An Alternative Proof of Scheiderer's Theorem on the Hasse Principle for Principal Homogeneous Spaces

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Abstract. We give an alternative proof of the Hasse principle for principal homogeneous spaces defined over fields of virtual cohomological dimension at most one which is based on a special decomposition of elements in Chevalley groups.

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1. Introduction

Let Y be a smooth irreducible projective curve defined over the real number field $\mathbb{R}$ and $k = \mathbb{R}(Y)$ be the field of $\mathbb{R}$-rational functions on Y. For a point $P \in Y(\mathbb{R})$ we denote the completion of k at the point P by k_P. The present paper is devoted to the Hasse principle for the existence of a rational point on principal homogeneous spaces of a connected linear algebraic group G defined over k. It was Colliot-Thélène who conjectured ([CT], Conjecture 2.9) that for any such space X the Hasse principle holds relative to all local fields k_P, $P \in Y(\mathbb{R})$, i.e. $X(k) \neq \emptyset$ iff $X(k_P) \neq \emptyset$ for each $P \in Y(\mathbb{R})$. Since principal homogeneous spaces of G are in natural one-to-one correspondence with elements of the set $H^1(k, G)$ the latter statement is equivalent to the following: the natural map of pointed sets

$$H^1(k,G) \longrightarrow \prod_{P \in Y(\mathbb{R})} H^1(k_P, G) \tag{1}$$

has trivial kernel ([S]).

In [CT] Colliot-Thélène proved the Hasse principle for algebraic k-tori and reduced the general case to that of a simple simply connected algebraic group G. The case of an arbitrary connected k-group G has been studied by Scheiderer ([Sch1]).

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To prove the Hasse principle he first made an important observation (which eventually turned out to be crucial) that local objects k_P can be replaced by real closures k_ξ of k, $\xi \in \Omega_k$, where Ω_k denotes the set of all orderings of k. Indeed, using the description of orderings of k and the so-called Artin-Lang homomorphism theorem ([Srl], Theorem 3.1) it is easy to show that the condition $X(k_P) \neq \emptyset$ for each real point P on Y implies $X(k_\xi) \neq \emptyset$ for each ordering ξ of k and hence the triviality of the kernel of (1) follows immediately from the triviality of the kernel of

$$\theta:\ H^1(k,G) \longrightarrow \prod_{\xi\in\Omega_k} H^1(k_\xi, G) \tag{2}$$

The question whether θ is injective makes sense not only for the function fields of curves but also for an arbitrary field k and it turned out that θ is indeed injective if k has virtual cohomological dimension (vcd) at most 1 (recall that function fields in one variable over $\mathbb{R}$ are such). We have even more.

Theorem 1. (Scheiderer, [Sch1]) *Let K be any field of virtual cohomological dimension ≤ 1. Then the Hasse principle holds for any homogeneous K-space X of a connected linear algebraic K-group G.*

Scheiderer's proof can be divided into two parts. In the first one it is proved that for X as in the theorem (here G may even be not connected) there exists a principal homogeneous space Z which is everywhere locally trivial and dominates X. The strategy of the proof in this part going back to Springer ([S], [Sp]) consists of replacing X by a homogeneous space which dominates X and has a smaller stabilizer. It is worth mentioning that in this part most arguments do not use specific properties of K and so most of them are valid over an arbitrary perfect field.

The second part of Scheiderer's proof is devoted to the case of a principal homogeneous space. To treat such a space Scheiderer first constructs a locally constant sheaf of sets $\mathcal{H}^1(G)$ on Ω_K whose stalks are just the sets $H^1(K_\xi, G)$. Then he shows that there exists a natural bijection between the set of global sections of $\mathcal{H}^1(G)$ and $H^1(K,G)$. As a whole the proof in this part is quite complicated. It is based on using étale machinery and, in particular, strongly relies on results of the book [Sch2].

The aim of this paper is to provide a simpler and shorter self-contained proof which is based only on the Bruhat decomposition in semisimple algebraic groups and the so-called strong approximation property (SAP) of fields (see § 3). We show that in fact the Hasse principle follows immediately modulo two facts. Informally speaking one of them says that the kernel of the natural map $H^1(K,T) \to H^1(K,G)$, where G is an (absolutely) simple simply connected linear K-group and T is a K-torus splitting over $K(\sqrt{-1})$, can be parametrized by "good" rational functions (see § 2) and the other says that any field of virtual cohomological dimension ≤ 1 is an SAP field.

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2. Algebraic groups splitting over quadratic extensions

Throughout the section K denotes an arbitrary field of characteristic 0. Let G be an (absolutely) simple simply connected algebraic group of rank n defined over K

and splitting over quadratic extension $L = K(\sqrt{d}\,)$. Let

$$\Theta = \mathrm{Gal}\,(L/K) = \langle \tau \mid \tau^2 = 1\rangle.$$

Consider a Borel L-subgroup B such that $T = B \cap \tau(B)$ is a maximal torus which will be assumed for simplicity to be K-anisotropic. Since T is splitting over L, one has

$$T \simeq \mathrm{R}^{(1)}_{L/K}(\mathrm{G}_m) \times \ldots \times \mathrm{R}^{(1)}_{L/K}(\mathrm{G}_m).$$

To prove the Hasse principle we need to describe $\mathrm{Ker}\,[H^1(\Theta, T(L)) \to H^1(\Theta, G(L))]$. This description can be easily extracted from [Ch]. However this paper is written in Russian and the translation made by the AMS is unreadable and contains a lot of misprints. So for the sake of expository completeness and the reader's convenience we include here details.

First recall some basic facts about the structure of the group $G(L)$ (for details see [St1]). Let $\Sigma = R(T,G)$ be the root system of G relative to T. The Borel subgroup B determines an ordering on the set Σ and hence a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. If $\alpha = \sum n_i\alpha_i \in \Sigma^+$, then the number $\mathrm{ht}(\alpha) = \sum n_i$ is called the height of α. If $\{X_\alpha, \alpha \in \Sigma; H_{\alpha_1}, \ldots, H_{\alpha_n}\}$ is a Chevalley basis of the Lie algebra of G, then $G(L)$ is generated by the corresponding root subgroups $G_\alpha = \langle x_{\pm\alpha}(t) \mid t \in L\rangle$, where

$$x_\alpha(t) = \sum_{n=0}^{\infty} t^n X_\alpha^n\,/n!\ ,$$

and the torus T is generated by $T_\alpha = T \cap G_\alpha = \langle h_\alpha(t)\rangle$, where $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$ and $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$.

Furthermore, since G is simply connected the following relations hold in G (cf. [St1], Lemma 28 b), Lemma 20 c), Lemma 15):

A) $T = \langle h_{\alpha_1}(t_1)\rangle \times \cdots \times \langle h_{\alpha_n}(t_n)\rangle$ and for $\alpha \in \Sigma$ we have

$$h_\alpha(t) = \prod_{i=1}^{n} h_{\alpha_i}(t)^{n_i}, \qquad \textit{where } H_\alpha = \sum_{i=1}^{n} n_i H_{\alpha_i}; \tag{3}$$

B) For $\alpha, \beta \in \Sigma$ let $\langle \beta, \alpha\rangle = 2\,(\beta,\alpha)/(\alpha,\alpha)$. Then we have

$$h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta(t^{\langle\beta,\alpha\rangle}u) \tag{4}$$

C) For all $u, v \in L$ such that $1 + uv \neq 0$ we have

$$x_{-\alpha}(u)x_\alpha(v) = x_\alpha(v(1+uv)^{-1})h_\alpha(1+uv)^{-1}x_{-\alpha}(u(1+uv)^{-1}) \tag{5}$$

D) For all $\alpha, \beta \in \Sigma$, $\beta \neq -\alpha$, we have

$$x_\alpha(v)x_\beta(u)x_\alpha(v)^{-1}x_\beta(u)^{-1} = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{i,j}v^iu^j) \tag{6}$$

where the product on the right hand side is taken over all roots of the form $i\alpha + j\beta$ and the $c_{i,j}$ are integers which depend on α, β and on the chosen ordering of the roots but do not depend on v and u.

Since T is K-defined, τ acts on the root system Σ. More exactly, for any $\alpha \in \Sigma$ the character $\alpha + \tau(\alpha)$ is K-defined and hence is zero, i.e. $\tau(\alpha) = -\alpha$, since, by assumption, T is K-anisotropic. It follows that there exists $c_\alpha \in L^*$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$; in particular, the subgroup G_α is K-defined.

The constants c_α actually lie in K and $c_{-\alpha} = c_\alpha^{-1}$. Indeed, for rank one groups, i.e. of the form SL $(1, D)$, where D is a quaternion K-algebra, this fact can be verified directly. The general case easily reduces to the rank one case since G_α is a simple simply connected K-group of rank 1. Thus, we have

LEMMA 1. *There exists constant $c_\alpha \in K^*$ such that for any $u \in L$ one has $\tau(x_\alpha(u)) = x_{-\alpha}(c_\alpha \tau(u))$. Moreover, $G_\alpha \simeq SL(1, D)$, where D is a quaternion algebra over K of the form $D = (d, c_\alpha)$.*

PROOF: Straightforward computations. □

LEMMA 2. *The positive roots $\Sigma^+ = \{\beta_1, \dots, \beta_m\}$ can be ordered in such a way that the following two properties hold:*
1) for any pair of roots β_i, β_j, for which $i < j$ and $\beta_i + \beta_j = \beta_k \in \Sigma^+$, the root β_k is between β_i and β_j, i.e. $i < k < j$;
2) if Σ is a root system of type either A_{2n-1} or D_n or E_6 and σ is the outer automorphism of Σ induced by the non-trivial automorphism of order 2 (resp. 3) of the corresponding Dynkin diagram, then for any root $\beta_i \in \Sigma^+$ the roots β_i and $\sigma(\beta_i)$ (resp. β_i, $\sigma(\beta_i)$, $\sigma^2(\beta_i)$) are neighbours.

PROOF. a) Let $\Sigma = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le 2n\}$ be a root system of type A_{2n-1}. Let $\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{2n-1} = \varepsilon_{2n-1} - \varepsilon_{2n}$ be a basis of Σ and Σ_1 be the subsystem generated by the roots $\alpha_2, \dots, \alpha_{2n-2}$. By induction, we can pick an ordering $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$ with the required properties. Let $\gamma = \alpha_1 + \dots + \alpha_{2n-1}$. We number the remaining roots $\Sigma^+ \setminus \{\Sigma_1^+ \cup \gamma\} = \{\beta_{k+1}, \dots, \beta_{m-1}\}$ in the order of decreasing height. If β_i denotes the last root among $\{\beta_{k+1}, \dots, \beta_{m-1}\}$ such that ht $(\beta_i) \ge n$, then the ordering

$$\Sigma^+ = \{\beta_1, \dots, \beta_k, \beta_{k+1}, \dots, \beta_i, \gamma, \beta_{i+1}, \dots, \beta_{m-1}\}$$

is as required.

b) Σ is a root system of type $A_{2n}, B_n, C_n, D_n, E_7$. It follows from the description of root systems of these types that there exists a subsystem Σ_1 generated by $n-1$ simple roots, say $\alpha_1, \dots, \alpha_{n-1}$, such that any root $\beta \in \Sigma^+ \setminus \Sigma_1^+$ can be written as a sum $\beta = m_1\alpha_1 + \dots + m_{n-1}\alpha_{n-1} + \alpha_n$. If Σ is of type D_n and $|\sigma| = 2$, we may assume in addition that the set $\{\alpha_1, \dots, \alpha_{n-1}\}$ is stable under σ. The root system Σ_1 has rank $n-1$ and so by induction, there exists an ordering of the required type on the set $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$. We number the remaining roots $\Sigma^+ \setminus \Sigma_1^+ = \{\beta_{k+1}, \dots, \beta_m\}$ in the order of decreasing height. Then the ordering $\{\beta_1, \dots, \beta_m\}$ is as required.

c) Σ is a root system of type E_6, E_8, F_4, G_2. Here one can argue as in case a). Namely, there exists a subsystem Σ_1 generated by simple roots $\alpha_1, \dots, \alpha_{n-1}$ such that any root $\beta \in \Sigma^+ \setminus \Sigma_1^+$ is of the form $\beta = m_1\alpha_1 + \dots + m_{n-1}\alpha_{n-1} + \alpha_n$ except for the maximal root $\tilde{\alpha}$ and $\tilde{\alpha}$ is of the form $\tilde{\alpha} = m_1\alpha_1 + \dots + m_{n-1}\alpha_{n-1} + 2\alpha_n$. Let $b = \mathrm{ht}\,(\tilde{\alpha})$. Again, applying induction we can find an ordering $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$ with the desired properties and then we number the roots $\Sigma^+ \setminus \{\Sigma_1^+ \cup \tilde{\alpha}\} = \{\beta_{k+1}, \dots, \beta_{m-1}\}$ in the order of decreasing height. If Σ has type E_6, we may assume in addition that β and $\sigma(\beta)$ are neighbours for all $\beta \in \Sigma^+$. Let β_i be the last root among $\{\beta_{k+1}, \dots, \beta_{m-1}\}$

such that $\mathrm{ht}\,(\beta_i) \geq b/2$. We claim that the ordering

$$\Sigma^+ = \{\beta_1, \dots, \beta_k,\, \beta_{k+1}, \dots, \beta_i,\, \tilde{\alpha},\, \beta_{i+1}, \dots, \beta_{m-1}\}$$

has the desired properties. Indeed, if $\beta_j = \beta_s + \beta_t$, where $s < t$ and $j \in \{k+1, \dots, m-1\}$, then clearly β_s belongs to Σ_1^+. It follows that β_j lies between β_s and β_t, since $\mathrm{ht}\,(\beta_j) \geq \mathrm{ht}\,(\beta_s), \mathrm{ht}\,(\beta_t)$. Now let $\tilde{\alpha} = \beta_s + \beta_t$, $s < t$,. Then $s, t \in \{k+1, \dots, m-1\}$ and $\mathrm{ht}\,(\beta_s) \geq b/2$, $\mathrm{ht}\,(\beta_t) < b/2$ (we use the fact that $\mathrm{ht}(\tilde{\alpha})$ is odd), implying $\tilde{\alpha}$ is also between β_s and β_t.

d) Σ has type D_4 and $|\sigma| = 3$. Let $\alpha_1, \dots, \alpha_4$ be simple roots such that σ permutes $\alpha_1, \alpha_3, \alpha_4$. Then the required ordering is as follows: first we place α_2, then all roots of the height 2, then the maximal root and then the roots of heights 3, 4, 1 respectively. □

Corollary 1. *Let β_i, β_j, $j < i$, be any two positive roots. Then for any positive root β_k of the form $\beta_k = r\beta_j - l\beta_i$, $r, l > 0$, one has $k < j$. Analogously, for any negative root of the form $-\beta_k = r\beta_j - l\beta_i$, $r, l > 0$, one has $k > i$.*

Proof. We distinguish three cases.

a) $\langle \beta_i, \beta_j \rangle_{\mathbb{Q}} \cap \Sigma$ has type A_2. Then $r = l = 1$ and hence if $\beta_k = \beta_j - \beta_i$ is a positive root then $\beta_k + \beta_i = \beta_j$, implying $k < j < i$. Analogously, if $\beta_j - \beta_i = -\beta_k$ then we have $j < i < k$.

b) $\langle \beta_i, \beta_j \rangle_{\mathbb{Q}} \cap \Sigma$ has type B_2. Then either $r = l = 1$ or $r = 1$ and $l = 2$ or $r = 2$ and $l = 1$. The case $r = l = 1$ was already handled in part a). Now let $\beta_k = \beta_j - 2\beta_i$. Then $\beta_j - \beta_i = \beta_s$ is also a positive root implying $s < j$. Futhermore, $\beta_k = \beta_s - \beta_i$ and $s < j < i$. So again we have $k < s < j$. The remaining cases can be handled in a similar way.

c) $\langle \beta_i, \beta_j \rangle_{\mathbb{Q}} \cap \Sigma$ has type G_2. Here the proof is similar to that of case b) and we omit it. □

Proposition 1. *Fix an order in Σ^+ as in Lemma 2. Then the regular map*

$$\omega : \mathrm{G}_m^n \times \mathbb{A}^{2m} \longrightarrow G, \qquad (t_1, \dots, t_n, u_1, \dots, u_m, v_1, \dots, v_m) \longrightarrow$$
$$\prod_{i=1}^{n} h_{\alpha_i}(t_i)\, x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_m}(u_m) x_{\beta_m}(v_m)$$

is birational over L.

Remark 1. This statement is also true in positive characteristic. There is the only place which require additional work: one need additionally to check that ω is a separable map.

Proof. Both sides have the same dimension and hence it suffices to prove the injectivity of ω on some Zariski open subset, since $\mathrm{char}\, K = 0$.

First we show that for any integer i and any parameters $u_1, \dots, u_i$ and $v_1, \dots, v_i$ from some Zariski open subset the element

$$A_i = x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_i}(u_i) x_{\beta_i}(v_i)$$

of the group G can be written in the form

$$A_i = \prod_{k=1}^{n} h_{\alpha_k}(f_k) \prod_{j=1}^{m} x_{-\beta_j}(r_j) \prod_{j=1}^{i-1} x_{\beta_j}(s_j)\, x_{\beta_i}(v_i),$$

where f_k, r_j, s_j are rational functions depending on $u_1, \dots, u_i$, $v_1, \dots, v_{i-1}$.

If $i = 1$ there is nothing to prove. By induction, we may write A_{i-1} in the form

$$\prod_{k=1}^{n} h_{\alpha_k}(f_k) \prod_{j=1}^{m} x_{-\beta_j}(r_j) \prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1}).$$

To write $A_i = A_{i-1}\, x_{-\beta_i}(u_i)\, x_{\beta_i}(v_i)$ in the same form we have to transpose $x_{-\beta_i}(u_i)$ with each factor in the product $\prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1})$. By (6) and by Corollary 1, every time doing so we obtain additional factors $x_{\beta_s}(\,)$ or $x_{-\beta_s}(\,)$, where $s < i-1$ in the first case and $s > i$ in the second case. Collecting together all these factors corresponding to negative roots we can write the element $\prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1})\, x_{-\beta_i}(u_i)$ in the form

$$\prod_{k=1}^{n} h_{\alpha_i}(\tilde{f}_k) \prod_{j=1}^{m} x_{-\beta_j}(\tilde{r}_j) \prod_{j=1}^{i-1} x_{\beta_j}(\tilde{s}_j)$$

and so our claim follows.

Now we are ready to prove the injectivity of ω. Suppose that

$$\omega(t_1, \dots, t_n,\, u_1, \dots, u_m,\, v_1, \dots, v_m) = \omega(\tilde{t}_1, \dots, \tilde{t}_n,\, \tilde{u}_1, \dots, \tilde{v}_m) \tag{7}$$

From the above argument and the Bruhat decomposition we get immediately $v_m = \tilde{v}_m$. To show that $u_m = \tilde{u}_m$, we use (4), (5). Namely, it follows from (4), (5) that the left hand side of (7) may be written in the form

$$\prod_{i=1}^{n} h_{\alpha_i}(f_i)\, [x_{\beta_1}(s_1)\, x_{-\beta_1}(r_1)] \cdots [x_{\beta_{m-1}}(s_{m-1})\, x_{-\beta_{m-1}}(r_{m-1})]$$

$$x_{\beta_m}[v_m(1 + u_m v_m)]\, x_{-\beta_m}[u_m(1 + u_m v_m)^{-1}]\,,$$

where $f_1, \dots, f_n$, $s_1, \dots, s_{m-1}$, $r_1, \dots, r_{m-1}$ are rational functions. Rewriting the right hand side of (7) in the same form we conclude that

$$u_m(1 + u_m v_m)^{-1} = \tilde{u}_m(1 + \tilde{u}_m \tilde{v}_m)^{-1},$$

hence $u_m = \tilde{u}_m$. After cancelling the factor $x_{-\beta_m}(u_m)\, x_{\beta_m}(v_m)$ in (7) the same argument shows that $v_{m-1} = \tilde{v}_{m-1}$, $u_{m-1} = \tilde{u}_{m-1}$ and so on. □

Now we are in position to formulate the main result of the section.

THEOREM 2. *Let $g \in G(L)$ be such that $g^{1-\tau} \in T(L)$. Then there exist quaternion algebras $D_1, \dots, D_m$ over K and elements $w_1, \dots, w_m \in K$ which are reduced norm of $D_1, \dots, D_m$ respectively and elements $t_1, \dots, t_n \in L$ such that*

$$g^{1-\tau} = \prod_{i=1}^{n} h_{\alpha_i}(t_i \tau(t_i)) \prod_{i=1}^{m} h_{\beta_i}(w_i)$$

PROOF. If $g^{1-\tau} \in T(L)$, then for any $x \in G(K)$ one has $g^{1-\tau} = (gx)^{1-\tau}$. Since $G(K)$ is Zariski dense in G, we may always assume that our element g is in "generic" position by which we mean point in some Zariski open subset $U \subset G$ which can be easily specified from the argument. So let

$$g = \prod_{i=1}^{n} h_{\alpha_i}(t_i)\, x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_m}(u_m) x_{\beta_m}(v_m)$$

where $t_i,\ u_i,\ v_i \in L$. Denote $t = \prod_{i=1}^{n} h_{\alpha_i}(t_i)$ and $g_i = x_{-\beta_i}(u_i)x_{\beta_i}(v_i)$, $i = 1, \dots, m$. Let also $t' = g^{1-\tau}$, so that

$$t \cdot g_1 \cdots g_m = t' \cdot \tau(t) \cdot \tau(g_1) \cdots \tau(g_m) \tag{8}$$

By Lemma 1, we have $\tau(g_i) \in G_{\beta_i}$. Then applying Proposition 1 we conclude that g_m and $\tau(g_m)$ coincide modulo $T_{\beta_m}(L) = T(L) \cap G_{\beta_m}$ and so the element $g_m^{\tau-1}$ is of the form $h_{\beta_m}(w_m)$ for some parameter w_m. We claim that $w_m \in K$ and it is a reduced norm of the quaternion K-algebra $D_m = (d, d_{\beta_m})$, where $d_{\beta_m} = c_{\beta_m}$. Indeed, by construction the cocycle $(g_m^{\tau-1}) \in Z^1(\Theta, T_{\beta_m}(L))$ is trivial in $Z^1(\Theta, G_{\beta_m}(L))$ and by Lemma 1, $G_{\beta_m} \simeq SL(1, D_m)$, hence our claim follows.

Substituting $\tau(g_m) = h_{\beta_m}(w_m) \cdot g$ in (8) and cancelling g, we have then

$$t \cdot g_1 \cdots g_{m-1} = t' \cdot \tau(t) \cdot h_{\beta_m}(w_m) \cdot [h_{\beta_m}(w_m)^{-1}\tau(g_1)h_{\beta_m}(w_m)] \cdots$$
$$\cdots [h_{\beta_m}(w_m)^{-1}\tau(g_{m-1})h_{\beta_m}(w_m)]$$

Applying again Proposition 1 and arguing analogously we have

$$[h_{\beta_m}(w_m)^{-1}\tau(g_{m-1})h_{\beta_m}(w_m)] = h_{\beta_{m-1}}(w_{m-1}) \cdot g_{m-1}$$

for some parameter w_{m-1}, which is again a reduced norm of the quaternion K-algebra $D_{m-1} = (d, d_{\beta_{m-1}})$, where

$$d_{\beta_{m-1}} = c_{\beta_{m-1}} w_m^{\langle \beta_{m-1}, \beta_m \rangle}.$$

To see it, let $\tilde{g}_{m-1} = h_{\beta_m}(w_m)^{-1}\tau(g_{m-1})h_{\beta_m}(w_m)$. Using (4) we have

$$\tilde{g}_{m-1} = x_{\beta_{m-1}}(c_{\beta_{m-1}}^{-1} w_m^{-\langle \beta_{m-1}, \beta_m \rangle}\tau(u_m)) \cdot x_{\beta_{m-1}}(c_{\beta_{m-1}} w_m^{\langle \beta_{m-1}, \beta_m \rangle}\tau(v_m)).$$

It follows that $(h_{\beta_{m-1}}(w_{m-1})) = (\tilde{g}_{m-1} \cdot g_{m-1}^{-1})$ can be viewed as a trivial cocycle in an K-group of rank 1 whose K-structure, i.e. action of τ, is given by the constant $d_{\beta_{m-1}}$. This fact combined with Lemma 1 implies w_{m-1} is a reduced norm of D_{m-1}, as claimed, and so on. Theorem 2 is proved. □

In § 4 we will also deal with a simple simply connected algebraic K-group G which is quasi-split over a quadratic extension L/K and for such a group we also need to describe elements of the form $g^{1-\tau} \in T(L)$, where $g \in G(L)$.

Clearly, K-groups of type ${}^2A_{2n}$ split over a quadratic extension of K. Since this case has been already handled, we may assume that G is an outer form of type not A_{2n}. As above, let B be an L-Borel subgroup B of G such that $T = B \cap \tau(B)$ is a maximal K-anisotropic torus.

Let F/K be the minimal extension over which G is an inner form and let $E = F{\cdot}L$. Let τ and σ be non-trivial automorphisms of E/K such that $\tau|_F = 1$ and $\sigma|_L = 1$ respectively. In the case ${}^{3,6}D_4$ by σ we denote any automorphism of order 3.

Clearly, σ induces an outer automorphism of the root system $\Sigma = \mathrm{R}\,(T, G)$ which will be denoted by the same letter. Let $\Lambda = \{\gamma_1, \dots, \gamma_s\} \subset \Sigma^+$ (resp. Λ') be a set of representatives of all orbits of σ in Σ^+ (resp. in Π). We divide Λ into two parts: $\Lambda_1 = \{\gamma_i \in \Lambda \mid \sigma(\gamma_i) = \gamma_i\}$ and $\Lambda_2 = \Lambda \setminus \Lambda_1$. Let also $\Lambda_i' = \Lambda' \cap \Lambda_i$, $i = 1, 2$. For $\gamma_i \in \Lambda_1$ (resp. Λ_2) we denote by H_i the subgroup in G generated by G_{γ_i} (resp. G_{γ_i}, $G_{\sigma(\gamma_i)}$ and $G_{\sigma^2(\gamma_i)}$, if $|\sigma| = 3$).

Lemma 3. *H_i is a simple simply connected K-group of type A_1 (resp. $A_1 \times A_1$ or $A_1 \times A_1 \times A_1$) if $\gamma_i \in \Lambda_1$ (resp. $\gamma_i \in \Lambda_2$ and $|\sigma| = 2$ or $|\sigma| = 3$).*

PROOF. It suffices to note that τ acts on Σ as either -1, if Σ has type D_{2n}, or $-\sigma$ otherwise, since it permutes positive and negative roots. Moreover, the combination $\beta_i \pm \sigma(\beta_i)$ is not a root, hence G_{γ_i} and $G_{\sigma(\gamma_i)}$ commute. $\square$

THEOREM 3. *Let $g \in G(L)$ be such that $g^{1-\tau} \in T(L)$. Then there exist quaternion algebras $D_1, \dots, D_s$ and elements $w_1, \dots, w_s$ which are reduced norm of $D_1, \dots, D_s$ respectively and elements $t_1, \dots, t_p$ such that:*

1) If Σ is not of type ${}^{3,6}D_4$, then

$$g^{1-\tau} = \prod_{\alpha_i \in \Lambda_1'} h_{\alpha_i}(t_i\tau(t_i)) \prod_{\alpha_i \in \Lambda_2'} h_{\alpha_i}(t_i\tau(t_i))h_{\sigma(\alpha_i)}[\sigma(t_i)(\tau\circ\sigma)(t_i)]\cdot$$
$$\prod_{\gamma_i \in \Lambda_1} h_{\gamma_i}(w_i) \prod_{\gamma_i \in \Lambda_2} h_{\gamma_i}(w_i)h_{\sigma(\gamma_i)}(\sigma(w_i))$$

2) If Σ is of type ${}^{3,6}D_4$, then

$$g^{1-\tau} = \prod_{\alpha_i \in \Lambda_2'} h_{\alpha_i}(t_i\tau(t_i))h_{\sigma(\alpha_i)}[\sigma(t_i)(\tau\circ\sigma)(t_i)]h_{\sigma^2(\alpha_i)}[\sigma^2(t_i)(\tau\circ\sigma^2)(t_i)]\cdot$$
$$\prod_{\alpha_i \in \Lambda_1'} h_{\alpha_i}(t_i\tau(t_i)) \prod_{\gamma_i \in \Lambda_1} h_{\gamma_i}(w_i) \prod_{\gamma_i \in \Lambda_2} h_{\gamma_i}(w_i)h_{\sigma(\gamma_i)}(\sigma(w_i))h_{\sigma^2(\gamma_i)}(\sigma^2(w_i))$$

Here D_i is over K (resp. over F) and $w_i \in K$ (resp. F), if $\gamma_i \in \Lambda_1$ (resp. $\gamma_i \in \Lambda_2$), and $t_i \in L$ (resp. E), if $\alpha_i \in \Lambda_1'$ (resp. $\alpha_i \in \Lambda_2'$).

PROOF. As in the L-split case first we may assume that g is in "generic" position and so by property 2 in Lemma 2 and by Proposition 1, it can be written in the form $g = t g_1 \cdots g_s$, where $t \in T$, $g_i \in H_i$, $i = 1, \dots, s$. Then the rest of the proof works exactly as in the L-split case, since by Lemma 3 all subgroups H_i are of the form $\mathrm{R}_{K'/K}(\mathrm{SL}(1, D))$, where D is a quaternion algebra over K' and K' is either F or K. $\square$

3. SOME COHOMOLOGICAL COMPUTATIONS

From now on we assume that $\mathrm{vcd}\,(K) \leq 1$ and we let $L = K(\sqrt{-1})$. We also assume that the set Ω_K of all orderings on K is non-empty; this means, in particular, that $\mathrm{char}\, K = 0$. Recall ([Srl]) that there is a canonical topology on Ω_K under which Ω_K is compact and totally disconnected.

REMARK 2. If $\Omega_K = \emptyset$, then -1 is a sum of squares in K and so $\mathrm{cd}\,(K) = \mathrm{cd}\,(K(\sqrt{-1})) \leq 1$ ([S], Ch. 2, Prop. 10′). Therefore, if $\Omega_K = \emptyset$, then by Steinberg's theorem ([St2]) one has $H^1(K, G) = 1$ for any connected linear algebraic K-group G.

To reduce the proof of the Hasse principle to the case of simply connected semisimple groups we need two auxiliary cohomological statements (Propositions 2 and 4 below) which are very particular cases of the general Theorem 12.13 in [Sch2]. Since we do not need to consider such a generality as in [Sch2] we include here the straightforward proofs of these statements.

Let A be a discrete Γ-module, where $\Gamma = \mathrm{Gal}\,(\overline{K}/K)$, and let

$$\varphi_i : H^i(K, A) \to \prod_{\xi \in \Omega_K} H^i(K_\xi, A)$$

be the canonical map induced by res_{K_ξ}. We want to describe $\operatorname{Ker}\varphi_i$, $i \geq 2$, and $\operatorname{Im}\varphi_1$. To do so first remind that there is not a canonical way of choosing a real closure of K at $\xi \in \Omega_K$. If K_ξ and K'_ξ are two real closures of K at ξ, then by the theorem of Artin-Schreier ([Srl] Ch. 3, Theorem 2.1) there is a unique K-isomorphism $K_\xi \simeq K'_\xi$, hence there is an element $g \in \Gamma$ such that $g\tau_\xi g^{-1} = \tau'_\xi$, where τ_ξ (resp. τ'_ξ) is the involution (= element of order 2) in Γ corresponding to K_ξ (resp. K'_ξ) (in other words, there is a natural one-to-one correspondence between points of the set Ω_K and conjugacy classes of involutions in Γ).

The element g induces a natural map $\lambda_{i,g} : H^i(K_\xi, A) \to H^i(K'_\xi, A)$ and obviously we have $res_{K'_\xi} = \lambda_{i,g} \circ res_{K_\xi}$. It follows that the question on whether φ_i is injective does not depend on a choice of real closures K_ξ, $\xi \in \Omega_K$.

Clearly, any cocycle from $Z^1(K_\xi, A)$ is determined by the single element $a \in A$ such that $a\tau_\xi(a) = 1$. We will say that an element $\{a_\xi\}_{\xi\in\Omega_K} \in \prod_{\xi\in\Omega_K} H^1(K_\xi, A)$ is locally constant if there are a decomposition $\Omega_K = U_1 \cup \ldots \cup U_l$ into disjoint clopen (= open and closed) sets and elements $\{a_1, \ldots, a_l\}$ of A for which the following condition holds: for any $\xi \in U_i$ there are a cocycle c_ξ representing a_ξ and $g_\xi \in \Gamma$ such that the cocycle $\lambda_{1,g_\xi}(c_\xi)$ is determined by a_i. Analogously, for any $i \geq 1$ one defines the subset of elements in $\prod_{\xi\in\Omega_K} H^i(K_\xi, A)$ which are locally constant. We denote this subset by $\left(\prod_{\xi\in\Omega_K} H^i(K_\xi, A)\right)^{lc}$. Since for any $\zeta \in H^i(K, A)$ the element $\varphi_i(\zeta)$ is locally constant we denote by the same letter the canonical map

$$\varphi_i : H^i(K, A) \longrightarrow \left(\prod_{\xi\in\Omega_K} H^i(K_\xi, A)\right)^{lc} \subset \prod_{\xi\in\Omega_K} H^i(K_\xi, A)$$

Proposition 2. *If A is a finite discrete Γ-module, then the maps φ_i are injective for all integers $i \geq 2$.*

Proof. Since $H^i(L, A) = 1$, $i \geq 2$, the *"res-cores"* argument shows that $H^i(K, A)$ has exponent 2. So replacing A, if necessary, by its 2-Sylow subgroup we may assume that A is a 2-group. First examine the case $A = \mathbb{Z}/2\mathbb{Z}$.

Lemma 4. *Let $A = \mathbb{Z}/2\mathbb{Z}$. Then φ_i is surjective if $i \geq 1$ and injective if $i \geq 2$.*

Proof. Recall ([L], §17) that a field F is said to be an SAP field (strong approximation property) if for any two disjoint closed subsets $A, B \subset \Omega_F$ there exists an element $f \in F$ such that f is positive at all orderings in A, but negative at all orderings in B. We need

Proposition 3. ([L], Theorem 17.9) *If $\operatorname{vcd}(K) \leq 1$, then K is a SAP field.*

Surjectivity of φ_i, $i \geq 1$. In view of the periodicity of $H^i(K_\xi, \mathbb{Z}/2\mathbb{Z})$ it suffices to consider the cases $i = 1, 2$. If $i = 1$ then $H^1(K, \mathbb{Z}/2\mathbb{Z}) = K^*/K^{*2}$, hence the surjectivity of φ_1 follows immediately from Proposition 3. Furthermore, any element from $H^2(K, \mathbb{Z}/2\mathbb{Z})$ splits over L and so can be represented by a quaternion algebra having L as a maximal subfield. Then clearly, the surjectivity of φ_2 again follows from Proposition 3.

Injectivity of φ_i, $i \geq 2$. The proof is similar to that of [B-P], Lemma 2.3. Namely, by Arason's theorem ([A1], Satz 3), local triviality of $\zeta \in H^i(K, \mathbb{Z}/2\mathbb{Z})$ implies that $\zeta \cup (-1)^r = 0$ for some integer r, where $\cup$ denotes the cup product. On the other hand from the exact sequence

$$H^i(L,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{cor} H^i(K,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup(-1)} H^{i+1}(K,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{res} H^{i+1}(L,\mathbb{Z}/2\mathbb{Z})$$

([A2], Corollary 4.6) and from the equalities

$$H^i(L,\mathbb{Z}/2\mathbb{Z}) = H^{i+1}(L,\mathbb{Z}/2\mathbb{Z}) = 1, \quad i \geq 2$$

we conclude that the product $\cup(-1)$ is an isomorphism. Therefore, $\zeta = 1$, as required. Lemma 4 is proved. □

We come back to an arbitrary finite 2-primary module A. Let Γ_2 be a Sylow 2-subgroup of Γ. Since the restriction map $H^i(K,A) \to H^i(\Gamma_2,A)$ is injective, after replacing Γ by Γ_2 we may assume that Γ is a pro-2-group. But for such a group any irreducible module is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ ([S], §4, Proposition 20). Therefore there exists a submodule $A' \subset A$ such that $A/A' = \mathbb{Z}/2\mathbb{Z}$. It induces the commutative diagram

$$\begin{array}{ccccc}
H^i(K,\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^{i+1}(K,A') & \longrightarrow \\
\Big\downarrow \theta_1 & & \Big\downarrow \theta_2 & \\
\Big(\prod_{\xi\in\Omega_K} H^i(K_\xi,\mathbb{Z}/2\mathbb{Z})\Big)^{lc} & \longrightarrow & \Big(\prod_{\xi\in\Omega_K} H^{i+1}(K_\xi,A')\Big)^{lc} & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
H^{i+1}(K,A) & \longrightarrow & H^{i+1}(K,\mathbb{Z}/2\mathbb{Z}) \\
\Big\downarrow \theta_3 & & \Big\downarrow \theta_4 \\
\Big(\prod_{\xi\in\Omega_K} H^{i+1}(K_\xi,A)\Big)^{lc} & \longrightarrow & \Big(\prod_{\xi\in\Omega_K} H^{i+1}(K_\xi,\mathbb{Z}/2\mathbb{Z})\Big)^{lc}
\end{array}$$

By what has been proved above, θ_1 (resp. θ_4) is surjective (resp. injective) and by induction, θ_2 is injective. It follows that θ_3 is injective as well. Proposition 2 is proved. □

Proposition 4. *If A is a finite discrete Γ-module, then φ_1 is surjective.*

Proof. Since φ_i, $i \geq 2$, are injective, one can easily verify that if the statement holds both for a submodule $A' \subset A$ and the quotient A/A', then it also holds for A. So we may assume, if necessary, that A is irreducible. It suffices to prove that for a given $\xi \in \Omega_K$ and an element $a \in A$ for which $a\tau_\xi(a) = 1$ there exist a small clopen neighbourhood $U \subset \Omega_K$ of ξ and a cocycle $\zeta \in Z^1(K,A)$ such that for a proper real closure $K_{\xi'}$ of K at ξ' the cocycle $\mathrm{res}_{K_{\xi'}}(\zeta)$ is determined by the element a if $\xi' \in U$, and is trivial otherwise.

We need the following simple property of orderings of K (see [Srl]):

if F/K is an extension of odd degree then for any ordering $\xi \in \Omega_K$ there is an extension of ξ to F; moreover, the restriction map $\phi : \Omega_F \to \Omega_K$ is a local homeomorphism.

Let E be a finite Galois extension of K over which A is a trivial module and let $F \subset E$ be the subfield corresponding to a Sylow 2-subgroup of $\mathrm{Gal}(E/K)$. Denote $\Delta = \mathrm{Gal}(\overline{K}/F)$. Let $\phi^{-1}(\xi) = \{\xi_1,\dots,\xi_t\} \subset \Omega_F$, where, as above, $\phi : \Omega_F \to \Omega_K$ is the restriction map.

By construction, $\phi(\xi_i) = \xi$. So we can pick a small clopen neighbourhood $U \subset \Omega_K$ of ξ and disjoint small clopen neighbourhoods $U_i \subset \Omega_F$ of ξ_i, $i = 1, \dots, t$, such that the restriction map $\phi|_{U_i} : U_i \to U$ is a homeomorphism and $\phi^{-1}(U) = U_1 \cup \ldots \cup U_t$. Taking smaller neighbourhoods, if necessary, one can additionally assume that for any $\xi' \in U_1$ there is an involution $\tau_{\xi'} \in \Delta$ corresponding to ξ' for which the following property holds:

$$\begin{array}{c}\text{if } g \in \Gamma\backslash\Delta \text{ be such that } \tilde{\tau}_{\xi'} = g\,\tau_{\xi'}\,g^{-1} \in \Delta \text{ then the point of } \Omega_F \\ \text{corresponding to the involution } \tilde{\tau}_{\xi'} \text{ does not lie in } U_1.\end{array} \tag{9}$$

Indeed, let $I_\Delta \subset \Delta$ be a subset of involutions and $\tau \in I_\Delta$ be an involution which corresponds to ξ_1. Assume the contrary. Since I_Δ, Γ are compact and totally disconnected there exist then in Δ a sequence of involutions $(\tau_1, \tau_2, \dots)$ converging to τ and a converging sequence of elements $(g_1, g_2, \dots)$ in $\Gamma\backslash\Delta$ such that $g_i\,\tau_i\,g_i^{-1} \in \Delta$. Letting $g = \lim g_i$, one has $g \in \Gamma\backslash\Delta$ and $\tau' = g\,\tau\,g^{-1} \in \Delta$. But by assumption, the point ξ' of Ω_F corresponding to τ' lies in U_1 and $\phi(\xi') = \xi$. This means that $\xi' = \xi_1$, hence there is $\delta \in \Delta$ such that $\tau' = \delta\,\tau\,\delta^{-1}$, implying $g^{-1}\delta$ lies in the centralizer $C_\Gamma(\tau)$. But every involution in Γ is self-centralizing, i.e. $C_\Gamma(\tau) = \langle\tau\rangle$,– a contradiction.

The map φ_1 is clearly surjective for the field F, since A can be viewed as $\mathrm{Gal}(E/F)$-module and $\mathrm{Gal}(E/F)$ is a 2-group, implying that any irreducible $\mathrm{Gal}(E/F)$-module is of the form $\mathbb{Z}/2\mathbb{Z}$. Therefore, we can pick $\zeta' \in Z^1(F, A)$ such that for proper real closures the cocycle $res_{F_{\xi'}}(\zeta')$ is determined by the element a if $\xi' \in U_1$ and is trivial otherwise. We claim that the cocycle $\zeta = cor_K^F(\zeta')$ has the same property. To verify it we need

Proposition 5. ([Br], Ch. III, Proposition 9.5) *Let A be a Γ-module and $\Theta \subset \Delta \subset \Gamma$ be subgroups. If $[\Gamma : \Delta] < \infty$ and $z \in H^*(\Delta, A)$ then we have*

$$\operatorname{res}_\Theta^\Gamma \circ \operatorname{cor}_\Delta^\Gamma(z) = \sum_{g\in\Lambda} \operatorname{cor}_{\Theta\cap g\Delta g^{-1}}^{\Theta} \circ \operatorname{res}_{\Theta\cap g\Delta g^{-1}}^{g\Delta g^{-1}}(\hat{g}(z)),$$

where Λ is a set of representatives of double cosets $\Theta\,g\,\Delta$ and

$$\hat{g} : H^*(\Delta, A) \to H^*(g\,\Delta\,g^{-1}, A)$$

is the natural map induced by pair $(\mathrm{int}(g^{-1}),\, g)$.

To prove our claim first take $\eta \in U$. Let $\xi' = \phi^{-1}(\eta) \cap U_1$ and let $\tau_{\xi'} \in \Delta$ be an involution corresponding to ξ' and satisfying (9). Then applying Proposition 5 we have

$$\operatorname{res}_{K_{\xi'}}(\zeta) = \sum \operatorname{res}_{\Theta_{\xi'}\cap g\Delta g^{-1}}^{g\Delta g^{-1}}(\hat{g}(\zeta')) = \sum \operatorname{res}_{g^{-1}\Theta_{\xi'}\,g\cap\Delta}^{\Delta}(\zeta') = \operatorname{res}_{\Theta_{\xi'}}^{\Delta}(\zeta')$$

where $\Theta_{\xi'} = \langle\tau_{\xi'}\rangle$, hence $\operatorname{res}_{K_{\xi'}}(\zeta)$ is defined by a. Analogously, one shows that $\operatorname{res}_{K_\eta}(\zeta)$ is trivial if $\eta \notin U$. Proposition 4 is proved. □

Corollary 2. *Let A be a commutative connected linear algebraic K-group. Then φ_2 is injective.*

Proof. One has $H^i(L, A) = 1$, $i \geq 1$. So $H^i(K, A)$ has exponent 2 and hence the map $H^i(K, {}_2A) \to H^i(K, A)$ is surjective, where ${}_2A$ consists of all elements of A killed by 2. By Proposition 4, it gives the surjectivity of φ_1 for A. Then the result follows from the injectivity of φ_2 for ${}_2A$. □

Corollary 3. *The Hasse principle holds for algebraic K-tori.*

Proof. Let T be a K-torus. There exists K-quasi-split torus S and its connected K-subtorus H such that $T = S/H$. Then the commutative diagram

$$\begin{array}{ccccc}
H^1(K,S)=1 & \longrightarrow & H^1(K,T) & \longrightarrow & H^2(K,H) \\
\Big\downarrow{\scriptstyle\theta_1} & & \Big\downarrow{\scriptstyle\theta_2} & & \Big\downarrow{\scriptstyle\theta_3} \\
\prod_{\xi\in\Omega_K} H^1(K_\xi,S)=1 & \longrightarrow & \prod_{\xi\in\Omega_K} H^1(K_\xi,T) & \longrightarrow & \prod_{\xi\in\Omega_K} H^2(K_\xi,H)
\end{array}$$

shows that the injectivity of θ_2 follows from that of θ_3. □

4. The Hasse principle for principal homogeneous spaces

Let us keep the notations of § 3. In particular, we assume that K is a field with $\mathrm{vcd}\,(K) \le 1$, $L = K(\sqrt{-1}\,)$ and $\Omega_K \ne \emptyset$. Let also τ be the non-trivial element of $\mathrm{Gal}\,(L/K)$. Using the results of the previous sections we may produce a simple proof of the triviality of the kernel of (2).

a) Let G' be a connected linear algebraic K-group, $Z \le G'$ be a finite central K-subgroup and let $G = G'/Z$.

Lemma 5. *If the Hasse principle holds for G' then it also holds for G.*

Proof. Consider the commutative diagram

$$\begin{array}{cccccccc}
H^1(K,Z) & \longrightarrow & H^1(K,G') & \longrightarrow & H^1(K,G) & \xrightarrow{\lambda_1} & H^2(K,Z) \\
\Big\downarrow{\scriptstyle\theta_1} & & \Big\downarrow{\scriptstyle\theta_2} & & \Big\downarrow{\scriptstyle\theta_3} & & \Big\downarrow{\scriptstyle\theta_4} \\
\prod_{\xi\in\Omega_K} H^1(K_\xi,Z) & \longrightarrow & \prod_{\xi\in\Omega_K} H^1(K_\xi,G') & \longrightarrow & \prod_{\xi\in\Omega_K} H^1(K_\xi,G) & \xrightarrow{\lambda_2} & \prod_{\xi\in\Omega_K} H^2(K_\xi,Z)
\end{array}$$

By assumption and by Proposition 2, the maps θ_2, θ_4 are injective. Then from the above diagram and from Proposition 4 we have $\mathrm{Ker}\,\theta_3 = 1$. □

b) *Reduction to semisimple groups.* Since unipotent K-groups have trivial cohomology we may assume without loss of generality that G is reductive. Then $G = T \cdot H$ is an almost direct product of the central torus T and the semisimple group $H = [G,G]$. Let $G' = T \times H$. Clearly, the kernel of the natural morphism $G' \to G$ is finite and by induction and by Corollary 3, the Hasse principle holds for H and T. So by Lemma 5, it holds for G as well.

c) *Reduction to simple simply connected groups.* One can again apply Lemma 5 to a simply connected covering G' of G.

d) Let G be an (absolutely) simple simply connected K-group. By Steinberg's theorem ([St2]), G has a Borel subgroup B over L. We may assume that $T = B \cap \tau(B)$ is a maximal K-torus of G. Since $H^1(L,G) = 1$, the map $H^1(L/K, G(L)) \to H^1(K,G)$ is surjective. By Lemma 6.28 [Pl-R], the map $H^1(L/K, T(L)) \to H^1(L/K, G(L))$ is surjective as well, hence any class $[\zeta] \in$

$H^1(K,G)$ can be represented by a cocycle $\zeta' \in Z^1(L/K, T(L))$. Let S be a maximal K-split subtorus of T.

First let $S \neq 1$. Then $C_G(S)$ is a proper connected subgroup of G. Since $C_G(S)$ is a reductive part of some parabolic K-subgroup, one has $\mathrm{Ker}\,(H^1(E, C_G(S)) \to H^1(E,G)) = 1$ for any extension E/K ([Pr-R], Lemma 5.1). So if in addition $\zeta \in \mathrm{Ker}\,\theta$, then for each $\xi \in \Omega_K$ the element $\mathrm{res}_{K_\xi}(\zeta')$ is trivial as an element of $H^1(K_\xi, C_G(S))$, hence the claim follows by induction.

e) $S = 1$, *i.e.* T *is a* K*-anisotropic torus.* By Steinberg's theorem, G is either split or quasi-split over L. We examine the L-splitting case only, since the L-quasi-splitting case can be handled analogously. Identify $Z^1(\Theta, T(L))$ with $(K^*)^n$. Arguing as in d) we get that any element from $\mathrm{Ker}\,\theta$ can be represented by a cocycle $\zeta \in Z^1(\Theta, T(L))$. We claim that there exist a maximal K-torus $T' \subset G$ isomorphic to T over K and a cocycle $\zeta' \in Z^1(\Theta, T'(L))$ equivalent to ζ in $Z^1(\Theta, G(L))$ such that ζ' is everywhere locally positive. By Corollary 3, the last would mean that ζ' is trivial as an element of $H^1(\Theta, T'(L))$, hence ζ is trivial in $H^1(\Theta, G(L))$ as well.

To show it, we proceed as in Theorem 2. Namely, we construct inductively quaternion algebras $D_1, \ldots, D_m$ over K and elements $g_i \in G_{\beta_i}(L)$ such that for $g = g_1 \cdots g_m$ the element $g^{1-\tau} \in T(L)$ and the components of the cocycles $(g^{1-\tau})$ and ζ everywhere locally have the same signs.

As in Theorem 2, we begin with $D_m = (-1, d_{\beta_m})$, where $d_{\beta_m} = c_{\beta_m}$. For $\xi \in \Omega_K$ let $g_\xi \in G(\overline{K}_\xi)$ be such that $\zeta = (g_\xi^{1-\tau})$ (note that T is still anisotropic over K_ξ). We may assume that g_ξ is in "generic" position and so we may write g_ξ as a product $g_\xi = t_\xi\, g_{\xi,1} \cdots g_{\xi,m}$, where $t_\xi \in T$, $g_{\xi,i} \in G_{\beta_i}$, $i = 1, \ldots, m$.

We have already known that $\tau(g_{\xi,m}) = h_{\beta_m}(w_{\xi,m})\, g_{\xi,m}$ for some parameter $w_{\xi,m} \in K_\xi$. By virtue of the facts that our field K has the property SAP and the Hasse principle holds for groups of type A_1 ([B-P], [Sch1]) we can pick $w_m \in K$, which has everywhere locally the same sign as $w_{\xi,m}$, and $g_m \in G_{\beta_m}(L)$ such that $h_{\beta_m}(w_m) = g_m^{1-\tau}$.

Next consider the quaternion K-algebra $D_{m-1} = (-1, d_{\beta_{m-1}})$, where

$$d_{\beta_{m-1}} = c_{\beta_{m-1}} w_m^{\langle \beta_{m-1}, \beta_m \rangle}.$$

Let $w_{\xi,m-1} \in K_\xi$ be such that $h_{\beta_{m-1}}(w_{\xi,m-1})\, h_{\beta_m}(w_{\xi,m}) = (g_{\xi,m-1}\, g_{\xi,m})^{1-\tau}$ Again we can pick $w_{m-1} \in K$ such that for all $\xi \in \Omega_K$ the elements w_{m-1} and $w_{\xi,m-1}$ have the same sign. By construction, the equation $h_{\beta_{m-1}}(w_{m-1})\, h_{\beta_m}(w_m) = (x\, g_m)^{1-\tau}$, where $x \in G_{\beta_{m-1}}(L)$, has solution everywhere locally, so it has solution g_{m-1} globally, and so on.

Thus, there exists $g \in G(L)$ such that the components of both cocycles $(g\tau(g^{-1}))$ and ζ have the same signs in K_ξ for each $\xi \in \Omega_K$. To complete the proof of the theorem it remains to notice that the cocycle $\zeta' = \tau(g)^{-1} \zeta\, g$ is equivalent to ζ in $Z^1(\Theta, G(L))$, takes values in the K-defined and L-splitting torus $T' = \tau(g)^{-1} T \tau(g)$ and ζ' is everywhere locally positive. $\square$

Remark 3. The same argument shows that θ is still injective if we replace Ω_K by a dense set of orderings.

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