j \in I_1^k \\ a_{ij} & \quad \text{otherwise.} \end{cases}$$

   We claim that B has the ϵ-property.

(i) The system $B\vec{x} \leq \vec{0}$ has a solution, for if $x_1, \dots x_n$ is a solution of the system $A\vec{x} \leq \vec{0}$, then define a solution of the system $B\vec{y} \leq \vec{0}$, $\vec{y} = (y_1, \dots, y_n)$ by

$$y_j = \begin{cases} \frac{1}{c_{1j}} x_j & \text{if} \quad 1 \leq j \leq k-1 \\ \frac{1}{c_1} x_j & \text{if} \quad j \in I_1^k \\ x_j & \quad\quad \text{otherwise.} \end{cases}$$

If we multiply $\vec{y}$ by the least common multiple of c_{1j}, c_1 we obtain a solution of the system $B\vec{x} \leq \vec{0}$ in $\mathbb{N}$.

(ii) Let $1 \leq p \leq n$ be given such that for every $\epsilon > 0$ $A^p(\epsilon)$ has *cpi* and let $I_0^p, \dots, I_l^p$ be the blocks and $c_{rj}^p(\epsilon), c_r^p(\epsilon)$ the corresponding coefficients, such that for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0^p} a_{ij}^p(\epsilon) \leq 0$$

and

$$\sum_{j \in \cup_{w \leq r} I_w^p} c_{rj}^p(\epsilon) a_{ij}^p(\epsilon) + c_r^p(\epsilon) \sum_{j \in I_{r+1}^p} a_{ij}^p(\epsilon) \leq 0.$$

Now we will show that $B^p(\epsilon)$ has *cpi* for all $\epsilon > 0$. Let $I_0'^p = I_0^k \cup I_1^k$ and $I_r'^p = I_{r-1}^p - I_0'^p$. Then for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0'^p} b_{ij} \leq 0$$

and

$$\sum_{j \in \cup_{w \leq r-1} I_w^p} c_{r-1j}^p(\epsilon) a_{ij}^p(\epsilon) + c_{r-1}^p(\epsilon) \sum_{j \in I_r^p} a_{ij}^p(\epsilon) \leq 0.$$

It follows that

$$\sum_{j \in I_0'^p} b_{ij}^p(\epsilon) + \sum_{j \in ((\cup_{w \leq r-1} I_w^p) \cap I_0'^p)} c_{r-1j}^p(\epsilon) a_{ij}^p(\epsilon) + \sum_{j \in I_{r-1}^p \cap I_0^p} c_r^p(\epsilon) a_{ij}^p(\epsilon) + \sum_{j \in (\cup_{w \leq r-1} I_w^p) - I_0'^p} c_{r-1j}^p(\epsilon) a_{ij}^p(\epsilon) + \sum_{j \in I_r'^p} c_r^p(\epsilon) a_{ij}^p(\epsilon) \leq 0.$$

Hence B has the ϵ-property. Thus B has *cpi* by induction. Let the corresponding partition of blocks be $I_0^B, \dots, I_l^B$ and let c_{rj}^B, c_r^B be the corresponding coefficients. We claim that A has *cpi*.

Let $I_0 = I_0^k, I_1 = I_1^k, I_r = I_{r-2}^B - (I_0^k \cup I_1^k)$. Obviously for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0} a_{ij} \leq 0.$$

For $2 \leq r \leq l-1$, for all $1 \leq i \leq m$ we have

$$\sum_{j \in \cup_{w \leq r-2} I_w^B} c_{r-2j}^B b_{ij} + c_{r-1}^B \Big(\sum_{j \in I_{r-1}^B} b_{ij} \Big) \leq 0.$$

Thus for all $1 \le i \le m$ the following is true

$$\begin{aligned}&\sum_{j\in I_0^k} a_{ij} + \sum_{j\in I_1^k} a_{ij} + \sum_{j\in(\cup_{w\le r-2} I_w^B \cap (I_0^k\cup I_1^k))} c_{r-2j}^B b_{ij}\\ &+ \sum_{j\in I_{r-1}^B\cap(I_0^k\cup I_1^k)} c_{r-2}^B b_{ij} + \sum_{j\in\cup_{w\le r-2} I_w^B-(I_0^k\cup I_1^k)} c_{r-2j}^B b_{ij} + c_{r-2}^B(\sum_{j\in I_{r+1}} b_{ij})\\ &\le 0.\end{aligned}$$

Hence A has *cpi*.

(b) $k \in I_1^k$.

Without loss of generality we can assume that $I_1^k = \{k,\dots,r\}$. For all $1 \le i \le m$ we know that $\sum_{j=1}^{k-1} a_{ij} \le 0$. It is no restriction to assume that

$$\sum_{j=1}^{k} a_{ij} \begin{cases} = 0 & \text{for} \quad 1 \le i \le m_1 \\ < 0 & \text{for} \quad m_1 < i \le m \end{cases}$$

for some $m_1 \le m$. In claim 1.11. we have shown that it is enough to consider the first m_1 rows of A. Let

$$B = (a^{(1)},\dots,a^{(k-1)})$$

be the matrix which consists of the first $k-1$ columns of A. Let

$$B'(\epsilon) = \begin{pmatrix} a_{11} & \dots & a_{1k-1} & (\sum_{j=k}^r a_{1j} - \epsilon) \\ \vdots & \vdots & \vdots & \vdots \\ a_{m_1 1} & \dots & a_{m_1 k-1} & (\sum_{j=k}^r a_{m_1 j} - \epsilon) \end{pmatrix}.$$

Obviously adding up the columns of B we get the zero vector. Further for all $\epsilon > 0$ the system $B'(\epsilon)\vec{x} \le \vec{0}$ has a solution. Hence we can apply lemma 1.10. to show that the system $B'(0)\vec{x} \le \vec{0}$ has a solution in $\mathbb{N}$. Assume that $c_{11},\dots,c_{1k-1},c_1$ is such a solution, hence for all $1 \le i \le m$ we have

$$\sum_{j=1}^{k-1} a_{ij}c_{1j} + c_1 \sum_{k}^{r} a_{ij} \le 0.$$

Then we consider the matrix $B = (b_{ij})_{1\le i\le m, 1\le j\le n}$

$$b_{ij} = \begin{cases} c_{1j}a_{ij} & \text{for} \quad 1 \le j \le k-1 \\ c_1 a_{ij} & \text{for} \quad k \le j \le r \\ a_{ij} & \qquad \text{otherwise.} \end{cases}$$

As in case a) it is now possible to show that B has the ϵ-property. Then by induction B has *cpi* which again implies as in case a) that A has *cpi*.

$\square_{lemma}$ 1.3.

Lemma 1.4. *Let $a, b \in \mathbb{Q}$ and let the following system of inequalities be given:*

$$(*) \qquad a \le \frac{x_1}{x_2} \le b.$$

Let

1. $1 < a \leq b$ *or*

2. $0 < a \leq b < 1$.

Then $(*)$ *is not partition regular over* $\mathbb{N}$.

PROOF OF LEMMA 1.4.:

1. Assume that $1 < a \leq b$.

   Let $n \in \mathbb{N}$ be minimal such that $a^n > b$. Consider the following coloring: $\Delta^{a,b} : \mathbb{N} \to [n+1]$ which is defined by

$$(**) \quad \Delta^{a,b}(x) = (\lfloor log_a(x) \rfloor mod(n+1)) + 1.$$

   In the following we will show that $(*)$ has no monochromatic solution for $\Delta^{a,b}$.

   Assume on the contrary that x_1, x_2 form a solution of $(*)$ which is monochromatic with respect to $\Delta^{a,b}$. Let $log_a(x_1) = \mu_{x_1}$ and $log_a(x_2) = \mu_{x_2}$. Then we have

$$\mu_{x_1} \equiv \mu_{x_2} mod(n+1).$$

   Say $\mu_{x_1} = k_{x_1}(n+1) + r$ and $\mu_{x_2} = k_{x_2}(n+1) + r$ for some $0 \leq r \leq n$. Because x_1, x_2 forms a solution of $(*)$ we have

$$a \leq \frac{x_1}{x_2} \leq b$$

   and thus

$$a \leq \frac{x_1}{x_2} < \frac{a^{\mu_{x_1}+1}}{a^{\mu_{x_2}}} = a^{(k_{x_1} - k_{x_2})(n+1)+1}.$$

   Therefore we have

$$(k_{x_1} - k_{x_2})(n+1) + 1 > 1$$

   and hence

$$k_{x_1} - k_{x_2} > 0.$$

   On the other hand we have:

$$a^n > b \geq \frac{x_1}{x_2} \geq \frac{a^{\mu_{x_1}}}{a^{\mu_{x_2}+1}} = a^{(k_{x_1} - k_{x_2})(n+1)-1},$$

   which implies

$$(k_{x_1} - k_{x_2})(n+1) - 1 < n$$

   and hence

$$k_{x_1} - k_{x_2} < 1.$$

   which is in contradiction to $(**)$.

2. Assume that $0 < a \leq b < 1$. Consider the following system of inequalities which is equivalent to $(*)$:

$$\frac{1}{a} \geq \frac{x_2}{x_1} \geq \frac{1}{b}.$$

   Then we have $1 < \frac{1}{b} \leq \frac{1}{a}$ and we can follow the arguments of case 1.

$\square_{lemma \quad 1.4.}$

LEMMA 1.5. *Let $z \in \mathbb{N}$ be given. Let $n \geq 2$ and let $f_i(x_2, \ldots, x_n) : \mathbb{R}^{n-1} \to \mathbb{R}$, $g_i(x_2, \ldots, x_n) : \mathbb{R}^{n-1} \to \mathbb{R}$ for $1 \leq i \leq z$ be given. Consider the following system of inequalities:*

$$f_i(x_2, \ldots, x_n) \leq \frac{x_1}{x_2} \leq g_i(x_2, \ldots, x_n). \qquad (*)$$

Let $(*)$ satisfy the following conditions:

1. $\exists i_1, 1 \leq i_1 \leq z, \exists \epsilon_1, 0 < \epsilon_1 < 1, \exists c_1 \in \mathbb{N}$ *and* $\exists \Delta^1 : \mathbb{N} \to [c_1]$ *such that* $(*)$ *has no solution* $x_1, \ldots, x_n$ *which is monochromatic with respect to* Δ^1 *and*
$$f_{i_1}(x_2, \ldots, x_n) \leq \epsilon_1.$$
2. $\exists i_2, 1 \leq i_2 \leq z, \exists \epsilon_2, \epsilon_3, 0 < \epsilon_2, \epsilon_3 < 1, \exists c_2 \in \mathbb{N}$ *and* $\exists \Delta^2 : \mathbb{N} \to [c_2]$ *such that* $(*)$ *has no solution* $x_1, \ldots, x_n$ *which is monochromatic with respect to* Δ^2 *and*
$$f_{i_2}(x_2, \ldots, x_n) \leq 1 + \epsilon_2$$
or there is no solution $x_1, \ldots, x_n$ *which is monochromatic with respect to* Δ^2 *and*
$$g_{i_2}(x_1, \ldots, x_n) \geq \epsilon_3.$$
3. $\exists k \in \mathbb{N}, \exists c_3 \in \mathbb{N}$ *and* $\exists \Delta^3 : \mathbb{N} \to [c_3]$ *such that* $(*)$ *has no solution* $x_1, \ldots, x_n$ *which is monochromatic with respect to* Δ^3 *and*
$$\frac{x_1}{x_2} \geq k.$$

Then there exists $c^* \in \mathbb{N}$ and a coloring $\Delta^* : \mathbb{N} \to [c^*]$, such that $(*)$ has no solution which is monochromatic for Δ^*.

PROOF OF LEMMA 1.5.: Let $\epsilon_1, \epsilon_2, \epsilon_3, k, c_1, c_2, c_3$ and $\Delta^1, \Delta^2, \Delta^3$ be defined as in the assumptions of lemma 1.13. Consider colorings of the form $\Delta^{a,b}$ which are defined as in the proof of lemma 1.11. $(**)$ with appropriate a and b, namely:

$$\Delta^4 = \Delta^{\frac{1}{1-\epsilon_3}, \frac{1}{\epsilon_1}} : \mathbb{N} \to [c_4],$$

where $c_4 \in \mathbb{N}$ is minimal such that $\frac{1}{1-\epsilon_3}^{(c_4-1)} > \frac{1}{\epsilon_1}$ and

$$\Delta^5 = \Delta^{1+\epsilon_2, k} : \mathbb{N} \to [c_5],$$

where $c_5 \in \mathbb{N}$ is minimal such that $(1+\epsilon_2)^{(c_5-1)} > k$.
Then define Δ^* as follows:

$$\Delta^* : \mathbb{N} \to \prod_{j=1}^{5} [c_j],$$

$$\Delta^*(x) = (\Delta^1(x), \Delta^2(x), \Delta^3(x), \Delta^4(x), \Delta^5(x)).$$

We claim that $(*)$ has no solution which is monochromatic for Δ^*.
Assume on the contrary that $x_1, \ldots, x_n$ is a solution of $(*)$ which is monochromatic with respect to Δ^*. Because $x_1, \ldots, x_n$ is monochromatic for Δ^* it is monochromatic for Δ^1. Hence we have

$$f_{i_1}(x_2, \ldots, x_n) \geq \epsilon_1,$$

which implies

$$\frac{x_1}{x_2} \geq \epsilon_1. \qquad (1)$$

Besides $x_1, \ldots, x_n$ is monochromatic for Δ^2. Hence we have

$$f_{i_2}(x_2, \ldots, x_n) \geq 1 + \epsilon_2$$

or

$$g_{i_2}(x_2, \ldots, x_n) \leq 1 - \epsilon_3,$$

which implies

$$\frac{x_1}{x_2} \geq 1 + \epsilon_2 \qquad (2)$$

or

$$\frac{x_1}{x_2} \leq 1 - \epsilon_3. \qquad (3)$$

Finally $x_1, \ldots, x_n$ is monochromatic for Δ_3 and therefore we have:

$$\frac{x_1}{x_2} \leq k. \qquad (4)$$

If we put together (1) and (3) and (2) and (4) respectively, we obtain:

$$\epsilon_1 \leq \frac{x_1}{x_2} \leq 1 - \epsilon_3 \qquad (5)$$

or

$$1 + \epsilon_2 \leq \frac{x_1}{x_2} \leq k. \qquad (6)$$

By lemma 1.12. (5) has no monochromatic solution for Δ^4 and (6) has no monochromatic solution for Δ^5. Hence $x_1, \ldots, x_n$ is not monochromatic for Δ^*. That is in contradiction to our assumption. $\square_{lemma}$ 1.13.

Now we are able to prove the second part of theorem 1.5., i.e. A has *cpi* if the system $A\vec{x} \leq \vec{0}$ is partition regular.

PROOF OF THEOREM 1.3. (SECOND PART): We will prove the theorem by induction on the number of columns of A. Note that a system, which is partition regular, necessarily has a solution.
$n = 1$:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

The system $(a_{i1}x_1 \leq 0)_{1\leq i\leq m}$ is partition regular. Hence it has a solution in $\mathbb{N}$, therefore for all $1 \leq i \leq m$ we have $a_{i1} \leq 0$ and thus A has *cpi* with $I_0 = \{1\}$.
In order to demonstrate the idea of the proof we additionally consider the case $n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix}$$

We distinguish the following cases:

1. For each row I $1 \leq i \leq m$ the first entry is less or equal zero, i.e. $a_{i1} \leq 0$.

   Let $I_0 = \{1\}$ and $I_1 = \{2\}$. Assume that $y_1, y_2 \in \mathbb{N}$ form a solution of the system $A\vec{x} \leq \vec{0}$. Then for all $1 \leq i \leq m$ we have

   $$\sum_{j\in I_0} c_{1j}a_{ij} + c_1 \sum_{j\in I_1} a_{ij} = c_{11}a_{i1} + c_1 a_{i2} \leq 0$$

   if we choose $c_{11} = y_1$ and $c_1 = y_2$.

2. For each row i $1 \leq i \leq m$ the first entry is greater or equal zero, i.e. $a_{i1} \geq 0$.

   In this case for all $1 \leq i \leq 0$ we have $a_{i2} \leq 0$

   Then A has *cpi* with blocks $I_0 = \{2\}$ and $I_1 = \{1\}$.

3. There exist $s, t \in [m]$ such that $a_{s1} < 0$ and $a_{t1} > 0$.

   Then the system $A\vec{x} \leq \vec{0}$ can be transformed as follows:

   $$-\frac{a_{t2}}{a_{t1}} \leq \frac{x_1}{x_2} \leq -\frac{a_{s2}}{a_{s1}}$$

   for all t with $a_{t1} < 0$ and for all s with $a_{s1} > 0$ and

   $$a_{t2}x_2 \leq 0 \quad \text{for all } t \text{ with } a_{t1} = 0.$$

   By lemma 1.12. we know that one of the following cases holds:

   (a) $-\frac{a_{t2}}{a_{t1}} \leq 0$ for all t with $a_{t1} < 0$ and $-\frac{a_{s2}}{a_{s1}} \geq 0$ for all s with $a_{s1} > 0$ and (obviously) $a_{t2} \leq 0$ for all t with $a_{t1} = 0$. In this case for all $1 \leq i \leq m$ we obtain

   $$a_{t2} \leq 0.$$

   Thus A has *cpi* with blocks $I_0 = \{2\}$ and $I_1 = \{1\}$.

   (b) $-\frac{a_{t2}}{a_{t1}} \leq 1$ for all t with $a_{t1} < 0$ and $-\frac{a_{s2}}{a_{s1}} \geq 1$ for all s with $a_{s1} > 0$ and hence for all $1 \leq t \leq m$ with $a_{t1} \neq 0$ we have

   $$a_{t1} + a_{t2} \leq 0$$

   and obviously for all $1 \leq t \leq m$ with $a_{t1} = 0$ we have

   $$a_{t2} \leq 0$$

and hence

$$a_{t1} + a_{t2} \le 0.$$

Thus A has *cpi* with $I_0 = \{1, 2\}$ in this case.

Hence we are done in the case $n = 2$.
Let us assume that the theorem is true for all matrices A with less than n columns for some (fixed) $n \ge 2$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}.$$

To prove the theorem we distinguish the following cases:

1. There exists $1 \le j^* \le n$ such that for all $1 \le i \le m$ the j^*th entry satisfies $a_{ij^*} < 0$.

   In this case let $I_0 = \{j^*\}$ and $I_1 = [n] - \{j^*\}$ and choose $c_{1j^*} > max_{1 \le i \le m}\{\frac{\sum_{s=1, s \ne j^*}^{n} |a_{is}|}{|a_{ij^*}|}\}, c_1 = 1$.

2. There exists $1 \le j^* \le n$ such that for all $1 \le i \le m$ the j^*th entry satisfies $a_{ij^*} \le 0$.

   Without loss of generality assume $j^* = 1$ and $a_{i1} < 0$ for $1 \le i \le m_1$ and $a_{i1} = 0$ for $m_1 < i \le m$ for some $m_1 \le m$. Then we have:

$$A = \begin{pmatrix} a_{11} < 0 & \\ \vdots & * \\ a_{1m_1} < 0 & \\ 0 & \\ \vdots & A' \\ 0 & \end{pmatrix}.$$

   Hence A is partition regular if and only if A' is partition regular. By induction A' has *cpi*. Let the corresponding blocks be $I'_0, \dots I'_r$ for a suitable $r \in \mathbb{N}$ and for $1 \le k \le r$ and for $j \in \cup_{s \le k} I_s$ let the coefficients be c'_{kj}, c'_k. Then A has *cpi* with blocks $I_0 = \{1\}, I_s = I'_{s-1}$ for $1 \le s \le r$ and coefficients

$$c_{k1} = \frac{max_{1 \le i \le m_1}\{\sum_{j \in \cup_{s \le k} I'_s} c'_{kj} a_{ij} + c'_k \sum_{j \in I'_{k+1}} a_{ij}\}}{min_{1 \le i \le m_1} |a_{1i}|}$$

   for $2 \le k \le r$ and

$$c_{11} = \frac{max_{1 \le i \le m_1} \sum_{j \in I'_1} a_{ij}}{min_{1 \le i \le m_1} |a_{1i}|},$$

   $c_1 = 1$ and $c_{kj} = c'_{k-1j}$ for all $j \ne 1$ and all $1 \le k \le r$.

3. There exists j^* such that for all $1 \leq i \leq m$ we have $a_{ij^*} \geq 0$.

   In this case obviously $A' = A - \{a^{(j^*)}\}$, the matrix which we obtain from A by omitting the column j^*, is partition regular and has *cpi* by induction. Let the blocks of A' be $I'_0, \dots I'_r$ and define for all $1 \leq s \leq r$ $I_s = I'_s$ and $I_{r+1} = \{j^*\}$. Further let $y_1, \dots, y_n \in \mathbb{N}$ be a solution of the system $A\vec{x} \leq \vec{0}$. Then A has *cpi* with coefficients $c_{rj} = y_j$ for $j \neq j^*$, and $c_r = y_{j^*}$.

4. Each column has both positive and negative entries.

   Let $1 \leq k < l \leq n$ be given. Then the system $A\vec{x} \leq \vec{0}$ can be transformed as follows:

$$(*)\begin{cases} -\dfrac{a_{sk}}{a_{sl}} - \displaystyle\sum_{j=1, j\notin\{k,l\}}^{n} \dfrac{a_{sj}}{a_{sl}}\dfrac{x_j}{x_k} \leq \dfrac{x_l}{x_k} \leq -\dfrac{a_{tk}}{a_{tl}} - \displaystyle\sum_{j=1, j\notin\{k,l\}}^{n} \dfrac{a_{sj}}{a_{sl}}\dfrac{x_j}{x_k} \\ \text{for all } s,t \text{ with } a_{sl} < 0 \text{ and } a_{tl} > 0, \\ \sum_{j=1, j\neq l}^{n} a_{ij}x_j \leq 0 \\ \text{for all } i \text{ with } a_{il} = 0. \end{cases}$$

   By lemma 1.13. we know that one of the following cases holds:

   (a) For all $\epsilon > 0$ the following system of inequalities is partition regular:

$$-\frac{a_{sk}}{a_{sl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sl}}\frac{x_j}{x_k} \leq \epsilon \quad \text{for all } s \text{ with } a_{sl} < 0$$

$$-\frac{a_{tk}}{a_{tl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tl}}\frac{x_j}{x_k} \geq 0 \quad \text{for all } t \text{ with } a_{tl} > 0$$

   and

$$\sum_{j=1, j\neq l}^{n} a_{ij}x_j \leq 0 \quad \text{for all } i \text{ with } a_{il} = 0.$$

   That means that for every $\epsilon > 0$ the system

$$A_l^k(\epsilon)\vec{y} \leq \vec{0}$$

   is partition regular and has *cpi* by induction. Hence by remark 1.8. $A^k(\epsilon)$ has *cpi* for all $\epsilon > 0$.

   (b) For all $r > 0$ and each coloring of the natural numbers with finitely many colors the system $(*)$ has a monochromatic solution $x_1, \dots, x_n$ such that

$$\frac{x_l}{x_k} > r,$$

   which is equivalent to

$$\frac{x_k}{x_l} < \frac{1}{r}.$$

We transform the system $A\vec{x} \leq \vec{0}$ as in $(*)$ exchanging k and l. Then we obtain:

$$-\frac{a_{sl}}{a_{sk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sk}} \frac{x_j}{x_l} \leq \frac{x_k}{x_l} \leq -\frac{a_{tl}}{a_{tk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tk}} \frac{x_j}{x_l}$$

for all s, t with $a_{sk} < 0$ and $a_{tk} > 0$ and

$$\sum_{j=1, j\neq k}^{n} a_{ij} x_j \leq 0$$

for all i with $a_{ik} = 0$. Therefore the following system is partition regular for each $r > 0$:

$$\begin{cases} -\frac{a_{sl}}{a_{sk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sk}} \frac{x_j}{x_l} \leq \frac{1}{r} \\ \text{for all } 1 \leq s \leq m \text{ with } a_{sk} < 0 \\ -\frac{a_{tl}}{a_{tk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tk}} \frac{x_j}{x_l} \geq 0 \\ \text{for all } 1 \leq t \leq m \text{ with } a_{tk} > 0 \text{ and} \\ \sum_{j=1, j\neq k}^{n} a_{ij} x_j \leq 0 \\ \text{for all } 1 \leq i \leq m \text{ with } a_{ik} = 0. \end{cases}$$

Hence the system $A_k^l(\frac{1}{r})$ is partition regular for every $r > 0$ and has *cpi* by induction. Therefore by remark 1.8. the system $A^l(\frac{1}{r})$ has *cpi* for every $r > 0$.

(c) For all $\epsilon > 0$ the following system is partition regular:

$$(*) \begin{cases} -\frac{a_{sk}}{a_{sl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sl}} \frac{x_j}{x_k} \leq 1 + \epsilon \\ \text{for all } 1 \leq s \leq m \text{ with } a_{sl} < 0 \\ -\frac{a_{tk}}{a_{tl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tl}} \frac{x_j}{x_k} \geq 1 - \epsilon \\ \text{for all } 1 \leq t \leq m \text{ with } a_{tl} > 0 \\ \sum_{j=1, j\neq l}^{n} a_{ij} x_j \leq 0 \\ \text{for all } 1 \leq i \leq m \text{ with } a_{il} = 0. \end{cases}$$

Then for every $\epsilon > 0$ the system $A^{(k)+(l)}\vec{y} \leq \vec{0}$ is partition regular and has *cpi* by induction, therefore by remark 1.8. $A^l(\epsilon)$ and $A^k(\epsilon)$ have *cpi*.

The system $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$ because otherwise it could not be partition regular and hence A has the ϵ-property. Therefore by lemma 1.9. A has *cpi*.

$\square_{theorem}$ 1.5.

In the following we will generalize the set of partitioned numbers. We will first state results over $\mathbb{Z}$ and $\mathbb{Q}$ and finally we will consider real matrices and generalize the set of partitioned numbers to the reals.

DEFINITION 1.5. *Let $K \subset \mathbb{R} - \{0\}$ be a set. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a matrix with entries in $\mathbb{R}$. A has the column property for systems of inequalities (cpi) over K if there exists $l \in \mathbb{N}$ and a partition $[n] = I_0 \cup I_1 \cup \ldots \cup I_l$ of the column indices such that*

1. *There exists $c \in K$ such that for all $1 \leq i \leq m$ we have $c \sum_{j \in I_0} a_{ij} \leq 0$ and*
2. *for all $k < l, j \in \cup_{s \leq k} I_s$ there exist $c_k, c_{kj} \in K$ such that for all $1 \leq i \leq m$ we have*
$$\sum_{j \in \cup_{s \leq k} I_s} c_{jk} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij} \leq 0.$$

And correspondingly we define:

DEFINITION 1.6. *Let $K \subset \mathbb{R} - \{0\}$ be a set. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a real matrix. Let $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$. The system $A\vec{x} \leq \vec{b}$ is called partition regular over K, if for every $c \in \mathbb{N}$ and every c-coloring of $K \quad \Delta : K \to [c]$ there exists a solution $x_1, \ldots x_n \in K$ of $A\vec{x} \leq \vec{b}$ such that $\Delta \big|_{\{x_1 \ldots x_n\}} = const.$*

LEMMA 1.6. *Let $K \subset \mathbb{R} - \{0\}$ and $K = K_1 \cup K_2$ such that $K_1 \cap K_2 = \emptyset$. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a real matrix. Then the following statements are equivalent:*

1. *The system $A\vec{x} \leq \vec{0}$ is partition regular over K.*
2. *The system $A\vec{x} \leq \vec{0}$ is partition regular over K_1 or the system is partition regular over K_2.*

PROOF OF LEMMA 1.6.: If the system $A\vec{x} \leq \vec{0}$ is partition regular over K_1 or over K_2 then it is clearly partition regular over K. For the opposite direction assume that the system $A\vec{x} \leq \vec{0}$ is neither partition regular over K_1 nor over K_2, i. e. there exists $c_1 \in \mathbb{N}$ and a coloring $\Delta_1 : K_1 \to [c_1]$ and there exists $c_2 \in \mathbb{N}$ and a coloring $\Delta_2 : K_2 \to [c_2]$, such that $A\vec{x} \leq \vec{0}$ has no monochromatic solution in K_1 for Δ_1 and no monochromatic solution in K_2 with respect to Δ_2. Define the following coloring: $\Delta : K \to [max\{c_1, c_2\}] \times [2]$ by

$$\Delta(x) = \begin{cases} (\Delta_1(x), 1) & \text{if} \quad x \in K_1 \\ (\Delta_2(x), 2) & \text{if} \quad x \in K_2. \end{cases}$$

Obviously the system $A\vec{x} \leq \vec{0}$ has no monochromatic solution with respect to the coloring Δ which is a contradiction to the partition regularity. $\square_{lemma \quad 1.6.}$

If we use lemma 1.16. together with theorem 1.5. we obtain the following theorem:

THEOREM 1.4. *Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a rational matrix. The system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z} - \{0\}$ if and only if A has cpi either over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$.*

PROOF OF THEOREM 1.4.: By lemma 1.16. we know that the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z} - \{0\}$ iff it is either partition regular over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$. The first case is equivalent to A having *cpi* over $\mathbb{N}$ by theorem 1.5. In the second case consider $(*)$ $-A\vec{x} \leq \vec{0}$ where $-A = (-a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$. $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z}^- - \{0\}$ iff $-A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{N}$. This is equivalent to $-A$ having *cpi* over $\mathbb{N}$, which is equivalent to A having *cpi* over $\mathbb{Z}^- - \{0\}$.

$\square_{theorem\ 1.4.}$

THEOREM 1.5. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix. Then the following statements are equivalent:*

1. *The system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Q} - \{0\}$.*
2. *A has cpi over $\mathbb{Q}^+ - \{0\}$ or over $\mathbb{Q}^- - \{0\}$.*
3. *A has cpi over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$.*

PROOF OF THEOREM 1.5. :
1. implies 2.:
It is enough to show that if $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Q}^+ - \{0\}$ then it has *cpi* over $\mathbb{Q}^+ - \{0\}$. This can be shown following the arguments of the second part of the proof of theorem 1.5. using $Q^+ - \{0\}$ instead of $\mathbb{N}$.
2. implies 3.:
Assume that A has *cpi* over $\mathbb{Q}^+ - \{0\}$, i. e. there exists a partition of the columns of A into blocks $[n] = I_0 \cup \ldots \cup I_l$ such that

1. There exists $q \in \mathbb{Q}^+ - \{0\}$ such that for all $1 \leq i \leq m$ we have $q\sum_{j\in I_0} a_{ij} \leq 0$, i. e. $\sum_{j\in I_0} a_{ij} \leq 0$.
2. For $k < l, j \in \cup_{s\leq k} I_s$ there exist $c_{kj}, c_k \in \mathbb{Q}^+ - \{0\}$ such that for all $1 \leq i \leq m$ we have
$$\sum_{j\in\cup_{s\leq k} I_s} c_{kj}a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij} \leq 0.$$
By multiplying the above inequality with the common divisor of c_{kj}, c_k we obtain positive integer coefficients.

3. implies 1.:
If A has *cpi* over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$ then by theorem 1.17. the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z} - \{0\}$. Hence it is partition regular over $\mathbb{Q} - \{0\}$.

$\square_{theorem\ 1.5.}$

THEOREM 1.6. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a real matrix. Then the following statements are equivalent:*

1. *The system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{R} - \{0\}$.*
2. *A has cpi over $\mathbb{R}^+ - \{0\}$ or over $\mathbb{R}^- - \{0\}$.*

Proof of theorem 1.6. :
1. implies 2.:
It is enough to show that if the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{R}^+ - \{0\}$ then A has *cpi* over $\mathbb{R}^+ - \{0\}$. This can be shown following the arguments of the second part of the proof of theorem 1.5. using $\mathbb{R}^+ - \{0\}$ instead of $\mathbb{N}$.
2. implies 1. :
Again it is enough to show that if A has *cpi* over $\mathbb{R}^+ - \{0\}$ then the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{R}^+ - \{0\}$. To prove this we employ a generalized environment lemma using the multidimensional version of van der Waerden's Theorem which is due independently to Gallai (see [10]) and Witt [16] instead of van der Waerden's Theorem [15]:

Lemma 1.7. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a real matrix such that the system $A\vec{x} \leq \vec{0}$ is partition regular over $R^+ - \{0\}$. Let $t \in \mathbb{N}$ and $W \subset \mathbb{R}$, $W = \{w_1, \ldots w_t\}$ be given. Let $c \in \mathbb{N}$. Then for every c-coloring $\Delta : \mathbb{R}^+ - \{0\} \to [c]$ there exists $\vec{x} = (x_0, \ldots, x_n) \in (\mathbb{R}^+ - \{0\})^n$ and there exists $r \in R^+ - \{0\}$ such that*

1. *$A\vec{x} \leq \vec{0}$ and*
2. *For all j, k with $1 \leq j \leq n$, $1 \leq k \leq t$ we have $\Delta(x_j + rw_k) = const.$*

Proof of lemma 1.7.: Assume that A is partition regular. Hence by compactness [6] there exists a finite set $V = V(A, c) \subset \mathbb{R}^+ - \{0\}$ such that for every c-coloring of V there exists a monochromatic solution of the system $A\vec{x} \leq \vec{0}$ in V. Let $V = \{v_1, \ldots, v_t\}$.
Let $\Delta : \mathbb{R}^+ - \{0\} \to [c]$ be an arbitrary coloring. Define a coloring $\Delta^* : \mathbb{R}^+ - \{0\} \to [c^t]$ by

$$\Delta^*(x) = (\Delta(xv_i))_{1\leq i\leq t}.$$

Define a finite set $W = \{w | w = \prod_{s=1}^{n} v_{j_s}, j_s \in [t]\}$. By Gallai-Witt's Theorem there exists a homothetic copy of the set W which is monochromatic with respect to Δ^*, say $W' = a' + r'W = \{a' + r'w | w \in W\}$. Consider another coloring Δ^{**} :V$\to [c]$ which is defined by $\Delta^{**}(x) = \Delta(a'x)$. By definition of V there exists a monochromatic solution of the system $A\vec{x} \leq \vec{0}$ in V with respect to Δ^{**}, say $x'_1, \ldots, x'_n$. Then $(x'_1 a', \ldots, x'_n a')$ is a solution and for all $1 \leq j \leq n$ we have $\Delta(x'_j a') = const.$
Let $r = r' x'_1 \ldots x'_n$. Then we have:

$$x'_i a' + rv_j = x'_i (a' + r' v_j x'_1 \cdot \ldots \cdot x'_{i-1} x'_{i+1} \cdot \ldots \cdot x'_n)$$

and by the definition of W

$$(v_j x'_1 \cdot \ldots \cdot x'_{i-1} x'_{i+1} \cdot \ldots \cdot x'_n) \in W.$$

Hence for all $1 \leq i \leq n, 1 \leq j \leq t$ we finally have

$$\Delta(x'_i a') = \Delta(x'_i a' + rv_j).$$

$\Box_{lemma}$ 1.7.

Now we are able to prove the second part of theorem 1.19.:

Let A be a real matrix which has *cpi* over $\mathbb{R}^+ - \{0\}$. Let $[n] = I_0 \cup \ldots \cup I_l$ be the corresponding partition. We prove theorem 1.19. by main induction over the number of colors and by subsidiary induction over the number of blocks. In both cases the start of the induction is easy to obtain: The system $A\vec{x} \leq \vec{0}$ has a solution (just take the coefficients c_{jl-1}, c_l). If only one color is used every solution is monochromatic. If $l = 0$ every singleton provides a solution.

Let $A_k = (a^{(j)} | j \in \cup_{j \leq k} I_k)$ be the submatrix of A which only consists of the columns belonging to the first k blocks. Assume that A_k is partition regular over $\mathbb{R}^+ - \{0\}$ for some $k \geq 0$ and assume that for every coloring with $c-1$ colors the system $(*)$ $A_{k+1}\vec{x} \leq \vec{0}$ has a monochromatic solution, i. e. by compactness there exists a finite set $V_{c-1} \subset \mathbb{R}^+ - \{0\}$ such that for every $(c-1)$-coloring $(*)$ has a monochromatic solution in V_{c-1}.

Let $\Delta : \mathbb{R}^+ - \{0\} \to [c]$ be an arbitrary coloring. We define W, a finite subset of $\mathbb{R}$, by $W = \{w = vu | v \in V, u \in \{c_{jk}, c_k | 1 \leq j \leq n, 1 \leq k < l\}\}$. We apply lemma 1.20. to A_k and W. Thus there exists a solution $(y_i)_{i \in \cup_{s \leq k} I_s}$ of the system $A_k\vec{y} \leq \vec{0}$ and $r \in \mathbb{R}^+ - \{0\}$ such that for all $i \in \cup_{s \leq k} I_s$ and all $w \in W$ we have $\Delta(y_i + rw) = const.$ Combining *cpi* and the fact that the y_i form a solution for every $v \in V$ we obtain:

$$\sum_{j \in \cup_{s \leq k} I_s} a_{ij}(y_j + c_{kj}rv) + \sum_{j \in I_{k+1}} a_{ij} c_k rv \leq 0.$$

Without loss of generality we may assume that $\Delta(y_i + rc_{kj}v) = c$ for all $i \in \cup_{s \leq k} I_s$ and $v \in V$.
If now one of the numbers $c_k rv$ is also colored in c we have found a monochromatic solution of the system $A_{k+1}\vec{x} \leq \vec{0}$. Otherwise the coloring

$$\Delta^* : V \to [c-1]$$

defined by

$$\Delta^*(x) = \Delta(xrc_k)$$

is well defined. Therefore by induction on the number of colors and the definition of V there exists a monochromatic solution of $A_{k+1}\vec{x} \leq \vec{0}$ with respect to Δ^*, say $(x_i^*)_{i \in \cup_{s \leq k+1} I_s}$. Then $(x_i^* rc_k)_{i \in \cup_{s \leq k+1} I_s}$ forms a solution which is monochromatic with respect to Δ. $\square_{theorem\ 1.6.}$

In his dissertation [10] Rado also considered systems of inhomogeneous equations. As well as for homogeneous systems the columns property plays an important role for the characterization of partition regular systems of inhomogeneous inequalities. We are able to give a complete characterization of those systems which are partition regular over the natural numbers, over the set of integers and over the rationals.

Theorem 1.7. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a rational matrix, let $\vec{b} = (b_1, \ldots, b_m) \in \mathbb{Q}^m$. The system of inequalities $A\vec{x} \subseteq 0$ is partition regular over $\mathbb{N}$ if and only if one of the following conditions is satisfied:

1. There exists $a \in \mathbb{N}$ such that $A \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \leq \overrightarrow{b}$

2. A has cpi and there exists $\overrightarrow{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ and there exists $I \subseteq [m]$, such that $\sum_{j=1}^{n} a_{ij}x_j \begin{cases} < 0 & \text{for} \quad i \in I \\ \leq 0 & \text{for} \quad i \in [m] - I. \end{cases}$

   and there exists $a \in \mathbb{Z}$ such that for all $i \in [m] - I$ we have $\sum_{j=1}^{n} a_{ij}a \leq b_i$

The proof of theorem 1.21 is a little bit tricky and in its main parts very technical. The interested reader can find the complete proof in [17].

THEOREM 1.8. Set $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix and $\vec{b} = (b_1, \ldots, b_m) \in \mathbb{Q}^m$. The system $A\vec{x} \leq \vec{b}$ is partition regular over $\mathbb{Q} - \{0\}$ if and only if $A\vec{x} \in \vec{b}$ is partition regular over $\mathbb{N}$ or the system $-\vec{A}\vec{x} \leq \vec{b}$ with $-A = (-a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ is partition regular over $\mathbb{N}$.

If we partition the set $\mathbb{Q} - \{0\}$ the situation is different:

THEOREM 1.9. Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix, let $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{Q}^n$. The system $(x)A\vec{x} \leq \vec{b}$ is partition regular over $\mathbb{Q}$ if and only if one of the following cases is valid:

1. There exists $a^* \in \mathbb{Q}$ such that for all $1 \leq i \leq m$ we have $\sum_{j=1}^{n} a_{ij}a^* \leq b_i$

2. There exists $I \subseteq [m]$ such that $b_i \geq 0$ for $i \in I, b_i > 0$ for $i \in [m]J - I$ and the matrix $A_I = (a_{ij})_{i\in I, 1\leq j\leq n}$ has cpi over $\mathbb{Q}^+ - \{0\}$.

3. A has cpi over $\mathbb{Q}^+ - \{0\}$ and there exists $I \subseteq [m]$ and there exists $\vec{x}(x_1 - x_n) \in (\mathbb{Q}^+ - \{0\}^n$ such that
   $\sum_{j=1}^{n} a_{ij}x_j \begin{cases} < 0 & \text{for} \quad i \in I \\ \leq 0 & \text{for} \quad i \in [m] - I. \end{cases}$

   and there exists $a^x \in \mathbb{Q}^+ - \{0\}$ such that for all $i \in [m] - I$ we have $\sum_{j=1}^{n} a_{ij}a^* \leq b_i$.

4. $-A = (-a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ fulfills condition 1, 2, or 3.

2. (m, p, c)-SETS

In 1973 Deuber [1] gave a semantical characterization of partition regular system of linear equations. The nature of this characterization is somewhat different form Rado's approach. Deuber described the arithmetic structure of the sets of solutions of partition regular linear systems $A\vec{x} = \vec{0}$. The central definition is that of $(m, p, c,)$-sets, which are m-fold arithmetic progressions together with c-fold differences:

DEFINITION 2.1. *Let $m, p, c \in \mathbb{N}$. A set $D \subseteq \mathbb{N}$ is an (m,p,c)-set if there exist $d_0, \ldots, d_m \in \mathbb{N}$ such that $D = D_{p,c}(d_0 \ldots d_m)$ consists of all numbers of the following list:*

$$\begin{array}{ccccccccc} cd_0 & + & l_1 d_1 & + & l_2 d_2 & + & \ldots & + & l_m d_m, \\ & & cd_1 & + & l_2 d_2 & + & \ldots & + & l_m d_m, \\ & & & & cd_2 & + & \ldots & + & l_m d_m, \\ & & & & & & & & \vdots \\ & & & & & & & & cd_m, \end{array}$$

where $l_i \in [-p, p]$, i. e.

$$D_{p,c}(d_0, \ldots, d_m) = \{cd_i + \sum_{j=i+1}^{m} l_j d_j | i \leq m, l_j \in [-p, p]\}.$$

In particular a $(1, k, c)$-set is a $(2k+1)$-term arithmetic progressions together with its differences. Deuber proved the following theorem [1]:

THEOREM 2.1. (DEUBER 1973) *A linear system $A\vec{x} = \vec{0}$ is partition regular if and only if there exist positive integers m, p, c such that every $(m, p, c)-$set D contains a solution of $A\vec{x} = \vec{0}$.*

(m, p, c)-sets not only describe the arithmetic structure of sets of solutions of partition regular systems of linear equations but they can also be used to characterize sets of solutions of systems of linear inequalities.

THEOREM 2.2. *Let $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$ be a rational matrix. Let $A\vec{x} \leq \vec{0}$ be a partition regular system of linear inequalities. Then there exist $m, p, c \in \mathbb{N}$ such that every (m, p, c)-set contains a solution of the system $A\vec{x} \leq \vec{0}$.*

PROOF OF THEOREM 2.2.: By theorem 1.5. we know that A has *cpi*, i. e. there exists $m \in \mathbb{N}$ and a partition $I_0 \cup \ldots \cup I_m = [n]$ such that

1. for all $1 \leq i \leq l$ we have $\sum_{j \in I_0} a_{ij} \leq 0$ and
2. for $k \leq m$ and $j \in \cup_{s \leq k} I_s$ there exist $c_{kj}, c_k \in \mathbb{N}$ such that for every $k < m$ and for all $1 \leq i \leq l$ we have

$$\sum_{j \in \cup_{s \leq k} I_s} c_{kj} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij} \leq 0.$$

Let c be the least common multiple of $\{c_k | 1 \leq k < m\}$. Multiply each inequality by $\frac{c}{c_k}$ such that for all $1 \leq i \leq l$ we have

$$\sum_{j \in \cup_{s \leq k} I_s} c'_{kj} a_{ij} + c \sum_{j \in I_{k+1}} a_{ij} \leq 0.$$

Further let $p = max_{1\le i\le l,1\le k<m}|c'_{kj}|$. We claim that these m, p, c have the desired properties. Let $A_k = (a_{ij})_{1\le i\le m, j\in\cup_{s\le k} I_s}$ be the submatrix of A which only consists of the columns of A belonging to the blocks one up to k. We will prove the claim by induction on m.

Let $m = 0$. Hence $A = A_0$, i. e. for all $1 \le i \le l$ we have $\sum_{j=1}^n a_{ij} \le 0$. Thus every singleton forms a solution of the system $A\vec{x} \le \vec{0}$ and $D_{pc}(d_0) = \{cd_0\} \neq \emptyset$. Assume that the statement is true for some $k \ge 0$. Consider a $(k+1, p, c)$-set $D = D_{p,c}(d_0, \dots, d_{k+1})$. By induction we know that the (k, p, c)-set $D_{p,c}(d_0, \dots, d_k)$ contains a solution of the system $A_k\vec{x} \le \vec{0}$. Let $(y_i)_{i\in\cup_{s\le k} I_s}$ be such a solution, i. e. $y_i \in D_{p,c}(d_0, \dots, d_k)$ and for all $1 \le i \le l$ we have

$$\sum_{j\in\cup_{s\le k} I_s} a_{ij} y_j \le 0,$$

which implies

$$\underbrace{\sum_{j\in\cup_{s\le k} I_s} a_{ij} y_j}_{\le 0} + \underbrace{d_{k+1}(\sum_{j\in\cup_{s\le k} I_s} c_{kj} a_{ij} + c \sum_{j\in I_{k+1}} a_{ij})}_{\le 0} \le 0.$$

Hence for all $1 \le i \le l$ we have

$$\sum_{j\in\cup_{s\le k} I_s} a_{ij}(y_j + d_{k+1} c_{kj}) + \sum_{j\in I_{k+1}} c d_{k+1} a_{ij} \le 0.$$

For $y_j \in D_{p,c}(d_0, \dots, d_k)$ and $|c_{kj}| \le p$ we have

$$y_i + c_{kj} d_{k+1} \in D_{p,c}(d_0, \dots, d_{k+1}) \text{ and}$$

$$c d_{k+1} \in D_{p,c}(d_0, \dots, d_{k+1}).$$

Hence we found a solution of the system $A_{k+1}\vec{x} \le \vec{0}$ in the arbitrary chosen $(k+1, p, c)$-set $D_{p,c}(d_0, \dots, d_{k+1})$. $\Box_{Theorem\ 2.2.}$

Theorem 2.3. *Let $A = (a_{ij})_{1\le i\le m,1\le j\le n}$ be a rational matrix. If there exist $m, p, c \in \mathbb{N}$ such that every $(m, p, c)-$set contains a solution of the system $A\vec{x} \le \vec{0}$ then the system $A\vec{x} \le \vec{0}$ is partition regular.*

Proof of Theorem 2.3.: Let $m, p, c \in \mathbb{N}$ be given such that every $(m, p, c)-$set contains a solution of the system $A\vec{x} \le \vec{0}$. By Deuber's theorem [1] we know that for every coloring Δ of the natural numbers with finitely many colors there exist $d_0 \dots d_m$ such that the (m, p, c)-set $D = D_{p,c}(d_0, \dots, d_m)$ is monochromatic with respect to Δ. For every (m, p, c)-set contains a solution of the system $A\vec{x} \le \vec{0}$, so does D and hence $A\vec{x} \le \vec{0}$ is partition regular. $\Box_{theorem\ 3.4.}$

Deuber [1] also proved a partition theorem for $(m, p, c)-$sets in order to resolve the following conjecture Rado stated 1933 [10].

Call a subset $S \subseteq \mathbb{N}$ *partition regular* if every partition regular system of linear equations can be solved in S. Rado conjectured that coloring a partition regular set S there is one color class which is again partition regular.

THEOREM 2.4. (DEUBER 1973) *Let m, p, c and r be positive integers. Then there exist positive integers n, q, d such that for every (n, q, d)-set $D \subseteq \mathbb{N}$ and every r-coloring $\Delta \to [r]$ there exists a monochromatic $(m, p, c)-$set $D' \subseteq D$.*

We can enlarge the definition of a partition regular set [1] to systems of linear inequalities:

DEFINITION 2.2. *Call a subset $S \subseteq \mathbb{N}$ partition regular for systems of inequalities (pri) if every partition regular system of inequalities $A\vec{x} \leq \vec{0}$ can be solved in S.*

Note that for matrices A and B having *cpi* over $\mathbb{N}$ also the direct sum

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

has *cpi* over $\mathbb{N}$.

THEOREM 2.5. *For every coloring of a pri set with finitely many colors at least one of the color classes again is partition regular for inequalities.*

PROOF OF THEOREM 2.5.: Assume that the statement is false, i. e. there exists a set $S \subseteq \mathbb{N}$ which is *pri* and there exists $r \in \mathbb{N}$ and a coloring $\Delta : S \to [r]$ such that no color class of Δ is *pri*. Thus for each color class i there exists a matrix A_i such that the system $A_i\vec{x} \leq \vec{0}$ is partition regular but has no solution in $\Delta^{-1}(i)$. Consider the system

$$(*) \qquad \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & \ddots & 0 \\ 0 & \dots & & 0 & A_r \end{pmatrix} \vec{x} \leq \vec{0}.$$

$(*)$ is partition regular therefore there exist $m, p, c \in \mathbb{N}$ such that every $(m, p, c)-$set contains a solution of $(*)$. By Deuber's theorem [1] there exist $n, q, d \in \mathbb{N}$ such that each coloring of an arbitrary $(n, q, d)-$set with finitely many colors contains a monochromatic $(m, p, c)-$set. For S is *pri*, it contains a $(n, q, d)-$set. Hence there is some $(m, p, c)-$set in S which is monochromatic with respect to Δ and thus there exists a monochromatic solution of $(*)$ in S which contradicts the definition of $(*)$. $\square_{theorem}$ 2.5.

3. CANONICAL RESULTS

In this chapter we want to extend our considerations to colorings with an unlimited number of colors. Call a coloring Δ of a set S *canonical* if Δ is either

1. monochromatic, i. e. for all $s, t \in S$ it holds $\Delta(s) = \Delta(t)$ or
2. distinct, i.e. for all $s, t \in S$ with $s \neq t$ it holds $\Delta(s) \neq \Delta(t)$.

In 1950 Erdös and Rado [4] proved a canonical version of Ramsey's theorem:

Theorem 3.1. (Erdös, Rado 1950) *If an infinite set S is colored then some infinite subset T is canonically colored. For all $k \in \mathbb{N}$ if $|S| > (k-1)^2 + 1$ and S colored there exists a subset $T \subseteq S$, $|T| = k$ which is canonically colored.*

Later Erdös and Graham [3] proved a canonical version of van der Waerden's theorem, i. e. for every $k \in \mathbb{N}$ and every coloring of the positive integers there exists a canonically colored k-term arithmetic progression. In 1986 Lefmann [7] extended the Erdös-Graham canonical theorem for arithmetic progressions to a canonical partition theorem for $(m, p, c)-$sets and partition regular systems of linear equations.

Let $D = D_{p,c}(d_0, \ldots, d_m) = \{cd_i + \sum_{j=i+1}^{m} l_j d_j | i \leq m, l_j \in [-p, p]\}$. Say that the elements of the form $cd_i + l_{i+1} d_{i+1} + \ldots + l_m x_m$ *belong to the ith row of the $(m, p, c)-$set* $D_{p,c}(d_0, \ldots, d_m)$. Let us further say that $\Delta : D_{p,c}(d_0, \ldots, d_m) \to \omega$ is a *row-coloring* provided that any two numbers $a, b \in D_{p,c}(d_0, \ldots, d_m)$ are colored the same if and only if they belong to the same row of $D_{p,c}(d_0, \ldots, d_m)$.
Lefmann proved the following theorem [7]:

Theorem 3.2. (Lefmann 1986) *Let $m, p, c \in \mathbb{N}$. Then there exists a least positive integer $L(m, p, c)$ with the following property: For every coloring $\Delta : [L(m, p, c)] \to \omega$ there exists a (m, p, c)-set $D_{p,c}(d_0, \ldots, d_m) \subseteq [L(m, p, c)]$ such that $\Delta\big|_{D_{p,c}(d_0,\ldots,d_m)}$ either is a canonical coloring or a row-coloring.*

As a corollary Lefmann [7] proved a canonical version of Rado's theorem:

Corollary 3.1. (Lefmann) *Let $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$ be an integer valued matrix having the column property, i. e. the system of linear equations $A\vec{x} = \vec{0}$ is partition regular. Let $I_0 \cup \ldots \cup I_m = [n]$ be the corresponding partition of the columns of A into blocks. Then there exists a positive integer $N \in \mathbb{N}$ such that for every coloring $\Delta : [N] \to \omega$ there exists a solution $\vec{x} = (x_1 \ldots x_n)$ such that one of the following cases holds:*

1. *$\Delta\big|_{\{x_1,\ldots,x_n\}}$ is a canonical coloring.*
2. *Each two elements x_i, x_j of $\{x_1, \ldots, x_n\}$ are colored the same if and only if $\{i, j\} \subseteq I_k$ for some $k \leq m$.*

In the following we will prove a canonical theorem for systems of linear inequalities, which is similar to the above canonical version of Rado's theorem.

Theorem 3.3. *Let $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$ be a rational matrix and let the system $A\vec{x} \leq \vec{0}$ be partition regular, i. e. A has cpi. Let $I_0 \cup \ldots \cup I_m = [n]$ be the corresponding* *partition of the columns of A into blocks. Then for every coloring $\Delta : \mathbb{N} \to \omega$ of the natural numbers there exists a solution $\vec{x} = (x_1 \ldots, x_n) \in \mathbb{N}^n$ such that one of the following cases is valid:*

1. $\Delta \big|_{\{x_1,\ldots,x_n\}}$ *is a canonical coloring*
2. $\Delta(x_i) = \Delta(x_j)$ *for some* $i, j \in [n]$ *if and only if there exists some* $k \leq m$ *such that* $i, j \in I_k$.

Proof of theorem 3.3.: The system $A\vec{x} \leq \vec{0}$ is partition regular. Thus by theorem 3.3. there exist positive integers m, p, c such that every $(m, p, c)-$set contains a solution of the system $A\vec{x} \leq \vec{0}$. In the proof of lemma 3.3. in chapter 3 we saw that a solution of $A\vec{x} \leq \vec{0}$ in an arbitrary $(m, p, c)-$set D can be constructed in such a way that for $i \in I_l$ x_i comes from the lth row of D. Let $\Delta : \mathbb{N} \to \omega$ be given. Theorem 4.2. gives us a $(m, p, c)-$set $D_{p.c}(d_0, \ldots, d_m)$ such that $\Delta \big|_{D_{p,c}(d_0,\ldots,d_m)}$ either is a canonical or a row-coloring. Let $\vec{y} = (y_1 \ldots y_n)$ be a solution of the system $A\vec{x} \leq \vec{0}$ such that for all $1 \leq i \leq n$ we have $y_i \in D_{p,c}(d_0, \ldots, d_m)$ and for $i \in I_k$ y_i belongs to the kth row of $D_{p,c}(d_0, \ldots, d_m)$. If $D_{p,c}(d_0, \ldots, d_m)$ is canonically colored then $\Delta \big|_{\{y_1,\ldots,y_n\}}$ is a canonical coloring and if $\Delta \big|_{D_{p,c}(d_0,\ldots,d_m)}$ is a row coloring then $\Delta(y_i) = \Delta(y_j)$ if and only if y_i and y_j belong to the same row of $D_{p,c}(d_0, \ldots, d_m)$, i. e. if and only if i and j belong to the same block I_k for some $k \leq m$. $\Box_{theorem\ 3.3.}$

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