Hopf-Bifurcation in Systems with Spherical Symmetry
Part I : Invariant Tori

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Abstract.

A Hopf-bifurcation scenario with symmetries is studied. Here, apart from the well known branches of periodic solutions, other bifurcation phenomena have to occur as it is shown in the second part of the paper using topological arguments. In this first part of the paper we prove analytically that invariant tori with quasiperiodic motion bifurcate. The main methods used are orbit space reduction and singular perturbation theory.

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Contents

1 INTRODUCTION

An interesting problem in the theory of ordinary differential equations is the generalization of the two dimensional Hopf-bifurcation to higher dimensional systems with symmetry. In this connection, [GoSt] and [GoStSch] investigated problems on a vector space X that can be decomposed into a direct sum of absolutely irreducible representations of the group O(3) of the form $X = V_l \oplus \mathrm{i}V_l$. Here V_l denotes the space of homogeneous harmonic polynomials $P : \mathbb{R}^3 \to \mathbb{R}$ of degree l. This is the simplest case where purely imaginary eigenvalues (of high multiplicity) in the bifurcation point are possible. Using Lie-group theory, the authors showed the existence of branches of periodic solutions with certain symmetries. Here in addition to the spatial O(3)-symmetry a temporal S^1-symmetry occurs. This symmetry corresponds to a time shift along the periodic solutions. In order to obtain their results, the authors made a Lyapunov-Schmidt-reduction on the space of periodic functions. The reduced system then has $\mathrm{O}(3) \times \mathrm{S}^1$-symmetry and solutions correspond to periodic solutions of the original system with spatial-temporal symmetry. Under certain transversality assumptions, periodic solutions with symmetry $\bar{H} \subset \mathrm{O}(3) \times \mathrm{S}^1$ bifurcate if Dim Fix$(\bar{H}) = 2$ for the induced representation of the group $\mathrm{O}(3) \times \mathrm{S}^1$ on the space X (cf. [GoSt] resp. [GoStSch]). [Fi] has shown that it is sufficient that $\bar{H}$ is a maximal subgroup for periodic solutions with symmetry $\bar{H}$ to bifurcate. Using these methods, only the existence of periodic solutions can be investigated. Via normal form theory (cf. [EletAl]) one gets $\mathrm{O}(3) \times \mathrm{S}^1$-equivariant polynomial vector fields up to every finite order for our systems. This additional S^1-symmetry is due to the fact that the normal form commutes with the one parameter group $e^{L^{\mathrm{T}} t}$ which is generated by the linearization L in the bifurcation point. For a Hopf-bifurcation L has purely imaginary eigenvalues (of high multiplicity) and the group generated is a rotation. [IoRo], [HaRoSt] and [MoRoSt] did analytic calculations for the normal form up to fifth order in the case $l = 2$. They gave conditions for the stability of the five branches of periodic solutions predicted by [GoSt] resp. [GoStSch] in terms of coefficients of the normal form. Quasiperiodic solutions found by [IoRo] in the normal form up to third order can not be confirmed in this paper. We shall show a mechanism for quasiperiodic solutions to bifurcate in the fifth order.

Investigating the normal form due to [IoRo], one finds a region in parameter space where two of the branches of periodic solutions bifurcating supercritically are stable simultaneously. Using topological methods, [Le] showed that we have the following alternative in this region in parameter space: Either besides the known branches of periodic solutions other invariant objects bifurcate or recurrent structure between the different invariant sets (e.g. between the different group orbits of periodic solutions and the trivial solution) exists. Actually the results of these topological investigations were the starting point of analytical efforts to find other solutions (or recurrent structure) in this paper. In order to get our results, we shall proceed as follows.

First the representation of the group $\Gamma = \mathrm{O}(3) \times \mathrm{S}^1$ on the ten dimensional space $X = V_2 \oplus \mathrm{i}V_2$ is introduced. The lattice of isotropy subgroups of this representation is given according to [MoRoSt] and the results of [IoRo] are quoted. The smallest invariant subspace containing both solutions that are stable simultaneously has isotropy $\Sigma = (\mathbb{Z}_2, 1)$.

Then our considerations are being restricted to this six dimensional subspace. The normaliser of Σ is $\mathrm{N}(\Sigma) = \mathrm{O}(2) \times \mathrm{S}^1 \subset \Gamma$. This is the biggest subgroup of Γ leaving $\mathrm{Fix}(\Sigma)$ invariant as a subspace. Now we shall look at the representation of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ on $\mathrm{Fix}(\Sigma)$.

Dealing with differential equations with symmetries, one has to deal with group orbits of solutions because a solution $x(t)$ gives rise to solutions $\gamma\, x(t)$ with $\gamma \in \Gamma$. This redundancy, induced by the action of the group, will be removed by identifying points that lie on a group orbit. I.e. one studies the orbit space that is homeomorphic to the image of the Hilbert-map $\Pi : \mathrm{Fix}(\Sigma) \to \mathbb{R}^k : z \to \pi_i(z)$ (cf. [La2] and [Bi]). Here k denotes the minimal number of generators of the ring of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ invariant polynomials $P : \mathrm{Fix}(\Sigma) \to \mathbb{R}$ and π_i, $i = 1, \ldots, k$, is such a system of generators. Thus the original differential equation is reduced to a differential equation on $\Pi\big(\mathrm{Fix}(\Sigma)\big)$ of the form $\dot{\pi} = g(\pi)$, $\pi = (\pi_1, \ldots, \pi_k)$. In order to perform this reduction for a given equation, one, first of all, has to know the number of independent invariants and equivariants for a given representation. Then one, actually, has to calculate them. Statements on the number of independent invariants and equivariants and possible relations between them are given by the Poincaré-series. These are formal power series $\sum_{i=0}^{\infty} a_i\, t^i$ in t. Here a_i denotes the dimension of the vector space of homogeneous invariant polynomials of degree i resp. the dimension of the vector space of homogeneous equivariant mappings of degree i. These series can be determined just by knowledge of the representation of the group on the space.
The lattice of isotropy subgroups of the representation of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ on $\mathrm{Fix}(\Sigma)$ and the image of the Hilbert-map are determined. This is a stratified space which consists of manifolds (strata). Each stratum consists of images of points of some isotropy type of the representation of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ on $\mathrm{Fix}(\Sigma)$. Thus it is flow invariant with respect to the reduced vector field on $\Pi\big(\mathrm{Fix}(\Sigma)\big)$.
Afterwards we shall carry out the orbit space reduction for the normal form up to third order. The critical points of the reduced vector field in $\Pi\big(\mathrm{Fix}(\Sigma)\big)$ are determined. As expected by inspection of the lattice of isotropy subgroups of Γ on $V_2 \oplus \mathrm{i}V_2$, we shall find images of periodic solutions of isotropy $(\mathrm{O}(2), 1)$, $(\mathrm{D}_4, \mathbb{Z}_2)$, $\widetilde{\mathrm{SO}(2)}^2$, and $(\mathbb{T}, \mathbb{Z}_3)$. Moreover there exists some stratum F in $\Pi\big(\mathrm{Fix}(\Sigma)\big)$. Connected via a curve g of fixed points the fixed points having isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_4, \mathbb{Z}_2)$ in the original system lie on F. The preimage of F consists of points having isotropy $(\mathbb{Z}_2, 1)$ in the restricted system. Perturbations that respect the symmetry will, therefore, respect this stratum. The curve g is stable for the reduced vector field restricted to F. Small perturbations of the original vector field in fifth order of magnitude ε will, therefore, preserve a curve. By use of singular perturbation theory (cf. [Fe]), one gets a resulting drift on the curve. This explains the observation made by [IoRo] that the stability of the fixed points of isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_2, \mathbb{Z}_2)$ is determined in the fifth order.
Dependent on the relative choice of the coefficients of the third order normal form in the region of parameter space in question, there is a point on the curve g where the linear stability of the curve in the direction of the principle stratum changes. Linearization of the reduced vector field in this point yields a nontrivial two dimensional Jordan-block to the eigenvalue zero. The second dimension results from the

linearization along the curve. Finally the flow on the two dimensional center manifold in this point is determined for small ε. The persistence of the curve g for small ε, knowledge of the direction of the drift, the change of stability in the direction of the principle stratum, and the existence of a nontrivial two dimensional Jordan-block to the eigenvalue zero are sufficient to prove for small ε the bifurcation of a fixed point of the reduced equation in the direction of the principle stratum using the implicit function theorem. Fixed points of the reduced system on the stratum F correspond to periodic solutions, fixed points in the principle stratum correspond to quasiperiodic solutions in the original system.

2 Representation of the group $\mathrm{O}(3) \times \mathrm{S}^1$ on $V_2 \oplus \mathrm{i}V_2$

We investigate systems of ODE's of the form

$$\dot{x} \;=\; f(\lambda, x)$$

in the ten dimensional space

$$X \;=\; V_2 \oplus \mathrm{i}V_2.$$

Let V_2 be the five dimensional space of homogeneous harmonic polynomials

$$p \;:\; \mathbb{R}^3 \;\to\; \mathbb{R}$$

of degree two. We have

$$V_2 \;=\; \left\langle 2x_3^2 - (x_1^2 + x_2^2), x_1x_3, x_2x_3, x_1^2 - x_2^2, x_1x_2 \right\rangle.$$

Let us introduce the following coordinates $(z, \overline{z})$,

$$z \;=\; (z_{-2}, z_{-1}, z_0, z_1, z_2),\;\; z_m \in \mathbb{C},\;\; m = -2, \ldots, 2,$$

in the space X:

$$x \in X \quad \Leftrightarrow \quad x = \sum_{m=-2}^{2} z_m Y_m.$$

Here

$$\begin{aligned} Y_0 &= \sqrt{\tfrac{5}{16\pi}}\left(2x_3^2 - (x_1^2 + x_2^2)\right), \\ Y_{\pm 1} &= \sqrt{\tfrac{15}{8\pi}}\left(x_1x_3 \pm \mathrm{i}x_2x_3\right), \\ Y_{\pm 2} &= \sqrt{\tfrac{15}{32\pi}}\left((x_1^2 - x_2^2) \pm \mathrm{i}2x_1x_2\right) \end{aligned}$$

denote spherical harmonics. Moreover let

$$f \;:\; \mathbb{R} \times X \;\to\; X$$

be a smooth map that commutes with the following representation of the compact Lie-group

$$\Gamma \;=\; \mathrm{O}(3) \times \mathrm{S}^1$$

on the space X.
The group

$$\mathrm{O}(3) \quad = \quad \mathrm{SO}(3) \oplus \mathbb{Z}_2^c$$

with

$$\mathbb{Z}_2^c \quad = \quad \{\pm \mathrm{Id}\}$$

acts via the natural representation absolutely irreducible on V_2. For $p \in V_2$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} \gamma\, p(\cdot) &= p(\gamma^{-1}\cdot) \text{ for } \gamma \in \mathrm{SO}(3), \\ -\mathrm{Id}\, p(\cdot) &= p(\cdot). \end{aligned}$$

This representation is a special case of the representation of the group $\mathrm{O}(3)$ on the space V_l, $l \geq 1$. For l even the subgroup $\mathbb{Z}_2^c$ acts trivially in the natural representation. On the space X the group $\mathrm{O}(3)$ acts diagonally. For the general representation theory of $\mathrm{O}(3)$ we refer to [StiFä] and [GoStSch].
The group S^1 acts as a rotation in the coordinates

$$\begin{aligned} \phi\, z &= e^{i\phi} z, \\ \phi\, \overline{z} &= e^{-i\phi} \overline{z} \end{aligned}$$

with $\phi \in \mathrm{S}^1$.
So we have

$$f(\lambda, \gamma\, x) \quad = \quad \gamma\, f(\lambda, x), \ \forall \gamma \in \Gamma.$$

In their paper concerning Hopf-bifurcation with O(3)-Symmetry [GoSt] and [GoStSch] look at systems of the form

$$\dot{x} \quad = \quad f(\lambda, x)$$

with

$$x \in X \quad = \quad V_l \oplus \mathrm{i} V_l$$

and

$$f \quad : \quad \mathbb{R} \times X \quad \rightarrow \quad X$$

a smooth mapping. This direct sum of two absolutely irreducible representations of the group $\mathrm{O}(3)$ is the simplest case allowing imaginary eigenvalues, however of high multiplicity, in the bifurcation point. Let us assume:

- f is equivariant with respect to the diagonal representation of $\mathrm{O}(3)$ on X.
- $f(\lambda, 0) \equiv 0$.
- $(\mathrm{D}f)_{\lambda,0}$ has a pair of complex conjugate eigenvalues $\sigma(\lambda) \pm \mathrm{i}\rho(\lambda)$ with $\sigma(0) = 0$, $\dot{\sigma}(0) \neq 0$, and $\rho(0) = \omega$ of multiplicity $(2l+1) = \mathrm{Dim}(V_l)$ with smooth functions σ and ρ.

The authors now look at subgroups

$$\bar{H} \quad \subset \quad \Gamma.$$

Here the group $\mathrm{S}^1 \subset \Gamma$ acts as a time shift on the periodic solutions. Therefore subgroups $\bar{H}$ consist of spatial and temporal symmetries. For subgroups $\bar{H}$ with

$$\mathrm{Dim}\,\mathrm{Fix}(\bar{H}) \quad = \quad 2$$

with respect to the representation of the group Γ on $V_l \oplus \mathrm{i}V_l$, the authors prove the existence of exactly one branch of periodic solutions with small amplitude of period near $\frac{2\pi}{\omega}$ and the group of symmetries $\bar{H}$. In order to do this, the authors make a Lyapunov-Schmidt-reduction on the space of periodic functions. The reduced system has the full $\mathrm{O}(3) \times \mathrm{S}^1$-symmetry and solutions correspond to periodic solutions with spatial-temporal symmetries in the original system.

For $l = 2$ [IoRo] applied normal form theory (cf. [EletAl]) to these systems. Up to every finite order they got $\mathrm{O}(3) \times \mathrm{S}^1$-equivariant systems of the form described above. This additional S^1-symmetry up to every finite order is due to the fact that the normal form of f commutes with the one-parameter group $e^{(\mathrm{D}f)_{0,0}^{\mathrm{T}}\, t}$. Due to our conditions on the eigenvalues, this is just a complex rotation.

The following calculations are done using the normal form up to fifth order due to [IoRo]. The normal form up to fifth order is very lengthy and shall not be given here. The parts important for our calculations shall be cited when necessary.

Let G be a compact Lie-group acting on a space X. The most general form of a G-equivariant polynomial mapping $g : X \to X$ is

$$g(x) \quad = \quad \sum_{i=1}^{n} p_i(x)\, e_i(x).$$

Here

$$p_i \quad : \quad X \quad \to \quad \mathbb{R}$$

denote G-invariant polynomials and

$$e_i \quad : \quad X \quad \to \quad X$$

G-equivariant, polynomial mappings.

In order to determine the most general G-equivariant, polynomial mapping up to a fixed order, one, first of all, has to know the number of independend invariants and equivariants and possible relations between them. On this occasion the Poincaré-series described in the next chapter are useful. The next problem is to find the polynomials. In the case of the group $\mathrm{O}(3)$, using raising and lowering operators (cf. [Sa],[Mi]), one can check whether a specific polynomial is invariant or not. The raising and lowering operators are in close relationship to the infinitesimal generators of the Lie-algebra of the group. So the problem is to construct and check all possible polynomials resp. polynomial mappings. Dealing with high order polynomials and large dimensions of the problem, this is a very difficult task that is only accessible via symbolic algebra. At least, using the Poincaré-series, one knows when everything is found.

The lattice of isotropy subgroups of the representation of the group Γ on $V_2 \oplus \mathrm{i}V_2$ has been determined by [MoRoSt].

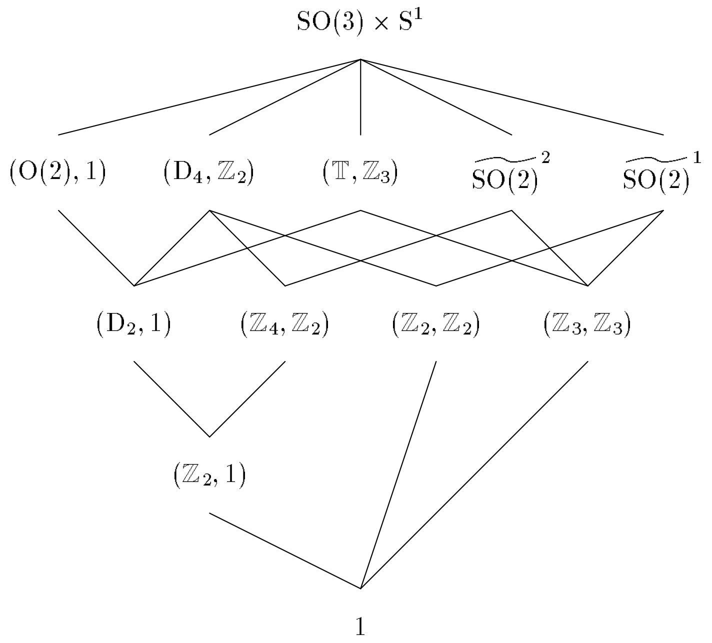

Figure 1: Lattice of isotropy subgroups of Γ on $V_2 \oplus \mathrm{i}V_2$.

The subgroups $\bar{H} \subset \Gamma$ are given as twisted subgroups

$$\bar{H} = (H, \Theta(H))$$

with $H \subset \mathrm{SO}(3)$ and $\Theta(H) \subset \mathrm{S}^1$. In this connection

$$\Theta : H \to \mathrm{S}^1$$

is a group homomorphism. Every isotropy subgroup $\bar{H} \subsetneq \Gamma$ can be written in this form (cf. [GoStSch]). In the case of the isotropy subgroups $\widetilde{\mathrm{SO}(2)}^1$ resp. $\widetilde{\mathrm{SO}(2)}^2$ we have $H = \mathrm{SO}(2) \subset \mathrm{SO}(3)$ and $\Theta(H) = \mathrm{S}^1$ with $\Theta(\phi) = \phi$ resp. $\Theta(\phi) = \phi^2$.
In [MoRoSt] the authors investigate Hamiltonian systems of the form

$$\dot{v} = J\,\mathrm{D}H(v)$$

with $v \in \mathbb{R}^{10} = V_2 \oplus \mathrm{i}V_2$,

$$J = \begin{pmatrix} 0 & -I_5 \\ I_5 & 0 \end{pmatrix},$$

and $\mathrm{O}(3) \times \mathrm{S}^1$ invariant Hamiltonian $H : \mathbb{R}^{10} \to \mathbb{R}$. This leads to restrictions on the coefficients of the normal form of the vector field. Like [IoRo] for the general vector field, [MoRoSt] analytically prove the existence of periodic solutions of isotropy

$(\mathrm{O}(2),1)$, $(\mathrm{D}_4,\mathbb{Z}_2)$, $(\mathbb{T},\mathbb{Z}_3)$, $\widetilde{\mathrm{SO}(2)}^1$, and $\widetilde{\mathrm{SO}(2)}^2$. These are exactly the subgroups of Γ having a two dimensional fixed point space for our representation, i.e. the subgroups for which [GoSt] and [GoStSch] predicted the bifurcation of periodic solutions using group theoretical methods. Moreover the authors give conditions for the stability of the different branches of periodic solutions by means of regions in the parameter space of the normal form.

In the following we shall look only at the situation where all solutions bifurcate supercritically. In this case there is a region in parameter space where the periodic solutions of isotropy $(\mathrm{O}(2),1)$ resp. $\widetilde{\mathrm{SO}(2)}^2$ are stable simultaneously, see [IoRo]. Using topological methods, [Le] showed that in this region in parameter space either other isolated invariant objects besides the trivial solution and the different group orbits of periodic solutions have to exist or there is recurrent structure between the trivial solution and the different group orbits of periodic solutions. Recurrent structure means that it is possible to go back via connecting orbits that connect different group orbits in the direction of the flow, from a specific group orbit to this group orbit itself.

In this paper we shall prove the existence of quasiperiodic solutions in the region in parameter space in question. The quasiperiodic solutions given by [IoRo] using the third order normal form cannot be confirmed. We shall prove that the quasiperiodic solutions bifurcate in fifth order from a curve of periodic solutions that is degenerate up to third order.

In order to reduce the dimension of the problem, we shall restrict our calculations in the following to the smallest invariant subspace containing the two stable solutions. This is a subspace of isotropy $(\mathbb{Z}_2,1)$ due to the lattice of isotropy subgroups. Next we want to fix a specific subgroup

$$\mathrm{O}(2) \quad\subset\quad \mathrm{SO}(3)$$

because it is well suited for our coordinates:

$$\mathrm{O}(2) \quad=\quad \left\{ r_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};\ \phi\in[0,2\pi) \right\}.$$

It acts (cf. [GoStSch]) in the following form on our coordinates z:

$$\begin{aligned} r_\phi\,(z_{-2},z_{-1},z_0,z_1,z_2) &= (e^{-2i\phi}z_{-2},e^{-i\phi}z_{-1},z_0,e^{i\phi}z_1,e^{2i\phi}z_2), \\ \kappa\,(z_{-2},z_{-1},z_0,z_1,z_2) &= (z_2,-z_1,z_0,-z_{-1},z_{-2}). \end{aligned}$$

Finally let

$$\Sigma \quad=\quad (\mathbb{Z}_2,1)$$

with

$$\mathbb{Z}_2 \quad=\quad \{1,r_\pi\}.$$

3 Restriction to $\mathrm{Fix}(\mathbb{Z}_2, 1)$

Lemma 3.0.1

$$\mathrm{Fix}(\Sigma) = \mathrm{Span}\{(z_{-2}, 0, z_0, 0, z_2)\} \cong \mathbb{C}^3.$$

Lemma 3.0.2

$$\Xi = \frac{\mathrm{N}(\Sigma)}{\Sigma} = \mathrm{O}(2) \times \mathrm{S}^1.$$

The group $\mathrm{O}(2) \times \mathrm{S}^1$ *acts on* $\mathbb{C}^3$*:*

$$\begin{aligned} r_\theta(z_{-2}, z_0, z_2) &= (e^{-\mathrm{i}\theta} z_{-2}, z_0, e^{\mathrm{i}\theta} z_2), \\ \kappa(z_{-2}, z_0, z_2) &= (z_2, z_0, z_{-2}), \\ \phi(z_{-2}, z_0, z_2) &= (e^{\mathrm{i}\phi} z_{-2}, e^{\mathrm{i}\phi} z_0, e^{\mathrm{i}\phi} z_2). \end{aligned}$$

The group $\mathrm{O}(2)$ *is generated by the rotations* r_θ *and the reflection* κ *and the group* S^1 *by the rotations* ϕ*.*

Proof: We have $\mathrm{N}_{\mathrm{SO}(3)}(\mathbb{Z}_2) = \mathrm{O}(2)$. The representation of $\mathrm{O}(2) \times \mathrm{S}^1$ on $\mathbb{C}^3$ is given by restriction of the representation of $\mathrm{SO}(3) \times \mathrm{S}^1$ on $\mathrm{Fix}(\Sigma)$. ⋈

Let $z = (z_{-2}, z_0, z_2) \in \mathbb{C}^3$. The definition

$$\sigma\overline{z} = \overline{\sigma z}, \ \sigma \in \Xi,$$

gives rise to an unitary representation of Ξ on the space

$$\mathbb{C}^3 \oplus \mathbb{C}^3 \supset \{(z, \overline{z}), \ z \in \mathbb{C}^3\} = \mathbb{R}^6.$$

3.1 Poincare-series, invariants, and equivariants

The number of generators of the ring of Ξ-invariant polynomials $P : \mathbb{R}^6 \to \mathbb{R}$ and of the module of Ξ-equivariant, polynomial mappings $Q : \mathbb{R}^6 \to \mathbb{R}^6$ over the ring of invariant polynomials can be determined using Poincaré-series.
For an unitary representation T of a compact Lie-group G on a vector space V we have

$$\begin{aligned} P_I(t) &= \int_G \frac{1}{\det(I - tT(g))} \mathrm{d}g = \sum_{i=0}^{\infty} c_i t^i, \\ P_{Eq}(t) &= \int_G \frac{\overline{\chi(g)}}{\det(I - tT(g))} \mathrm{d}g = \sum_{i=0}^{\infty} d_i t^i. \end{aligned}$$

Here $c_i, \ i > 0$, denotes the dimension of the vector space of homogeneous invariant polynomials of degree i and $d_i, \ i > 0$, the dimension of the vector space of homogeneous, equivariant mappings of degree i. Let $c_0 = d_0 = 1$. The integral appearing in the formulas is the Haar-integral associated to the compact Lie-group G (cf. [BrtD]), $\chi(g), \ g \in G$, denotes the character of g relative to the representation T. The theory of Poincaré-series is presented in [La2].

LEMMA 3.1.1

$$\begin{aligned} P_I(t) &= \frac{1+t^4}{(1-t^2)^2(1-t^4)^2}, \\ P_{Eq}(t) &= \frac{2t+3t^3+t^5}{(1-t^2)^2(1-t^4)^2}. \end{aligned}$$

Proof: The group $\Xi = \mathrm{O}(2) \times \mathrm{S}^1$ can be written as the disjoint union of two sets in the following form

$$\mathrm{O}(2) \times \mathrm{S}^1 = \mathrm{SO}(2) \times \mathrm{S}^1 \,\dot\cup\, \kappa\mathrm{SO}(2) \times \mathrm{S}^1.$$

Therefore the integrals appearing in the formulas split in two parts.
A. $\Xi_1 = \mathrm{SO}(2) \times \mathrm{S}^1$ acts on the space $\mathbb{C}^3 \oplus \mathbb{C}^3$. So we get

$$\begin{aligned} P_I^1(t) &= \int_{\Xi_1} \frac{1}{\det(I - tT(g))} \mathrm{d}g \\ &= \frac{1}{(2\pi)^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{2\pi} \frac{1}{\det(I - tT(\theta,\phi))} \mathrm{d}\theta\, \mathrm{d}\phi. \end{aligned}$$

For our representation we have

$$\begin{aligned} \det(I - tT(\theta,\phi)) &= (1 - te^{\mathrm{i}(\theta-\phi)})(1 - te^{-\mathrm{i}\phi})(1 - te^{-\mathrm{i}(\theta+\phi)})(1 - te^{\mathrm{i}(-\theta+\phi)}) \\ &\quad (1 - te^{\mathrm{i}\phi})(1 - te^{\mathrm{i}(\theta+\phi)}). \end{aligned}$$

A transformation of variables

$$e^{\mathrm{i}\theta} \to y_1,\ e^{\mathrm{i}\phi} \to y_2$$

leads to

$$\begin{aligned} P_I^1(t) &= \frac{1}{(2\pi\mathrm{i})^2} \oint_{y_1} \oint_{y_2} \frac{1}{y_1 y_2 \det(I - tT(y_1,y_2))} \mathrm{d}y_1\, \mathrm{d}y_2 \\ &= \frac{1}{(2\pi\mathrm{i})^2} \oint_{y_1} \oint_{y_2} \frac{y_1 y_2^2}{(y_2 - ty_1)(y_2 - t)(y_1 y_2 - t)(y_1 - ty_2)(1 - ty_2)(1 - ty_1 y_2)} \mathrm{d}y_1 \mathrm{d}y_2. \end{aligned}$$

Using the residue theorem twice, one gets

$$P_I^1(t) = \frac{1+t^4}{(1-t^2)^3(1-t^4)}.$$

B. For the set $\kappa\mathrm{SO}(2) \times \mathrm{S}^1$ we have

$$\det(I - tT(\kappa,\theta,\phi)) = (1 - te^{-\mathrm{i}\phi})^2(1 + te^{-\mathrm{i}\phi})(1 - te^{\mathrm{i}\phi})^2(1 + te^{\mathrm{i}\phi}).$$

A transformation of variables gives

$$P_I^2(t) = \frac{1}{2\pi\mathrm{i}} \oint_{y_2} \frac{y_2^2}{(y_2 - t)^2(y_2 + t)(1 - ty_2)^2(1 + ty_2)} \mathrm{d}y_2.$$

Using the residue theorem, one gets

$$P_I^2(t) \quad = \quad \frac{1+t^4}{(1-t^4)^2(1-t^2)}.$$

c. Because of the normalization of the Haar-integral, we have

$$\begin{aligned} P_I(t) &= \tfrac{1}{2}\left(P_I^1(t)+P_I^2(t)\right) \\ &= \frac{1+t^4}{(1-t^2)^2(1-t^4)^2} \end{aligned}$$

proving the first formula.
d. We want to calculate

$$P_{Eq}^1(t) \quad = \quad \int_{\Xi_1} \frac{\overline{\chi(g)}}{\det(I-tT(g))}\mathrm{d}g.$$

Here we get

$$\begin{aligned} \chi(\theta,\phi) &= \operatorname{Tr}(T(\theta,\phi)) \\ &= e^{\mathrm{i}(-\theta+\phi)}+e^{\mathrm{i}\phi}+e^{\mathrm{i}(\theta+\phi)}+e^{\mathrm{i}(\theta-\phi)}+e^{-\mathrm{i}\phi}+e^{-\mathrm{i}(\theta+\phi)} \\ &= \left(e^{\mathrm{i}\phi}+e^{-\mathrm{i}\phi}\right)\left(e^{\mathrm{i}\theta}+1+e^{-\mathrm{i}\theta}\right). \end{aligned}$$

This leads to

$$\begin{aligned} P_{Eq}^1(t) &= \frac{1}{(2\pi\mathrm{i})^2} \\ &\oint_{y_1}\oint_{y_2} \frac{y_2(1+y_1+y_1^2)(1+y_2^2)}{(y_2-ty_1)(y_2-t)(y_1y_2-t)(y_1-ty_2)(1-ty_2)(1-ty_1y_2)}\mathrm{d}y_1\,\mathrm{d}y_2 \\ &= 2\frac{3t+3t^3}{(1-t^2)^3(1-t^4)}. \end{aligned}$$

e. For the set $\kappa\mathrm{SO}(2)\times\mathrm{S}^1$ one correspondingly gets

$$\chi(\kappa,\theta,\phi) \quad = \quad e^{\mathrm{i}\phi}+e^{-\mathrm{i}\phi}.$$

This leads to

$$\begin{aligned} P_{Eq}^2(t) &= \frac{1}{2\pi\mathrm{i}}\oint_{y_2}\frac{y_2(1+y_2^2)}{(y_2-t)^2(y_2+t)(1-ty_2)^2(1+ty_2)}\mathrm{d}y_2 \\ &= 2\frac{t}{(1-t^2)^2(1-t^4)}. \end{aligned}$$

f. We therefore have

$$\begin{aligned} \tilde{P}_{Eq}(t) &= \tfrac{1}{2}\left(P_{Eq}^1(t)+P_{Eq}^2(t)\right) \\ &= 2\frac{2t+3t^3+t^5}{(1-t^2)^2(1-t^4)^2}. \end{aligned}$$

Doing this, we used the diagonal representation of Ξ on $\mathbb{C}^3 \oplus \mathbb{C}^3$. But we are interested in the subspace $\{(z, \overline{z}),\ z \in \mathbb{C}^3\} \subset \mathbb{C}^3 \oplus \mathbb{C}^3$ only. Therefore the number of equivariants given by the formula is twice as big as it should be counting also equivariants with one component being zero. $\bowtie$

The Poincaré-series can be interpreted in the following way.

LEMMA 3.1.2 *The polynomials*

$$\begin{aligned}
\pi_1 &= |z_0|^2, \\
\pi_2 &= |z_{-2}|^2 + |z_2|^2, \\
\pi_3 &= |z_{-2}|^2 |z_2|^2, \\
\pi_4 &= \frac{1}{2}\left(\overline{z_0}^2 z_{-2} z_2 + z_0^2 \overline{z_{-2}}\,\overline{z_2}\right), \\
\pi_5 &= \frac{\mathrm{i}}{2}\left(\overline{z_0}^2 z_{-2} z_2 - z_0^2 \overline{z_{-2}}\,\overline{z_2}\right)
\end{aligned}$$

are a minimal set of generators of the ring of invariant polynomials.

$$P \quad : \quad \mathbb{R}^6 \quad \to \quad \mathbb{R}.$$

The only relation between them is

$$\pi_4^2 + \pi_5^2 \quad = \quad \pi_1^2 \pi_3.$$

Proof: One easily sees that the given polynomials $\pi_1, \ldots, \pi_5$ are invariant, and just meet the given relation. Therefore the Poincaré-series of these polynomials is identical to the one calculated. Because of this there are no additional generators and relations. $\bowtie$

Introducing polar coordinates in the following form

$$z_j = r_j e^{\mathrm{i}\phi_j},\ j \in \{-2, 0, 2\},$$

and defining

$$\theta \quad = \quad 2\phi_0 - \phi_{-2} - \phi_2,$$

one gets

$$\pi_4 \quad = \quad r_0^2 r_{-2} r_2 \cos\theta$$

and

$$\pi_5 \quad = \quad r_0^2 r_{-2} r_2 \sin\theta.$$

Consequently the invariants π_4 and π_5 represent phase relations between the different coordinates.

LEMMA 3.1.3 *Let $\pi : \mathbb{R}^6 \to \mathbb{R}$ be an invariant polynomial for the representation of Ξ on $\mathbb{R}^6$.*
Then

$$p(z, \overline{z}) = \overline{\nabla_{z,\overline{z}} \pi(z, \overline{z})}$$

is a Ξ-equivariant polynomial mapping for this representation.

Proof: We have

$$p(\sigma(z,\overline{z})) = \overline{\nabla_{\sigma(z,\overline{z})}\pi(z,\overline{z})} = \overline{\nabla_{z,\overline{z}}\pi(z,\overline{z})}\,\overline{\sigma^{-1}} = \sigma p(z,\overline{z}).$$

The last equality is correct because the representation is unitary. ⋈

LEMMA 3.1.4 *The independent, Ξ-equivariant, polynomial mappings*

$$Q \quad : \quad \mathbb{R}^6 \ \to \ \mathbb{R}^6$$

up to fifth order are

$$e_1 = \begin{pmatrix} 0 \\ z_0 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} z_{-2} \\ 0 \\ z_2 \end{pmatrix}, \ e_3 = \begin{pmatrix} z_{-2}|z_2|^2 \\ 0 \\ z_2|z_{-2}|^2 \end{pmatrix},$$

$$e_4 = \frac{1}{2}\begin{pmatrix} z_0^2\overline{z_2} \\ 2z_{-2}z_2\overline{z_0} \\ z_0^2\overline{z_{-2}} \end{pmatrix}, \ e_5 = -\frac{\mathrm{i}}{2}\begin{pmatrix} z_0^2\overline{z_2} \\ -2z_{-2}z_2\overline{z_0} \\ z_0^2\overline{z_{-2}} \end{pmatrix}.$$

Here e_i, $i = 1,\ldots,5$, always denote the first component of the equivariant. The second is given by complex conjugation of the first one.

Proof: Using the previous lemma, one knows that the mappings $e_j = \overline{\nabla_{z,\overline{z}}\pi_j}$, $j = 1,\ldots,5$, are equivariant. Power series expansion of $P_{Eq}(t)$ leads to

$$P_{Eq}(t) = 2t + 7t^3 + 17t^5 + \mathrm{O}(t^7).$$

There are $2, 7$ resp. 18 different possibilities to construct equivariant mappings of degree $1, 3$ resp. 5 from invariant polynomials $\pi_1,\ldots,\pi_5$ and equivariant mappings $e_1,\ldots,e_5$ by multiplication of invariants with an equivariant. In the fifth order one gets the relation

$$e_1(\pi_4 - \mathrm{i}\pi_5) \quad = \quad \frac{1}{2}\pi_1(e_4 - \mathrm{i}e_5).$$

All other combinations can't be generated this way. Therefore the Poincaré-series belonging to $\pi_1,\ldots,\pi_5$ and $e_1,\ldots,e_5$ is identical to the calculated one up to fifth order. Because of this there are no further generators or relations up to fifth order. ⋈

3.2 ORBIT SPACE REDUCTION

The most general $\mathrm{O}(2)\times\mathrm{S}^1$-equivariant Hopf-bifurcation problem on $\mathbb{R}^6$ up to third order has the form

$$\dot{z} \quad = \quad (\lambda + \mathrm{i}\omega)(e_1 + e_2) + a_1\pi_1 e_1 + a_2\pi_1 e_2 + a_3\pi_2 e_1 + a_4\pi_2 e_2 + a_5 e_3 + a_6 e_4 + a_7 e_5,$$

$a_j \in \mathbb{C}$, $j = 1,\ldots,7$, $\lambda,\omega \in \mathbb{R}$, and $z = (z_{-2}, z_0, z_2)$.
We want to study bifurcation problems on $\mathbb{R}^6$ resulting from a $\mathrm{SO}(3)\times\mathrm{S}^1$-equivariant

problem on $V_2 \oplus iV_2$. This gives the following restrictions for the coefficients $a_1, \ldots, a_7$:

$$\begin{aligned}\dot{z} &= (\lambda + i\omega)(e_1 + e_2) + \left(a - \frac{1}{2}b - \sqrt{\frac{3}{2}}c\right)\pi_1 e_1 + \left(a - \sqrt{\frac{8}{3}}c\right)\pi_1 e_2 \\ &\quad + \left(a - \sqrt{\frac{8}{3}}c\right)\pi_2 e_1 + a\pi_2 e_2 - \left(b + \sqrt{6}c\right)e_3 + \left(-b + \sqrt{\frac{2}{3}}c\right)e_4 \\ &\quad +0e_5. \end{aligned} \tag{3.2.1}$$

Here $a, b, c \in \mathbb{C}$ denote the corresponding coefficients from the normal form of [IoRo]. This is obtained by comparison of the normal form of [IoRo] restricted to the subspace with the general equation. Define coefficients $\alpha, \beta, \gamma \in \mathbb{C}$:

$$\begin{aligned} \alpha &= a - \tfrac{1}{2}b - \sqrt{\tfrac{3}{2}}c, & a &= \gamma, \\ \beta &= a - \sqrt{\tfrac{8}{3}}c, & b &= -2\alpha + \tfrac{3}{2}\beta + \tfrac{1}{2}\gamma, \\ \gamma &= a, & c &= \sqrt{\tfrac{3}{8}}(\gamma - \beta). \end{aligned}$$

Then the vector field has the form

$$\begin{aligned}\dot{z} &= (\lambda + i\omega)(e_1 + e_2) + \alpha\pi_1 e_1 + \beta(\pi_1 e_2 + \pi_2 e_1) + \gamma\pi_2 e_2 \\ &\quad +2(\alpha - \gamma)e_3 + 2(\alpha - \beta)e_4 \\ &= \Big((\lambda + i\omega) + \alpha\pi_1 + \beta\pi_2\Big)e_1 + \Big((\lambda + i\omega) + \beta\pi_1 + \gamma\pi_2\Big)e_2 \\ &\quad +2(\alpha - \gamma)e_3 + 2(\alpha - \beta)e_4 \end{aligned} \tag{3.2.2}$$

with $\lambda, \omega \in \mathbb{R}$.

Let $\dot{x} = f(x)$ be a differential equation on a vector space X. Let the mapping f be equivariant with respect to the representation of the compact Lie-group G on X. Since

$$\dot{(gx)} = g\dot{x} = gf(x) = f(gx), \ \forall g \in G,$$

$gx(t),\ g \in G$, is a solution if $x(t)$ is a solution. This means one has to deal with group orbits Gx of solutions. Let G_x denote the isotropy of a point x. Then we have

$$\frac{G}{G_x} \cong Gx.$$

Here $\frac{G}{G_x}$ and Gx are compact manifolds and we have (cf. [Di])

$$\operatorname{Dim} Gx = \operatorname{Dim} G - \operatorname{Dim} G_x.$$

In order to get rid of the redundancy in our system induced by the group G, one studies the orbit space $\frac{X}{G}$. Here points lying on a group orbit are identified:

$$x \simeq y \iff x = gy \text{ with } x, y \in X \text{ and } g \in G.$$

The orbit space is homeomorphic to the image of the Hilbert-map $\Pi(X)$

$$\begin{aligned} \Pi \ : \ X &\to \mathbb{R}^k \\ x &\to (\pi_i(x)) \end{aligned}$$

(cf. [La2], [Bi]). Here k denotes the minimal number of generators of the ring of G-invariant polynomials $P : X \to \mathbb{R}$ and $\pi_i,\ i = 1,\dots,k$, is such a system of generators. The original differential equation is reduced to a differential equation on $\Pi(X)$ of the form

$$\dot{\pi} = g(\pi) \text{ with } \pi = (\pi_1,\dots,\pi_k).$$

The reduced equation can be calculated as follows:

$$\dot{\pi_i} = \nabla_x \pi_i\, \dot{x} = \nabla_x \pi_i\, f(x),\ i = 1,\dots,k.$$

The advantage of this reduction lies in the fact that in general the dimension of the reduced problem is smaller than the original one. Furthermore symmetry induced periodic solutions in the original system correspond to fixed points in the reduced system and can be dealt with more easily analytically. The disadvantage is that the orbit space in general is no vector space but a stratified space.

In our case the differential equation up to third order (Equation (3.2.2)) is given in the form

$$\dot{z} \;\; = \;\; \sum_{j=1}^{5} q_j\, e_j.$$

Here

$$q_j : \mathbb{R}^6 \to \mathbb{C},\ j = 1,\dots,5,$$

are invariant polynomials. So one gets

$$\begin{aligned} \dot{\pi_i} &= \nabla_z \pi_i\, \dot{z} + \nabla_{\overline{z}} \pi_i\, \dot{\overline{z}} \\ &= \overline{e_i}\, \dot{z} + e_i\, \dot{\overline{z}} \\ &= 2\mathrm{Re}\ (\overline{e_i}\, \dot{z}) \\ &= 2\mathrm{Re}\ \left(\sum_{j=1}^{5} q_j\, \overline{e_i} e_j \right). \end{aligned}$$

The products $\overline{e_i} e_j,\ i \le j \in \{1,\dots,5\}$, are

$$\begin{array}{ll} \overline{e_1}e_1 = \pi_1 & \overline{e_2}e_2 = \pi_2 \\ \overline{e_1}e_2 = 0 & \overline{e_2}e_3 = 2\pi_3 \\ \overline{e_1}e_3 = 0 & \overline{e_2}e_4 = \pi_4 + \mathrm{i}\pi_5 \\ \overline{e_1}e_4 = \pi_4 - \mathrm{i}\pi_5 & \overline{e_2}e_5 = -\mathrm{i}\pi_4 + \pi_5 \\ \overline{e_1}e_5 = \mathrm{i}\pi_4 + \pi_5 & \end{array}$$

$$\begin{array}{lll} \overline{e_3}e_3 = \pi_2\pi_3 & \overline{e_4}e_4 = \frac{1}{4}\pi_1^2\pi_2 + \pi_1\pi_3 & \overline{e_5}e_5 = \frac{1}{4}\pi_1^2\pi_2 + \pi_1\pi_3. \\ \overline{e_3}e_4 = \frac{1}{2}\pi_2(\pi_4 + \mathrm{i}\pi_5) & \overline{e_4}e_5 = -\frac{\mathrm{i}}{4}\pi_1^2\pi_2 + \mathrm{i}\pi_1\pi_3 & \\ \overline{e_3}e_5 = \frac{1}{2}\pi_2(-\mathrm{i}\pi_4 + \pi_5). & & \end{array}$$

For $i > j \in \{1,\dots,5\}$ we have

$$\overline{e_i}e_j \;\; = \;\; \overline{\overline{e_j}e_i}.$$

So the following lemma is proved.

LEMMA 3.2.1 *The Vector Field (3.2.2) yields the following reduced vector field on the orbit space*

$$\begin{aligned}
\dot{\pi}_1 &= 2(\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4\Big((\alpha-\beta)^r \pi_4 + (\alpha-\beta)^i \pi_5\Big) \\
\dot{\pi}_2 &= 2(\lambda + \beta^r \pi_1 + \gamma^r \pi_2)\pi_2 + 8(\alpha-\gamma)^r \pi_3 + 4\Big((\alpha-\beta)^r \pi_4 - (\alpha-\beta)^i \pi_5\Big) \\
\dot{\pi}_3 &= 4(\lambda + \beta^r \pi_1 + \alpha^r \pi_2)\pi_3 + 2\pi_2\Big((\alpha-\beta)^r \pi_4 - (\alpha-\beta)^i \pi_5\Big) \\
\dot{\pi}_4 &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_4 + 2(\alpha-\beta)^i(-\pi_1+\pi_2)\pi_5 \\
&\quad + (\alpha-\beta)^r \pi_1(\pi_1\pi_2 + 4\pi_3) \\
\dot{\pi}_5 &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_5 + 2(\alpha-\beta)^i(\pi_1-\pi_2)\pi_4 \\
&\quad + (\alpha-\beta)^i \pi_1(-\pi_1\pi_2 + 4\pi_3).
\end{aligned}$$

Here $\alpha^r, \beta^r, \gamma^r$ resp. $\alpha^i, \beta^i, \gamma^i$ denote the real resp. imaginary parts of α, β, γ.

3.3 LATTICE OF ISOTROPY SUBGROUPS

All isotropy subgroups $G \subsetneq \mathrm{O}(2) \times \mathrm{S}^1$ can be written as twisted subgroups in the form

$$G = H^{\Theta} = \{(h, \Theta(h)) \in \mathrm{O}(2) \times \mathrm{S}^1 \mid h \in H\}$$

(cf. [GoSt], [GoStSch]). Here $H \subset \mathrm{O}(2)$ denotes a closed subgroup of $\mathrm{O}(2)$ and

$$\Theta \quad : \quad \mathrm{O}(2) \quad \to \quad \mathrm{S}^1$$

is a group homomorphism. For a closed subgroup $H \subset \mathrm{O}(2)$ let

$$H' = \langle g^{-1}h^{-1}gh \mid g, h \in H \rangle$$

denote the commutator of H and

$$H^{ab} = \frac{H}{H'}$$

the abelianisation of H. Since $\Theta(H) \subset \mathrm{S}^1$ is abelian, the possible twist typs $\Theta(H)$ of H can be concluded from the abelianisation H^{ab}. One gets the following table.

H	H'	H^{ab}	$\Theta(H)$
$\mathrm{O}(2)$	$\mathrm{SO}(2)$	$\mathbb{Z}_2$	$1, \mathbb{Z}_2$
$\mathrm{SO}(2)$	1	$\mathrm{SO}(2)$	$1, \mathrm{S}^1$
$\mathrm{D_n}$	$\mathbb{Z}_{\frac{n}{2}}$, n even $\mathbb{Z}_{\mathrm{n}}$, n odd	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$, n even $\mathbb{Z}_2$, n odd	$1, \mathbb{Z}_2$
$\mathbb{Z}_{\mathrm{n}}$	1	$\mathbb{Z}_{\mathrm{n}}$	$1, \mathbb{Z}_{\mathrm{d}}, d \mid n$

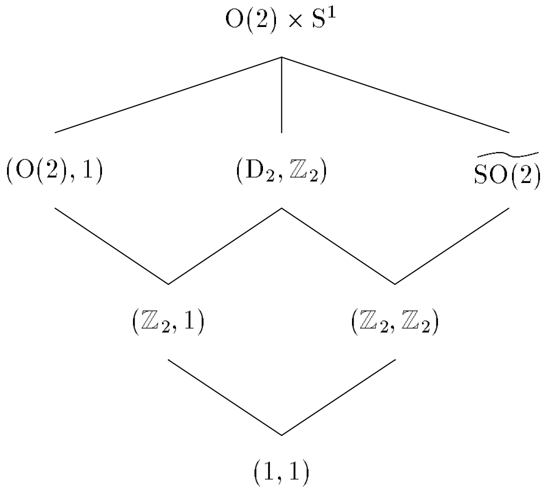

Figure 2: Lattice of isotropy subgroups of $\mathrm{O}(2) \times \mathrm{S}^1$ on $\mathbb{R}^6$

Lemma 3.3.1 *For our representation of the group* $\mathrm{O}(2) \times \mathrm{S}^1$ *on the space* $\mathbb{R}^6$ *one gets the following lattice of isotropy subgroups.*
The following table contains generating elements, representatives and the dimension of the associated fixed point space for every group H^{Θ}.

H^{Θ}	generators	representative	$\mathrm{Dim}\,\mathrm{Fix}(H^{\Theta})$
$\mathrm{O}(2) \times \mathrm{S}^1$	$\mathrm{O}(2) \times \mathrm{S}^1$	$(0, 0, 0)$	0
$(\mathrm{O}(2), 1)$	$(\mathrm{O}(2), 1)$	$(0, z_0, 0)$	2
$\widetilde{\mathrm{SO}(2)}$	$\langle(\phi, \phi),\ \phi \in \mathrm{S}^1\rangle$	$(z_{-2}, 0, 0)$	2
$(\mathrm{D}_2, \mathbb{Z}_2)$	$\langle(\kappa, 1), (\pi, \pi)\rangle$	$(z_2, 0, z_2)$	2
$(\mathbb{Z}_2, \mathbb{Z}_2)$	$\langle(\pi, \pi)\rangle$	$(z_{-2}, 0, z_2)$	4
$(\mathbb{Z}_2, 1)$	$\langle(\kappa, 1)\rangle$	(z_2, z_0, z_2)	4
$(1, 1)$	$\{(1, 1)\}$	(z_{-2}, z_0, z_2)	6

Proof: The dimension of the fixed point space of a potential isotropy subgroup

$$H^{\Theta} \quad \subset \quad \mathrm{O}(2) \times \mathrm{S}^1$$

is given by the trace formula (cf. [GoSt], [GoStSch])

$$\mathrm{Dim}\,\mathrm{Fix}\,H^{\Theta} \quad = \quad \int_{H^{\Theta}} \mathrm{Tr}(h, \theta(h))\,\mathrm{d}h.$$

The values of $\mathrm{Tr}(h,\theta(h))$, $h \in \mathrm{O}(2)$, $\Theta(h) \in \mathrm{S}^1$, are known by Section 3.1. Since we use the diagonal representation of the group $\mathrm{O}(2) \times \mathrm{S}^1$ on $\mathbb{C}^3 \oplus \mathbb{C}^3 \supset \mathbb{R}^6$ the formula yields the real dimension of the fixed point space.

A. Let $\Theta(H) = 1$. Then

$$\begin{aligned}
\mathrm{Dim\,Fix}\,(\mathrm{O}(2),1) &= \frac{1}{2}\left(\frac{1}{2\pi}\int_{\delta=0}^{2\pi} 2(1+2\cos\delta)\,\mathrm{d}\delta + \int_{\delta=0}^{2\pi} 2\,\mathrm{d}\delta\right) = 2,\\
\mathrm{Dim\,Fix}\,(\mathrm{SO}(2),1) &= \frac{1}{2\pi}\int_{\phi=0}^{2\pi} 2(1+2\cos\phi)\,\mathrm{d}\phi = 2,\\
\mathrm{Dim\,Fix}\,(\mathrm{D_n},1) &= \frac{1}{2n}\left(\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} 2\right) = \begin{cases} 4 & n=1,\\ 2 & n\geq 2,\end{cases}\\
\mathrm{Dim\,Fix}\,(\mathbb{Z}_{\mathrm{n}},1) &= \frac{1}{n}\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right) = 2.
\end{aligned}$$

The subspaces $\{(0,z_0,0)\}$ resp. $\{(z_2,z_0,z_2)\}$ have isotropy $(\mathrm{O}(2),1)$ resp. $(\mathbb{Z}_2,1)$ and, consequently, $(\mathrm{O}(2),1)$ resp. $(\mathbb{Z}_2,1)$ are isotropy subgroups with two resp. four dimensional fixed point spaces. Let $\mathbb{Z}_2 = \mathrm{D}_1$ denote the $\mathbb{Z}_2$ generated by κ. The other groups with trivial twist are no isotropy subgroups.

B. Let $\Theta(H) = \mathrm{S}^1$. Possible twists are

$$\begin{array}{rcccc} \Theta_k & : & \mathrm{SO}(2) & \to & \mathrm{S}^1\\ & & \phi & \to & k\phi \end{array}$$

with $k \in \mathbb{N}$. Then we have

$$\mathrm{Dim\,Fix}\,\widetilde{\mathrm{SO}(2)}^{k} = \frac{1}{2\pi}\int_{\phi=0}^{2\pi} 2(1+2\cos\phi)\cos k\phi\,\mathrm{d}\phi = \begin{cases} 2 & k=1,\\ 0 & k>1.\end{cases}$$

The subspace $\{(z_{-2},0,0)\}$ has isotropy $\widetilde{\mathrm{SO}(2)}$ and, therefore, $\widetilde{\mathrm{SO}(2)}$ is an isotropy group with two dimensional fixed point space.

C. Let $\Theta(H) = \mathbb{Z}_2$. Then

$$\mathrm{Dim\,Fix}\,(\mathrm{O}(2),\mathbb{Z}_2) = \frac{1}{2}\left(\frac{1}{2\pi}\int_{\delta=0}^{2\pi} 2(1+2\cos\delta)\,\mathrm{d}\delta - \int_{\delta=0}^{2\pi} 2\,\mathrm{d}\delta\right) = 0.$$

In the case $(\mathrm{D_n},\mathbb{Z}_2)$ there are several possibilities. Let first n be even. Here we have three possible twists.
To begin with let

$$H^{\Theta_{1,n}} = \left\langle \left(\frac{2\pi}{n},\pi\right), (\kappa,1)\right\rangle.$$

Then

$$\begin{aligned}
\mathrm{Dim\,Fix}\,H^{\Theta_{1,n}} &= \frac{1}{2n}\left(\sum_{j=1}^{n} 2(-1)^j\left(1+2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} 2(-1)^j\right)\\
&= \begin{cases} 2 & n=2,\\ 0 & n\geq 4.\end{cases}
\end{aligned}$$

Defining

$$H^{\Theta_{2,n}} = \left\langle \left(\frac{2\pi}{n}, \pi\right), (\kappa, \pi) \right\rangle,$$

we have

$$\begin{aligned} \operatorname{Dim}\operatorname{Fix} H^{\Theta_{2,n}} &= \frac{1}{2n}\left(\sum_{j=1}^{n} 2(-1)^j \left(1 + 2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} 2(-1)^{j+1}\right) \\ &= \begin{cases} 2 & n = 2, \\ 0 & n \geq 4. \end{cases} \end{aligned}$$

Finally let

$$H^{\Theta_{3,n}} = \left\langle \left(\frac{2\pi}{n}, 1\right), (\kappa, \pi) \right\rangle.$$

Then

$$\operatorname{Dim}\operatorname{Fix} H^{\Theta_{3,n}} = \frac{1}{2n}\left(\sum_{j=1}^{n} 2\left(1 + 2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} -2\right) = 0.$$

Setting

$$(\mathrm{D}_2, \mathbb{Z}_2) = \langle(\pi, \pi), (\kappa, 1)\rangle = H^{\Theta_{1,2}},$$

we have

$$\left(-\frac{\pi}{2}, 1\right) H^{\Theta_{2,2}} \left(\frac{\pi}{2}, 1\right) = H^{\Theta_{1,2}}.$$

Therefore both groups are conjugated.
The subspace $\{(z_2, 0, z_2)\}$ has isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$ and, therefore, $(\mathrm{D}_2, \mathbb{Z}_2)$ is an isotropy group with two dimensional fixed point space.
If n is odd, then

$$\operatorname{Dim}\operatorname{Fix}(\mathrm{D_n}, \mathbb{Z}_2) = \frac{1}{2n}\left(\sum_{j=1}^{n} 2\left(1 + 2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} -2\right) = \begin{cases} 2 & n = 1, \\ 0 & n \geq 3. \end{cases}$$

$(\mathrm{D}_1, \mathbb{Z}_2) = \langle(\kappa, \pi)\rangle$ is extended by $H^{\Theta_{2,2}}$ and, consequently, is no isotropy group.
In the case $(\mathbb{Z}_\mathrm{n}, \mathbb{Z}_2)$, in particular n has to be even, we have

$$\operatorname{Dim}\operatorname{Fix}(\mathbb{Z}_\mathrm{n}, \mathbb{Z}_2) = \frac{1}{n}\sum_{j=1}^{n} 2(-1)^j \left(1 + 2\cos\frac{2\pi}{n}j\right) = \begin{cases} 4 & n = 2, \\ 0 & n \geq 4. \end{cases}$$

The subspace $\{(z_{-2}, 0, z_2)\}$ has isotropy $(\mathbb{Z}_2, \mathbb{Z}_2) = \langle(\pi, \pi)\rangle$ and, therefore, $(\mathbb{Z}_2, \mathbb{Z}_2)$ is an isotropy group with four dimensional fixed point space.
D. Finally we have to study the case $(\mathbb{Z}_\mathrm{n}, \mathbb{Z}_\mathrm{d})$ with $d|n$ and $n \geq 2$. Possible nontrivial twists for $\mathbb{Z}_\mathrm{n}$ are

$$\begin{array}{cccc} \Theta_k : & \mathbb{Z}_\mathrm{n} & \to & \mathrm{S}^1 \\ & \frac{2\pi}{n}j & \to & \frac{2\pi}{n}jk \end{array}$$

with $1 \leq k < n$. This gives

$$
\begin{aligned}
\operatorname{Dim}\operatorname{Fix}(\mathbb{Z}_{\mathrm{n}}, \Theta_k(\mathbb{Z}_{\mathrm{n}})) &= \frac{1}{n}\sum_{j=1}^{n} 2\left(1 + 2\cos\frac{2\pi}{n}j\right)\cos\frac{2\pi}{n}jk \\
&= \begin{cases} 4 & n = 2,\ k = 1 \\ 2 & n \geq 3,\ k \in \{1, n-1\} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Studying the representations T_k of $\mathrm{D_n} = \langle \sigma, \kappa \rangle$ on $\mathbb{C}^2$ with

$$
T_k(\sigma) = \begin{pmatrix} e^{-\mathrm{i}\frac{2\pi}{n}k} & 0 \\ 0 & e^{\mathrm{i}\frac{2\pi}{n}k} \end{pmatrix}
$$

and

$$
T_k(\kappa) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

the last equality follows. The representations T_k are irreducible for $n \geq 3$. The representations T_1 and T_{n-1} are conjugated since

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-\mathrm{i}\frac{2\pi}{n}} & 0 \\ 0 & e^{\mathrm{i}\frac{2\pi}{n}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\frac{2\pi}{n}} & 0 \\ 0 & e^{-\mathrm{i}\frac{2\pi}{n}} \end{pmatrix}.
$$

Orthogonality relations for these representations (cf. [La2]) yield the equality.
The case $(\mathbb{Z}_2, \mathbb{Z}_2)$ has been dealt with in part c of the proof, the other cases correspond to conjugated twists of typ

$$
\begin{array}{rccc}
\Theta_k : & \mathbb{Z}_{\mathrm{n}} & \rightarrow & \mathrm{S}^1 \\
& \frac{2\pi}{n}j & \rightarrow & \pm\frac{2\pi}{n}j.
\end{array}
$$

These are extended by the isotropy group $\widetilde{\mathrm{SO}(2)}$. $\bowtie$

LEMMA 3.3.2 *For the isotropy groups $H^{\Theta} \subset \mathrm{SO}(3) \times \mathrm{S}^1$ introduced in the first chapter we have*

H^{Θ}	$\frac{H^{\Theta} \cap \mathrm{N}(\Sigma)}{\Sigma}$
$(\mathbb{Z}_2, 1)$	$(1, 1)$
$(\mathbb{Z}_4, \mathbb{Z}_2)$	$(\mathbb{Z}_2, \mathbb{Z}_2)$
$(\mathrm{D}_2, 1)$	$(\mathbb{Z}_2, 1)$
$(\mathrm{O}(2), 1)$	$(\mathrm{O}(2), 1)$
$(\mathrm{D}_4, \mathbb{Z}_2)$	$(\mathrm{D}_2, \mathbb{Z}_2)$
$\widetilde{\mathrm{SO}(2)}^2$	$\widetilde{\mathrm{SO}(2)}$
$(\mathbb{T}, \mathbb{Z}_3)$	$(\mathbb{Z}_2, 1)$.

Note that

$$
H^{\Theta} \subset N(\Sigma)
$$

for all isotropy groups H^{Θ} except for the group $(\mathbb{T}, \mathbb{Z}_3)$. The group $(\mathbb{T}, \mathbb{Z}_3)$ does not correspond to a special isotropy typ in the $\mathrm{O}(2) \times \mathrm{S}^1$-equivariant system. But the restricted Vector Field (3.2.2) leaves the corresponding two dimensional fixed point space lying in $\mathrm{Fix}\,(\Sigma)$ invariant.

LEMMA 3.3.3

$$\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3) \;=\; \left\{ \left(\frac{\mathrm{i}}{\sqrt{2}} z_0, z_0, \frac{\mathrm{i}}{\sqrt{2}} z_0 \right),\; z_0 \in \mathbb{C} \right\}.$$

Proof: Using the representation of $\mathrm{SO}(3)$ on the space $V_2 \oplus \mathrm{i}V_2$ introduced in the first chapter, one gets the following representation of the group

$$\mathbb{T} \;=\; \langle \pi, \tau \rangle \;\subset\; \mathrm{SO}(3)$$

with

$$\tau \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

on the subspace $\{(z_2, z_0, z_2)\} \subset \mathbb{R}^6$:

$$\begin{aligned} \pi(z_2, z_0, z_2) &= (z_2, z_0, z_2), \\ \tau(z_2, z_0, z_2) &= \left(-\frac{1}{2} z_2 - \frac{1}{2}\sqrt{\frac{3}{2}} z_0, \sqrt{\frac{3}{2}} z_2 - \frac{1}{2} z_0, -\frac{1}{2} z_2 - \frac{1}{2}\sqrt{\frac{3}{2}} z_0 \right). \end{aligned}$$

If an element has the form

$$\left\{ \left(\frac{\mathrm{i}}{\sqrt{2}} z_0, z_0, \frac{\mathrm{i}}{\sqrt{2}} z_0 \right),\; z_0 \in \mathbb{C} \right\},$$

then

$$\left(\tau, e^{\mathrm{i}\frac{2\pi}{3}} \right) (z_2, z_0, z_2) \;=\; (z_2, z_0, z_2).$$ ⋈

3.4 CRITICAL POINTS OF THE REDUCED VECTOR FIELD

LEMMA 3.4.1 *The image of the Hilbert-map $\Pi(\mathbb{R}^6)$ is sketched in Figure 3. One has to imagine circles of radius*

$$\pi_4^2 + \pi_5^2 = \pi_1^2 \pi_3$$

attached to points of the sketch. We have the following assignment

$(\pi_1, \ldots, \pi_5) \in \Pi(\mathbb{R}^6)$	*isotropy typ*
π_1-*axis*	$(\mathrm{O}(2), 1)$
π_2-*axis*	$\widetilde{\mathrm{SO}(2)}$
$\pi_1 = 0,\ \pi_3 = \frac{1}{4}\pi_2^2$	$(\mathrm{D}_2, \mathbb{Z}_2)$
$\pi_1 = 0,\ 0 < \pi_3 < \frac{1}{4}\pi_2^2$	$(\mathbb{Z}_2, \mathbb{Z}_2)$
$\pi_1 > 0,\ \pi_3 = \frac{1}{4}\pi_2^2$	$(\mathbb{Z}_2, 1)$
$\pi_1 > 0,\ 0 \leq \pi_3 < \frac{1}{4}\pi_2^2$	$(1, 1)$.

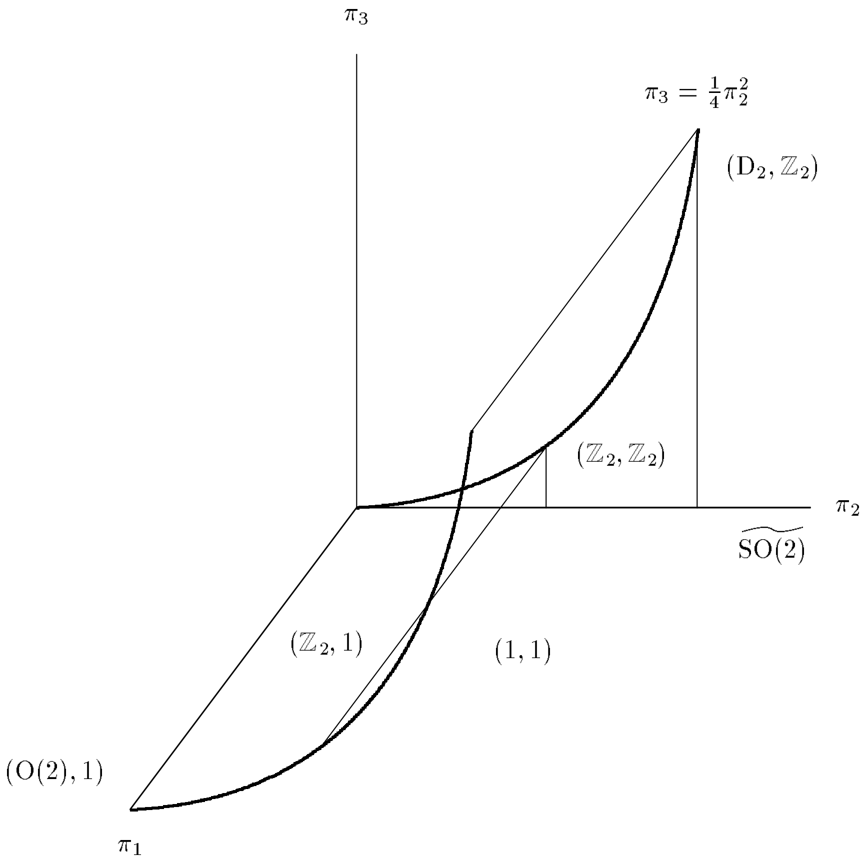

Figure 3: Image of the Hilbert-map

REMARK 3.4.2 *In the following the image of the Hilbert-map* $\Pi(\mathbb{R}^6)$ *shall be denoted Hilbert-set. Since the invariants* π_1, π_2, *and* π_3 *by definition mean radii, only non-negative values are possible. In* (π_1, π_2, π_3)*-space the Hilbert-set is a wedge (cf. Figure 3) limited at the top by the surface* $\pi_3 = \frac{1}{4}\pi_2^2$, *at the bottom by the surface* $\pi_3 = 0$, *and at the back by the surface* $\pi_1 = 0$.

Proof: By definition of the invariants in Lemma 3.1.2 we have

$$\pi_1,\ \pi_2,\ \pi_3 \geq 0.$$

A calculation using Lagrange-multipliers yields the possible values of π_3

$$0 \leq \pi_3 \leq \frac{1}{4}\pi_2^2.$$

The relation

$$\pi_4^2 + \pi_5^2 = \pi_1^2 \pi_3$$

has to be satisfied by Lemma 3.1.2. ⋈

REMARK 3.4.3 *Points with isotropy* $(\mathrm{O}(2),1)$ *and* $(\mathrm{D}_2,\mathbb{Z}_2)$ *and images of points with isotropy* $(\mathbb{T},\mathbb{Z}_3)$ *in the original system (cf. Lemma 3.4.5) under the Hilbert-map satisfy the relation*

$$\Delta = \frac{1}{4}\pi_2^2 - \pi_3 = 0.$$

In the following we shall study the reduced vector field (cf. Lemma 3.2.1) on the Hilbert-set $\Pi(\mathbb{R}^6)$.

LEMMA 3.4.4 *Let*

$$\Delta = \frac{1}{4}\pi_2^2 - \pi_3.$$

Then

$$\dot{\Delta} = 4\,\Delta\,(\lambda + \beta^r\pi_1 + \gamma^r\pi_2)\,.$$

Proof: The stratum

$$\Delta = 0$$

corresponds to points with a certain isotropy and, therefore, is flow invariant. Thus we have $\dot{\Delta} = 0$ for $\Delta = 0$ and there exists a relation of the form

$$\dot{\Delta} = \Delta\, r(\pi_1,\ldots,\pi_5).$$

A simple calculation gives the precise relation. ⋈

LEMMA 3.4.5 *The orbit space reduction maps* Fix $(\mathbb{T},\mathbb{Z}_3)$ *to the invariant curve*

$$\left(\pi_1, \pi_1, \frac{1}{4}\pi_1^2, -\frac{1}{2}\pi_1^2, 0\right) \subset \Pi(\mathbb{R}^6),\ \ \pi_1 > 0,$$

located on the stratum $\Delta = 0$.

Proof: The proof follows directly from the Lemmata 3.1.2 and 3.3.3. ⋈

In the following let the parameter of the Hopf-bifurcation λ be positive:

$$\lambda > 0.$$

We are only interested in supercritical bifurcations.

The restriction of the reduced vector field (cf. Lemma 3.2.1) to the statum $\Delta = 0$ is

$$\begin{aligned}
\dot{\pi_1} &= 2(\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4\Big((\alpha-\beta)^r \pi_4 + (\alpha-\beta)^i \pi_5\Big) && (3.4.3)\\
\dot{\pi_2} &= 2(\lambda + \beta^r \pi_1 + \alpha^r \pi_2)\pi_2 + 4\Big((\alpha-\beta)^r \pi_4 - (\alpha-\beta)^i \pi_5\Big) && (3.4.4)\\
\dot{\pi_3} &= \frac{1}{2}\pi_2\dot{\pi_2} && (3.4.5)\\
\dot{\pi_4} &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_4 + 2(\alpha-\beta)^i(-\pi_1+\pi_2)\pi_5 \\
&\quad +(\alpha-\beta)^r \pi_1\pi_2(\pi_1+\pi_2) && (3.4.6)\\
\dot{\pi_5} &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_5 + 2(\alpha-\beta)^i(\pi_1-\pi_2)\pi_4 \\
&\quad +(\alpha-\beta)^i \pi_1\pi_2(-\pi_1+\pi_2). && (3.4.7)
\end{aligned}$$

Here α^r, β^r resp. α^i, β^i denote the real resp. imaginary parts of α, β.

LEMMA 3.4.6 *Let $\alpha^r, \beta^r < 0$ and $\alpha^r \neq \beta^r$. Then the set of critical points of the Equations 3.4.3 to 3.4.7 on the stratum $\Delta = 0$ is given by a curve*

$$g(\pi_1) = \left(\pi_1, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r}\right), \frac{1}{4}\pi_2^2, \frac{1}{2}\pi_1\pi_2, 0\right), \ 0 \le \pi_1 \le -\frac{\lambda}{\alpha^r},$$

parametrised by π_1 and

$$h(\pi_1) = \left(\pi_1, \pi_1, \frac{1}{4}\pi_1^2, -\frac{1}{2}\pi_1^2, 0\right), \ \pi_1 = -\frac{\lambda}{2\beta^r}.$$

The curve $g(\pi_1)$, $0 \le \pi_1 \le -\frac{\lambda}{\alpha^r}$, connects a critical point with isotropy $(\mathrm{O}(2), 1)$,

$$g\left(-\frac{\lambda}{\alpha^r}\right) = \left(-\frac{\lambda}{\alpha^r}, 0, 0, 0, 0\right),$$

with a critical point with isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$,

$$g(0) = \left(0, \pi_2 = -\frac{\lambda}{\alpha^r}, \frac{1}{4}\pi_2^2, 0, 0\right).$$

The critical point $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$, lies in $\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3))$, the image of points with isotropy $(\mathbb{T}, \mathbb{Z}_3)$ in the original system under the Hilbert-map.

Proof: By addition resp. subtraction of Equations 3.4.3 and 3.4.4 one gets the following equations

$$\begin{aligned}
0 &= \lambda(\pi_1+\pi_2) + \alpha^r(\pi_1^2+\pi_2^2) + 2\beta^r\pi_1\pi_2 + 4(\alpha-\beta)^r\pi_4, && (3.4.8)\\
0 &= \lambda(\pi_1-\pi_2) + \alpha^r(\pi_1^2-\pi_2^2) + 4(\alpha-\beta)^i\pi_5. && (3.4.9)
\end{aligned}$$

Let $(\alpha-\beta)^i \neq 0$ then

$$\begin{aligned}
\pi_4 &= -\frac{\lambda(\pi_1+\pi_2) + \alpha^r(\pi_1^2+\pi_2^2) + 2\beta^r\pi_1\pi_2}{4(\alpha-\beta)^r},\\
\pi_5 &= -\frac{\lambda(\pi_1-\pi_2) + \alpha^r(\pi_1^2-\pi_2^2)}{4(\alpha-\beta)^i}.
\end{aligned}$$

Inserting this in Equations 3.4.6 and 3.4.7 gives

$$\begin{aligned} 0 &= -\frac{(\pi_1+\pi_2)\Big(\lambda+\alpha^r(\pi_1+\pi_2)\Big)\Big(\lambda+\beta^r(\pi_1+\pi_2)\Big)}{(\alpha-\beta)^r}, \\ 0 &= -\frac{(\pi_1-\pi_2)\Big(\lambda+\alpha^r(\pi_1+\pi_2)\Big)}{2(\alpha-\beta)^r(\alpha-\beta)^i} \\ &\quad \Big(2\lambda(\alpha-\beta)^r+(\pi_1+\pi_2)(\alpha^{r2}-\beta^{r2}+(\alpha-\beta)^{i2}\Big). \end{aligned} \tag{3.4.10}$$

Looking for nontrivial critical points, one, therefore, has to study two cases.
Let $\pi_1+\pi_2=-\frac{\lambda}{\alpha^r}$. Since we assume $\lambda>0$, only the choice $\alpha^r<0$ gives solutions that lie in $\Pi(\mathbb{R}^6)$. By insertion one gets the curve

$$g(\pi_1)=\left(\pi_1,\pi_2=-\left(\pi_1+\frac{\lambda}{\alpha^r}\right),\frac{1}{4}\pi_2^2,\frac{1}{2}\pi_1\pi_2,0\right),\ 0\leq\pi_1\leq-\frac{\lambda}{\alpha^r},$$

of critical points. Lemma 3.4.1 gives the associated orbit types.
Now let $\pi_1+\pi_2=-\frac{\lambda}{\beta^r}$. Only the choice $\beta^r<0$ gives solutions that lie in $\Pi(\mathbb{R}^6)$ as above. By insertion in Equation 3.4.10 one gets the condition

$$0 = \frac{\Big((\alpha-\beta)^{r2}+(\alpha-\beta)^{i2}\Big)\lambda^2(\lambda+2\beta^r\pi_2)}{2\beta^{r3}(\alpha-\beta)^i}.$$

In order to get critical points, one has to choose

$$\pi_1=\pi_2=-\frac{\lambda}{2\beta^r}.$$

By insertion one obtains the critical point

$$h(\pi_1)=\left(\pi_1,\pi_1,\frac{1}{4}\pi_1^2,-\frac{1}{2}\pi_1^2,0\right),\ \pi_1=-\frac{\lambda}{2\beta^r},$$

lying in $\Pi(\mathrm{Fix}(\mathbb{T},\mathbb{Z}_3))$ (cf. Lemma 3.4.5). It shall be shown that there are no other critical points with radius

$$\pi_1+\pi_2 = -\frac{\lambda}{\beta^r}.$$

Therefore the group orbit of periodic orbits with isotropy $(\mathbb{T},\mathbb{Z}_3)$ in the original system can only intersect the stratified space in the curve given in Lemma 3.4.5.
Now let $(\alpha-\beta)^i=0$. Equations 3.4.8 and 3.4.9 yield

$$\begin{aligned} 0 &= \lambda(\pi_1+\pi_2)+\alpha^r(\pi_1^2+\pi_2^2)+2\beta^r\pi_1\pi_2+4(\alpha-\beta)^r\pi_4, \\ 0 &= (\pi_1-\pi_2)\Big(\lambda+\alpha^r(\pi_1+\pi_2)\Big). \end{aligned}$$

Consequently we have to study two cases.
Let $\pi_1=\pi_2$. Then

$$\pi_4 = -\frac{\Big(\lambda+(\alpha+\beta)^r\pi_1\Big)\pi_1}{2(\alpha-\beta)^r}.$$

By insertion in Equation 3.4.6 one gets

$$0 \quad = \quad -\frac{\pi_1(\lambda + 2\beta^r \pi_1)(\lambda + 2\alpha^r \pi_1)}{(\alpha - \beta)^r}.$$

The choice $\pi_1 = \pi_2 = -\frac{\lambda}{2\alpha^r}$ and the relation

$$\pi_4^2 + \pi_5^2 = \pi_1^2 \pi_3 = \frac{1}{4}\pi_1^4$$

give the critical point

$$\left(\pi_1, \pi_1, \frac{1}{4}\pi_1^2, \frac{1}{2}\pi_1^2, 0\right), \;\; \pi_1 = -\frac{\lambda}{2\alpha^r},$$

that lies on the curve $g(\pi_1)$.
The case $\pi_1 = \pi_2 = -\frac{\lambda}{2\beta^r}$ again yields the solution $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$.
Finally we have to study the case $\pi_1 + \pi_2 = -\frac{\lambda}{\alpha^r}$. We get

$$\begin{aligned} \pi_4 &= -\frac{\pi_1}{2\alpha^r}(\lambda + \alpha^r \pi_1) \\ &= \frac{1}{2}\pi_1 \pi_2. \end{aligned}$$

The relation

$$\pi_4^2 + \pi_5^2 \quad = \quad \frac{1}{4}\pi_1^2 \pi_2^2$$

yields $\pi_5 = 0$. So again we get the curve $g(\pi_1)$. ⋈

LEMMA 3.4.7

$$\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3)) \cap g(\pi_1) = \emptyset, \;\; 0 \leq \pi_1 \leq -\frac{\lambda}{\alpha^r}.$$

The critical point $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$, (cf. Lemma 3.4.6) that lies in $\Pi(\mathrm{Fix}(\mathbb{T}, \mathbb{Z}_3))$ is isolated in the Hilbert-set $\Pi(\mathbb{R}^6)$.

Proof: For points lying on the curve $g(\pi_1)$ we have $\pi_1 + \pi_2 = -\frac{\lambda}{\alpha^r}$. Points in $\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3))$ satisfy the condition $\pi_1 = \pi_2$ (cf. Lemma 3.4.5). For a potential intersection this means $\pi_1 = \pi_2 = -\frac{\lambda}{2\alpha^r}$. We have

$$g\left(-\frac{\lambda}{2\alpha^r}\right) \quad = \quad \left(-\frac{\lambda}{2\alpha^r}, -\frac{\lambda}{2\alpha^r}, \frac{1}{16}\frac{\lambda^2}{\alpha^{r2}}, +\frac{1}{8}\frac{\lambda^2}{\alpha^{r2}}, 0\right)$$

whereas

$$\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3)) \cap \left(\pi_1 = -\frac{\lambda}{2\alpha^r}\right) \quad = \quad \left(-\frac{\lambda}{2\alpha^r}, -\frac{\lambda}{2\alpha^r}, \frac{1}{16}\frac{\lambda^2}{\alpha^{r2}}, -\frac{1}{8}\frac{\lambda^2}{\alpha^{r2}}, 0\right).$$

On the stratum $\Delta = 0$ the critical point $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$, (cf. Lemma 3.4.6) that lies on $\Pi(\mathrm{Fix}(\mathbb{T}, \mathbb{Z}_3))$, therefore, is isolated. We shall show in Lemma 3.4.8 that there are no further critical points in the Hilbert-set in the region $\Delta \neq 0$ near $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$. ⋈

Now we are looking for critical points of the reduced vector field (cf. Lemma 3.2.1) in $\Pi(\mathbb{R}^6)$ that do not lie on the stratum $\Delta = 0$. Such a critical point has to meet the condition (cf. Lemma 3.4.4)

$$\dot{\Delta} = 4\,\Delta\,(\lambda + \beta^r \pi_1 + \gamma^r \pi_2) = 0.$$

Since we assumed $\Delta \neq 0$, this means

$$\lambda + \beta^r \pi_1 + \gamma^r \pi_2 \quad = \quad 0. \tag{3.4.11}$$

So we get the following equations:

$$\begin{aligned}
0 &= 2(\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4\Big((\alpha-\beta)^r \pi_4 + (\alpha-\beta)^i \pi_5\Big) && (3.4.12)\\
0 &= 8(\alpha-\gamma)^r \pi_3 + 4\Big((\alpha-\beta)^r \pi_4 - (\alpha-\beta)^i \pi_5\Big) && (3.4.13)\\
0 &= \frac{1}{2}\pi_2\dot{\pi}_2 && (3.4.14)\\
0 &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_4 + 2(\alpha-\beta)^i(-\pi_1+\pi_2)\pi_5 \\
&\quad +(\alpha-\beta)^r \pi_1(\pi_1\pi_2 + 4\pi_3) && (3.4.15)\\
0 &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_5 + 2(\alpha-\beta)^i(\pi_1-\pi_2)\pi_4 \\
&\quad +(\alpha-\beta)^i \pi_1(-\pi_1\pi_2 + 4\pi_3) && (3.4.16)\\
\pi_2 &= -\frac{\lambda + \beta^r \pi_1}{\gamma^r}. && (3.4.17)
\end{aligned}$$

Here $\alpha^r, \beta^r, \gamma^r$ resp. α^i, β^i again denote the real resp. imaginary parts of α, β, γ. In the following we shall assume

$$\beta^r < \alpha^r < \gamma^r < 0.$$

In Lemma 3.5.1 we shall show that only for this choice of the coefficients the solutions with isotropy $(\mathrm{O}(2), 1)$ resp. $\widetilde{\mathrm{SO}(2)}$ can be stable simultaneously. Investigations using the topological Conlex-index suggested to study this case. In the following lemma the solution with isotropy $\widetilde{\mathrm{SO}(2)}$ is being described.

LEMMA 3.4.8 *Let $\beta^r < \alpha^r < \gamma^r < 0$. Then*

$$\left(0, -\frac{\lambda}{\gamma^r}, 0, 0, 0\right)$$

is the only critical point of the reduced vector field in $\Pi(\mathbb{R}^6)$ with $\Delta \neq 0$. This solution has isotropy $\widetilde{\mathrm{SO}(2)}$.

Proof: First let $(\alpha-\beta)^i \neq 0$. By addition resp. subtraction of Equations 3.4.12 and 3.4.13 we get

$$\begin{aligned}
0 &= (\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4(\alpha-\beta)^r \pi_4 + 4(\alpha-\gamma)^r \pi_3,\\
0 &= (\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4(\alpha-\beta)^i \pi_5 - 4(\alpha-\gamma)^r \pi_3.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\pi_4 &= -\frac{(\lambda+\alpha^r\pi_1+\beta^r\pi_2)\pi_1+4(\alpha-\gamma)^r\pi_3}{4(\alpha-\beta)^r},\\
\pi_5 &= -\frac{(\lambda+\alpha^r\pi_1+\beta^r\pi_2)\pi_1-4(\alpha-\gamma)^r\pi_3}{4(\alpha-\beta)^i}.
\end{aligned}$$

Insertion in Equation 3.4.15 yields

$$\frac{(\beta-\gamma)^r\Big(\lambda(\alpha-\gamma)^r+\alpha^r\pi_1(\beta-\gamma)^r\Big)(-\lambda\pi_1-\beta^r\pi_1^2+4\gamma^r\pi_3)}{(\alpha-\beta)^r\gamma^{r\,2}} = 0.$$

Let

$$\pi_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r} > 0.$$

Using Equation 3.4.17 we get

$$\pi_2 = \frac{\lambda(\alpha-\beta)^r}{\alpha^r(\beta-\gamma)^r} > 0.$$

Together with Equation 3.4.16 this yields

$$0 = \frac{(\alpha-\gamma)^r\Big((\alpha-\beta)^{r\,2}+(\alpha-\beta)^{i\,2}\Big)\lambda\Big(-\lambda^2(\alpha-\beta)^{r\,2}+4\alpha^{r\,2}\pi_3(\beta-\gamma)^{r\,2}\Big)}{2\alpha^{r\,3}(\alpha-\beta)^r(\alpha-\beta)^i(\beta-\gamma)^{r\,2}}.$$

So we have

$$\pi_3 = \frac{\lambda^2(\alpha-\beta)^{r\,2}}{4\alpha^{r\,2}(\beta-\gamma)^{r\,2}} = \frac{1}{4}\pi_2^2.$$

This solution lies on the stratum $\Delta = 0$.
Now let

$$\pi_3 = \frac{\pi_1(\lambda+\beta^r\pi_1)}{4\gamma^r} = -\frac{1}{4}\pi_1\pi_2.$$

Insertion in Equation 3.4.16 yields

$$\begin{aligned}
0 &= -2(\alpha-\beta)^{i\,2}\pi_1^2\pi_2-\frac{(\alpha-\beta)^{i\,2}\pi_1\Big(\lambda+(\beta+\gamma)^r\pi_1\Big)^2}{2\gamma^{r\,2}}\\
&\quad-\frac{\pi_1\Big(\lambda(\alpha+\beta-2\gamma)^r+\pi_1(\alpha+\beta)^r(\beta-\gamma)^r\Big)^2}{2\gamma^{r\,2}}.
\end{aligned}$$

Since all elements of the sum are nonpositive in $\Pi(\mathbb{R}^6)$ the sum can only be zero if all elements are zero. This is only possible if $\pi_1 = 0$. This yields

$$\left(0,-\frac{\lambda}{\gamma^r},0,0,0\right),$$

the solution with isotropy $\widetilde{\mathrm{SO}(2)}$.

Second let $(\alpha-\beta)^i=0$. Again by addition resp. subtraction of the Equations 3.4.12 and 3.4.13 we get

$$\begin{aligned}\pi_3 &= \frac{(\lambda+\alpha^r\pi_1+\beta^r\pi_2)\pi_1}{4(\alpha-\gamma)^r},\\ \pi_4 &= -\frac{2(\alpha-\gamma)^r\pi_3}{(\alpha-\beta)^r}.\end{aligned} \tag{3.4.18}$$

Furthermore we have

$$\pi_2 = -\frac{\lambda+\beta^r\pi_1}{\gamma^r}.$$

Insertion in Equation 3.4.16 yields

$$0 = \Big(2\lambda+(\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_5.$$

In order to solve this equation, we have to look at several cases.
Let $\pi_5=0$. Then

$$\pi_4^2+\pi_5^2 = \pi_1^2\pi_3$$

and Equation 3.4.18 yields

$$\frac{4(\alpha-\gamma)^{r2}}{(\alpha-\beta)^{r2}}\pi_3^2 = \pi_1^2\pi_3.$$

For $\pi_3\neq 0$ we get

$$\pi_1=-\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}>0,\ \pi_2=\frac{\lambda(\alpha-\beta)^r}{\alpha^r(\beta-\gamma)^r}>0,\ \pi_3=\frac{1}{4}\pi_2^2.$$

Therefore the solution lies on the stratum $\Delta=0$.
The choice $\pi_3=0$ and Equation 3.4.16 yield the solution

$$\left(0,-\frac{\lambda}{\gamma^r},0,0,0\right)$$

with isotropy $\widetilde{\mathrm{SO}(2)}$. For $\pi_5\neq 0$ and

$$0 = 2\lambda+(\alpha+\beta)^r(\pi_1+\pi_2)$$

Equation 3.4.15 gives

$$0 = \pi_1(\pi_1\pi_2+4\pi_3).$$

Choosing $\pi_1=0$ again yields the solution with isotropy $\widetilde{\mathrm{SO}(2)}$.
For $\pi_1\neq 0$ one gets the solution

$$\begin{aligned}\pi_1 &= -\frac{\lambda(\alpha+\beta-2\gamma)^r}{(\alpha+\beta)^r(\beta-\gamma)^r}>0,\\ \pi_2 &= \frac{\lambda(\alpha-\beta)^r}{(\alpha+\beta)^r(\beta-\gamma)^r}>0,\\ \pi_3 &= \frac{\lambda^2(\alpha-\beta)^r(\alpha+\beta-2\gamma)^r}{4(\alpha+\beta)^{r2}(\beta-\gamma)^{r2}}<0\end{aligned}$$

that does not lie in $\Pi(\mathbb{R}^6)$. $\bowtie$

Figure 4 sketches the position of the critical points of the reduced vector field (cf. Lemma 3.2.1) in the Hilbert-set $\Pi(\mathbb{R}^6)$ known by Lemmata 3.4.6 and 3.4.8 under the assumption

$$\beta^r < \alpha^r < \gamma^r < 0.$$

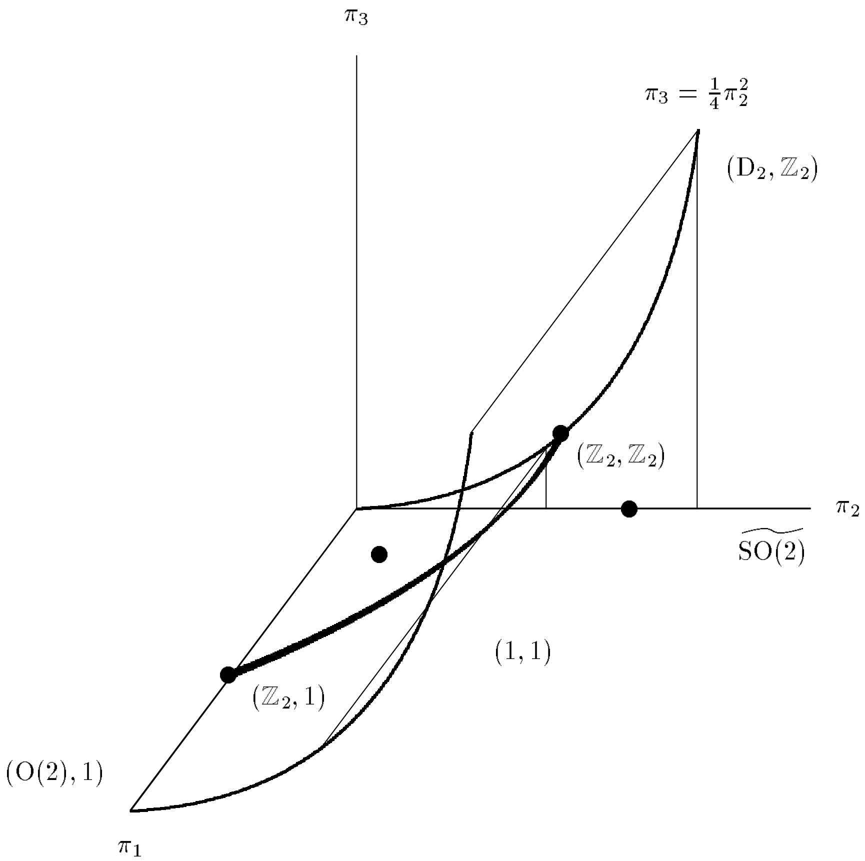

Figure 4: Critical points of the reduced vector field in the Hilbert-set

We now study the curve

$$g(\pi_1) = \left(\pi_1, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r}\right), \frac{1}{4}\pi_2^2, \frac{1}{2}\pi_1\pi_2, 0\right), \quad 0 < \pi_1 < -\frac{\lambda}{\alpha^r},$$

of critical points of the Equations (3.4.3) to (3.4.7) (cf. Lemma 3.4.6).

LEMMA 3.4.9 *The preimage of a point $g(\pi_1), \pi_1 \in \left(0, -\frac{\lambda}{\alpha^r}\right)$, in $\mathbb{R}^6$ is a two-torus. It is fibered with periodic solutions.*

Proof: The curve g of critical points lies on the statum

$$\Delta = \frac{1}{4}\pi_2^2 - \pi_3 = 0.$$

Introducing polar coordinates in the form

$$z_j = r_j\, e^{\mathrm{i}\phi_j}, \quad j \in \{-2, 0, 2\},$$

yields

$$r_{-2} \quad = \quad r_2.$$

The choice of

$$\pi_1 = r_0^2 \in \left(0, -\frac{\lambda}{\alpha^r}\right)$$

and the condition

$$\pi_1 + \pi_2 \quad = \quad -\frac{\lambda}{\alpha^r}$$

determine the radii. Let

$$\theta \quad = \quad 2\phi_0 - \phi_{-2} - \phi_2.$$

Then the conditions for π_4 resp. π_5 yield, in polar coordinates, the phase relations

$$\cos\theta = 1, \quad \sin\theta = 0$$

and, thus,

$$\theta \quad = \quad 0 \bmod 2\pi.$$

So one angle is determined, two are still available, the preimage is a 2-torus. Points on the surface $\Delta = 0$ have the (conjugated) isotropy $(\mathbb{Z}_2, 1)$. Therefore it is possible just to look at points of the form (z_2, z_0, z_2) in order to determine the resulting flow on the preimage of a point on the curve of fixed points. Thus we have the additional condition

$$\phi_{-2} \quad = \quad \phi_2.$$

Using $\theta = 0 \bmod 2\pi$, one sees that

$$\phi_0 \quad = \quad \phi_2 \bmod \pi.$$

Inserting this into the differential equation yields

$$\dot{\phi}_0 = \omega_0 = \omega + \alpha^i\left(r_0^2 + 2r_2^2\right).$$

Thus the 2−torus is fibered with periodic solutions of period near $\frac{2\pi}{\omega}$. $\bowtie$

3.5 Stability of the critical points of the reduced vector field

In Lemmata 3.4.6 and 3.4.8 we have shown that in the case of supercritical bifurcation $(\lambda > 0)$ the coefficients $\alpha^r, \beta^r, \gamma^r$ have to be negative in order that the corresponding solutions lie in the image of the Hilbert-map $\Pi(\mathbb{R}^6)$. The following lemma gives a condition on the choice of the coefficients relative to each other.

LEMMA 3.5.1 *Only by choosing the coefficients*

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta+3\gamma)^r, \gamma^r\right),$$

the critical points with isotropy $(\mathrm{O}(2),1)$ *resp.* $\widetilde{\mathrm{SO}(2)}$ *of the reduced vector field (cf. Lemma 3.2.1) can be stable simultaneously. The stability of the solution with isotropy* $(\mathrm{O}(2),1)$ *is determined by higher order terms because of the existence of a curve of critical points (cf. Lemma 3.4.6).*

Proof: The calculations of [IoRo] yield (using our parameters) up to third order the following conditions for the stability of the periodic solutions with isotropy $(\mathrm{O}(2),1)$ resp. $\widetilde{\mathrm{SO}(2)}$ in the original, ten dimensional system:

isotropy	nontrivial Floquet-exponents
$(\mathrm{O}(2),1)$	$-2\lambda < 0, -\frac{2\lambda}{\alpha^r}(\alpha-\beta)^r < 0, -\frac{2\lambda}{\alpha^r}(-4\alpha+\beta+3\gamma)^r < 0$
$\widetilde{\mathrm{SO}(2)}$	$-2\lambda < 0, -\frac{2\lambda}{\gamma^r}(\alpha-\gamma), cc, \frac{\lambda}{\gamma^r}(\gamma-\beta), cc, \frac{3\lambda}{2\gamma^r}(\gamma-\beta), cc.$

Here cc denotes the complex conjugate of the preceding number.
So we get the conditions

$$\beta^r < \alpha^r < \gamma^r < 0$$

and

$$\beta^r + 3\gamma^r < 4\alpha^r.$$

The ansatz

$$\alpha^r = t\,\beta^r + (1-t)\,\gamma^r, \quad t \in (0,1),$$

yields

$$(\beta-\gamma)^r(1-4t) < 0$$

and, therefore, we have

$$t \in \left(0, \frac{1}{4}\right).$$

This means

$$\alpha^r \in \left(\frac{1}{4}(\beta+3\gamma)^r, \gamma^r\right).$$

Especially

$$\frac{(\alpha-\gamma)^r}{(\beta-\gamma)^r} \in \left(0, \frac{1}{4}\right). \qquad \bowtie$$

Now we want to determine the linearization of the reduced vector field (cf. Lemma 3.2.1) along the curve $g(\pi_1)$ of critical points (cf. Lemma 3.4.6). For the general linearization L one gets

$$L = \left(\begin{array}{ccccc} 2\lambda + 4\pi_1\alpha^r + 2\pi_2\beta^r & 2\pi_1\beta^r & 0 & 4(\alpha-\beta)^r & 4(\alpha-\beta)^i \\ 2\pi_2\beta^r & 2\lambda + 2\pi_1\beta^r + 4\pi_2\gamma^r & 8(\alpha-\gamma)^r & 4(\alpha-\beta)^r & -4(\alpha-\beta)^i \\ 4\pi_3\beta^r & 4\pi_3\alpha^r + 2\big(\pi_4(\alpha-\beta)^r - \pi_5(\alpha-\beta)^i\big) & 4(\lambda+\beta^r\pi_1+\alpha^r\pi_2) & 2\pi_2(\alpha-\beta)^r & -2\pi_2(\alpha-\beta)^i \\ 2\pi_4(\alpha+\beta)^r - 2\pi_5(\alpha-\beta)^i + (2\pi_1\pi_2+4\pi_3)(\alpha-\beta)^r & 2\pi_4(\alpha+\beta)^r + 2\pi_5(\alpha-\beta)^i + \pi_1^2(\alpha-\beta)^r & 4\pi_1(\alpha-\beta)^r & 4\lambda + 2(\pi_1+\pi_2)(\alpha+\beta)^r & 2(-\pi_1+\pi_2)(\alpha-\beta)^i \\ 2\pi_5(\alpha+\beta)^r + 2\pi_4(\alpha-\beta)^i + (-2\pi_1\pi_2+4\pi_3)(\alpha-\beta)^i & 2\pi_5(\alpha+\beta)^r - 2\pi_4(\alpha-\beta)^i - \pi_1^2(\alpha-\beta)^i & 4\pi_1(\alpha-\beta)^i & 2(\pi_1-\pi_2)(\alpha-\beta)^i & 4\lambda + 2(\pi_1+\pi_2)(\alpha+\beta)^r \end{array} \right).$$

We are interested in the eigenvalues of L along the curve $g(\pi_1)$ with reference to $\Pi(\mathbb{R}^6) \subset \mathbb{R}^5$. Thus we have to determine the tangent space at points of the curve in $\Pi(\mathbb{R}^6)$. It is given by the relation

$$\pi_4^2 + \pi_5^2 \quad = \quad \pi_1^2\pi_3.$$

The curve itself lies on the stratum

$$\Delta = \frac{1}{4}\pi_2^2 - \pi_3 = 0.$$

So we get the following lemma.

Lemma 3.5.2 *The tangent space at the stratum $\Delta = 0$ along the curve*

$$g(\pi_1),\ 0 < \pi_1 < -\frac{\lambda}{\alpha^r},$$

is spanned by the vectors

$$\begin{aligned} t_1 &= \left(1, -1, -\frac{1}{2}\pi_2, \frac{1}{2}(\pi_2-\pi_1), 0\right), \\ t_2 &= \left(\pi_1, \pi_2, \frac{1}{2}\pi_2^2, \pi_1\pi_2, 0\right), \\ t_3 &= \big(2\alpha^r(\alpha-\beta)^i, -2\alpha^r(\alpha-\beta)^i, -\pi_2\alpha^r(\alpha-\beta)^i, \alpha^r(\pi_2-\pi_1)(\alpha-\beta)^i, \\ &\quad -\alpha^r(\pi_1+\pi_2)(\alpha-\beta)^r\big). \end{aligned}$$

The vectors t_1, t_2, t_3 are eigenvectors of L to the eigenvalues

$$\begin{aligned} ew_1 &= 0, \\ ew_2 &= -2\lambda \ = \ 2\alpha^r(\pi_1+\pi_2), \\ ew_3 &= \frac{2(\alpha-\beta)^r\lambda}{\alpha^r} \ = \ -2(\alpha-\beta)^r(\pi_1+\pi_2). \end{aligned}$$

The curve $g(\pi_1)$, $0<\pi_1<-\frac{\lambda}{\alpha^r}$, *is stable on the stratum* $\Delta=0$.

Proof: The relations $\pi_4^2+\pi_5^2=\pi_1^2\pi_3$ and $\Delta=0$ yield the following vectors normal to the tangent space at the surface $\Delta=0$ in $\Pi(\mathbb{R}^6)\subset\mathbb{R}^5$:

$$\begin{aligned} n_1 &= \left(-2\pi_1\pi_3, 0, -\pi_1^2, 2\pi_4, 2\pi_5\right), \\ n_2 &= \left(0, \frac{1}{2}\pi_2, -1, 0, 0\right). \end{aligned}$$

The orthogonal complement to $Span(n_1,n_2)$ is spanned by the vectors t_1,t_2,t_3. A simple calculation shows that these vectors are eigenvectors to the given eigenvalues. The eigenvector t_1 points along the curve of critical points. Therefore the associated eigenvalue is zero. By definition of the curve $g(\pi_1)$ we have

$$\pi_1+\pi_2=-\frac{\lambda}{\alpha^r}.$$

Therefore the curve $g(\pi_1)$, $0<\pi_1<-\frac{\lambda}{\alpha^r}$, is stable on the stratum $\Delta=0$. $\bowtie$

Now we want to determine the linearization of the reduced vector field along the curve $g(\pi_1)$ of fixed points in the direction of the principal stratum. We shall show that there exists a point $\tilde{\pi}_1$ on the curve $g(\pi_1)$ in which the stability of the curve changes from stable to unstable in the direction of the principal stratum. In this point the linearization L of the vector field of the reduced equation has a nontrivial two dimensional Jordan-block with respect to the eigenvalue zero.
Let

$$t = (0,1,0,0,0).$$

Then $n_1 t=0$ and $n_2 t\neq 0$ for $\pi_2\neq 0$. Thus the vectors t_1,t_2,t_3,t span the tangent space at the Hilbert-set $\Pi(\mathbb{R}^6)$ along the curve $g(\pi_1)$, $0<\pi_1<-\frac{\lambda}{\alpha^r}$. One gets

$$Lt = a\,t_1+b\,t_2+c\,t_3+d\,t$$

with

$$\begin{aligned} a &= -2\pi_1\frac{(\alpha-\beta)^{r\,2}+(\alpha-\beta)^{i\,2}}{(\alpha-\beta)^r}, \\ b &= 2\alpha^r, \\ c &= \pi_1\frac{(\alpha-\beta)^i}{\alpha^r(\alpha-\beta)^r}, \\ d &= 4\frac{\lambda(\alpha-\gamma)^r+\pi_1\alpha^r(\beta-\gamma)^r}{\alpha^r}. \end{aligned}$$

Restricted to the tangent space at the curve $g(\pi_1)$, $0<\pi_1<-\frac{\lambda}{\alpha^r}$, according to our choice of the vectors t_1,t_2,t_3,t, L has the form $\tilde{L}$:

$$\tilde{L} = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & -2\lambda & 0 & b \\ 0 & 0 & \frac{2(\alpha-\beta)^r\lambda}{\alpha^r} & c \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Especially the fourth eigenvalue is

$$\begin{aligned} ew_4 &= d \\ &= 4\frac{\lambda(\alpha-\gamma)^r + \pi_1\alpha^r(\beta-\gamma)^r}{\alpha^r} \;=\; -4\Big(\pi_1(\alpha-\beta)^r + \pi_2(\alpha-\gamma)^r\Big). \end{aligned}$$

For

$$\tilde\pi_1 \;=\; -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}$$

we have $ew_4(\tilde\pi_1) = 0$. Choosing the coefficients according to Lemma 3.5.1 yields

$$0 < \tilde\pi_1 < -\frac{\lambda}{4\alpha^r}.$$

The point $g(\tilde\pi_1)$ is exactly the intersection point of the curve $g(\pi_1)$, $0 \le \pi_1 \le -\frac{\lambda}{\alpha^r}$, with the surface $\lambda + \beta^r\pi_1 + \gamma^r\pi_2 = 0$ (cf. Lemma 3.4.4). Only on this surface we can have critical points of the reduced vector field (cf. Lemma 3.2.1) outside the stratum $\Delta = 0$ (cf. Lemma 3.4.6).
For

$$h(\tilde\pi_1) \;=\; \begin{pmatrix} -\frac{(\alpha-\gamma)^r\lambda\left((\alpha-\beta)^{r2}-(\alpha-\beta)^{i2}\right)}{(\alpha-\beta)^{r2}(\beta-\gamma)^r} \\ \frac{(\alpha-\gamma)^r\lambda\left((\alpha-\beta)^{r2}-(\alpha-\beta)^{i2}\right)}{(\alpha-\beta)^{r2}(\beta-\gamma)^r} \\ \frac{\lambda^2\left((\alpha-\beta)^{r3}-(\alpha-\gamma)^r(\alpha-\beta)^{i2}\right)}{2\alpha^r(\alpha-\beta)^r(\beta-\gamma)^{r2}} \\ -\frac{(\alpha-\gamma)^r\lambda^2\left(2(\alpha-\beta)^{r3}+(-2\alpha+\beta+\gamma)^r(\alpha-\beta)^{i2}\right)}{2\alpha^r(\alpha-\beta)^{r2}(\beta-\gamma)^{r2}} \\ \frac{(\alpha-\gamma)^r(\alpha-\beta)^i\lambda^2}{2\alpha^r(\alpha-\beta)^r(\beta-\gamma)^r} \end{pmatrix}$$

we have

$$h(\tilde\pi_1) \;=\; \alpha^r t_2(\tilde\pi_1) - \frac{(\alpha-\beta)^i\tilde\pi_1}{2(\alpha-\beta)^{r2}}t_3(\tilde\pi_1) + \lambda t.$$

Consequently, $h(\tilde\pi_1) \in \operatorname{Span}\Big(t_1(\tilde\pi_1), t_2(\tilde\pi_1), t_3(\tilde\pi_1), t\Big)$, and one sees that $L\,h(\tilde\pi_1) = j\,t_1(\tilde\pi_1)$ with

$$j \;=\; \frac{2\lambda^2(\alpha-\gamma)^r\Big((\alpha-\beta)^{r2}+(\alpha-\beta)^{i2}\Big)}{\alpha^r(\alpha-\beta)^r(\beta-\gamma)^r} \;<\; 0.$$

So we have shown the following lemma.

LEMMA 3.5.3 *In the point $g(\tilde\pi_1)$, $\tilde\pi_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}$, the linearization L of the vector field of the reduced equation (cf. Lemma 3.2.1) has a nontrivial, two dimensional Jordan-block with respect to the eigenvalue zero.*

Up to now we have studied the reduced vector field resulting from the normal form up to third order (cf. [IoRo]). It has been shown in this section that this vector field is degenerate. In the next section we shall use fifth order terms to investigate this degeneracy.

3.6 FIFTH ORDER TERMS

Restricted to $\mathrm{Fix}(\mathbb{Z}_2, 1)$ the normal form (cf. [IoRo]) yields the following fifth order terms (FOT). Proceed as in Chapter 3.2 to get these terms.

$$\begin{aligned}
\mathrm{FOT} \;=\; & (\delta_1 \pi_1^2 + \delta_2 \pi_1 \pi_2 + \delta_3 \pi_2^2 + \delta_4 \pi_3 + \delta_5 \pi_4)\, e_1 \\
& + (\delta_6 \pi_1^2 + \delta_7 \pi_1 \pi_2 + \delta_8 \pi_2^2 + \delta_9 \pi_3 + \delta_{10} \pi_4 + \delta_{11} i \pi_5)\, e_2 \\
& + (\delta_{12} \pi_1 + \delta_{13} \pi_2)\, e_3 + (\delta_{14} \pi_1 + \delta_{15} \pi_2)\, e_4 - \frac{1}{2} \delta_{11} \pi_2 i e_5 .
\end{aligned}$$

The coefficients $\delta_1, \ldots, \delta_{15} \in \mathbb{C}$ result from a transformation of the coefficients $d_1, \ldots, d_9 \in \mathbb{C}$ of the normalform (cf. [IoRo])

$$\begin{aligned}
d_3 &\to \sqrt{6}\, d_3 \\
d_4 &\to -d_4 \\
d_5 &\to -\sqrt{6}\, d_5 \\
d_6 &\to \sqrt{\tfrac{3}{2}}\, d_6 \\
d_7 &\to \sqrt{\tfrac{3}{2}}\, d_7 \\
d_9 &\to \tfrac{3}{8}\, d_9
\end{aligned}$$

as follows

$$\begin{aligned}
\delta_1 &= d_1 + \frac{1}{4} d_2 - 3 d_3 + \frac{1}{2} d_4 - \frac{3}{2} d_5 + d_6 + d_7 - d_8 \\
\delta_2 &= 2 d_1 - 7 d_3 + \frac{1}{2} d_4 - 2 d_5 - 2 d_7 + 3 d_8 \\
\delta_3 &= d_1 - 4 d_3 - 2 d_8 + d_9 \\
\delta_4 &= d_2 + 2 d_5 + 12 d_6 + 4 d_7 - 4 d_9 \\
\delta_5 &= d_2 + 2 d_5 - 4 d_6 - 4 d_7 \\
\delta_6 &= d_1 + \frac{1}{4} d_2 - 4 d_3 + \frac{1}{2} d_5 - 2 d_7 + d_8 \\
\delta_7 &= 2 d_1 - 4 d_3 + 4 d_7 - 2 d_8 \\
\delta_8 &= d_1 \\
\delta_9 &= d_2 - 6 d_5 \\
\delta_{10} &= d_2 - 2 d_5 + 4 d_7 - 2 d_8 \\
\delta_{11} &= -4 d_5 + 4 d_7 + 2 d_8 \\
\delta_{12} &= -6 d_3 + d_4 - 4 d_5 + 12 d_6 - 2 d_8 \\
\delta_{13} &= -6 d_3 + d_4 \\
\delta_{14} &= 2 d_3 + d_4 - 4 d_5 - 4 d_6 + 2 d_8 \\
\delta_{15} &= 2 d_3 + d_4 - 2 d_5 + 2 d_7 - 3 d_8 .
\end{aligned}$$

In the following we want to study the vector field perturbed in fifth order of the form

$$\dot{\pi} \;=\; f(\pi) + \varepsilon\, \mathrm{RFOT}(\pi), \;\; \varepsilon \ll 1.$$

By reduction of the fifth order terms (FOT) to the orbit space one gets the perturbation RFOT (reduced fifth order terms) with components RFOT1, ..., RFOT5.

$$\begin{aligned}
\mathrm{RFOT1} &= 2\pi_1(\delta_1^r\pi_1^2+\delta_2^r\pi_1\pi_2+\delta_3^r\pi_2^2+\delta_4^r\pi_3)+\pi_4\Big(2(\delta_5+\delta_{14})^r\pi_1\\
&\quad+(\delta_{11}+2\delta_{15})^r\pi_2\Big)+\pi_5\Big(2\delta_{14}^i\pi_1+(\delta_{11}+2\delta_{15})^i\pi_2\Big)\\
\mathrm{RFOT2} &= 4\delta_{12}^r\pi_1\pi_3+2\pi_2\Big(\delta_6^r\pi_1^2+\delta_7^r\pi_1\pi_2+\delta_8^r\pi_2^2+(\delta_9+2\delta_{13})^r\pi_3\Big)\\
&\quad+\pi_4\Big(2\delta_{14}^r\pi_1+(2\delta_{10}-\delta_{11}+2\delta_{15})^r\pi_2\Big)-\pi_5\Big(2\delta_{14}^i\pi_1\\
&\quad+(\delta_{11}+2\delta_{15})^i\pi_2\Big)\\
\mathrm{RFOT3} &= 2\pi_3\Big(2\delta_6^r\pi_1^2+(2\delta_7+\delta_{12})^r\pi_1\pi_2+(2\delta_8+\delta_{13})^r\pi_2^2+2\delta_9^r\pi_3\Big)\\
&\quad+\pi_4\left(\pi_2\left(\delta_{14}^r\pi_1-\left(\frac{1}{2}\delta_{11}-\delta_{15}\right)^r\pi_2\right)+4\delta_{10}^r\pi_3\right)\\
&\quad-\pi_5\left(\pi_2\left(\delta_{14}^i\pi_1-\left(\frac{1}{2}\delta_{11}-\delta_{15}\right)^i\pi_2\right)+4\delta_{11}^i\pi_3\right)\\
\mathrm{RFOT4} &= 2\pi_4\left((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2\right.\\
&\quad\left.+\left(\delta_3+\delta_8+\frac{1}{2}\delta_{13}\right)^r\pi_2^2+(\delta_4+\delta_9)^r\pi_3+(\delta_5+\delta_{10})^r\pi_4\right)\\
&\quad-2\pi_5\left((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2\right.\\
&\quad\left.+\left(\delta_3-\delta_8-\frac{1}{2}\delta_{13}\right)^i\pi_2^2+(\delta_4-\delta_9)^i\pi_3-\delta_{11}^r\pi_5\right)\\
&\quad-2\pi_4\pi_5(\delta_5-\delta_{10}+\delta_{11})^i+\pi_1\left(\frac{1}{2}\delta_{14}^r\pi_1^2\pi_2-\left(\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{15}\right)^r\pi_1\pi_2^2\right.\\
&\quad\left.+2\delta_{14}^r\pi_1\pi_3+(\delta_{11}+2\delta_{15})^r\pi_2\pi_3\right)\\
\mathrm{RFOT5} &= 2\pi_4\left((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2+\left(\delta_3-\delta_8-\frac{1}{2}\delta_{13}\right)^i\pi_2^2\right.\\
&\quad\left.+(\delta_4-\delta_9)^i\pi_3+(\delta_5-\delta_{10})^i\pi_4\right)\\
&\quad+2\pi_5\left((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2\right.\\
&\quad\left.+\left(\delta_3+\delta_8+\frac{1}{2}\delta_{13}\right)^r\pi_2^2+(\delta_4+\delta_9)^r\pi_3-\delta_{11}^i\pi_5\right)\\
&\quad+2\pi_4\pi_5(\delta_5+\delta_{10}-\delta_{11})^r+\pi_1\left(-\frac{1}{2}\delta_{14}^i\pi_1^2\pi_2+\left(\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{15}\right)^i\pi_1\pi_2^2\right.\\
&\quad\left.+2\delta_{14}^i\pi_1\pi_3+(\delta_{11}+2\delta_{15})^i\pi_2\pi_3\right).
\end{aligned}$$

Lemma 3.6.1 *Restriction to the stratum $\Delta = 0$ yields*

$$
\begin{aligned}
\Delta\mathrm{RFOT1} \;&=\; 2\pi_1\left(\delta_1^r\pi_1^2+\delta_2^r\pi_1\pi_2+\left(\delta_3+\frac{1}{4}\delta_4\right)^r\pi_2^2\right)\\
&\quad+\pi_4\Big(2(\delta_5+\delta_{14})^r\pi_1+(\delta_{11}+2\delta_{15})^r\pi_2\Big)\\
&\quad+\pi_5\Big(2\delta_{14}^i\pi_1+(\delta_{11}+2\delta_{15})^i\pi_2\Big)\\
\Delta\mathrm{RFOT2} \;&=\; 2\pi_2\left(\delta_6^r\pi_1^2+\left(\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2+\left(\delta_8+\frac{1}{4}\delta_9+\frac{1}{2}\delta_{13}\right)^r\pi_2^2\right)\\
&\quad+\pi_4\Big(2\delta_{14}^r\pi_1+(2\delta_{10}-\delta_{11}+2\delta_{15})^r\pi_2\Big)\\
&\quad-\pi_5\Big(2\delta_{14}^i\pi_1+(\delta_{11}+2\delta_{15})^i\pi_2\Big)\\
\Delta\mathrm{RFOT3} \;&=\; \frac{1}{2}\pi_2\,\Delta\mathrm{RFOT2}\\
\Delta\mathrm{RFOT4} \;&=\; 2\pi_4\Bigg((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2\\
&\quad+\left(\delta_3+\frac{1}{4}\delta_4+\delta_8+\frac{1}{4}\delta_9+\frac{1}{2}\delta_{13}\right)^r\pi_2^2+(\delta_5+\delta_{10})^r\pi_4\Bigg)\\
&\quad-2\pi_5\Bigg((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2\\
&\quad+\left(\delta_3+\frac{1}{4}\delta_4-\delta_8-\frac{1}{4}\delta_9-\frac{1}{2}\delta_{13}\right)^i\pi_2^2-\delta_{11}^r\pi_5\Bigg)\\
&\quad-2\pi_4\pi_5(\delta_5-\delta_{10}+\delta_{11})^i+\pi_1\pi_2\Bigg(\frac{1}{2}\delta_{14}^r\pi_1^2\\
&\quad-\left(\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{14}-\frac{1}{2}\delta_{15}\right)^r\pi_1\pi_2+\left(\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{15}\right)^r\pi_2^2\Bigg)\\
\Delta\mathrm{RFOT5} \;&=\; 2\pi_4\Bigg((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2\\
&\quad+\left(\delta_3+\frac{1}{4}\delta_4-\delta_8-\frac{1}{4}\delta_9-\frac{1}{2}\delta_{13}\right)^i\pi_2^2+(\delta_5-\delta_{10})^i\pi_4\Bigg)\\
&\quad+2\pi_5\Bigg((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2\\
&\quad+\left(\delta_3+\frac{1}{4}\delta_4+\delta_8+\frac{1}{4}\delta_9+\frac{1}{2}\delta_{13}\right)^r\pi_2^2-\delta_{11}^i\pi_5\Bigg)\\
&\quad+2\pi_4\pi_5(\delta_5+\delta_{10}-\delta_{11})^r+\pi_1\pi_2\Bigg(-\frac{1}{2}\delta_{14}^i\pi_1^2\\
&\quad+\left(\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{14}-\frac{1}{2}\delta_{15}\right)^i\pi_1\pi_2+\left(\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{15}\right)^i\pi_2^2\Bigg).
\end{aligned}
$$

Here ΔRFOT1, ... , ΔRFOT5 denote the components of the reduced fifth order terms (RFOT) restricted to the statum $\Delta = 0$.

3.7 SINGULAR PERTURBATION THEORY

For the moment we want to restrict our considerations to the stratum $\Delta = 0$. The curve $g(\pi_1)$, $0 \le \pi_1 \le -\frac{\lambda}{\alpha^r}$, of critical points of the reduced vector field $\dot{\pi} = f(\pi)$ (cf. Equations 3.4.3 to 3.4.7) is located on this stratum (cf. Lemma 3.4.6). According to Lemma 3.5.2 this curve is asymptotically stable for our choice of the coefficients

$$\beta^r < \alpha^r < \gamma^r < 0.$$

Now we want to study the perturbed vector field (cf. Lemma 3.6.1)

$$\dot{\pi} \;\; = \;\; f^{\varepsilon}(\pi) \;\; = \;\; f(\pi) + \varepsilon\, \Delta\mathrm{RFOT}(\pi), \;\; \varepsilon \ll 1. \tag{3.7.19}$$

We have the following propostion.

PROPOSITION 3.7.1 *For the perturbed Vector Field 3.7.19 and* $0 < |\varepsilon| < \varepsilon_0$ *there persists an invariant curve* g_ε *near* g *on the stratum* $\Delta = 0$. *This curve* g_ε *is parametrised over* π_1. *The vector field on* g_ε *has the form*

$$r(\pi_1) \;\; = \;\; 2\frac{\pi_1\pi_2}{\pi_1+\pi_2}\left(16\pi_1^2 + 16\frac{\lambda}{\alpha^r}\pi_1 + 3\frac{\lambda^2}{\alpha^{r2}}\right) d$$

with

$$0 < \pi_1 < -\frac{\lambda}{\alpha^r}, \;\; \pi_1 + \pi_2 = -\frac{\lambda}{\alpha^r},$$

$$d = \left((d_6 + d_7 - d_8)^r + \frac{(\alpha-\beta)^i}{(\alpha-\beta)^r}(d_6 + d_7 - d_8)^i\right).$$

Proof: In Lemma 3.5.2 we showed that the curve $g(\pi_1)$, $0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, is normally hyperbolic. Thus an invariant curve g_ε near g persists under small perturbations. The curve g_ε will no longer consist of critical points but there will be a resulting flow on g_ε. This flow is determined in the lowest order by projection of the perturbation onto the curve g.
Let

$$E \;\; = \;\; \left\{ g(\pi_1) \,|\, 0 < \pi_1 < -\frac{\lambda}{\alpha^r} \right\}$$

be the curve of critical points of the vector field $f^0(\pi)$ on the stratum

$$F = \left\{ \pi \in \Pi(\mathbb{R}^6) \,|\, \Delta(\pi) = 0 \right\}.$$

For a point $\pi \in E$ let

$$Tf^0(\pi) \;\; : \;\; T_\pi F \;\; \to \;\; T_\pi F$$

denote the linearization of f^0 in π. By construcion $T_\pi E$ lies in the kernel of $Tf^0(\pi)$. So a linear map

$$Qf^0(\pi) \;\; : \;\; \frac{T_\pi F}{T_\pi E} \;\; \to \;\; \frac{T_\pi F}{T_\pi E}$$

is induced on the quotient space. The eigenvalues of $Qf^0(\pi)$ have been determined in Lemma 3.5.2 and are both negative. Thus for every $\pi \in E$ $T_\pi E$ has a unique complement N_π that is invariant under $Tf^0(\pi)$. Let P^E denote the projection onto TE defined by the splitting

$$TF_{|E} \quad = \quad TE \oplus N$$

On E we define the vector field

$$f_R(\pi) \quad = \quad P^E \frac{\partial}{\partial \varepsilon} f^\varepsilon(\pi)_{|\varepsilon=0}.$$

We have the extended vector field

$$f^\varepsilon(\pi) \times \{0\} \text{ on } F \times (-\varepsilon_0, \varepsilon_0).$$

In this system, according to [Fe], a two dimensional center manifold C exists for small $\varepsilon_0 \ll 1$. The second dimension has its origin in the extension of the system in ε-direction.
On C near $E \times \{0\}$ a smooth vector field

$$f_C \quad = \quad \begin{cases} \frac{1}{\varepsilon} f^\varepsilon(\pi) \times 0, & \varepsilon \neq 0 \\ f_R(\pi) \times 0, & \varepsilon = 0 \end{cases}$$

is defined. The center manifold is fibered in ε-direction with invariant curves g_ε. The flow on g_ε has the form

$$\dot{\pi} \quad = \quad \varepsilon f_R(\pi) + O(\varepsilon^2).$$

We want to determine the vector field

$$f_R(\pi) \quad = \quad P^E \frac{\partial}{\partial \varepsilon} f^\varepsilon(\pi)_{|\varepsilon=0}.$$

The vectors t_1, t_2, t_3 that span the tangent space to the stratum F along the curve g, are known (cf. Lemma 3.5.2). The vector t_1 is the tangent vector along the curve g. Now we want to write the terms of higher order ΔRFOT along the curve g in the form

$$\Delta\text{RFOT}(\pi) \quad = \quad a(\pi)t_1 + b(\pi)t_2 + c(\pi)t_3.$$

This gives the projection P^E we are looking for and we have

$$f_R(\pi) \quad = \quad a(\pi).$$

As the restriction of the vector field $\frac{\partial}{\partial\varepsilon} f^\varepsilon(\pi)_{|\varepsilon=0}$ along the curve g one gets

$$
\begin{aligned}
r_1 &= \pi_1\left(2\delta_1^r\pi_1^2 + (2\delta_2+\delta_5+\delta_{14})^r\pi_1\pi_2 + \left(2\delta_3+\frac{1}{2}\delta_4+\frac{1}{2}\delta_{11}+\delta_{15}\right)^r\pi_2^2\right)\\
r_2 &= \pi_2\Bigg((2\delta_6+\delta_{14})^r\pi_1^2 + \left(2\delta_7+\delta_{10}-\frac{1}{2}\delta_{11}+\delta_{12}+\delta_{15}\right)^r\pi_1\pi_2\\
&\quad + \left(2\delta_8+\frac{1}{2}\delta_9+\delta_{13}\right)^r\pi_2^2\Bigg)\\
r_3 &= \frac{1}{2}\pi_2\, r_2\\
r_4 &= \pi_1\pi_2\Bigg(\left(\delta_1+\delta_6+\frac{1}{2}\delta_{14}\right)^r\pi_1^2\\
&\quad + \left(\delta_2+\frac{1}{2}\delta_5+\delta_7+\frac{1}{2}\delta_{10}-\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{12}+\frac{1}{2}\delta_{14}+\frac{1}{2}\delta_{15}\right)^r\pi_1\pi_2\\
&\quad + \left(\delta_3+\frac{1}{4}\delta_4+\delta_8+\frac{1}{4}\delta_9+\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{13}+\frac{1}{2}\delta_{15}\right)^r\pi_2^2\Bigg)\\
r_5 &= \pi_1\pi_2\Bigg(\left(\delta_1-\delta_6-\frac{1}{2}\delta_{14}\right)^i\pi_1^2\\
&\quad + \left(\delta_2+\frac{1}{2}\delta_5-\delta_7-\frac{1}{2}\delta_{10}+\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{12}+\frac{1}{2}\delta_{14}-\frac{1}{2}\delta_{15}\right)^i\pi_1\pi_2\\
&\quad + \left(\delta_3+\frac{1}{4}\delta_4-\delta_8-\frac{1}{4}\delta_9+\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{13}+\frac{1}{2}\delta_{15}\right)^i\pi_2^2\Bigg).
\end{aligned}
$$

We always have

$$\pi_1+\pi_2 = -\frac{\lambda}{\alpha^r}$$

and get the following equations

$$
\begin{aligned}
r_1 &= a+b\pi_1+2c\alpha^r(\alpha-\beta)^i && (3.7.20)\\
r_2 &= -a+b\pi_2-2c\alpha^r(\alpha-\beta)^i && (3.7.21)\\
r_4 &= \frac{1}{2}a(\pi_2-\pi_1)+b\pi_1\pi_2+c\alpha^r(\pi_2-\pi_1)(\alpha-\beta)^i && (3.7.22)\\
r_5 &= -c\alpha^r(\pi_1+\pi_2)(\alpha-\beta)^r. && (3.7.23)
\end{aligned}
$$

Thus

$$c = -\frac{r_5}{\alpha^r(\pi_1+\pi_2)(\alpha-\beta)^r},$$

$$r_1+r_2 = b(\pi_1+\pi_2)$$

and

$$r_1-r_2 = 2a+b(\pi_1-\pi_2)+4c\alpha^r(\alpha-\beta)^i.$$

Finally we get

$$b = \frac{r_1 + r_2}{\pi_1 + \pi_2}$$

and

$$\begin{aligned} a &= \frac{1}{2}(r_1 - r_2) - \frac{1}{2}b(\pi_1 - \pi_2) - 2c\alpha^r(\alpha-\beta)^i \\ &= \frac{r_1\pi_2 - r_2\pi_1}{\pi_1 + \pi_2} - 2c\alpha^r(\alpha-\beta)^i. \end{aligned}$$

Insertion of r_1, r_2, c and retranslation of the coefficients $\delta_1, \ldots, \delta_{15}$ into the coefficients $d_1, \ldots, d_9$ finishes the proof. $\bowtie$

In the following let

$$d \neq 0.$$

Proposition 3.7.2 *On the invariant curve g_ε (cf. Propostion 3.7.1) for the perturbed Vector Field 3.7.19 exactly two critical points persist for $0 < |\varepsilon| < \tilde{\varepsilon}_0 < \varepsilon_0$. In the entire ten dimensional system these critical points have isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_4, \mathbb{Z}_2)$. The latter corresponds to the isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$ in the reduced system. Their stability in $\mathbb{R}^6$ is determined by the sign of*

$$d = \left((d_6 + d_7 - d_8)^r + \frac{(\alpha-\beta)^i}{(\alpha-\beta)^r}(d_6 + d_7 - d_8)^i\right).$$

Especially a connection between the group orbits of solutions with isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_4, \mathbb{Z}_2)$ persists for small ε in $\mathbb{R}^6$.
The position of the critical points, their isotropy in the entire system, and the direction of the resulting flow on g_ε is given in Figure 5.

Proof: On the curve $g_\varepsilon(\pi_1), 0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, near g there are two critical points of the Fenichel vector field $r(\pi_1)$ (cf. Proposition 3.7.1) with

$$\pi_1 \in \left\{-\frac{\lambda}{4\alpha^r}, -\frac{3\lambda}{4\alpha^r}\right\}.$$

Linearization of the vector field $r(\pi_1)$ in these critical points yields

π_1	$\frac{\mathrm{d}r}{\mathrm{d}\pi_1}$
$-\frac{\lambda}{4\alpha^r}$	$-3d\frac{\lambda^2}{\alpha^{r2}}$
$-\frac{3\lambda}{4\alpha^r}$	$3d\frac{\lambda^2}{\alpha^{r2}}$,

and, thus, they are hyperbolic. Here d is defined as in Proposition 3.7.1. Therefore these critical points persist for $|\varepsilon| < \tilde{\varepsilon}_0 < \varepsilon_0$ in the perturbed Vector Field 3.7.19. We shall show that the persisting critical points lie on the group orbits of solutions with isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_4, \mathbb{Z}_2)$ with reference to the entire system. First let

$$\tilde{\pi} \quad = \quad -\frac{\lambda}{4\alpha^r}.$$

Using the representation of the group element

$$\tau \quad = \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

introduced in Lemma 3.3.3 and

$$\pi \quad \in \quad \mathrm{S}^1,$$

we have

$$\tau\, \pi\, (0, z, 0) \quad = \quad \tau\, (0, -z, 0) \; = \; \left(\frac{1}{2}\sqrt{\frac{3}{2}} z, \frac{1}{2} z, \frac{1}{2}\sqrt{\frac{3}{2}} z \right).$$

Points of the form $(0, z, 0)$ with

$$|z|^2 \quad = \quad -\frac{\lambda}{\alpha^r}$$

are mapped to the critical point of isotropy $(\mathrm{O}(2), 1)$ in the reduced system by the Hilbert-map. Thus

$$\begin{aligned} \Pi(\tau\pi\, (0, z, 0)) \quad = \quad & \left(\pi_1 = -\frac{\lambda}{4\alpha^r}, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r} \right), \pi_3 = \frac{1}{4}\pi_2^2, \right. \\ & \left. \pi_4 = \frac{1}{2}\pi_1 \pi_2, \pi_5 = 0 \right). \end{aligned}$$

Therefore

$$\Pi\big(\tau\, \pi\, (0, z, 0)\big) \quad = \quad g(\tilde{\pi}).$$

Second let

$$\tilde{\pi} \quad = \quad -\frac{3\lambda}{4\alpha^r}.$$

Correspondingly the Hilbert-map maps points of the form $(z, 0, z)$ with

$$|z|^2 \quad = \quad -\frac{\lambda}{2\alpha^r}$$

to the critical point of isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$ in the reduced system. With $r_\pi \in \mathrm{O}(2)$ we have

$$r_\pi\, \tau\, (z, 0, z) \quad = \quad r_\pi \left(-\frac{1}{2} z, \sqrt{\frac{3}{2}} z, -\frac{1}{2} z \right) \quad = \quad \left(\frac{1}{2} z, \sqrt{\frac{3}{2}} z, \frac{1}{2} z \right).$$

Thus

$$\Pi(r_\pi\,\tau\,(z,0,z)) \;=\; \left(\pi_1 = -\frac{3\lambda}{4\alpha^r}, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r}\right), \pi_3 = \frac{1}{4}\pi_2^2,\right.$$
$$\left.\pi_4 = \frac{1}{2}\pi_1\pi_2, \pi_5 = 0\right) \;=\; g(\tilde{\pi}).$$

Since the perturbation respects the symmetry, the critical points persisting for small ε on the curve have the same isotropies.
Besides these two critical points there are no critical points on g_ε for small $\varepsilon > 0$. Since the two critical points are hyperbolic, in a neighbourhood of these points no further critical points exist by the implicit function theorem. If there were critical points (x_n, ε_n) in the remaining part of g_ε, for a sequence $(\varepsilon_n) \to 0$ the accumulation point $(\bar{x}, 0)$ would have to be a critical point of the resulting vector field in contradiction to Proposition 3.7.1. ⋈

Figure 5 shows the resulting flow on the invariant curve $g_\varepsilon(\pi_1)$, $0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, in a schematic way for $d > 0$ and small $\varepsilon > 0$. Choosing $d < 0$ will change the direction of the arrows. The isotropies of the solutions in the entire ten dimensional system are indicated in the sketch.
For $\varepsilon = 0$ (i.e. $g_\varepsilon = g$) $g(0)$ resp. $g(-\frac{\lambda}{\alpha^r})$ are fixed points of isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$ resp. $(\mathrm{O}(2), 1)$ (cf. Lemma 3.4.6). The curve itself consists of fixed points.

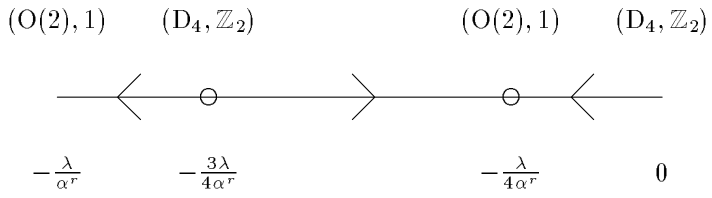

Figure 5: Resulting flow on g_ε

3.8 Invariant tori

In this section we want to show that for small $\varepsilon > 0$ a fixed point bifurcates from the critical point $\tilde{\pi}_1$ in the direction of the principal stratum. The critical point $\tilde{\pi}_1$ lies on the curve g on the stratum $\Delta = 0$. According to Lemma 3.5.3 the linearization of the vector field of the reduced equation (cf. Lemma 3.2.1) has a nontrivial Jordan-block to the eigenvalue zero in the point

$$g(\tilde{\pi}_1),\;\; \tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}.$$

The position of this point on the invariant curve

$$g(\pi_1),\;\; 0 < \pi_1 < -\frac{\lambda}{\alpha^r},$$

depends on the relative choice of the coefficients

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta^r + 3\gamma^r), \gamma^r\right),$$

according to Figure 6. Making the ansatz

$$\alpha^r \;=\; t\,\beta^r + (1-t)\,\gamma^r, \quad t \in \left(0, \frac{1}{4}\right),$$

this follows as in the proof of Lemma 3.5.1. Thus

$$\tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r} = -\frac{\lambda}{\alpha^r}t, \quad t \in \left(0, \frac{1}{4}\right).$$

Figure 6: Possible region of the point $\tilde{\pi}_1$

We want to determine the form of the resulting vector field on the local two dimensional center manifold W^c_{loc} near the point $g(\tilde{\pi}_1)$. The center manifold W^c_{loc} is tangential to Span (t_1, h) (cf. Lemma 3.5.3) and intersects the stratum $\Delta = 0$ in a part of the invariant curve $g(\pi_1)$ near $g(\tilde{\pi}_1)$. Let t_1 be the tangent vector in the direction of the curve $g(\pi_1)$ and h be the hauptvector associated to the Jordan-block of the linearization. By definition of the vectors t_1, t, h in Lemma 3.5.3 h points in the direction of the principal stratum.
We introduce x-coordinates in the direction of $(-t_1)$ along the invariant curve $g(\pi_1)$ and y-coordinates in the direction of $(-h)$ with origin in $g(\tilde{\pi}_1)$. Therefore the vector field on W^c_{loc} has the form

$$\begin{aligned} \dot{x} &= -y + H(x,y) \\ \dot{y} &= y\,G(x,y). \end{aligned} \tag{3.8.24}$$

We are only interested in the region $y \leq 0$ that describes a part of the Hilbert-set $\Pi(\mathbb{R}^6)$ according to our choice of the coordinates. The $(-y)$-term in the x-equation models the Jordan-block, the minus sign follows from the equation

$$Lh \;=\; j\,t_1$$

with $j < 0$ according to Lemma 3.5.3. The y-term in the y-equation describes the flow invariance of the curve $y = 0$, i.e. of the stratum $\Delta = 0$.

The function $H(x,y)$ has the following properties

$$\begin{aligned} H(x,y) &= O(x^2,xy,y^2), \\ H(x,0) &\equiv 0, \\ \frac{\partial H}{\partial x}(x,0) &\equiv 0. \end{aligned}$$

The last two properties are due to the fact that points of the form $(x,0)$ are critical points of the system 3.8.24 by construction. The linearization of the Vector Field 3.8.24 in such a point $(x,0)$ yields

$$A = \begin{pmatrix} 0 & -1+\frac{\partial H}{\partial y}(x,0) \\ 0 & G(x,0) \end{pmatrix}.$$

Consequently the eigenvalues are zero in the direction of the curve of fixed points and $G(x,0)$ in the direction of the principal stratum. This eigenvalue has been calculated in Lemma 3.5.3, and has the form

$$e_4 = 4\frac{\lambda(\alpha-\gamma)^r + \pi_1\,\alpha^r\,(\beta-\gamma)^r}{\alpha^r}.$$

Therefore in our coordinates we have

$$G(x,0) = a\,x + O(x^2)$$

with $a>0$. The invariant curve changes the stability in the direction of the principal stratum in the first order from stable to unstable in the point $(0,0)$ (transversality condition).
Now let's look at the extended system

$$\begin{aligned} \dot{\pi} &= f(\pi) + \varepsilon\,\mathrm{RFOT}(\pi) \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{3.8.25}$$

Here near the point

$$(g(\tilde{\pi}_1),0)$$

there exists a local center manifold. This manifold is fibered in ε-direction with two dimensional invariant manifolds $W^c_{loc,\varepsilon}$. For $\varepsilon=0$ the manifold $W^c_{loc,0}$, tangential to Span $\{t_1,h\}$, intersects the stratum $\Delta=0$ in a part of the curve g near $g(\tilde{\pi}_1)$ transversally. This property is preserved for small

$$\varepsilon < \overline{\varepsilon} < \tilde{\varepsilon}.$$

On the two dimensional center manifolds $W^c_{loc,\varepsilon}$ again we introduce, now ε-dependent, coordinates x_ε in the direction of g_ε and y_ε in the direction of the principal stratum. We shall continue writing x resp. y for x_ε resp. y_ε.
Now let $\tilde{\pi}_1$ be tuned in such a way such that the Fenichel-drift in $g_\varepsilon(\tilde{\pi}_1)$, $|\varepsilon|<\overline{\varepsilon}$, is not zero. Then the flow on the corresponding center manifold $W^c_{loc,\varepsilon}$ has the form

$$\begin{aligned} \dot{x} &= -y+\varepsilon+H(x,y,\varepsilon) \\ \dot{y} &= y\,G(x,y,\varepsilon). \end{aligned} \tag{3.8.26}$$

The sign of ε depends on the direction of the resulting Fenichel-drift. We want to assume the solution of isotropy $(\mathrm{O}(2), 1)$ to be stable. Therefore according to Proposition 3.7.2 we have to choose $d > 0$ and the resulting Fenichel-drift has the form indicated in Figure 5. For the choice of parameters

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta + 3\gamma)^r, \gamma^r\right),$$

we have

$$\tilde{\pi}_1 \quad \in \quad \left(0, -\frac{\lambda}{4\alpha^r}\right)$$

and, thus, we have to choose $\varepsilon < 0$.
The functions $G(x, y, \varepsilon)$ resp. $H(x, y, \varepsilon)$ have the following properties

$$\begin{aligned} G(x, y, \varepsilon) &= O(x, y, \varepsilon), \\ G(x, y, 0) &= G(x, y) \end{aligned}$$

resp.

$$\begin{aligned} H(x, y, \varepsilon) &= O(x^2, xy, y^2, \varepsilon x, \varepsilon y, \varepsilon^2), \\ H(x, y, 0) &= H(x, y). \end{aligned}$$

Proposition 3.8.1 *Let*

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta + 3\gamma)^r, \gamma^r\right).$$

Then there exists $\overline{\varepsilon} > 0$ and a unique curve

$$\Big(x(\varepsilon), y(\varepsilon) \leq 0\Big), \quad -\overline{\varepsilon} < \varepsilon \leq 0,$$

of critical points of the flow on the center manifold $W^c_{loc,\varepsilon} \cap \Pi(\mathbb{R}^6)$ with

$$\Big(x(0), y(0)\Big) = (0, 0).$$

The critical points are saddles.

Proof: We are looking for critical points of the Vector Field 3.8.26. Therefore we first solve the equation

$$P(x, y, \varepsilon) \quad = \quad -y + \varepsilon + H(x, y, \varepsilon) \;=\; 0.$$

We have

$$P(0, 0, 0) \quad = \quad 0$$

and

$$\frac{\partial P}{\partial y}(0, 0, 0) \quad = \quad -1$$

since $H(x,y,\varepsilon)$ is of second order. Using the implicit function theorem, locally near $(x,\varepsilon) = (0,0)$ one gets a unique surface $y = y(x,\varepsilon)$ with $y(0,0) = 0$ and $P(x, y(x,\varepsilon), \varepsilon) = 0$. Furthermore

$$y(x,\varepsilon) \quad = \quad \varepsilon + O(x^2, x\varepsilon, \varepsilon^2)$$

and

$$\frac{\partial y}{\partial x}(0,0) \quad = \quad 0.$$

Now we want to solve the equation

$$G(x, y(x,\varepsilon), \varepsilon) \quad = \quad 0.$$

We have

$$G(0,0,0) \quad = \quad 0$$

and

$$\frac{\partial G}{\partial x}(0,0,0) \quad = \quad a > 0$$

because of the transversality property of G and the condition $\frac{\partial y}{\partial x}(0,0) = 0$. Therefore, again by the implicit function theorem, there exists a unique curve

$$(x(\varepsilon), y(\varepsilon)),\ 0 \leq |\varepsilon| < \overline{\varepsilon},\ \varepsilon \leq 0,$$

of critical points of the Vector Field 3.8.26. Furthermore

$$x \quad = \quad O(\varepsilon).$$

Thus the curve $y(\varepsilon)$ has the form

$$y(\varepsilon) \quad = \quad \varepsilon + O(\varepsilon^2).$$

The sign of $y(\varepsilon)$ is determined by the sign of ε for small ε. Here we have $\varepsilon < 0$ and, therefore, $y(\varepsilon) < 0$. Consequently the curve lies in the Hilbert-set $\Pi(\mathbb{R}^6)$.
The linear stability of the critical point $(x(\varepsilon), y(\varepsilon))$, $-\overline{\varepsilon} < \varepsilon \leq 0$, is to be determined. The linearization of the Vector Field 3.8.26 in the point $(x(\varepsilon), y(\varepsilon))$ yields

$$\begin{aligned} D \quad &= \quad \begin{pmatrix} \frac{\partial H}{\partial x}(x(\varepsilon), y(\varepsilon), \varepsilon) & -1 + \frac{\partial H}{\partial y}(x(\varepsilon), y(\varepsilon), \varepsilon) \\ y(\varepsilon)\, \frac{\partial G}{\partial x}(x(\varepsilon), y(\varepsilon), \varepsilon) & G(x(\varepsilon), y(\varepsilon), \varepsilon) + y(\varepsilon)\, \frac{\partial G}{\partial y}(x(\varepsilon), y(\varepsilon), \varepsilon) \end{pmatrix} \\ &= \quad \begin{pmatrix} O(\varepsilon) & -1 + O(\varepsilon) \\ \varepsilon\, a(\varepsilon) + O(\varepsilon^2) & O(\varepsilon) \end{pmatrix}. \end{aligned}$$

We have $\frac{\partial G}{\partial x}(0,0,0) = a > 0$. Thus

$$\frac{\partial G}{\partial x}(x(\varepsilon), y(\varepsilon), \varepsilon) \quad = \quad a(\varepsilon) \ > \ 0$$

with $a(0) = a$ for small ε. So we get two eigenvalues of D of the following form

$$\rho_{1,2} \quad = \quad O(\varepsilon) \pm \sqrt{O(\varepsilon^2) - \varepsilon\, a(\varepsilon)}$$

with $\varepsilon < 0$ and $a(\varepsilon) > 0$. For small ε the $\sqrt{-\varepsilon}$-term is dominating, the critical point is a saddle. $\bowtie$

The bifurcating critical point lies in the principal stratum. The preimages are two 2-tori. Since there are no additional, symmetry given phase relations (cf. Lemma 3.4.9) in general we have quasiperiodic solutions.

3.9 Stability of the invariant tori

We want to know the stability of the group orbit of the quasiperiodic solutions (cf. Proposition 3.8.1) in the entire ten dimensional system. This information is useful for calculating the Conley-index of this group orbit (cf. [Le]). We shall determine the Floquet-exponents of the periodic solutions that correspond to the critical points on the curve

$$g(\pi_1), \quad 0 < \pi_1 < -\frac{\lambda}{\alpha^r}.$$

According to our choice of the coefficients only the interval

$$0 < \pi_1 < -\frac{\lambda}{4\alpha^r}$$

is of interest. Here, in dependence on the relative choice of the coefficients, critical points of the reduced system bifurcate (cf. Proposition 3.8.1).

The periodic solutions are rotating waves. In a rotating coordinate system one gets a static problem which is accessible more easily. We make the ansatz

$$\begin{aligned} z_0 &= (r_0 + \rho_0)\, e^{\mathrm{i}(\omega_0 t + \phi_0)} \\ z_{\pm 2} &= (r_2 + \rho_{\pm 2})\, e^{\mathrm{i}(\omega_0 t + \phi_{\pm 2})} \\ z_{\pm 1} &= y_{\pm 1}\, e^{\mathrm{i}\omega_0 t} \end{aligned}$$

with

$$\omega_0 \quad = \quad \omega + \alpha^i \left(r_0^2 + 2r_2^2\right)$$

and

$$r_0^2 + 2r_2^2 \quad = \quad -\frac{\lambda}{\alpha^r}.$$

In the lowest order one gets the following systems which decouple for symmetry reasons:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} y_1 \\ \overline{y}_1 \\ y_{-1} \\ \overline{y}_{-1} \end{pmatrix} \quad = \quad \begin{pmatrix} s & t & t & s \\ \overline{t} & \overline{s} & \overline{s} & \overline{t} \\ t & s & s & t \\ \overline{s} & \overline{t} & \overline{t} & \overline{s} \end{pmatrix} \begin{pmatrix} y_1 \\ \overline{y}_1 \\ y_{-1} \\ \overline{y}_{-1} \end{pmatrix}$$

with

$$
\begin{aligned}
s &= r_0^2\left(-\alpha+\frac{1}{4}\beta+\frac{3}{4}\gamma\right)+r_2^2\left(-2\alpha+\frac{3}{2}\beta+\frac{1}{2}\gamma\right),\\
t &= 2\sqrt{\frac{3}{8}}(\gamma-\beta)r_0r_2
\end{aligned}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_{-2}\\ \rho_0\\ \rho_2\\ \phi_{-2}\\ \phi_0\\ \phi_2\end{pmatrix} = \left(\begin{matrix}
2r_2^2\gamma^r-r_0^2(\alpha-\beta)^r & 2r_0r_2\alpha^r & -2r_2^2\gamma^r+r_0^2(\alpha-\beta)^r+4r_2^2\alpha^r\\
2r_0r_2\alpha^r & 2r_0^2\alpha^r & 2r_0r_2\alpha^r\\
-2r_2^2\gamma^r+r_0^2(\alpha-\beta)^r+4r_2^2\alpha^r & 2r_0r_2\alpha^r & 2r_2^2\gamma^r-r_0^2(\alpha-\beta)^r\\
2r_2\gamma^i-\frac{r_0^2}{r_2}(\alpha-\beta)^i & 2r_0\alpha^i & -2r_2\gamma^i+\frac{r_0^2}{r_2}(\alpha-\beta)^i+4r_2\alpha^i\\
2r_2\alpha^i & 2r_0\alpha^i & 2r_2\alpha^i\\
-2r_2\gamma^i+\frac{r_0^2}{r_2}(\alpha-\beta)^i+4r_2\alpha^i & 2r_0\alpha^i & 2r_2\gamma^i-\frac{r_0^2}{r_2}(\alpha-\beta)^i
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
r_0^2r_2(\alpha-\beta)^i & -2r_0^2r_2(\alpha-\beta)^i & r_0^2r_2(\alpha-\beta)^i\\
-2r_0r_2^2(\alpha-\beta)^i & 4r_0r_2^2(\alpha-\beta)^i & -2r_0r_2^2(\alpha-\beta)^i\\
r_0^2r_2(\alpha-\beta)^i & -2r_0^2r_2(\alpha-\beta)^i & r_0^2r_2(\alpha-\beta)^i\\
-(\alpha-\beta)^r r_0^2 & 2(\alpha-\beta)^r r_0^2 & -(\alpha-\beta)^r r_0^2\\
2(\alpha-\beta)^r r_2^2 & -4(\alpha-\beta)^r r_2^2 & 2(\alpha-\beta)^r r_2^2\\
-(\alpha-\beta)^r r_0^2 & 2(\alpha-\beta)^r r_0^2 & -(\alpha-\beta)^r r_0^2
\end{matrix}\right)\begin{pmatrix}\rho_{-2}\\ \rho_0\\ \rho_2\\ \phi_{-2}\\ \phi_0\\ \phi_2\end{pmatrix}.
$$

One gets the following eigenvalues

$$
\begin{aligned}
\mu_{1,2} &= 0,\\
\mu_3 &= 2(s+t)^r,\\
\mu_4 &= 2(s-t)^r.
\end{aligned}
$$

Our choice of coordinates yields

$$
\begin{aligned}
\left(-\alpha+\frac{1}{4}\beta+\frac{3}{4}\gamma\right)^r &< 0,\\
\left(-\alpha+\frac{3}{4}\beta+\frac{1}{4}\gamma\right)^r &< 0,\\
2\sqrt{\frac{3}{8}}(\gamma-\beta)^r &> 0,
\end{aligned}
$$

and, therefore,

$$
\mu_4 < 0.
$$

Finally we want to show

$$\mu_3 \quad < \quad 0.$$

By insertion on gets

$$\mu_3 \quad = \quad 2\left((\gamma-\beta)^r\left(\frac{1}{2}r_0^2+2\sqrt{\frac{3}{8}}r_0\sqrt{-\frac{\lambda}{2\alpha^r}-\frac{r_0^2}{2}}\right)-\frac{\lambda}{2\alpha^r}\left(-2\alpha+\frac{3}{2}\beta+\frac{1}{2}\gamma\right)^r\right).$$

The ansatz

$$r_0^2 \quad = \quad -t\frac{\lambda}{\alpha^r}, \quad t\in\left(0,\frac{1}{4}\right),$$

yields

$$\mu_3 \quad = \quad -2\frac{\lambda}{\alpha^r}\left((\gamma-\beta)^r\left(\frac{1}{2}t+2\sqrt{\frac{3}{8}}\sqrt{\frac{t(1-t)}{2}}\right)+\frac{1}{2}\left(-2\alpha+\frac{3}{2}\beta+\frac{1}{2}\gamma\right)^r\right).$$

In the admissible region we have

$$\mu_3 \quad < \quad 0.$$

The eigenvalues of the second system are (cf. Lemmata 3.5.2 and 3.5.3),

$$\begin{aligned}
\mu_{1,2,3} &= 0,\\
\mu_4 &= -2\lambda<0,\\
\mu_5 &= 2\frac{\lambda}{\alpha^r}(\alpha-\beta)^r<0,\\
\mu_6 &= -2\left(r_0^2(\alpha-\beta)^r+2r_2^2(\alpha-\gamma)^r\right).
\end{aligned}$$

Therefore in the bifurcation point

$$\tilde{\pi}_1 \quad = \quad -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}$$

there are six trivial and four negative Floquet-exponents. In the entire system the solution has isotropy $(\mathrm{D}_2,1)$ in the bifurcation point. Thus the group orbit is four dimensional. Therefore four trivial exponents are symmetry given. The sign of the Floquet-exponents of the periodic solution corresponds to the sign of the eigenvalues of the associated fixed point in the stratified space. Dealing with fixed point bifurcation in the stratified space the group orbit of the bifurcating solution inherits the stability of the bifurcation point. The double zero eigenvalue at the bifurcation point splits into one positive and one negative eigenvalue (cf. Proposition 3.8.1). Therefore the bifurcating fixed point is hyperbolic. In the entire system the bifurcating fixed point has isotropy $(\mathbb{Z}_2,1)$.
The following lemma is shown.

LEMMA 3.9.1 *The bifurcating group orbit (cf. Proposition 3.8.1) has the unstable dimension one.*

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