On the Cuspidal Divisor Class Group of a Drinfeld Modular Curve

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Abstract. The theory of theta functions for arithmetic groups Γ that act on the Drinfeld upper half-plane is extended to allow degenerate parameters. This is used to investigate the cuspidal divisor class groups of Drinfeld modular curves. These groups are finite for congruence subgroups Γ and may be described through the corresponding quotients of the Bruhat-Tits tree by Γ. The description given is fairly explicit, notably in the most important special case of Hecke congruence subgroups Γ over a polynomial ring.

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Introduction.

Drinfeld modular curves are the substitutes in positive characteristics of classical modular curves. Like these, they have a rich structure where various mathematical disciplines interact: number theory, algebraic geometry, (non-Archimedean) function theory, representation theory and automorphic forms, and others. They encode important pieces of the arithmetic of global function fields, notably those related to two-dimensional Galois representations and elliptic curves, in a way similar to the correspondence ascribed to Shimura, Taniyama and Weil and partially proven by A. Wiles.

By their very construction, these curves come equipped with a uniformization through the Drinfeld upper half-plane Ω, a one-dimensional rigid analytic symmetric space. Hence many questions about such a curve M_Γ may be attacked by function theoretic means, through the construction and investigation of analytic functions on Ω (analogues of elliptic modular forms, or of theta functions) that satisfy functional equations under Γ, the group that uniformizes $M_\Gamma = \Gamma \setminus \Omega$.

Leaving aside Tate's elliptic curves, the first appearance of non-Archimedean uniformized curves is in work of Mumford [16] and of Manin-Drinfeld [14], where the acting group Γ is a Schottky group, that is, a finitely generated free group consisting of hyperbolic elements. For the corresponding *Mumford curves*, Gerritzen and van der Put in their monograph [11] obtained a very satisfactory description of the minimal model, the Jacobian, the Abel-Jacobi map, ...

A similar program for Drinfeld modular curves was started in [10], whose main results were the construction of the Jacobian J_Γ of M_Γ through non-Archimedean theta functions $\theta_\Gamma(\omega, \eta, z)$ and, as an application, the analytic description of "Weil uniformizations" of elliptic curves over global functions fields. Apart from the fact that a Drinfeld modular curve is defined over a global field (which gives an abundance of arithmetic structure), the crucial difference to Mumford curves is that $M_\Gamma = \Gamma \setminus \Omega$ by construction is an affine curve, and has to be "compactified" to a smooth projective curve $\overline{M}_\Gamma$ by adding a finite number of "cusps" of Γ. Several natural questions (with important arithmetical applications) arise, about the

- structure of the group $\mathcal{C}$ generated in the Jacobian J_Γ by the cusps;
- degeneration of the theta functions $\theta_\Gamma(\omega, \eta, z)$ if the parameters $\omega, \eta \in \Omega$ approach cusps of Γ;
- relationship between $\mathcal{C}$ and the minimal model of $\overline{M}_\Gamma$.

It turns out that these questions have satisfactory answers in terms of the associated *almost finite* graphs $\Gamma \setminus \mathcal{T}$, which can be mechanically calculated from the initial data that define Γ, e.g., from congruence conditions.

In order to give more precise statements, we now introduce some notation.

We start with a function field K in one variable with exact field of constants $\mathbb{F}_q$, the finite field with $q = p^r$ elements. In K, we fix a place "∞", and we let $A \subset K$ be the Dedekind subring of elements regular away from ∞. Then A is a discrete and cocompact subring of the completion K_∞. We finally need C, the completed algebraic closure of K_∞. By an arithmetic subgroup of $\mathrm{GL}(2, K)$, we understand a subgroup commensurable with $\mathrm{GL}(2, A)$. Such a group Γ acts with finite stabilizers on $\Omega = C - K_\infty$, and M_Γ will be the uniquely determined algebraic curve whose space of C-points is given by $\Gamma \setminus \Omega$. The cusps are given as the orbits $\Gamma \setminus \mathbb{P}^1(K)$ on the projective line $\mathbb{P}^1(K)$. It is customary to recall here the obvious analogy of the data K, A, K_∞, C, Ω, $\mathrm{GL}(2, A)$ with $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, H = complex upper half-plane, $\mathrm{SL}(2, \mathbb{Z})$ (or rather $H^\pm = \mathbb{C} - \mathbb{R}$ and $\mathrm{GL}(2, \mathbb{Z})$), respectively.

In [10], we studied theta functions $\theta_\Gamma(\omega, \eta, z)$, which are defined as certain infinite products depending on parameters $\omega, \eta \in \Omega$. These functions are meromorphic on Ω with zeros (resp. poles) at the orbits of ω (resp. η); they transform according to a character $c(\omega, \eta) : \Gamma \longrightarrow C^*$, have a nice behavior at the boundary $\partial\Omega = \mathbb{P}^1(K)$ of Ω, and give rise to a pairing $\overline{\Gamma} \times \overline{\Gamma} \longrightarrow K_\infty^*$ on the maximal torsion-free Abelian quotient $\overline{\Gamma}$ of Γ. The analytic space Ω has a canonical covering through standard rational subsets of $\mathbb{P}^1(C)$, the nerve of which equals the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{GL}(2, K_\infty)$. There results a $\mathrm{GL}(2, K_\infty)$-equivariant map $\lambda : \Omega \longrightarrow \mathcal{T}(\mathbb{R})$ that allows to describe many properties of M_Γ and of related objects in terms of the graph $\Gamma \setminus \mathcal{T}$. The main results of the present paper go into this direction. They are:

- Theorem 3.8 and its corollaries, which give the link between theta functions, cuspidal divisors on $\overline{M}_\Gamma$, and harmonic Γ-invariant cochains on $\mathcal{T}$;

- the description, given in sections 4 and 5, of the cuspidal divisor class group $\mathcal{C}(\Gamma)$ of $\overline{M}_\Gamma$ and of the canonical map from $\mathcal{C}(\Gamma)$ to $\Phi_\infty(\Gamma)$ = group of connected components of the Néron model of J_Γ at ∞ (here Γ is assumed to be a congruence subgroup);
- the determination of the subgroup generated by the $\theta_\Gamma(\omega,\eta,z)$ $(\omega,\eta\in\mathbb{P}^1(K))$ in the group of all theta functions for Γ (Thm. 5.4), valid for Hecke congruence subgroups Γ of $\mathrm{GL}(2,A)$, where A is a polynomial ring.

These results depend on an extension of the theory developed in [10] to the case of theta functions $\theta_\Gamma(\omega,\eta,z)$ whose parameters ω,η are allowed to lie in the boundary of Ω. This is carried out in section two: proof of convergence, functional equation, behavior at the boundary. Roughly speaking, theta functions with degenerate parameters behave similar to those with $\omega,\eta\in\Omega$, and analytic dependence on the parameters holds at least for the associated multipliers $c(\omega,\eta)$. That part of the theory, as well as the links (given in section three) with harmonic cochains on $\mathcal{T}$ and cuspidal divisor groups on $\overline{M}_\Gamma$, works in the context of arbitrary groups Γ commensurable with $\mathrm{GL}(2,A)$, and may thus be used also for the study of non-congruence subgroups. From section four on we specialize to congruence subgroups Γ and use the known finiteness of $\mathcal{C}(\Gamma)$ in this case (i.e., the analogue of Manin-Drinfeld's theorem, cf. [2], [5]) to express it through the graph $\Gamma\setminus\mathcal{T}$. $\mathcal{C}(\Gamma)$ agrees (modulo finite groups annihilated by $q^{\deg\,\infty}-1$) with $\underline{H}/\underline{H}_!\oplus\underline{H}_!^\perp$, where $\underline{H}=\underline{H}(\mathcal{T},\mathbb{Z})^\Gamma$ is the group of Γ-invariant $\mathbb{Z}$-valued harmonic cochains on $\mathcal{T}$, $\underline{H}_!$ is the subgroup of cochains with compact support mod Γ, and $\underline{H}_!^\perp$ its ortho-complement in $\underline{H}$.

A refinement of the above in the important special case of Hecke congruence subgroups $\Gamma_0(n)$ over $A=\mathbb{F}_q[T]$ is given in section five. Here we use in a crucial way the known results (cf. [9]) about the structure of the graph $\Gamma_0(n)\setminus\mathcal{T}$. We conclude, in section six, with a worked-out example (hopefully instructive), where the canonical map $\mathrm{can}_\infty:\mathcal{C}(\Gamma)\longrightarrow\Phi_\infty(\Gamma)$ fails to be injective or surjective even for a Hecke congruence group Γ with prime conductor. The existence of a non-trivial kernel of can_∞ is reflected in congruence properties of a corresponding "Eisenstein quotient" of J_Γ, an elliptic curve in the example treated.

The notation of the present paper is largely compatible to that of [10], to which it is a sequel. Thus without further explanation, for a group G acting on a set X and $x\in X$, G_x is the stabilizer, Gx the orbit, $G\setminus X$ the set of all orbits, G^{ab} the maximal Abelian quotient of G. We often write gx for $g(x)$, $g\in G$. As far as misconceptions are unlikely, we do not distinguish between matrices in $\mathrm{GL}(2)$ and their classes in $\mathrm{PGL}(2)$, and between varieties over C or K_∞, their associated analytic spaces, and their sets of C-valued points.

1. Background [10].

(1.1) We let K be the function field of a smooth projective geometrically connected curve $\mathfrak{C}$ over $\mathbb{F}_q$ (q = power of the rational prime p) and $\infty\in\mathfrak{C}$ a closed point fixed once for all. Attached to these data, we dispose of

- the subring A of K of functions regular away from ∞;
- the completion K_∞ of K at ∞;
- the completed algebraic closure $C=C_\infty$ of K_∞;

- Drinfeld's upper half-plane $\Omega = C - K_\infty$, on which $\mathrm{GL}(2, K_\infty)$ acts through $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$;
- the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{GL}(2, K_\infty)$.

Recall that $\mathcal{T}$ is a $(q_\infty + 1)$-regular tree ($q_\infty = q^{\deg\ \infty}$ = size of residue class field $\mathbb{F}_q(\infty)$) provided with a $\mathrm{GL}(2, K_\infty)$-action and an equivariant map λ from Ω to the real points $\mathcal{T}(\mathbb{R})$ of $\mathcal{T}$.

The group $\mathrm{GL}(2, K)$ acts from the right on the space K^2 of row vectors. For an A-lattice (= projective A-submodule of rank two) $Y \hookrightarrow K^2$, we let $\mathrm{GL}(Y) = \{\gamma \in \mathrm{GL}(2, K) \mid Y\gamma = Y\}$.

(1.2) An *arithmetic subgroup* Γ of $\mathrm{GL}(2, K)$ is a subgroup commensurable with some $\mathrm{GL}(Y)$, i.e., $\Gamma \cap \mathrm{GL}(Y)$ has finite index in both Γ and $\mathrm{GL}(Y)$, and which acts without inversion on $\mathcal{T}$. A *congruence subgroup* is some Γ that satisfies $\mathrm{GL}(Y, \mathfrak{n}) \subset \Gamma \subset \mathrm{GL}(Y)$, where $0 \neq \mathfrak{n} \subset A$ is an ideal and $\mathrm{GL}(Y, \mathfrak{n})$ is the kernel of the reduction map $\mathrm{GL}(Y) \longrightarrow \mathrm{GL}(Y/\mathfrak{n}Y)$. According to [20] II Thm. 12, there are "many" subgroups of finite index of $\mathrm{GL}(Y)$ that are not congruence subgroups, although it is not easy to display examples.

Now fix some arithmetic subgroup Γ as above. The following facts, in the case of congruence subgroups, are proved and/or described in more detail in [10] I - III; their generalization to arbitrary arithmetic subgroups is obvious .

(1.2.1) Γ acts with finite stabilizers on Ω and $\mathcal{T}$. Hence e.g. the quotient $\Gamma \setminus \Omega$ may be defined as an analytic space.

(1.2.2) Γ has finite covolume in $\mathrm{GL}(2, K_\infty)$ modulo its center.

(1.2.3) The quotient $\Gamma \setminus \mathcal{T}$ is (in an essentially unique fashion, *loc. cit.*) the union of a finite graph and a finite number of half-lines $\bullet - - - \bullet - - - \bullet - - - \bullet \cdots$, the *ends* of $\Gamma \setminus \mathcal{T}$.

(1.2.4) There exists a smooth connected affine algebraic curve M_Γ / C (which may even be defined over a finite field extension $K' \subset K_\infty$ of K) whose set $M_\Gamma(C)$ of C-points agrees with $\Gamma \setminus \Omega$ as an analytic space. The M_Γ or their canonical smooth compactifications $\overline{M}_\Gamma$ are what we here call Drinfeld modular curves.

(1.2.5) There are canonical bijections between the sets of

(A) ends of $\Gamma \setminus \mathcal{T}$,

(B) cusps $\overline{M}_\Gamma(C) - M_\Gamma(C)$ of $\overline{M}_\Gamma$, and

(C) orbits $\Gamma \setminus \mathbb{P}^1(K)$ on the projective line $\mathbb{P}^1(K)$.

In the sequel, we will not distinguish between (a), (b), (c) and label it by *cusp*(Γ). Its cardinality is denoted by $c = c(\Gamma)$.

(1.2.6) The genus $g = g(\Gamma)$ of $\overline{M}_\Gamma$ agrees with the number of $\dim_{\mathbb{Q}} H_1(\Gamma \setminus \mathcal{T}, \mathbb{Q})$ of independent cycles of the graph $\Gamma \setminus \mathcal{T}$, which in turn equals the rank $\mathrm{rk}(\Gamma^{ab})$ of the factor commutator group Γ^{ab} of Γ.

Let $\overline{\Gamma} = \Gamma^{ab}/\mathrm{tor}(\Gamma^{ab}) \cong \mathbb{Z}^{g(\Gamma)}$ and Γ_f be the subgroup of Γ generated by the elements of finite order. It follows from [20] I Thm. 13, Cor. 1 that

(1.2.7) (i) Γ/Γ_f is free in g generators,
(ii) $\mathrm{tor}(\Gamma^{ab})$ is generated by the image of Γ_f in Γ^{ab}, and
(iii) the canonical map $\overline{\Gamma} \longrightarrow (\Gamma/\Gamma_f)^{ab}$ is an isomorphism.

(1.3) Let $X(\mathcal{T})$ and $Y(\mathcal{T})$ be the sets of vertices, of oriented edges of $\mathcal{T}$, respectively. As in [10], $\underline{H}(\mathcal{T},\mathbb{Z})$ is the right $\mathrm{GL}(2,K_\infty)$-module of $\mathbb{Z}$-valued harmonic cochains in $\mathcal{T}$, i.e., of maps $\varphi : Y(\mathcal{T}) \longrightarrow \mathbb{Z}$ that satisfy $\varphi(\overline{e}) = -\varphi(e)$ ($\overline{e} = e$ oriented inversely) and

$$\sum_{e\in Y(\mathcal{T}) \text{ with origin } v} \varphi(e) \ = 0 \qquad (v \in X(\mathcal{T})). \tag{1.3.1}$$

Further, $\underline{H}(\mathcal{T},\mathbb{Z})^\Gamma$ denotes the Γ-invariants in $\underline{H}(\mathcal{T},\mathbb{Z})$ and $\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma \subset \underline{H}(\mathcal{T},\mathbb{Z})^\Gamma$ the subgroup of those φ with finite support modulo Γ. It follows from (1.2.3) and simple graph-theoretical arguments that $\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ is free Abelian of rank $g = g(\Gamma)$, and is a direct factor of the free Abelian group $\underline{H}(\mathcal{T},\mathbb{Z})^\Gamma$ of rank $g+c-1$. In fact, there is a canonical injection with finite p-free cokernel (*loc. cit.* sect. 3, 6)

$$j : H_1(\Gamma \setminus \mathcal{T},\mathbb{Z}) \xrightarrow{\cong} \overline{\Gamma} \hookrightarrow \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma,$$

which turns out to be bijective in important cases.

(1.4) *A holomorphic theta function* for Γ is an invertible holomorphic function $f : \Omega \longrightarrow C$ that for each $\gamma \in \Gamma$ satisfies

$$f(\gamma z) = c_f(\gamma)\ f(z)$$

with some $c_f(\gamma) \in C^*$, and is holomorphic non-zero at the cusps of Γ ([10] 5.1). For *meromorphic theta* functions, we allow poles and zeros on Ω, but not at the cusps. The homomorphism $c_f : \Gamma \longrightarrow \Gamma^{ab} \longrightarrow C^*$ that maps γ to $c_f(\gamma)$ is the *multiplier* of the (holomorphic or meromorphic) theta function f. The main construction of such functions is as follows. Let ω,η be fixed elements of Ω, and put

$$\theta_\Gamma(\omega,\eta,z) = \prod_{\gamma\in\tilde{\Gamma}} \left(\frac{z-\gamma\omega}{z-\gamma\eta}\right). \tag{1.4.1}$$

Note that the product is not over Γ but over its quotient $\tilde{\Gamma}$ by its center (the latter being isomorphic with a subgroup of $A^* = \mathbb{F}_q^*$), which acts effectively on Ω. The next theorem collects the principal properties of the θ_Γ. In the case of congruence subgroups Γ, it is the synopsis of several results proved in [10], mainly Thm. 5.4.1, Thm. 5.4.12, Thm. 5.7.1 and their corollaries. The reader will easily convince himself that the arguments given there apply verbatim to the case of general arithmetic subgroups as defined in (1.2).

1.5 Theorem. (i) *The product* (1.4.1) *for* $\theta(\omega,\eta,z) = \theta_\Gamma(\omega,\eta,z)$ *converges locally uniformly* (*loc. cit.* (5.2.2)) *in* $z \in \Omega$ *and defines a meromorphic theta function for* Γ. *It is invertible (holomorphic nowhere zero) if the orbits* $\Gamma\omega$, $\Gamma\eta$ *agree, and has its only zeroes and poles at* $\Gamma\omega$, $\Gamma\eta$, *of order* $\sharp\tilde{\Gamma}_\omega$, $\sharp\tilde{\Gamma}_\eta$, *respectively, if* $\Gamma\omega \neq \Gamma\eta$.
(ii) *The multiplier* $c(\omega,\eta,\cdot) : \Gamma \longrightarrow C$ *of* $\theta(\omega,\eta,\cdot)$ *factors through* $\overline{\Gamma}$.
(iii) *Given* $\alpha \in \Gamma$, *the holomorphic theta function* $u_\alpha(z) = \theta(\omega,\alpha\omega,z)$ *is well-defined independently of* $\omega \in \Omega$, *and depends only on the class of* α *in* $\overline{\Gamma}$. *Further,* $u_{\alpha\beta} = u_\alpha u_\beta$.
(iv) $c(\omega,\eta,\alpha) = \frac{u_\alpha(\eta)}{u_\alpha(\omega)}$, *and in particular, is holomorphic in* ω *and* η.
(v) *Let* $c_\alpha(\ \cdot\ ) = c(\omega,\alpha\omega,\cdot)$ *be the multiplier of* u_α. *The rule* $(\alpha,\beta) \longmapsto c_\alpha(\beta)$ *defines*

a symmetric bilinear map on $\overline{\Gamma} \times \overline{\Gamma}$, *which takes its values in* $K_\infty^* \hookrightarrow C^*$.
(vi) *Let* $v_\infty : K_\infty^* \longrightarrow \mathbb{Z}$ *be the valuation and* $(\alpha, \beta) := -v_\infty(c_\alpha(\beta))$. *Then* $(.\,,.) : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow \mathbb{Z}$ *is positive definite.*

As a consequence of (vi), the map $\overline{c} : \overline{\Gamma} \longrightarrow \mathrm{Hom}(\overline{\Gamma}, C^*)$ induced by $\alpha \longmapsto c_\alpha$ is injective, and the analytic group variety $\mathrm{Hom}(\overline{\Gamma}, C^*)/\overline{c}(\overline{\Gamma})$ carries the structure of an Abelian variety J_Γ defined over K_∞.

1.6 THEOREM ([10] Thm. 7.4.1). J_Γ *equals the Jacobian variety of the curve* $\overline{M}_\Gamma$, *and the Abel-Jacobi map with base point* $[\omega] \in \Gamma \setminus \Omega = M_\Gamma(C)$ *is given by* $[\eta] \longmapsto$ *class of* $c(\omega, \eta, \cdot)$ *modulo* $\overline{c}(\overline{\Gamma})$.

Again, the proof given in *loc. cit.* (including its ingredients (6.5.4) and (6.4.4) carries over to the case of a general arithmetic Γ.

2. THETA FUNCTIONS WITH DEGENERATE PARAMETERS.

(2.1) We show how functions $\theta_\Gamma(\omega, \eta, z)$ with similar properties can be defined when the parameters ω, η are allowed to take values in

$$\overline{\Omega} = \Omega \cup \mathbb{P}^1(K). \tag{2.1.1}$$

Here Γ is any arithmetic subgroup of $\mathrm{GL}(2, K)$ and $\tilde{\Gamma} \hookrightarrow \mathrm{PGL}(2, K)$ its factor group modulo the center. For $\omega, \eta \in \overline{\Omega}$ we define the rational function $F(\omega, \eta, z)$ in $z \in \mathbb{P}^1(C)$ as

$$\begin{array}{cl} \frac{z-\omega}{z-\eta} & \text{if } \omega \neq \infty \neq \eta \\ (1-\frac{z}{\eta})^{-1} & \text{if } \omega = \infty,\ \eta \neq 0, \infty \\ 1-\frac{z}{\omega} & \text{if } \eta = \infty,\ \omega \neq 0, \infty \\ z^{-1} & \text{if } \omega = \infty,\ \eta = 0 \\ z & \text{if } \eta = \infty,\ \omega = 0 \\ 1 & \text{if } \omega = \eta = \infty. \end{array} \tag{2.1.2}$$

Hence, up to cancelling, $F(z) = F(\omega, \eta, z)$ has a simple zero at ω, a simple pole at η, and is normalized such that $F(\infty) = 1$ (resp. $F(0) = 1$, resp. $F(1) = 1$) whenever the first of these conditions makes sense. We further put

$$\theta_\Gamma(\omega, \eta, z) = \prod_{\gamma \in \tilde{\Gamma}} F(\gamma\omega, \gamma\eta, z), \tag{2.1.3}$$

which specializes to (1.4.1) if both ω and η are in Ω.

(2.2) Our first task will be to establish the locally uniform convergence of the product. We let "$|\,.\,|$": $C \longrightarrow \mathbb{R}_{\geq 0}$ be the extension of the normalized absolute value on K_∞ to C and "$|\,.\,|_i$": $C \longrightarrow \mathbb{R}_{\geq 0}$ the "imaginary part" map, i.e., $|z|_i = \inf\{|z-x| \mid x \in K_\infty\}$. Besides several obvious properties, it also satisfies

$$|\gamma z|_i = \frac{\det\ \gamma}{|cz+d|^2} |z|_i \tag{2.2.1}$$

for $z \in \Omega$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, K_\infty)$. We will perform the relevant estimates on the sets

$$U_n = \{z \in \Omega \mid |z| \leq q_\infty^n,\ |z|_i \geq q_\infty^{-n}\}. \tag{2.2.2}$$

These are affinoid subsets of $\mathbb{P}^1(C)$, and $\Omega = \bigcup_{n \in \mathbb{N}} U_n$ is an admissible covering.

2.3 Proposition. *Let $\omega, \eta \in \overline{\Omega}$ be fixed. The product* (2.1.3) *for $\theta_\Gamma(\omega,\eta,z)$ converges locally uniformly for $z \in \Omega$ and defines a meromorphic function on Ω. If both ω, η are in $\mathbb{P}^1(K)$ or if $\Gamma_\omega = \Gamma_\eta$, it is even invertible on Ω. Otherwise, $\theta_\Gamma(\omega,\eta,z)$ has zeroes of order $\sharp\tilde{\Gamma}_\omega$ at $\Gamma\omega$, poles of order $\sharp\tilde{\Gamma}_\eta$ at $\Gamma\eta$, and no further zeroes or poles on Ω.*

Proof. It is easily seen that the assertion is stable under replacing Γ by a commensurable group. Since any Γ is commensurable with $\mathrm{GL}(2,A)$, we may assume $\Gamma = \mathrm{GL}(2,A)$. Now for $\omega, \eta \in \Omega$, the result is [10] Prop. 5.2.3. Hence suppose that at least one of ω and η lies in $\mathbb{P}^1(K)$. Without restriction, $\omega \in \mathbb{P}^1(K)$, $\omega \neq \eta$, and $\omega \neq \infty \neq \eta$. We need the following facts, which result from (2.2.1) and/or elementary calculations:

(2.3.1) $$\{\gamma \in \Gamma \mid \gamma U_n \cap U_n \neq \emptyset\} \text{ is finite for each } n \in \mathbb{N};$$

(2.3.2) $$\frac{z-\gamma\omega}{z-\gamma\eta} - 1 = \frac{(\det\gamma)(\eta-\omega)}{(z-\gamma\eta)(c\omega+d)(c\eta+d)}$$
$$(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,\ \gamma\omega \neq \infty \neq \gamma\eta);$$

(2.3.3) $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ define the same element in $\Gamma_\infty \setminus \Gamma$ if and only if $(c',d') = u(c,d)$ with some $u \in \mathbb{F}_q^*$;

(2.3.4) $$|z - \gamma\eta| \geq q_\infty^{-n} \text{ whenever } z \in U_n,\ \gamma\eta \notin U_n.$$

Combining (2.3.1) and (2.3.4) yields the existence of $c_1(n,\omega,\eta) > 0$ such that

(2.3.5) $$\frac{|\det\ \gamma|\ |\eta-\omega|}{|z-\gamma\eta|} \leq c_1(n,\omega,\eta)$$
uniformly on U_n for almost all $\gamma \in \Gamma$.

In view of (2.3.2), we must estimate $|(c\omega+d)(c\eta+d)|$ from below.

2.3.6 Claim. For given $c_2 > 0$, the number of classes of pairs (c,d) as in (2.3.3) (i.e., of classes of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_\infty \setminus \Gamma$) such that $|(c\omega+d)(c\eta+d)| < c_2$ holds, is finite.

Proof of claim. First, exclude the finite (!) number of pairs (c,d) with $c\omega+d = 0$ or $c\eta+d = 0$. There exists $c_3(\omega) > 0$ such that the non-vanishing elements $c\omega+d$ of the fractional ideal $A\omega + A \subset K$ satisfy

(2.3.7) $$|c\omega+d| \geq c_3(\omega).$$

Hence, if $\eta \in \Omega$, the claim follows from:

(2.3.8) For any $c_4 > 0$, the number of pairs (c,d) with $|c\eta+d| < c_4$ is finite.

If $\eta \in K$, we consider the map $(c,d) \longmapsto (c\omega+d, c\eta+d)$ from $A \times A$ to $K_\infty \times K_\infty$, which by $\omega \neq \eta$ is injective. Its image is an A-lattice, which implies:

(2.3.9) Given $c_5, c_6 > 0$, the simultaneous inequalities $|c\omega+d| \leq c_5$, $|c\eta+d| \leq c_6$ are possible for a finite number of pairs only.

Since the possible values of $|c\omega+d|$, $|c\eta+d|$ are discrete and bounded from below (cf. (2.3.7)), the assertion (2.3.6) follows.

Next we observe:

(2.3.10) If (c,d) as above, $n \in \mathbb{N}$ and $c_7 > 0$ are fixed, then $|z-\gamma\omega| \geq c_7$ uniformly in $z \in U_n$ for almost all $\gamma \in \Gamma$ of the form $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Now (2.3.2) together with (2.3.5), (2.3.6) and (2.3.10) yields the following:

(2.3.11) Given $\epsilon > 0$ and $n \in \mathbb{N}$, almost all of the factors of type $\frac{z-\gamma\omega}{z-\gamma\eta}$ that appear in (2.1.3) satisfy
$$\left|\frac{z-\gamma\omega}{z-\gamma\eta} - 1\right| < \epsilon$$
uniformly in $z \in U_n$.

It remains to verify the analogous statement for the other factors in (2.1.3). They are of type

$$(2.3.12)\qquad \begin{array}{lcl} \text{(a)} & (1-\frac{z}{\gamma\eta})^{-1} & \text{if } \gamma\omega = \infty,\ \gamma\eta \neq 0, \infty \\ \text{(b)} & (1-\frac{z}{\gamma\omega}) & \text{if } \gamma\eta = \infty,\ \gamma\omega \neq 0, \infty \\ \text{(c)} & z^{-1} & \text{if } \gamma\omega = \infty,\ \gamma\eta = 0 \\ \text{(d)} & z & \text{if } \gamma\eta = \infty,\ \gamma\omega = 0. \end{array}$$

Now cases (c) and (d) can occur only finitely many times since $\Gamma_\infty \cap \Gamma_0$ is finite. Cases (a) and (b) are similar, so we restrict to (b). Let γ_0 be such that $\gamma_0\eta = \infty$. The other such elements of Γ are the $\gamma\gamma_0$, where $\gamma \in \Gamma_\infty = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in \mathbb{F}_q^*,\, b \in A\}$. Thus we have to show that $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\gamma_0\omega = \frac{a}{d}\gamma_0\omega + \frac{b}{d}$ tends with b to infinity in absolute value, which is clear. Hence the product (2.1.3) converges uniformly on each U_n to a meromorphic function with the asserted divisor. □

From now on, we omit the subscript Γ in $\theta(\omega,\eta,z) = \theta_\Gamma(\omega,\eta,z)$.

2.4 PROPOSITION. *For $\alpha \in \Gamma$, $\theta(\omega,\eta,z)$ satisfies a functional equation*
$$\theta(\omega,\eta,\alpha z) = c(\omega,\eta,\alpha)\ \theta(\omega,\eta,z)$$
with $c(\omega,\eta,\alpha) \in C^*$ independent of $z \in \Omega$.

Proof. We let $h(\omega,\eta,\alpha)$ be the quotient of $F(\omega,\eta,\alpha z)$ by $F(\alpha^{-1}\omega,\alpha^{-1}\eta,z)$. Since the two rational functions have the same divisors, $h(\omega,\eta,\alpha)$ is well-defined and constant. Now
$$\begin{aligned} \theta(\omega,\eta,\alpha z) &= \prod_{\gamma\in\tilde{\Gamma}} F(\gamma\omega,\gamma\eta,\alpha z) \\ &= \prod h(\gamma\omega,\gamma\eta,\alpha)\cdot\prod F(\alpha^{-1}\gamma\omega,\alpha^{-1}\gamma\eta,z) \\ &= \prod h(\gamma\omega,\gamma\eta,\alpha)\ \theta(\omega,\eta,z), \end{aligned}$$
whence the convergence of $c(\omega,\eta,\alpha) := \prod_{\gamma\in\tilde{\Gamma}} h(\gamma\omega,\gamma\eta,\alpha)$ results from that of $\theta(\omega,\eta,z)$, i.e., from (2.3). □

(2.5) The next step is to describe the behavior of $\theta(\omega,\eta,z)$ at the boundary, i.e., at $s\in\mathbb{P}^1(K)=\overline{\Omega}-\Omega$. As usual, possibly replacing Γ by its conjugate $\gamma\Gamma\gamma^{-1}$, where $\gamma\in\mathrm{GL}(2,K)$ satisfies $\gamma\infty=s$, it suffices to discuss the case $s=\infty$. The stabilizer $\tilde{\Gamma}_\infty$ in $\tilde{\Gamma}$ is represented by matrices $\left(\begin{smallmatrix}a&b\\0&1\end{smallmatrix}\right)$, where a runs through a subgroup W_∞ (of order w_∞, say) of $\mathbb{F}_q^*$, and b through an infinite-dimensional $\mathbb{F}_p$-vector space $\mathfrak{b}\subset K$ commensurable with a fractional A-ideal. In particular, $\mathfrak{b}\in C$ is discrete, which ensures the convergence of the infinite product written below. Put

$$t_\infty(z)=e_{\mathfrak{b}}^{-1}(z), \tag{2.5.1}$$

where $e_{\mathfrak{b}}:\ C\longrightarrow C$ is the function

$$e_{\mathfrak{b}}(z)=z\prod_{0\neq b\in\mathfrak{b}}(1-\frac{z}{b}).$$

For the essential properties of such functions, see e.g. [12] I, IV. We need the observation:

(2.5.2) $e_{\mathfrak{b}}$ is $\mathbb{F}$-linear, where $\mathbb{F}\subset\mathbb{F}_q$ is the subfield generated by W_∞. Hence for $a\in W_\infty$, $t_\infty(az)=a^{-1}t_\infty(z)$ and $t_\infty^{w_\infty}(az)=t_\infty^{w_\infty}(z)$.

It results from the fact that $\mathfrak{b}$ is even an $\mathbb{F}$-vector space.

(2.5.3) The subspace $\Omega_c=\{z\in\Omega\ \mid\ |z|_i\geq c\}$ of Ω is stable under $\tilde{\Gamma}_\infty$ and $\tilde{\Gamma}_\infty^u=\{\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\ \mid\ b\in\mathfrak{b}\}$, and for a suitable $c\gg 0$, t_∞ identifies $\tilde{\Gamma}_\infty^u\setminus\Omega_c=\mathfrak{b}\setminus\Omega_c$ with a small pointed ball $B_\epsilon(0)-\{0\}=\{t\in C\ \mid\ 0<|t|\leq\epsilon\}$. Again for $c\gg 0$, $\tilde{\Gamma}_\infty\setminus\Omega_c$ is an open subspace of $\Gamma\setminus\Omega\hookrightarrow\Gamma\setminus\overline{\Omega}$ (since $\gamma\Omega_c\cap\Omega_c\neq\emptyset$ implies $\gamma\in\Gamma_\infty$, cf. (2.2.1)), and $t_\infty^{w_\infty}$ is a uniformizer around the point ∞. This allows to define holomorphy, meromorphy, vanishing order at ∞, ... for functions on Ω_c invariant under $\tilde{\Gamma}_\infty^u$ or $\tilde{\Gamma}_\infty$. (For more details, see e.g. [5] V or [10] 2.7.)

As results from (2.4) and (2.3), $\theta(\omega,\eta,z)$ is invariant under $\tilde{\Gamma}_\infty^u$ and has neither zeroes nor poles on $\mathfrak{b}\setminus\Omega_c$, provided c is large (or ϵ is small) enough. It has therefore a Laurent expansion with respect to t_∞. Now the factors of type $\frac{z-\gamma\omega}{z-\gamma\eta}$ in (2.1.3) tend to 1 uniformly in γ if $|z|_i\longrightarrow\infty$, i.e., if $|t_\infty(z)|\longrightarrow 0$, hence they contribute $1+o(t_\infty)$ to the Laurent expansion. Therefore,

$$\begin{array}{l}\theta(\omega,\eta,z)\text{ is invertible around }t_\infty=0\text{ if}\\ \text{neither }\Gamma\omega\text{ nor }\Gamma\eta\text{ contains }\infty.\end{array} \tag{2.5.4}$$

(2.5.5) Suppose that $\infty\in\Gamma\eta\neq\Gamma\omega$. Without restriction, we may even assume $\eta=\infty$. The factors of type (b) and (d) in (2.3.12) yield

$$\prod_{\substack{\gamma\in\tilde{\Gamma}_\infty\\ \gamma\omega=0}}z\prod_{\substack{\gamma\in\tilde{\Gamma}_\infty\\ \gamma\omega\neq 0}}(1-\frac{z}{\gamma\omega})=\prod_{\substack{\gamma\in\tilde{\Gamma}_\infty\\ \gamma\omega=0}}z\prod_{\substack{\gamma\in\tilde{\Gamma}_\infty\\ \gamma\omega\neq 0}}(1-\frac{z}{a\omega+b}),$$

writing $\gamma\in\tilde{\Gamma}_\infty$ in the form $\left(\begin{smallmatrix}a&b\\0&1\end{smallmatrix}\right)$ as above. That product defines an entire function $f:\ C\longrightarrow C$ with its zeroes at the points z_0 of shape $z_0=a\omega+b$, each of the same order $\sharp\{\left(\begin{smallmatrix}a&b\\0&1\end{smallmatrix}\right)\in\tilde{\Gamma}\ \mid\ a\omega+b=z_0\}$.

Let first $\boxed{\omega\notin\mathfrak{b}}$. Since an entire function is determined up to constants by its divisor, we have, using (2.5.2):

$$\begin{aligned}
\text{const. } f(z) &= \prod_{a\in W_\infty} e_{\mathfrak{b}}(z-a\omega) \\
&= \prod_a (e_{\mathfrak{b}}(z) - a e_{\mathfrak{b}}(\omega)) \\
&= \prod_a (t_\infty^{-1}(1+o(t_\infty))) \\
&= t_\infty^{-w_\infty}(1+o(t_\infty)).
\end{aligned}$$

Next, let $\boxed{\omega \in \mathfrak{b}}$. Then f has zeroes of order w_∞ at the points of $\mathfrak{b}$, which gives

$$\text{const. } f(z) = e_{\mathfrak{b}}(z)^{w_\infty} = t_\infty^{-w_\infty}.$$

It is straight from definitions that for $a \in W_\infty$ (i.e., $\left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \in \tilde{\Gamma}_\infty$),

$$\theta(\omega,\eta,az) = \theta(\omega,\eta,z)$$

holds. Hence, by (2.5.2), the Laurent expansion of $\theta(\omega,\eta,z)$ w.r.t. t_∞ is actually a series in $t_\infty^{w_\infty}$. Therefore, under our condition $\infty \in \Gamma\eta \neq \Gamma\omega$, $\theta(\omega,\eta,z)$ has a simple pole at the cusp represented by ∞ w.r.t. its correct uniformizer $t_\infty^{w_\infty}$. Analogous assertions hold if $\infty \in \Gamma\omega \neq \Gamma\eta$, or if $\Gamma\omega = \Gamma\eta$ (in which case the possible zeroes and poles at the cusps cancel).

We collect what has been proven.

2.6 PROPOSITION. *The function $\theta(\omega,\eta,\cdot)$ has a meromorphic continuation to the boundary $\mathbb{P}^1(K)$ of Ω. With respect to the uniformizer $t_s^{w_s}$ at the cusp $[s]$ of $\overline{M}_\Gamma$ represented by $s \in \mathbb{P}^1(K)$, it*

has a simple zero, if $s \in \Gamma\omega \neq \Gamma\eta$,
has a simple pole, if $s \in \Gamma\eta \neq \Gamma\omega$,
is invertible, if $\Gamma\omega = \Gamma\eta$ (whether or not $s \in \Gamma\omega = \Gamma\eta$). □

Here of course, w_s is the *weight* of $[s]$, i.e., the size of the non-p part W_s of $\tilde{\Gamma}_s$ (cf. (2.5)).

2.7 COROLLARY. *The holomorphic function $u_\alpha(z) := \theta(\omega,\alpha\omega,z)$ on $\overline{\Omega}$ ($\omega \in \overline{\Omega}$, $\alpha \in \Gamma$ fixed) does not depend on the choice of ω.*

Proof. In view of (2.6), it suffices to verify this for $z \in \Omega$. If the parameters ω,η are in Ω, we get as in [10] Thm. 5.4.1 (iv):

$$\begin{aligned}
\frac{\theta(\omega,\alpha\omega,z)}{\theta(\eta,\alpha\eta,z)} &= \prod_{\gamma\in\tilde{\Gamma}} \left(\frac{z-\gamma\omega}{z-\gamma\alpha\omega}\right)\left(\frac{z-\gamma\alpha\eta}{z-\gamma\eta}\right) = \prod_{\gamma\in\tilde{\Gamma}} \left(\frac{z-\gamma\omega}{z-\gamma\eta}\right)\left(\frac{z-\gamma\alpha\eta}{z-\gamma\alpha\omega}\right) \\
&= \theta(\omega,\eta,z)\theta(\eta,\omega,z) = 1
\end{aligned}$$

The reader will easily verify through a case-by-case consideration that the same cancelling takes place if ω,η are allowed to take values in $\mathbb{P}^1(K)$. □

2.8 DEFINITION. A *cuspidal theta function* for Γ is an invertible holomorphic function f on Ω that for each $\gamma \in \Gamma$ satisfies

$$f(\gamma z) = c_f(\gamma) f(z)$$

with some $c_f(\gamma) \in C^*$, and is meromorphic at the cusps. This means that, compared to (1.4), we allow zeroes and poles at the cusps.

The prototype of a cuspidal theta function is $\theta(\omega, \eta, \cdot)$, where both ω and η are in $\mathbb{P}^1(K)$.

2.9 Lemma. *Let $\omega, \eta \in \overline{\Omega}$, $\alpha, \gamma \in \Gamma$. The factors $F(\ .\ ,\ .\ ,\ .\ )$ of* (2.1.2) *satisfy*

$$\frac{F(\gamma\omega, \gamma\eta, \alpha z)}{F(\gamma\omega, \gamma\eta, z)} = \frac{F(\gamma^{-1}\alpha z, \gamma^{-1} z, \omega)}{F(\gamma^{-1}\alpha z, \gamma^{-1} z, \eta)}$$

(identity of rational functions in $z \in \mathbb{P}^1(C)$).

Proof. We may assume that $\omega \neq \eta$. Let

$$D(a, b, c, d) := \frac{a-c}{b-c} / \frac{a-d}{b-d} \quad (a, b, c, d \in \mathbb{P}^1(C))$$

be the cross-ratio which, through the usual conventions, delivers a well-defined element of $P^1(C)$ if at least three of a, b, c, d are different. Going through the cases, it is easily seen that $F(a, b, c)/F(a, b, d) = D(c, d, a, b)$, and hence the assertion follows from the invariance of $D(a, b, c, d)$ under projective transformations, in particular, under the Klein group of order 4. □

2.10 Corollary. *Let $\alpha \in \Gamma$ be fixed. The multiplier $c(\omega, \eta, \alpha)$ satisfies $c(\omega, \eta, \alpha) = \frac{u_\alpha(\eta)}{u_\alpha(\omega)}$. In particular, it is holomorphic on Ω and at the cusps, considered as a function in ω with η fixed (resp. in η with ω fixed).*

Proof. Let $\omega, \eta \in \overline{\Omega}$ be given. Then

$$\begin{aligned} c(\omega, \eta, \alpha) &= \frac{\theta(\omega, \eta, \alpha z)}{\theta(\omega, \eta, z)} = \prod_{\gamma \in \tilde{\Gamma}} \frac{F(\gamma\omega, \gamma\eta, \alpha z)}{F(\gamma\omega, \gamma\eta, z)} \\ &= \prod_{\gamma \in \tilde{\Gamma}} \frac{F(\gamma^{-1}\alpha z, \gamma^{-1} z, \omega)}{F(\gamma^{-1}\alpha z, \gamma^{-1} z, \eta)} = \frac{u_\alpha(\eta)}{u_\alpha(\omega)}, \end{aligned}$$

where the last equality follows from (2.7). □

2.11 Corollary. *Let $\omega, \eta \in \overline{\Omega}$. The constant $c(\omega, \eta, \alpha)$ and the function u_α depend only on the class of α in $\overline{\Gamma} = \Gamma^{ab}/\mathrm{tor}(\Gamma^{ab})$.*

Proof. By (2.10), the statement about $c(\omega, \eta, \alpha)$ follows from that on u_α. But $u_\alpha = \theta(\omega, \alpha\omega, \cdot)$ may be described with an arbitrary base point $\omega \in \Omega$, so the result follows from (1.5) (iii). □

2.12 Remark. As in Shimura's book [21], we may provide $\overline{\Omega}$ with a topology coming from the strong topology on $\mathbb{P}^1(C)$. To do so, it suffices to describe a fundamental system of neighborhoods for $s \in \mathbb{P}^1(K)$. By the usual homogeneity argument, we may even assume $s = \infty$, in which case the system of sets $\{\infty\} \cup \Omega_c$ $(c \in \mathbb{N})$ is as desired. It is then natural to expect that our theta functions satisfy

$$\lim_{\omega \to \omega_0,\ \eta \to \eta_0} \theta(\omega, \eta, z) = \theta(\omega_0, \eta_0, z) \tag{2.12.1}$$

with respect to that topology. This is easy to verify if e.g. all of $\omega_0, \eta_0, z \notin \Gamma\omega_0 \cup \Gamma\eta_0$ belong to Ω. On the other hand, for $\omega, \eta \in \Omega$, $\theta(\omega, \eta, z)$ is normalized such that it takes the value 1 at $z = \infty$, whereas $\theta(\infty, \eta, z)$ has a simple zero at $z = \infty$ if $\eta \notin \Gamma\infty$.

This rules out the possibility of (2.12.1) if one of the parameters ω_0, η_0 belongs to the boundary. The best we can hope for is the continuous dependence on parameters of the multiplier instead of the theta functions themselves.

2.13 Corollary. Let $\omega_0, \eta_0 \in \overline{\Omega}$, $\alpha \in \Gamma$. *Then*

$$\lim_{\omega \to \omega_0,\ \eta \to \eta_0} c(\omega, \eta, \alpha) = c(\omega_0, \eta_0, \alpha),$$

where the double limit with respect to the topology defined in (2.12) *is taken in arbitrary order.*

Proof. Apply (2.10). □

We finally note the observation, which is immediate from the product for $\theta(\omega, \eta, \cdot)$:

(2.14) The multiplier $c(\omega, \eta, \cdot) : \overline{\Gamma} \longrightarrow C^*$ has values in K_∞^* if both ω, η are in $\mathbb{P}^1(K)$.

3. Relationship with harmonic cochains.

Recall Marius van der Put's exact sequence ([24], [1])

$$0 \longrightarrow C^* \longrightarrow \mathcal{O}_\Omega(\Omega)^* \xrightarrow{r} \underline{H}(\mathcal{T}, \mathbb{Z}) \longrightarrow 0 \tag{3.1}$$

of right $\mathrm{GL}(2, K_\infty)$-modules, where the middle term is the group of invertible functions on Ω. As is explained in [10], the map r plays the role of logarithmic derivation. We briefly sketch the construction of r, and refer to *loc. cit.* for details and notations.

Let $f \in \mathcal{O}_\Omega(\Omega)^*$ and e be an oriented edge of $\mathcal{T}$ with origin v and terminus w. Then $|f|$ is constant on the rational subdomains $\lambda^{-1}(v)$ and $\lambda^{-1}(w)$ of Ω determined by v and w. Both of these are isomorphic with a projective line $\mathbb{P}^1(C)$ with $q_\infty + 1$ disjoint open balls deleted. The value of $r(f)$ on e is then

$$r(f)(e) = \log \frac{|f|_{\lambda^{-1}(w)}}{|f|_{\lambda^{-1}(v)}}, \tag{3.1.1}$$

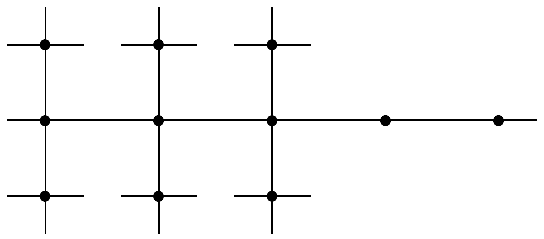
where here and in the sequel, $\log = \log_{q_\infty}$ is the logarithm to base q_∞.

Let Γ be any arithmetic subgroup of $\mathrm{GL}(2, K)$. We put $\Theta_h(\Gamma) \subset \Theta_c(\Gamma)$ for the groups of holomorphic and cuspidal theta functions for Γ as defined in (1.4) and (2.8), respectively. We have a commutative diagram

$$\begin{tikzcd} & \overline{\Gamma} \arrow[dl, "\overline{u}"'] \arrow[dr, "j"] & \\ \Theta_h(\Gamma)/C^* \arrow[rr, "\overline{r}_h"] \arrow[d, hook] & & \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \arrow[d, hook] \\ \Theta_c(\Gamma)/C^* \arrow[rr, "\overline{r}_c"] & & \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma, \end{tikzcd} \tag{3.2}$$

where $\overline{u}$ is derived from $\alpha \longmapsto u_\alpha$ and the horizontal maps from r. Recall that j is injective with finite prime-to-p cokernel ([10] 6.44; the proof given there applies to general arithmetic groups), and is bijective at least if $\tilde{\Gamma}$ has no prime-to-p torsion, or if K is a rational function field, ∞ the usual place at infinity, and Γ is a congruence subgroup of $\mathrm{GL}(2, A)$ [9].

(3.3) Next, we let $\mathfrak{b} \subset K$, $\tilde{\Gamma}_\infty$, $\tilde{\Gamma}_\infty^u$, $e_{\mathfrak{b}}$, t_∞ etc. be as in (2.5). The function $e_{\mathfrak{b}}$ is invertible on Ω and so $r(e_{\mathfrak{b}})$ is defined. The quotient graph $\tilde{\Gamma}_\infty^u \setminus \mathcal{T} = \mathfrak{b} \setminus \mathcal{T}$ has the following shape:



where the distinguished end points to ∞.

Since $r(e_{\mathfrak{b}}) \in \underline{H}(\mathcal{T}, \mathbb{Z})$ is invariant under $\tilde{\Gamma}_\infty^u$, it follows from the way how edges of $\mathcal{T}$ are identified mod $\mathfrak{b}$ (see e.g. proof of Proposition 3.5.1 in [10]) that for edges sufficiently close to ∞, the function $r(e_{\mathfrak{b}})$ grows by a factor q_∞ for each step towards ∞. In view of (3.1.1), this allows to describe the growth of $e_{\mathfrak{b}}(z)$ (or the decay of $t_\infty = e_{\mathfrak{b}}^{-1}(z)$) if $z \longrightarrow \infty$ in the topology introduced in (2.12). It is given by

$$c_1 q_\infty^{c_2|z|_i} \leq \log\ |e_{\mathfrak{b}}(z)| \leq c_1' q_\infty^{c_2|z|_i} \quad (|z|_i \gg 0) \tag{3.3.1}$$

for suitable constants $0 < c_1 < c_1'$, $c_2 > 0$ depending on $\mathfrak{b}$. (These constants can be made explicit if the need arises, see e.g. [7] for the case of $A = \mathbb{F}_q[T]$.) Note that multiplying z by the inverse π_∞^{-1} of a uniformizer π_∞ of K_∞ corresponds to shifting $\lambda(z)$ by one towards ∞, using again the terminology of [10].

Similarly, if $f \in \mathcal{O}_\Omega(\Omega)^*$ is invariant under $\tilde{\Gamma}_\infty^u$, its logarithmic derivative $r(f)$ may be considered as a function on edges of $\mathfrak{b} \setminus \mathcal{T}$, which implies that f must satisfy similar estimates

$$c_3 q_\infty^{c_4|z|_i} \leq \log\ |f(z)| \leq c_3' q_\infty^{c_4|z|_i}$$

for $|z|_i$ large. Hence, multiplying $f(z)$ by a suitable power t_∞^k of t_∞, the resulting $t_\infty^k f(z)$ will be bounded around $t_\infty = 0$, and $f(z)$ is meromorphic at ∞. The same reasoning applies to the other cusps. Thus:

(3.3.2) If $f \in \mathcal{O}_\Omega(\Omega)^*$ is invariant under the unipotent radical $\tilde{\Gamma}_s^u$ of $\tilde{\Gamma}_s$ then f is meromorphic at the cusp represented by $s \in \mathbb{P}^1(K)$.

3.4 Proposition. *The maps $\overline{r}_h$ and $\overline{r}_c$ in* (3.2) *are bijective.*

Proof. For $\overline{r}_h$, this is [10] 6.4.3. Injectivity of $\overline{r}_c$ follows directly from (3.1). Thus let $\varphi \in \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma$ equal $r(f)$ with $f \in \mathcal{O}_\Omega(\Omega)^*$. Then f satisfies $f(\gamma z) = c_f(\gamma) f(z)$ for $\gamma \in \Gamma$. The map $\gamma \longmapsto c_f(\gamma)$ is a homomorphism, which vanishes on p-groups of type $\tilde{\Gamma}_s^u$. By (3.3.2), f is meromorphic at the cusps, and is therefore a cuspidal theta function. $\square$

(3.5) We let $\Theta_c'(\Gamma) \subset \Theta_c(\Gamma)$ be the subgroup of cuspidal theta functions f whose multiplier $c_f : \tilde{\Gamma}^{ab} \longrightarrow C^*$ factors over $\overline{\Gamma} = \Gamma^{ab}/\mathrm{tor}(\Gamma^{ab}) = \tilde{\Gamma}^{ab}/\mathrm{tor}(\tilde{\Gamma}^{ab})$. Since the prime-to-$p$ torsion of $\tilde{\Gamma}^{ab}$ is always finite ([20] II, sect. 2, Ex. 2), the inclusion

$$\begin{array}{rcl} \Theta_c(\Gamma)/\Theta_c'(\Gamma) & \hookrightarrow & \mathrm{Hom}(\mathrm{tor}(\tilde{\Gamma}^{ab}), C^*) \\ f & \longmapsto & c_f \mid \mathrm{tor}(\tilde{\Gamma}^{ab}) \end{array} \tag{3.5.1}$$

shows that the index $[\Theta_c(\Gamma) : \Theta'_c(\Gamma)]$ is always finite and not divisible by p. Note that $\mathrm{Hom}(\mathrm{tor}(\tilde{\Gamma}^{ab}), C^*)$ is trivial if $\tilde{\Gamma}$ has no prime-to-p torsion, as follows e.g. from (1.2.7) (ii). Hence $\Theta_c(\Gamma) = \Theta'_c(\Gamma)$ in this case.

3.6 LEMMA. Let $j : \overline{\Gamma} \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ be the canonical inclusion. We have

$$j(\overline{\Gamma}) = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \cap r(\Theta'_c(\Gamma)).$$

Proof. The inclusion of $j(\overline{\Gamma})$ in $r(\Theta'_c(\Gamma))$ comes from (1.5) (ii), i.e., the fact that c_α factors through $\overline{\Gamma}$. The opposite inclusion is [10] Cor. 7.5.3. □

(3.7) We next interpret the quotient $r(\Theta'_c(\Gamma))/j(\overline{\Gamma})$ as the group of cuspidal divisors of degree zero on the curve $\overline{M}_\Gamma$. Recall that $cusp(\Gamma) = \Gamma \setminus \mathbb{P}^1(K)$ is the set of cusps, of order $c = c(\Gamma)$, and for each $[s] \in cusp(\Gamma)$, $w_s = [\tilde{\Gamma}_s : \tilde{\Gamma}_s^u]$ is its weight. We put

$$D_\infty := D_\infty(\Gamma) := \mathbb{Z}[cusp(\Gamma)]$$

for the group of cuspidal divisors on $\overline{M}_\Gamma$. At $[s]$, each $f \in \Theta_c(\Gamma)$ has an expansion w.r.t. t_s, and even w.r.t. $t_s^{w_s}$ if $f \in \Theta'_c(\Gamma)$. We let $\mathrm{ord}_{[s]}(f)$ be the order of f w.r.t. t_s (which clearly depends only on the class $[s]$ of s) and

$$\mathrm{div}(f) = \sum_{[s] \in cusp(\Gamma)} \frac{\mathrm{ord}_{[s]} f}{w_s} [s] \in D_\infty \otimes \mathbb{Q}. \tag{3.7.1}$$

3.8 THEOREM. *The map $f \longmapsto \mathrm{div}(f)$ induces an isomorphism*

$$\overline{\mathrm{div}} : r(\Theta'_c(\Gamma))/j(\overline{\Gamma}) \xrightarrow{\cong} D^0_\infty,$$

where $D^0_\infty \hookrightarrow D_\infty$ is the subgroup of divisors of zero degree.

Proof. For $f \in \Theta'_c(\Gamma)$, $\mathrm{div}(f)$ lies in D_∞, as follows from (2.5.2). Clearly, div restricted to $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ (or more precisely, to those f such that $r(f) \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$) is trivial, hence $\overline{\mathrm{div}}$ is well-defined. It is surjective by (2.6) and injective since, by (3.4) and (3.6), $r(\Theta'_c(\Gamma)/j(\overline{\Gamma})$ is free Abelian of rank $c(\Gamma) - 1$. □

3.9 COROLLARY. *$\Theta'_c(\Gamma)$ is the group generated by the constants C^* and the functions $\theta(\omega, \eta, \cdot)$ with $\omega, \eta \in \mathbb{P}^1(K)$.*

Proof. Obvious from (3.8), (3.6), (3.4), and (2.11). □

For what follows, we write Θ'_c for $\Theta'_c(\Gamma)$, and abbreviate $\underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma$ and $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ by $\underline{H}$ and $\underline{H}_!$, respectively. Let l be the least common multiple of the weights w_s, $[s] \in cusp(\Gamma)$.

3.10 COROLLARY. The index of $(\underline{H}_! + r(\Theta'_c))/\underline{H}_! \xrightarrow{\cong} r(\Theta'_c)/j(\overline{\Gamma}) \xrightarrow{\cong} D^0_\infty$ in $\underline{H}/\underline{H}_!$ is a divisor of $\displaystyle l^{-1} \prod_{[s] \in cusp(\Gamma)} w_s$, and the quotient group is annihilated by $q - 1$.

Proof. We may extend $\overline{\mathrm{div}}$ to a map from $\underline{H}/\underline{H}_!$ into the elements of degree zero of $\oplus_{[s]} w_s^{-1}\mathbb{Z}[s] \hookrightarrow D_\infty \otimes \mathbb{Q}$. The inverse image of D^0_∞ is precisely $(\underline{H}_! + r(\Theta'_c))/\underline{H}_!$, as follows from (3.8). The assertion now results from chasing in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^0_\infty & \longrightarrow & D_\infty & \stackrel{\deg}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\oplus w_s^{-1}\mathbb{Z}[s])^0 & \longrightarrow & \oplus w_s^{-1}\mathbb{Z}[s] & \longrightarrow & l^{-1}\mathbb{Z} & \longrightarrow & 0 \end{array} \tag{3.10.1}$$

and noting that the w_s are divisors of $q-1$. $\square$

(3.11) Since $\underline{H}_!$ is a space of functions with finite support on the edges of the graph $\Gamma \setminus \mathcal{T}$, it is provided with a natural bilinear form

$$(.\,,.) : \underline{H}_! \times \underline{H}_! \longrightarrow \mathbb{Q}.$$

If $\tilde{\Gamma}_e$ is the stabilizer of $e \in Y(\mathcal{T})$, the volume of the corresponding edge of $\Gamma \setminus \mathcal{T}$ is $\frac{1}{2}\sharp(\tilde{\Gamma}_e)^{-1}$. Two remarks are in order.

(3.11.1) $(.\,,.)$ as defined above is the restriction of the Petersson scalar product on $\underline{H}_!(\mathcal{T},\mathbb{C})^{\Gamma}$, which is a space of automorphic forms. In fact, the restriction of $(.\,,.)$ to $\overline{\Gamma} \xrightarrow{\cong} j(\overline{\Gamma}) \hookrightarrow \underline{H}_!$ agrees with the pairing $(.\,,.)$ in (1.5) (vi) ([10] 5.7.1), and in particular, takes its values in $\mathbb{Z}$.

(3.11.2) There exists a natural extension of $(.\,,.)$ to a pairing labeled by the same symbol

$$(.\,,.) : \underline{H}_! \times \underline{H} \longrightarrow \mathbb{Q}.$$

It is characterized through its restriction to $j(\overline{\Gamma}) \times r(\Theta_c')$, where it satisfies

$$(r(u_\alpha), r(f)) = -v_\infty(c_f(\alpha)), \tag{3.11.3}$$

compare (3.2) and (1.5) (vi). Finally, we put

$$\underline{H}_!^{\perp} := \{\varphi \in \underline{H}(\mathcal{T},\mathbb{Z})^{\Gamma} \mid (\underline{H}_!, \varphi) = 0\}. \tag{3.11.4}$$

Then $\underline{H}_!^{\perp}$ is a direct factor of $\underline{H}$ and "almost complementary" to $\underline{H}_!$, i.e., $\underline{H}/\underline{H}_! \oplus \underline{H}_!^{\perp}$ is finite. We will see at once that this group is closely related to the cuspidal divisor class group of $\overline{M}_\Gamma$.

4. The cuspidal divisor class group.

From now on, we assume that Γ is a congruence subgroup of some $\mathrm{GL}(Y)$. The next result follows from determining the divisors of certain modular units (analogues of classical Weber or Fricke functions) and expressing them through partial zeta functions. This has been carried out in detail in the special cases where

a) the base ring A is a polynomial ring $\mathbb{F}_q[T]$ and $\Gamma \subset \mathrm{GL}(2,A)$ is an arbitrary congruence subgroup [2], or

b) the base ring A is subject only to the conditions given in (1.1), but $\Gamma = \mathrm{GL}(Y)$ is the full linear group of a rank-two A-lattice Y [5].

The proof of the general case (A and Γ without further restrictions) will follow e.g. by combining the methods of [2] and [5]. The necessary ingredients are sketched in [5] VI.5.13, but still some work has to be done to complete the argument. A rather short proof which avoids the difficult calculations of *loc. cit.* will be given in [8].

4.1 Theorem. *Let Γ be a congruence subgroup of $GL(2,K)$. The cuspidal divisors of degree zero on $\overline{M}_\Gamma$ generate a finite subgroup $\mathcal{C}(\Gamma)$ of the Jacobian J_Γ of $\overline{M}_\Gamma$.*

The corresponding result for classical modular curves has been proven by Manin and Drinfeld [14]; a different proof has been given by Kubert and Lang [13]. Our aim is now to give a more accurate description of $\mathcal{C} = \mathcal{C}(\Gamma)$.

4.2 PROPOSITION. *Let f be a modular unit, i.e., a meromorphic function on $\overline{M}_\Gamma$ with its divisor supported by the cusps. Then $r(f) \in \underline{H}_!^\perp = \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma)^\perp$. Conversely, if $f \in \Theta'_c(\Gamma)$ is such that $r(f) \in \underline{H}_!^\perp$ then $f^{q_\infty - 1}$ is a modular unit.*

Proof. Since f is invertible on Ω, $r(f)$ is defined, and $r(f) \in \underline{H}_!^\perp$ follows from (3.11.3). Let $f \in \Theta'_c$ be such that $r(f) \in \underline{H}_!^\perp$, and let $\chi = c_f$ be its multiplier. By (4.1) there exists $n \in \mathbb{N}$ and a modular unit g such that f^n/g is holomorphic on Ω and at the cusps. From [10] 7.5.3 $f^n/g = \text{const.}\, u_\alpha$ for some $\alpha \in \Gamma$, hence $\chi^n = c_\alpha$. Since $r(f) \perp j(\overline{\Gamma})$, we have $|c_\alpha(\beta)| = 1$ for all $\beta \in \Gamma$, which gives $c_\alpha = 1$. Therefore, χ has finite order, which by (3.9) and (2.14) is a divisor of $q_\infty - 1$. $\square$

(4.3) We let P_∞ be the divisors of modular units, i.e., the principal divisors on $\overline{M}_\Gamma$ supported by the cusps. The map $\mathrm{div}(f) \longmapsto r(f)$ identifies P_∞ with a subgroup of $\underline{H}_!^\perp \hookrightarrow \underline{H}$, which by abuse of language will be labeled by the same symbol P_∞. By the above,

$$(q_\infty - 1)(\underline{H}_!^\perp \cap r(\Theta'_c)) \subset P_\infty \subset \underline{H}_!^\perp \cap r(\Theta'_c), \tag{4.3.1}$$

and the group $\mathcal{C}$ of cuspidal divisor classes is

$$\mathcal{C} = D^0_\infty / P_\infty \xrightarrow{\cong} r(\Theta'_c)/(j(\overline{\Gamma}) \oplus P_\infty). \tag{4.3.2}$$

We therefore have an exact sequence

$$0 \longrightarrow U \longrightarrow \mathcal{C} \longrightarrow V \longrightarrow 0, \tag{4.3.3}$$

where $U = \underline{H}_!^\perp \cap r(\Theta'_c)/P_\infty$ is isomorphic with a quotient of $(\mathbb{Z}/(q_\infty - 1)\mathbb{Z})^{c(\Gamma)-1}$ and $V = r(\Theta'_c)/(j(\overline{\Gamma}) \oplus \underline{H}_!^\perp \cap r(\Theta'_c)) \hookrightarrow \underline{H}/\underline{H} \oplus \underline{H}_!^\perp$. The following diagram displays the inclusions.

(4.4)

$$\begin{tikzcd}[column sep=small]
 & & \underline{H} = r(\Theta_c) \arrow[ddll, dash] \arrow[d, dash] \arrow[ddrr, dash] & & & \hookrightarrow \mathrm{Hom}(\mathrm{tor}(\widetilde{\Gamma}^{ab}), C^*) \\
 & & r(\Theta'_c) \arrow[ddl, dash] \arrow[ddrr, dash] & & & \\
\underline{H}_!^\perp \arrow[d, dash] & & & & \underline{H}_! \arrow[d, dash] & D^0_\infty \\
(q_\infty - 1)(\underline{H}_!^\perp \cap r(\Theta'_c)) \subset P_\infty \subset \underline{H}_!^\perp \cap r(\Theta'_c) & & & & \underline{H}_! \cap r(\Theta'_c) = j(\overline{\Gamma}) &
\end{tikzcd}$$

4.5 REMARKS. (i) As follows from (1.2.7), the vertical inclusions are bijective if Γ has no non-p torsion, in which case $V = \underline{H}/\underline{H}_! \oplus \underline{H}_!^\perp$.
(ii) In the general case, both U and the cokernel of V in $\underline{H}/\underline{H}_! \oplus \underline{H}_!^\perp$ have prime-to-p order. Hence the p-parts of $\mathcal{C}$ and of $\underline{H}/\underline{H}_! \oplus \underline{H}_!^\perp$ always agree.
(iii) We know of no single example of a congruence group Γ such that $j(\overline{\Gamma}) \neq \underline{H}_! = \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$. The two groups agree at least if $A = \mathbb{F}_q[T]$ (see [9]). However, there are examples, given in the next section, where $r(\Theta'_c)$ and even $r(\Theta'_c) + \underline{H}_!$ differs from $\underline{H}$.

The description for the Jacobian J_Γ of $\overline{M}_\Gamma$ given in (1.6) is valid over each complete subextension of C/K_∞, in particular, over K_∞ itself. We let $\phi_\infty(\Gamma)$ be the group of connected components of the Néron model $\mathcal{J}_\Gamma$ of J_Γ/K_∞.

4.6 Theorem. *$\phi_\infty(\Gamma)$ is canonically isomorphic with $\mathrm{Hom}(\overline{\Gamma},\mathbb{Z})/i(\overline{\Gamma})$, where $i:\overline{\Gamma}\hookrightarrow \mathrm{Hom}(\overline{\Gamma},\mathbb{Z})$ comes from the pairing $(.\,,.)$ on $\overline{\Gamma}$.*

Proof. Easy consequence of the construction of J_Γ ([10] sect. 7) and Mumford's results [17] on degenerating Abelian varieties. Details are given in [6] Cor. 2.11. The assumption of $A=\mathbb{F}_q[T]$ made in that paper is not used in an essential fashion. $\square$

There is a canonical map can_∞ from $\mathcal{C}=\mathcal{C}(\Gamma)$ to $\phi_\infty(\Gamma)$, which to each divisor class $[D]$ associates the component of the reduction of $[D]$ at infinity. Combining what we know about these groups ((4.3), (4.4), (4.6)) yields the following description of can_∞.

4.7 Corollary. The map $\mathrm{can}_\infty:\mathcal{C}(\Gamma)\longrightarrow\phi_\infty(\Gamma)$ is given by

$$\begin{array}{rcl}\mathcal{C}(\Gamma)\xrightarrow{\cong} r(\Theta_c')/(j(\overline{\Gamma})\oplus P_\infty) & \longrightarrow & \mathrm{Hom}(\overline{\Gamma},\mathbb{Z})/i(\overline{\Gamma})\xrightarrow{\cong}\phi_\infty(\Gamma)\\ \text{class of } r(f) & \longmapsto & \text{class of } (-v_\infty\circ c_f).\end{array}$$

Here $c_f:\overline{\Gamma}\longrightarrow K_\infty^*$ is the multiplier of f and $v_\infty:K_\infty^*\longrightarrow\mathbb{Z}$ the valuation.

Obviously, the kernel of can_∞ is $j(\overline{\Gamma})\oplus(\underline{H}_!^\perp\cap r(\Theta_c'))/j(\overline{\Gamma})\oplus P_\infty$, i.e., the group U of (4.3.3). As we will see, can_∞ need neither be injective nor surjective.

We finally recall the fact that each congruence subgroup Γ' contains a congruence subgroup Γ without prime-to-p torsion. For such Γ, (4.5) (i) applies, and (4.7) becomes

$$(4.8)\qquad \mathcal{C}(\Gamma)\xrightarrow{\cong}\underline{H}/\underline{H}_!\oplus P_\infty\xrightarrow{\text{proj.}}\underline{H}/\underline{H}_!\oplus\underline{H}_!^\perp\hookrightarrow\mathrm{Hom}(\underline{H}_!,\mathbb{Z})/i(\underline{H}_!)\xrightarrow{\cong}\phi_\infty(\Gamma).$$

Hence in this case, $\phi_\infty(\Gamma)$ as well as the image $\phi_\infty^{cusp}(\Gamma):=\mathrm{can}_\infty(\mathcal{C}(\Gamma))$ of the cuspidal divisor classes may be described entirely in terms of the almost finite graph $\Gamma\setminus\mathcal{T}$. Note that assertions similar to (4.6) - (4.8) are valid also in the case of a general arithmetic group Γ (i.e., without the assumption of being a congruence subgroup), except for the finiteness of $\mathcal{C}(\Gamma)$. By analogy with the number field case [18], that latter is unlikely to hold.

5. The case of Hecke congruence subgroups over a polynomial ring.

We now assume that A equals the polynomial ring $\mathbb{F}_q[T]$ and Γ is the Hecke congruence subgroup $\Gamma_0(n)=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}(2,A)\mid c\equiv 0\mod n\}$ for a certain $n\in A$. A lot of material about these groups, including structural properties of $\Gamma\setminus\mathcal{T}$, formulae for $g(\Gamma)$, $c(\Gamma)$ etc., may be found in [9]. Note in particular that (*loc. cit.*, Thm. 3.3)

$$(5.1)\qquad H_1(\Gamma\setminus\mathcal{T},\mathbb{Z})\cong\overline{\Gamma}\xrightarrow[j]{\cong}\underline{H}_!=\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma.$$

(5.2) We start with a few examples that illustrate how $\mathrm{can}_\infty:\mathcal{C}(\Gamma)\longrightarrow\phi_\infty(\Gamma)$ may be calculated. Let $q=2$. Apart from the general advantage that $g(\Gamma)$ and $c(\Gamma)$ are then small, $q=2$ forces that

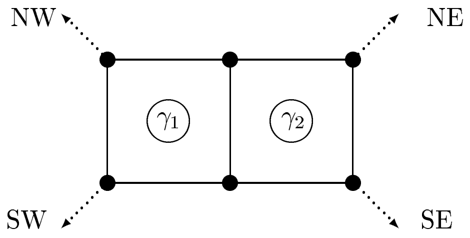
(5.2.1) the group U of (4.3.3) is trivial, hence

(5.2.2) $\mathrm{can}_\infty:\mathcal{C}(\Gamma)$ is injective, and

(5.2.3) $\mathcal{C}(\Gamma)=\underline{H}/\underline{H}_!\oplus\underline{H}_!^\perp$, due to (3.10).

5.3 Examples.

(5.3.1) $\Gamma=\Gamma_0(n)$, $n=T(T^2+T+1)\in\mathbb{F}_2[T]$. The graph $\Gamma\setminus\mathcal{T}$ looks:



Here $\cdots >$ indicates a cusp. Let γ_1, γ_2 be the two cycles of length 4, oriented counter-clockwise, and $\varphi_1, \varphi_2, \varphi_3$ the $\mathbb{Z}$-valued harmonic cochains flowing from the SW, the SE, the NE cusp, respectively, to the NW cusp, going the way round counter-clockwise. Then $\{\gamma_1, \gamma_2\}$ and $\{\gamma_1, \gamma_2, \varphi_1, \varphi_2, \varphi_3\}$ are $\mathbb{Z}$-bases of $\underline{H}_!$ and $\underline{H}$, respectively. With respect to these bases, the pairing $(\,.\,,.\,) : \underline{H}_! \times \underline{H} \longrightarrow \mathbb{Z}$ is given by

	γ_1	γ_2	φ_1	φ_2	φ_3
γ_1	4	-1	2	1	1
γ_2	-1	4	3	2	1

.

We get $\sharp\phi_\infty(\Gamma) = \left|\det\begin{pmatrix} 4, & -1 \\ -1, & 4 \end{pmatrix}\right| = 15$, and after an elementary computation, $\sharp\mathcal{C}(\Gamma) = [\underline{H} : \underline{H}_! \oplus \underline{H}_!^{\perp}] = 15$, too. Hence can_∞ is bijective.

N.B. J_Γ splits into two elliptic curves with 3 resp. 5 rational points over $K = \mathbb{F}_2(T)$, which are therefore all "cuspidal" ([6] 4.4).

(5.3.2) Drawings of the graphs $\Gamma \setminus \mathcal{T}$ $(\Gamma = \Gamma_0(n))$ for the next examples may be found in [19]. For these, the matrix of $(\,.\,,.\,) : \underline{H}_! \times \underline{H} \longrightarrow \mathbb{Z}$ and thus $\mathcal{C}$ and ϕ_∞ may be calculated as above. We restrict to giving the results. In all cases, can_∞ is bijective (which, however, is not typical: see (5.3.3)!).

$n \in \mathbb{F}_2[T]$	$g(\Gamma)$	$c(\Gamma)$	$\mathcal{C}(\Gamma)$
$T^2(T+1)$	1	6	$\mathbb{Z}/6\mathbb{Z}$
T^3	1	4	$\mathbb{Z}/4\mathbb{Z}$
T^3+T+1	2	2	$\mathbb{Z}/7\mathbb{Z}$
$(T^2+T+1)^2$	2	5	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$
T^4	3	6	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$

(5.3.3) $\Gamma = \Gamma_0(n)$, where (i) $n = T^4 + T^3 + 1$ or (ii) $n = T^4 + T + 1$, which both are irreducible over $\mathbb{F}_2$. In both cases, $g(\Gamma) = 4$, $c(\Gamma) = 2$, $\sharp\mathcal{C}(\Gamma) = 5$ (see also (5.6)). However, $\phi_\infty(\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/80\mathbb{Z}$ for (i) and $\phi_\infty(\Gamma) \cong \mathbb{Z}/45\mathbb{Z}$ for (ii). Hence can_∞ is *not* surjective in these cases.

We let now again $\mathbb{F}_q$ be an arbitrary finite field, n a monic polynomial of degree d in $A = \mathbb{F}_q[T]$, and $\Gamma = \Gamma_0(n)$. We give an intrinsic description of the group $\Theta_c'(\Gamma)$ of (3.5).

5.4 Theorem. *Let n have h different monic prime divisors in A. Then $\Theta_c'(\Gamma)$ has index $(q-1)^{2^{h-1}}$ in $\Theta_c(\Gamma)$.*

Proof. Without restriction, we may assume $q > 2$.

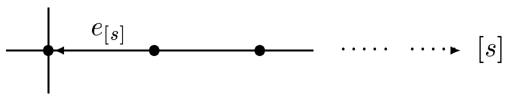
(i) By (5.1) and (3.6), $\underline{H}_! \subset r(\Theta_c')$, hence $\Theta_c/\Theta_c' \xrightarrow{\cong} \underline{H}/r(\Theta_c')$. Consider the commutative diagram with exact rows:

$$
\text{(5.4.1)} \qquad
\begin{tikzcd}
0 \arrow[r] & r(\Theta'_c)/\underline{H}_! \arrow[r] \arrow[d, "\wr|"'] & \underline{H}/\underline{H}_! \arrow[r] \arrow[d, hook, "\overline{\mathrm{div}}"] & \underline{H}/r(\Theta'_c) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathrm{Div}^0_\infty \arrow[r] & \Big(\bigoplus\limits_{[s]\in cusp(\Gamma)} w_s^{-1}\mathbb{Z}[s]\Big)^0 \arrow[r] & \Big(\bigoplus w_s^{-1}\mathbb{Z}[s]\,/\mathbb{Z}[s]\Big)^0 \arrow[r] & 0
\end{tikzcd}
$$

The right hand arrow α is injective; it suffices therefore to calculate its image.
(ii) From [9] 2.15 we know that Γ has precisely 2^h cusps $[s]$ with $w_s = q-1$ (the *regular* cusps), and for the other (*irregular*) cusps, $w_s = 1$. Hence the lower right group in (5.4.1) equals $(\mathbb{Z}/(q-1)\mathbb{Z})^{reg(\Gamma),0}$, the subgroup of elements of degree zero in $(\mathbb{Z}/(q-1)\mathbb{Z})^{reg(\Gamma)}$, where $reg(\Gamma)$ is the set of regular cusps. Using this identification,

$$\alpha:\ r(\Theta_c)/r(\Theta'_c) = \underline{H}/r(\Theta'_c) \hookrightarrow (\mathbb{Z}/(q-1)\mathbb{Z})^{reg(\Gamma),0}$$

associates with each $r(f) \in \underline{H}$ the 2^h-tuple $(\ldots, \mathrm{ord}_{[s]}f \mod q-1, \ldots)$.
(iii) We have to introduce some more notation. Suppose from now on that $d := \deg\ n \geq 2$. (The case $d = 1$, which leads to $g(\Gamma) = 0$, $c(\Gamma) = 2$, $\Gamma \setminus \mathcal{T}$ isomorphic with a straight line $\cdots - - - \bullet - - - \bullet - - - \bullet \cdots$, is easily dealt with directly. The result follows in this case also from (5.7).)
Then to each cusp $[s]$ there corresponds a maximal half-line $hl[s]$ of $\Gamma \setminus \mathcal{T}$. We let $e_{[s]}$ be the first edge of $hl[s]$, oriented away from $[s]$, and call it the base edge of $[s]$.



5.4.3 Claim. For each $f \in \Theta_c$, we have $\mathrm{ord}_{[s]}f = r(f)(e_{[s]})$.

For the proof of this fact, it suffices to verify $r(t_s)(e_{[s]}) = 1$, where t_s is the corresponding uniformizer, cf. (2.5). As usual, possibly replacing Γ by a conjugate, we may assume $s = \infty$, in which case the assertion is a consequence of

- Proposition 1.14 of [7],
- the way how vertices and edges of $\mathcal{T}$ are identified under Γ_∞,

and the trivial but crucial fact:

- each fractional ideal $\mathfrak{b}$ of K has a direct complement of the form (π_∞^r) in K_∞. Here π_∞ is a uniformizer at ∞, e.g. $\pi_\infty = T^{-1}$.

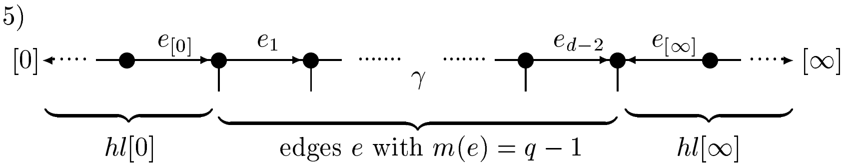
(iv) Let $\varphi \in \underline{H} = \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma$. The harmonicity condition (1.3.1) for φ as a function on $\Gamma \setminus \mathcal{T}$ reads

$$\sum_{\substack{e\in Y(\Gamma\setminus\mathcal{T})\\ o(e)=v}} m(e)\varphi(e) = 0 \tag{5.4.4}$$

for each vertex v of $\Gamma \setminus \mathcal{T}$, where the multiplicity $m(e)$ $(1 \leq m(e) \leq q+1)$ takes care of the identification of edges of $\mathcal{T}$ modulo Γ. Clearly, $\sum\limits_{o(e)=v} m(e) = q+1$.

(v) The next statements result from the description of $\Gamma \setminus \mathcal{T}$ given in [9]. As usual, $[0]$ and $[\infty]$ denote the cusps represented by $(0:1)$ and $(1:0)$, respectively. Their corresponding half-lines $hl[0]$ and $hl[\infty]$ in $\Gamma \setminus \mathcal{T}$ are connected by a path γ consisting of a sequence of $d-2$ edges $e_1, \ldots, e_{d-2}$ of valence 3. The edges $e = \overline{e}_{[0]}, e_1, \overline{e}_1, \ldots, e_{d-2}, \overline{e}_{d-2}, \overline{e}_{[\infty]}$ enter with multiplicity $m(e) = 1$ into (5.4.4), whereas the $d-1$ edges connecting $hl[0] \cup \gamma \cup hl[\infty]$ with the rest of $\Gamma \setminus \mathcal{T}$ have multiplicity $q-1$, always with respect to vertices on γ. This is the picture:

(5.4.5)



(vi) By the above, for any $\varphi \in \underline{H}$ we have

$$\varphi(e_{[0]}) \equiv \varphi(e_1) \equiv \cdots \equiv \varphi(e_{d-2}) \equiv -\varphi(e_{[\infty]}) \mod q-1.$$

The group W of Atkin-Lehner-involutions (which acts on $\overline{M}_\Gamma$ as well as on $\Gamma \setminus \mathcal{T}$) acts transitively on $reg(\Gamma)$, and some pair $([s],[s'])$ of regular cusps lies in the W-orbit of $([0],[\infty])$ if and only if $[s'] = w[s]$, where $w = w_n$ is the total involution induced from the matrix $\left(\begin{smallmatrix} 0 & 1 \\ n & 0 \end{smallmatrix}\right) \in \mathrm{GL}(2,K)$. Hence for any $\varphi \in \underline{H} \in reg(\Gamma)$,

$$\varphi(e_{[s]}) \equiv -\varphi(e_{w[s]}) \mod q-1 \tag{5.4.6}$$

holds. On the other hand, it is obvious from (5.4.5) that for each pair $([s], w[s])$ of w-conjugate regular cusps there exists a harmonic cochain $\varphi \in \underline{H}$ such that $\varphi(e_{[s]}) = 1$, $\varphi(e_{w[s]}) = -1$. Hence the image of α in $(\mathbb{Z}/(q-1)\mathbb{Z})^{reg(\Gamma),0}$ (see (5.4.2)) agrees with the free $\mathbb{Z}/(q-1)\mathbb{Z}$-submodule of rank $\frac{1}{2}\sharp\ reg(\Gamma) = 2^{h-1}$ defined by the congruence condition (5.4.6), which finally yields the result. $\square$

5.5 COROLLARY. *With notations as in* (5.4)*, the cokernel* $\phi_\infty/\phi_\infty^{cusp}$ *of* $\mathrm{can}_\infty : \mathcal{C}(\Gamma) \longrightarrow \phi_\infty(\Gamma)$ *has order a multiple of* $(q-1)^{2^{h-1}} : [\underline{H}_!^\perp : \underline{H}_!^\perp \cap r(\Theta_c')]$.

Proof. With identifications as in (4.7), $\phi_\infty^{cusp} = r(\Theta_c')/\underline{H}_! \oplus \underline{H}_!^\perp \cap r(\Theta_c') \hookrightarrow \underline{H}/\underline{H}_! \oplus \underline{H}_!^\perp \hookrightarrow \phi_\infty$. The stated value is the index of ϕ_∞^{cusp} in $\underline{H}/\underline{H}_! \oplus \underline{H}_!^\perp$. $\square$

For the remainder of this section, we suppose in addition that n is *prime*. The cuspidal divisor class group $\mathcal{C} = \mathcal{C}(\Gamma)$ of $\Gamma = \Gamma_0(n)$ has been determined in [3] and, with different methods, in [7]. The result is

5.6 THEOREM. *In the above situation,* $\mathcal{C}$ *is cyclic of order* $\frac{q^d-1}{q^2-1}$ *if* $d = \deg\ n$ *is even and* $\frac{q^d-1}{q-1}$ *if* d *is odd.*

Here $c(\Gamma) = 2$ with the two cusps $[0]$ and $[\infty]$. A meromorphic function f on $\overline{M}_\Gamma$ with divisor $\sharp(\mathcal{C})([0]-[\infty])$ may be constructed as follows. Let $\Delta : \Omega \longrightarrow C$ be the Drinfeld discriminant (see e.g. [7]) and $\Delta_n(z) = \Delta(nz)$. Then Δ/Δ_n is a modular function (i.e., invariant) for Γ and $\mathrm{div}(\Delta/\Delta_n) = (q^d-1)([0]-[\infty])$ (*loc. cit.* (3.11)). Let now

$$\begin{array}{rcll} r & := & (q^2-1)(q-1) & \text{for even } d \\ & = & (q-1)^2 & \text{for odd } d. \end{array}$$

Using the machinery of Drinfeld modular forms, it is further shown in [7] 3.18:

5.7 Theorem. *Δ/Δ_n admits an r-th root in $\mathcal{O}_\Omega(\Omega)^*$, and r is maximal with this property.*

(5.8) Let D_n be such an r-th root. It transforms under Γ through a certain character $\omega_n : \Gamma \longrightarrow \mathbb{F}_q^* \hookrightarrow C^*$ of precise order $q-1$ (*loc. cit.* 3.21, 3.22). Therefore, D_n^{q-1} (but no smaller power of D_n) is Γ-invariant, and it has the asserted divisor $\sharp(\mathcal{C})([0]-[\infty])$ on $\overline{M}_\Gamma$. Put finally

$$t := \gcd(q-1, \sharp(\mathcal{C})). \tag{5.8.1}$$

Then yet

$$\operatorname{div}(D_n^{(q-1)/t} = \frac{\sharp(\mathcal{C})}{t}([0]-[\infty])$$

is an integral divisor, whose class generates the subgroup U_t of order t in $\mathcal{C}$. A look at (4.7) shows that U_t is contained in the kernel of can_∞, with which it must agree in view of (5.7).

5.9 Theorem. *Let n be an irreducible monic polynomial of degree d in $A = \mathbb{F}_q[T]$, let $\Gamma = \Gamma_0(n)$ be the Hecke congruence subgroup, and t as given in* (5.8.1).

(i) *There is an exact sequence $0 \longrightarrow U_t \longrightarrow \mathcal{C} \xrightarrow{\mathrm{can}_\infty} \phi_\infty$, where U_t is the unique subgroup of order t in $\mathcal{C} = \mathcal{C}(\Gamma)$.*

(ii) *The cokernel $\phi_\infty/\phi_\infty^{cusp}$ of can_∞ has order a multiple of t.*

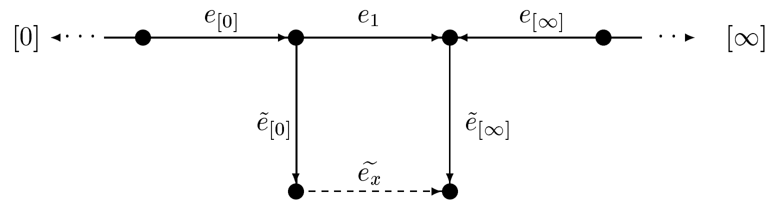
Proof. (i) has been shown. (ii) comes from (5.5), noting that $[\underline{H}_!^\perp : \underline{H}_!^\perp \cap r(\Theta_c')] = (q-1)/t$. $\square$

Pairs (q,d) where $t > 1$ are for example (4,3), (7,3), (13,3) with $t = 3$ and (3,4), (5,4) with $t = 2$. In the final section, we work out an example with $(q,d) = (7,3)$.

6. An example.

We consider in detail the case where n is a prime of degree 3 in $A = \mathbb{F}_q[T]$. The graph $\Gamma \setminus \mathcal{T}$ looks ([4] 5.3, $\Gamma := \Gamma_0(n)$):

(6.1)



Here - - - - - -► stands for q edges $\tilde{e}_x$ indexed by $x \in \mathbb{F}_q$. The multiplicities $m(e)$ (see (5.4.4)) of all drawn edges and their inverses are 1 except for $\tilde{e}_{[0]}$ and $\tilde{e}_{[\infty]}$, which enter with multiplicity $q-1$ into the harmonicity condition w.r.t. their origins. Hence e.g.

$$(q-1)\varphi(\tilde{e}_{[\infty]}) - \varphi(e_1) - \varphi(e_{[\infty]}) = 0$$

for $\varphi \in \underline{H}$. The scalar product on $\underline{H}_! = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is such that each pair $\{e, \overline{e}\}$ of inversely oriented edges contributes volume 1 except for $\{e_1, \overline{e}_1\}$, which has volume $q-1$. For each $x \in \mathbb{F}_q$, let φ_x be the unique element of $\underline{H}_!$ with

$$\varphi_x(\tilde{e}_{[\infty]}) = -1,\ \varphi_x(\tilde{e}_y) = \delta_{x,y} \quad (y \in \mathbb{F}_q).$$

Let further $\psi \in \underline{H}$ be such that

$$\psi(e_{[0]}) = 1 = \psi(e_1) = -\psi(e_{[\infty]})$$

and ψ vanishes off the line from [0] to $[\infty]$. Next, let $\delta \in \underline{H}$ be defined as

$$\delta = \sum_{x \in \mathbb{F}_q} \varphi_x + (q^2+q+1)\psi.$$

Then, as is easily verified:

(6.2) (i) $\{\varphi_x \mid x \in \mathbb{F}_q\}$ is a basis of $\underline{H}_!$.

(ii) $\{\varphi_x \mid x \in \mathbb{F}_q\} \cup \{\psi\}$ is a basis of $\underline{H}$.

(iii) $\underline{H}_!^\perp = \mathbb{Z}\delta$

(iv) $r(\Theta_c') = \underline{H}_! + (q-1)\mathbb{Z}\psi$ (use (5.4)!)

(v) $\underline{H}_!^\perp \cap r(\Theta_c') = \frac{q-1}{t}\mathbb{Z}\delta$ ($t := \gcd(q-1, q^2+q+1)$)

(vi) $P_\infty = (q-1)\mathbb{Z}\delta$ (see (4.3)).

Furthermore,

(6.3) (i) $\mathcal{C} = r(\Theta_c')/\underline{H}_! \oplus P_\infty \xrightarrow{\cong} \mathbb{Z}/(q^2+q+1)\mathbb{Z}$
$\varphi \longmapsto \varphi(e_{[0]})$

(in accordance with (5.6)) and

(ii) $\sharp(\phi_\infty) = q^2+q+1 = \sharp(\mathcal{C})$ (from calculating the determinant of $(.,.) : \underline{H}_! \times \underline{H}_! \longrightarrow \mathbb{Z}$), but $\mathrm{can}_\infty : \mathcal{C} \longrightarrow \phi_\infty$ has kernel and cokernel each of size t. (It is easy to show that in this case, ϕ_∞ is cyclic, too.)

(6.4) As is explained in [10], the splitting of the Jacobian $J := J_0(n)$ of $\overline{M}_\Gamma$ corresponds to the splitting of $\underline{H}_! \otimes \mathbb{Q}$ under the Hecke algebra, which can be calculated by the formulae in [4], or by the approach via modular symbols proposed in [23]. Let now, more specifically

(6.4.1) $q = 7$ and $n = T^3 - 2 \in \mathbb{F}_7[T]$, which gives $\sharp(\mathcal{C}) = 57$ and $t = \gcd(6, 57) = 3$. In that case, $\underline{H}_! \otimes \mathbb{Q}$ splits under the Hecke algebra into an irreducible piece of dimension 6 and the eigenspace generated by (see [4], table 10.3)

(6.4.2) $$\varphi = \sum_{x \in \mathbb{F}_7} a_x \varphi_x \text{ with } (a_0, \ldots, a_6) = (4, 1, 1, -2, 1, -2, -2).$$

This means, there exists an elliptic curve E/K, uniquely determined up to isogeny, with good reduction outside of the two places ∞, (n) of $K = \mathbb{F}_7(T)$, multiplicative reduction at (n) and split multiplicative reduction at ∞, which has a "Weil uniformization" $\pi : \overline{M}_\Gamma \longrightarrow E$, and whose reduction at $(T-x)$ has $8 + a_x$ rational points over $A/(T-x) = \mathbb{F}_7$. We have

(6.4.3) $(\varphi, \varphi) = 39$, $m := \min\{(\varphi, \alpha) > 0 \mid \alpha \in \underline{H}_!\} = 3$, hence ([6] 3.19, 3.20) $\deg \pi = 39/3 = 13$ and $v_\infty(j_E) = -3$ for the j-invariant j_E of E, π supposed to be a "strong Weil uniformization". Comparing with [4] table 9.3, case 3a and performing the unramified quadratic twist to get split multiplicative reduction at ∞ yields the following equation for E:

(6.4.4) $$Y^2 = X^3 + aX + b$$

with $a = -3T(T^3+2)$, $b = -2T^6 + 3T^3 + 1$. It can be shown by routine methods that (6.4.4) in fact yields the strong Weil curve in the given isogeny class, and that

$$E(K) = \{0, (3T^2, \pm 4(T^3-2))\} \cong \mathbb{Z}/3\mathbb{Z}. \tag{6.4.5}$$

(We should note here that the equation given in [23] p. 289, dealing with the same example, does not describe the isogeny factor E of J but its unramified quadratic twist. Hence some conclusions derived there must be slightly modified.)

Similar to (4.7), there is a map $\mathrm{can}_{\infty,E} : \mathcal{C}_E \longrightarrow \phi_{\infty,E}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \stackrel{\mathrm{can}_\infty}{\longrightarrow} & \phi_\infty \\ \downarrow & & \downarrow \\ \mathcal{C}_E & \stackrel{\mathrm{can}_{\infty,E}}{\longrightarrow} & \phi_{\infty,E}, \end{array} \tag{6.4.6}$$

where $\mathcal{C}_E$ is the image of the map $\mathcal{C} \longrightarrow E(K)$ derived from π and $\phi_{\infty,E}$ the group of connected components of E at ∞, isomorphic with $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/3\mathbb{Z}$. Further, as results from the calculation of Hecke operators, $\mathcal{C} \longrightarrow E(K)$ is non-trivial, hence $\mathcal{C} \longrightarrow \mathcal{C}_E = E(K) \cong \mathbb{Z}/3\mathbb{Z}$, and E is the quotient of J corresponding to the *Eisenstein prime number* $l = 3$ ([15], [22]). Since, by (5.9), can_∞ kills the subgroup of order $t = 3$ in $\mathcal{C}$, (6.4.6) forces $\mathrm{can}_{\infty,E}$ to be trivial. In other words:

(6.4.7) The rational 3-division points (6.4.5) of E map to the connected component of the Néron model at ∞.

Of course, this is easy to see directly. An equivalent form of stating this fact is as follows: Let $f \in \Theta_c(\Gamma)$ be such that $r(f) = \delta$, and regard $\varphi \in \underline{H}_! \stackrel{\cong}{\longleftarrow} \overline{\Gamma}$ as the class of some element of Γ. Then f^6 is a modular unit and, up to scaling, a 6-th root of Δ/Δ_n. Its third root f^2 belongs to $\Theta_c'(\Gamma)$ and transforms under Γ through a character $\chi = c_{f^2}$, and $\chi(\varphi)$ is a *non-trivial* third root of unity.

(6.5) The above example (and similar ones) suggests to refine the investigation (begun in [3] and, much more deeply, in [22]) of the Eisenstein ideal, the Eisenstein quotient of J etc., i.e., of data defined by means of the cuspidal divisor class group $\mathcal{C}(\Gamma)$, by taking into account the Hecke module $\phi_\infty(\Gamma)$ and the map $\mathrm{can}_\infty : \mathcal{C}(\Gamma) \longrightarrow \phi_\infty(\Gamma)$.

References

[1] J. Fresnel, M. van der Put: Géométrie analytique Rigide et Applications. Progr. Math. 18, Birkhäuser, Basel-Boston 1981.

[2] E.-U. Gekeler: Modulare Einheiten für Funktionenkörper. J. Reine Angew. Math. 348, 94-115 (1984).

[3] E.-U. Gekeler: Über Drinfeld'sche Modulkurven vom Hecke-Typ. Comp. Math. 57, 219-236 (1986).

[4] E.-U. Gekeler: Automorphe Formen über $\mathbb{F}_q(T)$ mit kleinem Führer. Abh. Math. Sem. Univ. Hamburg 55, 111-146 (1985).

[5] E.-U. Gekeler: Drinfeld Modular Curves. Lecture Notes Math. 1231, Springer 1986.

[6] E.-U. Gekeler: Analytical Construction of Weil Curves over Function Fields. J. Théorie des Nombres Bordeaux 7, 27-49 (1995).

[7] E.-U. Gekeler: On the Drinfeld Discriminant Function. Comp. Math. 106, 181-202 (1997).

[8] E.-U. Gekeler: A note on the finiteness of certain cuspidal divisor class groups, in preparation.

[9] E.-U. Gekeler, U. Nonnengardt: Fundamental Domains of some Arithmetic Groups over Function Fields. Int. J. Math. 6, 689-708 (1995).

[10] E.-U. Gekeler, M. Reversat: Jacobians of Drinfeld Modular Curves. J. Reine Angew. Math. 476, 27-93 (1996).

[11] L. Gerritzen, M. van der Put: Schottky Groups and Mumford Curves. Lecture Notes Math. 817, Springer 1980.

[12] D. Goss: Basic Structures of Function Field Arithmetic. Ergeb. Math. 35, Springer 1996.

[13] D. Kubert, S. Lang: Modular units. Grundlehren Math. Wiss. 244, Springer 1981.

[14] Y. Manin, V.G. Drinfeld: Periods of p-adic Schottky Groups. J. Reine Angew. Math. 262/263, 239-247 (1973).

[15] B. Mazur: Modular Curves and the Eisenstein Ideal. Publ. Math. IHES 47, 33-186 (1977).

[16] D. Mumford: An Analytic Construction of Degenerating Curves over Local Fields. Comp. Math. 24, 129-174 (1972).

[17] D. Mumford: An Analytic Construction of Degenerating Abelian Varieties over Complete Rings. Comp. Math. 24, 239-272 (1972).

[18] A. Scholl: Fourier Coefficients of Eisenstein Series on Non-Congruence Subgroups. Math. Proc. Camb. Phil. Soc. 99, 11-17 (1986).

[19] A. Schweizer: Zur Arithmetik der Drinfeld'schen Modulkurven $X_0(\mathfrak{n})$. Thesis, Saarbrücken 1996.

[20] J.-P. Serre: Trees. Springer 1980.

[21] G. Shimura: Introduction to the Arithmetic Theory of Automorphic Functions. Publ. Math. Soc. Japan, Tokyo-Princeton 1971.

[22] A. Tamagawa: The Eisenstein Quotient of the Jacobian Variety of a Drinfeld Modular Curve. Publ. RIMS, Kyoto Univ. 31, 203-246 (1995).

[23] J. Teitelbaum: Modular Symbols for $\mathbb{F}_q(T)$. Duke Math. J. 68, 271-295 (1992).

[24] M. van der Put: Les Fonctions thêta d'une Courbe de Mumford. Groupe d'Etude d'Analyse Ultramétrique 1981/82, Paris 1982.

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