Stability of C^*-Algebras is Not a Stable Property

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Abstract. We show that there exists a C^*-algebra B such that $M_2(B)$ is stable, but B is not stable. Hence stability of C^*-algebras is not a stable property. More generally, we find for each integer $n \geq 2$ a C^*-algebra B so that $M_n(B)$ is stable and $M_k(B)$ is not stable when $1 \leq k < n$. The C^*-algebras we exhibit have the additional properties that they are simple, nuclear and of stable rank one.

The construction is similar to Jesper Villadsen's construction in [7] of a simple C^*-algebra with perforation in its ordered K_0-group.

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1 Introduction

A C^*-algebra A is said to be stable if $A \cong A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space. The problem of deciding which C^*-algebras are stable relates to structure problems of simple C^*-algebras. For example, as shown in [3, Proposition 5.2], if all non-unital hereditary sub-C^*-algebras of a given C^*-algebra A are stable, and if A is simple and not of type I, then A must be purely infinite. It was also remarked in [3, Proposition 5.1] that an AF-algebra is stable if and only if it admits no bounded (densely defined) traces, and it was asked if a similar characterization might hold in general. In more detail, is a C^*-algebra A stable if and only if A admits no bounded (quasi-)trace and no quotient of A is unital?

It is a consequence of the examples produced in this article that the answer to this question is no. Indeed, let A be a C^*-algebra such that $M_2(A)$ is stable and A is not stable. Then $M_2(A)$ admits no bounded (quasi-)trace, and no quotient of $M_2(A)$ is unital. This is easily seen to imply that A admits no bounded (quasi-)trace, and that no quotient of A is unital.

Jesper Villadsen gave in [7] the first examples of simple C^*-algebras whose ordered K_0-groups have perforation. As shown in Proposition 3.3, the examples constructed here must also have perforation in their K_0-group (at least when they admit an approximate unit consisting of projections). We shall in this article make extensive use of the techniques developed by Villadsen.

2 A preliminary result

Let A be a C^*-algebra and consider the set $\Gamma(A)$ consisting of those integers $n \geq 1$ where $M_n(A)$ is stable. The result below shows that this set must be either empty, $\mathbb{N}$, or equal to $\{n, n+1, n+2, \dots\}$ for some $n \geq 2$. Clearly, the empty set and $\mathbb{N}$ arise as $\Gamma(A)$ for appropriate C^*-algebras A. The main result of this article (Theorem 5.3) shows that the remaining sets are also realized.

Proposition 2.1 *Let A be a σ-unital C^*-algebra, let $n \geq 1$ be an integer, and suppose that $M_n(A)$ is stable. Then $M_{n+1}(A)$ is stable.*

Proof: By [3, Theorem 2.1 and Proposition 2.2] it suffices to show that one for all positive elements $a \in M_{n+1}(A)$ and all $\varepsilon > 0$ can find positive elements $b, c \in M_{n+1}(A)$ with $\|a-b\| \leq \varepsilon$, $\|bc\| \leq \varepsilon$, and $b \sim c$ (i.e. $b = x^*x$ and $c = xx^*$ for some $x \in M_{n+1}(A)$). To obtain this it suffices to find positive elements $e, f \in M_{n+1}(A)^+$ with $e \sim f$, $e \perp f$, and ea close to a. Indeed, if $e = x^*x$ and $f = xx^*$, then set $y = xa^{1/2}$, and note that y^*y is close to a and that $(yy^*)(y^*y)$ is small.

Now,

$$a = \begin{pmatrix} a_1 & z \\ z^* & a_2 \end{pmatrix},$$

where $a_1 \in M_n(A)^+$, $a_2 \in A^+$ and $z \in M_{n,1}(A)$. Let $\varepsilon > 0$, and let $\varphi_\varepsilon \colon \mathbb{R}^+ \to [0,1]$ be a continuous function which is zero on $[0, \varepsilon/2]$ and equal to 1 on $[\varepsilon, \infty)$. Set

$$e' = \begin{pmatrix} \varphi_\varepsilon(a_1) & 0 \\ 0 & \varphi_\varepsilon(a_2) \end{pmatrix}.$$

Then $e'a$ is close to to a if $\varepsilon > 0$ is small.

Since $M_n(A)$ is stable, we can find positive elements $e_1, f_1, f_2 \in M_n(A)$ and $e_2 \in A$ such that $e_1 \sim f_1$, $e_2 \sim f_2$ (in the sense that $e_2 = x^*x$ and $f_2 = xx^*$ for some $x \in M_{n,1}(A)$), e_1, f_1, f_2 are mutually orthogonal, e_1 is close to $\varphi_\varepsilon(a_1)$, and e_2 is close to $\varphi_\varepsilon(a_2)$. Set

$$e = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 + f_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then ea is close to a, $e \sim f$, and $e \perp f$ as desired. $\square$

3 Stability and the scale of K_0

We investigate in this section the connection between the scaled ordered group of a C^*-algebra and stability of matrix algebras over the C^*-algebra. Recall that if A is a C^*-algebra, then

$$K_0(A)^+ = \{[p]_0 \mid p \in P(A \otimes \mathcal{K})\} \subseteq K_0(A), \quad \Sigma(A) = \{[p]_0 \mid p \in P(A)\} \subseteq K_0(A)^+,$$

where $P(A \otimes \mathcal{K})$ and $P(A)$ denote the set of projections in $A \otimes \mathcal{K}$, respectively, A.

One can in some cases see from the triple $(K_0(A), K_0(A)^+, \Sigma(A))$ if A is stable. A C^*-algebra A is said to have the *cancellation property* if $p + r \sim q + r$ implies that $p \sim q$ for all projections $p, q, r \in A \otimes \mathcal{K}$ with $p \perp r$ and $q \perp r$. If A has the cancellation property, then $[p]_0 = [q]_0$ in $K_0(A)$ implies $p \sim q$ for all projections $p, q \in A \otimes \mathcal{K}$. Recall also that A has the cancellation property if A is of stable rank one (see [1, Proposition 6.5.1]).

Proposition 3.1 *Let A be a C^*-algebra with the cancellation property and with a countable approximate unit consisting of projections. Then A is stable if and only if $\Sigma(A) = K_0(A)^+$.*

Proof: The "only if" part is trivial. To show the "if" part, assume that $\Sigma(A) = K_0(A)^+$. By [3, Theorem 3.3] it suffices to show that for each projection $p \in A$ there exists a projection $q \in A$ with $p \sim q$ and $p \perp q$. Let a projection $p \in A$ be given. By the assumptions that A has an approximate unit consisting of projections, and $\Sigma(A) = K_0(A)^+$, there exist projections $e, f \in A$ such that $[e]_0 = 2[p]_0 = [p \oplus p]_0$, $e \leq f$ and $p \leq f$. Since A has the cancellation property, this implies that $e \sim p \oplus p$, which again implies that $e = e_1 + e_2$, where $e_1 \sim e_2 \sim p$. Now, $[f - p]_0 = [f - e_1]_0$, and so $p \sim e_2 \leq f - e_1 \sim f - p$. Hence p is equivalent to a subprojection q of $f - p$ as desired. $\square$

Definition 3.2 *A triple (G, G^+, Σ) will be called a* scaled, ordered abelian group *if (G, G^+) is an ordered abelian group, and Σ is an upper directed, hereditary, full subset of G^+, ie.,*

(i) $\forall x_1, x_2 \in \Sigma \; \exists x \in \Sigma : x_1 \leq x, \; x_2 \leq x,$

(ii) $\forall x \in G^+ \; \forall y \in \Sigma : x \leq y \implies x \in \Sigma,$

(iii) $\forall x \in G^+ \; \exists y \in \Sigma \; \exists k \in \mathbb{N} : x \leq ky.$

Let (G, G^+) be an ordered abelian group, and let Σ_1 and Σ_2 be upper directed, hereditary, full subsets of G^+. Define $\Sigma_1 \hat{+} \Sigma_2$ to be the set of all elements $x \in G^+$ for which there exist $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$ with $x \leq x_1 + x_2$. Observe that $\Sigma_1 \hat{+} \Sigma_2$ is an upper directed, hereditary, full subset of G^+. Denote the k-fold sum $\Sigma \hat{+} \Sigma \hat{+} \cdots \hat{+} \Sigma$ by $k \hat{\cdot} \Sigma$. Using that Σ is upper directed we see that $y \in k \hat{\cdot} \Sigma$ if and only if $0 \leq y \leq kx$ for some $x \in \Sigma$.

If A is a stably finite C^*-algebra with the cancellation property and with an approximate unit consisting of projections, then $(K_0(A), K_0(A)^+, \Sigma(A))$ is a scaled, ordered abelian group. If A has these properties, then

$$\big(K_0(M_k(A)), K_0(M_k(A))^+, \Sigma(M_k(A))\big) \cong (K_0(A), K_0(A)^+, k \hat{\cdot} \Sigma(A)). \tag{1}$$

Suppose that $n \geq 2$ and that (G, G^+, Σ) is a scaled, ordered Abelian group such that $(n-1) \hat{\cdot} \Sigma \neq G^+$ and $n \hat{\cdot} \Sigma = G^+$, and suppose that A is a C^*-algebra of stable rank one and with an approximate unit of projections such that $(K_0(A), K_0(A)^+, \Sigma(A)) \cong (G, G^+, \Sigma)$. Then it follows from Proposition 3.1 and (1) that $M_n(A)$ is stable and $M_k(A)$ is not stable for $1 \leq k < n$.

Recall that an ordered Abelian group (G, G^+) is called *weakly unperforated* if $ng \in G^+ \setminus \{0\}$ for some $n \in \mathbb{N}$ and some $g \in G$ implies $g \in G^+$.

PROPOSITION 3.3 *Let (G, G^+, Σ) be a weakly unperforated, scaled, ordered, Abelian group, and suppose that $n \hat{\cdot} \Sigma = G^+$ for some $n \in \mathbb{N}$. Then $\Sigma = G^+$.*

Proof: Let g be an element of G^+, and choose a non-zero element $u \in G^+$. Since $n \hat{\cdot} \Sigma = G^+$, there is an element $x \in \Sigma$ with $nx \geq ng + u$. Now, $n(x-g) \geq u > 0$, and this entails that $x - g \geq 0$, by the assumption that (G, G^+) is weakly unperforated. By the hereditary property of Σ we get that $g \in \Sigma$. Thus $\Sigma = G^+$. $\square$

We give below an explicit example of a scaled, ordered Abelian group (G, G^+, Σ) with $\Sigma \hat{+} \Sigma = G^+$ and $\Sigma \neq G^+$. Note that this ordered group necessarily must be perforated (by Proposition 3.3 above).

It is not known if every (countable) scaled ordered Abelian group is the scaled ordered Abelian group of a C^*-algebra — the problem here lies in realizing the given order structure, not in realizing the given scale. We can therefore not immediately conclude from the example below that there exists a non-stable C^*-algebra B where $M_2(B)$ is stable. Actually, it is not known (to the author) if the ordered Abelian group constructed below is the ordered K_0-group of any C^*-algebra.

EXAMPLE 3.4 Let $\mathbb{Z}_2$ denote the group $\mathbb{Z}/2\mathbb{Z}$, and let $\mathbb{Z}_2^{(\infty)}$ denote the group of all sequences $t = (t_j)_{j=1}^{\infty}$, with $t_j \in \mathbb{Z}_2$ and where $t_j \neq 0$ only for finitely many j. For each $t \in \mathbb{Z}_2^{(\infty)}$, let $d(t)$ be the number of elements in $\{j \in \mathbb{N} \mid t_j \neq 0\}$. Set

$$G_2 = \mathbb{Z} \oplus \mathbb{Z}_2^{(\infty)}, \qquad G_2^+ = \{(k,t) \mid d(t) \leq k\}, \qquad \Sigma_2 = \{(k,t) \mid d(t) = k\}.$$

Then (G_2, G_2^+, Σ_2) is a scaled, ordered Abelian group with $\Sigma_2 \neq G_2^+$ and $\Sigma_2 \hat{+} \Sigma_2 = G_2^+$. To see this, let $e_j \in \mathbb{Z}_2^{(\infty)}$ be the element which is a generator of $\mathbb{Z}_2$ at the jth coordinate and zero elsewhere, set $g_j = (1, e_j) \in G^+$, and set $h_j = g_1 + g_2 + \cdots + g_j$. Then

$$\Sigma_2 = \bigcup_{j=1}^{\infty} \{x \in G^+ \mid x \leq h_j\}. \tag{2}$$

The claims made about (G_2, G_2^+, Σ_2) are now easy to verify.

Notice that $\Sigma_2 + \Sigma_2 \neq \Sigma_2 \hat{+} \Sigma_2$, since for example $(3, e_1 + e_2) \notin \Sigma_2 + \Sigma_2$. This was pointed out to me by Jacob Hjelmborg, and it shows that the sum of two scales is not a scale in general. $\square$

EXAMPLE 3.5 Let $n \geq 2$ be an integer. Let $\mathbb{Z}_n^{(\infty)}$ be the Abelian group of all sequences $(t_j)_{j=1}^{\infty}$ with $t_j \in \mathbb{Z}_n$ $(= \mathbb{Z}/n\mathbb{Z})$, and $t_j \neq 0$ only for finitely many j. Let $e_j \in \mathbb{Z}_n^{(\infty)}$ be a generator of the jth copy of $\mathbb{Z}_n$. Then each $t \in \mathbb{Z}_n^{(\infty)}$ is a sum $t = \sum_{j=1}^{\infty} r_j e_j$ with $0 \leq r_j < n$ and where $r_j = 0$ for all but finitely many j. Set $d(t) = \sum_{j=1}^{\infty} r_j$, and set

$$G_n = \mathbb{Z} \oplus \mathbb{Z}_n^{(\infty)}, \qquad G_n^+ = \{(k,t) \mid d(t) \leq k\}, \qquad \Sigma_n = \bigcup_{j=1}^{\infty} \{x \in G^+ \mid x \leq h_j\},$$

where $g_j = (1, e_j)$ and $h_j = g_1 + g_2 + \cdots + g_j$. Then (G_n, G_n^+, Σ_n) is a scaled, ordered, Abelian group, $(n-1) \hat{\cdot} \Sigma_n \neq G_n^+$ and $n \hat{\cdot} \Sigma_n = G_n^+$. $\square$

Adopt the following (standard) notation. If $e \in M_n(A)$ and $f \in M_m(A)$ are projections, then let $e \oplus f$ denote the projection $\mathrm{diag}(e, f) \in M_{n+m}(A)$. Write $e \sim f$ if there is an element $v \in M_{m,n}(A)$ with $e = v^*v$ and $f = vv^*$, and write $e \precsim f$ if $e \sim f_0$ for some subprojection f_0 of f. Denote the k-fold direct sum $e \oplus e \oplus \cdots \oplus e$ by $e \otimes 1_k$. If A has the cancellation property (see the introduction to this section), and if $e, f \in A$ are projections, then $[e]_0 \leq [f]_0$ if and only if $e \precsim f$.

PROPOSITION 3.6 *Let A be a C^*-algebra, let $n \geq 2$ be an integer, and suppose that A contains projections $e, p_1, p_2, p_3, \ldots$ that satisfy*

(i) *$e \otimes 1_n \sim p_j \otimes 1_n$ for all j,*

(ii) *e is not equivalent to a subprojection of $(p_1 \oplus p_2 \oplus \cdots \oplus p_j) \otimes 1_{n-1}$ for any j.*

Set $q_j = p_1 \oplus p_2 \oplus \cdots \oplus p_j$, and embed all matrix algebras over A coherently into $A \otimes \mathcal{K}$ so that q_j belongs to $A \otimes \mathcal{K}$ for all j. Set

$$B = \overline{\bigcup_{j=1}^{\infty} q_j (A \otimes \mathcal{K}) q_j}. \tag{3}$$

Then $M_k(B)$ is not stable for $1 \leq k < n$, but $M_n(B)$ is stable.

Let H be the subgroup of $K_0(B)$ generated by the K_0-classes of the projections $e, p_1, p_2, p_3, \ldots$. Assume that B has the cancellation property. Then

$$(H, H \cap K_0(B)^+, H \cap \Sigma(B)) \cong (G_n, G_n^+, \Sigma_n), \tag{4}$$

where the triple on the right hand-side is the scaled, ordered, Abelian group defined in Example 3.5.

Proof: Observe that

$$M_k(B) = \overline{\bigcup_{j=1}^{\infty} (q_j \otimes 1_k)(A \otimes \mathcal{K})(q_j \otimes 1_k)},$$

for each k, and that $\{q_j \otimes 1_k\}_{j=1}^{\infty}$ is an approximate unit for $M_k(B)$.

To show that $M_k(B)$ is not stable for $1 \leq k < n$ it suffices by Proposition 2.1 to show that $M_{n-1}(B)$ is not stable.

If $M_{n-1}(B)$ were stable, then there would exist a projection $q \in M_{n-1}(B)$ such that $q \sim p_1 \otimes 1_{n-1}$ and $q \perp p_1 \otimes 1_{n-1}$. (This is rather easy to see directly, and one can also obtain this from [3, Theorem 3.3].) Since $\{q_j \otimes 1_{n-1} - p_1 \otimes 1_{n-1}\}_{j=1}^{\infty}$ is an approximate unit for $(1 - p_1 \otimes 1_{n-1}) M_{n-1}(B)(1 - p_1 \otimes 1_{n-1})$, there is a j, so that q is equivalent to a subprojection of $q_j \otimes 1_{n-1} - p_1 \otimes 1_{n-1}$ $(= (p_2 \oplus p_3 \oplus \cdots \oplus p_j) \otimes 1_{n-1})$. By assumption (i),

$$\begin{aligned} e &\precsim e \otimes 1_n \sim p_1 \otimes 1_n \precsim (p_1 \otimes 1_{n-1}) \oplus (p_1 \otimes 1_{n-1}) \precsim (p_1 \otimes 1_{n-1}) \oplus q \\ &\precsim (p_1 \oplus p_2 \oplus \cdots \oplus p_j) \otimes 1_{n-1}, \end{aligned}$$

in contradiction with assumption (ii).

We proceed to show that $M_n(B)$ is stable. By (i), $q_j \otimes 1_n$ is equivalent to the direct sum of $e \otimes 1_n$ with itself j times. It follows quite easily from this that $M_n(B)$ is stable. We can also use [3, Theorem 3.3] to obtain this conclusion by showing that there for each projection p in $M_n(B)$ exists a projection q in $M_n(B)$ with $p \sim q$ and $p \perp q$. One can here reduce to the case where p is a subprojection of $q_j \otimes 1_n$ for some j, and the result then follows from the fact that $q_{2j} \otimes 1_n - q_j \otimes 1_n \sim q_j \otimes 1_n$.

Assume now that B has the cancellation property. To establish the isomorphism (4), note first that $n([p_j]_0 - [e]_0) = 0$ by (i). Retaining the notation from Example 3.5, we define a group homomorphism $\varphi\colon G_n \to H$ by $\varphi(1,0) = [e]_0$ and $\varphi(0, e_j) = [p_j]_0 - [e]_0$. φ is clearly surjective. For any $(k,t) \in G_n$ with $t = \sum_{j=1}^N r_j e_j$, $0 \le r_j < n$,

$$\varphi(k,t) = k[e]_0 + \sum_{j=1}^N r_j\big([p_j]_0 - [e]_0\big) = \big(k - d(t)\big)[e]_0 + \sum_{j=1}^N r_j[p_j]_0.$$

It follows that $\varphi(k,t) \ge 0$ if $(k,t) \ge 0$. Conversely, if (k,t) is not positive, then $k - d(t) \le -1$, and so

$$\varphi(k,t) = \big(k - d(t)\big)[e]_0 + \sum_{j=1}^N r_j[p_j]_0 \le (n-1)\big([p_1]_0 + [p_2]_0 + \cdots + [p_N]_0\big) - [e]_0.$$

By (ii) and the assumption that B has the cancellation property, the element on the right-hand side of this inequality is not positive. All in all we have shown that $\varphi(k,t) \ge 0$ if and only if $(k,t) \ge 0$. This entails that φ is injective and that $\varphi(G_n^+) = H \cap K_0(B)^+$.

Since $\{q_j\}_{j=1}^\infty$ is an approximate unit for B, an element $g \in K_0(B)$ lies in $\Sigma(B)$ if and only if $0 \le g \le [q_j]_0$ for some j. Notice that $\varphi(h_j) = [q_j]_0$. Hence $\varphi(k,t) \in \Sigma(B)$ if and only if $0 \le (k,t) \le h_j$ for some j, and this shows that $\varphi(\Sigma_n) = H \cap \Sigma(B)$. □

Remark 3.7 Corollary 4.2 and Proposition 5.2 contain for each prime number n examples of C^*-algebras with projections $e, p_1, p_2, p_3, \dots$ satisfying (i) and (ii) of Proposition 3.6. The C^*-algebras in Proposition 5.2 have the cancellation property (being of stable rank one).

Remark 3.8 One can replace condition (i) in Proposition 3.6 by a weaker condition such as for example $e \precsim p_j \otimes 1_n$ for all j, and still obtain that the C^*-algebra B defined in (3) has the property that $M_k(B)$ is not stable for $1 \le k < n$ and $M_n(B)$ is stable. However, with this weaker condition one would not have a description of the scaled ordered group as in (4).

4 The commutative case

We realize for each positive prime number n projections $e, p_1, p_2, p_3, \dots$ satisfying conditions (i) and (ii) of Proposition 3.6, with respect to that n, inside a C^*-algebra which is stably isomorphic to a commutative C^*-algebra. At the same time, Lemma 4.1 below, is a key ingredient in Section 5.

If $\pi\colon X_1 \to X_2$ is a continuous function, then π^* will denote the map from the cohomology groups of X_2 to the cohomology groups of X_1, and the same symbol will be used to denote the map from vector bundles over X_2 to vector bundles over X_1. By naturality of the Euler class, $\mathrm{e}(\pi^*(\xi)) = \pi^*(\mathrm{e}(\xi))$ for all complex vector bundles ξ over Y.

The proof of Lemma 4.1 below is almost identical to the proof of [6, Theorem 3.4]. The statements of Lemma 4.1 and of [6, Theorem 3.4] are, however, quite different. Therefore, and for the convenience of the reader, we include a proof of Lemma 4.1.

Let $\mathbb{D}$ denote the unit disk in the complex plane. Consider for each integer $n \geq 2$ the equivalence relation $\sim$ on $\mathbb{D}$ given by: $z \sim w$ if $z = w$ or if $|z| = |w| = 1$ and $z^n = w^n$. Put $Y_n = \mathbb{D}/\sim$.

LEMMA 4.1 *Let n be a positive prime number, and put $X = Y_n^{n-1}$. There exists a complex line bundle ω over X with the following properties. Let m be a positive integer, let $\pi_1, \pi_2, \ldots, \pi_m \colon X^m \to X$ be the coordinate maps, and set*

$$\xi_k^{(m)} = \pi_1^*(\omega) \oplus \pi_2^*(\omega) \oplus \cdots \oplus \pi_k^*(\omega), \qquad 1 \leq k \leq m,$$

which is a complex vector bundle over X^m of dimension k. Let θ_d denote the trivial complex vector bundle (over X or X^m) of (complex) dimension d. Then

(i) *$n\omega \cong \theta_n$,*

(ii) *if $(n-1)\xi_k^{(m)} \oplus \theta_{d_1} \cong \eta \oplus \theta_{d_2}$ for some complex vector bundle η over X^m, and some positive integers d_1 and d_2, then $d_1 \geq d_2$, and*

(iii) *$\omega \oplus \eta \cong \theta_n$ for some $(n-1)$-dimensional complex vector bundle η over X.*

Proof: Recall that $H^2(Y_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. There is a complex line bundle ζ over Y_n with non-trivial Euler class $\mathrm{e}(\zeta) \in H^2(Y_n; \mathbb{Z})$, and with $n\zeta \cong \theta_n$. Let $\nu_1, \nu_2, \ldots, \nu_{n-1} \colon X = Y_n^{n-1} \to Y_n$ be the coordinate projections, and set

$$\omega = \nu_1^*(\zeta) \otimes \nu_2^*(\zeta) \otimes \cdots \otimes \nu_{n-1}^*(\zeta).$$

Then ω is a complex line bundle over X, and successive applications of the isomorphism $n\zeta \cong \theta_n = n\theta_1$, yield $n\omega \cong \theta_n$. Hence (i) holds, and (iii) is a trivial consequence of (i).

To prove claim (ii) we first show that the Euler class, $\mathrm{e}((n-1)\xi_k^{(m)})$, is non-zero. The Euler class of ω is given by

$$\mathrm{e}(\omega) = \sum_{j=1}^{n-1} \nu_j^*(\mathrm{e}(\zeta)), \tag{5}$$

cf. [4, Proposition V.3.10]. By the product formula for the Euler class, cf. [4, Proposition V.3.10],

$$\mathrm{e}((n-1)\xi_k^{(m)}) = \prod_{j=1}^{k} \pi_j^*(\mathrm{e}(\omega)^{n-1}). \tag{6}$$

Since $\mathrm{e}(\zeta)^2 \in H^4(Y_n; \mathbb{Z})$ and $H^4(Y_n; \mathbb{Z}) = 0$, it follows from (5) and (6) that

$$\mathrm{e}(\omega)^{n-1} = (n-1)! \prod_{i=1}^{n-1} \nu_i^*(\mathrm{e}(\zeta)).$$

Let $\rho_1, \rho_2, \dots, \rho_k \colon X^k \to X$ and $\pi \colon X^m \to X^k$ be the projections maps. Then $\pi_j = \rho_j \circ \pi$, and $\pi^* \colon H^{2k}(X^k;\mathbb{Z}) \to H^{2k}(X^m;\mathbb{Z})$ is an injection. The map

$$\mu \colon H^2(Y_n;\mathbb{Z}) \otimes H^2(Y_n;\mathbb{Z}) \otimes \cdots \otimes H^2(Y_n;\mathbb{Z}) \to H^{2k(n-1)}(X^k;\mathbb{Z})$$

given by

$$\mu(x_{1,1} \otimes x_{1,2} \otimes \cdots \otimes x_{k,n-1}) = \prod_{j=1}^{k} \prod_{i=1}^{n-1} (\rho_j^* \circ \nu_i^*)(x_{i,j}),$$

is injective by the Künneth formula. Now,

$$\begin{aligned}
\mathrm{e}((n-1)\xi_k^{(m)}) &= \prod_{j=1}^{k} \pi_j^*(\mathrm{e}(\omega)^{n-1}) \\
&= \prod_{j=1}^{k} \pi_j^*\big((n-1)! \prod_{i=1}^{n-1} \nu_i^*(\mathrm{e}(\zeta))\big) \\
&= (n-1)!^k \pi^*\big(\prod_{j=1}^{k} \prod_{i=1}^{n-1} (\rho_j^* \circ \nu_i^*)(\mathrm{e}(\zeta)\big) \\
&= (\pi^* \circ \mu)\big((n-1)!^k \mathrm{e}(\zeta) \otimes \mathrm{e}(\zeta) \otimes \cdots \otimes \mathrm{e}(\zeta)\big).
\end{aligned}$$

The element $\mathrm{e}(\zeta) \otimes \mathrm{e}(\zeta) \otimes \cdots \otimes \mathrm{e}(\zeta)$ has order n in $H^2(Y_n;\mathbb{Z}) \otimes H^2(Y_n;\mathbb{Z}) \otimes \cdots \otimes H^2(Y_n;\mathbb{Z})$. Because n is assumed to be prime, and because $\pi^* \circ \mu$ is injective, we get that $\mathrm{e}((n-1)\xi_k^{(m)}) \neq 0$.

Assume (ii) were false. Then $(n-1)\xi_k^{(m)} \oplus \theta_{d_1} \cong \eta \oplus \theta_{d_2}$ for some η and some positive integers $d_1 < d_2$. Hence $(n-1)\xi_k^{(m)}$ would be stably isomorphic to $\eta \oplus \theta_{d_2 - d_1}$. The Euler class is invariant under stable isomorphism, and the Euler class of a trivial bundle (of dimension ≥ 1) is zero, and so by the product formula we get $\mathrm{e}((n-1)\xi_k^{(m)}) = 0$, a contradiction. □

George Elliott pointed out to me that one obtains the following corollary from Lemma 4.1:

COROLLARY 4.2 *Let n be a positive prime number, let Z be the infinite Cartesian product of Y_n with itself. Then there exist projections $e, p_1, p_2, p_3, \dots$ in $M_n(C(Z))$ satisfying*

(i) *$e \otimes 1_n \sim p_j \otimes 1_n$ for all j,*

(ii) *e is not equivalent to a subprojection of $(p_1 \oplus p_2 \oplus \cdots \oplus p_j) \otimes 1_{n-1}$ for any $j \geq 1$.*

Proof: Let ω be the complex line bundle over $X = Y_n^{n-1}$ from Lemma 4.1 and use Lemma 4.1 (iii) to find a projection $p \in C(X, M_n(\mathbb{C})) = M_n(C(X))$ that corresponds to ω. Identify Z with $\prod_{j=1}^{\infty} X$, and let $\pi_j \colon Z \to X$, $j \in \mathbb{N}$, be the coordinate maps. Put $p_j = p \circ \pi_j \in C(Z, M_n(\mathbb{C})) = M_n(C(Z))$, and let $e \in M_n(C(Z))$ be a one-dimensional constant projection. It follows from Lemma 4.1 (i) that $p_j \otimes 1_n \sim e \otimes 1_n$ for all j. To see (ii), view $M_n(C(Z))$ as the inductive limit,

$$M_n(C(X)) \to M_n(C(X^2)) \to M_n(C(X^3)) \to \cdots \to M_n(C(Z)),$$

so that $e, p_1, p_2, \dots, p_j \in M_n(C(X^j))$. Then, by Lemma 4.1 (ii), for each k and for each $m \geq k$, e is not equivalent to a subprojection of $(p_1 \oplus p_2 \oplus \dots \oplus p_k) \otimes 1_{n-1}$ in (a matrix algebra over) $M_n(C(X^m))$. This implies that (ii) holds. □

Combining Corollary 4.2 with Proposition 3.6 we get for each prime number n a hereditary sub-C^*-algebra B of $C(Z) \otimes \mathcal{K}$ such that $M_k(B)$ is not stable for $1 \leq k < n$, and $M_n(B)$ is stable. Proceeding as in the proof of Theorem 5.3 one can find such examples B for all integers $n \geq 2$.

5 The simple case

We use an inductive limit construction, like the one Villadsen used in [7], to obtain projections as in Proposition 3.6 inside a *simple* C^*-algebra.

Fix a positive prime number n. Let $\{k_j\}_{j=1}^{\infty}$ be a sequence of positive integers chosen large enough so that

$$\sum_{j=1}^{\infty} (1 - \prod_{i=j}^{\infty} \frac{k_i}{1 + k_i}) < \frac{1}{n-1}. \tag{7}$$

Define inductively another sequence of integers $\{m_j\}_{j=1}^{\infty}$ by $m_1 = 1$ and $m_{j+1} = m_j(k_j + 1)$.

Let $Y_n = \mathbb{D}/\sim$ be as defined in Section 4, and put $X = Y_n^{n-1}$. Define inductively a sequence of spaces $\{X_j\}_{j=1}^{\infty}$ by setting $X_1 = X$ and $X_{j+1} = X_j^{k_j} \times X^{m_{j+1}}$. Set

$$A_j = M_{2^{n-1} m_j}(C(X_j)) = C(X_j, M_{2^{n-1} m_j}(\mathbb{C})).$$

Choose $x_j \in X_j$ appropriately (in a way which will be made precise later), and define *-homomorphisms $\varphi_j \colon A_j \to A_{j+1}$ by

$$\varphi_j(f)(x) = \mathrm{diag}((f \circ \pi_1^j)(x), (f \circ \pi_2^j)(x), \dots, (f \circ \pi_{k_j}^j)(x), f(x_j)), \quad x \in X_{j+1}, \quad f \in A_j,$$

where $\pi_1^j, \pi_2^j, \dots, \pi_{k_j}^j \colon X_{j+1} = X_j^{k_j} \times X^{m_{j+1}} \to X_j$ are the projections from the first factor of X_{j+1}.

Let $(A, \mu_j \colon A_j \to A)$ be the inductive limit of the sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots.$$

It will be convenient to have an expression for the composed connecting maps $\varphi_{i,j} \colon A_j \to A_i$ for $i > j$. For this purpose set

$$k_{i,j} = \prod_{n=j}^{i-1} k_n, \qquad l_{i,j} = \prod_{n=j}^{i-1} (k_n + 1) - \prod_{n=j}^{i-1} k_n, \qquad m_{i,j} = \sum_{n=j+1}^{i} m_n k_{i,n}, \tag{8}$$

(with the convention that $k_{i,i} = 1$). Then $X_i = X_j^{k_{i,j}} \times X^{m_{i,j}}$, and the composed connecting maps are up to unitary equivalence given by

$$\begin{aligned} &\varphi_{i,j}(f)(x) \\ &\quad = \mathrm{diag}\big((f \circ \pi_1^{i,j})(x), (f \circ \pi_2^{i,j})(x), \dots, (f \circ \pi_{k_{i,j}}^{i,j})(x), f(x_1^{i,j}), f(x_2^{i,j}), \dots, f(x_{l_{i,j}}^{i,j})\big). \end{aligned}$$

The maps $\pi_1^{i,j}, \pi_2^{i,j}, \dots, \pi_{k_{i,j}}^{i,j} \colon X_i = X_j^{k_{i,j}} \times X^{m_{i,j}} \to X_j$ are here the projections onto the first $k_{i,j}$ coordinates of X_i, the set

$$X_j^i := \{x_1^{i,j}, x_2^{i,j}, \dots, x_{l_{i,j}}^{i,j}\} \subseteq X_j$$

is for $i \geq j+2$ equal to $X_j^{i-1} \cup \{\pi_1^{i,j}(x_i), \pi_2^{i,j}(x_i), \dots, \pi_{k_{i,j}}^{i,j}(x_i)\}$, where each element of the first set is repeated $k_i + 1$ times, and $X_j^{j+1} = \{x_j\}$.

Choose the points $x_j \in X_j$ such that $\bigcup_{r=j+1}^{\infty} X_j^r$ is dense in X_j for each $j \in \mathbb{N}$. Since each X_j^i is finite and since no X_j has isolated points this will entail that $\bigcup_{r=i}^{\infty} X_j^r$ is dense in X_j for each $j \in \mathbb{N}$ and for every $i > j$.

By [2, Proposition 1] and [7, Proposition 10] we get:

Proposition 5.1 *The C^*-algebra A is simple and has stable rank one.*

With the C^*-algebra A and the prime number n as above, we have:

Proposition 5.2 *There exist projections $e, p_1, p_2, p_3, \dots$ in A so that*

(i) $p_j \otimes 1_n \sim e \otimes 1_n$ *for all* $j \geq 1$, *and*

(ii) e *is not equivalent to a subprojection of* $(p_1 \oplus p_2 \oplus \cdots \oplus p_j) \otimes 1_{n-1}$ *for any* $j \geq 1$.

Proof: By Lemma 4.1 (iii) there exists a projection $q \in A_1 = M_{2^{n-1}}(C(X))$ which corresponds to the complex line bundle ω. Let $\rho_1, \rho_2, \dots, \rho_{m_j} \colon X_j = X_{j-1}^{k_{j-1}} \times X^{m_j} \to X$ be coordinate projections corresponding to the last factor of X_j. Set $q_1 = q$, set

$$q_j = \operatorname{diag}(q \circ \rho_1, q \circ \rho_2, \dots, q \circ \rho_{m_j}) \in A_j,$$

for $j \geq 2$, and set $p_j = \mu_j(q_j) \in A$. Let $e_1 \in A_1$ be a constant projection of dimension 1, so that e_1 corresponds to the trivial complex line bundle θ_1, and set $e = \mu_1(e_1) \in A$.

It follows from Lemma 4.1 (i) that $q \otimes 1_n \sim e_1 \otimes 1_n$. This implies that $q_j \otimes 1_n$ is equivalent to a constant projection. Since $\varphi_{j,1}(e_1) \otimes 1_n$ is a constant projection (in $M_n(A_j)$) of the same dimension as $q_j \otimes 1_n$, we find that $q_j \otimes 1_n \sim \varphi_{j,1}(e_1) \otimes 1_n$ in $M_n(A_j)$. Hence

$$p_j \otimes 1_n = \mu_j(q_j \otimes 1_n) \sim \mu_j(\varphi_{j,1}(e_1) \otimes 1_n) = e \otimes 1_n$$

in $M_n(A)$.

For $i \geq j$, put

$$f_{i,j} = \varphi_{i,1}(q_1) \oplus \varphi_{i,2}(q_2) \oplus \cdots \oplus \varphi_{i,j}(q_j).$$

Then $p_1 \oplus p_2 \oplus \cdots \oplus p_j = \mu_i(f_{i,j})$, and $f_{i,j} = \varphi_{i,j}(f_{j,j})$. Observe that $X_j = X^{d_j}$, where $d_1 = 1$ and $d_{j+1} = n_j k_j + m_{j+1}$. By inspection of the formula for the composed connecting maps $\varphi_{j,l}$, we find that the projection $f_{j,j}$ corresponds to the vector bundle $\xi_{d_j}^{(d_j)} \oplus \theta_{c_j}$, where $c_j = \sum_{r=1}^{j} m_r l_{j,r}$, cf. (8). From this we get that the projection $f_{i,j}$ corresponds to the vector bundle $\xi_{a_{i,j}}^{(d_i)} \oplus \theta_{b_{i,j}}$ over X_i, where $a_{i,j} = k_{i,j} d_j$ and $b_{i,j} = \sum_{r=1}^{j} m_r l_{i,r}$, possibly after a permutation of the coordinates of X_i.

The trivial projection $\varphi_{i,1}(e_1)$ has dimension m_i and corresponds therefore to the trivial vector bundle θ_{m_i}. Now,

$$\begin{aligned}
\frac{1}{m_i} b_{i,j} &= \frac{1}{m_i}\sum_{r=1}^{j} m_r l_{i,r} \\
&= \frac{1}{m_i}\sum_{r=1}^{j}\prod_{s=1}^{r-1}(1+k_s)\Big(\prod_{s=r}^{i-1}(1+k_s)-\prod_{s=r}^{i-1}k_s\Big) \\
&= \sum_{r=1}^{j}\Big(1-\prod_{s=r}^{i-1}\frac{k_s}{1+k_s}\Big) \\
&\leq \sum_{r=1}^{\infty}\Big(1-\prod_{s=r}^{\infty}\frac{k_s}{1+k_s}\Big) \;<\; \frac{1}{n-1},
\end{aligned}$$

where the last inequality follows from (5). This shows that $(n-1)b_{i,j} < m_i$. By Lemma 4.1 (ii), there exists no vector bundle η over X_i such that

$$\eta \oplus \theta_{m_i} \cong (n-1)\xi_{a_{i,j}}^{(d_i)} \oplus \theta_{(n-1)b_{i,j}} \;\; (= (n-1)(\xi_{a_{i,j}}^{(d_i)} \oplus \theta_{b_{i,j}})),$$

or, equivalently, $\varphi_{i,1}(e_1)$ is not equivalent to a subprojection of $f_{i,j} \otimes 1_{n-1}$. Since this holds for all $i > j$, e is not equivalent to a subprojection of $(p_1 \oplus p_2 \oplus \cdots \oplus p_j) \otimes 1_{n-1}$, and this completes the proof. $\square$

Theorem 5.3 *For each integer $n \geq 2$ there exists a C^*-algebra B such that $M_n(B)$ is stable, and $M_k(B)$ is not stable for $1 \leq k < n$. Moreover, B can be chosen to be simple, nuclear, with stable rank one and with an approximate unit consisting of projections.*

Proof: Consider first the case where n is prime. Let B be the C^*-algebra defined in display (3) in Proposition 3.6 corresponding to the C^*-algebra A and to the projections $e, p_1, p_2, p_3, \dots$ found in Proposition 5.2. Then B is a hereditary subalgebra of $A \otimes \mathcal{K}$, and since A is simple, nuclear and has stable rank one, it follows that B also has these properties (see [5, Theorem 3.3] for the last claim). The sequence $\{q_j\}_{j=1}^{\infty}$ is an approximate unit for B. By Proposition 3.6, $M_k(B)$ is not stable for $1 \leq k < n$ and $M_n(B)$ is stable.

Suppose now that $n \geq 2$ is an arbitrary integer. Observe that all integers $\geq (n-1)^2$ belong to the set

$$\bigcup_{m=1}^{\infty} ((n-1)m, nm].$$

Choose a prime number $p \geq (n-1)^2$. Then there exists an integer $m \geq 1$ so that $(n-1)m < p \leq nm$. By the first part of the proof there exists a C^*-algebra D with $M_p(D)$ stable and $M_k(D)$ not stable for $1 \leq k < p$. Set $B = M_m(D)$. Then B is simple, nuclear, and has stable rank one and an approximate unit consisting of projections because D has these properties. Moreover, $M_k(B) = M_{km}(D)$, and so, by Proposition 2.1, $M_k(B)$ is stable if and only if $km \geq p$, which, by the choice of p and m, happens if and only if $k \geq n$. $\square$

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