Semigroup Crossed Products and Hecke Algebras Arising from Number Fields

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Abstract. Recently Bost and Connes considered a Hecke C^*-algebra arising from the ring inclusion of $\mathbb{Z}$ in $\mathbb{Q}$, and a C^*-dynamical system involving this algebra. Laca and Raeburn realized this algebra as a semigroup crossed product, and studied it using techniques they had previously developed for studying Toeplitz algebras. Here we associate Hecke algebras to general number fields, realize them as semigroup crossed products, and analyze their representations.

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Introduction

In their work on phase transitions in number theory, Bost and Connes considered the Hecke algebra $\mathcal{H}(\Gamma, \Gamma_0)$ of a particular group–subgroup pair (Γ, Γ_0), and gave a presentation of this algebra involving a unitary representation of the additive group $\mathbb{Q}/\mathbb{Z}$ and an isometric representation of the multiplicative semigroup $\mathbb{N}^*$ [3]. From this presentation, Laca and Raeburn recognized $\mathcal{H}(\Gamma, \Gamma_0)$ as a dense subalgebra of a semigroup crossed product of the form $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^*$, and then applied techniques they had previously developed for studying Toeplitz algebras to obtain information about $\mathcal{H}(\Gamma, \Gamma_0)$ and its representations [8].

The fascinating ideas of Bost and Connes raise many possibilities for fruitful interaction between number theory and operator algebras, and in particular promise to provide new and intriguing examples of dynamical systems. Here we investigate a family of semigroup crossed products similar to $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^*$, but with $\mathbb{Q}$ replaced by a finite extension K of $\mathbb{Q}$, and the subring $\mathbb{Z}$ of $\mathbb{Q}$ replaced by the ring $\mathcal{O}$ of integers in K. We construct an action α of the multiplicative semigroup of nonzero integers $\mathcal{O}^\times$ on the C^*-algebra of the additive group $K/\mathcal{O}$, and show that all the main

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results of [8] carry over to an arbitrary number field K. This has not been completely routine: in particular, to construct some of the key representations and prove our main theorem we had to look very closely at the compact dual $(K/\mathcal{O})^\frown$ of the discrete Abelian group $K/\mathcal{O}$, and our results here may be of independent interest.

The main theorem of [8], motivated by our earlier approach to uniqueness theorems for semigroups of non-unitary isometries [1, 7], is a characterization of faithful representations of the crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^*$. Thus the crossed product has several faithful realizations: on $\ell^2(\mathbb{Q}/\mathbb{Z})$, extending the regular representation of $C^*(\mathbb{Q}/\mathbb{Z})$; on $\ell^2(\mathbb{N}^*)$, extending the Toeplitz representation of $\mathbb{N}^*$; and on $\ell^2(\Gamma_0\backslash\Gamma)$, arising from the canonical representation of $\mathcal{H}(\Gamma,\Gamma_0)$ in the commutant of the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}1$. For our action α of $\mathcal{O}^\times$ by endomorphisms of $C^*(K/\mathcal{O})$, it is easy enough to construct the regular representation on $\ell^2(K/\mathcal{O})$. We shall find a group–subgroup pair $(\Gamma_K,\Gamma_{\mathcal{O}})$ whose Hecke algebra is isomorphic to our crossed product and hence gives a representation on $\ell^2(\Gamma_{\mathcal{O}}\backslash\Gamma_K)$, and, through our analysis of $(K/\mathcal{O})^\frown$, find faithful representations of $C^*(K/\mathcal{O})$ on $\ell^2(\mathcal{O}^\times)$ which are compatible with the Toeplitz representation of $\mathcal{O}^\times$. Our main theorem implies that all these realizations of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ are faithful.

We begin in §1 by constructing the action α of $\mathcal{O}^\times$ on $C^*(K/\mathcal{O})$. For $a \in \mathcal{O}^\times$, α_a is determined on generators δ_y for $C^*(K/\mathcal{O})$ by averaging in the group algebra the generators δ_x corresponding to solutions of the equation $ax = y$ in $\mathcal{O}$; thus α is almost by definition a right inverse for the action of $\mathcal{O}^\times$ induced by multiplication on $K/\mathcal{O}$. We then discuss the crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$, which is universal for covariant representations of the system $(C^*(K/\mathcal{O}),\mathcal{O}^\times,\alpha)$, and the dual action of $(K^*)^\frown$, which integrates to give a faithful expectation of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ onto $C^*(K/\mathcal{O})$. We can immediately write down several representations of the crossed product, including the regular representation on $\ell^2(K/\mathcal{O})$.

In §2 we construct the Hecke algebra realization $\mathcal{H}(\Gamma_K,\Gamma_{\mathcal{O}})$ of the crossed product, and give a presentation of this algebra similar to that given by Bost and Connes in the case $K = \mathbb{Q}$. The isomorphism of $\mathcal{H}(\Gamma_K,\Gamma_{\mathcal{O}})$ into $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ gives a natural representation of the crossed product on $\ell^2(\Gamma_{\mathcal{O}}\backslash\Gamma_K)$, which we call the Hecke representation. It is interesting to note that, by identifying a subrepresentation with the GNS-representation of a faithful state on $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$, we can see directly that the Hecke representation is faithful. This approach bypasses the appeal to the theory of groupoid C^*-algebras in [3], and our own main theorem.

Our main technical innovations are in §3, where we discuss characters of $K/\mathcal{O}$. In [3] and [8], essential use was made of the injective character $r \mapsto \exp 2\pi i r$ on $\mathbb{Q}/\mathbb{Z}$. In general there are no injective characters, and one is forced to look for a family of characters which can play the same rôle. We show that there is a nonempty set $\mathcal{X}_K$ of characters χ with two important properties: $\chi(\mathfrak{a}^{-1}/\mathcal{O}) \neq 1$ for every nontrivial ideal $\mathfrak{a}$ in $\mathcal{O}$, and $\{r \mapsto \chi(br) : b \in \mathcal{O}\}$ is dense in $(K/\mathcal{O})^\frown$. The key step in the proof that $\mathcal{X}_K \neq \emptyset$ is the construction of projections which behave as one would expect $\alpha_{\mathfrak{a}}(1)$ to behave — if we knew that the action α extended to an action of the semigroup of ideals in $\mathcal{O}$. Using the characters in $\mathcal{X}_K$, we can construct representations of the crossed product on $\ell^2(\mathcal{O}^\times)$ extending the Toeplitz representation.

The characterization of faithful representations of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ is Theorem 4.1. This theorem and its proof have a long history: the strategy is that used by Cuntz in [4], which has been streamlined over the years, and adapted to the present situation in [1, 7]. The crucial ingredient is an estimate, whose proof uses in several key places the properties of the characters in $\mathcal{X}_K$. Thus the end result is substantially deeper than its analogue in [8]; in addition, the presence of units in $\mathcal{O}^\times$, which is necessary for the construction of the action α, complicates the proof of the estimate. We finish §4 with a discussion of the various representations and their interrelations.

In our last section, we consider a field K with class number 1. Now the ring $\mathcal{O}$ is a principal ideal domain, and one can realize the semigroup of ideals in $\mathcal{O}$ as a subsemigroup S of $\mathcal{O}^\times$. There is therefore a similar dynamical system $(C^*(K/\mathcal{O}), S, \alpha)$ which does not involve units. The corresponding version of Theorem 4.1 is therefore slightly easier to prove, and is a direct generalization of the main theorem of [8].

While we were preparing the final version of this paper, we received a preprint from David Harari and Eric Leichtnam, in which they extend the original Bost-Connes analysis to more general fields K [5]. They associate a Hecke algebra to a class of fields more general than ours; however, they have used a principal ideal domain larger than the ring $\mathcal{O}$ of integers, which is principal only if K has class number 1. Berndt Brenken has recently told us that he has been looking at the Hecke algebras of more general almost normal inclusions from the point of view of semigroup crossed products.

Background

This paper is addressed primarily at operator algebraists, so general facts about C^*-algebras have been used freely. However, it is an attractive feature of the semigroup-crossed-product approach to Toeplitz algebras that it is relatively elementary: it requires only the basic theory of C^*-algebras and familiarity with the group C^*-algebras of discrete groups. Many of the results in the first two sections have purely algebraic analogues, involving the action α of the semigroup $\mathcal{O}^\times$ by *-endomorphisms of the group *-algebra $\mathbb{C}(K/\mathcal{O}) := \operatorname{span}\{\delta_x : x \in K/\mathcal{O}\}$.

Our notation concerning number fields is as follows. Throughout K will denote a finite extension of the rational numbers $\mathbb{Q}$, called a *number field*. Every number field has an associated *ring of integers* $\mathcal{O}$, consisting of the solutions in K of monic polynomials with coefficients in $\mathbb{Z}$; for example, $\mathbb{Z}$ is the ring of integers of $\mathbb{Q}$. We write $\mathcal{O}^\times$ for the multiplicative semigroup of nonzero integers, and $\mathcal{O}^*$ for the multiplicative group of units, or invertible elements, in $\mathcal{O}$. The only units in $\mathbb{Z}$ are ± 1, but this is certainly not true for general rings of integers: for example, real quadratic number fields have their group of units isomorphic to $\mathbb{Z}$. The field K can be recovered from $\mathcal{O}$ as its field of fractions: in other words, every number in K has the form a/b for some $a \in \mathcal{O}$ and $b \in \mathcal{O}^\times$.

The *norm* is a multiplicative homomorphism from ideals in $\mathcal{O}$ to $\mathbb{N}$, given by $N(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$ for an ideal $\mathfrak{a} \subseteq \mathcal{O}$. If $\mathfrak{a}$ is principally generated, so $\mathfrak{a} = a\mathcal{O}$ for some $a \in \mathcal{O}$, then this norm coincides with the absolute value of the standard number-theoretic norm $N(a)$ of the element a [11, Prop. 3.5.1]. We shall write either $N_\mathfrak{a}$ or $N(\mathfrak{a})$ to denote the norm of the ideal $\mathfrak{a}$, and for principal ideals, $N_a = |N(a)|$ will denote the norm of the ideal $a\mathcal{O}$. In §3, we shall need to use the extension of the norm to fractional ideals, but we shall discuss the key points then.

1. The semigroup dynamical system $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$

Because $\mathcal{O}$ is a subring of K, multiplication by elements of $\mathcal{O}^\times$ gives an action of the semigroup $\mathcal{O}^\times$ as endomorphisms of the additive group $K/\mathcal{O}$. The universality of the group algebra construction allows us to lift this to an action β of $\mathcal{O}^\times$ by endomorphisms of the group C^*-algebra: thus, by definition, we have $\beta_a(\delta_x) = \delta_{ax}$ for $x \in K/\mathcal{O}$, $a \in \mathcal{O}^\times$. These are endomorphisms rather than automorphisms: as the next Lemma shows, multiplication by $a \in \mathcal{O}^\times$ is not injective at the group or group-algebra level unless a is a unit.

Lemma 1.1. *If $a \in \mathcal{O}^\times$ and $y \in K/\mathcal{O}$, the equation $ax = y$ has N_a solutions in $K/\mathcal{O}$. We write $[x : ax = y]$ for the set of solutions.*

Proof. Multiplication by a induces an isomorphism of the group $[x : ax = 0] = \frac{1}{a}\mathcal{O}/\mathcal{O}$ onto $\mathcal{O}/a\mathcal{O}$, and hence $[x : ax = 0]$ is a finite set with N_a elements. If x' is one solution of $ax' = y$, then

$$[x : ax = y] = [x : ax = ax'] = [x + x' : ax = 0] = x' + [x : ax = 0], \tag{1.1}$$

which also has N_a elements. □

When the equation $ax = y$ has more than one solution in $K/\mathcal{O}$, division by a does not give a well-defined endomorphism of $K/\mathcal{O}$. Nevertheless, one can define an endomorphism of the C^*-algebra $C^*(K/\mathcal{O})$ by averaging over the set of all solutions, and this endomorphism α_a is a right inverse for β_a. It is important to realize that the construction of α_a is not possible on $K/\mathcal{O}$ itself: one must pass to the group algebra $C^*(K/\mathcal{O})$ (or $\mathbb{C}(K/\mathcal{O})$) before the averaging makes sense.

Proposition 1.2. *Let K be a number field with ring of integers $\mathcal{O}$. The formula*

$$\alpha_a(\delta_y) = \frac{1}{N_a} \sum_{[x:ax=y]} \delta_x \tag{1.2}$$

defines an action of $\mathcal{O}^\times$ by endomorphisms of $C^*(K/\mathcal{O})$. For every $a \in \mathcal{O}^\times$, $\alpha_a(1)$ is a projection, and

$$\alpha_a(1)\alpha_b(1) = \alpha_{ab}(1) \quad \textit{whenever} \quad a\mathcal{O} + b\mathcal{O} = \mathcal{O}. \tag{1.3}$$

The action α is a right inverse for the action β defined by $\beta_a : \delta_y \mapsto \delta_{ay}$, so $\beta_a \circ \alpha_a =$ id, while $\alpha_a \circ \beta_a$ is multiplication by $\alpha_a(1)$.

The action α restricts to an action of $\mathcal{O}^\times$ by *-endomorphisms of the group *-algebra $\mathbb{C}(K/\mathcal{O})$.

Proof. For $y, y' \in K/\mathcal{O}$ and $a \in \mathcal{O}^\times$,

$$\begin{aligned}
\alpha_a(\delta_y)\alpha_a(\delta_{y'}) &= \left(\frac{1}{N_a}\sum_{[x:ax=y]} \delta_x\right)\left(\frac{1}{N_a}\sum_{[x':ax'=y']} \delta_{x'}\right) \\
&= \frac{1}{N_a^2}\sum_{[x:ax=y]}\sum_{[x':ax'=y']} \delta_x\delta_{x'} = \frac{1}{N_a^2}\sum_{[x:ax=y]}\sum_{[x':ax'=y']} \delta_{x+x'} \\
&= \frac{1}{N_a}\sum_{[x'':ax''=y+y']} \delta_{x''} = \alpha_a(\delta_y\delta_{y'}),
\end{aligned}$$

where the fourth equality holds because addition induces a N_a–to–one surjective map from $[x : ax = y] \times [x' : ax' = y']$ onto $[x'' : ax'' = y + y']$.

Thus $x \mapsto \alpha_a(\delta_x)$ is a homomorphism of $K/\mathcal{O}$ into $C^*(K/\mathcal{O})$, and it clearly preserves adjoints. Hence $\alpha_a(1) = \alpha_a(\delta_0)$ is a projection in the C^*-algebra $C^*(K/\mathcal{O})$, and $x \mapsto \alpha_a(\delta_x)$ is a homomorphism of $K/\mathcal{O}$ into the unitary group of the C^*-algebra $\alpha_a(1)C^*(K/\mathcal{O})\alpha_a(1)$. The universal property of $C^*(K/\mathcal{O})$ now implies that α_a extends to a homomorphism of $C^*(K/\mathcal{O})$ into itself — that is, to an endomorphism of the C^*-algebra $C^*(K/\mathcal{O})$. It follows similarly from the universal property of $\mathbb{C}(K/\mathcal{O})$ that the same formula gives *-endomorphisms α_a of $\mathbb{C}(K/\mathcal{O})$.

Next assume $a, b \in \mathcal{O}^\times$ and $z \in K/\mathcal{O}$, and calculate

$$\begin{aligned}\alpha_a(\alpha_b(\delta_z)) &= \alpha_a\left(\frac{1}{N_b}\sum_{[y:by=z]}\delta_y\right) = \frac{1}{N_a\,N_b}\sum_{[y:by=z]}\left(\sum_{[x:ax=y]}\delta_x\right)\\ &= \frac{1}{N_{ab}}\sum_{[x:abx=z]}\delta_x = \alpha_{ab}(\delta_z),\end{aligned}$$

where the third equality holds because $N_aN_b = N_{ab}$ and $[x : abx = z]$ is the disjoint union of the sets $[x : ax = y]$ with y ranging in $[y : by = z]$. We have now proved that α is an action by endomorphisms of $C^*(K/\mathcal{O})$, and the same calculations show that it restricts to an action on $\mathbb{C}(K/\mathcal{O})$.

To prove (1.3), multiply

$$\begin{aligned}\alpha_a(1)\alpha_b(1) &= \left(\frac{1}{N_a}\sum_{[x:ax=0]}\delta_x\right)\left(\frac{1}{N_b}\sum_{[y:by=0]}\delta_y\right)\\ &= \frac{1}{N_aN_b}\sum_{[x:ax=0]\times[y:by=0]}\delta_{x+y}\\ &= \frac{1}{N_{ab}}\sum_{[z:abz=0]}\delta_z = \alpha_{ab}(1);\end{aligned}$$

for the third equality, note that, by the Chinese Remainder Theorem, $a\mathcal{O} + b\mathcal{O} = \mathcal{O}$ implies $\mathcal{O}/ab\mathcal{O} \cong \mathcal{O}/a\mathcal{O} \times \mathcal{O}/b\mathcal{O}$, which in turn implies $\frac{1}{ab}\mathcal{O}/\mathcal{O} \cong \frac{1}{a}\mathcal{O}/\mathcal{O} \times \frac{1}{b}\mathcal{O}/\mathcal{O}$.

It is easy to check that $\beta_a(\alpha_a(\delta_y)) = \delta_y$ for any $y \in K/\mathcal{O}$. To see that $\alpha_a \circ \beta_a$ is multiplication by $\alpha_a(1)$, we compute:

$$\alpha_a(\beta_a(\delta_y)) = \frac{1}{N_a}\sum_{[x:ax=ay]}\delta_x = \frac{1}{N_a}\sum_{[x':ax'=0]}\delta_{x'+y} = \frac{1}{N_a}\left(\sum_{[x':ax'=0]}\delta_{x'}\right)\delta_y = \alpha_a(1)\delta_y,$$

where the second equality holds as in (1.1). □

Remark 1.3. Since $\beta_a \circ \alpha_a = \mathrm{id}$, α_a is injective and β_a is surjective for each $a \in \mathcal{O}^\times$. If a is a unit, $\alpha_a(1) = 1$, so $\alpha_a \circ \beta_a = \mathrm{id}$, and units act by automorphisms. Conversely, $\alpha_a(1) = 1$ only for $a \in \mathcal{O}^*$, so only units act by automorphisms. These automorphisms leave the projections $\alpha_a(1)$ fixed, because for every $a \in \mathcal{O}^\times$ and $u \in \mathcal{O}^*$, we have $\alpha_{ua}(1) = \alpha_{au}(1) = \alpha_a(\alpha_u(1)) = \alpha_a(1)$.

Definition 1.4. A *covariant representation* of the system $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$ is a pair (π, V), in which π is a unital representation of $C^*(K/\mathcal{O})$ on a Hilbert space H, and V is an isometric representation of $\mathcal{O}^\times$ on H, satisfying the covariance condition

$$\pi(\alpha_a(f)) = V_a\pi(f)V_a^* \quad \text{for } a \in \mathcal{O}^\times \text{ and } f \in C^*(K/\mathcal{O}).$$

We can use the same covariance condition to define an *algebraic covariant representation* of the system $(\mathbb{C}(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$ with values in a unital *-algebra.

This covariance condition combines with the left inverse β to give the following useful identities:

Lemma 1.5. *Suppose (π, V) is a covariant representation for $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$. If $a, b \in \mathcal{O}^\times$ and $x \in K/\mathcal{O}$, then*

1. $V_a\pi(\delta_x) = \pi(\alpha_a(\delta_x))V_a$, $\pi(\delta_x)V_a^* = V_a^*\pi(\alpha_a(\delta_x))$,
2. $\pi(\delta_x)V_a = V_a\pi(\beta_a(\delta_x))$, $V_a^*\pi(\delta_x) = \pi(\beta_a(\delta_x))V_a^*$,
3. *and if in addition $a\mathcal{O} + b\mathcal{O} = \mathcal{O}$, then $V_a^*V_b = V_bV_a^*$.*

Proof. Since $V_a^*V_a = 1$, claim (1) is immediate from covariance. Use (1) and facts about β to compute $V_a\pi(\beta_a(\delta_x)) = V_a\pi(\delta_{ax}) = \pi(\alpha_a(\delta_{ax})V_a = \pi(\alpha_a(\beta_a(\delta_x)))V_a = \pi(\alpha_a(1)\delta_x)V_a = \pi(\delta_x)\pi(\alpha_a(1))V_a = \pi(\delta_x)V_a$, since $\alpha_a(1) = V_aV_a^*$ by covariance. The second equality in (2) is shown similarly. To see (3), multiply (1.3) by V_a^* on the left and V_b on the right. ☐

Example 1.6. We construct a covariant representation (λ, L) on $\ell^2(K/\mathcal{O})$, in which λ is the left regular representation of $C^*(K/\mathcal{O})$ on $\ell^2(K/\mathcal{O})$.

The isometric representation L of the semigroup $\mathcal{O}^\times$ is defined by the formula

$$L_a\epsilon_y = \frac{1}{N_a^{1/2}} \sum_{[x:ax=y]} \epsilon_x,$$

where $\{\epsilon_y : y \in K/\mathcal{O}\}$ is the usual orthonormal basis of $\ell^2(K/\mathcal{O})$. First we need to check that these are actually isometries, and for this it suffices to show that L_a maps this orthonormal basis into orthogonal unit vectors. That they are unit vectors is an easy calculation. If $ax = y \neq y' = ax'$ in $K/\mathcal{O}$ then $x \neq x'$ in $K/\mathcal{O}$, so the sums for $L_a\epsilon_y$ and $L_a\epsilon_{y'}$ are over disjoint sets, and hence orthogonal.

The same type of calculation used to show $\alpha_a \circ \alpha_b = \alpha_{ab}$ yields $L_aL_b = L_{ab}$, and one checks easily that that $L_a^*\epsilon_x = (1/N_a^{1/2})\epsilon_{ax}$, which can then be used to compute

$$\begin{aligned} L_a\lambda(\delta_x)L_a^*\epsilon_y &= \frac{1}{N_a^{1/2}} L_a\epsilon_{ay+x} = \frac{1}{N_a} \sum_{[z:az=ay+x]} \epsilon_z = \frac{1}{N_a} \sum_{[z:a(z-y)=x]} \epsilon_z \\ &= \frac{1}{N_a} \sum_{[z':az'=x]} \epsilon_{z'+y} = \lambda(\alpha_a(\delta_x))\epsilon_y. \end{aligned}$$

Therefore the pair (λ, L) is a covariant representation of the system $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$.

Definition 1.7. Because we have just constructed a non-trivial covariant representation, we know from Proposition 2.1 of [7] that the system $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$ has a *crossed product.* This is a C^*-algebra B generated by a *universal covariant representation* (i, v) of $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$ in B: for every other covariant representation (π, V), there is a representation $\pi \times V$ of B such that $(\pi \times V) \circ i = \pi$ and $(\pi \times V) \circ v = V$. The triple (B, i, v) is unique up to isomorphism [7, Proposition 2.1]. Since the representation λ in the example is faithful, and $\lambda = (\lambda \times L) \circ i$, the homomorphism i is injective on $C^*(K/\mathcal{O})$.

We can similarly define the algebraic crossed product $(\mathbb{C}(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times, i, v)$ to be the *-algebra generated by a universal algebraic covariant representation. The construction of [7, Proposition 2.1] can be easily modified to show that there is such a representation.

LEMMA 1.8. *The vector space* $\operatorname{span}\{v_a^* i(\delta_x) v_b : x \in K/\mathcal{O}, a, b \in \mathcal{O}^\times\}$ *is a dense* **-subalgebra of* $C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times$*. We also have*

$$\operatorname{span}\{v_a^* i(\delta_x) v_b : x \in K/\mathcal{O}, a, b \in \mathcal{O}^\times\} = \operatorname{span}\{i(\delta_x) v_a^* v_b : x \in K/\mathcal{O}, a, b \in \mathcal{O}^\times\}$$

Proof. The vector space certainly contains every $i(\delta_x)$ and v_a, and is obviously closed under taking adjoints, so it is enough to to show that the product of two spanning elements is a linear combination of such elements. To prove this let $x, y \in K/\mathcal{O}$ and $a, b, c, d \in \mathcal{O}^\times$. Then, since $v_b v_c = v_c v_b$, we have

$$\begin{aligned}
(v_a^* i(\delta_x) v_b)(v_c^* i(\delta_y) v_d) &= v_a^* i(\delta_x) v_c^* (v_b v_c)(v_b v_c)^* v_b i(\delta_y) v_d \\
&= v_a^* v_c^* i(\alpha_c(\delta_x) \alpha_{bc}(1) \alpha_b(\delta_y)) v_b v_d && \text{by Lemma 1.5 (1)} \\
&= (v_a v_c)^* i(\alpha_{bc} \circ \beta_{bc}(\alpha_c(\delta_x) \alpha_b(\delta_y))) v_b v_d && \text{by Proposition 1.2} \\
&= (v_a v_c)^* i(\alpha_{bc}(\beta_b(\delta_x)(\beta_c(\delta_y)))(v_b v_d) \\
&= (v_a v_c)^* i(\alpha_{bc}(\delta_{bx} + \delta_{cy}))(v_b v_d),
\end{aligned}$$

which we can see is in the linear span of $\{v_a^* i(\delta_x) v_b : x \in K/\mathcal{O}, a, b \in \mathcal{O}^\times\}$ by considering the formula (1.2) defining α. The last equality follows from Lemma 1.5. □

Remark 1.9. The labeling of the spanning elements by the ordered triples $(v_a, i(\delta_x), v_b)$ is not one-to-one. If $bc = ad$ and $bx = dy + n + mb/a$ for $m, n \in \mathcal{O}$, then, using Lemma 1.5(2) repeatedly,

$$\begin{aligned}
v_a^* i(\delta_x) v_b &= v_a^* v_b i(\delta_{bx}) \\
&= v_a^* v_b i(\delta_{mb/a}) i(\delta_{dy}) && \text{by assumption, since } i(\delta_n) = 1 \\
&= v_a^* i(\delta_{m/a}) v_b i(\delta_{dy}) \\
&= i(\delta_{am/a}) v_a^* v_b i(\delta_{dy}) \\
&= v_c^* v_d i(\delta_{dy}) \\
&= v_c^* i(\delta_y) v_d,
\end{aligned}$$

where the fifth equality holds because $i(\delta_m) = 1$ and $v_a^* v_b = v_a^* v_c^* v_c v_b = v_c^* v_a^* v_b v_c = v_c^* v_a^* v_a v_d = v_c^* v_d$.

From the discussion of the Hecke algebra in §2 it will follow that $v_a^* i(\delta_x) v_b = v_c^* i(\delta_y) v_d$ implies $b/a = d/c$ and $bx \equiv dy \pmod{\mathcal{O} + \frac{b}{a}\mathcal{O}}$. It will also follow that the set $\{v_a^* i(\delta_x) v_b : x \in K/\mathcal{O}, a, b \in \mathcal{O}^\times\}$ is linearly independent, hence a linear basis for the dense subalgebra $\mathbb{C}(K/\mathcal{O}) \rtimes \mathcal{O}^\times$ of $C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times$.

PROPOSITION 1.10. *Let K be a number field with ring of integers $\mathcal{O}$. There is a strongly continuous action $\widehat{\alpha}$ of the compact group $\widehat{K^*}$ on $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ such that*

$$\widehat{\alpha}_\gamma\big(v_a^* i(\delta_x) v_b\big) = \gamma(a^{-1} b) v_a^* i(\delta_x) v_b$$

for all $\gamma \in \widehat{K^*}$, $a, b \in \mathcal{O}^\times$ and $x \in K/\mathcal{O}$; $\widehat{\alpha}$ is called the dual action*.*

Proof. For fixed γ, the map $w : a \mapsto \gamma(a) v_a$ gives another covariant pair (i, w), which is easily seen to be universal. Thus we can deduce from the uniqueness of the crossed product that there is an automorphism $\widehat{\alpha}_\gamma$ of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ with the required behavior on generators. The continuity of $\gamma \mapsto \widehat{\alpha}_\gamma(c)$ is easy to check when c belongs to $\operatorname{span}\{v_a^* i(\delta_x) v_b\}$, and because automorphisms of C^*-algebras are norm-preserving, this extends to $c \in C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$. □

COROLLARY 1.11. *There is a faithful positive linear map Φ of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ onto $C^*(K/\mathcal{O})$ (strictly speaking, onto its image $i(C^*(K/\mathcal{O}))$ in the crossed product) such that*

$$\Phi\big(v_a^* i(\delta_x) v_b\big) = \begin{cases} v_a^* i(\delta_x) v_a & \text{if } b = a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Define

$$\Phi(c) := \int_{\widehat{K^*}} \widehat{\alpha}_\gamma(c)\, d\gamma;$$

this gives a norm-decreasing projection of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ onto the fixed-point algebra for the action $\widehat{\alpha}$, which is faithful in the sense that $\Phi(b^*b) = 0$ only if $b = 0$. Because $\int \gamma(a^{-1}b)\, d\gamma = 0$ unless $a^{-1}b = 1$, Φ has the required form on generators. The covariance of (i, v) implies that $v_a^* i(\delta_x) v_a = i(\beta_a(\delta_x)) = i(\delta_{ax})$, so Φ does indeed have range $i(C^*(K/\mathcal{O}))$. One can check by representing $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ on Hilbert space that Φ is positive (in fact, completely positive of norm 1). □

Example 1.12. Composing the expectation Φ with the canonical trace $\tau : z \mapsto z(0)$ on $C^*(K/\mathcal{O})$ gives a state $\tau \circ \Phi$ on $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$. This state is faithful on positive elements because both τ and Φ are. Thus the GNS-representation $\pi_{\tau\circ\Phi}$ is a faithful representation of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$. (We observe that when $K = \mathbb{Q}$, $\tau \circ \Phi$ is the KMS_1 state of [3, Theorem 5], which is shown there to be a factor state of type III.)

2. THE HECKE ALGEBRA OF A NUMBER FIELD

The universal property defining the crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ can be restated as a presentation in terms of generators and relations similar to the modification in [8, Corollaries 2.9 and 2.10] of [3, Proposition 18]. To do this we need to extend the definition of covariance to say that a pair (U, V) consisting of an isometric representation V of $\mathcal{O}^\times$ and a unitary representation U of $K/\mathcal{O}$ is covariant if

$$\frac{1}{N_a} \sum_{[x : ax = y]} U(x) = V_a U(y) V_a^*, \qquad \text{for } a \in \mathcal{O}^\times \text{ and } y \in K/\mathcal{O}.$$

Since $C^*(K/\mathcal{O})$ is universal for unitary representations of $K/\mathcal{O}$, a pair (U, V) is covariant in this sense precisely when (π_U, V) is a covariant representation of the dynamical system.

PROPOSITION 2.1. *The crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ is the universal C^*-algebra generated by elements $\{u(y) : y \in K/\mathcal{O}\}$, $\{v_a : a \in \mathcal{O}^\times\}$ subject to the relations:*

1. $v_a^* v_a = 1$ *for* $a \in \mathcal{O}^\times$,
2. $v_a v_b = v_{ab}$ *for* $a, b \in \mathcal{O}^\times$,
3. $u(0) = 1,\ u(x)^* = u(-x),\ u(x)u(y) = u(x + y)$ *for* $x, y \in K/\mathcal{O}$, *and*
4. $\displaystyle\frac{1}{N_a} \sum_{[x : ax = y]} u(x) = v_a u(y) v_a^*$, *for* $a \in \mathcal{O}^\times$ *and* $x, y \in K/\mathcal{O}$.

Similarly, the algebraic crossed product $\mathbb{C}(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ *is the universal involutive algebra generated by such elements and relations.*

Proof. Relations (1) and (2) say that v is an isometric representation of $\mathcal{O}^\times$, (3) says that u is a unitary representation of $K/\mathcal{O}$, and (4) is the covariance condition. Clearly, a universal representation of the above relations is a universal covariant pair for the system $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$, and vice versa. $\square$

In Example 1.6 we gave a concrete representation of these relations. In this section we obtain another, by real-Ising the crossed product as a Hecke algebra, and using the regular representation of this Hecke algebra.

Recall that a subgroup Γ_0 of a group Γ is almost normal if the orbits for the left action of Γ_0 on the right coset space Γ/Γ_0 are finite. Consider the subgroup

$$\Gamma_{\mathcal{O}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathcal{O} \right\} \quad \text{of}$$

$$\Gamma_K = \left\{ \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} : x, y \in K, x \neq 0 \right\}.$$

LEMMA 2.2. $\Gamma_{\mathcal{O}}$ *is an almost normal subgroup of* Γ_K.

Proof. The right coset of $\gamma = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \in \Gamma_K$ is $\gamma\Gamma_{\mathcal{O}} = \begin{pmatrix} 1 & y+\mathcal{O} \\ 0 & x \end{pmatrix}$, so

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \gamma\Gamma_{\mathcal{O}} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y+\mathcal{O} \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & ax+y+\mathcal{O} \\ 0 & x \end{pmatrix}.$$

Thus the orbit has as many points as there are classes of $ax+y$ modulo $\mathcal{O}$. If $x = b/c$ with $b, c \in \mathcal{O}$, then $a \equiv a' \pmod{c}$ implies $ax + y \equiv a'x + y \pmod{\mathcal{O}}$, so there are at most N_c points in the orbit. $\square$

The generalized Hecke algebra $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ is defined in [3, §1] as a convolution *-algebra of $\Gamma_{\mathcal{O}}$-biinvariant functions on Γ_K. As a complex vector space, $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ is the space of functions $f : \Gamma_K \to \mathbb{C}$ which are constant on double cosets, so $f(\gamma_0\gamma\gamma_0') = f(\gamma)$ for $\gamma_0, \gamma_0' \in \Gamma_{\mathcal{O}}$ and $\gamma \in \Gamma_K$, and which are supported on finitely many of these double cosets. The convolution product is

$$(f * g)(\gamma) \quad = \sum_{\gamma_1 \in \Gamma_{\mathcal{O}} \backslash \Gamma_K} f(\gamma\gamma_1^{-1}) g(\gamma_1),$$

where the sum is over left-cosets, and the involution is $f^*(\gamma) = \overline{f(\gamma^{-1})}$. With these operations, $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ is a unital *-algebra.

It is convenient to think of $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ as the linear span of characteristic functions of double cosets, indicated by square brackets, with the multiplication rule:

$$\begin{aligned} [\Gamma_{\mathcal{O}}\gamma_1\Gamma_{\mathcal{O}}] * [\Gamma_{\mathcal{O}}\gamma_2\Gamma_{\mathcal{O}}](\gamma) &= \sum_{\gamma' \in \Gamma_{\mathcal{O}} \backslash \Gamma_K} [\Gamma_{\mathcal{O}}\gamma_1\Gamma_{\mathcal{O}}](\gamma\gamma'^{-1})[\Gamma_{\mathcal{O}}\gamma_2\Gamma_{\mathcal{O}}](\gamma') \qquad (2.1) \\ &= \#\,\mathrm{LC}\left\{ (\Gamma_{\mathcal{O}}\gamma_1^{-1}\Gamma_{\mathcal{O}})\gamma \cap (\Gamma_{\mathcal{O}}\gamma_2\Gamma_{\mathcal{O}}) \right\}, \end{aligned}$$

where the sum is taken over representatives γ' of the left cosets $\Gamma_{\mathcal{O}} \backslash \Gamma_K$, and $\#\,\mathrm{LC}$ counts the number of left cosets in a left-invariant subset of Γ_K. The last equality holds because the term of the sum corresponding to a left coset γ' is 0 unless

$\gamma{\gamma'}^{-1} \in \Gamma_{\mathcal{O}}\gamma_1\Gamma_{\mathcal{O}}$ and $\gamma' \in \Gamma_{\mathcal{O}}\gamma_2\Gamma_{\mathcal{O}}$, in which case it is 1. Involution is determined by conjugate-linearity and $[\Gamma_{\mathcal{O}}\gamma\Gamma_{\mathcal{O}}]^* = [\Gamma_{\mathcal{O}}\gamma^{-1}\Gamma_{\mathcal{O}}]$, and the unit is $[\Gamma_{\mathcal{O}}]$.

Consider the maps $\mu : \mathcal{O}^\times \to \mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ and $e : K \to \mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ defined by

$$\mu_a = \frac{1}{N_a^{1/2}}\left[\Gamma_{\mathcal{O}}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\Gamma_{\mathcal{O}}\right] \tag{2.2}$$

$$e(r) = \left[\Gamma_{\mathcal{O}}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\Gamma_{\mathcal{O}}\right]. \tag{2.3}$$

The map e factors through $K/\mathcal{O}$ because $\Gamma_{\mathcal{O}}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\Gamma_{\mathcal{O}} = \begin{pmatrix} 1 & r+\mathcal{O} \\ 0 & 1 \end{pmatrix}$, and the same notation will be used for the corresponding map of $K/\mathcal{O}$ into $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$. The following generalization of [3, Proposition 18] shows that the Hecke algebra is generated by these elements, and that they are universal generators. More precisely, it says that the pair (e, μ) is covariant and that $\pi_e \times \mu$ is a *-algebra isomorphism of $\mathbb{C}(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ onto $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$.

Theorem 2.3. *Let K be a number field with ring of integers $\mathcal{O}$. The elements μ_a and $e(x)$ defined in (2.2) and (2.3), with $a \in \mathcal{O}^\times$ and $x \in K/\mathcal{O}$, generate the Hecke algebra $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$, and satisfy the relations*

$\mathcal{H}1.$ $\mu_a^*\mu_a = 1$ *for* $a \in \mathcal{O}^\times$,
$\mathcal{H}2.$ $\mu_a\mu_b = \mu_{ab}$ *for* $a, b \in \mathcal{O}^\times$,
$\mathcal{H}3.$ $e(0) = 1$, $e(x)^* = e(-x)$ *and* $e(x)e(y) = e(x+y)$ *for* $x, y \in K/\mathcal{O}$, *and*
$\mathcal{H}4.$ $\frac{1}{N_a}\sum_{[x:ax=y]} e(x) = \mu_a e(y)\mu_a^*$, *for* $a \in \mathcal{O}^\times$ *and* $y \in K/\mathcal{O}$.

Moreover, $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ is the universal *-algebra over $\mathbb{C}$ with these generators and relations; it is spanned by the set $\{\mu_a^* e(x)\mu_b : a, b \in \mathcal{O}^\times, x \in K\}$.

Proof. To prove $(\mathcal{H}3)$, first observe that

$$\Gamma_{\mathcal{O}}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\Gamma_{\mathcal{O}} = \Gamma_{\mathcal{O}}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\Gamma_{\mathcal{O}} = \begin{pmatrix} 1 & r+\mathcal{O} \\ 0 & 1 \end{pmatrix},$$

so for these elements, left cosets, right cosets and double cosets coincide. Let $r, s \in K$, $\gamma = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \in \Gamma_K$, and compute as in (2.1):

$$\begin{aligned} e(r)e(s)(\gamma) &= \left[\begin{pmatrix} 1 & r+\mathcal{O} \\ 0 & 1 \end{pmatrix}\right] * \left[\begin{pmatrix} 1 & s+\mathcal{O} \\ 0 & 1 \end{pmatrix}\right](\gamma) \\ &= \#\,\mathrm{LC}\left\{\begin{pmatrix} 1 & -r+\mathcal{O} \\ 0 & 1 \end{pmatrix}\gamma \cap \begin{pmatrix} 1 & s+\mathcal{O} \\ 0 & 1 \end{pmatrix}\right\} \\ &= \#\,\mathrm{LC}\left\{\begin{pmatrix} 1 & y - rx + x\mathcal{O} \\ 0 & x \end{pmatrix} \cap \begin{pmatrix} 1 & s+\mathcal{O} \\ 0 & 1 \end{pmatrix}\right\} \\ &= \begin{cases} 1 & \text{if } x = 1 \text{ and } y \equiv r+s \pmod{\mathcal{O}} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

because if $x = 1$ and $y - r \equiv s \pmod{\mathcal{O}}$, the intersection is the (single) left coset $\begin{pmatrix} 1 & s+\mathcal{O} \\ 0 & 1 \end{pmatrix}$. Thus $e(r)e(s) = e(r+s)$. The remaining identities are easily verified.

To see $(\mathcal{H}1)$ and $(\mathcal{H}2)$, notice that $\Gamma_{\mathcal{O}}\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}\Gamma_{\mathcal{O}} = \begin{pmatrix}1 & \mathcal{O}\\ 0 & a\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}\Gamma_{\mathcal{O}}$, so the support of μ_a is a right coset, and the support of μ_a^* is the left coset $\Gamma_{\mathcal{O}}\begin{pmatrix}1 & \frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}$. Thus, for $\gamma = \begin{pmatrix}1 & y\\ 0 & x\end{pmatrix}$, we have

$$\begin{aligned}
\mu_a^*\mu_a(\gamma) &= \frac{1}{N_a}\left[\begin{pmatrix}1 & \frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}\right] * \left[\begin{pmatrix}1 & \mathcal{O}\\ 0 & a\end{pmatrix}\right](\gamma)\\
&= \frac{1}{N_a}\#\,\mathrm{LC}\left\{\begin{pmatrix}1 & \mathcal{O}\\ 0 & a\end{pmatrix}\gamma \cap \begin{pmatrix}1 & \mathcal{O}\\ 0 & a\end{pmatrix}\right\}\\
&= \frac{1}{N_a}\#\,\mathrm{LC}\left\{\begin{pmatrix}1 & y+x\mathcal{O}\\ 0 & ax\end{pmatrix} \cap \begin{pmatrix}1 & \mathcal{O}\\ 0 & a\end{pmatrix}\right\}\\
&= \begin{cases}1 & \text{if } x = 1 \text{ and } y \in \mathcal{O}\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

because if $x = 1$ and $y \in \mathcal{O}$ the intersection $\begin{pmatrix}1 & \mathcal{O}\\ 0 & a\end{pmatrix}$ contains exactly N_a left cosets. This proves $\mu_a^*\mu_a = \left[\begin{pmatrix}1 & \mathcal{O}\\ 0 & 1\end{pmatrix}\right] = [\Gamma_{\mathcal{O}}] = 1$. A similar computation proves $(\mathcal{H}2)$.

Before proving the covariance condition $(\mathcal{H}4)$, we compute $\mu_a e(r)$:

$$\begin{aligned}
\mu_a e(r)(\gamma) &= \frac{1}{N_a^{1/2}}\#\,\mathrm{LC}\left\{\begin{pmatrix}1 & \frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}\gamma \cap \begin{pmatrix}1 & r+\mathcal{O}\\ 0 & 1\end{pmatrix}\right\}\\
&= \frac{1}{N_a^{1/2}}\#\,\mathrm{LC}\left\{\begin{pmatrix}1 & y+\frac{x}{a}\mathcal{O}\\ 0 & \frac{1}{a}x\end{pmatrix} \cap \begin{pmatrix}1 & r+\mathcal{O}\\ 0 & 1\end{pmatrix}\right\}\\
&= \begin{cases}1/N_a^{1/2} & \text{if } x = a \text{ and } y \equiv r \pmod{\mathcal{O}}\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Thus $\mu_a e(r) = \frac{1}{N_a^{1/2}}\left[\begin{pmatrix}1 & r+\mathcal{O}\\ 0 & a\end{pmatrix}\right]$ and

$$\begin{aligned}
\mu_a e(r)\mu_a^*(\gamma) &= \frac{1}{N_a}\left[\begin{pmatrix}1 & r+\mathcal{O}\\ 0 & a\end{pmatrix}\right] * \left[\begin{pmatrix}1 & \frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}\right](\gamma)\\
&= \frac{1}{N_a}\#\,\mathrm{LC}\left\{\begin{pmatrix}1 & -\frac{r}{a}+\frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}\gamma \cap \begin{pmatrix}1 & \frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}\right\}\\
&= \frac{1}{N_a}\#\,\mathrm{LC}\left\{\begin{pmatrix}1 & y-\frac{rx}{a}+\frac{x}{a}\mathcal{O}\\ 0 & \frac{x}{a}\end{pmatrix} \cap \begin{pmatrix}1 & \frac{1}{a}\mathcal{O}\\ 0 & \frac{1}{a}\end{pmatrix}\right\}\\
&= \begin{cases}1/N_a & \text{if } x = 1 \text{ and } y - r/a \in \frac{1}{a}\mathcal{O}\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

This gives

$$\mu_a e(r)\mu_a^* = \frac{1}{N_a}\left[\begin{pmatrix}1 & \frac{1}{a}(r+\mathcal{O})\\ 0 & 1\end{pmatrix}\right],$$

which implies ($\mathcal{H}4$) because the right-hand-side is the sum of N_a characteristic functions of double cosets, one for each class in $r/a + (1/a)\mathcal{O} \pmod{\mathcal{O}}$; in other words,

$$\mu_a e(r)\mu_a^* = \frac{1}{N_a}\sum_{[x:ax=r]}\left[\begin{pmatrix} 1 & x+\mathcal{O} \\ 0 & 1 \end{pmatrix}\right] = \frac{1}{N_a}\sum_{[x:ax=r]} e(x) = \pi_e(\alpha_a(\delta_r)).$$

Now that we have verified ($\mathcal{H}1$)–($\mathcal{H}4$), the universal property of the algebraic crossed product gives a *-algebra homomorphism $\pi_e \times \mu$ of $\mathbb{C}(K/\mathcal{O}) \rtimes \mathcal{O}^\times$ into the Hecke algebra $\mathcal{H}(\Gamma_K, \Gamma_\mathcal{O})$, and it only remains to prove that $\pi_e \times \mu$ is one-to-one and onto.

Consider a single monomial $\mu_a^* e(r)\mu_b$. A computation similar to the one above gives

$$\mu_a^* e(r)(\gamma) = \frac{1}{N_a^{1/2}}\left[\begin{pmatrix} 1 & r+\frac{1}{a}\mathcal{O} \\ 0 & \frac{1}{a} \end{pmatrix}\right],$$

and further calculation shows

$$\mu_a^* e(r)\mu_b(\gamma) = \frac{1}{N_{ab}^{1/2}}\#\,\mathrm{LC}\left\{\begin{pmatrix} 1 & y-rb+\frac{b}{a}\mathcal{O} \\ 0 & ax \end{pmatrix} \cap \begin{pmatrix} 1 & \mathcal{O} \\ 0 & b \end{pmatrix}\right\}.$$

Thus we must have $x = b/a$ and $y \in rb + \frac{b}{a}\mathcal{O} + \mathcal{O}$. Since $\begin{pmatrix} 1 & \mathcal{O} \\ 0 & b \end{pmatrix}$ is *not* a (single) left coset, we must count carefully to find the number of left cosets in this intersection. We notice, first, that $ab\mathcal{O} \subseteq b\mathcal{O} \cap a\mathcal{O} \subseteq a\mathcal{O}$, $\frac{b}{a}\mathcal{O} \cap \mathcal{O}$ is an ideal in $\mathcal{O}$ and $(\frac{b}{a}\mathcal{O} \cap \mathcal{O})/b\mathcal{O} \cong (b\mathcal{O} \cap a\mathcal{O})/ab\mathcal{O}$, and, second, that $a\mathcal{O}/(b\mathcal{O} \cap a\mathcal{O}) \cong \mathcal{O}/(\frac{b}{a}\mathcal{O} \cap \mathcal{O})$, so that $|a\mathcal{O}/(b\mathcal{O} \cap a\mathcal{O})| = N(\frac{b}{a}\mathcal{O} \cap \mathcal{O})$. From the isomorphism theorems we have

$$|a\mathcal{O}/(b\mathcal{O} \cap a\mathcal{O})|\,|(b\mathcal{O} \cap a\mathcal{O})/ab\mathcal{O}| = |a\mathcal{O}/ab\mathcal{O}| = |\mathcal{O}/b\mathcal{O}| = |N(b)| = N_b,$$

and from the multiplicativity of the norm, we deduce that the number of left cosets is $N_b/N(\frac{b}{a}\mathcal{O} \cap \mathcal{O})$. We divide by $N_{ab}^{1/2}$ and manipulate to get

$$\mu_a^* e(r)\mu_b = \frac{N(\frac{b}{a})^{1/2}}{N(\frac{b}{a}\mathcal{O} \cap \mathcal{O})}\left[\begin{pmatrix} 1 & rb+\frac{b}{a}\mathcal{O}+\mathcal{O} \\ 0 & \frac{b}{a} \end{pmatrix}\right].$$

The support of the right hand side is a single double-coset. To see this, multiply one of its elements on the left and on the right by $\Gamma_\mathcal{O}$ to get

$$\begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & rb \\ 0 & \frac{b}{a} \end{pmatrix}\begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & rb+\frac{b}{a}\mathcal{O}+\mathcal{O} \\ 0 & \frac{b}{a} \end{pmatrix}.$$

Since every double coset has this form, and since $N(\frac{b}{a})^{1/2} \neq 0$, the linear span of the elements $\mu_a^* e(r)\mu_b$ is all of $\mathcal{H}(\Gamma_K, \Gamma_\mathcal{O})$. Moreover, if two such elements $\mu_a^* e(x)\mu_b$ and $\mu_c^* e(y)\mu_d$ do not have disjoint support, they are supported on the same double coset, in which case $b/a = d/c$ and $\mu_a^* e(x)\mu_b = \mu_c^* e(y)\mu_d$. Thus the set $\{\mu_a^* e(x)\mu_b : a, b \in \mathcal{O}^\times\ x \in K/\mathcal{O}\}$ is linearly independent, because distinct elements have disjoint support.

Since the representation $\pi_e \times \mu$ maps $\{v_a^* u(x) v_b : x \in K/\mathcal{O}, a, b \in \mathcal{O}^\times\}$ injectively onto a linear basis for the Hecke algebra, it follows that $\{v_a^* u(x) v_b : x \in K/\mathcal{O}$ and $a, b \in \mathcal{O}^\times\}$ is a linear basis for the algebraic crossed product and that

$$\pi_e \times \mu : \mathbb{C}(K/\mathcal{O}) \rtimes \mathcal{O}^\times \to \mathcal{H}(\Gamma_K, \Gamma_\mathcal{O})$$

is a *-algebra isomorphism. The result now follows from Proposition 2.1. □

The Hecke algebra $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ acts as convolution operators on the Hilbert space $\ell^2(\Gamma_{\mathcal{O}} \backslash \Gamma_K)$, and then the Hecke C^*-algebra $C^*(\Gamma_K, \Gamma_{\mathcal{O}})$ is by definition the closure of $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ in the operator norm, [3, Proposition 3], [2]. Thus, the generators $e(r)$ and μ_a, viewed as unitaries and isometries on $\ell^2(\Gamma_{\mathcal{O}} \backslash \Gamma_K)$, give a covariant representation (π_e, μ) of $(C^*(K/\mathcal{O}), u, v)$ such that $C^*(\Gamma_K, \Gamma_{\mathcal{O}}) = (\pi_e \times \mu)(C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times)$. It will follow from our main theorem in §4 that this *Hecke representation* is faithful; i.e. that the Hecke C^*-algebra is the universal C^*-algebra of the relations $(\mathcal{H}1)$–$(\mathcal{H}4)$.

We can also establish directly that the Hecke representation is faithful by embedding the faithful representation of Example 1.12 as a subrepresentation. Indeed, the subspace of $\ell^2(\Gamma_{\mathcal{O}} \backslash \Gamma_K)$ consisting of biinvariant functions is invariant under the Hecke representation (π_e, μ), and the corresponding subrepresentation turns out to be the GNS-representation of the state $\tau \circ \Phi$.

Proposition 2.4. *The representation of the Hecke algebra as convolution operators on $\ell^2(\Gamma_{\mathcal{O}} \backslash \Gamma_K / \Gamma_{\mathcal{O}})$ is unitarily equivalent to the GNS-representation of $\tau \circ \Phi$.*

Proof. By uniqueness of the GNS-representation, it is enough to show that the vector $[\Gamma_{\mathcal{O}}] \in \ell^2(\Gamma_{\mathcal{O}} \backslash \Gamma_K / \Gamma_{\mathcal{O}})$ is cyclic for the left convolution action of $\mathcal{H}(\Gamma_K, \Gamma_{\mathcal{O}})$ and that the corresponding vector state $\omega_{\Gamma_{\mathcal{O}}}$ is equal to $\omega \circ \Phi$. Since $[\Gamma_{\mathcal{O}}]$ is an identity for convolution, its cyclic component contains every biinvariant function supported on finitely many double cosets; this proves that $[\Gamma_{\mathcal{O}}]$ is cyclic.

To show that $\omega_{\Gamma_{\mathcal{O}}} = \tau \circ \Phi$, notice first that, because the fixed point algebra of the dual action $\hat{\alpha}$ of $\widehat{K^\times}$ is exactly $C^*(K/\mathcal{O})$, any state ω of $C^*(K/\mathcal{O})$ has a *unique* $\hat{\alpha}$-invariant extension to $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$, namely $\omega \circ \Phi$. So it suffices to prove that the vector state $\omega_{\Gamma_{\mathcal{O}}}$ is $\hat{\alpha}$-invariant and agrees with τ on $C^*(K/\mathcal{O})$. If $a \neq b$, then the support of $\mu_a^* e(r) \mu_b [\Gamma_{\mathcal{O}}]$ is disjoint from $\Gamma_{\mathcal{O}}$, and hence $\omega_{\Gamma_{\mathcal{O}}}(\mu_a^* e(r) \mu_b) = \langle \mu_a^* e(r) \mu_b [\Gamma_{\mathcal{O}}], [\Gamma_{\mathcal{O}}], \rangle = 0$. Similarly, if $r \neq 0$ the support of $e(r)[\Gamma_{\mathcal{O}}]$ is disjoint from $[\Gamma_{\mathcal{O}}]$, and hence $\omega_{\Gamma_{\mathcal{O}}}(e(r)) = \langle e(r)[\Gamma_{\mathcal{O}}], [\Gamma_{\mathcal{O}}] \rangle = 0$. Since we trivially have $\omega_{\Gamma_{\mathcal{O}}}(e(0)) = 1$, this proves that $\omega_{\Gamma_{\mathcal{O}}}$ is $\hat{\alpha}$-invariant and agrees with τ on $C^*(K/\mathcal{O})$, as required. ☐

Corollary 2.5. *Let K be a number field with ring of integers $\mathcal{O}$. Then the Hecke representation $\pi_e \times \mu$ is faithful on $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ and the Hecke C^*-algebra $C^*(\Gamma_K, \Gamma_{\mathcal{O}})$ is the universal C^*-algebra of the relations $(\mathcal{H}1)$–$(\mathcal{H}4)$.*

3. Characters of $K/\mathcal{O}$

In [8] the character $\varkappa(r) = \exp(2\pi i r)$ gave an embedding of $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{T}$ which was essential to the characterization of faithful covariant representations. There is no such embedding in general:

Lemma 3.1. *If K is a nontrivial extension of $\mathbb{Q}$, there are no injective characters of $K/\mathcal{O}$.*

Proof. Suppose that K is an extension of degree $[K : \mathbb{Q}] = n > 1$, and choose an integer $a \in \mathbb{Z} \cap \mathcal{O}^\times$ with $a \neq \pm 1$. Then the subgroup $\frac{1}{a}\mathcal{O}/\mathcal{O}$ of $K/\mathcal{O}$ has order $N_a = a^n$ [11, 2.6(3)]. On the other hand, every $x \in \mathcal{O}$ satisfies $x = ax/a = 0$ in $\frac{1}{a}\mathcal{O}/\mathcal{O}$, so the order of $\chi(x/a)$ divides a for every character χ. Thus $\chi(\frac{1}{a}\mathcal{O}/\mathcal{O})$ is a subgroup of the a^{th}-roots of unity and χ cannot be injective. ☐

For $\chi \in (K/\mathcal{O})\widehat{}$ and $b \in \mathcal{O}$, define a character χ^b on $K/\mathcal{O}$ by $\chi^b(x) := \chi(bx)$. Our key technical Lemma says that for every number field K there exists $\chi \in (K/\mathcal{O})\widehat{}$ such that $\{\chi^b : b \in \mathcal{O}\}$ is dense in $(K/\mathcal{O})\widehat{}$ (Corollary 3.5, Lemma 3.6); these characters play the role of the injective characters of $\mathbb{Q}/\mathbb{Z}$. We begin by recording a general fact.

Lemma 3.2. *Let χ be a character on $K/\mathcal{O}$, and let $a, b \in \mathcal{O}^\times$. Then*

$$\sum_{[x:ax=0]} \chi(bx) = 0 \text{ if and only if } \chi(bx) \neq 1 \text{ for some } x \in [x : ax = 0]. \tag{3.1}$$

Proof. The set $\{\chi(bx) : ax = 0\}$ is a group of roots of unity, and hence, unless this group is trivial, its elements sum to zero. $\square$

In dealing with semigroup crossed products $A \rtimes_\alpha S$, one often needs to know that $\prod_{a\in F}(1 - \alpha_a(1))$ is nonzero for every finite set of elements F of S (see [7, Theorem 3.7], for example). In the present setting, something stronger is needed. The problem is that $\alpha_a(1)\alpha_b(1)$ is not necessarily of the form $\alpha_c(1)$ for $c \in \mathcal{O}^\times$. To get around this, we would like to make sense of $\alpha_{\mathfrak{a}}(1)$ for ideals $\mathfrak{a}$ in $\mathcal{O}$, in such a way that $\alpha_a(1)\alpha_b(1) = \alpha_{\mathfrak{a}}(1)$ with $\mathfrak{a}$ the not-necessarily-principal ideal generated by a and b. The ideals in $\mathcal{O}$ form a semigroup including $\mathcal{O}^\times/\mathcal{O}^*$ as the subsemigroup of principal ideals, but we have been unable to find a suitable action α of this semigroup on $C^*(K/\mathcal{O})$. However, we can define projections $P_{\mathfrak{a}}$ which have the properties we require of $\alpha_{\mathfrak{a}}(1)$. Once we have established these properties in Proposition 3.4, we can show the existence of the required characters on $K/\mathcal{O}$ (Corollary 3.5, Lemma 3.6).

We need some basic facts about *fractional ideals*. A fractional ideal $\mathfrak{f}$ of a number field K is a nonzero finitely-generated $\mathcal{O}$-submodule of K such that $d\mathfrak{f} \subset \mathcal{O}$ for some $d \in \mathcal{O}^\times$. Ideals in $\mathcal{O}$ are certainly fractional ideals, with $d = 1$; these are called *integral ideals* when it is necessary to distinguish them. Products and inverses of fractional ideals are defined by

$$\mathfrak{f}\mathfrak{g} = \{\sum_{i=1}^n f_i g_i : f_i \in \mathfrak{f}, g_i \in \mathfrak{g}\}$$
$$\mathfrak{f}^{-1} = \{x \in K : x\mathfrak{f} \subset \mathcal{O}\},$$

and are fractional ideals too. Since the ring of integers $\mathcal{O}$ is a Dedekind domain, these operations make the set of fractional ideals into a multiplicative group $\mathcal{I}_K$ with identity element the ideal $\mathcal{O}$; moreover, every element in $\mathcal{I}_K$ can be factored uniquely into a product of integer powers of prime ideals in $\mathcal{O}$. Hence $\mathcal{I}_K$ is a free Abelian group with the set $\mathcal{P}$ of prime ideals as generators [11, Theorem 3.4.3].

The intersection $\mathfrak{f} \cap \mathfrak{g}$ of two fractional ideals, which is sometimes denoted $[\mathfrak{f}, \mathfrak{g}]$, is a greatest lower bound in terms of ideal inclusion; similarly, $\mathfrak{f} + \mathfrak{g}$, which is sometimes denoted $(\mathfrak{f}, \mathfrak{g})$, is the least upper bound. The notation of lcm and gcd is meaningful; if $\mathfrak{f}$ and $\mathfrak{g}$ are two fractional ideals with factorizations

$$\mathfrak{f} = \prod_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}^{n_{\mathfrak{p}}(\mathfrak{f})} \quad \text{and} \quad \mathfrak{g} = \prod_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}^{n_{\mathfrak{p}}(\mathfrak{g})},$$

then

$$[\mathfrak{f}, \mathfrak{g}] = \mathfrak{f} \cap \mathfrak{g} = \prod_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}^{\max(n_{\mathfrak{p}}(\mathfrak{f}), n_{\mathfrak{p}}(\mathfrak{g}))},$$

and

$$(\mathfrak{f}, \mathfrak{g}) = \mathfrak{f} + \mathfrak{g} = \prod_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}^{\min(n_{\mathfrak{p}}(\mathfrak{f}), n_{\mathfrak{p}}(\mathfrak{g}))}.$$

Notice that with these factorizations, if $\mathfrak{f}$ is integral, all the exponents $n_\mathfrak{p}$ are nonnegative, and if $\mathfrak{f}$ is the inverse of an integral ideal, $n_\mathfrak{p} \leq 0$ for all $\mathfrak{p}$. Thus any fractional ideal can be written as $\mathfrak{f} = \frac{\mathfrak{a}}{\mathfrak{b}}$, with $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}$, and we can define the norm of a fractional ideal by $N(\mathfrak{f}) = N(\mathfrak{a})/N(\mathfrak{b})$ [6, pp. 17,24]. However, if $\mathfrak{f}$ is not integral this norm no longer represents a cardinality.

If $\mathfrak{a}$ is an integral ideal, then $\mathfrak{a}^{-1}$ contains $\mathcal{O}$. Let $d \in \mathcal{O}$ be an integer such that $d\mathfrak{a}^{-1} \subseteq \mathcal{O}$. Since we trivially have $d\mathcal{O} \subseteq d\mathfrak{a}^{-1}$, the isomorphism theorems give

$$|\mathcal{O}/d\mathcal{O}| = \left|\mathcal{O}/d\mathfrak{a}^{-1}\right| \left|d\mathfrak{a}^{-1}/d\mathcal{O}\right|;$$

since $d\mathfrak{a}^{-1}/d\mathcal{O} \cong \mathfrak{a}^{-1}/\mathcal{O}$, we deduce that

$$\left|\mathfrak{a}^{-1}/\mathcal{O}\right| = \left|d\mathfrak{a}^{-1}/d\mathcal{O}\right| = \frac{N_d}{N(d\mathfrak{a}^{-1})} = N(\mathfrak{a}).$$

Lemma 3.3. *Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are integral ideals in $\mathcal{O}$. Then*

$$0 \to (\mathfrak{a}+\mathfrak{b})^{-1}/\mathcal{O} \xrightarrow[x\mapsto(x,-x)]{} \mathfrak{a}^{-1}/\mathcal{O} \times \mathfrak{b}^{-1}/\mathcal{O} \xrightarrow[(x,y)\mapsto x+y]{} (\mathfrak{a}\cap\mathfrak{b})^{-1}/\mathcal{O} \to 0$$

is an exact sequence of finite Abelian groups.

Proof. From the factorization into prime ideals it is easy to see that $\mathfrak{a}^{-1} + \mathfrak{b}^{-1} = (\mathfrak{a} \cap \mathfrak{b})^{-1}$ and $\mathfrak{a}^{-1} \cap \mathfrak{b}^{-1} = (\mathfrak{a} + \mathfrak{b})^{-1}$. Hence addition gives a natural surjective homomorphism $(x, y) \in \mathfrak{a}^{-1} \times \mathfrak{b}^{-1} \mapsto x + y \in (\mathfrak{a} \cap \mathfrak{b})^{-1}$ with kernel $\{(x, -x) : x \in (\mathfrak{a} + \mathfrak{b})^{-1}\}$. Taking quotients by $\mathcal{O}$ gives the sequence. $\square$

We are now ready to define the projections $P_\mathfrak{a}$ in $C^*(K/\mathcal{O})$.

Proposition 3.4. *For each integral ideal $\mathfrak{a}$ in $\mathcal{O}$ let*

$$P_\mathfrak{a} = \frac{1}{N(\mathfrak{a})} \sum_{x\in\mathfrak{a}^{-1}/\mathcal{O}} \delta_x, \tag{3.2}$$

where the sum is taken over any set of representatives of $\mathfrak{a}^{-1}/\mathcal{O}$. Then

(i) $P_{(a)} = \alpha_a(1)$ *for every $a \in \mathcal{O}^\times$,*
(ii) $P_\mathfrak{a}$ *is a projection for every $\mathfrak{a}$,*
(iii) $P_\mathfrak{a} \geq P_\mathfrak{b}$ *whenever $\mathfrak{a}|\mathfrak{b}$ (i.e. whenever $\mathfrak{b} \subset \mathfrak{a}$),*

and, for every finite collection $\{\mathfrak{a}_i\}_{1\leq i\leq n}$ of integral ideals,

(iv) $\prod_i P_{\mathfrak{a}_i} = P_{\cap_i \mathfrak{a}_i}$, *and*
(v) $\prod_i (1 - P_{\mathfrak{a}_i}) \neq 0$ *whenever $\mathfrak{a}_i \neq \mathcal{O}$ for $1 \leq i \leq n$.*

Proof. Claim (i) is verified directly from the definition. Since multiplication and intersection are associative operations, to prove (iv) it is enough to consider two ideals $\mathfrak{a}$ and $\mathfrak{b}$:

$$\begin{aligned} P_\mathfrak{a} P_\mathfrak{b} &= \frac{1}{N(\mathfrak{a})N(\mathfrak{b})} \sum_{x\in\mathfrak{a}^{-1}/\mathcal{O}} \sum_{y\in\mathfrak{b}^{-1}/\mathcal{O}} \delta_{x+y} \\ &= \frac{N(\mathfrak{a}+\mathfrak{b})}{N(\mathfrak{a})N(\mathfrak{b})} \sum_{z\in(\mathfrak{a}\cap\mathfrak{b})^{-1}/\mathcal{O}} \delta_z \\ &= \frac{1}{N(\mathfrak{a}\cap\mathfrak{b})} \sum_{z\in(\mathfrak{a}\cap\mathfrak{b})^{-1}/\mathcal{O}} \delta_z \\ &= P_{\mathfrak{a}\cap\mathfrak{b}}, \end{aligned}$$

where the second equality holds by Lemma 3.3. Since $\mathfrak{a}^{-1}/\mathcal{O}$ contains $-x$ whenever it contains x, $P_\mathfrak{a}$ is self adjoint, and setting $\mathfrak{a}=\mathfrak{b}$ in (iv) gives $P_\mathfrak{a}^2 = P_\mathfrak{a}$, proving (ii). If $\mathfrak{b}|\mathfrak{a}$ then $\mathfrak{a}\cap\mathfrak{b}=\mathfrak{b}$, so (iii) follows from (iv).

It remains to prove (v). Observe first that replacing each $\mathfrak{a}_i$ by one of its prime factors gives a smaller projection because of (iii); repeated primes are irrelevant because the $P_{\mathfrak{a}_i}$ are idempotents. Thus it suffices to prove that $\prod_{\mathfrak{a}\in F}(1-P_\mathfrak{a})\neq 0$ for any finite set F of distinct prime ideals. Multiplying out and using (iv) gives

$$\prod_{\mathfrak{a}\in F}(1-P_\mathfrak{a}) = \sum_{A\subset F}\prod_{\mathfrak{a}\in A}(-P_\mathfrak{a}) = \sum_{A\subset F}(-1)^{|A|}P_{\cap A},$$

where $\cap A$ indicates the intersection of all the members of A, which in this case equals their product because they are all prime. This projection is in $\mathbb{C}(K/\mathcal{O})$, and, viewing it as a function on $K/\mathcal{O}$, it makes sense to evaluate it at $0\in K/\mathcal{O}$:

$$\begin{aligned}\prod_{\mathfrak{a}\in F}(1-P_\mathfrak{a})(0) &= \sum_{A\subset F}(-1)^{|A|}P_{\cap A}(0)\\ &= \sum_{A\subset F}(-1)^{|A|}\frac{1}{N(\cap A)}\sum_{x\in(\cap A)^{-1}/\mathcal{O}}\delta_x(0)\\ &= \sum_{A\subset F}\prod_{\mathfrak{a}\in A}(-\frac{1}{N(\mathfrak{a})}), \qquad \text{because } N(\cap A)=\prod_{\mathfrak{a}\in A}N(\mathfrak{a}),\\ &= \prod_{\mathfrak{a}\in F}(1-\frac{1}{N(\mathfrak{a})})\neq 0,\end{aligned}$$

because $N(\mathfrak{a})>1$ for every integral ideal $\mathfrak{a}\neq\mathcal{O}$. □

Corollary 3.5. *Let $f\mapsto\hat{f}$ denote the Fourier transform isomorphism of $C^*(K/\mathcal{O})$ onto $C(\widehat{K/\mathcal{O}})$. Then*

$$\mathcal{X}_K := \bigcap\{\operatorname{supp}\widehat{1-P_\mathfrak{a}} : \mathfrak{a} \textit{ is a nontrivial ideal in } \mathcal{O}\}$$

is a nonempty compact G_δ subset of $\widehat{K/\mathcal{O}}$.

Proof. The space $\widehat{K/\mathcal{O}}$ is compact, and the family $\{\operatorname{supp}(1-P_\mathfrak{a})\hat{}\}$ has the finite intersection property by Proposition 3.4(v). □

The following lemma shows that the characters in $\mathcal{X}_K$ have the required properties.

Lemma 3.6. *Let $\chi\in\widehat{K/\mathcal{O}}$. Then*

1. *$\chi\in\mathcal{X}_K$ if and only if $\chi(\mathfrak{a}^{-1}/\mathcal{O})\neq\{1\}$ for every non-trivial ideal $\mathfrak{a}\subseteq\mathcal{O}$,*
2. *if $\chi\in\mathcal{X}_K, a,b\in\mathcal{O}^\times$, and $\chi(bx)=1$ for all $x\in\frac{1}{a}\mathcal{O}/\mathcal{O}$, then $a|b$, and*
3. *if $\chi\in\mathcal{X}_K$, then $\{\chi^b : b\in\mathcal{O}\}$ is dense in $\widehat{K/\mathcal{O}}$.*

Proof. Suppose $\chi\in\mathcal{X}_K$. By the definition of the set $\mathcal{X}_K$, $\hat{P}_\mathfrak{a}(\chi)\neq 1$, so it must be zero, which means $\sum_{x\in\mathfrak{a}^{-1}/\mathcal{O}}\chi(x)=0$. Equivalently, the group $\chi(\mathfrak{a}^{-1}/\mathcal{O})$ of roots of unity is non-trivial by (3.1), giving (1). To see (2), note that

$$\tfrac{1}{a}(a\mathcal{O}+b\mathcal{O}) = \tfrac{1}{a}\{ax+by : x,y\in\mathcal{O}\} = \{x+\tfrac{b}{a}y : x,y\in\mathcal{O}\}.$$

Suppose a does not divide b, and set $\mathfrak{a}^{-1} = \frac{1}{a}(a\mathcal{O} + b\mathcal{O})$: this makes sense since by dividing ideals we can compute

$$\tfrac{1}{a}\left(a\mathcal{O} + b\mathcal{O}\right) = \tfrac{1}{a}\left(\tfrac{ab\mathcal{O}}{a\mathcal{O}\cap b\mathcal{O}}\right) = (a)^{-1}\left(\tfrac{a\mathcal{O}\cap b\mathcal{O}}{ab\mathcal{O}}\right)^{-1},$$

and so $\mathfrak{a} = (a\mathcal{O}\cap b\mathcal{O})/b\mathcal{O}$ is an integral ideal. If $\chi \in \mathcal{X}_K$, then from (1) we have

$$\chi(\{\frac{by}{a} : y \in \mathcal{O}\}) = \chi(\{x + \frac{by}{a} : x, y \in \mathcal{O}\}) = \chi(\mathfrak{a}^{-1}) \neq \{1\},$$

so (2) is proved.

Let $\chi \in \mathcal{X}_K$. The map $b \mapsto \chi^b$ from $\mathcal{O}$ to the characters on $K/\mathcal{O}$ is a group homomorphism. We claim that the homomorphism $b \mapsto \chi^b|_{\frac{1}{a}\mathcal{O}/\mathcal{O}}$ has kernel $a\mathcal{O}$. We see that a is in the kernel, since $\chi^a(\frac{1}{a}\mathcal{O}) = \chi(\mathcal{O}) = \{1\}$. Suppose b is in the kernel. Then $\chi(bx) = 1$ for all $x \in \frac{1}{a}\mathcal{O}/\mathcal{O}$, so (2) implies that $a|b$; thus $b \in a\mathcal{O}$, and the claim is true. Thus we have an injective homomorphism of $\mathcal{O}/a\mathcal{O}$ into $(\frac{1}{a}\mathcal{O}/\mathcal{O})\widehat{\ }$, and since these are finite Abelian groups of the same cardinality N_a, the homomorphism must also be surjective. Thus every character on $\frac{1}{a}\mathcal{O}/\mathcal{O}$ is the restriction of some χ^b. Since $K/\mathcal{O} = \cup\{\frac{1}{a}\mathcal{O}/\mathcal{O} : a \in \mathcal{O}^\times\}$, we have

$$\widehat{K/\mathcal{O}} = \varprojlim \widehat{\tfrac{1}{a}\mathcal{O}/\mathcal{O}},$$

and we can deduce that $\{\chi^b : b \in \mathcal{O}\}$ is dense in $\widehat{K/\mathcal{O}}$. □

Remark 3.7. The referee suggested that it should also be possible to prove the existence of characters with the required properties using Fourier analysis on the adele group A of K, as in [6]. In fact, this method is used by Harari and Leichtnam [5]. The approach presented here is more elementary, and in particular bypasses the application of the strong approximation theorem.

The characters in $\mathcal{X}_K$ will play a very important rôle in the proof of our main theorem. We can also use them to construct new covariant representations of the system $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$ involving the usual Toeplitz representation T of $\mathcal{O}^\times$ on $\ell^2(\mathcal{O}^\times)$, which is defined in terms of the usual basis $\{\varepsilon_b : b \in \mathcal{O}^\times\}$ for $\ell^2(\mathcal{O}^\times)$ by $T_a(\varepsilon_b) := \varepsilon_{ab}$.

Proposition 3.8. *Suppose $\chi \in \mathcal{X}_K$. Then $\tau_\chi(x) : \varepsilon_b \mapsto \chi^b(x)\varepsilon_b$ extends to a faithful representation of $C^*(K/\mathcal{O})$ such that the pair (τ_χ, T) is covariant.*

Proof. The operator $\tau_\chi(\delta_x)$ is multiplication by the circle-valued function $b \mapsto \chi^b(x)$ on $\ell^2(\mathcal{O}^\times)$, so τ_χ is a unitary representation of $K/\mathcal{O}$; we use the same symbol for the corresponding representation of $C^*(K/\mathcal{O})$. For $f \in C^*(K/\mathcal{O})$, $\tau_\chi(f)$ is multiplication by the function $b \mapsto \widehat{f}(\chi^b)$, and since $\{\chi^b : b \in \mathcal{O}^\times\}$ is dense in $(K/\mathcal{O})\widehat{\ }$ by Lemma 3.6, τ_χ is faithful.

To check the covariance condition, fix $b \in \mathcal{O}^\times$. Compute first

$$T_a\tau_\chi(y)T_a^*\varepsilon_b = \begin{cases} T_a\tau_\chi(y)\varepsilon_{b/a} & \text{if } a|b \\ 0 & \text{if } a \not| b \end{cases} = \begin{cases} \chi((b/a)y)\varepsilon_b & \text{if } a|b \\ 0 & \text{if } a \not| b, \end{cases}$$

and then

$$\tau_\chi(\alpha_a(y))\varepsilon_b = \frac{1}{N_a}\sum_{[x:ax=y]}\tau_\chi(x)\varepsilon_b = \left(\frac{1}{N_a}\sum_{[x:ax=y]}\chi(bx)\right)\varepsilon_b.$$

Let z be a fixed element of $[z : az = y]$. Then

$$\begin{aligned}\frac{1}{N_a}\sum_{[x:ax=y]}\chi(bx) &= \frac{1}{N_a}\sum_{[x':ax'=0]}\chi(b(x'+z)) = \chi(bz)\frac{1}{N_a}\sum_{[x':ax'=0]}\chi(bx') \\ &= \begin{cases}\chi(bz) & \text{if } a|b \\ 0 & \text{if } a \not| b,\end{cases}\end{aligned}$$

by Lemma 3.6(2) and (3.1). Since $a|b$ implies $\chi(bz) = \chi((b/a)az) = \chi((b/a)y)$, covariance follows. □

4. Representations of the crossed product

In this section we prove our main theorem — the characterization of faithful representations of the crossed product — and then discuss the various specific representations we have constructed earlier.

Theorem 4.1. *Let K be a number field with ring of integers $\mathcal{O}$. A covariant representation $\pi \times V$ of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ is faithful if and only if π is faithful.*

The strategy of the proof is familiar: the crux is to show that deleting the terms with $a \neq b$ from finite sums $\sum_{a,b\in F}\pi(f)V_a^*V_b$ gives a norm-decreasing expectation of $\pi \times V(C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times)$ onto $\pi(C^*(K/\mathcal{O}))$. For this, we want a projection $Q = \pi(q)$ such that compressing by Q kills the off-diagonal terms while retaining the norm of the remaining sum of diagonal terms (see Lemma 4.3 below). The presence of invertible elements (units) in the semigroup $\mathcal{O}^\times$ makes this trickier than it was in [8], and we begin with a lemma which will help deal with units. Both the next two lemmas depend crucially on the characters constructed in the previous section.

Lemma 4.2. *Suppose $\chi \in \mathcal{X}_K$, $c \in \mathcal{O}^\times$ and H is a finite set of units in $\mathcal{O}$. Then there is a projection $q \in C^*(K/\mathcal{O})$ such that $q\alpha_u(q) = 0$ for all $u \in H$ and $\widehat{q}(\chi^c) = 1$.*

Proof. We begin by observing that the units in $\mathcal{O}$ act as automorphisms of $C^*(K/\mathcal{O})$ (the inverse of α_u is $\beta_{u^{-1}}$), and hence α induces an action of $\mathcal{O}^*$ on the spectrum $(K/\mathcal{O})^\frown$ of $C^*(K/\mathcal{O})$. Indeed, we have $u \cdot \theta(x) := \theta(\alpha_u^{-1}(x)) = \theta(ux) = \theta^u(x)$ for every θ in $(K/\mathcal{O})^\frown$. We claim that $\mathcal{O}^*$ acts freely on the set $\{\chi^b : b \in \mathcal{O}^\times\}$. To see why, suppose $u \in \mathcal{O}^*$ satisfies $u \cdot \chi^b = \chi^b$ — or, equivalently, $\chi^{ub} = \chi^b$. Then for all $x \in K/\mathcal{O}$, we have

$$1 = \chi^{ub}(x)\chi^b(x)^{-1} = \chi((u-1)bx).$$

By Lemma 3.6, this implies that every $a \in \mathcal{O}^\times$ divides $(u-1)b$, and this is only possible if $u = 1$. This justifies the claim.

The claim implies that the characters $\{u\cdot\chi^c = \chi^{uc} : u \in H\}$ are distinct elements of $(K/\mathcal{O})^\frown$. Since the discrete group $K/\mathcal{O} = \cup_a \frac{1}{a}\mathcal{O}/\mathcal{O}$ is a directed union of finite subgroups, the dual $(K/\mathcal{O})^\frown$ is a topological inverse limit of finite groups, and hence is a totally disconnected compact Hausdorff space. Thus we can find a compact neighborhood N of χ^c such that $(u \cdot N) \cap N = \emptyset$ for all $u \in H$. Its characteristic function $1_N \in C((K/\mathcal{O})^\frown)$ is the Fourier transform of a projection $q \in C^*(K/\mathcal{O})$ with the required properties. □

Recall from Lemma 1.8 that the crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ is the closed linear span of $\{i(f)v_a^*v_b : f \in \mathbb{C}(K/\mathcal{O}) \text{ and } a, b \in \mathcal{O}^\times\}$.

Lemma 4.3. *Let $\sum_{a,b\in F} i(f_{a,b})v_a^*v_b$ be a finite linear combination with $f_{a,b} \in C^*(K/\mathcal{O})$, and let $\epsilon > 0$. Then there exists a projection $q = q(\epsilon) \in C^*(K/\mathcal{O})$ such that*

$$i(q)i(f_{a,b})v_a^*v_b i(q) = 0 \qquad \text{if } a \neq b, \text{ and} \tag{4.1}$$

$$\Big\| q\Big(\sum f_{a,a}\Big)q\Big\| \geq \Big\|\sum f_{a,a}\Big\| - \epsilon. \tag{4.2}$$

Proof. Let $\chi \in \mathcal{X}_K$ and let $g = \sum f_{a,a} \in C^*(K/\mathcal{O})$. By Lemma 3.6(3) there exists $c \in \mathcal{O}^\times$ such that $|\widehat{g}(\chi^c)| \geq \|\widehat{g}\| - \epsilon$. Consider the projection

$$q_1 = \alpha_c(1)\prod_{a\not| b}(1 - \beta_b \circ \alpha_{ac}(1))\prod_{b\not| a}(1 - \beta_a \circ \alpha_{bc}(1)).$$

If $a \in F$ is not associate to $b \in F$ then either $a \not| b$ or $b \not| a$. Suppose first $b \not| a$. Then $i(q_1)i(f_{a,b})v_a^*v_b i(q_1)$ has a factor

$$\begin{aligned}
&i((\alpha_c(1) - \alpha_c(1)\beta_a(\alpha_{bc}(1))))v_a^*v_b i(\alpha_c(1)) = \\
&\quad= v_a^* i((\alpha_{ac}(1) - \alpha_{ac}(1)\alpha_a \circ \beta_a(\alpha_{bc}(1)))\alpha_{bc}(1))v_b \quad \text{by Lemma 1.5(1),}\\
&\quad= v_a^* i((\alpha_{ac}(1) - \alpha_{ac}(1)\alpha_a(1)\alpha_{bc}(1))\alpha_{bc}(1))v_b\\
&\quad= v_a^* i((\alpha_{ac}(1) - \alpha_{ac}(1)\alpha_{bc}(1))\alpha_{bc}(1))v_b\\
&\quad= 0.
\end{aligned}$$

The case $a \not| b$ reduces to this one by taking adjoints.

We now consider $H := \{u \in \mathcal{O}^* \setminus \{1\} : \text{there exists } a \in F \text{ with } ua \in F\}$. By Lemma 4.2, there is a projection q_2 such that $q_2\alpha_u(q_2) = 0$ for all $u \in H$ and $\widehat{q_2}(\chi^c) = 1$. We claim that the projection $q := q_1q_2$ has the required properties. Indeed, the calculation in the previous paragraph shows that $i(q)v_a^*v_b i(q) = 0$ when $a, b \in F$ are not associate. If a is associate to b, then $b = ua$ for some $u \in H$, and $v_a^*v_b = v_u$; now the property $q_2\alpha_u(q_2) = 0$ forces $i(q)v_a^*v_b i(q) = i(q)v_u i(q) = 0$.

By construction, χ^c is in the support of $\widehat{q_2}$, so to finish the proof of (4.2) we need to show that $\widehat{q_1}(\chi^c) = 1$. Since χ^c is always in the support of $\alpha_c(1)\widehat{}$, it suffices to prove that $(\beta_a \circ \alpha_{bc}(1))\widehat{}(\chi^c) = 0$ whenever $b \not| a$ in $\mathcal{O}^\times$.

$$\begin{aligned}
(\beta_a \circ \alpha_{bc}(1))\widehat{}(\chi^c) &= \frac{1}{N_{bc}}\sum_{[x:bcx=0]}\widehat{\beta_a(\delta_x)}(\chi^c)\\
&= \frac{1}{N_{bc}}\sum_{[x:bcx=0]}\chi(cax).
\end{aligned}$$

By Lemma 3.6(2), at least one of the summands is $\neq 1$, because bc does not divide ac. Thus the sum vanishes by (3.1). $\square$

Recall from Corollary 1.11 that we have a faithful linear map $\Phi : C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times \to C^*(K/\mathcal{O})$, constructed by averaging over the compact orbits of $(K^*)\widehat{}$.

Proposition 4.4. *Let (π, V) be covariant for $(C^*(K/\mathcal{O}), \mathcal{O}^\times, \alpha)$. If π is faithful, the map*

$$\phi : \pi(f)V_a^*V_b \mapsto \begin{cases} \pi(f) & \text{if } a = b\\ 0 & \text{if } a \neq b\end{cases}$$

extends by linearity and continuity to a projection of norm 1 *from* $C^*(\pi, V)$ *onto* $C^*(\pi)$*, such that the following diagram commutes*

$$\begin{array}{ccc} C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times & \xrightarrow{\pi \times V} & C^*(\pi, V) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} \\ C^*(K/\mathcal{O}) & \xrightarrow{\pi} & C^*(\pi). \end{array} \tag{4.3}$$

Proof. Let $\sum_{a,b \in F} \pi(f_{a,b}) V_a^* V_b$ be a linear combination of the spanning monomials and fix $\epsilon > 0$. Let q be the projection from Lemma 4.3, and take $Q := \pi(q)$. Since π is faithful, it is isometric. Thus

$$\begin{aligned} \Big\| \sum_{a,b \in F} \pi(f_{a,b}) V_a^* V_b \Big\| &\geq \Big\| Q \sum_{a,b \in F} \pi(f_{a,b}) V_a^* V_b Q \Big\| \\ &= \Big\| \sum_a Q \pi(f_{a,a}) V_a^* V_a Q \Big\| \\ &= \Big\| \sum_a q f_{a,a} q \Big\| \\ &\geq \Big\| \sum_a f_{a,a} \Big\| - \epsilon \\ &= \Big\| \sum_a \pi(f_{a,a}) \Big\| - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, this gives the existence of the contractive projection ϕ. That the diagram commutes is easily verified on the spanning set. □

Proof. [Proof of Theorem 4.1.] Since there is a covariant representation (λ, L) with λ faithful, and this representation factors through (i, v), i must be faithful. Thus if $\pi \times V$ is faithful, so is $\pi = (\pi \times V) \circ i$. For the other direction, suppose π is faithful and $\pi \times V(b) = 0$. Then $\pi(\Phi(b^* b)) = \phi(\pi \times V)(b^* b) = 0$, and the faithfulness of Φ on positive elements implies $b = 0$. □

Next we consider the various covariant representations of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$:

1. The representation $\lambda \times L$ on $\ell^2(K/\mathcal{O})$ (Example 1.6).
2. The GNS-representation associated to the state $\tau \circ \Phi$ on $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$, which is already known to be faithful (Example 1.12).
3. The Hecke representation on $\ell^2(\Gamma_{\mathcal{O}} \backslash \Gamma_K)$ (see §2).
4. The representations $\tau_\chi \times T$ from Proposition 3.8.
5. A one-dimensional representation: the trivial character on $K/\mathcal{O}$ and the trivial representation of $\mathcal{O}^\times$ on $\mathbb{C}$ form a covariant pair.

Corollary 4.5. *The representations* (1), (3) *and* (4) *of* $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ *are all faithful.*

As things stand, it is not obvious that these representations are different. In fact (λ, L) is quite different: the dual action is not unitarily implemented. Our proof of this shows more: the representations $\{\lambda \times \gamma L : \gamma \in (K^*)\widehat{\ }\}$ are a family of mutually inequivalent irreducible representations.

Proposition 4.6. *Suppose that* U *is a non-zero bounded operator on* $\ell^2(K/\mathcal{O})$*, and that there exists* $\gamma \in \widehat{K^*}$ *such that*

1. *$U\lambda_x = \lambda_x U$ for all $x \in K/\mathcal{O}$, and*
2. *$UL_a = \gamma(a)L_a U$ for all $a \in \mathcal{O}^\times$.*

Then U is a scalar multiple of 1 and $\gamma = 1$.

Proof. Let $u_x := (U\varepsilon_0|\varepsilon_x)$. Then $\sum_{x\in K/\mathcal{O}} |u_x|^2 = \|U\varepsilon_0\|^2 < \infty$. Condition (1) implies

$$\begin{aligned} (U\varepsilon_y|\varepsilon_x) &= (U\lambda_y\varepsilon_0|\varepsilon_x) = (\lambda_y U\varepsilon_0|\varepsilon_x) = (U\varepsilon_0|\lambda_y^*\varepsilon_x) && (4.4)\\ &= (U\varepsilon_0|\lambda_{-y}\varepsilon_x) = (U\varepsilon_0|\varepsilon_{x-y}) = u_{x-y}. && (4.5)\end{aligned}$$

(We think of $U \sim \sum u_y\lambda_y$ as the Fourier series of U, which by (1) belongs to the maximal Abelian algebra $\lambda(K/\mathcal{O})''$.) We claim that, for each fixed $n \in \mathbb{N} \subset \mathcal{O}$ and each $x \in K/\mathcal{O}$, we have

$$\sum_{[y:ny=x]} u_y = \gamma(n)u_x.$$

To see this, we use (2) and calculate:

$$\begin{aligned} \gamma(n)u_x &= \big(\gamma(n)U\varepsilon_0|\varepsilon_x\big) = \big(L_n^* UL_n\varepsilon_0|\varepsilon_x\big) = \big(UL_n\varepsilon_0|L_n\varepsilon_x\big)\\ &= \Big(U\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \varepsilon_{i/n}\Big)\Big|\frac{1}{\sqrt{n}}\sum_{[y:ny=x]}\varepsilon_x\Big)\\ &= \frac{1}{n}\sum_i\sum_{[y:ny=x]} u_{y-i/n}.\end{aligned}$$

Now $\{y - i/n : ny = x, 1 \le i \le n\}$ is n copies of $[y : ny = x]$, so

$$\gamma(n)u_x = \frac{1}{n}\sum_{[y:ny=x]} nu_y = \sum_{[y:ny=x]} u_y,$$

as claimed.

Now suppose that $u_x \neq 0$ for some $x \neq 0$, and fix $n \in \mathbb{N}$. Recall that the ℓ^2- and ℓ^1-norms on $\mathbb{C}^n$ are related by $\|z\|_2 \ge \|z\|_1/\sqrt{n}$. Thus the claim implies that

$$|u_x| = \Big|\sum_{[y:ny=x]} u_y\Big| \le \sum_{[y:ny=x]} |u_y| \le \sqrt{n}\Big(\sum_{[y:ny=x]} |u_y|^2\Big)^{1/2}.$$

We deduce that

$$\sum_{y\in K/\mathcal{O}} |u_y|^2 \ge \sum_{n\in\mathbb{N}}\Big(\sum_{[y:ny=x]} |u_y|^2\Big) \ge \sum_n \frac{|u_x|^2}{n} = |u_x|^2\Big(\sum_n \frac{1}{n}\Big) = \infty,$$

contradicting $\sum |u_y|^2 = \|U\varepsilon_0\|^2 < \infty$. □

COROLLARY 4.7. *The representations $\{(\lambda, \chi L) : \chi \in \widehat{K^*}\}$ are irreducible and mutually inequivalent.*

Proof. For the first assertion, take $\gamma = 1$ in the proposition, and multiply both sides by $\chi(a)$. To see that $(\lambda, \chi_1 L)$ is not equivalent to $(\lambda, \chi_2 L)$, apply the proposition with $\gamma = \chi_1^{-1}\chi_2$. □

COROLLARY 4.8. *The automorphisms in the dual action $\widehat{\alpha}$ of $\widehat{K^*}$ on $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ are not implemented by unitaries in the representation $\lambda \times L$.*

Remark 4.9. That the dual action is not implemented distinguishes the representations $\lambda \times \gamma L$ from the others in the list. For example, because the state $\omega \circ \Phi$ is invariant under the dual action $\widehat{\alpha}$, there is a unitary representation U of $(K^*)\widehat{}$ on $H_{\omega\circ\Phi}$ such that $(\pi_{\omega\circ\Phi}, U)$ is a covariant representation of $(C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times, (K^*)\widehat{}, \widehat{\alpha})$. It is also easy to check that the representation $U : (K^*)\widehat{} \to B(\ell^2(\mathcal{O}^\times))$ defined by $U_\gamma \varepsilon_a = \gamma(a)\varepsilon_a$ gives a covariant representation $(\tau \times T, U)$.

To see that the dual action is unitarily implemented in the Hecke representation, define $U : (K^*)\widehat{} \to B(\ell^2(\Gamma_{\mathcal{O}}\backslash\Gamma_K))$ by

$$U_\gamma : \left[\begin{pmatrix} 1 & y + x\mathcal{O} \\ 0 & x \end{pmatrix}\right] \mapsto \gamma(x) \left[\begin{pmatrix} 1 & y + x\mathcal{O} \\ 0 & x \end{pmatrix}\right].$$

The necessary relations $U_\gamma e(r) = e(r)U_\gamma$ and $U_\gamma \mu_a = \gamma(a)\mu_a U_\gamma$ follow easily by observing that

$$\operatorname{supp}\left(e(r) * \left[\begin{pmatrix} 1 & y + x\mathcal{O} \\ 0 & x \end{pmatrix}\right]\right) \subset \begin{pmatrix} 1 & * \\ 0 & x \end{pmatrix}, \text{ and}$$
$$\operatorname{supp}\left(\mu_a * \left[\begin{pmatrix} 1 & y + x\mathcal{O} \\ 0 & x \end{pmatrix}\right]\right) \subset \begin{pmatrix} 1 & * \\ 0 & ax \end{pmatrix}.$$

Remark 4.10. The representation $\lambda \times L$ is the GNS-representation corresponding to the vector state $\phi : c \mapsto (\lambda \times L(c)\varepsilon_0|\varepsilon_0)$. Since $\tau \circ \Phi = \int_{\widehat{K^*}} \phi \circ \widehat{\alpha}_\gamma \, d\gamma$, it is tempting to guess that $\pi_{\tau\circ\Phi}$ is the direct integral of the representations $\lambda \times \gamma L = (\lambda \times L) \circ \widehat{\alpha}_\gamma$. However, because each $\lambda \times \gamma L$ is irreducible, the direct integral representation on $L^2((K^*)\widehat{}, \ell^2(K/\mathcal{O}))$ has commutant $L^\infty((K^*)\widehat{})$, and is therefore type I. On the other hand, in the case $K = \mathbb{Q}$, $\tau \circ \Phi$ is the KMS$_1$-state described in [3, §1], and this is known to be a factor state of type III$_1$ [3, Theorem 5].

5. Fields of class number 1

The ideal class group of a field K is the quotient of the group F of fractional ideals by the subgroup P of principally generated ideals; it is a finite Abelian group whose cardinality is called the class number h_K of the field [11, §4.3]. The group of principal ideals is always isomorphic to $K^*/\mathcal{O}^*$, so we have an exact sequence

$$1 \to \mathcal{O}^* \to K^* \to F \to F/P \to 1$$

of Abelian groups. Since fractional ideals factor uniquely as products of prime ideals, when $h_K = |F/P| = 1$, $K^*/\mathcal{O}^*$ is the free Abelian group generated by the prime ideals. It is possible in this case to choose a multiplicative section S in $\mathcal{O}^\times$ consisting of one associate for each class in $\mathcal{O}^\times$: select an arbitrary prime generator from each prime ideal, and take S to consist of 1 and the products of the selected generators.

Throughout this section, K will be a number field with $h_K = 1$, and S will be such a subsemigroup of $\mathcal{O}^\times$. The semigroup S is lattice ordered in the sense of [10, 7], with $a \vee b$ defined to be the unique representative in S of the ideal generated generated by a and b. Restricting α to S gives another semigroup dynamical system $(C^*(K/\mathcal{O}), S, \alpha)$ associated to a number field of class number 1.

In the case of $K = \mathbb{Q}$, selecting the positive primes gives the section $\mathbb{N}^*$, and the dynamical system $(C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)$ is the one studied in [8]. In fact S is always non-canonically isomorphic to $\mathbb{N}^* \cong \oplus_{p\in\mathcal{P}}\mathbb{N}$, so in some sense the dynamical systems

$(C^*(K/\mathcal{O}), S, \alpha)$ involve different actions of the same lattice-ordered semigroup. However, the inclusion of $\mathbb{Z}$ in $\mathcal{O}$ induces a canonical inclusion of $\mathbb{N}^*$ in S, which takes each prime generator of $\mathbb{N}^*$ to the unique product in S of (the representatives in S of) its prime factors, and this is not an isomorphism unless $K = \mathbb{Q}$.

The pairs (λ, L) and (τ_χ, T) restrict to covariant representations of $(C^*(K/\mathcal{O}), S, \alpha)$ which are faithful on $C^*(K/\mathcal{O})$, so it follows from [7, Proposition 2.1] that the system has a unique crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha S$. The following version of our main theorem is a direct generalization of [8, Theorem 3.7].

THEOREM 5.1. *Suppose K is a number field with $h_K = 1$, and $(C^*(K/\mathcal{O}), S, \alpha)$ is the dynamical system constructed above. Then a representation $\pi \times V$ is faithful on $C^*(K/\mathcal{O}) \rtimes_\alpha S$ if and only if π is faithful.*

This theorem can be proved by modifying the proof of Theorem 4.1. The crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha S$ carries a dual action of $(K^*)\widehat{\ }$, and averaging over this dual action gives a faithful expectation of $C^*(K/\mathcal{O}) \rtimes_\alpha S$ onto $C^*(K/\mathcal{O})$ (as in Proposition 1.10 and Corollary 1.11). The analogue of Lemma 4.3 is easier: if $\sum_{a,b\in F} f_{a,b} v_a^* v_b$ is a finite sum in $C^*(K/\mathcal{O}) \rtimes_\alpha S$, then no two different elements of F are associates, and we can take for q the projection q_1 constructed in the first paragraph of the proof of Lemma 4.3. Now the proofs of Proposition 4.4 and Theorem 4.1 carry over verbatim, giving Theorem 5.1.

It is interesting to note that Theorem 5.1 is substantially deeper than in the special case $K = \mathbb{Q}$ [8, Theorem 3.7]; it depends crucially on the existence of characters χ such that $\{\chi^b : b \in \mathcal{O}\}$ is dense in $(K/\mathcal{O})\widehat{\ }$, which was much easier in the case of $\mathbb{Q}$ (compare Corollary 3.5 and Lemma 3.6(3) with [8, Lemma 2.5]).

Remark 5.2. The crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha S$ is the Hecke C^*-algebra $C^*(\Gamma_S, \Gamma_\mathcal{O})$ of the almost normal inclusion

$$\Gamma_\mathcal{O} = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} \subset \Gamma_S = \begin{pmatrix} 1 & K \\ 0 & SS^{-1} \end{pmatrix}.$$

To see this, note that $\Gamma_\mathcal{O} \backslash \Gamma_S / \Gamma_\mathcal{O}$ is a subset of $\Gamma_\mathcal{O} \backslash \Gamma_K / \Gamma_\mathcal{O}$, so $\mathcal{H}(\Gamma_S, \Gamma_\mathcal{O})$ naturally embeds in $\mathcal{H}(\Gamma_K, \Gamma_\mathcal{O})$. As in the proof of Theorem 2.3, the characteristic function of every double coset is $\mu_a^* e(x) \mu_b$ for some $a, b \in S$ and $x \in K/\mathcal{O}$, so $\mathcal{H}(\Gamma_S, \Gamma_\mathcal{O})$ is generated by $\{\mu_a : a \in S\}$ and $\{e(x) : x \in K/\mathcal{O}\}$; they still satisfy the relations $(\mathcal{H}1)$–$(\mathcal{H}4)$ for $a, b \in S$, and are linearly independent because they have disjoint support. Hence $\mathcal{H}(\Gamma_S, \Gamma_\mathcal{O})$ is the universal *-algebra with such generators and relations. Theorem 5.1 therefore implies that the completion $C^*(\Gamma_S, \Gamma_\mathcal{O})$ is isomorphic to $C^*(K/\mathcal{O}) \rtimes_\alpha S$.

Remark 5.3. Because the semigroup S is lattice-ordered, we can write down an alternative spanning set for the crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha S$:

$$C^*(K/\mathcal{O}) \rtimes_\alpha S = \overline{\operatorname{span}}\{i(x) v_a v_b^* : x \in K/\mathcal{O},\ a, b \in S \text{ with } (a, b) = 1\}.$$

To see this, first note that because ideals are principal, Proposition 3.4 yields

$$\alpha_a(1)\alpha_b(1) = \alpha_{a\vee b}(1),$$

which is equivalent to $v_a v_a^* v_b v_b^* = v_{a\vee b} v_{a\vee b}^*$. Multiplying on the left by v_a^*, right by v_b gives

$$v_a^* v_b = v_a^* v_{a\vee b} v_{a\vee b}^* v_b = v_{a^{-1}(a\vee b)} v_{b^{-1}(a\vee b)}^*;$$

this suffices to prove the claim because $(a^{-1}(a \vee b), b^{-1}(a \vee b)) = 1$.

Remark 5.4. It follows from Theorem 5.1 that $C^*(K/\mathcal{O}) \rtimes_\alpha S$ embeds as a subalgebra of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$. In fact we can recover $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ from this subalgebra by taking the crossed product by the action γ of $\mathcal{O}^*$ satisfying

$$\gamma_u(i(f)v_a^*v_b) = i(\alpha_u(f))v_a^*v_b.$$

To see this, first observe that the unitary elements v_u implement the automorphisms γ_u, so there is a homomorphism π of $(C^*(K/\mathcal{O}) \rtimes_\alpha S) \rtimes \mathcal{O}^*$ into $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$. On the other hand, because $\mathcal{O}^\times$ is the direct product of $\mathcal{O}^*$ and S, we can combine the embeddings of $\mathcal{O}^*$ and S in $(C^*(K/\mathcal{O}) \rtimes_\alpha S) \rtimes \mathcal{O}^*$ into one homomorphism of $\mathcal{O}^\times$, which is covariant with the embedding of $C^*(K/\mathcal{O})$, and hence gives a homomorphism ρ of $C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{O}^\times$ into the iterated crossed product. It is easy to check that π and ρ are inverses of each other.

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