Supplement on Curved Flats in the Space of Point Pairs and Isothermic Surfaces: A Quaternionic Calculus

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Abstract. A quaternionic calculus for surface pairs in the conformal 4-sphere is elaborated. It introduces a rich algebraic structure and allows the use of global frames while, at the same time, incorporates the classical "geometric" model of Möbius geometry providing geometric clarity. This way, it provides the foundation for the development of new techniques in Möbius differential geometry.

A field where the quaternionic calculus already proved particularly useful is the geometry of transformations of isothermic surfaces: in the second half of the paper, the relation of Darboux and Christoffel pairs of isothermic surfaces and curved flats in the symmetric space of point pairs is discussed and some applications are sketched. In particular, a new viewpoint on relations between surfaces of constant mean curvature in certain spaces of constant curvature, and on Bryant's Weierstrass type representation for surfaces of constant mean curvature 1 in hyperbolic 3-space is presented.

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1. Introduction

It is well known that the orientation preserving Möbius transformations of the "conformal 2-sphere" $S^2 \cong \mathbb{C} \cup \{\infty\}$ can be described as fractional linear transformations $z \mapsto \frac{a_{11}z+a_{12}}{a_{21}z+a_{22}}$ where $a = (a_{ij}) \in Sl(2,\mathbb{C})$. The reason for this fact is that the conformal 2-sphere $S^2 \cong \mathbb{C}P^1$ can be identified with the complex projective line. Introducing homogeneous coordinates $p = v_p\mathbb{C}$, $v_p \in \mathbb{C}^2$, on $\mathbb{C}P^1$ the special linear group $Sl(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ by projective transformations — which are, for 1-dimensional projective spaces, identical with Möbius transformations — via $v_p\mathbb{C} \mapsto Av_p\mathbb{C} = v_q\mathbb{C}$. Thus, in affine coordinates one has

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \simeq \begin{pmatrix} \frac{a_{11}z+a_{12}}{a_{21}z+a_{22}} \\ 1 \end{pmatrix}.$$

This (algebraic) model of Möbius geometry in dimension 2 complements the ("geometric") model commonly used in differential geometry: here, the conformal 2-sphere (or, more general, the conformal n-sphere) is considered as a quadric in the real projective 3-space $\mathbb{R}P^3$ and the group of Möbius transformations is isomorphic to the group of projective transformations of $\mathbb{R}P^3$ that map the "absolute quadric" S^2 onto itself (cf.[3]). Equipping the space of homogeneous coordinates of $\mathbb{R}P^3$ with a Lorentz scalar product that has the points of S^2 as isotropic (null) lines, the Möbius group can be identified with the pseudo orthogonal group of this Minkowski space $\mathbb{R}^4_1$.

Several attempts have been made to generalize the described algebraic model to higher dimensions — in particular to dimensions 3 and 4, by using quaternions (cf.[14],[15]): analogous to the above model, the conformal 4-sphere is identified with the quaternionic projective line, $S^4 \cong \mathbb{H}P^1$, with $Sl(2,\mathbb{H})$ acting on it by Möbius transformations. In order to use such an "algebraic model" in Möbius *differential* geometry, it is not enough to describe the underlying space and the Möbius group acting on it, though. One also needs a convenient description for (hyper-) spheres since the geometry of surfaces in Möbius geometry is often closely related to the geometry of an enveloped sphere congruence (cf.[3]). For example, Willmore surfaces in S^3 can be related to minimal surfaces in the space of 2-spheres in S^3, and the geometry of isothermic surfaces is related to that of "sphere surfaces" with flat normal bundle, "Ribaucour sphere congruences".

One way is to identify a hypersphere $s \subset \mathbb{H}P^1$ with the inversion at this sphere. The problem with this approach is, that only the orientation preserving Möbius transformations are naturally described in the algebraic model — but, inversions are orientation reversing Möbius transformations. Adjoining the (quaternionic) conjugation as a basic orientation reversing Möbius transformation and working with the larger group of *all* Möbius transformations, works relatively fine for 2-dimensional Möbius geometry, but turns into a nightmare[1)] in dimension 4 since the quaternions form a non commutative field.

Another way is to identify a sphere $s \subset S^4 \cong \mathbb{H}P^1$ with that quaternionic hermitian form on the space $\mathbb{H}^2$ of homogeneous coordinates that has this sphere s as a

[1)] Identifying 2-spheres in $S^3 \subset S^4 \cong \mathbb{H}P^1$ with inversions in S^4 provides a solution in the codimension 1 case, though: as the composition of two inversions at hyperspheres, the inversion at a 2-sphere in S^4 is orientation preserving.

null cone. After discussing some basics in quaternionic linear algebra we will follow this approach — to obtain not only a description for the space of spheres but also to establish the relation with the classical "geometric" model of Möbius geometry: the space of quaternionic hermitian forms will canonically turn into a real six dimensional Minkowski space, the classical model space.

This (second) way, we combine the advantages of both models for Möbius differential geometry: on one side, we introduce a rich algebraic structure which provides a significant simplification of calculations and, at the same time, we also obtain a calculus that will be more suitable to discuss the global geometry of surfaces in Möbius geometry, as well as the geometry of discrete nets. On the other side, we keep a close connection to the classical model of Möbius geometry which will make it easier to understand the results geometrically. In particular, our calculus will provide an ideal setting for the study of surface *pairs*, maps into the (symmetric) space of point pairs in $I\!H P^1$ — in Möbius differential geometry, surfaces often occur naturally in pairs, as envelopes of certain distinguished sphere congruences: for example, Willmore surfaces come in dual pairs as envelopes of their common central sphere congruences, and isothermic surfaces permit pairings via Darboux (and Christoffel) transforms.

The latter will be examined in the remaining part of the paper, on one side to see the calculus at work, on the other side to demonstrate some new results: here, our quaternionic calculus provides very elegant characterizations for Darboux and Christoffel pairs of isothermic surfaces that led to the discovery of the Riccati type equation (cf.[11]) for the Darboux transformation of isothermic surfaces — an equation that apparently cannot be derived in the classical calculus (cf.[2]). This is one reason, why the presented calculus was necessary to develop the definition of the discrete version of the Darboux transformation for discrete isothermic nets and the (geometric) definition of discrete cmc nets (cf.[10]). The mentioned characterizations rely on the relation between Darboux pairs of isothermic surfaces and curved flats in the space of point pairs — since this space will turn out to be symmetric the notion of curved flats makes sense. Although this relation was already established in [6] for the codimension 1 case, it might be of interest to see that it also holds in the higher codimension case[2)] of Darboux pairs in $I\!H P^1$ (cf.[13]). Even though our calculus also provides a framework to discuss global aspects of isothermic surfaces (cf.[12]) we will only focus on their local geometry: there is a variety of possible definitions of "globally isothermic surfaces" whose degree of generality and whose consequences are yet to be worked out. However, computer experiments seem to indicate that Darboux (and Christoffel) transforms of isothermic surfaces only exist locally, in general. And, worse, near certain types of umbilics even their local existence is not clear — resp. depends on the chosen definition of a "globally isothermic surface" ...

In the last section, we study minimal and constant mean curvature surfaces in 3-dimensional spaces of constant curvature. These are "special" isothermic surfaces, and a suitable Christoffel transform in $I\!R^3$ can be determined algebraically (in the general case, an integration has to be carried out). Examining the effect of the spectral parameter that comes with a curved flat, we obtain a new interpretation for the relations between surfaces of constant curvature in certain space forms. In fact, these

[2)] Most recently, these results were generalized to arbitrary codimension using an extension of the presented calculus [5].

relations can be interpreted in terms of Bianchi's "T-transformation" for isothermic surfaces [2]. For example, the well known relation between minimal surfaces in the (metric) 3-sphere and surfaces of constant mean curvature in Euclidean space, as well as the relation between minimal surfaces in Euclidean 3-space and surfaces of constant mean curvature 1 in hyperbolic 3-space are discussed. In case of the constant mean curvature 1 surfaces in hyperbolic 3-space, a new form of Bryant's Weierstrass type representation [4] is given. In this context, the classical Enneper-Weierstrass representation for minimal surfaces in Euclidean 3-space is described as a Goursat type transform of the (multiply covered) plane — similar to the way certain surfaces of constant Gauss curvature are described as a Bäcklund transforms of a line. Finally, the classical Goursat transformation for minimal surfaces is generalized for isothermic surfaces in Euclidean space.

2. The Study determinant

Throughout this paper we will use various well known models [1] for the non commutative field of quaternions:

$$\begin{aligned} \mathbb{H} &\cong \{a+v \,|\, a \in \mathbb{R} \cong \mathrm{Re}\mathbb{H}, v \in \mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}\} \\ &\cong \{a_0 + a_1 i + a_2 j + a_3 k \,|\, a_0, a_1, a_2, a_3 \in \mathbb{R}\} \\ &\cong \{x + y\, j \,|\, x, y \in \mathbb{C}\} \\ &\cong \{A \in M(2\times 2, \mathbb{C}) \,|\, \mathrm{tr}A \in \mathbb{R}, A + A^* \in \mathbb{R}I\}. \end{aligned}$$

Herein, we can identify i, j, k with the standard basis vectors of $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$: if $v, w \in \mathrm{Im}\mathbb{H}$ are two "vectors" their product $v\,w = -v\cdot w + v \times w$ which is equivalent to the familiar identities $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$. Obviously, the first model will turn out particularly useful when focusing on the geometry of 3-space while the decomposition $\mathbb{H} \cong \mathbb{C} + \mathbb{C}\, j$ will prove useful in the context of surfaces, 2-dimensional submanifolds, since their tangent planes (and normal planes) carry a natural complex structure. We will switch between these models as it appears convenient.

As the quaternions can be thought of as a Euclidean 4-space, $\mathbb{R}^4 \cong \mathbb{H}$, the (conformal) 4-sphere $S^4 \cong \mathbb{R}^4 \cup \{\infty\}$ can be identified with the quaternionic projective line: $S^4 \cong \mathbb{H}P^1 = \{\text{lines through } 0 \text{ in } \mathbb{H}^2\}$. Thus, a point $p \in S^4$ of the conformal 4-sphere is described by its homogeneous coordinates $v_p \in \mathbb{H}^2$; and its stereographic projection onto Euclidean 4-space $\mathbb{R}^4 \cong \{v \in \mathbb{H}^2 \,|\, v_2 = 1\}$ is obtained by normalizing the second component of v_p.

Since the quaternions form a non commutative field, we have to agree whether the scalar multiplication in a quaternionic vector space is from the right or left: in this paper, $\mathbb{H}^2$ will be considered a *right vector space* over the quaternions. This way, quaternionic linear transformations can be described by the multiplication (of column vectors) with (quaternionic) matrices from the *left*: $A(v\lambda) = (Av)\lambda$. For a quaternionic 2-by-2 matrix $A \in M(2 \times 2, \mathbb{H})$ we introduce the Study determinant[3] [1] (cf. Study's "Nablafunktion" [14])

$$\begin{aligned} \mathcal{D}(A) &:= \det(A^*A) \\ &= |a_{11}|^2|a_{22}|^2 + |a_{12}|^2|a_{21}|^2 - (\bar{a}_{11}a_{12}\bar{a}_{22}a_{21} + \bar{a}_{21}a_{22}\bar{a}_{12}a_{11}). \end{aligned}$$

[3] Note, that the notion of determinant makes sense for self adjoint matrices $A \in M(2\times 2, \mathbb{H})$.

This is exactly the determinant of the complex 4-by-4 matrix corresponding to A when using the complex matrix model for the quaternions. Thus, $\mathcal{D}$ clearly satisfies the usual multiplication law, $\mathcal{D}(AB) = \mathcal{D}(A)\mathcal{D}(B)$, and vanishes exactly when A is singular. The multiplication law implies that $\mathcal{D}$ is actually an invariant of the linear transformation described by a matrix: $\mathcal{D}(U^{-1}AU) = \mathcal{D}(A)$ for any basis transformation $U : \mathbb{H}^2 \to \mathbb{H}^2$. Also note that $0 \le \mathcal{D}(A) \in \mathbb{R}$.

Definition. *The general and special linear groups of $\mathbb{H}^2$ will be denoted by*

$$\begin{array}{rcl} Gl(2,\mathbb{H}) & := & \{A \in M(2\times 2,\mathbb{H}) \,|\, \mathcal{D}(A) \neq 0\} \\ Sl(2,\mathbb{H}) & := & \{A \in M(2\times 2,\mathbb{H}) \,|\, \mathcal{D}(A) = 1\}. \end{array}$$

With the help of Study's determinant, the inverse of a quaternionic 2-by-2 matrix $A \in Gl(2,\mathbb{H})$ can be expressed directly as

$$A^{-1} = \frac{1}{\mathcal{D}(A)} \begin{pmatrix} |a_{22}|^2\bar a_{11} - \bar a_{21}a_{22}\bar a_{12} & |a_{12}|^2\bar a_{21} - \bar a_{11}a_{12}\bar a_{22} \\ |a_{21}|^2\bar a_{12} - \bar a_{22}a_{21}\bar a_{11} & |a_{11}|^2\bar a_{22} - \bar a_{12}a_{11}\bar a_{21} \end{pmatrix}.$$

Note also, that $Sl(2,\mathbb{H})$ is a 15-dimensional Lie group — it will turn out to be a double cover of the identity component of the Möbius group of S^4.

Considering $\mathcal{D} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$ as a function of two (column) vectors we see that $\mathcal{D}(v, v+w) = \mathcal{D}(v,w)$ and $\mathcal{D}(v, w\lambda) = |\lambda|^2\mathcal{D}(v,w)$ — similar formulas holding for the first entry since $\mathcal{D}$ is symmetric: $\mathcal{D}(v,w) = \mathcal{D}(w,v)$. Reformulating our previous statement, we also obtain that $\mathcal{D}(v,w) = 0$ if and only if v and w are linearly dependent[4)]. Particularly, if v and w are points in an affine quaternionic line, say the Euclidean 4-space $\{v \in \mathbb{H}^2 \,|\, v_2 = 1\}$, then $\mathcal{D}(v,w) = |v_1 - w_1|^2$ measures the distance between v and w with respect to a Euclidean metric. This fact can be used to express the cross ratio of four points in Euclidean 4-space (cf.[10]) in terms of the Study determinant[5)]:

$$|DV(h_1,h_2,h_3,h_4)|^2 = \frac{\mathcal{D}\begin{pmatrix} h_1 & h_2 \\ 1 & 1\end{pmatrix}\mathcal{D}\begin{pmatrix} h_3 & h_4 \\ 1 & 1\end{pmatrix}}{\mathcal{D}\begin{pmatrix} h_2 & h_3 \\ 1 & 1\end{pmatrix}\mathcal{D}\begin{pmatrix} h_4 & h_1 \\ 1 & 1\end{pmatrix}}.$$

The expression on the right hand is obviously invariant under individual rescalings of the vectors which shows that the cross ratio is, in fact, an invariant of four points in the quaternionic projective line $\mathbb{H}P^1$.

3. Quaternionic hermitian forms

will be a key tool in our calculus for Möbius geometry: any quaternionic hermitian form $s : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}$,

$$\begin{array}{rcl} s(v, w_1\lambda + w_2\mu) & = & s(v,w_1)\lambda + s(v,w_2)\mu \\ s(v_1\lambda + v_2\mu, w) & = & \bar\lambda s(v_1,w) + \bar\mu s(v_2,w) \\ s(w,v) & = & \overline{s(v,w)}, \end{array}$$

4) All these properties are also easily checked directly, without using the complex matrix representation of the quaternions.

5) For a more complete discussion of the *complex* cross ratio of four points in space consult [10].

is determined by its values on a basis (e_1, e_2) of $\mathbb{H}^2$, $s_{ij} = s(e_i, e_j)$. Since s is hermitian, $s_{11}, s_{22} \in \mathbb{R}$ and $s_{21} = \bar{s}_{12} \in \mathbb{H}$, the quaternionic hermitian forms on $\mathbb{H}^2$ form a 6-dimensional (real) vector space. Clearly, $Gl(2, \mathbb{H})$ operates on this vector space via $(A, s) \mapsto As := [(v, w) \mapsto s(Av, Aw)]$, or, in the matrix representation of s, via $(A, s) \mapsto A^*sA$. A straightforward calculation shows that $\det(As) = \mathcal{D}(A)\det(s)$. This enables us to introduce a Lorentz scalar product

$$\langle s, s\rangle := -\det(s) = |s_{12}|^2 - s_{11}s_{22}$$

on the space $\mathbb{R}^6_1$ of quaternionic hermitian forms, which is well defined up to a scale[6] (or, the choice of a basis in $\mathbb{H}^2$). Fixing a scaling of this Lorentz product, the special linear transformations act as isometries on $\mathbb{R}^6_1$ — $Sl(2, \mathbb{H})$ is a double cover of the identity component[7] of $SO_1(6)$, which itself is isomorphic to the group of orientation preserving Möbius transformations of S^4. Thus, restricting our attention to Euclidean 4-space $\{e_1h + e_2 \,|\, h \in \mathbb{H}\}$, the orientation preserving Möbius transformations appear as fractional linear transformations (cf.[14],[15])

$$\begin{pmatrix} h \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} h \\ 1 \end{pmatrix} \simeq \begin{pmatrix} (a_{11}h + a_{12})(a_{21}h + a_{22})^{-1} \\ 1 \end{pmatrix}.$$

If $s \neq 0$ lies in the light cone of $\mathbb{R}^6_1$, $\langle s, s\rangle = 0$, then the corresponding quadratic form $v \mapsto s(v, v)$ annihilates exactly *one* direction $v\mathbb{H} \subset \mathbb{H}^2$: $0 = s(v, v)$ vanishes iff $0 = |s_{11}v_1 + s_{12}v_2|^2$ or $0 = |s_{21}v_1 + s_{22}v_2|^2$ since at least one, s_{11} or s_{22} does not vanish. Hence, we can identify a point $p = v\mathbb{H} \in \mathbb{H}P^1$ of the quaternionic projective line — the 4-sphere — with the null line of quaternionic hermitian forms in the Minkowski $\mathbb{R}^6_1$ that annihilate this point. In homogeneous coordinates, this identification can be given by[8]

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} |v_2|^2 & -v_1\bar{v}_2 \\ -v_2\bar{v}_1 & |v_1|^2 \end{pmatrix} = s_v. \tag{1}$$

Note, that with this identification, $\langle s_v, s\rangle = -s(v, v)$ for any quaternionic hermitian form $s \in \mathbb{R}^6_1$. If $s = s_w$ is an isotropic form too, then $\langle s_v, s_w\rangle = -\mathcal{D}(v, w)$.

If, on the other hand, $\langle s, s\rangle = 1$ we obtain — depending on whether $s_{11} = 0$ or $s_{11} \neq 0$ in the chosen basis (e_1, e_2) of $\mathbb{H}^2$ —

$$s = \begin{pmatrix} 0 & -n \\ -\bar{n} & 2d \end{pmatrix} \qquad \text{or} \qquad s = \frac{1}{r}\begin{pmatrix} 1 & -m \\ -\bar{m} & |m|^2 - r^2 \end{pmatrix}$$

with suitable n resp. $m \in \mathbb{H}$ and d resp. $r \in \mathbb{R}$: the null cone of s is a plane with unit normal n and distance d from the origin or a sphere with center m and radius r in Euclidean 4-space $\{e_1h + e_2 \,|\, h \in \mathbb{H}\}$. Consequently, we identify the Lorentz sphere $S^5_1 \subset \mathbb{R}^6_1$ with the space of spheres and planes in Euclidean 4-space, or with the space of spheres in S^4 — as the readers familiar with the classical model (cf.[3]) of Möbius geometry might already have suspected. The incidence of a point $p \in S^4 \cong \mathbb{H}P^1$ and a sphere $s \subset S^4$, i.e. $s \in S^5_1$, is equivalent to $s(p, p) = 0$ in our quaternionic model.

A key concept in

[6] At this point, we notice that the geometrically significant space is the projective 5-space $\mathbb{R}P^5$ with absolute quadric $Q = \{\mathbb{R}x \,|\, \langle x, x\rangle = 0\}$, not its space of homogeneous coordinates, $\mathbb{R}^6_1$.

[7] Using a basis of quaternionic hermitian forms, it is an unpleasant but straightforward calculation to establish a Lie algebra isomorphism $\mathfrak{sl}(2, \mathbb{H}) \leftrightarrow \mathfrak{o}_1(6)$.

[8] Note the analogy with the Veronese embedding.

4. Möbius differential geometry

is that of (hyper-) sphere congruences and envelopes of sphere congruences:

Definition. *An immersion* $f : M \to S^4$ *is called an envelope of a hypersphere congruence* $s : M \to S^5_1$ *if, at each point* $p \in M$, f *touches the corresponding sphere* $s(p)$: $f(p) \in s(p)$ *and* $d_pf(T_pM) \subset T_{f(p)}s(p)$.

According to our previous discussion, the first condition — the incidence of $f(p)$ and the corresponding sphere $s(p)$ — is equivalent to $s(f,f) = 0$ in our quaternionic model. Calculating, for a moment, in a Euclidean setting — i.e. $s = \frac{1}{r}\begin{pmatrix} 1 & -m \\ -\bar{m} & |m|^2 - r^2 \end{pmatrix}$ — we find $s(f,df) + s(df,f) = \frac{2}{r}(f-m)\cdot df$. Thus[9],

Lemma. *An immersion* $f : M \to I\!\!HP^1$ *envelopes a sphere congruence* $s : M \to S^5_1$ *if and only if* $s(f,f) = 0$ *and* $s(f,df) + s(df,f) = 0$.

Before going on, we introduce the symmetric space of point pairs: given two (distinct) points of the quaternionic projective line $I\!\!HP^1$, we may identify these points with a quaternionic linear transformation P which maps a (fixed) basis (e_1, e_2) of $I\!\!H^2$ to their homogeneous coordinates — or, in coordinates, with a matrix having for columns the homogeneous coordinates of the two points. This linear transformation P is obviously not uniquely determined by the two points in $I\!\!HP^1$: any gauge transform $P \cdot H$ of P with H in the isotropy subgroup $K := \{H \in Gl(2, I\!\!H) \,|\, He_1 = e_1\lambda, He_2 = e_2\mu\}$ determines the same point pair. Thus, the space $\mathcal{P}$ of point pairs in the conformal 4-sphere $I\!\!HP^1$ is a homogeneous space, $\mathcal{P} = Gl(2, I\!\!H)/K$. Moreover, the decomposition $\mathfrak{gl}(2, I\!\!H) = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\begin{array}{rcl} \mathfrak{k} & = & \{X \in \mathfrak{gl}(2, I\!\!H) \,|\, Xe_1 = e_1\lambda, Xe_2 = e_2\mu\} \\ \mathfrak{p} & = & \{X \in \mathfrak{gl}(2, I\!\!H) \,|\, Xe_1 = e_2\lambda, Xe_2 = e_1\mu\} \end{array} \tag{2}$$

is a Cartan decomposition since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ so that $\mathcal{P}$ is, in fact, a symmetric space.

Now, if $F = (f, \hat{f}) : M \to Gl(2, I\!\!H)$ is a framing (lift) of a point pair map $M \to \mathcal{P}$, a simple calculation using (1) shows that

$$Ff = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F\hat{f} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

if the relative scaling of f and $\hat{f}$ is chosen such that F takes values in the special linear group $Sl(2, I\!\!H)$. Since $Sl(2, I\!\!H)$ acts by isometries on the space $I\!\!R^6_1$ of quaternionic hermitian forms, for any sphere congruence $s : M \to S^5_1$ containing the points of f and $\hat{f}$, we have

$$Fs = \begin{pmatrix} 0 & s_0 \\ \bar{s}_0 & 0 \end{pmatrix}$$

[9] Note, that with the identification (1) of points in $I\!\!HP^1$ with isotropic quaternionic hermitian forms, $s(f,df) + s(df,f) = -\langle s, df \rangle$ which gives the link with the classical model of Möbius geometry.

with a suitable function $s_0 : M \to S^3 \subset I\!H$ taking values in the unit quaternions. Passing to another set of homogeneous coordinates by means of a gauge transformation $(f, \hat f) \mapsto (f\lambda, \hat f\hat\lambda)$ results in $s_0 \mapsto \bar\lambda s_0 \hat\lambda$. Thus, depending on a given sphere congruence s, we may fix the homogeneous coordinates of f and $\hat f$ such that $s_0 \equiv 1$ — leaving us with a scaling freedom $(f, \hat f) \mapsto (f\lambda, \hat f\bar\lambda^{-1})$ with $\lambda \in I\!H$. A second sphere congruence $\tilde s$ (orthogonal to the first one) can be used to further fix the scalings via $\tilde s_0 \equiv i$ up to $\lambda \in \mathbb{C}$. Giving a complete set of four accompanying orthogonal sphere congruences and fixing a third one, $\hat s$, to satisfy $\hat s_0 \equiv j$ leaves us with the familiar real scaling freedom, $\lambda \in I\!R$ (cf.[3]). These choices of accompanying spheres, and accordingly these choices of homogeneous coordinates for a point pair map $(f, \hat f)$ are the only aspect of the presented calculus that will generally not work globally.

Writing down the derivatives $df = f\varphi + \hat f\psi$ and $d\hat f = f\hat\psi + \hat f\hat\varphi$ of f and $\hat f$, we obtain the connection form

$$\Phi := F^{-1}dF = \begin{pmatrix} \varphi & \hat\psi \\ \psi & \hat\varphi \end{pmatrix} : TM \to \mathfrak{gl}(2, I\!H)$$

of a framing $F : M \to Gl(2, I\!H)$. A gauge transformation $(f, \hat f) \mapsto (f\lambda, \hat f\hat\lambda)$ of the frame will result in a change

$$\begin{pmatrix} \varphi & \hat\psi \\ \psi & \hat\varphi \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1}\varphi\lambda & \lambda^{-1}\hat\psi\hat\lambda \\ \hat\lambda^{-1}\psi\lambda & \hat\lambda^{-1}\hat\varphi\hat\lambda \end{pmatrix} + \begin{pmatrix} \lambda^{-1}d\lambda & 0 \\ 0 & \hat\lambda^{-1}d\hat\lambda \end{pmatrix} \tag{3}$$

of the connection form Φ. The integrability conditions $0 = d^2 f = d^2\hat f$ yield the Maurer-Cartan equation $0 = d\Phi + \Phi \wedge \Phi$ for the connection form: the Gauss-Ricci equations for f resp. $\hat f$,

$$\begin{array}{rcl} 0 & = & d\varphi + \varphi\wedge\varphi + \hat\psi\wedge\psi \\ 0 & = & d\hat\varphi + \hat\varphi\wedge\hat\varphi + \psi\wedge\hat\psi, \end{array} \tag{4}$$

and the Codazzi equations,

$$\begin{array}{rcl} 0 & = & d\psi + \psi\wedge\varphi + \hat\varphi\wedge\psi \\ 0 & = & d\hat\psi + \hat\psi\wedge\hat\varphi + \varphi\wedge\hat\psi. \end{array} \tag{5}$$

Note, that since the quaternions are not commutative, generally $\varphi\wedge\varphi \neq 0$. Moreover, $d(\lambda\varphi) = d\lambda\wedge\varphi + \lambda d\varphi$, $d(\varphi\lambda) = d\varphi\,\lambda - \varphi\wedge d\lambda$ and $\overline{\varphi\wedge\psi} = -\bar\psi\wedge\bar\varphi$ for any quaternion valued 1-forms φ and ψ and function $\lambda : M \to I\!H$.

If $s : M \to I\!R^6_1$ is a map into the vector space of quaternionic hermitian forms, then its derivative, $ds : TM \to I\!R^6_1$ is a 1-form with values in the quaternionic hermitian forms. If $Fs \equiv const$, this derivative can be expressed in terms of the connection form Φ of F: since $d(Fs) = 0$,

$$F\,ds = -F[s(.,\Phi) + s(\Phi,.)] \simeq -[Fs\cdot\Phi + \Phi^*\cdot Fs] \tag{6}$$

when using the matrix representation for quaternionic hermitian forms.

5. Curved flats and Isothermic surfaces

The concept of curved flats in symmetric spaces was first introduced by D. Ferus and F. Pedit [9] as a natural generalization of developable surfaces: a curved flat is an envelope of a congruence of flats in a symmetric space or, more technical, a submanifold of a symmetric space (with semisimple isometry group) whose tangent spaces are maximal abelian subalgebras in the tangent spaces of that symmetric space. In [6] it was then applied to the geometry of isothermic surfaces in 3-space. To demonstrate our quaternionic calculus at work, we are going to discuss curved flats in the symmetric space $\mathcal{P}$ of point pairs in $I\!\!HP^1$. As in the codimension 1 case, these will turn out to be Darboux pairs of isothermic surfaces in 4-space: given a point pair map $(f,\hat{f}):M\to\mathcal{P}$, we choose a framing $F:M\to Sl(2,I\!\!H)$ and write its connection form $\Phi=\Phi_{\mathfrak{k}}+\Phi_{\mathfrak{p}}:TM\to\mathfrak{sl}(2,I\!\!H)=\mathfrak{k}\oplus\mathfrak{p}$. Then[10],

Definition. *A map $(f,\hat{f}):M\to\mathcal{P}$ into the symmetric space of point pairs is called a curved flat if $\Phi_{\mathfrak{p}}\wedge\Phi_{\mathfrak{p}}=0$.*

Note, that the defining equation is invariant under gauge transformations (3) of F, i.e. does not depend on a choice of homogeneous coordinates. Thus, the notion of a curved flat is a well defined notion for a point pair map $(f,\hat{f}):M\to\mathcal{P}$.

In order to understand the geometry of a curved flat $(f,\hat{f}):M^2\to\mathcal{P}$ in the symmetric space of point pairs we will first express its connection form in a simpler form, and then interpret it geometrically in a second step[11]. We start with an $Sl(2,I\!\!H)$-framing $F:M^2\to Sl(2,I\!\!H)$ and write its connection form

$$\Phi=\begin{pmatrix}\varphi_1+\varphi_2 j & \hat{\psi}_1+\hat{\psi}j\\ \psi_1+\psi j & \hat{\varphi}_1+\hat{\varphi}_2 j\end{pmatrix}$$

in terms of complex valued 1-forms. Using a rescaling $(f,\hat{f})\mapsto(f\lambda,\hat{f}\hat{\lambda})$ we can achieve $\psi_1=0$; then, the curved flat equations read (we assume $\psi\neq0$) $\hat{\psi}_1=0$ and $\hat{\psi}\wedge\bar{\psi}=0$. A rescaling $(f,\hat{f})\mapsto(f\bar{\lambda},\hat{f}\lambda^{-1})$ with a complex valued function λ results in $(\psi,\hat{\psi})\mapsto(\lambda^2\psi,\bar{\lambda}^{-2}\hat{\psi})$; as any 1-form on M^2 has an integrating factor, we may assume $d\psi=0$, i.e. $\psi=dw$. Since $\hat{\psi}\wedge\bar{\psi}=0$, $\hat{\psi}=\bar{a}^4d\bar{w}$ with a suitable function $a:M\to\mathbb{C}$. From the Codazzi equations, $da\wedge dw=0$ — thus, by a holomorphic change $z_w=a^2$ of coordinates, $\psi=a^{-2}dz$ and $\hat{\psi}=\bar{a}^2d\bar{z}$, or, after rescaling again with $\lambda=a$, $\psi=dz$ and $\hat{\psi}=d\bar{z}$. Now, the Codazzi equations also yield $\hat{\varphi}_2\wedge dz=\bar{\varphi}_2\wedge d\bar{z}$ and $\hat{\varphi}_2\wedge d\bar{z}=\bar{\varphi}_2\wedge dz$. Thus, $\varphi_2=q_1dz-\bar{q}_2d\bar{z}$ and $\hat{\varphi}_2=-\bar{q}_1dz+q_2d\bar{z}$ with suitable functions $q_1,q_2:M\to\mathbb{C}$. This way, $\varphi_2\wedge\bar{\varphi}_2=\hat{\varphi}_2\wedge\bar{\hat{\varphi}}_2$ such that $d\varphi_1=d\hat{\varphi}_1$ from the Gauss-Ricci equations. With the ansatz $\hat{\varphi}_1-\varphi_1=2a$, we find that a rescaling $(f,\hat{f})\mapsto(f\lambda,\hat{f}\lambda^{-1})$ with $\lambda=e^a$ yields $\varphi_1=\hat{\varphi}_1$. At the same time, $(\psi,\hat{\psi})\mapsto(e^u\psi,e^{-u}\hat{\psi})$ with $u=a+\bar{a}$. So, we end up with a connection form

$$\Phi=\begin{pmatrix}i\eta+(q_1dz-\bar{q}_2d\bar{z})j & e^{-u}d\bar{z}\,j\\ e^u dz\,j & i\eta+(-\bar{q}_1dz+q_2d\bar{z})j\end{pmatrix}\tag{7}$$

[10] For simplicity of notation, we reduce the definition to the case under investigation.

[11] Note that, from this point on, we will restrict to local geometry: as Darboux pairs of isothermic surfaces generally only exist locally so do curved flats in the space of point pairs. Also, some of the presented arguments require the dondegeneracy of the curvature line net of the surfaces.

where $u : M \to \mathbb{R}$, $q_1, q_2 : M \to \mathbb{C}$ and $\eta : TM \to \mathbb{R}$ is a real valued 1-form — remember that we have chosen an $Sl(2,\mathbb{H})$-framing from the beginning.

In order to interpret this connection form geometrically, we first note that all sphere congruences

$$s_c := F^{-1}\begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} : M \to S^5_1$$

with $c = e^{i\vartheta}$ are enveloped by the two maps f and $\hat{f}$:

$$Fds_c = -\begin{pmatrix} 0 & 2[-\mathrm{Re}(\bar{c}q_1)dz + \mathrm{Re}(cq_2)d\bar{z}]j \\ 2[\mathrm{Re}(\bar{c}q_1)dz - \mathrm{Re}(cq_2)d\bar{z}]j & 0 \end{pmatrix}$$

Thus, in the $\mathbb{R}^6_1$-model of Möbius geometry, the s_c can be viewed as common normal fields of f and $\hat{f}$. Using the identification (1) of points in $\mathbb{H}P^1$ and isotropic lines in $\mathbb{R}^6_1$, we obtain

$$df = F^{-1}\begin{pmatrix} 0 & e^u dz\, j \\ -e^u dz\, j & 0 \end{pmatrix} \quad \text{and} \quad d\hat{f} = F^{-1}\begin{pmatrix} 0 & -e^{-u} d\bar{z}\, j \\ e^{-u} d\bar{z}\, j & 0 \end{pmatrix}$$

as the derivatives (6) of f and $\hat{f}$. Calculating the induced metrics

$$\langle df, df\rangle = e^{2u}|dz|^2 \quad \text{and} \quad \langle d\hat{f}, d\hat{f}\rangle = e^{-2u}|dz|^2$$

of f and $\hat{f}$, and their second fundamental forms with respect to s_c,

$$\begin{array}{rcl} -\langle df, ds_c\rangle & = & e^u[-2\mathrm{Re}(\bar{c}q_1)|dz|^2 + \mathrm{Re}(cq_2)(dz^2 + d\bar{z}^2)], \\ -\langle d\hat{f}, ds_c\rangle & = & e^{-u}[-2\mathrm{Re}(cq_2)|dz|^2 + \mathrm{Re}(\bar{c}q_1)(dz^2 + d\bar{z}^2)], \end{array}$$

we see that f and $\hat{f}$ have well defined principal curvature directions (independent of the normal direction s_c) which do correspond on both surfaces ($\{s_c \,|\, c \in S^1\}$ is a "Ribaucour sphere pencil"), and that f and $\hat{f}$ induce conformally equivalent metrics on M. Moreover, $z : M \to \mathbb{C}$ are conformal curvature line coordinates on both surfaces, i.e. both surfaces are isothermic. Consequently, $(f, \hat{f}) : M \to \mathcal{P}$ is a "Darboux pair" of isothermic surfaces in 4-space[12)]:

Definition. *Two surfaces are said to form a Darboux pair if they envelope a (non-trivial) congruence of 2-spheres (two orthogonal congruences of 3-spheres in 4-space) such that the curvature lines on both surfaces correspond and the induced metrics in corresponding points are conformally equivalent.*

Conversely, if $(f, \hat{f}) : M \to \mathcal{P}$ envelope two congruences of orthogonal spheres, say $s_1, s_i : M \to S^5_1$, then the connection form

$$\Phi = \begin{pmatrix} \varphi_1 + \varphi_2 j & \hat{\psi}\, j \\ \psi\, j & \hat{\varphi}_1 + \hat{\varphi}_2 j \end{pmatrix}$$

12) This geometric description of Darboux pairs of isothermic surfaces can obviously be used to define isothermic surfaces and Darboux pairs of any codimension — as the one below for Christoffel pairs can (cf.[13]). Note, that the flatness of the normal bundle of a surface — which is necessary to make sense of the notion of curvature lines — is a conformal notion, i.e. it is invariant under conformal changes of the ambient space's metric.

with complex 1-forms $\psi, \hat\psi : TM \to \mathbb{C}$. Assuming the curvature lines of f and $\hat f$ to correspond, and their induced metrics to be conformally equivalent, we can introduce common curvature line coordinates: $\psi = e^u\omega$ and $\hat\psi = e^{-u}\omega$, or $\hat\psi = e^{-u}\bar\omega$. In both cases, from the Gauss-Ricci equations $\mathrm{Re}[d(\varphi_1 - \hat\varphi_1)] = 0$, so that after a suitable real rescaling of f and $\hat f$, $\mathrm{Re}(\varphi_1 - \hat\varphi_1) = 0$. Then, in the first case, the Codazzi equations imply $u \equiv const$: the sphere congruences enveloped by f and $\hat f$ lie in a fixed linear complex, consequently f and $\hat f$ are congruent in some space of constant curvature (cf.[3], [6]) — and are not considered to form a Darboux pair. In the other case, the Codazzi equations yield $d\omega = 0$ — we have *conformal* curvature line parameters, i.e. f and $\hat f$ are isothermic; we could also have concluded this from the fact that f and $\hat f$ obviously form a curved flat:

Theorem. *A surface pair $(f, \hat f) : M^2 \to \mathcal{P}$ is a curved flat if and only if f and $\hat f$ form a Darboux pair. Two surfaces forming a Darboux pair are isothermic.*

The $\mathfrak{k}$-part — see (2) — of the Maurer-Cartan equation of a $Gl(2, \mathbb{H})$-framing reads $0 = d\Phi_{\mathfrak{k}} + \Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}$. Thus, for a curved flat, $\Phi_{\mathfrak{k}} = H^{-1}dH$ with a suitable $H : M \to K$: if λ and $\hat\lambda$ are given by

$$\lambda^{-1}d\lambda = i\eta + (q_1 dz - \bar q_2 d\bar z)j \quad \text{and} \quad \hat\lambda^{-1}d\hat\lambda = i\eta + (-\bar q_1 dz + q_2 d\bar z)j$$

then a gauge transformation $(f, \hat f) \mapsto (f\lambda^{-1}, \hat f\hat\lambda^{-1})$ of our previous framing with connection form (7) leaves us with

$$\Phi = \begin{pmatrix} 0 & \lambda(e^{-u}d\bar z\, j)\hat\lambda^{-1} \\ \hat\lambda(e^u dz\, j)\lambda^{-1} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \hat\omega \\ \omega & 0 \end{pmatrix}.$$

The Codazzi equations for this new framing simply read $d\omega = d\hat\omega = 0$ showing that $\bar\omega = df_0$ and $\hat\omega = d\hat f_0$ with suitable maps $f_0, \hat f_0 : M \to \mathbb{H}$. Here, we identify the two copies of the quaternions sitting in $\mathfrak{p} = \mathbb{H} \oplus \mathbb{H}$ as the eigenspaces of $\mathrm{ad}_C : \mathfrak{p} \to \mathfrak{p}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, by means of the real endomorphism $X \mapsto X^*$ of $\mathfrak{p}$. Note, that since the 1-forms $\lambda^{-1}d\lambda, \hat\lambda^{-1}d\hat\lambda : TM \to \mathrm{Im}\mathbb{H}$ take values in the imaginary quaternions, $|\lambda| = |\hat\lambda| \equiv 1$. Consequently, the induced metrics of $f_0 : M \to \mathbb{H}$ and $\hat f_0 : M \to \mathbb{H}$, $\mathbb{H} \cong \mathbb{R}^4$ considered as a Euclidean space, are

$$df_0 \cdot df_0 = e^{2u}|dz|^2 \quad \text{and} \quad d\hat f_0 \cdot d\hat f_0 = e^{-2u}|dz|^2.$$

Moreover, with the common unit normal fields $n_c = -\lambda c\hat\lambda^{-1}$ of f_0 and $\hat f_0$, where $c = e^{i\vartheta}$, their second fundamental forms become

$$\begin{array}{rcl} -df_0 \cdot dn_c &=& e^u[-2\mathrm{Re}(cq_1)|dz|^2 + \mathrm{Re}(\bar c q_2)(dz^2 + d\bar z^2)], \\ -d\hat f_0 \cdot d\hat n_c &=& e^{-u}[-2\mathrm{Re}(\bar c q_2)|dz|^2 + \mathrm{Re}(cq_1)(dz^2 + d\bar z^2)]. \end{array} \tag{8}$$

Thus, f_0 and $\hat f_0$ are two isothermic surfaces that carry common curvature line coordinates — and, $\hat f_0$ and $\bar f_0$ have parallel tangent planes. Hence, we define[13]:

[13] If $f_0, \hat f_0 : M^2 \to \mathrm{Im}\mathbb{H}$, this definition yields the classical notion of a Christoffel pair (cf.[6]).

DEFINITION. *Two (non homothetic) surfaces $f_0, \hat{f}_0 : M^2 \to I\!H$ with parallel tangent planes in corresponding points are said to form a Christoffel pair if the curvature lines on both surfaces correspond and the induced metrics are conformally equivalent.*

Conversely, if two surfaces $f_0, \hat{f}_0 : M^2 \to I\!H$ carry conformally equivalent metrics and have parallel tangent planes in corresponding points $f_0(p)$ and $\hat{f}_0(p)$ then[14), $df_0 = \lambda e^u \psi j \hat{\lambda}^{-1}$ and $d\hat{f}_0 = \pm\lambda e^{-u}\psi j \hat{\lambda}^{-1}$, or $d\hat{f}_0 = \lambda e^{-u}\bar{\psi} j \hat{\lambda}^{-1}$ with a real valued function u, a complex 1-form $\psi : TM \to \mathbb{C}$ and suitable quaternionic functions $\lambda, \hat{\lambda} : M \to I\!H$ — where $|\lambda| = |\hat{\lambda}| \equiv 1$ without loss of generality. In the first case, the integrability conditions yield $0 = du \wedge \psi$ showing that $u \equiv const$. Consequently, $\hat{f}_0$ is homothetic to f_0 — and f_0 and $\hat{f}_0$ are not considered to form a Christoffel pair. In the second case, $d\bar{f}_0 \wedge d\hat{f}_0 = d\hat{f}_0 \wedge d\bar{f}_0 = 0$. Hence, the surface pair $f_0, \hat{f}_0 : M \to I\!H$ gives rise to a curved flat by integrating $\Phi := \begin{pmatrix} 0 & d\hat{f}_0 \\ d\bar{f}_0 & 0 \end{pmatrix}$ — we obtain the following

THEOREM. *Two surfaces $f_0, \hat{f}_0 : M^2 \to I\!H$ form a Christoffel pair if and only if $d\bar{f}_0 \wedge d\hat{f}_0 = d\hat{f}_0 \wedge d\bar{f}_0 = 0$. Two surfaces forming a Christoffel pair are isothermic.*

Curved flats — or, Darboux pairs of isothermic surfaces — naturally arise in 1-parameter families [9]: if $\Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ denotes one of the connection forms associated to a curved flat $(f, \hat{f}) : M^2 \to \mathcal{P}$, then, with a real parameter $\varrho \in I\!R$, all the connection forms

$$\Phi_\varrho := \Phi_{\mathfrak{k}} + \varrho^2 \Phi_{\mathfrak{p}} : TM^2 \to \mathfrak{sl}(2, I\!H) = \mathfrak{k} \oplus \mathfrak{p} \tag{9}$$

are integrable and give rise to curved flats $(f_\varrho, \hat{f}_\varrho) : M^2 \to \mathcal{P}$; in fact, if the connection forms (9) are integrable for more than one value of ϱ^2, then the associated point pair maps are necessarily curved flats. From (3), we learn that this 1-parameter family of curved flats does not depend on the framing chosen to describe the curved flat $(f, \hat{f})$. Moreover, sending the parameter $\varrho \to 0$, and rescaling $(f_\varrho, \hat{f}_\varrho) \mapsto (\varrho^{-1} f_\varrho, \varrho \hat{f}_\varrho)$ or $(f_\varrho, \hat{f}_\varrho) \mapsto (\varrho f_\varrho, \varrho^{-1} \hat{f}_\varrho)$ at the same time, provides us with

$$(f_{\varrho=0}, \hat{f}_{\varrho=0}) = \begin{pmatrix} 1 & 0 \\ \bar{f}_0 & 1 \end{pmatrix} \quad \text{or} \quad (f_{\varrho=0}, \hat{f}_{\varrho=0}) = \begin{pmatrix} 1 & \hat{f}_0 \\ 0 & 1 \end{pmatrix}.$$

Hence, we may think of the Christoffel pair $(f_0, \hat{f}_0)$ — that is, as before, associated to a 1-parameter family of curved flats by integrating

$$\Phi_\varrho = \begin{pmatrix} 0 & \varrho^2 d\hat{f}_0 \\ \varrho^2 d\bar{f}_0 & 0 \end{pmatrix}$$

— as a limiting case for the Darboux pairs $(f_\varrho, \hat{f}_\varrho)$. Comparison with (3) shows that the spectral parameter ϱ corresponds to the scaling ambiguity of the members of a Christoffel pair: one of the surfaces of a Christoffel pair is determined by the other only up to a homothety (and translation).

We will use those facts to discuss perturbation methods (cf.[16]) for the construction of constant mean curvature surfaces and, in particular, for Bryant's Weierstrass type representation [4] for

[14) If p is not an umbilic for either surface, it *follows* that the principal curvature directions of both surfaces correspond. In case one of the surfaces is totally umbilic we need also to assume that the curvature lines on both surfaces coincide — otherwise we might find two associated minimal surfaces.

6. Constant mean curvature surfaces

in hyperbolic space forms. We restrict our attention to codimension 1 by assuming that our surfaces lie in a fixed conformal 3-sphere, say s_1. Thus the connection form (7) of a Darboux pair $(f,\hat{f}): M^2 \to \mathbb{H}P^1$ takes the form

$$\Phi = \begin{pmatrix} i[\eta + \frac{1}{2}(e^u H dz - e^{-u}\hat{H} d\bar{z})j] & e^{-u} d\bar{z}\, j \\ e^u dz\, j & i[\eta + \frac{1}{2}(e^u H dz - e^{-u}\hat{H} d\bar{z})j] \end{pmatrix} \tag{10}$$

where the (real) functions $H, \hat{H}$ can be interpreted as the mean curvature functions of the members f_0 and $\hat{f}_0$ of the limiting Christoffel pair: from (10) we see that a rescaling $(f,\hat{f}) \mapsto (f\lambda, \hat{f}\lambda)$ will provide us with $\Phi_{\mathfrak{k}} = 0$, such that $df_0, d\hat{f}_0 : TM \to \mathrm{Im}\mathbb{H}$. The second fundamental forms (8) with respect to the remaining common normal field $n_i = -\lambda i \lambda^{-1} = -\hat{n}_i$ become

$$\begin{array}{rcl} -df_0 \cdot dn_i & = & He^{2u}|dz|^2 - \frac{1}{2}\hat{H}(dz^2 + d\bar{z}^2)], \\ -d\hat{f}_0 \cdot d\hat{n}_i & = & \hat{H}e^{-2u}|dz|^2 - \frac{1}{2}H(dz^2 + d\bar{z}^2)]. \end{array}$$

The Codazzi equations (5) yield $\eta = \frac{i}{2}(-u_z dz + u_{\bar{z}} d\bar{z})$ and from (4) we recover the classical Gauss equation $0 = u_{z\bar{z}} + \frac{1}{4}(H^2 e^{2u} - \hat{H}^2 e^{-2u})$ holding for both surfaces f_0 and $\hat{f}_0$, and the classical Codazzi equations $dH \wedge e^u dz = d\hat{H} \wedge e^{-u} d\bar{z}$. Hence, $H \equiv const$ if and only if $\hat{H} \equiv const$, reflecting the fact that a pair of parallel constant mean curvature surfaces, or a minimal surface and its Gauss map form Christoffel pairs (cf.[11]).

Calculating the derivative of the sphere congruence s_i enveloped by the two surfaces f and $\hat{f}$ — which form the Darboux pair associated with the Christoffel pair $(f_0, \hat{f}_0)$ — we find

$$F ds_i = \begin{pmatrix} 0 & (He^u dz - \hat{H}e^{-u} d\bar{z})j \\ (-He^u dz + \hat{H}e^{-u} d\bar{z})j & 0 \end{pmatrix} = H \cdot F\, df + \hat{H} \cdot F\, d\hat{f}.$$

Hence, the vector $N := s_i - Hf - \hat{H}\hat{f}$ is constant as soon as one of the mean curvatures, H or $\hat{H}$, is. In order to interpret this fact geometrically, we have to distinguish two cases:

If $H\hat{H} \neq 0$, i.e. $(f_0, \hat{f}_0)$ is equivalent to a pair of parallel constant mean curvature surfaces, then $\langle N, \frac{2}{\hat{H}} f \rangle \equiv 1$ and $\langle N, \frac{2}{H} \hat{f} \rangle \equiv 1$ — and consequently (cf.[3]), the two surfaces $\frac{1}{\hat{H}} f, \frac{1}{H}\hat{f} : M^2 \to s_1 \simeq S^3 \subset \mathbb{H}P^1$ can be interpreted as surfaces in the space $M^3_N := \{y \in \mathbb{R}^6_1 \,|\, \langle N, y \rangle = 1, \langle s_1, y \rangle = 0, \langle y, y \rangle = 0\}$ of constant sectional curvature $\kappa = -\langle N, N \rangle = -(1 - H\hat{H})$. Their induced metrics are

$$\langle d(\tfrac{2}{\hat{H}} f), d(\tfrac{2}{\hat{H}} f) \rangle = \tfrac{4}{\hat{H}^2} e^{2u} |dz|^2 \quad \text{and} \quad \langle d(\tfrac{2}{H}\hat{f}), d(\tfrac{2}{H}\hat{f}) \rangle = \tfrac{4}{H^2} e^{-2u} |dz|^2$$

while, with the unit normal fields $t = s_i - \frac{2}{\hat{H}} f$ and $\hat{t} = s_i - \frac{2}{H}\hat{f}$ in that space M^3_N, their second fundamental forms become

$$\begin{array}{rcl} -\langle d(\frac{2}{\hat{H}} f), dt \rangle & = & \frac{4}{\hat{H}^2} e^{2u} (1 - \frac{1}{2} H\hat{H})\, |dz|^2 + (dz^2 + d\bar{z}^2) \\ -\langle d(\frac{2}{H}\hat{f}), d\hat{t} \rangle & = & \frac{4}{H^2} e^{-2u} (1 - \frac{1}{2} H\hat{H})\, |dz|^2 + (dz^2 + d\bar{z}^2) \end{array}$$

— showing that both surfaces have the same constant mean curvature $1 - \frac{1}{2}H\hat{H}$. As a special case, $H = 1$ and $\hat{H} = 2$, this provides the well known relation between constant mean curvature surfaces in Euclidean space $\mathbb{R}^3$ and minimal surfaces in the 3-sphere S^3.

If $H\hat{H} = 0$, one of the surfaces f_0 or $\hat{f}_0$ is a minimal surface, say $\hat{H} = 0$, while the other is homothetic to its Gauss map, say $n = H\, f_0$. Now, the surface $\frac{2}{H}\hat{f} : M^2 \to M^3_N$ lies in hyperbolic space, $\kappa = -1$, while f is the hyperbolic Gauss map (cf.[4]) of $\frac{2}{H}\hat{f}$ since $\langle N, f\rangle \equiv 0$, i.e. f takes values in the infinity boundary $N \in S^5_1$ of M^3_N. As before, the mean curvature of $\frac{2}{H}\hat{f} : M^2 \to M^3_N$ is easily calculated to be constant $= 1$. This is how Bryant's Weierstrass type representation [4] for surfaces of constant mean curvature 1 in hyperbolic 3-space H^3 can be obtained in this context: we write the differential $d\hat{f}_0 = \frac{1}{2}(i + gj)\bar{\omega}j(i + gj)$ of a minimal immersion $\hat{f}_0 : M^2 \to \mathbb{R}^3$ (and its Christoffel transform, its Gauss map $f_0 = (i + gj)i(i + gj)^{-1} : M^2 \to S^2$) in terms of a holomorphic 1-form $\omega : TM^2 \to \mathbb{C}$ and the (meromorphic) stereographic projection $g : M \to \mathbb{C}$ of its Gauss map. Then, the constant mean curvature surface $\hat{f} : M^2 \to H^3$ (and its hyperbolic Gauss map $f : M^2 \to N \simeq S^2$) are obtained by integrating the connection form[15)]

$$\Phi = \begin{pmatrix} 0 & \frac{1}{2}(i + gj)\bar{\omega}j(i + gj) \\ -2(i + gj)^{-1}dg\, j(i + gj)^{-1} & 0 \end{pmatrix}, \tag{11}$$

to the framing $(f, \hat{f}) \simeq F : M^2 \to Gl(2, \mathbb{H})$ where $dF = F\Phi$ — thus (locally) characterizing Bryant's Weierstrass type representation of surfaces of constant mean curvature 1 in hyperbolic space as Bianchi's T-transform [2] of minimal surfaces in Euclidean space. In fact, introducing the spectral parameter (9), surfaces of constant mean curvature c in hyperbolic space forms of curvature $\kappa = -c^2$ arise by "perturbation" of minimal surfaces in Euclidean 3-space (cf.[16]).

Parametrizing a minimal surface patch $\hat{f}_0$ in terms of curvature line parameters, $z = x + iy$, the above representation of $\hat{f}_0$ becomes the classical Enneper-Weierstrass representation, i.e. $\omega = \frac{dz}{g'}$. Performing a Möbius transformation on the Gauss map g (resp. f_0 — its Christoffel transform) and integrating the Enneper-Weierstrass representation again (i.e. taking the Christoffel transform of the Möbius transformed Gauss map) yields the classical Goursat transformation of the minimal surface patch. But, a closer look at the connection form (11) suggests that the Enneper-Weierstrass representation itself can be interpreted as a Goursat type transformation of a planar patch: considering $gj, \int \bar{\omega}j : M^2 \to \mathbb{C}j$ as a (highly degenerate) Christoffel pair, the corresponding minimal surface $\hat{f}_0$ is obtained as a Christoffel transformation of $f_0 = \frac{1}{1+|g|^2}[(1 - |g|^2)i + 2gj]$, the stereographic projection of gj ("the" Christoffel transform of $\int \bar{\omega}j$) into S^2. This Goursat type transformation can (obviously) be generalized to arbitrary Christoffel pairs of isothermic surfaces: if $f_0, \hat{f}_0 : M^2 \to \mathbb{H}$ form a Christoffel pair, then, for any (constant) $a \in \mathbb{H}$, the quaternionic 1-forms

15) With the ansatz $F = \begin{pmatrix} 2(\overline{x_{21}g + x_{22}})(i + gj)^{-1} & j(x_{21}i - x_{22}j) \\ 2j(\overline{x_{11}g + x_{12}})(i + gj)^{-1} & -(x_{11}i - x_{12}j) \end{pmatrix}$, the common form of Bryant's representation is obtained as $xx^* : M^2 \to H^3 \subset \{y \in Gl(2, \mathbb{C}) \mid y = y^*\} \cong \mathbb{R}^4_1$ where the scalar product on H^3 is induced by the Lorentz scalar product $|y|^2 = -\det(y)$ on $\mathbb{R}^4_1$.

$(a+\bar{f}_0)^{-1}d\bar{f}_0(a+\bar{f}_0)^{-1}$ and $(a+\bar{f}_0)d\hat{f}_0(a+\bar{f}_0)$ are closed — and consequently give rise to a new Christoffel pair.

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