The Minimum Principle from a Hamiltonian Point of View

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Abstract. Let G be a complex Lie group and $G_{\mathbb{R}}$ a real form of G. For a $G_{\mathbb{R}}$-stable domain of holomorphy X in a complex G-manifold we consider the question under which conditions the extended domain $G \cdot X$ is a domain of holomorphy. We give an answer in term of $G_{\mathbb{R}}$-invariant strictly plurisubharmonic functions on X and the associate Marsden-Weinstein reduced space which is given by the Kaehler form and the moment map associated with the given strictly plurisubharmonic function. Our main application is a proof of the so called extended future tube conjecture which asserts that $G \cdot X$ is a domain of holomorphy in the case where X is the N-fold product of the tube domain in $\mathbb{C}^4$ over the positive light cone and G is the connected complex Lorentz group acting diagonally.

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Let $G_{\mathbb{R}}$ be a connected real form of a complex Lie group G and X a $G_{\mathbb{R}}$-stable domain in a complex G-manifold Z such that $G{\cdot}X = Z$. In this paper we consider the following question. Under which conditions on X is Z the natural domain of definition of the $G_{\mathbb{R}}$-invariant holomorphic functions on X? If Z is an open submanifold of a Stein manifold, then there is an envelope of holomorphy for Z. Consequently, every $G_{\mathbb{R}}$-invariant holomorphic function on X which extends to Z also extends to the envelope of holomorphy of Z. Thus one also has to ask under which additional requirements is Z a Stein manifold.

In order that an invariant holomorphic function extends to $Z = G \cdot X$ it is sufficient that X is orbit connected, i.e., for every $z \in Z$ the set $\{g \in G;\ g \cdot z \in X\}$ is connected (see [H]). Thus under this condition the main question is whether Z is a Stein manifold. Now if Z is a domain in a Stein manifold V, then Z itself is a Stein manifold if one can find a plurisubharmonic function Ψ on Z which goes to $+\infty$ at every boundary point of $\partial Z \subset V$. There is a natural way to construct

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G-invariant plurisubharmonic functions out of $G_{\mathbb{R}}$-invariant functions on X which was first proposed by Loeb in [L]. In this paper Loeb used an extended version of Kiselman's minimum principle ([K]) in order to construct invariant plurisubharmonic functions. The main idea is the following. Assume that there is a nice quotient $\pi : Z \to Z/G$ and let ϕ be a smooth $G_{\mathbb{R}}$-invariant plurisubharmonic function on X which is a strictly plurisubharmonic exhaustion on each fibre of $\pi|X$. Then the fibre wise minimum of ϕ defines a function ψ on Z/G which is a candidate for a plurisubharmonic function on Z/G. This procedure can be described in terms of Hamiltonian actions as follows.

Assume for simplicity that ϕ is strictly plurisubharmonic. Then $\omega := 2i\partial\bar{\partial}\phi$ defines an invariant Käher form on X and $\mu(x)(\xi) = d\phi(J\xi_X)$ is the associated moment map $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$. In this situation $\mu^{-1}(0)$ is the set of fibre wise critical points of ϕ which in good cases are exactly the points such that the restriction of ϕ to the fibre attains its minimum. Again under some additional assumption, it then follows from the principle of symplectic reduction that the reduced space $\mu^{-1}(0)/G_{\mathbb{R}}$ has a symplectic structure which in fact is Kählerian and moreover is given by the function ψ which is induced on $\mu^{-1}(0)/G_{\mathbb{R}}$ by $\phi|\mu^{-1}(0)$. It turns out that in the situation under consideration the procedures given by symplectic reduction and minimum principle are compatible. This is well known in the case where $G_{\mathbb{R}}$ is a compact Lie group (see e.g. [H-H-L], where a much more general result is proved) and we give here precise conditions such that it also works for a non compact group $G_{\mathbb{R}}$.

The application of Loeb's Minimum Principle is limited mainly to the case of free $G_{\mathbb{R}}$-actions. For the more general case of proper actions it seems that the Hamiltonian point of view is much more adequate. Moreover, for applications it is necessary to consider also domains X of G-spaces Z which do not admit a geometrical quotient Z/G. A typical example is given by the so called extended future tube which we will describe next.

Let $< \, , \, >$ denote the Lorentz product on $\mathbb{R}^4$ and also its $\mathbb{C}$-bilinear extension to $\mathbb{C}^4$. The future tube $\mathcal{T}$ is by definition the tube domain in $\mathbb{C}^4 = \mathbb{R}^4 + i\mathbb{R}^4$ over the positive light cone $C^+ = \{y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4;\ y_0 > 0,\ < y, y >= (y_0)^2 - (y_1)^2 - (y_2)^2 - (y_3)^2 > 0\}$, i.e.,

$$\mathcal{T} = \{z \in \mathbb{C}^4;\ \operatorname{Im} z \in C^+\}\,.$$

This domain is invariant under the action of the connected component $G_{\mathbb{R}}$ of the identity of the homogeneous Lorentz group $\mathrm{O}_{\mathbb{R}}(1,3)$. Now consider the N-fold product $\mathcal{T}^N$ with the diagonal action of $G_{\mathbb{R}}$. The extended future tube $(\mathcal{T}^N)^{\mathbb{C}}$ is by definition the orbit of $\mathcal{T}^N$ under the action of the complexified group G of $G_{\mathbb{R}}$. In other words

$$(\mathcal{T}^N)^{\mathbb{C}} = G \cdot \mathcal{T}^N = \{(g \cdot z_1, \ldots, g \cdot z_N);\ g \in G,\ z_j \in \mathcal{T}\}\,.$$

Note that G is the group $\mathrm{SO}_4(\mathbb{C})$ which is defined by the quadratic form $< \, , \, >$. Although there is no geometric quotient of Z, we have a quotient $\pi : (\mathbb{C}^4)^N \to (\mathbb{C}^4)^N /\!\!/ G$ which is given by the invariant holomorphic functions on $(\mathbb{C}^4)^N$ and it is a fundamental fact that the extended tube $(\mathcal{T}^N)^{\mathbb{C}}$ is saturated with respect to π ([H-W], see §3 for additional remarks). In this case it turns out that this invariant theoretical quotient has sufficiently many good properties in order to apply the main result of this paper which we formulate now.

Let V be a Stein G-manifold such that there exists almost a quotient $\pi : V \to V/\!/G$. More precisely we will assume that $V/\!/G$ is a complex space, $\pi : V \to V/\!/G$ is a G-invariant surjective holomorphic map and for an analytically Zariski open π-saturated subset V^0 of V the restriction map $\pi : V^0 \to V^0/\!/G$ is a holomorphic fibre bundle with typical fibre G/H. Thus $V^0/\!/G = V^0/G$ is a geometric quotient. Let X be a $G_{\mathbb{R}}$-stable domain in V such that $Z := G \cdot X$ is saturated with respect to $\pi : V \to V/\!/G$.

Theorem 1. *Let $\phi : X \to \mathbb{R}$ be a smooth non-negative $G_{\mathbb{R}}$-invariant plurisubharmonic function and assume that*

(i) *The fibres of π restricted to $X^0 := X \cap V^0$ are connected,*

(ii) *the restriction of $\phi^0 := \phi|X^0$ to the fibres of π restricted to X^0 is strictly plurisubharmonic,*

(iii) *ϕ^0 is proper mod $G_{\mathbb{R}}$ along $\pi|Z^0$ where $Z^0 := V^0 \cap Z$ and*

(iv) *ϕ is a weak exhaustion of X over $V/\!/G$,*

Then $Z = G \cdot X$ is a Stein manifold.

In the case where $G_{\mathbb{R}}$ acts properly on X^0 condition (iii) means that the map $\phi^0 \times \pi|X^0 : X^0 \to \mathbb{R} \times (Z^0/\!/G)$ induces a proper map $X^0/G_{\mathbb{R}} \to \mathbb{R} \times (Z^0/\!/G)$. By a weak exhaustion of X over $V/\!/G$ we mean a function which goes to $+\infty$ on a sequence if the corresponding sequence in $V/\!/G$ converges to a boundary point of $Z/\!/G$ in $V/\!/G$.

In the case where the G-action on Z^0 is assumed to be free, the theorem can be proved rather directly by applying Loeb's minimum principle. For a compact group it is a consequence of the methods presented in [H-H-K] (see also [H-H-L]).

In the last section we recall some previously known facts proved in [H-W] together with a more recent result in [Z] about the orbit geometry of the extended future tube in order to verify that the conditions of Theorem 1 are satisfied in the case of the extended future tube. This leads to a conceptual proof of the so called extended future tube conjecture in the last section.

Theorem 2. *The extended future tube is a domain of holomorphy.*

This result has conjecturally been known in constructive quantum field theory for more then thirty years. For its relevance and other publications concerning problems related to it we refer the reader to the literature ([B-L-T], [H-S], [J], [S-W], [S-V]).

There is a proof of Theorem 2 in [Z] which due to several mistakes and gaps is difficult to understand.

1. Hamiltonian actions on Kähler spaces.

Let $G_{\mathbb{R}}$ be a real connected Lie group and X a complex $G_{\mathbb{R}}$-space, i.e., $G_{\mathbb{R}}$ acts on X by holomorphic transformations such that the action $G_{\mathbb{R}} \times X \to X$, $(g, x) \to g \cdot x$, is real analytic. If ω is a smooth $G_{\mathbb{R}}$-invariant Kähler structure on X, then a $G_{\mathbb{R}}$-equivariant smooth map μ from X into the dual $\mathfrak{g}_{\mathbb{R}}^*$ of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is said to be an equivariant moment map if

$$d\mu_\xi = \imath_{\xi_X}\omega$$

holds on every $G_{\mathbb{R}}$-stable complex submanifold Y of X. Here ω denotes the Kähler form on Y induced by the Kählerian structure on X (see [H-H-L]), $\mu_\xi := <\mu, \xi>$ is the component of μ in the direction of $\xi \in \mathfrak{g}_{\mathbb{R}}$, ξ_X is the vector field on X induced by ξ and $\imath_{\xi_X}\omega$ denotes the one form given by contraction, i.e., $\eta \to \omega(\xi, \eta)$.

Example. If ω is given by a smooth strictly plurisubharmonic $G_{\mathbb{R}}$-invariant function ϕ, i.e., $\omega = 2i\partial\bar{\partial}\phi$ on every smooth part of X, then

$$\mu_\xi(x) := d\phi(J\xi_X) = (i(\partial - \bar{\partial})\phi)(\xi_X) = d^c\phi(\xi_X)$$

defines an equivariant moment map. This follows from invariance of ϕ, since in this case we have

$$d\mu_\xi = d\imath_{\xi_X} d^c\phi = -\imath_{\xi_X} dd^c\phi = \imath_{\xi_X} 2i\partial\bar{\partial}\phi\,.$$

Here we use the formula

$$\mathcal{L}_\xi\alpha = \imath_\xi d\alpha + d\imath_\xi\alpha$$

for all vector fields ξ and differential forms α where $\mathcal{L}_\xi$ denotes the Lie derivative in the direction of ξ.

Later we will need the following fact about the zero level set of μ.

LEMMA. *Assume that X is smooth and that $G_{\mathbb{R}}$ acts properly on X. If the dimension of the $G_{\mathbb{R}}$-orbits in $\mu^{-1}(0)$ is constant, then $\mu^{-1}(0)$ is a submanifold of X.*

Proof. Since the action is assumed to be proper, there is a local normal form for the moment map (see e.g. [A] or [H-L]). The statement is an easy consequence of this fact (see e.g. [A]. In [S-L] the argument is given for a compact group $G_{\mathbb{R}}$). □

Remark 1. It can be shown that the converse of the Lemma also holds. We will not use this fact here.

Remark 2. The properness assumption is very often satisfied. Since one may assume that $G_{\mathbb{R}}$ acts effectively, $G_{\mathbb{R}}$ is a Lie subgroup of the group I of isometries of the Riemannian manifold X. The group of isometries acts properly on X and consequently the $G_{\mathbb{R}}$-action on X is proper if and only if $G_{\mathbb{R}}$ is a closed subgroup of I. This is the case if and only if there is a point $x \in X$ such that $G_{\mathbb{R}} \cdot x$ is closed and the isotropy group $(G_{\mathbb{R}})_x := \{g \in G_{\mathbb{R}};\, g \cdot x = x\}$ is compact.

Remark 3. If $G_{\mathbb{R}}$ acts such that the isotropy groups are discrete, then μ has maximal rank. Thus in this case $\mu^{-1}(0)$ is obviously a submanifold of X. Moreover $T_x(\mu^{-1}(0)) = \ker d\mu(x)$ for all $x \in \mu^{-1}(0)$.

2. HAMILTONIAN ACTIONS ON INVARIANT DOMAINS

Let G be a connected complex Lie group and Z a holomorphic G-space, i.e., the action $G \times Z \to Z$ is assumed to be a holomorphic map. Let $G_{\mathbb{R}}$ be a connected real form of G. By an invariant domain in Z we mean in the following a $G_{\mathbb{R}}$-stable connected open subspace X of Z. In the homogeneous case we have the following

LEMMA 1. *Let X be an invariant domain in Z and assume that Z is G-homogeneous. If the zero level set of $\mu : X \to \mathfrak{g}_{\mathbb{R}}$ is not empty, then $\mu^{-1}(0)$ is a Lagrangian submanifold of X and each connected component of $\mu^{-1}(0)$ is a $G_{\mathbb{R}}$-orbit.*

Proof. For $z_0 \in X$ let N be an open convex neighborhood of $0 \in \mathfrak{g}_{\mathbb{R}}$ such that $U := G_{\mathbb{R}} \cdot \exp iN \cdot z_0 \subset X$. Since $G_{\mathbb{R}} \cdot \exp iN$ is a neighborhood of $G_{\mathbb{R}}$ in G, the set U is a neighborhood of $G_{\mathbb{R}} \cdot z_0$ in X. The proof of Lemma 1 is a consequence of the following

Claim. $U \cap \mu^{-1}(0) = G_{\mathbb{R}} \cdot z_0$ for $z_0 \in \mu^{-1}(0)$.

In order to proof the claim, let $z \in U \cap \mu^{-1}(0)$ be given. Then there are $h \in G_{\mathbb{R}}$ and $\xi \in N$ such that $z = h \exp i\xi \cdot z_0 \in U \cap \mu^{-1}(0)$. Thus $\exp i\xi \cdot z_0 \in \mu^{-1}(0) \cap U$ and $z_t := \exp it\xi \cdot z_0 \in U$ for $t \in [0,1]$. Note that $J\xi_X(x) = \frac{d}{dt}|_{t=0} \exp it\xi \cdot x$ is the gradient flow of μ_ξ with respect to the Riemannian metric induced by ω. Thus, if z_t is not constant, then $t \to \mu_\xi(z_t)$ is strictly increasing. This contradicts $\mu_\xi(z_0) = 0 = \mu_\xi(z_1)$. Therefore $z_0 = \exp it\xi \cdot z_0$ for all $t \in \mathbb{R}$. This implies $z = h \cdot z_1 = h \cdot z_0 \in G_{\mathbb{R}} \cdot z_0$.

It is a consequence of the claim that every $G_{\mathbb{R}}$-orbit is closed in X. Therefore every component of $\mu^{-1}(0)$ is a $G_{\mathbb{R}}$-orbit. It remains to show that these orbits are Lagrangian. Since $\mu(G_{\mathbb{R}} \cdot z_0) = 0$ we have

$$0 = d\mu_\xi(\eta_X(z_0)) = \omega(\xi_X(z_0), \eta_X(z_0))$$

for all $\xi, \eta \in \mathfrak{g}_{\mathbb{R}}$. This means that $G_{\mathbb{R}} \cdot z_0$ is an isotropic submanifold of X. In particular, $\dim_{\mathbb{R}} G_{\mathbb{R}} \cdot z_0 \leq \dim_{\mathbb{C}} X$. In general the tangent space $T_{z_0}(G_{\mathbb{R}} \cdot z_0)$ spans $T_{z_0}X$ over $\mathbb{C}$. Thus $\dim_{\mathbb{R}} G_{\mathbb{R}} \cdot z_0 \geq \dim_{\mathbb{C}} G \cdot z_0 = \dim_{\mathbb{C}} X$. This shows that $\dim_{\mathbb{R}} G_{\mathbb{R}} \cdot z_0 = \frac{1}{2} \dim_{\mathbb{R}} X$. Hence $G_{\mathbb{R}} \cdot z_0$ is Lagrangian. □

Every Lagrangian submanifold of a Kähler manifold is totally real. Thus, if Z is G-homogeneous, then $\mu^{-1}(0)$ is a totally real submanifold of X. Note that the $G_{\mathbb{R}}$-orbits in $\mu^{-1}(0)$ are closed since they are connected components of the zero fibre of μ. Now if $G_{\mathbb{R}}$ is such that $0 \in \mathfrak{g}_{\mathbb{R}}^*$ is the only $G_{\mathbb{R}}$-fixed point, then $x \in \mu^{-1}(0)$ if and only if the orbit $G_{\mathbb{R}} \cdot x$ is isotropic. This condition holds for example for a semisimple Lie group.

It almost never happens that there is a $G_{\mathbb{R}}$-invariant Kähler form ω which is defined on Z. For example, if $G_{\mathbb{R}}$ is a simple non compact Lie group or more generally a semisimple Lie group without compact factors, then there does not exist a $G_{\mathbb{R}}$-invariant Kähler form on a non trivial holomorphic G-manifold Z. In order to see this, recall that since $G_{\mathbb{R}}$ is semisimple there is a moment map $\mu : Z \to \mathfrak{g}_{\mathbb{R}}^*$. Now let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition where $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup of $G_{\mathbb{R}}$. Then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is the Lie algebra of the maximal compact subgroup U of G. For $\xi \in i\mathfrak{p}$ the image of the one-parameter group $\gamma : t \to \exp it\xi$ lies in U and therefore there is a basis of $\mathfrak{p}$ consisting of ξ's such that the image of γ is compact, i.e., isomorphic to S^1. But γ is the flow of the gradient vector field of μ_ξ and therefore $t \to \mu_\xi(\gamma(t) \cdot z)$ is strictly increasing for every $z \in Z$. This implies that γ acts trivially on Z. Since G is semisimple and contains no compact factor, G itself is the smallest complex subgroup of G which contains $\exp \mathfrak{p}$. Thus G acts trivially on Z.

A geometric interpretation of the zero fibre $\mu^{-1}(0)$ of an equivariant moment map $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$ associated to a smooth $G_{\mathbb{R}}$-invariant strictly plurisubharmonic function $\phi : X \to \mathbb{R}$ (see Section 1, Example) can be given in the case where X is an invariant domain in Z as follows. For $x \in X$ let $\Omega(x) := \{g \cdot x; g \in G \text{ and } g \cdot x \in X\}$ be the local G-orbit of G through x in X where $(g, x) \to g \cdot x$ denotes the G-action on Z. Then by $G_{\mathbb{R}}$-invariance of ϕ we have

$$\mu^{-1}(0) = \{x \in X;\ \ x \text{ is a critical point of } \phi|\Omega(x)\}\,.$$

We consider now invariant domains X in G-homogeneous spaces Z such that there is a moment map associated to $\phi : X \to \mathbb{R}$ more closely. In order to do that we first introduce the notion of an exhaustion mod $G_{\mathbb{R}}$.

Let F be a complex space with a proper $G_{\mathbb{R}}$-action and let $F/G_{\mathbb{R}}$ be the space of $G_{\mathbb{R}}$-orbits endowed with the quotient topology. A $G_{\mathbb{R}}$-invariant function $f : F \to \mathbb{R}$ is said to be proper mod $G_{\mathbb{R}}$ if the induced map $\bar{f} : F/G_{\mathbb{R}} \to \mathbb{R}$ is proper. The map f is said to be an exhaustion mod $G_{\mathbb{R}}$ if $\bar{f}$ is an exhaustion, i.e., if for all $r \in \mathbb{R}$ the set $\{q \in F/G_{\mathbb{R}};\ \bar{f}(q) < r\}$ is relatively compact in F. Note that a $G_{\mathbb{R}}$-invariant continuous function which is bounded from below is proper mod $G_{\mathbb{R}}$ if and only if it is an exhaustion mod $G_{\mathbb{R}}$.

LEMMA 2. *Let Z be G-homogeneous and assume that the $G_{\mathbb{R}}$-action on X is proper. Let $\phi : X \to \mathbb{R}$ be a smooth strictly plurisubharmonic $G_{\mathbb{R}}$-invariant function which is an exhaustion mod $G_{\mathbb{R}}$. Then there is a $z_0 \in X$ such that*

$$G_{\mathbb{R}} \cdot z_0 = \mu^{-1}(0) = \{z \in X;\ \phi(z) \textit{ is a minimal value of } \phi\}\,.$$

Proof. Since ϕ is plurisubharmonic and an exhaustion mod $G_{\mathbb{R}}$ there is a point $z_0 \in X$ which is a minimum for ϕ. In particular, $\mu^{-1}(0)$ is not empty where μ denotes the moment map associated with ϕ. We have to prove that $\mu^{-1}(0)$ is connected. By Lemma 1, every connected component of the set $M_\phi = \mu^{-1}(0)$ of critical points of ϕ is a $G_{\mathbb{R}}$-orbit. We claim that the $G_{\mathbb{R}}$-orbits are non degenerate in the sense that the Hessian of ϕ in normal directions is positive definite. This is seen as follows.

The vector fields $J\xi_X$, $\xi \in \mathfrak{g}_{\mathbb{R}}$ span the normal space at $x \in M_\phi$ and

$$(J\xi_X)(J\xi_X(\phi)) = \imath_{J\xi_X} d\mu_\xi = \omega(\xi_X, J\xi_X)\,.$$

Hence the Hessian at $x \in M_\phi$ is positive in the normal directions. Since ϕ is proper mod $G_{\mathbb{R}}$ and the gradient vector field of ϕ with respect to the $G_{\mathbb{R}}$-invariant Kähler metric given by $2i\partial\bar{\partial}\phi$ is $G_{\mathbb{R}}$-invariant, Lemma 2 follows from standard arguments in Morse Theory. □

In the situation of Lemma 2 every critical point of ϕ is a minimum and the set of these points is a $G_{\mathbb{R}}$-orbit and coincides with $\mu^{-1}(0)$.

We will now generalize the results in the homogeneous case to spaces Z which possess a geometric G-quotient and X is a weakly orbit connected invariant domain in Z. Here a $G_{\mathbb{R}}$-stable subset X of Z is said to be weakly orbit connected if for every $x \in X$ the local G-orbit $\Omega(x) := \{g \cdot x \in X;\ g \in G\}$ is connected.

Remark 1. A $G_{\mathbb{R}}$-invariant set X in Z is said to be orbit connected if for every $x \in X$ the set $\Omega_x := \{g \in G;\ g \cdot x \in X\}$ is connected. This is a stronger concept then weakly orbit connectedness.

Let Z be a holomorphic G-space such that there is a geometric quotient $\pi : Z \to Z/G$. By this we mean that the orbit space Z/G is a complex space such that the quotient map $\pi : Z \to Z/G$ is holomorphic. Moreover we assume that the structure sheaf of Z/G is the sheaf of invariants, i.e., for an open subset Q of Z/G a function $f : Q \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi : \pi^{-1}(Q) \to \mathbb{C}$ is holomorphic.

Now let $X \subset Z$ be an invariant domain which lies surjectively over Z/G or equivalently such that $Z = G \cdot X$. Assume that $G_{\mathbb{R}}$ acts properly on X and that X is weakly orbit connected. Let $\phi : X \to \mathbb{R}$ be a smooth $G_{\mathbb{R}}$-invariant strictly plurisubharmonic function which is an exhaustion mod $G_{\mathbb{R}}$ along π, i.e., $\pi^{-1}(C) \cap \{x \in X;\ \phi(x) \leq r\}/G_{\mathbb{R}} \subset X/G_{\mathbb{R}}$ is compact for every compact subset C in Z/G and $r \in \mathbb{R}$. We set $M_\phi = \mu^{-1}(0)$ where $\mu : X \to \mathfrak{g}_{\mathbb{R}}$ denotes the moment map associated with ϕ.

Proposition 1. *The map $\bar\imath : M_\phi/G_{\mathbb{R}} \to Z/G$ induced by the inclusion $\imath : M_\phi \to Z$ is a homeomorphism. If X is a manifold, then M_ϕ is smooth and*

$$T_x M_\phi = \ker d\mu(x)$$

holds for all $x \in M_\phi$.

Proof. The map $\bar\imath$ is continuous and by Lemma 2 it is also a bijection. We claim that $\bar\imath$ is proper. Since the $G_{\mathbb{R}}$-action on M_ϕ is proper, $M_\phi/G_{\mathbb{R}}$ is a locally compact topological space. Thus properness of $\bar\imath$ implies that $\bar\imath$ is a homeomorphism.

Let (q_n) be a sequence in $M_\phi/G_{\mathbb{R}}$ and x_n a point in M_ϕ which lies over q_n. Assume that $(\pi(x_n)) = (\bar\imath(q_n))$ has a limit in Z/G and let $x_0 \in M_\phi$ be a point which lies over $\lim \pi(x_n)$. If some subsequence of $\phi(x_n)$ goes to infinity, then we may assume $\phi(x_n) > \phi(x_0) + 1$ for all n. Since $\pi : Z \to Z/G$ is an open map, there are $g_n \in G$ such that $\lim g_n \cdot x_n = x_0$ for some subsequence. This is a contradiction since $\phi(x_n) < \phi(g_n \cdot x_n)$ for all n such that $g_n \cdot x_n \in X$. Thus, since ϕ is assumed to be an exhaustion mod $G_{\mathbb{R}}$ along π, there are $h_n \in G_{\mathbb{R}}$ such that a subsequence of $(h_n \cdot x_n)$ converges to x_0. This implies that a subsequence of (q_n) converges in M_ϕ. So far we proved that $\bar\imath$ is a homeomorphism.

Assume now that X is smooth. The existence of a geometric quotient implies that the dimension of the G-orbits in Z is constant and therefore this is also true for the $G_{\mathbb{R}}$-orbits in M_ϕ (Lemma 1). Thus M_ϕ is a submanifold of X (Section 1, Lemma). Since $T_x M_\phi$ is a subspace of $\ker d\mu(x)$ and $\ker d\mu(x) = T_x(G_{\mathbb{R}} \cdot x) + T_x(G \cdot x)^\perp$, the claim follows from the obvious dimension count as follows. Let $d := \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot x$ for $x \in M_\phi$. Note that d is the complex dimension of the π-fibres. Thus $\dim_{\mathbb{R}} M_\phi = \dim_{\mathbb{R}} M_\phi/G_{\mathbb{R}} + d = \dim_{\mathbb{R}} Z/G + d = \dim_{\mathbb{R}} T_x(G \cdot x)^\perp + \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot x$ implies that $T_x M_\phi = \ker d\mu(x)$ for all $x \in M_\phi$. □

Remark 2. Without a reference to an embedding into a holomorphic G-space one can show that $\mu^{-1}(0)/G_{\mathbb{R}}$ is a complex space in a natural way (see [A-H-H] and [A]).

If $G_{\mathbb{R}}$ does not act properly on X, then let $\overline{G}_{\mathbb{R}}$ be the closure of $G_{\mathbb{R}}$ in the group I of isometries of the Kähler manifold X. Since the $G_{\mathbb{R}}$-orbits in $M_\phi = \mu^{-1}(0)$ are closed (Lemma 1), it follows that they coincide with the $\overline{G}_{\mathbb{R}}$-orbits. Moreover ϕ is $\overline{G}_{\mathbb{R}}$-invariant and $M_\phi = \bar{\mu}^{-1}(0) =: \overline{M}_\phi$, where $\bar{\mu}$ is the moment map associated with ϕ. Now if one redefines an exhaustion mod $G_{\mathbb{R}}$ along π in terms of sequences in X, then also in this case M_ϕ is smooth and $T_x M_\phi = T_x(G_{\mathbb{R}} \cdot x) \oplus T_x(G \cdot x)^\perp = \ker d\mu(x)$ holds for all $x \in M_\phi$.

Proposition 1 can be generalized to the case where $\phi : X \to \mathbb{R}$ is only assumed to be plurisubharmonic and strictly plurisubharmonic on the fibres. More precisely we have the following consequence which can be thought of as a version of Loeb's minimum principle (see [L]).

COROLLARY 1. *Let $X \subset Z$ be a weakly orbit connected invariant domain with $\pi(X) = Z$ and $\phi : X \to \mathbb{R}$ a smooth $G_{\mathbb{R}}$-invariant plurisubharmonic function which is an exhaustion mod $G_{\mathbb{R}}$ along π such that the restriction of ϕ to the local G-orbits in X is a strictly plurisubharmonic exhaustion mod $G_{\mathbb{R}}$. If $\pi : Z \to Z/G$ is a holomorphic bundle, then*

(i) $M_\phi = \mu^{-1}(0)$ *is smooth where* $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$, $\mu_\xi = d\phi(J\xi_X)$,

(ii) $T_x(M_\phi) = \ker d\mu(x)$ *for all* $x \in M_\phi$.

(iii) $M_\phi/\overline{G}_{\mathbb{R}}$ *is homeomorphic to* Z/G *and the function* $\psi : Z/G \to \mathbb{R}$ *which is induced by* $\phi|M_\phi$ *is a smooth plurisubharmonic function.*

Proof. We may assume that $G_{\mathbb{R}}$ acts properly on X and, since the statements are local over Z/G that Z/G is a Stein manifold. Let $\rho : Z \to \mathbb{R}$ be the the pull back of a strictly plurisubharmonic function on Z/G. Then $\phi + \rho$ is $G_{\mathbb{R}}$-invariant, strictly plurisubharmonic and an exhaustion mod $G_{\mathbb{R}}$ on the local G-orbits in X. Since $d\rho(J\xi_X) = 0$ for all $\xi \in \mathfrak{g}_{\mathbb{R}}$, the moment map associated with $\phi + \rho$ is the same as the moment map associated with ϕ. Thus Proposition 1 implies directly (i), (ii) and the first part of (iii). It remains to show that $\psi : Z/G \to \mathbb{R}$ is a smooth plurisubharmonic function.

For the plurisubharmonicity of ψ we recall the calculation in [H-H-L], §2. For $z \in M_\phi$ we have $T_z(M_\phi) = \ker d\mu(z) = T_z(G_{\mathbb{R}} \cdot z) \oplus T_z(G \cdot z)^\perp$. We may assume that $Z = G/H \times \Delta$ where Δ is an open neighborhood of 0 in $\mathbb{C}^d \cong T_z(G \cdot z)^\perp$, and $\pi(z) = 0$ where π is given by the projection on the second factor. Furthermore there is a section $\eta : \Delta \to M_\phi$, $\eta(w) = (\sigma(w), w)$ and therefore we have $\psi(w) = \phi(\eta(w))$. A direct calculation shows that

$$\partial\bar{\partial}\,\psi(0) = \partial\bar{\partial}\,\phi(\eta(0))\,.$$

Here one has to use that $d\phi(z) = 0$ and that $d\sigma(0) = 0$. Thus ψ is plurisubharmonic and smooth. □

If ϕ is strictly plurisubharmonic, then the proof shows that ψ is also strictly plurisubharmonic. For a proper $G_{\mathbb{R}}$-action the space Z/G is then given by symplectic reduction $M_\phi/G_{\mathbb{R}}$ and the induced Kählerian structure on Z/G is determined by the function $\psi(q) = \inf_{x \in \pi^{-1}(q) \cap X} \phi(x)$ which is obtained by applying the minimum principle ([L]). Thus symplectic reduction and the minimum principle are compatible procedures.

For the remainder of this section we assume now that Z is a holomorphic G-manifold such that there is almost a quotient $Z/\!\!/G$. More precisely we will assume that $Z/\!\!/G$ is a complex space, $\pi : Z \to Z/\!\!/G$ is a surjective G-invariant holomorphic map and there is an analytically Zariski open π-saturated subset Z^0 of Z such that $\pi : Z^0 \to Z^0/\!\!/G$ is a geometric quotient, i.e., $Z^0/\!\!/G = Z^0/G$. Moreover, for the sake of simplicity we assume that $\pi : Z^0 \to Z^0/\!\!/G$ is a holomorphic fibre bundle.

Now let X be an invariant domain in Z with $\pi(X) = Z$ and assume that $X^0 := X \cap Z^0$ is weakly orbit connected. Let ϕ be a $G_{\mathbb{R}}$-invariant plurisubharmonic function such that $\phi^0 := \phi|X^0$ is smooth, strictly plurisubharmonic on the local G-orbits in X^0 and an exhaustion mod $G_{\mathbb{R}}$ along $\pi|Z^0$. Thus the restriction $\phi^0 := \phi|M_\phi^0$, $M_\phi^0 := M_\phi \cap Z^0$ induces a plurisubharmonic function $\psi^0 : Z^0/\!\!/G \to \mathbb{R}$.

Lemma 3. *There is a unique G-invariant plurisubharmonic function $\Psi : Z \to [-\infty, +\infty)$ which extends $\Psi^0 := \psi^0 \circ \pi|Z^0$.*

Proof. The function $\Psi(z) = \inf\limits_{g \in \Omega_z} \phi(g \cdot z)$ is upper semi-continuous on Z where $\Omega_z := \{g \in G;\, g \cdot z \in X\}$. Now $\Psi = \Psi^0$ on Z^0 (Lemma 2), and $Z \setminus Z^0$ is a proper analytic subset of Z. Thus Ψ is plurisubharmonic and by definition G-invariant. $\square$

Remark 3. If $Z/\!\!/G$ is smooth and π is an open map, then ψ^0 extends uniquely to a plurisubharmonic function ψ on $Z/\!\!/G$. Of course in this case we have $\psi(q) = \inf\limits_{x \in F_q} \phi(x)$, where $F_q := \pi^{-1}(q) \cap X$. If $\phi|F_q$ is an exhaustion mod $G_{\mathbb{R}}$, M_ϕ intersects every $G_{\mathbb{R}}$-stable closed analytic subset of F_q non trivially. But it might happen that $M_\phi \cap F_q$ is a union of several $G_{\mathbb{R}}$-orbits. On the other hand for $q \in Z^0/\!\!/G$ the intersection is exactly one $G_{\mathbb{R}}$-orbit.

Assume now in addition that Z is an open G-stable subspace of a holomorphic Stein G-manifold V which is saturated with respect to $\pi : V \to V/\!\!/G$. We say that $\phi : X \to \mathbb{R}$ is a weak exhaustion of X over $V/\!\!/G$ if $\limsup \phi(z_n) = +\infty$ for any sequence (z_n) in X such that $(\pi(z_n))$ converges to some q_0 in the boundary $\partial(Z/\!\!/G)$ in $V/\!\!/G$.

Theorem. *Let Z be a G-stable π-saturated open subspace of V, X an invariant domain in Z with $G \cdot X = Z$ and $\phi : X \to \mathbb{R}$ a $G_{\mathbb{R}}$-invariant plurisubharmonic function. Assume that*

(i) *X^0 is weakly orbit connected,*

(ii) *the restriction of $\phi^0 := \phi|X^0$ to the local G-orbits is strictly plurisubharmonic,*

(iii) *ϕ^0 is an exhaustion mod $G_{\mathbb{R}}$ along $\pi|Z^0$ and*

(iv) *ϕ is a weak exhaustion of X over $V/\!\!/G$,*

Then $Z = G \cdot X$ is a Stein manifold.

Proof. Let $z_0 \in \partial Z$ and $z_n \in Z$ be such that $z_0 = \lim z_n$. We have to show $\limsup \Psi(z_n) = +\infty$. Thus assume that $\Psi(z_n) < r$ for all n and some $r \in \mathbb{R}$.

There are $w_n \in G \cdot M_\phi^0 = Z^0$ such that $\Psi(w_n) < r$ and $z_0 = \lim w_n$. Let $w_n = g_n \cdot x_n$ where $g_n \in G$ and $x_n \in M_\phi^0$. Now $\Psi(w_n) = \Psi(x_n) = \phi(x_n) < r$ and, since $Z = G \cdot X$ is saturated, $\pi(x_n) = \pi(w_n) \to \pi(z_0) \in \partial(Z/\!\!/G)$. This contradicts the assumption that ϕ is a weak exhaustion. Thus Z is a domain in a Stein manifold with a plurisubharmonic weak exhaustion function and therefore Stein. $\square$

Remark 4. Elementary examples show that for a Stein $G_{\mathbb{R}}$-manifold some conditions are necessary in order that $G \cdot X$ is a Stein manifold. For example there is an $\mathrm{Sl}_2(\mathbb{R})$-invariant domain Ω of holomorphy in $\mathbb{C}^2$ such that $\mathrm{Sl}_2(\mathbb{C}) \cdot \Omega = \mathbb{C}^2 \setminus \{0\}$.

Now let G be complex reductive group and assume that the semistable quotient $\pi : Z \to Z/\!\!/G$ exists (see [H-M-P]). Thus $Z/\!\!/G$ is a complex space whose structure sheaf $\mathcal{O}_{Z/\!\!/G}(U) = \mathcal{O}_Z(\pi^{-1}(U)^G$ is the sheaf of invariants and every point in $Z/\!\!/G$ has an open Stein neighborhood such that the inverse image in Z is Stein. For example, if V is a holomorphic Stein G-manifold, then a semistable quotient $V/\!\!/G$ alway exists. Moreover it is shown in [H-M-P] that Z is a Stein space if and only if $Z/\!\!/G$ is a Stein space.

Assume that Z is connected and that some orbit of maximal dimension is closed. Then there exists a proper analytic subset A in $Z/\!\!/G$ such that $Z^o/\!\!/G = Z/\!\!/G \setminus A$ is a geometric quotient of $Z^o := \pi^{-1}(Z/\!\!/G \setminus A)$. In particular, every fibre of $\pi|Z^o$ is G-homogeneous or equivalently the dimension of the G-orbits in Z^o is constant. Every $x \in Z^o$ has a G-stable neighborhood U which is G-equivariantly biholomorphic to $G \times_H S$ where H is the isotropy group of G at x and S is a Stein space such that the connected component H^0 of the identity of H acts trivially on S. Here $G \times_H S$ denotes the bundle associated to the H-principal bundle $G \to G/H$. Thus locally $Z^o/\!\!/G$ is given by S/Γ where $\Gamma := H/H_0$ is a finite group. Moreover, there is an analytically Zariski open G-stable subset Z^{oo} of Z which is contained in Z^o such that the isotropy type is constant. This implies that Z^{oo} is a fibre bundle over $Z^{oo}/\!\!/G \subset Z/\!\!/G$.

3. Orbit geometry of the future tube.

In the following it will be convenient to introduce a linear coordinate change such that $< z, z >= (z_0)^2 - (z_1)^2 - (z_2)^2 - (z_3)^2$ has the form $z_0 z_1 - z_2 z_3$. Thus we set

$$Z := \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}$$

and obtain $\det Z =< z, z >$ and $\det \operatorname{Im} Z =< \operatorname{Im} z, \operatorname{Im} z >$ where $\operatorname{Im} Z := \frac{1}{2i}(Z - \bar{Z}^t)$.

Let $H := \{Z \in V;\ \operatorname{Im} Z > 0\}$ denote the generalized upper half plane where $V := \mathbb{C}^{2\times 2}$. Note that H is just the tube over the positive light cone in the new coordinates. Moreover H is stable with respect to the action of $G_{\mathbb{R}} := \mathrm{SL}_2(\mathbb{C})$ which is given by $G_{\mathbb{R}} \times H \to H,\ (g, Z) \to g * Z := gZ\bar{g}^t$. This action is not effective. The ineffectivity consists of $\Gamma = \{+I, -I\}$ and the quotient $\mathrm{SL}_2(\mathbb{C})/\Gamma$ is the connected component of the identity of the homogeneous Lorentz group.

Let $H^N := H \times \cdots \times H \subset V \times \cdots \times V =: V^N$ denote the N-fold product of H and set $G := (G_{\mathbb{R}})^{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ where $G_{\mathbb{R}}$ is embedded in G via $g \to (g, \bar{g})$. The diagonal $G_{\mathbb{R}}$ action on V^N extends to a holomorphic G action $G \times V^N \to V^N$, $((g, h), Z^1, \ldots, Z^N) \to (g, h) * (Z^1, \ldots, Z^N) := (gZ^1 h^t, \ldots, gZ^N h^t)$.

Theorem. *The extended future tube* $(H^N)^{\mathbb{C}} := G * H^N$ *is a domain of holomorphy.*

In the proof we will make an axiomatic use of the following statements

Fact 1 (see Streater Wightman [S-W], p. 66). The set H^N is orbit connected in V^N, i.e., $\{g \in G;\ g * Z \in H^N\}$ is connected for every $Z \in V^N$.

Fact 2. The extended future tube $G * H^N$ is saturated with respect to $\pi : V^N \to V^N /\!\!/ G$.

Fact 2 implies that the semistable quotient $G * H^N /\!\!/ G$ exists and is an open subset of $V^N /\!\!/ G$. The quotient map is given by restricting $\pi : V^N \to V^N /\!\!/ G$ to $G * H^N$.

There does not seem to be a proof in the literature of Fact 2 but there is a detailed proof for the whole complex orthogonal group in [H-W]. A slight modification of the proof there can be used for a proof of Fact 2. In order to be complete let us recall briefly the main steps. First we note that it is sufficient to show the following (see e.g. [H]).

Claim. If $Z \in H^N$, then the unique closed orbit $G * W$ in the closure of $\overline{G * Z}$ lies in $G * H^N$.

This can be seen as follows. Let $<\ ,\ >$ be the complex Lorenz product, i.e., the symmetric bilinear form on V which is associated to the quadratic form $\det : V \to V$. Thus V is just the standard representation of $\tilde{G} := \mathrm{O}_4(\mathbb{C})$. Note that $\tilde{G}$ has two connected components and the connected component of the identity is G. The functions $(Z^1, \ldots, Z^N) \to < Z^i, Z^j >$, form a set of generators for the algebra of the $\tilde{G}$-invariant polynomials on V^N. Thus the image of V^N in the set of symmetric $N \times N$-matrices of the map $\tilde{\pi}$ which sends $(Z^1, \ldots, Z^N)$ to the matrix $(< Z^i, Z^j >)$ is an affine variety which is isomorphic to $V^N /\!\!/ \tilde{G}$.

The matrices of rank 3 or 4 correspond to fibres of $\tilde{\pi}$ which are closed $\tilde{G}$-orbits. It follows that the G-orbit through every point $Z \in H^N$ such that the rank r of $\tilde{Z}$ is greater or equal to 3 is already closed. Now assume that $r \leq 2$. In this case the following is shown in [H-W]: There exists an $g \in \tilde{G}$, $\alpha_j \in \mathbb{C}$ and an $\omega \in V$ with $< \omega, \omega >= 0 =< \omega, W^j >$ such that

$$Z^j = g * W^j + \alpha_j \omega\ ,\ j = 1, \ldots, N\,.$$

The proof actually shows that one can choose $g \in G$, i.e., $\det g = 1$. Now an argument of Hall-Wightman ([H-W], p.21) implies that $g * W^j \in H$ for all j, i.e., $G * W \subset G * H^N$.

Fact 3. The function $\phi : H^N \to \mathbb{R}$, $\phi(Z^1, \ldots, Z^N) := \frac{1}{\det \operatorname{Im} Z^1} + \cdots + \frac{1}{\det \operatorname{Im} Z^N}$ is $G_{\mathbb{R}}$-invariant and strictly plurisubharmonic. Moreover, ϕ is a weak exhaustion of H^N.

The simplest way to see that ϕ is strictly plurisubharmonic is to note that $Z^j \to \frac{1}{\det \operatorname{Im} Z^j}$ it is given by the Bergmann kernel function on H. Since $\det \operatorname{Im} Z = 0$ for $Z \in \partial H$, ϕ is a weak exhaustion of H^N, i.e., $\phi(Z_k) \to +\infty$ if $\lim Z_k = Z_0 \in \partial(H^N) \subset V^N$.

Let $K_{\mathbb{R}} := \{(a, \bar{a});\ a \in \mathrm{SU}_2(\mathbb{C})\}$ be the maximal compact subgroup of $G_{\mathbb{R}}$. We set $V^0 := \{Z \in V;\ \det Z \neq 0\}$. Note that $V /\!\!/ G \cong \mathbb{C}$ and that after this identification the quotient map is given by $\det : V \to \mathbb{C}$. In particular, V^0 is saturated with respect to $V \to V/\!\!/G$.

LEMMA 1. *Let (W_n) be a sequence in H such that $(\pi(W_n))$ converges in $V/\!\!/G$. Then there exist $h_n \in G_{\mathbb{R}}$ such that a subsequence of $(h_n * W_n)$ converges in V.*

Proof. There exist $u_n \in K_{\mathbb{R}}$ such that

$$X_n := u_n * W_n =: \begin{pmatrix} x_n & z_n \\ 0 & y_n \end{pmatrix}.$$

Since $(\pi(W_n)) = (\pi(X_n))$ converges, it follows that $|\det X_n| = |x_n y_n| \leq R$ for some $R \geq 0$ and all n. Furthermore, $X_n \in H$ implies that $\frac{1}{4}|z_n|^2 < \operatorname{Im} x_n \operatorname{Im} y_n \leq |x_n y_n| = |\det X_n|$. Therefore (z_n) is bounded. Now $0 < |x_n y_n| \leq R$ implies that $|r_n^2 x_n| = |r_n^{-2} y_n|$ for some $r_n > 0$. In particular the sequence $(r_n^2 x_n, r_n^{-2} y_n)$ is bounded. Hence $h_n * W_n$ has a convergent subsequence where $h_n := r_n \cdot u_n \in G_{\mathbb{R}}$ and r_n is identified with $\left(\begin{pmatrix} r_n & 0 \\ 0 & \frac{1}{r_n} \end{pmatrix}, \begin{pmatrix} r_n & 0 \\ 0 & \frac{1}{r_n} \end{pmatrix} \right)$. ☐

Remark 1. Geometrically Lemma 1 asserts that H is relatively compact over $V/\!\!/G$ mod $G_{\mathbb{R}}$.

LEMMA 2. *Let (Z_n, W_n) be a sequence of points in $H \times H$ and assume that*

(i) *$\pi(Z_n, W_n)$ converges in $(V \times V)/\!\!/G$ and*

(ii) *$W_0 = \lim W_n$ exists in H.*

Then a subsequence of (Z_n) converges to a $Z_0 \in \overline{H}$.

Proof. Note that $V \times V^0$ is an open G-stable subset of $V \times V$ which is saturated with respect to $V \times V \to (V \times V)/\!\!/G$ and contains $H \times H$. The map $V \times V^0 \to V$, $(Z, W) \to ZW^{-1}$, is G-equivariant, where G acts on the image V by conjugation with the first component, i.e. by $\operatorname{int}(g, h) \cdot X = gXg^{-1}$. It is sufficient to show the following

Claim. A subsequence of (X_n) converges.

Since the image of $X_n := Z_n W_n^{-1}$ in $V/\!\!/\operatorname{int} G$ converges, the trace and the determinant of X_n and therefore the eigenvalues of X_n are bounded. Let $u_n = (a_n, \bar{a}_n) \in K_{\mathbb{R}}$ be such that $\operatorname{int} a_n \cdot X_n = (u_n * Z_n)(u_n * W_n)^{-1} = \begin{pmatrix} x_n & z_n \\ 0 & y_n \end{pmatrix}$. Since $K_{\mathbb{R}}$ is compact, we may assume that $X_n = \begin{pmatrix} x_n & z_n \\ 0 & y_n \end{pmatrix}$.

Let $W_n =: \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ and $W_0 =: \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. By assumption we have $W_0 \in H$. Therefore $\operatorname{Im} d_0 \neq 0$. From

$$Z_n = X_n W_n = \begin{pmatrix} x_n a_n + z_n c_n & x_n b_n + z_n d_n \\ y_n c_n & y_n d_n \end{pmatrix} \in H$$

it follows that

$$\frac{1}{|z_n|^2} (\operatorname{Im}(x_n a_n + z_n c_n) \operatorname{Im}(y_n d_n) - \frac{1}{4}|x_n b_n + z_n d_n - \bar{y}_n \bar{c}_n|^2) > 0$$

for $z_n \neq 0$. Since the eigenvalues x_n, y_n and a_n, b_n, c_n, d_n are bounded, $d_0 \neq 0$ implies that $|z_n|$ is bounded. Thus (X_n) has a convergent subsequence. □

Remark 2. The proofs of Lemma 1 and Lemma 2 use arguments which can be found at least implicitly in [Z] on p. 17.

In the above proof we used that $H \subset V^0$ which is implied by $\det \operatorname{Im} Z \leq |\det Z|$.

Corollary 1. *If $Z_n = (Z_n^1, \dots, Z_n^N) \in H^N$ are such that $(\pi(Z_n))$ converges in $V^N /\!\!/ G$ and (Z_n^N) converges in H, then (Z_n) has a convergent subsequence in $\overline{H}^N$.* □

Lemma 3. *ϕ is a weak exhaustion of X over $V /\!\!/ G$.*

Proof. Let $(Z_n) = ((Z_n^1, \dots, Z_n^N))$ be a sequence in H^N such that $q := \lim \pi(Z_n) \in \partial(G * H^N /\!\!/ G) \subset V^N /\!\!/ G$ exists. There are $h_n \in G_{\mathbb{R}}$ such that a subsequence of $(h * Z_n^N)$ converges to $W^N \in \overline{H}$ (Lemma 1). Now, if $W^N \in \partial H$, then $\limsup \phi(Z_n) = +\infty$. Thus assume that $W^N \in H$. It follows that $(h_n * Z_n)$ has a subsequence which converges to $W \in \overline{H}^N$ (Corollary 1). But W is not in H^N, since $q = \pi(W) \in \partial(G * H^N /\!\!/ G)$. Thus $W \in \partial H^N$ and therefore again $\limsup \phi(Z_n) = +\infty$ follows. □

Lemma 4. *The function ϕ is an exhaustion mod $G_{\mathbb{R}}$ along π.*

Proof. For $r > 0$ let $Z_n \in H^N$, $Z_n =: (Z_n^1, \dots, Z_n^N)$, be such that $\phi(Z_n) \leq r$ and assume that $\lim \pi(Z_n)$ exists in $G * H^N /\!\!/ G$. Thus there are $h_n \in G_{\mathbb{R}}$ such that $(h_n * Z_n^N)$ has a subsequence which converges to some $W^N \in \overline{H}$. If $W^N \in \partial H$, then $\phi(Z_n)$ goes to infinity. This contradicts $\phi(h_n * Z_n) \leq r$. Thus $W^N \in H$ and therefore $(h_n * Z_n)$ has a subsequence with limit $W = (W^1, \dots, W^N) \in \overline{H}^N$. The same argument as above implies that $W^j \in H$ for $j = 1, \dots, N$. □

Proof of the Theorem. From the invariant theoretical point of view the $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ action on V^N is the N-fold product of the standard representation of $\mathrm{SO}_4(\mathbb{C})$ on $\mathbb{C}^4$. It is well known that for any $N = 1, 2, \dots$ the generic G-orbit in V^N is closed. Let $(V^N)^0$ denote the set of points in V^N which lie in a generic closed orbit, i.e., $(V^N)^0$ is a union of the fibres of the quotient $V \to V /\!\!/ G$ which consist exactly of one G-orbit. Since the $G_{\mathbb{R}}$-action on H is proper, $G_{\mathbb{R}}$ acts properly on H^N. It follows from the results in §2 that there is a G-invariant plurisubharmonic function Ψ on $G * H^N$ which is a weak exhaustion. Thus $G * H^N$ is a domain of holomorphy. □

Corollary 2. *The image $G * H^N /\!\!/ G$ of H^N in $V^N /\!\!/ G$ is an open Stein subspace.* □

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