Bifurcation from Relative Equilibria of Noncompact Group Actions: Skew Products, Meanders, and Drifts

Bernold Fiedler, Björn Sandstede,
Arnd Scheel and Claudia Wulff

Received: October 4, 1996

Communicated by Alfred K. Louis

Abstract. We consider a finite-dimensional, typically noncompact Riemannian manifold M with a differentiable proper action of a possibly noncompact Lie group G. We describe G-equivariant flows in a tubular neighborhood U of a relative equilibrium $G \cdot u_0, u_0 \in M$, with compact isotropy H of u_0, by a skew product flow $\dot{g} = g\mathbf{a}(v), \ \dot{v} = \varphi(v)$. Here $g \in G, \ \mathbf{a} \in \mathrm{alg}(G)$. The vector v is in a linear slice V to the group action. The induced local flow on $G \times V$ is equivariant under the action of $(g_0, h) \in G \times H$ on $(g, v) \in G \times V$, given by $(g_0, h)(g, v) = (g_0 g h^{-1}, hv)$. The original flow on U is equivalent to the induced flow on $\{id\} \times H$-orbits in $G \times V$.

Applications to relative equivariant Hopf bifurcation in V are presented, clarifying phenomena like periodicity, meandering, and drifting. Specific illustrations involving Euclidean groups G are meandering spirals, in the plane, and drifting twisted scroll rings, in three-dimensional Belousov-Zhabotinsky media.

1991 Mathematics Subject Classification: 58F35, 57S20, 55R91

1 Introduction

Going beyond rigidly rotating spirals, meandering and drifting spiral wave patterns have been observed in Belousov-Zhabotinsky media [UNUM93], [JSW89], [BE93] and in low pressure CO-oxidation on platinum monocrystals [NvORE93]. Mathematically speaking, the wave patterns are described by concentration vectors $u = u(t, x)$ depending on time t and location $x \in \mathbb{R}^2$. The partial differential equations, which model the dynamics of the solutions $u(t, x)$, are equivariant with respect to the standard affine action of the planar Euclidean group $E(2)$.

The *Euclidean group* $E(N), N = 2, 3, \ldots$, is a semidirect product $E(N) = O(N) \times \mathbb{R}^N$ of the orthogonal group $O(N)$ with the Abelian translation group $\mathbb{R}^N$. The composition for $(R, S), (R', S') \in O(N) \times \mathbb{R}^N$ is defined by

$$(R, S) \circ (R', S') := (RR', S + RS'); \tag{1.1}$$

this rule is compatible with the standard affine representation

$$(R, S)x := Rx + S \tag{1.2}$$

on $x \in \mathbb{R}^N$. *Equivariance* of our dynamical system means that $u(t, \cdot)$ is a solution if, and only if, $(R, S)u(t, \cdot)$ is a solution for any (R, S). Here the linear representation of (R, S) in the state space X of solution x-profiles $u(t, \cdot)$ is given by

$$((R, S)u(t, \cdot))\,(x) := u(t, (R, S)^{-1}x). \tag{1.3}$$

The inverse $(R, S)^{-1}x$ is, of course, given explicitly by

$$(R, S)^{-1} = (R^{-1}, -R^{-1}S). \tag{1.4}$$

A *spiral wave* $u(t, \cdot)$ is a special time periodic solution, for which the time orbit is contained in a single group orbit. After a fixed shift of x-coordinates, it can be written as

$$u(t, \cdot) = (R(t), 0)u(0, \cdot). \tag{1.5}$$

The rotations $R(t) \in SO(N)$ are given as a periodic one-parameter subgroup

$$R(t) = \exp(\mathbf{r}_0 t) \tag{1.6}$$

generated by $\mathbf{r}_0$ in the Lie algebra $so(N)$ of anti-symmetric matrices. In the terminology of [Fie88], non-stationary spiral waves are called *rotating waves*; see also section 3. The term "spiral" arises from the above applied context, where the concentration patterns largely follow Archimedean spirals. Quite analogously, a *meandering wave* $u(t, \cdot)$ is a special solution of the form

$$u(t, \cdot) = (R(t), S(t))\,v(t, \cdot), \tag{1.7}$$

where this time $v(t, \cdot)$ is a nonstationary time periodic solution and the shifts $S(t)$ remain bounded. If the shifts $S(t)$ are unbounded, we call the solution $u(t, \cdot)$ *drifting.*

Numerically, meandering and drifting one-armed spirals have been observed in planar ($N = 2$) models by Barkley [Bar94]. Emphasizing the lack of a theoretical framework, based on Euclidean $E(2)$ equivariance, he also presented an ad-hoc heuristic ODE model exhibiting meandering and drifting solutions.

The first mathematically rigorous analysis of these phenomena has recently been achieved by Wulff, see [Wul96]. Her result is based on a careful Lyapunov-Schmidt reduction in a scale of Banach spaces. This resolves the difficulties of non-differentiability and, in some cases, *non-continuity* of the group action (1.3) on the infinite-dimensional Banach space $u(t, \cdot) \in X$. For technically related earlier results, restricted to compact group actions, see [Ren82] and [Ran82]. It has recently been shown, for the first time, that a center manifold reduction to a finite-dimensional globally group-invariant and locally time-invariant C^{k+1} manifold $M \subseteq X$ can also be achieved in an $E(2)$-equivariant context, if the nonlinearity of the differential equation governing the dynamics of the spiral waves is smooth; see [SSW96a], [SSW96b]. The reduction is based on the assumption that the linearization at the spiral wave does not exhibit continuous spectrum near the imaginary axis. Most notably, the group action becomes differentiable on M, albeit its possible noncontinuity on X. Communicated by one of the present authors, this idea is already being used successfully to investigate meandering of multi-armed spirals [GLM96]. The method of *center bundles,* there, is similar in spirit to a previous approach to bifurcation from relative equilibria of compact group actions [Kru90].

In the present paper we give an alternative, new description of the flow near relative equilibria inside a finite-dimensional Riemannian C^{k+1}-manifold M, typically noncompact, with a C^{k+1}-smooth action of a possibly noncompact Lie group G. Our principal aim is to represent the flow as a skew product flow on a trivial disk bundle $G \times V$ over G, see (1.19). The alternative approach by [GLM96], instead, works on a center bundle over the coset space G/H with respect to some discrete isotropy subgroup H. In our approach this amounts to working in the space $G \times_H V$ of H-orbits on $G \times V$, as defined in (1.15), (2.6) below.

Also, we will allow for general compact isotropies H, rather than requiring H to be finite or even trivial. In the following, the reader may find some background in Lie groups helpful; see for example [Bre72], [BtD85], [tD91], [Hel62], [Pal61], or [Die72].

To set up, we *assume* g in the Lie group G to act as a C^{k+1}-diffeomorphism $u \mapsto gu$ on the finite-dimensional Riemannian C^{k+1}-manifold M, such that the map

$$\begin{array}{rcl} \rho : G \times M & \to & M \\ (g,u) & \mapsto & gu = \rho(g,u) \end{array} \tag{1.8}$$

is C^{k+1}. Of course, we assume that G acts on M, that is $(gg')u = g(g'u)$ for all $g, g' \in G$ and $u \in M$. We also require the action to be *proper*, that is, the map $\tilde{\rho}(g,u) := (gu, u) \in M \times M$ is closed (mapping closed sets to closed sets) with compact preimages $\tilde{\rho}^{-1}(u_1, u_2)$, for any $u_1, u_2 \in M$. As a caveat, we note that $G = \mathbb{R}$ activing by shift on $BC_{unif}(\mathbb{R}, \mathbb{R})$, for example, does *not* define a proper $\mathbb{R}$ action. Still, the action of $G = SE(2)$ on a center manifold M is proper [SSW96b]. Picking $u_1 = u_2 = u_0$, in particular, we observe that the isotropy subgroup

$$H = H(u_0) := \{ g \in G \mid gu_0 = u_0 \} \tag{1.9}$$

is compact, for any $u_0 \in M$. Indeed, $H \times \{u_0\} = \tilde{\rho}^{-1}(u_0, u_0)$ is compact. Although M, G are allowed to be compact, in principle, we note here that the interesting new cases arise for noncompact M and G.

We fix u_0 and its isotropy H, henceforth. We construct the disk V of the trivial bundle $G \times V$ as a geometric cross section to the action of G near u_0. Using the Haar measure on the compact Lie group H, we may first assume the given Riemannian metric on M to be H-invariant, without loss of generality; see [Bre72], section VI.2. In particular, any $h \in H$ acts linearly and orthogonally on the tangent space $T_{u_0}M$ to M in u_0, by the derivative of $u \mapsto \rho(h,u)$ at $u = u_0$. Similarly, ρ induces a C^k-action of G on the C^k tangent bundle TM; we cannot assume G to act as an isometry on tangent spaces in general, if G is non-compact. It should be noted, however, that the special action (1.3) of the Euclidean group, arising in spiral wave motion, is an isometry in the usual L^p and $W^{k,p}$ spaces. In that case, G would automatically act as an isometry on a center manifold M; see [SSW96a], [SSW96b].

We will construct V as a linear version of a *slice* to the action of G in an arbitrarily small G-invariant neighborhood U, called a *tube*, around the G-orbit

$$G \cdot u_0 := \{gu_0 \mid g \in G\} \tag{1.10}$$

of u_0 as follows. Let $\mathrm{alg}(G) = T_{\mathrm{id}}\, G$ denote the Lie algebra of G and

$$T_{u_0}(Gu_0) = \mathrm{alg}(G) \cdot u_0 \tag{1.11}$$

the tangent space to the group orbit $G \cdot u_0$ at u_0. The Lie algebra of G acts on $u \in M$ by the derivative of $g \mapsto \rho(g, u)$ at $g = \mathrm{id}$. Now let the desired disk V of the bundle $G \times V$ be defined as the open ϵ_0-ball, centered at u_0, inside the orthogonal complement

$$V \subset (T_{u_0}(G \cdot u_0))^{\perp} \subseteq T_{u_0}M \tag{1.12}$$

to the orbit tangent space $T_{u_0}(G \cdot u_0)$ in $T_{u_0}M$. Note that the isotropy H of u_0 acts linearly and orthogonally on V, as it does on $T_{u_0}M$ and $T_{u_0}(G \cdot u_0)$.

To define the slice to the G-action and the G-invariant tube U around $G \cdot u_0$, let $\psi : (T_{u_0}M)_{\mathrm{loc}} \to M$ denote a local C^{k+1}-chart of M which is H-equivariant, that is

$$\psi(hv) = h\psi(v), \tag{1.13}$$

for all $v \in (T_{u_0}M)_{\mathrm{loc}}$ and $h \in H$. Here $(T_{u_0}M)_{\mathrm{loc}}$ denotes an ϵ_0-ball in $T_{u_0}M$. In fact we construct ψ^{-1}, first, such that $\psi^{-1}(u_0) = u_0$, and then achieve H-equivariance, by Haar measure, preserving the property that ψ^{-1} is a diffeomorphism; see for example [tD91], section I.5. Then $\psi(V) \subset M$ is a slice to the G-action at $u_0 \in \psi(V)$, and

$$U := G \cdot \psi(V) \tag{1.14}$$

is an open G-invariant tube around the G-orbit $G \cdot u_0$. For convenience, we also call the ϵ_0-disk $V \subset T_{u_0}M$ around u_0 a (linear) slice. We will take license to identify $u_0 \in V$ with the origin in $\mathbb{R}^l = T_{u_0}V$ sometimes.

To describe the dynamics in the tube U well, we consider the C^k-action of the direct product Lie group $G \times H$ on the Cartesian product $G \times V$, given by

$$(g_0, h)(g, v) := (g_0 g h^{-1}, hv). \tag{1.15}$$

Because the derivative of this action at (id, u_0) is surjective, by the choice (1.12) of V, the G-equivariant map

$$\begin{array}{rcl} \bar{\tau} : G \times V & \to & U \supset G \cdot u_0 \\ (g, v) & \mapsto & g\psi(v) \end{array} \tag{1.16}$$

is a submersion for small radius ϵ_0 of the disk V. In fact, the triple $(G \times V, U; \bar{\tau})$ identifies the trivial product $G \times V$ as a (generally nontrivial) C^{k+1} principal fiber bundle over U with fiber, alias structure group, H. For more details, we refer to section 2.

Returning to dynamics, consider a G-equivariant C^k vector field f on the "center" manifold M, that is

$$gf(u) = f(gu), \tag{1.17}$$

for all $u \in M, g \in G$. Of course, here we define $gf(u)$ by the induced (differential) C^k-action of G on the tangent space TM. We seek a representation of the (local) G-equivariant flow

$$\dot{u} = f(u) \tag{1.18}$$

on M near the G-orbit $G \cdot u_0$ by the *skew product flow*

$$\begin{array}{rcl} \dot{g} & = & g\mathbf{a}(v) \\ \dot{v} & = & \varphi(v) \end{array} \tag{1.19}$$

on $G \times V$. Here the maps $\mathbf{a} : V \to \mathrm{alg}(G)$ and $\varphi : V \to T_{u_0}V$ are requested to be of class C^k and H-equivariant in the following sense:

$$\begin{array}{rcl} \mathbf{a}(hv) & = & \mathrm{Ad}(h)\mathbf{a}(v) = h\mathbf{a}(v)h^{-1}, \\ \varphi(hv) & = & h\varphi(v), \end{array} \tag{1.20}$$

for all $h \in H$ and all $v \in V$. Here $\mathrm{Ad}(h)$ denotes the standard adjoint representation on the Lie algebra, and $h\varphi$ is again understood to be differential on the linear ball $V \subseteq T_{u_0}M$.

THEOREM 1.1 *Let f be a G-equivariant C^k vector field on the Riemannian C^{k+1}-manifold $M, k \geq 1$, with proper C^{k+1}-action of G on M. Let $u_0 \in M$.*

Then the isotropy H of u_0 is compact. Moreover, there exists a disk slice V, an open G-invariant tube U around the group orbit $G \cdot u_0$, and H-equivariant C^k-maps $\mathbf{a}, \varphi$, as in (1.20), such that the projection $u := \bar{\tau}(g, v) \in U$ of any solution (g, v) of the skew product system (1.19) satisfies the original differential equation (1.18) in U. The projection $\bar{\tau}$ is defined in (1.16).

Conversely, for the local $G \times H$-equivariant flow defined on $(g, v) \in G \times V$ by any C^k vector field (1.19), which is H-equivariant in the sense of (1.20), the projection $u := \bar{\tau}(g, v) \in U$ induces a G-equivariant C^k vector field f on U such that (1.17), (1.18) hold.

We do not think that this theorem is particularly surprising: our proof, given in section 2, is essentially based on a coordinatization of U by the space $G \times_H V$ of the orbits in $G \times V$ under the action of the group $\{\mathrm{id}\} \times H$. This point of view is due to [Pal61] and is concisely presented in the beautiful topology textbook [tD91], section I.5.

We do think, however, that our theorem is particularly useful: in the present paper, it enables us to analyze drifting and meandering solutions on the "center manifold" M. To be precise, we fix nomenclature.

DEFINITION 1.2 *Consider $u_0 \in M$ with isotropy H and a G-equivariant vector field f on M, as in the theorem, with lifted skew product $\dot{g} = g\mathbf{a}(v)$, $\dot{v} = \varphi(v)$ as in (1.19), (1.20).*

We call u_0 a relative equilibrium, *if*

$$\varphi(u_0) = 0 \tag{1.21}$$

In other words, $u_0 \in M$ is a relative equilibrium if, and only if, the time orbit of u_0 remains inside the group orbit $G \cdot u_0$:

$$\{(u(t)|t \in \mathbb{R}\} \subseteq G \cdot u_0. \tag{1.22}$$

Equivalently, $G \cdot u_0$ is a flow invariant manifold.

Next, take any solution $u(t) \in U$. Suppose that $u(t)$ is neither stationary nor periodic. Then, we call $u(t)$ meandering *if*

$$\{(g(t), v(t)) \mid t \in \mathbb{R}\} \subset G \times V \tag{1.23}$$

is globally defined and relatively compact. If, in contrast, the G-component

$$\{g(t) \mid t \in \mathbb{R}\} \tag{1.24}$$

is globally defined but not relatively compact, then we call $u(t)$ drifting.

Equilibria, as well as rotating waves (spirals) are examples of relative equilibria. The reference point $u_0 \in M$ is not required to be a relative equilibrium in theorem 1.1, although it will typically be in applications, and may be forced to be, by nontrivial H-equivariance of the skew product.

While the notion (1.22) of a relative equilibrium u_0 is intrinsically flow-defined, the definition (1.21) refers to a specific $G \times V$ lifting with respect to the isotropy H of u_0, as stated. For example, to apply condition (1.21) to any given point $\tilde{u}_0 \in U$ other than u_0, the vector field (1.19) has to be constructed with respect to $\tilde{u}_0$ instead of u_0. This subtlety, however, is irrelevant for small tubular neighborhoods U, as long as H is finite.

In the very special case $G = \{\mathrm{id}\}$ the maximal isolated invariant set, in the sense of [Con78], of an isolating neighborhood $V = U$ of u_0 consists precisely of the equilibria, the periodic solutions, and the meanders in $U = V$. A similar statement holds for the case of compact G.

As mentioned above, we prove our theorem in section 2. In section 3 we discuss H-equivariant Hopf bifurcation in V, in general. Section 4 collects some useful facts on actions of the Euclidean groups $SE(N)$ before we proceed sorting out drifts and meanders for $N = 2$, in section 5. We conclude, in section 6, with a slow-fast analysis of drifting circular filaments of scroll waves, so-called twisted scroll rings, in $N = 3$ dimensions.

ACKNOWLEDGMENT. We gratefully acknowledge helpful advice by Christian Leis. We are also much indebted to Marty Golubitsky for sharing his insight into the behavior of multi-armed spirals, as well as for some inciting competitiveness. Regina Löhr helped us with careful and very patient typesetting.

2 TUBES, SLICES, AND SKEW PRODUCTS

In this section we prove theorem 1.1. So, let its assumptions hold. We specifically recall that

$$\dot{u} = f(u) \tag{2.1}$$

is a G-equivariant C^k vector field on the Riemannian C^{k+1}-manifold M with proper C^{k+1}-action of the Lie group G on M. Given $u_0 \in M$ with isotropy H, tube U, and linear slice V, we will relate (2.1) to a $G \times H$-equivariant C^k skew product flow

$$\begin{array}{rcl} \dot{g} & = & g\mathbf{a}(v) \\ \dot{v} & = & \varphi(v) \end{array} \tag{2.2}$$

on $G \times V$. The $G \times H$-action on $G \times V$ is defined as

$$(g_0, h)(g, v) = (g_0 g h^{-1}, hv); \tag{2.3}$$

see in particular (1.15)–(1.20). Talking about the H-action on $G \times V$, below, we will mean the action of $\{\mathrm{id}\} \times H$. Similarly, G-action refers to $G \times \{\mathrm{id}\}$.

Our proof can be outlined as follows. First, we check $G \times H$-equivariance of (2.2). After a brief digression clarifying the structure of $G \times V$ over U as a principal H bundle, we project (2.2) from $G \times V$ down to the tube $U \subset M$ by the submersion

$$\bar{\tau}(g, v) = g\psi(v), \tag{2.4}$$

defined in (1.16), to obtain a G-equivariant C^k vector field (2.1) on U from the skew product (2.2). To complete the proof, we finally lift a given G-equivariant C^k vector field f on U back to a $G \times H$-equivariant C^k skew product (2.2) on $G \times V$, such that the skew product projects down to the prescribed f, by $\bar{\tau}$.

Checking $G \times H$-equivariance of the skew product (2.2) on $G \times V$ is easy: fix $(g_0, h) \in G \times H$ and $(g, v) \in G \times V$. Then (2.3), (1.20) imply

$$\begin{array}{l} (g_0, h)(g\mathbf{a}(v), \varphi(v)) = (g_0 g\mathbf{a}(v)h^{-1}, h\varphi(v)) = \\ = (g_0 g h^{-1} h\mathbf{a}(v)h^{-1}, h\varphi(v)) = ((g_0 g h^{-1})\mathbf{a}(hv), \varphi(hv)). \end{array} \tag{2.5}$$

In other words, $(g_0, h)(g(t), v(t))$ is a solution of the skew product (2.2) if, and only if, $(g(t), v(t))$ is. This proves $G \times H$-equivariance of the skew product on $G \times V$.

In passing, we note that the skew product (2.2) with equivariance condition (1.20) is the most general form of a $G \times H$-equivariant C^k vector field on $G \times V$. Indeed, (left) G-equivariance forces the $\dot{g}$ component to be of the form $g\mathbf{a}(v)$ with $\mathbf{a}(v) \in \mathrm{alg}(G)$. Moreover, the $\dot{v}$ component must be independent of g. Then H-equivariance provides the equivariance conditions (1.20).

We briefly digress now, to clarify the structure of $G \times V$ as an H principal C^{k+1} bundle over U with fiber, alias structure group, H. Our presentation essentially follows [Pal61] and the textbook [tD91].

Identifying H-orbits of the free H-action on $G \times V$, we obtain the H orbit space

$$G \times_H V \quad := \quad G \times V \;/\; \{\mathrm{id}\} \times H. \tag{2.6}$$

It turns out that the $G(\times\{\mathrm{id}\})$-equivariant C^{k+1}-submersion $\bar{\tau}$ factorizes over the H orbit space $G \times_H V$, such that

$$\bar{\tau} : G \times V \xrightarrow{p} G \times_H V \xrightarrow{\tau} U. \tag{2.7}$$

Here p is the canonical G-equivariant C^{k+1}-submersion which projects (g, v) onto its H-orbit; it induces the structure of a C^{k+1}-manifold on $G \times_H V$ because the free H-action on $G \times V$ is C^{k+1}. In fact, $(G \times V, G \times_H V, p)$ is a G-equivariant C^{k+1} principal fiber bundle with compact fiber, alias structure group, H. The G-equivariant map τ, called *tube map*, is a C^{k+1} diffeomorphism onto the open tube U around the group orbit $G \cdot u_0$. We emphasize that these results are by no means original. They are essentially due to [Pal61], and are concisely summarized in the textbook [tD91], sections I.5, II.6.

After our bundle digression, we now project the skew product (2.2) down to M with the submersion $\bar{\tau}$, aiming at the second part of our theorem. Let $u \in M$. Since the C^{k+1}-submersion $\bar{\tau} : G \times V \to U$ is surjective, there exists (g, v) such that $\bar{\tau}(g, v) = u$. By the bundle digression, any other (g', v') in $\bar{\tau}^{-1}(u)$ is on the same H-orbit: there exists $h \in H$ such that

$$(g', v') = (gh^{-1}, hv). \tag{2.8}$$

We define $f(u)$ via the differential $D\bar{\tau}(g, v)$ of $\bar{\tau}$ with respect to g and v,

$$f(u) := D\bar{\tau}(g, v) \cdot (g\mathbf{a}(v), \varphi(v)). \tag{2.9}$$

To show that f is well-defined, we use the action of H on $G \times V$. In fact, we prefer an explicit calculation even though we could also argue "elegantly" with the H bundle structure. We start from

$$\bar{\tau}(gh^{-1}, hv) = \bar{\tau}(g, v), \tag{2.10}$$

for all $h \in H$. Differentiating with respect to g and v, we obtain

$$D\bar{\tau}(gh^{-1}, hv) \cdot (g\mathbf{a}h^{-1}, h\varphi) = D\bar{\tau}(g, v) \cdot (g\mathbf{a}, \varphi), \tag{2.11}$$

for any $\mathbf{a} \in \mathrm{alg}(G), \varphi \in T_{u_0}V$. Therefore, $f(u)$ does not depend on the choice of $(g', v') \in \bar{\tau}^{-1}(u)$, because (2.8)–(2.11) and equivariance (1.20) imply

$$\begin{aligned} & D\bar{\tau}(g', v') \cdot (g'\mathbf{a}(v'), \varphi(v')) = \\ &= D\bar{\tau}(gh^{-1}, hv) \cdot (gh^{-1}\mathbf{a}(hv), \varphi(hv)) = \\ &= D\bar{\tau}(gh^{-1}, hv) \cdot (gh^{-1}h\mathbf{a}(v)h^{-1}, h\varphi(v)) = \\ &= D\bar{\tau}(gh^{-1}, hv) \cdot (g\mathbf{a}(v)h^{-1}, h\varphi(v)) = \\ &= D\bar{\tau}(g,\ v) \cdot (g\mathbf{a}(v), \varphi(v)) = f(u). \end{aligned} \tag{2.12}$$

This proves that $f(u)$ is indeed well-defined on $u \in U$, by (2.9).

Because $\bar{\tau} \in C^{k+1}$ and $\mathbf{a}, \varphi \in C^k$, it is obvious that f is a C^k vector field on the tube U. It remains to check G-equivariance (1.17) of f. Fixing $(g, v) \in \bar{\tau}^{-1}(u)$, this follows directly from G-equivariance of $\bar{\tau}$ and of the skew product $(g\mathbf{a}(v), \varphi(v))$. Explicitly, we have $\bar{\tau}(g_0 g, v) = g_0\bar{\tau}(g, v) = g_0 u$, and hence

$$\begin{aligned} f(g_0 u) &= D\bar{\tau}(g_0 g, v) \cdot (g_0 g\mathbf{a}(v), \varphi(v)) = \\ &= g_0 D\bar{\tau}(g, v) \cdot (g\mathbf{a}(v), \varphi(v)) = \\ &= g_0 f(u). \end{aligned} \tag{2.13}$$

This proves the second part of our theorem: the submersion $\bar{\tau}$ projects any $G \times H$-equivariant C^k vector field (2.2) on $G \times V$ down to a G-equivariant C^k vector field f on U.

It remains to, conversely, lift f from $U \subset M$ up to a skew product on $G \times V$, such that the lift projects back onto the prescribed f, by $\bar{\tau}$. Since the fiber is the isotropy group H, this is trivial if H happens to be discrete, that is, finite. Then we can simply lift the flow in U, and f, back to any sheet h_0 of the covering space $G \times V$ of U, by the local diffeomorphism $\bar{\tau}_{h_0}^{-1}$. Lifting f back to any other sheet h_1, locally near u_0, where $h_1 = h^{-1}h_0$ for some $h \in H$, we see that

$$(gh^{-1}, hv) = (\bar{\tau}_{h_0}^{-1} \circ \bar{\tau}_{h_1})(g, v) \tag{2.14}$$

induces, by linearization with respect to (g, v), the claimed H-equivariance of the lifted vector fields, where $h \in H$ acts freely as a permutation of the sheets in the covering space $G \times V$. The trivial case $H = \mathrm{id}$ was first presented by one of the authors, explaining meandering and drifting spirals [Fie95]; for a recent version see also [BHN96].

We now return to the general case of compact isotropy H. It is convenient to describe the lift of f in slightly more abstract notation. Let $w = (g_0, v) \in W := G \times V$ with left action $gw := (gg_0, v)$ of G and right action $wh := (g_0 h^{-1}, hv)$ of H describe the action of the direct product $G \times H$ on W. Note that G, H act freely, separately. It remains to construct a $G \times H$-equivariant C^k vector field F on the total space W of our principal H bundle

$$\bar{\tau} : G \times V \to U, \tag{2.15}$$

such that F projects down to f by $\bar{\tau}$, that is,

$$D\bar{\tau}(w)F(w) = f(\bar{\tau}(w)) \tag{2.16}$$

for all $w \in W$.

We first define F on the linear slice $w \in \{\mathrm{id}\} \times V \subseteq G \times V$. Let P_v denote the orthogonal projection, with respect to the $H \times H$-invariant Riemannian metric on W, in the tangent space $T_{(\mathrm{id},v)}W = \mathrm{alg}(G) \times V$ onto the orthogonal complement $\left(T_{(\mathrm{id},v)}((\mathrm{id}, v) \cdot H)\right)^{\perp}$ of the right H-action. So P_v projects onto the second summand of the orthogonal decomposition

$$T_{(\mathrm{id},v)}W = T_{(\mathrm{id},v)}((\mathrm{id}, v) \cdot H) \oplus \left(T_{(\mathrm{id},v)}(\mathrm{id}, v) \cdot H)\right)^{\perp}. \tag{2.17}$$

Then we define the lifted vector field F at $w = (\mathrm{id}, v)$ as

$$F(w) := P_v(D\bar{\tau}(w))^{-1} f(\bar{\tau}(w)). \tag{2.18}$$

Note that F is now well defined on $\{\mathrm{id}\} \times V$. Indeed $\bar{\tau}^{-1}(u) = wH$, for $u = \bar{\tau}(w)$, and

$$D\bar{\tau}(w) : T_w W \to T_{\bar{\tau}(w)} U \tag{2.19}$$

is surjective. Hence the kernel of $D\bar{\tau}(w)$ is given by

$$\ker D\bar{\tau}(w) = T_w(w \cdot H) \tag{2.20}$$

in the H principal fiber bundle $\bar{\tau} : G \times V \to U$, and P_v annihilates that kernel. Thus (2.18) defines $F(w)$ properly on $w \in \{\mathrm{id}\} \times V$. Moreover, F is of class C^k on $\{\mathrm{id}\} \times V$, as are $P_v, D\bar{\tau}$, and f.

We extend F to $W = G \times V$ by the left action of G on W, defining

$$F(g, v) := gF(\mathrm{id}, v) \in T_{(g,v)}W, \tag{2.21}$$

for all $g \in G$. The vector field F is still C^k, by smoothness of the free G-action. By construction, F is G-equivariant.

We verify the projection property (2.16) next. Because $\bar{\tau}, F$, and f are all G-equivariant, it is sufficient to verify (2.16) at $w = (\mathrm{id}, v)$, that is,

$$D\bar{\tau}(\mathrm{id}, v)F(\mathrm{id}, v) = f(\bar{\tau}(\mathrm{id}, v)). \tag{2.22}$$

This follows trivially from definition (2.18) of F at $w = (\mathrm{id}, v)$, because P_v projects onto a complement of $\ker D\bar{\tau}(w)$.

To complete the proof of theorem 1.1, it remains to show equivariance of F under the right action of H, that is

$$F(gh^{-1}, hv) = F(g, v) \cdot h, \tag{2.23}$$

for all $g \in G, h \in H, v \in V$. By left G-equivariance of F, this is equivalent to showing

$$F(hw \cdot h) - hF(w) \cdot h = 0 \tag{2.24}$$

for any $w = (\mathrm{id}, v) \in \{\mathrm{id}\} \times V, h \in H$. To show (2.24), we first differentiate the relation $\bar{\tau}(gw \cdot h) = g\bar{\tau}(w)$ with respect to w to obtain

$$D\bar{\tau}(gw \cdot h)(g\tilde{w} \cdot h) = gD\bar{\tau}(w)\tilde{w}, \tag{2.25}$$

for any $g \in G, h \in H$, and $(w, \tilde{w}) \in TW$. Putting $w = (\mathrm{id}, v), g := h$, and $\tilde{w} := F(w)$, this implies

$$\begin{aligned} &D\bar{\tau}(hw \cdot h)(hF(w) \cdot h) = hD\bar{\tau}(w)F(w) = \\ &= hf(\bar{\tau}(w)) = f(h\bar{\tau}(w)) = f(\bar{\tau}(hw)) = \\ &= f(\bar{\tau}(hw \cdot h)), \end{aligned} \tag{2.26}$$

so that $hF(w) \cdot h$ is indeed a candidate for $F(hw \cdot h)$ in (2.24): the difference lies in the kernel of $D\bar{\tau}(hw \cdot h)$.

To complete the proof of (2.24), and of theorem 1.1, we finally show that $hF(w) \cdot h$ is orthogonal to $\ker D\bar{\tau}(hw \cdot h)$, as is $F(hw \cdot h)$ by definition (2.18), at $hw \cdot h = (\mathrm{id}, hv)$. Indeed, by invariance of the Riemannian metric on W with respect to the action of the compact group $H \times H$, we conclude from (2.18) at w and (2.25) that

$$\begin{aligned} hF(w) \cdot h \quad &\in h(\ker D\bar{\tau}(w))^{\perp} \cdot h = \\ &= (h(\ker D\bar{\tau}(w)) \cdot h)^{\perp} = \\ &= (\ker D\bar{\tau}(hw \cdot h))^{\perp}. \end{aligned} \tag{2.27}$$

This completes the proof of $G \times H$-equivariance of F, and of theorem 1.1. □

We note that our orthogonality condition in (2.18) at $w \in \{\mathrm{id}\} \times V$ determines the lifted vector field F uniquely. We formalize this statement for $F(\mathrm{id}, v) = (\mathbf{a}(v), \varphi(v))$.

COROLLARY 2.1 *Let the assumptions of theorem 1.1 hold. Let* $\langle \cdot, \cdot \rangle_{\mathrm{alg}(G)}$ *denote an invariant scalar product on* $\mathrm{alg}(G)$ *under the adjoint action* $\mathrm{Ad}(h)$ *of* $h \in H$, *and let* $(\cdot, \cdot)_V$ *denote an* H*-invariant scalar product on the linear slice space* V.

Then the lifted vector field $F(\mathrm{id}, v) = (\mathbf{a}(v), \varphi(v))$ *can be chosen such that*

$$(\varphi(v), \eta v)_V = \langle \mathbf{a}(v), \eta \rangle_{\mathrm{alg}(G)}, \tag{2.28}$$

for any $v \in V, \eta \in \mathrm{alg}(H)$. *The above conditions, together with the vector field* f *on the base* U, *determine* F *uniquely.*

3 Equivariant periodic orbits in a slice

By theorem 1.1 we can discuss any local bifurcation from a relative equilibrium u_0 with isotropy H in the associated $G \times H$-equivariant skew product system

$$\begin{aligned} \dot{g} &= g\mathbf{a}(v), \\ \dot{v} &= \varphi(v). \end{aligned} \tag{3.1}$$

To interpret results in terms of $u = \bar{\tau}(g, v)$ in the tube U around $G \cdot u_0$, we just have to identify points $w = (g, v)$ on the same right H-orbit $w \cdot H \in G \times_H V$. In this section, we investigate some elementary consequences of our decomposition (3.1) in case the H-equivariant $\dot{v}$ equation possesses a periodic orbit. Such periodic orbits may arise by H-equivariant Hopf bifurcation from the H-invariant equilibrium $v = u_0$ of the $\dot{v}$ equation; for a detailed background using compactness of H see [GSS88] or [Fie88].

The spatio-temporal *symmetry* of any periodic solution $v(t)$ of $\dot{v} = \varphi(v)$, with minimal period normalized to 1, can be described by a triple (L, K, Θ) as follows. Let L denote the set of $h \in H$ mapping some point $v(t_1)$ to any point $v(t_2)$ on the periodic orbit. Denoting $\Theta(h) := t_2 - t_1$, equivariance of φ then implies

$$hv(t) = v(t + \Theta(h)), \tag{3.2}$$

for all real t. Moreover

$$\Theta : L \to S^1 := \mathbb{R}/\mathbb{Z} \tag{3.3}$$

is a homomorphism into the additively written circle group. Letting $K := \ker \Theta$, we have a normal subgroup of L, and $L/K \cong$ image(Θ). Note that the groups L, K, image(Θ) are closed. The kernel K is the isotropy of some, and hence all, $v(t)$ with $t \in \mathbb{R}$. Following [Fie88], we call $v(\cdot)$ a *discrete wave*, if image$(\Theta) = \mathbb{Z}_n = \{0, 1/n, \cdots, (n-1)/n\}$ is finite. A *rotating wave* has image$(\Theta) = S^1$.

The periodic solution $v(t)$ gives rise to solutions $g(t)$ of $\dot{g} = g\mathbf{a}(v)$. By left G-equivariance, any solution $g(t)$ with initial condition $g(0) = g_0$ is given by

$$g(t) = g_0 g_*(t), \tag{3.4}$$

where $g_*(t)$ denotes the fundamental solution

$$\begin{aligned} \dot{g}_*(t) &= g_*(t)\mathbf{a}(t) \\ g_*(0) &= \mathrm{id} \end{aligned} \tag{3.5}$$

with the abbreviation $\mathbf{a}(t) := \mathbf{a}(v(t))$.

Theorem 3.1 *Let $v(t)$ be a rotating wave solution of $\dot{v} = \varphi(v)$ in (3.1).*

Then there exist $\eta \in \mathrm{alg}(H), \mathbf{a}_0 \in \mathrm{alg}(G)$ such that

$$\begin{aligned} v(t) &= \exp(\eta t) v_0 \\ g_*(t) &= \exp((\mathbf{a}_0 + \eta)t) \exp(-\eta t). \end{aligned} \tag{3.6}$$

The projected solution $u(t) = \bar{\tau}(g_*(t), v(t))$ near the relative equilibrium u_0 is again a relative equilibrium and can be represented as

$$u(t) = \exp((\mathbf{a}_0 + \eta)t) u(0). \tag{3.7}$$

PROOF: To construct η, just note that $v_0 := v(0)$ is a relative equilibrium to the action of H on V because $v(t)$ is a rotating wave. In particular

$$\dot{v}(0) \in T_{v_0}(Hv_0) = \mathrm{alg}(H) \cdot v_0. \tag{3.8}$$

Pick $\eta \in \mathrm{alg}(H)$ such that $\eta v_0 = \dot{v}(0) = \varphi(v_0)$. Let $v_*(t) := \exp(\eta t)v_0$. Then $v_*(0) = v_0$ and H-equivariance (1.20) of φ implies

$$\begin{aligned} \dot{v}_*(t) &= \exp(\eta t)\eta v_0 = \exp(\eta t)\varphi(v_0) = \\ &= \varphi(\exp(\eta t)v_0) = \varphi(v_*(t)), \end{aligned} \tag{3.9}$$

for all t. Therefore $v(t) = v_*(t)$, for all t.

Let $\mathbf{a}_0 := \mathbf{a}(v(0))$ and define $g_*(t)$ as in (3.6). We have to show that $g_*(t)$ solves (3.5). Trivially $g_*(0) =$ id. Using H-equivariance (1.20) of $\mathbf{a}(v(t)) = \mathbf{a}(t)$ yields

$$\begin{aligned} \dot{g}_*(t) = \;& \exp((\mathbf{a}_0 + \eta)t)\ (\mathbf{a}_0 + \eta)\ \exp(-\eta t) - \\ & - \exp((\mathbf{a}_0 + \eta)t)\ \eta\ \exp(-\eta t) = \\ = \;& \exp((\mathbf{a}_0 + \eta)t) \exp(-\eta t) \exp(\eta t)\ \mathbf{a}(v(0))\ \exp(-\eta t) = \\ = \;& g_*(t)\ \mathbf{a}(\exp(\eta t)v(0)) = \\ = \;& g_*(t)\ \mathbf{a}(v(t)) = \\ = \;& g_*(t)\ \mathbf{a}(t). \end{aligned} \tag{3.10}$$

This proves (3.6). To prove (3.7), we remember that $\bar{\tau}$ is left G equivariant and collapses right H-orbits. Therefore (3.6) implies

$$\begin{aligned} u(t) &:= \bar{\tau}(g_*(t), v(t)) = \bar{\tau}(\exp(\mathbf{a}_0 + \eta)t)(\mathrm{id}, v_0) \cdot \exp(\eta t)) = \\ &= \exp((\mathbf{a}_0 + \eta)t)\bar{\tau}(\mathrm{id}, v_0) = \exp((\mathbf{a}_0 + \eta)t)u(0), \end{aligned}$$

and the theorem is proved. □

Note that the relative equilibrium $u(t)$ above can be stationary, periodic, meandering, or drifting, depending on the values of the infinitesimal generator $\mathbf{a}_0 + \eta \in \mathrm{alg}(G)$. In particular, the closure of the orbit $u(\cdot)$ can have large dimension, for example if G contains large dimensional tori. Although the motion of $u(\cdot)$ can then be quasiperiodic in time, the associated rotation numbers given by $a_0 + \eta$ vary smoothly with parameters, and phase locking does not occur.

Next let $v(t)$ be a discrete wave with symmetry $(L, K, \Theta), L/K \cong \mathbb{Z}_n$. We describe the spatio-temporal symmetry of the associated not necessarily periodic solution $u(t) = \bar{\tau}(g_*(t), v(t))$ by a triple $(\tilde{L}, \tilde{K}, \tilde{\Theta})$ similarly to the periodic case. Let $\tilde{L}$ denote the set of $g \in G$ such that $gu(t_1) = u(t_2)$, for some t_1, t_2. Letting $\tilde{\Theta}(g) := t_2 - t_1$, we obtain

$$gu(t) = u(t + \tilde{\Theta}(g)), \tag{3.11}$$

for all real t and $g \in \tilde{L}$, similarly to (3.2). Let $\Sigma := \mathbb{R}/p\mathbb{Z}$ if u is periodic with minimal period $p > 0$, and $\Sigma := \mathbb{R}$ if u is nonperiodic ($p = \infty$). Then

$$\tilde{\Theta} : \tilde{L} \to \Sigma \tag{3.12}$$

is a homomorphism with kernel, alias isotropy of any $u(t)$, denoted by $\tilde{K}$.

Theorem 3.2 *Let v be a discrete wave solution of $\dot{v} = \varphi(v)$ in (3.1) with symmetry* (L, K, Θ), image $(\Theta) = \mathbb{Z}_n$, *and minimal period 1, as above. Let $g_*(t)$ denote the associated solution of (3.5), a nonautonomous, 1-periodic, G-equivariant equation.*

Then, for any $k \in \mathbb{Z}, h_0 \in K, h \in L, t \in \mathbb{R}$, we have

$$\begin{array}{rcl} g_*(t) & = & h_0 g_*(t) h_0^{-1} \\ g_*(t+\Theta(h)+k) & = & g_*(k) g_*(\Theta(h)) h g_*(t) h^{-1}. \end{array} \tag{3.13}$$

For the stroboscope map $g_*(1)$ of (3.5) we obtain

$$\begin{array}{rcl} g_*(1) & = & (g_*(1/n)h_*)^n h_*^{-n} \\ g_*(k) & = & g_*(1)^k \end{array} \tag{3.14}$$

where $h_* \in L$ can be chosen arbitrarily such that $\Theta(h_*) = 1/n$ generates $\text{image}(\Theta) = \mathbb{Z}_n = \{0, 1/n, \cdots, (n-1)/n\}$. *In particular, the stroboscope maps $g_*(k)$ commute with $g_*(\Theta(h))h$, for all $k \in \mathbb{Z}$ and $h \in L$.*

For $g_*(\Theta(h)), \Theta(h) = k/n, k = 0, 1, \cdots, n-1$, we have the more explicit expression

$$g_*(k/n) = (g_*(1/n)h_*)^k \; h_*^{-k}. \tag{3.15}$$

The symmetry $(\tilde{L}, \tilde{K}, \tilde{\Theta})$ of the projected solution $u(t) = \bar{\tau}(g_*(t), v(t))$ with minimal "period" $0 < p \leq \infty$ satisfies

$$\begin{array}{rcl} \tilde{K} & = & K, \\ \tilde{L} & = & \{g_*(k)g_*(\Theta(h))h \mid k \in \mathbb{Z}, h \in L\}, \text{ and} \\ \tilde{\Theta}(g_*(k)g_*(\Theta(h))h) & = & \Theta(h) + k \qquad (\text{mod } p), \end{array} \tag{3.16}$$

where we fix representatives $0 \leq \Theta(h) < 1$. In particular we obtain for $k := 0, h := h_*$

$$u(1/n) = g_*(1/n) h_* u(0). \tag{3.17}$$

Proof: To prove (3.13), we first claim that

$$g_\sharp(t) := h^{-1} g_*(\Theta(h))^{-1} g_*(k)^{-1} g_*(t + \Theta(h) + k) h \tag{3.18}$$

solves the same initial value problem (3.5) as $g_*(t)$ does. Then $g_\sharp \equiv g_*$, of course. To prove the claim, differentiate (3.18) with an eye on H-equivariance (1.20) and 1-periodicity of $\mathbf{a}(t)$:

$$\begin{array}{rcl} \dot{g}_\sharp(t) & = & h^{-1} g_*(\Theta(h))^{-1} g_*(k)^{-1} \dot{g}_*(t+\Theta(h)+k) h = \\ & = & g_\sharp(t) h^{-1} \mathbf{a}(t+\Theta(h)+k) h = \\ & = & g_\sharp(t) h^{-1} \mathbf{a}(v(t+\Theta(h))) h = \\ & = & g_\sharp(t) \mathbf{a}(t). \end{array} \tag{3.19}$$

We now show that $g_\sharp(t)$ satisfies the same initial condition $g_\sharp(0) = \text{id}$ as $g_*(t)$, for any choice of $h \in L$. Consider the special case $h = \text{id}$, first. Then $\Theta(h) = 0$ and $g_\sharp(0) = \text{id}$ is trivial. In particular $g_\sharp(t) = g_*(t)$, in that case, proving

$$g_*(t) = g_*(k)^{-1} g_*(t+k) \tag{3.20}$$

for all $t \in \mathbb{R}, k \in \mathbb{Z}$. Now (3.20) with $t := \Theta(h)$ implies $g_\sharp(0) = \mathrm{id}$, for all choices of $h \in L$. This proves $g_\sharp(t) = g_*(t)$ in (3.18).

Inserting $h := h_0 \in K = \ker\Theta$ and $k := 0$ into (3.18) with $g_\sharp = g_*,\ g_*(0) = \mathrm{id}$, we immediately see that $g_*(t)$ and h_0 commute. Together with (3.18), $g_\sharp = g_*$, this proves (3.13).

The choice $t = 1$ in (3.20) yields

$$g_*(1+k) = g_*(k)g_*(1), \tag{3.21}$$

whence $g_*(k) = g_*(1)^k$, for all $k \in \mathbb{Z}$. This also follows directly, because multiplication by $g_*(1)$ is the time 1 stroboscope map for the nonautonomous equation $\dot{g} = g\mathbf{a}(t)$ with time period 1.

Inserting $k := 0, h := h_*$ with $\Theta(h_*) = 1/n$ in (3.13) yields

$$g_*(t + 1/n) = g_*(1/n)h_*g_*(t)h_*^{-1}. \tag{3.22}$$

An n-fold iteration of (3.22), evaluated at $t = 0$, yields the expression for $g_*(1)$ in (3.14). Together with (3.21), this proves (3.14).

Similarly, a k-fold iteration of (3.22) for $k = 0, 1, \cdots, n-1$ proves (3.15). Since $h_*^{-n} \in K$ commutes with h_* and, by (3.13), with $g_*(1/n)$, the stroboscope map $g_*(1)$ in (3.14) also commutes with $g_*(1/n)h_*$. Since $g_*(1)$ also commutes with K, and because h_* generates H/K, the stroboscope $g_*(1)$ and its iterates $g_*(k)$ also commute with all $g_*(\Theta(h))h, h \in L$.

To prove (3.16), let $g \in \tilde{L}$. Then $gu(t) = u(t + \vartheta)$ for some real ϑ and all $t \in \mathbb{R}$. Upstairs, there exists $h \in H$ such that

$$\begin{array}{rcl} (g_*(t+\vartheta), v(t+\vartheta)) & = & g(g_*(t), v(t)) \cdot h = \\ & = & (gg_*(t)h^{-1}, hv(t)) \end{array} \tag{3.23}$$

for some, and hence all, real t. Comparing the second components we see that $h \in L$ and there exists a unique $k \in \mathbb{Z}$ such that

$$\vartheta = \Theta(h) + k, \tag{3.24}$$

if we fix representatives $0 \le \Theta(h) < 1$. Comparing the first components, in view of (3.13), (3.24), we find

$$g_*(k)g_*(\Theta(h))h = g, \tag{3.25}$$

after cancellation of $g_*(t)h^{-1}$. Conversely, any such g lies in $\tilde{L}$, by (3.13), (3.23), (3.24). Letting $\tilde{\Theta}(g) = \vartheta \pmod p$, it only remains to prove $\tilde{K} = K$.

Note that $g \in \tilde{K} = \ker\tilde{\Theta}$ if, and only if, (3.23) holds with $\vartheta = 0$ and for some (hence all) t, say $t = 0$. Comparing components and using $g_*(0) = \mathrm{id}$, this is equivalent to $gh^{-1} = \mathrm{id}$ with $h \in K$. Hence $\tilde{K} = K$, and the proof is complete. □

The simple fact $\tilde{K} = K$, in our notation, implies that the isotropy groups occurring in the tube U are precisely the conjugates $gKg^{-1},\ g \in G$, of isotropy groups K occurring in the (linear) slice V. Concisely: the isotropy types in U and V coincide.

We emphasize that the spatio-temporal symmetry $\tilde{L}$ of $u(t)$, given in (3.16), is a group, and $\tilde{\Theta} : \tilde{L} \to \Sigma$ is a group homomorphism. For suitable H-equivariant choices of $\mathbf{a}(v)$, the element $g_*(1/n)$ can be thought of as an arbitrary element of the connected component G_0 of the identity in G. Indeed, H-equivariance does not impose any significant restriction on $\mathbf{a}(t), 0 \leq t < 1/n$, thus leaving sufficient freedom to prescribe a path $g_*(t) \in G$ from $g_*(0) = \mathrm{id}$ to $g_*(1/n)$. However, the skew product consequences of the *interplay* of the various spatio-temporal symmetries (L, K, Θ) in equivariant Hopf bifurcation certainly deserve further investigation.

4 Basic facts on Euclidean groups

We collect some background material concerning $G = E(N)$ (or $SE(N)$), the (special) Euclidean groups on $\mathbb{R}^N$. In lemma 4.1 below, we identify the compact subgroups of G as translation conjugates of purely orthogonal groups. In lemma 4.2 this is applied to distinguish meandering from drifting solutions. We recall the semidirect product structure $(S)E(N) = (S)O(N) \times \mathbb{R}^N$ and the composition rule, coming from the standard affine action on $\mathbb{R}^N$; see (1.1)–(1.4).

For computations involving the Lie algebras $se(N)$ it is convenient to represent $(R, S) \in SE(N) = SO(N) \times \mathbb{R}^N$ isomorphically as an element in $SL(N+1)$,

$$(R, S) \mapsto \begin{pmatrix} R & S \\ 0 & 1 \end{pmatrix}, \tag{4.1}$$

in block matrix notation. With this identification, an element $(\mathbf{r}, \mathbf{s})$ of the Lie algebra $se(N)$ becomes the $(N+1) \times (N \times 1)$ matrix

$$(\mathbf{r}, \mathbf{s}) \mapsto \begin{pmatrix} \mathbf{r} & \mathbf{s} \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

In particular, conjugation, iterates, the exponential map exp, the adjoint representation Ad of $E(N)$ on $se(N)$, and the commutator $[\cdot, \cdot]$ are given by

$$\begin{aligned} (R,S)(R',S')(R,S)^{-1} &= (RR'R^{-1}, (\mathrm{id} - RR'R^{-1})S + RS'); \\ (R,S)^n &= (R^n, (\mathrm{id} + R + \cdots + R^{n-1})S); \\ (\mathbf{r},\mathbf{s})^n &= (\mathbf{r}^n, \mathbf{r}^{n-1}\mathbf{s}); \\ \exp\,(\mathbf{r},\mathbf{s}) &= (\exp\,(\mathbf{r}), \mathbf{r}^{-1}(\exp\,(\mathbf{r}) - \mathrm{id})\mathbf{s}); \\ (R,S)(\mathbf{r},\mathbf{s}) &= (R\mathbf{r}, R\mathbf{s}); \\ (R,S)(\mathbf{r},\mathbf{s})(R,S)^{-1} &= (R\mathbf{r}R^{-1}, -R\mathbf{r}R^{-1}S + R\mathbf{s}); \\ [(\mathbf{r},\mathbf{s}),(\mathbf{r}',\mathbf{s}')] &= ([\mathbf{r},\mathbf{r}'], \mathbf{r}\mathbf{s}' - \mathbf{r}'\mathbf{s}). \end{aligned} \tag{4.3}$$

The notation in (4.3) is concise, but somewhat tricky. The first/last two relations hold in the group/algebra, respectively. Similarly, $\exp(\mathbf{r}, \mathbf{s})$ is in the group. The expressions for $(\mathbf{r}, \mathbf{s})^n, (R, S)(\mathbf{r}, \mathbf{s})$ are neither in the group nor in the algebra, in general, and are evaluated in $SL(N+1)$ in the sense of (4.1), (4.2). Similarly, all equations of (4.3) are easily checked in $SL(N+1)$.

In the settings of theorems 1.1, 3.1, 3.2, the compact isotropy subgroup H of G was playing a central role. We determine the compact subgroups H of $E(N)$ next. We use the equivariant projection

$$p : E(N) = O(N) \times \mathbb{R}^N \to O(N) \tag{4.4}$$

onto the first component.

LEMMA 4.1 *Let H be a compact subgroup of $E(N)$. Then H is conjugate to its projection $p(H) \leq O(N)$ by a fixed translation $S_0 \in \mathbb{R}^N$:*

$$H \quad = \quad (\mathrm{id}, S_0)\, p(H)\, (\mathrm{id}, -S_0) \tag{4.5}$$

PROOF: We will first prove that there exists a map

$$\sigma : p(H) \to \mathbb{R}^N \tag{4.6}$$

such that H has the form

$$H = p(H)^\sigma := \{(R, \sigma(R)) \mid R \in p(H)\}. \tag{4.7}$$

In a second step, we identify a fixed $S_0 \in \mathbb{R}^N$ such that

$$\sigma(R) = (\mathrm{id} - R)S_0, \tag{4.8}$$

for all $R \in p(H)$. Then (4.5) is proved.

To construct σ, let $(R, S), (R, S') \in H$ possess the same projection $R \in p(H)$. Then, for any integer n,

$$H \ni ((R, S)(R, S')^{-1})^n = (\mathrm{id}, n(S - S')). \tag{4.9}$$

Since H is compact, this implies $S' = S$ and $\sigma := S$ is well-defined. This proves (4.6), (4.7).

For the second step note that σ is at least continuous. Indeed, H is compact and the bijection $p : H \to p(H)$ is continuous, with inverse determined by σ. Therefore p is a homeomorphism, and σ is continuous.

Multiplying $(R, S), (R', S')$ in H yields the functional equation

$$\sigma(RR') = \sigma(R) + R\sigma(R'). \tag{4.10}$$

Note continuous dependence on R'. We integrate (4.10) over R' with respect to the left invariant Haar measure on the compact Lie group $p(H)$. With the abbreviation

$$S_0 := \int_{p(H)} \sigma(R')dR', \tag{4.11}$$

we obtain

$$\begin{aligned} \sigma(R) &= \int \sigma(R)dR' = \int \sigma(RR')dR' - \int R\sigma(R')dR' = \\ &= (\mathrm{id} - R)S_0. \end{aligned} \tag{4.12}$$

This proves the lemma. □

The lemma holds, more generally, for any compact subgroup H of the general affine group $GL(N) \times R^N$. The proof is the same, and the compact group $p(H) \leq GL(N)$ may in fact be assumed to act orthogonally.

Using the notation of section 3, we now consider a periodic solution $v(t)$ of $\dot{v} = \varphi(v)$ in the skew product, with period 1, and with associated fundamental solution

$g_*(t)$ of (3.5). We derive a criterion to decide whether the projected solution $u(t) = \bar{\tau}(g_*(t), v(t))$ is meandering or drifting, in the sense of definition 1.2.

Lemma 4.2 *Let $G = SE(N)$ or $E(N)$. Consider v of period 1, and g_*, u as above. Assume $u(t)$ is neither stationary nor periodic. Let*

$$(R_*, S_*) := g_*(1). \tag{4.13}$$

Then $u(t)$ is meandering, if S_* is orthogonal to the fix space of the rotation $R_* \in SO(N)$ in $\mathbb{R}^N$, that is,

$$S_* \perp \ker(\mathrm{id} - R_*) =: (\mathbb{R}^N)^{R_*}. \tag{4.14}$$

If, on the other hand, (4.14) does not hold, then $u(t)$ is drifting.

Proof: By definition 1.2, the nonstationary, nonperiodic solution $u(t)$ is meandering if the orbit $g_*(t), t \in \mathbb{R}$, is relatively compact, and drifting otherwise. By theorem 3.2 and the differential equation (3.5) for $g_*(t)$, this orbit is relatively compact if, and only if,

$$\begin{aligned} H' &:= \operatorname{clos}\{g_*(k) \mid k \in \mathbb{Z}\} \\ &= \operatorname{clos}\{g_*(1)^k \mid k \in \mathbb{Z}\} \end{aligned} \tag{4.15}$$

is a compact subgroup of $SE(2)$. (Note here that theorem 3.2 also applies to rotating waves $v(t)$, viewed as discrete waves with arbitrary $n \in \mathbb{N}$. By lemma 4.1, the group H' is compact if, and only if, it can be conjugated to its projection $p(H') \subseteq (S)O(N)$, by a pure translation $S_0 \in \mathbb{R}^N$. This is possible if, and only if, the translation component of

$$(\mathrm{id}, -S_0)(R_*, S_*)(\mathrm{id}, S_0) = (R_*, -S_0 + S_* + R_* S_0) \tag{4.16}$$

vanishes. Using orthogonality of R_*, this is equivalent to

$$S_* \in \operatorname{image}(\mathrm{id} - R_*) = \ker(\mathrm{id} - R_*)^{\perp}, \tag{4.17}$$

proving claim (4.14), and the lemma. □

We note a dichotomy with respect to dimension N, here, which was also observed by [AM96]. For even N, we have $(\mathbb{R}^N)^{R_*} = \{0\}$, for generic rotations R_*, and hence generic meandering. For odd N, in contrast, $\dim(\mathbb{R}^N)^{R_*} = 1$, generically, which implies generic drifting.

If the 1-periodic solution $v(t) \in V$ possesses spatio-temporal symmetry (L, K, Θ) with non-trivial pointwise isotropy K, we obtain a particularly simple criterion excluding drifts.

Lemma 4.3 *Let $G = SE(N)$ or $E(N)$, consider v, u, g_* as above, and let $g_*(1) = (R_*, S_*)$. Assume the compact isotropy group K of $v(t)$ to be contained in $O(N)$, after conjugation by a translation as in lemma 4.1.*

Then the translation component S_* of the stroboscope map $g_*(1)$ is fixed under K, that is

$$S_* \in (\mathbb{R}^N)^K. \tag{4.18}$$

In particular, drifting is excluded if

$$(\mathbb{R}^N)^K \perp (\mathbb{R}^N)^{R_*}. \tag{4.19}$$

Most trivially, of course, condition (4.19) holds if $(\mathbb{R}^N)^K = \{0\}$ *or* $(\mathbb{R}^N)^{R_*} = \{0\}$.
PROOF: Lemma 4.2 and (4.18) imply claim (4.19). To prove (4.18), we let $h_0 \in K \leq O(N)$. Since h_0 and $g_*(1)$ commute, by theorem 3.2, (3.13), this implies

$$\begin{aligned}(R_*, S_*) &= g_*(1) = h_0 g_*(1) h_0^{-1} = \\ &= (h_0 R_* h_0^{-1}, h_0 S_*).\end{aligned} \tag{4.20}$$

Therefore $h_0 S_* = S_*$, and the lemma is proved. □

The projected solution $u(t)$ satisfies

$$\begin{aligned}u(k) &= \bar{\tau}(g_*(k), v(k)) = (g_*(1))^k \bar{\tau}(\mathrm{id}, v(0)) = \\ &= g_*(1)^k u(0)\end{aligned} \tag{4.21}$$

for all stroboscope times $k \in \mathbb{Z}$. Let $g_*(1)^k = (R_*^k, S_*^k)$. Aside from a compact part, due to R_*^k, and possibly the isotropy H of u_0, the displacement of $u(0)$ is therefore given by the translation component S_*^k of the k-fold iterated stroboscope $g_*(1)^k$. From (4.3), we recall $S_*^k = (\mathrm{id} + R_* + \cdots + R_*^{k-1})S_*$ and $R_*^k = (R_*)^k$.

To analyze S_*^k, we consider the meandering case $S_* \perp \ker(\mathrm{id} - R_*)$ next, for the stroboscope map $g_*(1) = (R_*, S_*)$. Let $(\mathrm{id} - R_*)^\dagger$ denote the pseudo-inverse of $(\mathrm{id} - R_*)$, that is, the isomorphism inverting $(\mathrm{id} - R_*)$ within the R_*-invariant subspace $(\ker(\mathrm{id} - R_*))^\perp =$ image$(\mathrm{id} - R_*)$. Define

$$S_\dagger := (\mathrm{id} - R_*)^\dagger S_*. \tag{4.22}$$

LEMMA 4.4 *As in the above setting, let* $S_* \perp \ker(\mathrm{id} - R_*)$. *Then* $g_*(1)^k, k \in \mathbb{Z}$, *are all conjugate to the rotations* $(R_*^k, 0)$ *around the origin, by the fixed translation* $S_\dagger$:

$$\begin{aligned}g_*(1)^k &= (\mathrm{id}, S_\dagger)(R_*^k, 0)(\mathrm{id}, -S_\dagger) \\ &= (R_*^k, S_\dagger - R_*^k S_\dagger).\end{aligned} \tag{4.23}$$

In particular, the translation components $S_*^k \in \mathbb{R}^N$ *of* $(g_*(1))^k$ *all lie on a sphere around* $S_\dagger \in \mathbb{R}^N$ *with radius* $|S_\dagger|_2$.
PROOF: By (4.3), applied to $(R_*^k, S_*^k) = (g_*(1))^k$, $k > 0$, and geometric summation, we have

$$\begin{aligned}S_*^k &= (\mathrm{id} + R_* + \cdots + R_*^{k-1})S_* = \\ &= (\mathrm{id} - R_*^k)(\mathrm{id} - R_*)^\dagger S_* = (\mathrm{id} - R_*^k)S_\dagger \\ &= S_\dagger - R_*^k S_\dagger.\end{aligned} \tag{4.24}$$

In case $k < 0$, the same formula holds, by $(g_*(1))^k = ((g_*(1))^{-1})^{-1}$ and (1.4). This proves (4.23) and, by orthogonality of R_*^k, the lemma. □

The radius $|S_\dagger|_2$ defined in (4.22) and lemma 4.4 relates to the "radius" of a meandering solution $u(t) = g_*(t)v(t)$ as follows. Let $u_0(t) = (\exp(\mathbf{r}_0 t), 0)u_0$ be a primary rotating wave solution, as in the introduction (1.5), (1.6). Then $u_0(t)$ rotates around its core point centered at zero. For $v(0)$ near u_0, we can consider zero also as the core point of $u(0) = \mathrm{id}\ v(0)$. Then S_*^k, the translation component of $g_*(k) = g_*(1)^k$, is the core position of $u(k) = g_*(k)v(0)$, by 1-periodicity of $v(\cdot)$. Since S_*^k all lie on a sphere around $S_\dagger$ with radius $|S_\dagger|_2$, we can call the Euclidean length $|S_\dagger|_2$

the *stroboscope radius* of $u(t)$. In section 5, (5.7) we will see how $|S_\dagger|_2 \to \infty$, when a planar meandering spiral passes through a drift resonance, for which $S_* \neq 0$ and $R_* = \mathrm{id}$.

We caution our reader that our notion of a stroboscope radius requires $u_0(t)$ to rotate around the origin. Moreover, the precise value of $|S_\dagger|_2$ depends on our choice of $t = 0$ as a reference point within the period of v. Indeed, other choices lead to expressions

$$\tilde{S}_\dagger = R_*(t)^{-1}(S_\dagger - P_* S_*(t)), \tag{4.25}$$

$0 \leq t \leq 1$, replacing $S_\dagger$, with correspondingly modified radii $|\tilde{S}_\dagger|_2$. Here P_* projects onto $\ker(id - R_*)$, orthogonally. Note that (4.25) has period 1 in t, by definition (4.22) of $S_\dagger$. Bounded modifications as in (4.25), however, do not affect the asymptotics of $|S_\dagger|_2 \to \infty$, when passage through a drift resonance occurs.

□

5 THE PLANAR CASE E(2): MEANDERING AND DRIFTING MULTI-ARMED SPIRALS

First rigorous results on meandering and drifting one-armed spirals in the plane were obtained by [Wul96], using a Lyapunov-Schmidt procedure in scales of Banach spaces. First formal results on meandering and drifting *multi-armed* spirals in the plane were obtained by [GLM96], using a formal center bundle reduction in the spirit of [Kru90]. Using the rigorous center manifold reduction due to [SSW96a], [SSW96b], the skew product structure developed in the present paper applies. We recover results of [GLM96], and investigate the behavior of meander radii at drift resonance.

Throughout this section, $G = E(2)$, and H is a compact subgroup which we may consider to be a subgroup of $O(2)$, after conjugation by a fixed translation, without loss of generality. As in section 3, we consider H-equivariant Hopf bifurcation for $\dot{v} = \varphi(\lambda, v)$ in the slice $v \in V$ of our skew product (3.1). Let (L, K, Θ) denote the spatio-temporal symmetry of our periodic solution $v(t)$, with minimal period normalized to 1. We also normalize the primary relative equilibrium u_0 to become $v = 0$, without loss of generality. The case of a rigidly rotating "primary" spiral wave with n identical arms, in the setting of the introduction, now corresponds to a rotating wave u_0 with $H = \mathbb{Z}_n \leq SO(2)$.

We begin with a simple criterion excluding drifting solutions $u(t) := \bar{\tau}(g_*(t), v(t))$ for general $H \leq O(2)$.

COROLLARY 5.1 *In the above planar setting, assume the isotropy group K of $v(t)$ contains some nontrivial rotation, that is, $K \leq O(2)$ is neither trivial nor generated by a single reflection.*

Then $u(t)$ cannot drift, in the sense of definition 1.2.

PROOF: Suppose $K \leq O(2)$ contains some nontrivial rotation. Then K fixes only the origin, in $\mathbb{R}^2$, that is $(\mathbb{R}^2)^K = \{0\}$. By lemma 4.3, this excludes drifting. □

We look at meandering and drifting for spatio-temporal symmetries (H, K, Θ) of $v(\cdot)$ next. Throughout, we identify $\mathbb{R}^2 = \mathbb{C}$ and write $(R, S) \in SE(2)$ in complex notation:

$$R = e^{2\pi i \alpha}, \quad \alpha \in \mathbb{R}/\mathbb{Z}, \quad S \in \mathbb{C}. \tag{5.1}$$

We consider solutions $v(\cdot)$ with spatio-temporal symmetry (H, K, Θ) given by

$$\begin{array}{rcl} H &=& \mathbb{Z}_n = \{\mathrm{e}^{2\pi ik/n} \mid k = 0, \ldots, n-1\}, \quad n \geq 2, \\ K &=& \{1\}, \\ \Theta(e^{2\pi ik/n}) &=& mk/n \in S^1 = \mathbb{R}/\mathbb{Z}, \quad k = 0, \ldots, n-1. \end{array} \tag{5.2}$$

We require $K = \{1\}$, to give drifting a chance. Note that this is equivalent to requiring the integer $m \in \{1, \ldots, n-1\}$ to be relatively prime to n.

Meandering, meander radii, drifting, and drift resonance will follow from theorem 3.2 and lemma 4.4. We will express all these effects in terms of the fractional stroboscope map

$$g_*(1/n) = \left(\exp(2\pi i\alpha_{1/n}), S_{1/n}\right). \tag{5.3}$$

Also, we have to choose $h_* \in H$ such that $\Theta(h_*) = 1/n$ generates image(Θ). Of course, we have to choose

$$\begin{array}{rcl} h_* &=& \exp(2\pi im'/n), \text{ where} \\ m'm &\equiv& 1 \qquad (\mathrm{mod}\ n). \end{array} \tag{5.4}$$

In other words, m' is the unique multiplicative inverse of m, mod n.

COROLLARY 5.2 *With the above notation, the stroboscope map $g_*(1)$ is given explicitly by*

$$g^*(1) = \left(\exp(2\pi in\alpha_{1/n}), \left(\sum_{k=0}^{n-1} \exp(2\pi ik(\alpha_{1/n} + m'/n))\right) S_{1/n}\right). \tag{5.5}$$

The solution $u(t) = \bar{\tau}(g_*(t), v(t))$ satisfies

$$u\,(1/n) = (\exp(2\pi i(\alpha_{1/n} + m'/n)), S_{1/n})u(0). \tag{5.6}$$

In particular, the solution $u(t)$ is

(i) *periodic,* *if* $S_{1/n} = 0$ and $\alpha_{1/n} \in \mathbb{Q}$;
(ii) *periodic,* *if* $\alpha_{1/n} + m'/n \notin \mathbb{Z}$ and $\alpha_{1/n} \in \mathbb{Q}$;
(iii) *meandering,* *if* $(\alpha_{1/n} + m'/n) \notin \mathbb{Z}$ and $\alpha_{1/n} \notin \mathbb{Q}$;
(iv) *drifting,* *if* $\alpha_{1/n} + m'/n \in \mathbb{Z}$ and $S_{1/n} \neq 0$.

In case (iii), the meandering stroboscope radius r is given explicitly by

$$r = \frac{1}{2}\,|\sin((\alpha_{1/n} + m'/n)\pi)|^{-1} \cdot\ |S_{1/n}|_2. \tag{5.7}$$

PROOF: By theorem 3.2, (3.14), we compute the stroboscope map $g_*(1)$ as

$$\begin{array}{l} (\exp(2\pi i\alpha_*), S_*) := g_*(1) = (g_*(1/n)h_*)^n h_*^{-n} = \\ = (\exp(2\pi i(\alpha_{1/n} + m'/n)), S_{1/n})^n. \end{array} \tag{5.8}$$

In particular (4.3) implies

$$\begin{array}{rcl} \alpha_* &=& n\alpha_{1/n} \qquad (\mathrm{mod}\ 1) \\ S_* &=& \left(\displaystyle\sum_{k=0}^{n-1} \exp(2\pi ik(\alpha_{1/n} + m'/n))\right) S_{1/n}. \end{array} \tag{5.9}$$

This proves (5.5) and case (i).

We now have to distinguish two cases. If $\alpha_{1/n} + m'/n$ is integer, then

$$\begin{array}{rcll} \alpha_* & = & 0 & \pmod 1, \\ S_* & = & nS_{1/n}, & \end{array} \tag{5.10}$$

proving case (iv). If, on the other hand, $\alpha_{1/n}+m'/n \notin \mathbb{Z}$, then we can easily compute

$$S_* = \frac{\exp(2\pi i\alpha_*) - 1}{\exp(2\pi i(\alpha_{1/n} + m'/n)) - 1} \quad S_{1/n}. \tag{5.11}$$

In view of formula (4.3) for the iterates $g_*(n) = (g_*(1))^n = (\exp(2\pi i\alpha_*), S_*)^n$, this proves cases (ii), (iii).

In case (iii), we can apply lemma 4.4 to compute the meandering radius r, because $\ker(\mathrm{id} - R_*) = \{0\}$ for $\alpha_* = n\alpha_{1/n} \notin \mathbb{Z}$. Therefore (5.11) implies

$$\begin{array}{rcl} r = |S_\dagger|_2 & = & |(\mathrm{id} - R_*)^{-1} S_*|_2 = \\ & = & |S_{1/n}|_2 / |\exp(2\pi i(\alpha_{1/n} + m'/n)) - 1| = \\ & = & \frac{1}{2}\,|S_{1/n}|_2 / |\sin((\alpha_{1/n} + m'/n)\pi)|. \end{array} \tag{5.12}$$

This proves the corollary. □

In a Hopf bifurcation situation, it is easy to derive expansions for the various cases of the previous corollary. Indeed, consider a primary n-armed spiral $u_0(t) = \exp(i\omega_{\mathrm{rot}}t)u_0(0)$ with isotropy $H = \mathbb{Z}_n$ and minimal period $T_{\mathrm{rot}} = 2\pi/(n\omega_{\mathrm{rot}})$. Assume an additional pair $\pm 2\pi i$ of imaginary eigenvalues of the linearization (in rotating coordinates). Then we can parameterize

$$v(t) = \epsilon \mathrm{e}^{2\pi it} + O(\epsilon^2) \tag{5.13}$$

at parameter $\lambda = \lambda_0 + \lambda_2\epsilon^2 + O(\epsilon^3)$. The equation for $g_*(t)$ becomes

$$\dot{g}_* \quad = \quad g_*(\mathbf{a}_0 + \mathbf{a}_1 v(t) + \cdots), \tag{5.14}$$

where $\mathbf{a}_0 = \mathbf{a}(v = 0)$ and $\mathbf{a}_1 = D\mathbf{a}(v = 0)$. For simplicity of presentation, we focus on the rotational component $R_*(t) = \exp(2\pi i\alpha(t))$ of $g_*(t)$. Inserting the v-expansion (5.13) we obtain

$$2\pi\dot{\alpha}(t) = \omega_{\mathrm{rot}} + \cdots, \quad \alpha(0) = 0, \tag{5.15}$$

omitting time dependent terms of order ϵ. Note that, indeed, ω_{rot} is the rotation frequency of the rotating spiral $u_0(t)$. Solving (5.15), up to terms of order ϵ, we get for $g_*(1/n) = (\alpha_{1/n}, S_{1/n})$

$$\alpha_{1/n} = \alpha(1/n) = \frac{\omega_{\mathrm{rot}}}{2\pi n} + \cdots. \tag{5.16}$$

Letting $2\pi = \omega_{\mathrm{Hopf}}$ denote the (normalized) frequency of the nontrivial Hopf eigenvalues, the transition to the drift case (iv) occurs, for example, at

$$n\alpha_{1/n} + m' = \frac{\omega_{\mathrm{rot}}}{\omega_{\mathrm{Hopf}}} + m' \equiv 0 \pmod n. \tag{5.17}$$

From (5.7) we see how the meandering stroboscope radius blows up, at this resonance, provided $S_{1/n} \neq 0$.

To analyze $S_{1/n}$ in more detail, we write the differential equation for the component $S(t)$ of $g_*(t)$ as

$$\dot{S}(t) = \mathrm{e}^{2\pi i\alpha(t)} \cdot \xi(v(t)). \tag{5.18}$$

In view of the v-expansion (5.13), we can restrict our attention to the case $v \in \mathbb{C}$. Note that the spatio-temporal symmetry

$$v(t + \Theta(h)) = h \cdot v(t) \tag{5.19}$$

then forces $h \in H = \mathbb{Z}_n \subseteq \mathbb{C}$ to act on v as complex multiplication by h^m, to be consistent with (5.2) and (5.13). Writing the H-action in this complex notation, equivariance condition (1.20) becomes

$$\xi(h^m v) = h\xi(v). \tag{5.20}$$

Expanding, as far as necessary, by

$$\xi(v) = \sum_{k,l=0}^{\infty} \xi_{kl}\; v^k\; \bar{v}^l, \tag{5.21}$$

we see that $\xi_{kl} = 0$, unless

$$(k - l)m \equiv 1 \qquad (\mathrm{mod}\ n). \tag{5.22}$$

Requiring m coprime to n, still, this yields

$$k \equiv l + m' \qquad (\mathrm{mod}\ n) \tag{5.23}$$

with the mod n multiplicative inverse m' of m. The terms of leading order are $\xi_{m',0}v^{m'}$, if $0 < m' \leq n/2$, and $\xi_{0,n-m'}\bar{v}^{n-m'}$, in case $n/2 \leq m' < n$. Integrating the $\dot{S}$ equation, up to higher order in ϵ, yields

$$S(1/n) = \epsilon^{m'} \frac{\xi_{m',0}}{\omega_{\mathrm{rot}} + 2\pi m'} \left(\mathrm{e}^{i(\omega_{\mathrm{rot}} + 2\pi m')/n} - 1 \right) \neq 0. \tag{5.24}$$

for $\epsilon, \xi_{m',0} \neq 0$, in case $0 < m' < n/2$. The case $n/2 < m' < n$ reads

$$S(1/n) = \epsilon^{n-m'} \frac{\xi_{0,n-m'}}{\omega_{\mathrm{rot}} + 2\pi(m' - n)} \left(\mathrm{e}^{i(\omega_{\mathrm{rot}} + 2\pi m')/n} - 1 \right) \neq 0. \tag{5.25}$$

for $\epsilon, \xi_{0,n-m'} \neq 0$. For $m' = n/2$, the coefficients of $\xi_{m',0}$ and $\xi_{0,n-m'}$ add. Most notably, we see a stroboscopic radius of meandering r proportional to higher powers of ϵ, in these cases; see (5.7). A similar calculation for $H = \{\mathrm{id}\}, n = 1$, yields r proportional to ϵ.

6 Meandering and drifting in three dimensions: twisted scroll rings

Let $G = SE(3)$, in this section. We first consider a primary wave $u_0(t)$ with trivial isotropy $H = \{\mathrm{id}\}$. At the end of this section, we comment on the case $H = \mathbb{Z}_n$. Pictorially, we think of u_0 as a hypothetical one parameter family of one-armed spirals with a core filament aligned along a unit circle parallel to the (x, y)-plane. The spiral patterns occur, locally, in the bundle of normal planes to the core circle. Such patterns have been called *scroll waves* by [Win73]. Moreover, assume the spirals to possess a phase difference along the family of normal planes. For simplicity, we assume that phase difference to equal the angle difference of the core points on the unit circle (rather than equaling an integer multiple of that angle.) While that pattern rotates, horizontally, around the vertical z-axis, as a rotating wave, it also propagates, vertically, along the z-axis, at constant speed. We call such a hypothetical pattern (if it exists) a *twisted scroll ring* [PW85]. The so inclined reader may also visualize smoke rings, with an inner rotating structure. For another recent example involving rigid body motion (of submarines) with $SE(3)$ symmetry see [LM96]. More mathematically, we require

$$u_0(t) = \exp(\mathbf{a}_0 t) u_0(0), \tag{6.1}$$

where u_0 has trivial isotropy H, and $\mathbf{a}_0 = (\mathbf{r}_0, \mathbf{s}_0)$ in the Lie algebra of $SE(3)$ has the special form

$$\mathbf{r}_0 = \begin{pmatrix} i\omega_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{s}_0 \;= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.2}$$

We use complex notation in the horizontal (x, y)-plane, here, writing $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We assume $\omega_0 \neq 0$ for the horizontal rotation frequency. Technically speaking, we might call $u_0(t)$ a drifting and rotating relative equilibrium. Lemma 4.2 explains why we choose the translation $\mathbf{s}_0$ to be vertical to the rotation plane.

Because the isotropy H is trivial, the skew product

$$\begin{array}{rcl} \dot{g}_* & = & g_* \mathbf{a}(v), \quad g_*(0) = id, \\ \dot{v} & = & \varphi(v), \end{array} \tag{6.3}$$

describes the flow in a neighborhood U of $G \cdot u_0$. We consider a family of periodic solutions $v = v(\epsilon, t)$ of period normalized to 1, bifurcating from the trivial solution $v \equiv u_0$. The parameter λ, so necessary for such a Hopf bifurcation, is suppressed. Instead, we represent dependence of $\mathbf{a}(v)$ on $v = v(\epsilon, t)$ by a differentiable function

$$\mathbf{a}(\epsilon, t) := \mathbf{a}(v(\epsilon, t)) \tag{6.4}$$

in the Lie algebra, directly. Note that

$$\mathbf{a}_0 := \mathbf{a}(0, t) = \mathbf{a}(u_0) \tag{6.5}$$

does not depend on time, while $\mathbf{a}(\epsilon, \cdot)$ has (normalized) period 1 for $\epsilon > 0$.

As in any differential equation, we can differentiate the solution $g_* = g_*(\epsilon, t)$ with respect to ϵ. Writing

$$\begin{array}{rcl} \gamma(\epsilon, t) & := & (\partial_\epsilon g_*) g_*^{-1} \\ \eta(\epsilon, t) & := & g_*^{-1} \partial_\epsilon g_* \end{array} \tag{6.6}$$

with $\gamma, \eta \in \mathrm{alg}(G)$, the differential equations for γ, η, respectively, are

$$\begin{array}{rcl} \dot{\gamma} & = & g_*(\partial_\epsilon \mathbf{a})g_*^{-1}, \\ \dot{\eta} & = & [\eta, \mathbf{a}] + \partial_\epsilon \mathbf{a}, \end{array} \tag{6.7}$$

with initial conditions $\gamma = \eta = 0$ at $t = 0$ and $\dot{} = \partial_t$. For example, at $\epsilon = 0$ and $t = 1$, the derivative of the stroboscope map g_* with respect to ϵ becomes

$$\partial_\epsilon g_* = \gamma \cdot g_* = \int_0^1 \exp(\mathbf{a}_0 t')\partial_\epsilon \mathbf{a}(t') \exp(-\mathbf{a}_0 t')dt' \; g_*, \tag{6.8}$$

because $g_*(0, t) = \exp(\mathbf{a}_0 t)$.

What are the effects of this ϵ-expansion on the dynamics, alias on the iterates of the stroboscope map $g_*(\epsilon, 1) = (R(\epsilon), S(\epsilon))$? At $\epsilon = 0$, we have

$$\begin{array}{rcl} R(0) & = & \begin{pmatrix} \exp(i\omega_0) & 0 \\ 0 & 1 \end{pmatrix}, \\ S(0) & = & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C} \times \mathbb{R}. \end{array} \tag{6.9}$$

For small positive ϵ, by (6.8), we get a rotation axis of $R(\epsilon)$ near the z-axis, tilted by an angle proportionally to ϵ. Conjugating by a small rotation around a horizontal axis orthogonal to that angle, we can assume

$$R(\epsilon) \quad = \quad \begin{pmatrix} \exp(i\omega) & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.10}$$

with $\omega = \omega(\epsilon)$ near ω_0. Conjugating by yet another rotation around the z-axis, afterwards, we can assume

$$S(\epsilon) \quad = \quad \begin{pmatrix} \sigma(\epsilon) \\ 1 + s(\epsilon) \end{pmatrix}, \tag{6.11}$$

with small complex σ and small real s. Now we can iterate the stroboscope map $g_*(\epsilon, 1) = (R, S)$. Using (3.14) and (4.3),

$$g_*(\epsilon, n) = (R_n, S_n) = (R, S)^n = (R^n, \sum_{k=0}^{n-1} R^k S). \tag{6.12}$$

With (6.10) we obtain the rotation

$$R_n = \begin{pmatrix} \exp(i\omega n) & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.13}$$

Similarly, the translation $S_n = (\sigma_n, n + sn)$ is given by

$$\sigma_n = (\sum_{k=0}^{n-1} e^{i\omega k})\sigma. \tag{6.14}$$

Summarizing, the propagation speed of our original twisted scroll ring u_0 experiences periodic fluctuations, due to $v(t)$. The period near 1 has been scaled to 1, here. Lighted with a stroboscope at (normalized) integer times $t = n$, we observe identical shapes of the twisted scroll ring. It propagates along the (slightly tilted) z-axis at a slightly modified average speed $1 + s$. This oscillating propagation is a three-dimensional analogue of Hopf bifurcation from a traveling wave in one space dimension; for the latter see [Pos92]. In a plane perpendicular to the vertical propagation direction, our scroll ring performs a planar meandering motion of stroboscopic radius

$$r = \frac{1}{2}|\sin(\pi\omega_{\mathrm{rot}}/\omega_{\mathrm{Hopf}})|^{-1} \cdot |\sigma|, \tag{6.15}$$

as has been investigated in section 5. (We have returned to the notation $\omega_{\mathrm{rot}} = \omega$, $\omega_{\mathrm{Hopf}} \approx 2\pi$ used there). Typically, $|\sigma|$ will be of order ϵ. Note the horizontal drift resonance which occurs at integer values

$$\omega_{\mathrm{rot}}/\omega_{\mathrm{Hopf}} \in \mathbb{Z}. \tag{6.16}$$

At these values, the meandering propagation along a spiral around the z-axis becomes a slow sidewards drift, away from the z-axis.

Additional isotropies $H = \mathbb{Z}_n$, commuting with the primary rotation $\exp(\mathbf{r}_0 t)$ of $u_0(t)$ in (6.1), (6.2), can be incorporated. Note that H rotates around the vertical z-axis. For the horizontal planar meandering, the results of section 5 will reappear. Specifically, let (H, K, Θ) be the spatio-temporal symmetry of a bifurcating periodic solution $v(t)$ in the skew product. According to lemma 4.3, nontrivial rotations in K will force translations S_* in the stroboscope map $g_*(1) = (R_*, S_*)$ to point along the z-axis. Likewise, R_* near $\exp \mathbf{r}_0$ will rotate around the z-axis, unless $R_* = \mathrm{id}$. Indeed $R_* \in SO(3) \setminus \{\mathrm{id}\}$ commutes with K, by (3.13) and lemma 4.3, and hence R_* and K fix the same axis of rotation. Therefore horizontal meandering is impossible, if K contains a nontrivial rotation. Pure drifts $g_*(1) = (\mathrm{id}, S_*)$ can only point along the z-axis.

If $K = \{\mathrm{id}\}$ is trivial, transverse meandering perpendicular to the direction of propagation becomes possible. Indeed, let $g_*(1/n) = (R_{1/n}, S_{1/n})$. Again we conjugate the axis of $R_{1/n}$ to be vertical, so that

$$R_{1/n} = \begin{pmatrix} \exp(i\alpha_{1/n}) & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.17}$$

Then $S_{1/n}$ possesses a rather irrelevant vertical component, which only modifies the vertical propagation speed. The important horizontal component, however, produces periodicity, meandering, and drifting phenomena transversely to the propagation direction. Note how the ϵ-expansions (5.24), (5.25) force the transverse drifting to be of small radius, or the transverse drifting to be slow.

Arrows by American Indians and other early, even neolithic civilizations are a practical visualization of some of the results discussed here. In fact, elastic vibrations and interaction with the air flow could lead to destabilization of the straight flight path. However, the feathers can provide an isotropy K, if they prevent rotation around the axis of the arrow. This isotropy, in turn, prevents transverse drifting and fixes the direction of propagation to be, quite literally, "straight as an arrow". Even in the case of a rotating feathered arrow, transverse deviations caused by symmetry breaking bifurcations from the straight path will be slow, due to (5.24), (5.25).

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[Reference added in proof:]

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Bernold Fiedler
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin, Germany
fiedler@fu-berlin.de

Björn Sandstede
Div. of Applied Mathematics
Brown University
Providence, RI 02912, USA

Arnd Scheel
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin, Germany

Claudia Wulff
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin, Germany