An Invariant of Quadratic Forms over Schemes

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Received: September 29, 1996

Communicated by Alexander S. Merkurjev

Abstract. A ring homomorphism $e^0: W(X) \to EX$ from the Witt ring of a scheme X into a proper subquotient EX of the Grothendieck ring $K_0(X)$ is a natural generalization of the dimension index for a Witt ring of a field. In the case of an even dimensional projective quadric X, the value of e^0 on the Witt class of a bundle of an endomorphisms $\mathcal{E}$ of an indecomposable component $\mathcal{V}_0$ of the Swan sheaf $\mathcal{U}$ with the trace of a product as a bilinear form θ is outside of the image of composition $W(F) \to W(X) \to E(X)$. Therefore the Witt class of $(\mathcal{E}, \theta)$ is not extended.

Introduction

An important role in the quadratic form theory is played by the first (0-dimensional) cohomological invariant, the dimension index $e^0: W(F) \to \mathbb{Z}/2\mathbb{Z}$, which maps a Witt class of a symmetric bilinear space $(\mathcal{V}, \beta)$ over a field F onto $\dim \mathcal{V} \mod 2$. A straightforward generalization of this map for symmetric bilinear spaces over rings or schemes, which assigns to a Witt class the rank of its supporting module or bundle, is commonly used. We define a better invariant e^0 in Section 1 below. It is a variant of the construction used in [8] and [9]. The map e^0 defined in Section 1 assigns to a Witt class of a symmetric bilinear space $(\mathcal{V}, \beta)$ a class $[\mathcal{V}]$ of $\mathcal{V}$ in the group EX, attached functorially to a scheme X. The group EX consist of the self-dual (i.e., stable under dualization) elements of the Grothendieck group $K_0(X)$ up to the split self-dual ones (i.e., sums of a class and its dual). Thus the rank $\mod 2$ may be obtained by passing to the generic stalk. The group EX carries much more information on the Witt group $W(X)$ than $\mathbb{Z}/2\mathbb{Z}$, and so does the map e^0 defined here when compared to the rank $\mod 2$. In particular, we use it here to show that certain Witt classes are not extended, i.e., are not of the form $(V \otimes \mathcal{O}_X, \beta \otimes 1)$ for a symmetric bilinear space (V, β) over a base field.

In the Section 1 basic facts on dualization in the Grothendieck group, definition and elementary properties of the group EX and map e^0 are given. Theorem 1.1 describes EX for a smooth curve X. In the geometric case (algebraically closed base field) the group EX appears to coincide with the Witt group $W(X)$ of curve X itself.

Moreover, it is shown that Witt classes of line bundles of order two in Picard group are not extended from the base field.

Section 2 contains a number of examples to show that EX may be actually computed: the affine space - Proposition 2.1.1, the projective space over a field - Proposition 2.1.3, the projective space over a scheme - Proposition 2.1.5.

The main objective of this paper is to prove that on the projective quadric of even dimension $d \neq 2$ defined by a hyperbolic form, there exist nonextended Witt classes. For this purpose, a close look at the Swan computation of the K-theory of a quadric hypersurface is needed. Section 3 contains all needed facts on Clifford algebras and modules, the construction of the Swan bundle, its behavior under dualization, and how to find a canonical resolution of a regular bundle.

In Section 4, we develop a combinatorial method for operations with resolutions using generating functions. Next we use the classical computation of the Chow ring of a split quadric X to establish the ring structure of $K_0(X)$. Theorem 4.3 gives the description of EX for a split quadric.

Thus, in Section 5, we show in Theorem 5.1 that, in case of even dimension $d > 2$ of a quadric the bundle of endomorphisms of each indecomposable component of the Swan bundle carries a canonical symmetric bilinear form, whose Witt class is not extended from the base field, since its invariant e^0 has a value outside the image of the composite map $W(F) \to W(X) \to EX$.

The first version of this paper contained only an explicit computation for a quadric of dimension 4. The referee made several suggestions for simplification of proofs and computations. These remarks led author to the present more general results. The author would like to thank very much the referee for generous assistance. The author is glad to thank Prof. W. Scharlau for helpful discussions and Prof. K. Szymiczek, who suggested several improvements of the exposition.

1 The group EX and the invariant e^0

1.1 Notation.

If X is a scheme with the structural sheaf $\mathcal{O}_X$, and $\mathcal{M}$, $\mathcal{N}$ are coherent locally free sheaves of $\mathcal{O}_X$-modules (vector bundles on X), $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism, then we write

$$\mathcal{M}^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) \qquad \text{and} \qquad \phi^\wedge : \mathcal{N}^\wedge \to \mathcal{M}^\wedge$$

for the duals.

A *symmetric bilinear space* $(\mathcal{M}, \beta)$ consists of a coherent locally free sheaf $\mathcal{M}$ and a morphism $\beta : \mathcal{M} \to \mathcal{M}^\wedge$, which is self-dual, i.e. $\beta^\wedge = \beta$.

For a subbundle (a subsheaf which is locally a direct summand) $\iota : \mathcal{N} \to \mathcal{M}$ define its *orthogonal complement* $\mathcal{N}^\perp$ as a kernel of composition $\iota^\wedge \circ \beta$:

$$\mathcal{N}^\perp = \mathrm{Ker}(\mathcal{M} \xrightarrow{\beta} \mathcal{M}^\wedge \xrightarrow{\iota^\wedge} \mathcal{N}^\wedge).$$

Thus β induces an isomorphism $\mathcal{N}^\perp \cong (\mathcal{M}/\mathcal{N})^\wedge$.

There are two important special cases: the first, when $\mathcal{N}$ has trivial intersection with $\mathcal{N}^\perp$ or is non-singular, then β induces an isomorphism $\mathcal{N} \cong \mathcal{N}^\wedge$; the second, when $\mathcal{N} = \mathcal{N}^\perp$, and in this case $\mathcal{N}$ is said to be a *Lagrangian subbundle.*

A symmetric bilinear space $(\mathcal{M},\beta)$ is said to be *metabolic* if it possesses a Lagrangian subbundle, i.e., if there exists an exact sequence

$$0 \to \mathcal{N} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\iota^\wedge \circ \beta} \mathcal{N}^\wedge \to 0 \tag{1.1.1}$$

for some subbundle $\mathcal{N}$.

Direct sum and tensor product are defined in the set $B(X)$ of isomorphism classes of symmetric bilinear spaces, and in its Grothendieck ring $G(X)$ the set $M(X)$ of differences of classes of metabolic spaces forms an ideal.

The *Witt ring* $W(X)$ of X is the factor ring $G(X)/M(X)$. The *Witt class* of a symmetric bilinear space $(\mathcal{M},\beta)$ is its coset in $W(X)$. Two symmetric bilinear spaces $(\mathcal{M}_1,\beta_1)$ and $(\mathcal{M}_2,\beta_2)$ are *Witt equivalent*, $(\mathcal{M}_1,\beta_1) \approx (\mathcal{M}_2,\beta_2)$ iff their Witt classes are equal, or - equivalently - iff $(\mathcal{M}_1 \oplus \mathcal{M}_2, \beta_1 + (-\beta_2))$ is metabolic. Each Witt class (an element of $W(X)$) contains a symmetric bilinear space and $X \mapsto W(X)$ is a contravariant functor on schemes, namely for arbitrary morphism $f: Y \to X$ of schemes the inverse image functor f^* induces a ring homomorphism $f^*: W(X) \to W(Y)$. In fact, $f^*(\mathcal{M}^\wedge) = (f^*(\mathcal{M}))^\wedge$ and f^* is an exact functor. In the affine case $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, $f^\# : R \to S$ a ring homomorphism, $f^*: W(X) \to W(Y)$ is simply the scalar extension $S \otimes_R - : W(R) \to W(S)$. Important special cases are localization or taking a stalk at a point $x \in X$, i.e., the inverse image for $\operatorname{Spec} \mathcal{O}_{X,x} \to X$, and the extension, i.e., taking the inverse image for the structure map $f: X \to \operatorname{Spec} F$ for a variety X over a field F. In the latter case a Witt class of the form $(f^*\mathcal{M}, f^*\beta) = (\mathcal{M} \otimes_F \mathcal{O}_X, \beta \otimes 1)$ for genuine bilinear space $(\mathcal{M},\beta)$ over F is said to be *extended* or *induced* from the base field F.

1.2 Rank mod 2

In the affine case $X = \operatorname{Spec} R$, we write as usual $W(R)$ instead of $W(\operatorname{Spec} R)$. The classical situation is if $R = F$ is a field of characteristic different from two. In this case there is a ring homomorphism

$$e^0 : W(F) \to \mathbb{Z}/2\mathbb{Z}\,,\; e^0(\mathcal{M},\beta) = \dim\, \mathcal{M} \mod 2,$$

known as dimension index. One may put the definition of e^0 into a K-theoretical framework as follows:

The map $e: (\mathcal{M},\beta) \mapsto [\mathcal{M}]$ induces a ring homomorphism

$$G(F) \xrightarrow{e} K_0(F) \xrightarrow{\cong} \mathbb{Z}$$

which is surjective, since each vector space over F carries a symmetric bilinear form. Any metabolic form $(\mathcal{M},\beta)$ is *hyperbolic*, i.e., the sequence 1.1.1 splits, and

$$(\mathcal{M},\beta) \cong (\mathcal{N} \oplus \mathcal{N}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}).$$

Since each vector space is self-dual, $e(M(F)) = 2K_0(F) \cong 2\mathbb{Z}$, so e^0 is the induced ring homomorphism

$$W(F) \xrightarrow{e^0} K_0(F)/2K_0(F) \cong \mathbb{Z}/2\mathbb{Z}.$$

In general the forgetful functor $(\mathcal{M},\beta) \mapsto [\mathcal{M}]$ induces a ring homomorphism which in general neither is surjective nor maps $M(X)$ into $2K_0(X)$. We shall show below how to handle this using a proper subquotient of $K_0(X)$.

1.3 The involution $^\wedge$ and the group $E(X)$

Denote by $\mathbf{P}(X)$ the category of locally free coherent $\mathcal{O}_X$-modules. The dualization functor $^\wedge$ is an exact additive functor $^\wedge : \mathbf{P}(X) \to \mathbf{P}(X)^{op}$, which preserves tensor products and commutes with inverse image functors. Since

$$K_*(\mathbf{P}(X)) = K_*(\mathbf{P}(X)^{op}) = K_*(X),$$

the functor $^\wedge$ induces a homomorphism on K-groups, known also as the Adams operation ψ^{-1} . We shall denote it by $^\wedge$:

Definition 1.3.1. $^\wedge : K_*(X) \to K_*(X)$ is the homomorphism induced by the exact functor $^\wedge : \mathbf{P}(X) \to \mathbf{P}(X)^{op}$.

Proposition 1.3.2. *The homomorphism* $^\wedge : K_*(X) \to K_*(X)$ *enjoys the following properties:*

i) $^\wedge$ *is an involution,* $^\wedge \circ {}^\wedge = 1$;

ii) $^\wedge$ *is a graded ring automorphism of* $K_*(X)$: $(\alpha \cdot \beta)^\wedge = \alpha^\wedge \cdot \beta^\wedge$;

iii) if $f : Y \to X$ *is a morphism of schemes, then* $f^* \circ {}^\wedge = {}^\wedge \circ f^*$;

iv) if $i : Z \to X$ *is a closed immersion and* X *is regular of finite dimension, then* $(i^*(K_0(Z)))^\wedge = i^*(K_0(Z))$.

Proof. iv) Consider a finite resolution of $i^*(\mathcal{M})$ by vector bundles for a bundle $\mathcal{M}$ on Z. The stalk of this resolution at any point outside Z is exact, so its dual is exact. Hence the class of the alternating sum of the members of the resolution vanishes outside Z. □

We focus our attention on the Grothendieck group $K_0(X)$. The main object of this paper are the homology groups of the following complex:

$$\cdots \to K_0(X) \xrightarrow{1+^\wedge} K_0(X) \xrightarrow{1-^\wedge} K_0(X) \xrightarrow{1+^\wedge} K_0(X) \xrightarrow{1-^\wedge} \cdots \tag{1.3.1}$$

Definition 1.3.3.

$$EX = \mathrm{Ker}(1 - {}^\wedge)/\,\mathrm{Im}(1 + {}^\wedge)$$
$$E^-X = \mathrm{Ker}(1 + {}^\wedge)/\,\mathrm{Im}(1 - {}^\wedge).$$

We shall define a natural homomorphism $e^0 : W(X) \to EX$. The group E^-X will play only a technical role here, although one may consider a natural map $L_{2k+1}(X) \to E^-X$. The EX is the group of "symmetric" or "self-dual" elements in $K_0(X)$ modulo "split self-dual" elements, i.e., elements of the form $[\mathcal{M}] + [\mathcal{M}^\wedge]$. The following observations are obvious:

Proposition 1.3.4. *i)* $\operatorname{Ker}(1-\wedge)$ *is a subring of* $K_0(X)$ *and the groups* $\operatorname{Im}(1+\wedge)$, $\operatorname{Ker}(1+\wedge)$, $\operatorname{Im}(1-\wedge)$ *are* $\operatorname{Ker}(1-\wedge)$*-modules;*

ii) EX *is a ring and* E^-X *is an* EX*-module;*

iii) *an arbitrary morphism* $f: Y \to X$ *of schemes induces a ring homomorphism* $f^*: EX \to EY$ *and an* EX*-module homomorphism* $f^*: E^-X \to E^-Y$ *;*

iv) *for a regular Noetherian* X, EX *and* E^-X *carry a natural filtration, induced by the topological filtration of* $K_0(X) = K_0'(X)$ *;*

v) $2EX = 0$ *and* $2E^-X = 0$.

□

Note that the forgetful functor $(\mathcal{M},\beta) \mapsto [\mathcal{M}]$ induces a ring homomorphism $G(X) \mapsto K_0(X)$ which admits values in $\operatorname{Ker}(1-\wedge)$ and maps $M(X)$ onto $\operatorname{Im}(1+\wedge)$, since for a metabolic space $(\mathcal{M},\beta)$ there is exact sequence 1.1.1, i.e., the equality $[\mathcal{M}] = [\mathcal{N}] + [\mathcal{N}^\wedge]$ holds in $K_0(X)$.

Definition 1.3.5. $e^0: W(X) \to EX$ is the ring homomorphism induced by the forgetful functor $(\mathcal{M},\beta) \mapsto [\mathcal{M}]$.

This notion enjoys nice functorial properties.

Proposition 1.3.6. *Let* $f: X \to Y$ *be a morphism of schemes. Then the following diagram commutes:*

$$\begin{array}{ccc} W(X) & \xrightarrow{e^0} & EX \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ W(Y) & \xrightarrow{e^0} & EY \end{array}$$

□

Example 1.3.7. Let X be an irreducible scheme with the function field $F(X)$, and let $j: \operatorname{Spec} F(X) \to X$ be the embedding of the generic point. Then there is a commutative diagram

$$\begin{array}{ccc} W(F(X)) & \xrightarrow{e^0} & E(F(X)) = \mathbb{Z}/2\mathbb{Z} \\ {\scriptstyle j^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ W(X) & \xrightarrow{e^0} & EX \end{array}$$

and the composition $j^* \circ e^0 = e^0 \circ j^*$ is rank mod 2, usually used instead of e^0. Since $\mathcal{O}_X$ carries the standard symmetric bilinear form $< 1 >$, the surjection $j^*: EX \to \mathbb{Z}/2\mathbb{Z}$ splits canonically. The kernel of the map $j^*: EX \to \mathbb{Z}/2\mathbb{Z}$ has been used in [9]. It is easy to see that this kernel is a nilpotent ideal of a ring EX for a regular Noetherian X of finite dimension.

EXAMPLE 1.3.8. Retain the notation of example 1.3.7, and assume in addition that X is a variety over a field F, $\operatorname{char} F \neq 2$. Let $f : X \to \operatorname{SPEC} F$ be the structure map. Thus we have a commutative diagram:

$$\begin{tikzcd}
W(F(X)) \arrow[rrr, "e^0"] & & & \mathbb{Z}/2\mathbb{Z} \\
& W(X) \arrow[ul] \arrow[r, "e^0"] & EX \arrow[ur] & \\
W(F) \arrow[uu] \arrow[ur] \arrow[rrr, "e^0"] & & & \mathbb{Z}/2\mathbb{Z} \arrow[ul] \arrow[uu, "\mathrm{id}"']
\end{tikzcd}$$

The values of $e^0 \circ f^*$ are inside the direct summand $\mathbb{Z}/2\mathbb{Z}[\mathcal{O}_X]$ of EX. If we produce a variety X with nontrivial (i.e., having more than two elements) EX, and a symmetric bilinear space with a nontrivial value of e^0, then the Witt class of this space must be non-extended.

1.4 CURVES

The case $\dim X = 1$ is exceptional for several reasons, so we treat it here as an illustration. The following theorem covers the classical case of (spectra of) Dedekind rings.

THEOREM 1.1. *Let X be an irreducible regular Noetherian scheme of dimension one. Then*

i) $EX = \mathbb{Z}/2\mathbb{Z}[\mathcal{O}_X] \oplus I$, *where* $I \cdot I = 0$ *and* I *is canonically isomorphic to the group* ${}_2\mathrm{Pic}(X)$ *of the elements of order* ≤ 2 *in the Picard group;*

ii) E^-X *is canonically isomorphic to* $\mathrm{Pic}(X)/2\,\mathrm{Pic}(X)$;

iii) the map $e^0 : W(X) \to EX$ *is surjective.*

Proof. The rank map (i.e., the restriction to the generic point) yields the splitting

$$K_0(X) = \mathbb{Z} \cdot [\mathcal{O}_X] \oplus \mathrm{F}^1 K_0(X)$$

where $0 \subset \mathrm{F}^1 K_0(X) \subset K_0(X)$ is the topological filtration on $K_0(X)$. The map $^\wedge$ maps each direct summand onto itself.

Under assumptions on X the map $\bigwedge: \mathrm{F}^1 K_0(X) \to \mathrm{Pic}(X)$, induced by taking the highest exterior power of a bundle, is an isomorphism. An arbitrary element α of the group $\mathrm{F}^1 K_0(X)$ may be expressed as a difference of the classes of two bundles of the same rank r:

$$\alpha = [\mathcal{M}] - [\mathcal{N}].$$

The isomorphism $\bigwedge$ maps α onto the class of a line bundle $\mathcal{L}$,

$$\mathcal{L} = \bigwedge^r \mathcal{M} \otimes \bigwedge^r \mathcal{N}^\wedge$$

in $\mathrm{Pic}(X)$. The isomorphism $\bigwedge$ maps $[\mathcal{L}] - [\mathcal{O}_X]$ onto the class $\mathcal{L}$ in $\mathrm{Pic}(X)$, too. So, any element a of $\mathrm{F}^1 K_0(X)$ may be expressed as a difference of a line bundle and the trivial line bundle:

$$\alpha = [\mathcal{L}] - [\mathcal{O}_X].$$

Moreover, for arbitrary line bundles $\mathcal{L}_1,\ \mathcal{L}_2$

$$([\mathcal{L}_1] - [\mathcal{O}_X]) \cdot ([\mathcal{L}_2] - [\mathcal{O}_X]) = [\mathcal{L}_1 \otimes \mathcal{L}_2] - [\mathcal{O}_X].$$

Hence the involution $^\wedge$ acts on $\mathrm{F}^1 K_0(X)$ as taking the opposite, and it acts trivially on $\mathbb{Z} \cdot [\mathcal{O}_X]$. Therefore

$$\mathrm{Ker}(1 - {}^\wedge) = \mathbb{Z} \cdot [\mathcal{O}_X] \oplus 2\,\mathrm{F}^1 K_0(X) \quad , \quad \mathrm{Im}(1 + {}^\wedge) = 2\mathbb{Z} \cdot [\mathcal{O}_X],$$
$$\mathrm{Ker}(1 + {}^\wedge) = \mathrm{F}^1 K_0(X) \quad , \quad \mathrm{Im}(1 - {}^\wedge) = 2\,\mathrm{F}^1 K_0(X),$$

and assertions i), ii) follow.

To prove iii) note that a line bundle $\mathcal{L}$ which has order two in $\mathrm{Pic}(X)$ is isomorphic to its inverse $\mathcal{L}^\wedge$, so is automatically endowed with a nonsingular bilinear form $\mu : \mathcal{L} \to \mathcal{L}^\wedge$. This form must be symmetric locally at any point, hence is symmetric globally. Finally, e^0 maps the Witt class of $(\mathcal{L}, \mu) \oplus (\mathcal{O}_X, < 1 >)$ onto the class of $\mathcal{L}$ in $2\,\mathrm{Pic}(X)$ via $\bigwedge$. □

Remark 1.4.1. If R is a Dedekind ring, $X = \mathrm{Spec}\, R$, then $\mathrm{Pic}(X) = \mathrm{Pic}(R)$ is simply the ideal class group $H(R)$; the claim on the form of element of $F^1 K_0(X)$ is a consequence of the structural theorem for projective modules: if $\mathrm{rank}(P) = r$, then there exist fractional ideals $I_1, ..., I_r$ such that $P \cong I_1 \oplus ... \oplus I_r$; moreover, $P \cong R^{r-1} \oplus I_1 \cdot ... \cdot I_r \cong R^{r-1} \oplus \bigwedge^r P$. In this case $L^1(X) \cong \mathrm{Pic}(X)/2\,\mathrm{Pic}(X)$ and $L^1(X)$ is isomorphic to $E^- X$ via obvious generalization of e^0.

Remark 1.4.2. If ${}_2\mathrm{Pic}(X)$ is nontrivial, ${}_2\mathrm{Pic}(X) \neq 0$, then there exist non-extended Witt classes on X.

Corollary 1.4.3. *If X is a smooth projective curve of genus g over an algebraically closed field F, then*

i) if $\mathrm{char}\, F \neq 2$, *then* $EX \cong (\mathbb{Z}/2\mathbb{Z})^{1+2g}$;

ii) the degree map induces isomorphism $E^- X \cong \mathbb{Z}/2\mathbb{Z}$.

□

Remark 1.4.4. The result in Corollary 1.4.3. i) has been pointed out to author by W. Scharlau.

Remark 1.4.5. The proposition 2.1 of [3] states that for $F = \mathbb{C}$ the Witt group $W(X)$ of a smooth projective curve X is itself isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{1+2g}$, but the proof remains valid for an arbitrary algebraically closed field F provided $\operatorname{char} F \neq 2$. So under assumptions of Corollary 1.4.3.i) the map $e^0 : W(X) \to EX$ is an isomorphism.

2 The map $e^0 : W(X) \to EX$ for certain quasiprojective X.

2.1

We shall show now that the group EX may be actually computed, and compare the result with known Witt rings. The simplest case is following:

Proposition 2.1.1. *If R is a regular ring, and $X = \mathbb{A}^n_R$, the affine space, then the inverse image functor f^* for the structure map $f : X \to \operatorname{Spec} R$ induces isomorphisms $W(R) \to W(X)$, $ER \to EX$, $E^-R \to E^-X$.*

Proof. By the homotopy property of K-theory, the map $f^* : K_0(R) \to K_0(X)$ is an isomorphism and commutes with $^\wedge$, so the assertion on E and E^- follows. The assertion on $W(X)$ is a consequence of the Karoubi theorem, see [6], Ch. VI.2, Corollary 2.2.2. □

Now let X be a quasiprojective variety over a field F, $\operatorname{char} F \neq 2$, with the structure map $f : X \to \operatorname{Spec} F$. Consider the commutative diagram

$$\begin{array}{ccc} W(X) & \xrightarrow{e^0} & EX \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ W(F) & \xrightarrow[e^0]{} & EF \end{array} \tag{2.1.1}$$

We shall refer to "left f^*" and "right f^*" in 2.1.1 for various X.
Next, fix a projective embedding $i : X \to \mathbb{P}^n_F$ and denote:

$$1 = [\mathcal{O}_X] \text{ - the unit element in } K_0(X); \tag{2.1.2}$$

$$\mathcal{O}_X(-1) = i^* \mathcal{O}_{\mathbb{P}^n_F}(-1); \tag{2.1.3}$$

$$H = 1 - [\mathcal{O}_X(-1)] \text{ - the class of hyperplane section in } K_0(X). \tag{2.1.4}$$

We summarize some technicalities as follows:

Lemma 2.1.2. *If $d = \dim X$, then*

i) $H^{d+1} = 0$;

ii) $[\mathcal{O}_X(1)] = (1-H)^{-1} = \sum_{i=0}^{d} H^i$ *in* $K_0(X)$ *(here $H^0 = 1$)*;

iii) $H^\wedge = \dfrac{-H}{1-H} = -\sum_{i=1}^{d} H^i$;

iv) $(H^k)^\wedge = \left(\dfrac{-H}{1-H}\right)^k = (-1)^k H^k \sum_{i=0}^{d-k} \binom{k+i-1}{i} H^i$;

v) $(H^d)^\wedge = (-1)^d H^d$.

Proof. $H = 1 - [\mathcal{O}_X(-1)]$, so $[\mathcal{O}_X(-1)] = 1 - H$, $[\mathcal{O}_X(1)] = (1-H)^{-1}$, H being nilpotent. Thus $H^\wedge = 1 - [\mathcal{O}_X(1)] = ([\mathcal{O}_X(-1)] - 1) \cdot [\mathcal{O}_X(1)] = -H \cdot (1-H)^{-1}$ and $(H^k)^\wedge = (-H)^k(1-H)^{-k}$. □

In the case $i = \mathrm{id}$, $X = \mathbb{P}^d_F$, the family $1,\ H,\ \dots,\ H^d$ forms a basis of a free Abelian group $K_0(X)$, which allows us to compute EX, E^-X:

Proposition 2.1.3. *If $X = \mathbb{P}^d_F$, the projective space, then:*

i) both vertical arrows in the diagram 2.1.1 are isomorphisms;

ii) $E^-X = \mathbb{Z}/2\mathbb{Z} \cdot [H^d]$ for odd d and $E^-X = 0$ for even d.

Proof. The left f^* in the diagram 2.1.1 is an isomorphism by Arason's theorem [1]. Note that the statements on EX, E^-X are valid for $d = 0$, and - by Theorem 1.1 above - for $d = 1$. Consider $Y = \mathbb{P}^{d-1}_F$ and a closed embedding $k : Y \to X$ of Y as a hyperplane in X. There is an exact sequence

$$0 \to \mathbb{Z} \cdot H^d \to K_0(X) \xrightarrow{k^*} K_0(Y) \to 0$$

since $k^*\,\mathcal{O}_X(i) = \mathcal{O}_Y(i)$. Thus we have a short exact sequence of complexes:

$$\begin{array}{ccccccc}
\cdots \xrightarrow{1-\wedge} & K_0(Y) & \xrightarrow{1+\wedge} & K_0(Y) & \xrightarrow{1-\wedge} \cdots \\
 & {\scriptstyle k^*}\uparrow & & {\scriptstyle k^*}\uparrow & \\
\cdots \xrightarrow{1-\wedge} & K_0(X) & \xrightarrow{1+\wedge} & K_0(X) & \xrightarrow{1-\wedge} \cdots \\
 & \uparrow & & \uparrow & \\
\cdots \xrightarrow{1-(-1)^d} & \mathbb{Z} \cdot H^d & \xrightarrow{1+(-1)^d} & \mathbb{Z} \cdot H^d & \xrightarrow{1-(-1)^d} \cdots
\end{array}$$

and an induced exact sequence in homology. For even d this looks like

$$\cdots \to 0 \to E^-X \to E^-Y \to \mathbb{Z}/2\mathbb{Z} \cdot [H^d] \xrightarrow{\partial} EX \to EY \to 0 \to \cdots$$

and if - by induction - the proposition holds for Y, then ∂ maps the generator of $E^-Y = \mathbb{Z}/2\mathbb{Z} \cdot [H^{d-1}]$ onto $H^d \mod 2\mathbb{Z} \cdot H^d$, so the proposition holds for X: $E^-X = 0$, $k^* :\ EX \to EY$ is an isomorphism. In case of an odd d we have an exact sequence

$$\cdots \to 0 \to EX \to EY \xrightarrow{\partial} \mathbb{Z}/2\mathbb{Z} \cdot [H^d] \to E^-X \to E^-Y \to 0 \to \cdots$$

in homology. By induction $EY = \mathbb{Z}/2\mathbb{Z} \cdot [\mathcal{O}_X]$, $\partial = 0$, so $k^* :\ EX \to EY$ is an isomorphism. Thus $\mathbb{Z}/2\mathbb{Z} \cdot [H^d] \to E^-X$ is an isomorphism, since $E^-Y = 0$. □

Remark 2.1.4. The idea of this proof is due to the referee.

Proposition 2.1.5. *For an arbitrary variety Y let $X = \mathbb{P}^d_F \times Y$ and let $p_1 : X \to \mathbb{P}^d_F$, $p_2 :\ X \to Y$ be the projections. Then*

$$EX = (p_1^*(E(\mathbb{P}^d_F)) \otimes p_2^*(EY)) \oplus (p_1^*(E^-(\mathbb{P}^d_F)) \otimes p_2^*(E^-Y))$$
$$E^-X = (p_1^*(E(\mathbb{P}^d_F)) \otimes p_2^*(E^-Y)) \oplus (p_1^*(E^-(\mathbb{P}^d_F)) \otimes p_2^*(EY)).$$

Proof. By the projective bundle theorem p_1^*, p_2^* yield the identification $K_0(X) = K_0(\mathbb{P}_F^d) \otimes K_0(Y)$. Denote

$$A = \operatorname{Ker}(K_0(\mathbb{P}_F^d) \xrightarrow{1-\wedge} K_0(\mathbb{P}_F^d)), \qquad B = (1-\wedge)K_0(\mathbb{P}_F^d).$$

The complex 1.3.1 for $X = \mathbb{P}_F^d \times Y$ may be included into the short exact sequence of complexes:

$$\begin{tikzcd}
\cdots \arrow[r, "1-\wedge"] & B \otimes K_0(Y) \arrow[r, "1+\wedge"] & B \otimes K_0(Y) \arrow[r, "1-\wedge"] & \cdots \\
\cdots \arrow[r, "1-\wedge"] & K_0(X) \arrow[u, "(1-\wedge)\otimes 1"'] \arrow[r, "1+\wedge"] & K_0(X) \arrow[u, "(1-\wedge)\otimes 1"'] \arrow[r, "1-\wedge"] & \cdots \\
\cdots \arrow[r, "1-\wedge"] & A \otimes K_0(Y) \arrow[u] \arrow[r, "1+\wedge"] & A \otimes K_0(Y) \arrow[u] \arrow[r, "1-\wedge"] & \cdots
\end{tikzcd}$$

Note that $1 \pm \wedge$ restricted to $A \otimes K_0(Y)$ coincides with $1 \otimes (1 \pm \wedge)$ and induces $1 \otimes (1 \mp \wedge)$ on $B \otimes K_0(Y)$. Therefore the exact hexagon in homology

$$\begin{tikzcd}
& EX \arrow[dr] & \\
A \otimes EY \arrow[ur] & & B \otimes E^-Y \arrow[d, "\partial_2"] \\
B \otimes EY \arrow[u, "\partial_1"] & & A \otimes E^-Y \arrow[dl] \\
& E^-X \arrow[ul] &
\end{tikzcd}$$

breaks into short split exact sequences:

$$0 \to E(\mathbb{P}_F^d) \otimes E^-Y \to E^-X \to E^-(\mathbb{P}_F^d) \otimes EY \to 0 \tag{2.1.5}$$

$$0 \to E(\mathbb{P}_F^d) \otimes EY \to EX \to E^-(\mathbb{P}_F^d) \otimes E^-Y \to 0. \tag{2.1.6}$$

□

EXAMPLE 2.1.6. Put $d = 1$, $Y = \mathbb{P}_F^1$, i.e., $X = \mathbb{P}_F^1 \times \mathbb{P}_F^1$. Then

$$EX = \mathbb{Z}/2\mathbb{Z} \cdot [\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [H \boxtimes H] \tag{2.1.7}$$

$$E^-X = \mathbb{Z}/2\mathbb{Z} \cdot [H \boxtimes 1] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [1 \boxtimes H] \tag{2.1.8}$$

where $\boxtimes$ is induced by operation $\mathcal{F} \boxtimes \mathcal{G} = p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$. Since Witt ring is an invariant of birational equivalence in the class of smooth projective surfaces over a field F, $\operatorname{char} F \neq 2$ ([2], Theorem 3.4) and $X = \mathbb{P}_F^1 \times \mathbb{P}_F^1$ is birationally equivalent to $\mathbb{P}_F^2$, the left f^* in the diagram 2.1.1 is an isomorphism while the right f^* is not. This example shows that $e^0 : W(X) \to EX$ need not be surjective in general.

Remark 2.1.7. Probably there exists a skew symmetric bilinear space $(\mathcal{M}, \beta)$ on $X = \mathbb{P}^1_F \times \mathbb{P}^1_F$ such that $[\mathcal{M}] = [H \boxtimes H]$ in EX.

Remark 2.1.8. $X = \mathbb{P}^1_F \times \mathbb{P}^1_F$ may be embedded into $\mathbb{P}^3_F$ by Segre immersion as a quadric surface $x_0x_1 - x_2x_3 = 0$. In fact in the preliminary version of this paper this example was given using Swan's description of the K-theory of a quadric. The idea to use inverse images for projections was pointed out to author by the referee.

Remark 2.1.9. Note that we know $W(X)$ and EX for three quadrics of maximal index:

X	equation	$W(X)$	EX	E^-X
two points	$z_0^2 - z_1^2 = 0$	$W(F) \times W(F)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	0
$\mathbb{P}^1_F$	$z_0^2 - z_1^2 + z_2^2 = 0$	$W(F)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{P}^1_F \times \mathbb{P}^1_F$	$x_0x_1 - x_2x_3 = 0$	$W(F)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

We shall compute EX and E^-X for all projective quadrics of maximal index. To do this, some preparational work is required.

3 The Swan K-theory of a split projective quadric.

To compute EX and E^-X, we need some facts on dualization of vector bundles on quadrics. All needed information is known in fact, since indecomposable components of a Swan sheaf correspond to spinor representations. Nevertheless we give here complete proofs of the needed facts.

We shall apply the results of [11] in the simplest possible case of a split quadric: X is a projective quadric hypersurface over a field F, $\operatorname{char} F \neq 2$, defined by the quadratic form of maximal index.

3.1 Notation

Consider a vector space V with basis $v_0, v_1, \ldots, v_{d+1}$ over a field F, $\operatorname{char} F \neq 2$. Denote $z_0, z_1, \ldots, z_{d+1}$ the dual basis of $V^\wedge$. Let q be the quadratic form

$$q = \sum_{i=0}^{d+1} (-1)^i z_i^2.$$

Moreover, let $e_i = \frac{1}{2}(v_{2i} - v_{2i+1})$, $f_i = \frac{1}{2}(v_{2i} + v_{2i+1})$ for all possible values of i. Thus if d is even, $d = 2m$, then $e_0, f_0, e_1, f_1, \ldots, e_m, f_m$ form a basis of V with the dual basis $x_0, y_0, x_1, y_1, \ldots, x_m, y_m$ and

$$q = \sum_{i=0}^{m} x_i y_i.$$

If d is odd, $d = 2m+1$, then $f_0, e_1, f_1, \ldots, e_m, f_m, v_{d+1}$ form a basis of V with the dual basis $x_0, y_0, x_1, y_1, \ldots, x_m, y_m, z_{d+1}$ and

$$q = \sum_{i=0}^{m} x_i y_i + z_{d+1}^2.$$

We shall compute EX and $E^{-}X$ for a d-dimensional projective quadric X defined by equation $q = 0$ in $\mathbb{P}_F^{d+1}$, i.e., for

$$X = \operatorname{Proj} S(V^{\wedge})/(q) \cong \operatorname{Proj} F[z_0,\ z_1,\ \dots,\ z_{d+1}]/(q).$$

3.2 THE CLIFFORD ALGEBRA

In case of an odd $d = 2m+1$ the even part $C_0 = C_0(q)$ of the Clifford algebra $C(q)$ is isomorphic to the matrix algebra $M_N(F)$, where $N = 2^{m+1}$. In particular, $K_p(C_0) \cong K_p(F)$.

In case of an even $d = 2m$, the algebra C_0 has the center $F \oplus F \cdot \delta$, where $\delta = v_0 \cdot v_1 \cdot \ldots \cdot v_{d+1}$ and $\delta^2 = 1$. Thus $\frac{1}{2}(1+\delta)$, $\frac{1}{2}(1-\delta)$ are orthogonal central idempotents of C_0, so

$$C_0 = \frac{1}{2}(1+\delta)C_0 \oplus \frac{1}{2}(1-\delta)C_0$$

where each direct summand is isomorphic to the matrix algebra $M_{2^m}(F)$. For even $d = 2m$, consider the principal antiautomorphism $\Im : C_0 \to C_0$:

$$\Im(w_1 \cdot w_2 \cdot \ldots \cdot w_k) = (-1)^k w_k \cdot w_{k-1} \cdot \ldots \cdot w_1 \text{ for } w_1, w_2,\ \dots\ , w_k \in V.$$

Note that

$$\Im(\delta) = (-1)^{m+1}\delta. \tag{3.2.1}$$

Moreover, for every anisotropic vector $w \in V$, the reflection $\alpha \mapsto -w\alpha w^{-1}$ in V induces an automorphism ρ_w of C_0, which interchanges δ with its opposite:

$$\rho_w(\delta) = -\delta. \tag{3.2.2}$$

Regarding subscripts $i \mod 2$ denote

$$P_i = (1 + (-1)^i\delta)C_0 \text{ for even } d.$$

LEMMA 3.2.1. *In case of an even $d = 2m$:*

i) the involution $\Im$ of the algebra C_0 provides an identification of the left C_0-module $P_i{}^{\wedge} = \operatorname{Hom}_F(P_i, F)$ with the right C_0-module P_{i+m+1};

ii) for any anisotropic vector $w \in V$, the reflection ρ_w interchanges P_i's: $\rho_w(P_i) = P_{i+1}$.

□

3.3 SWAN K-THEORY OF A QUADRIC

Recall some basic facts and notation of [11]. Denote by C_1 the odd part of the Clifford algebra $C(q)$. We shall use $\mod 2$ subscripts in C_i. Recall the definition of the Swan bundle $\mathcal{U}$. Put

$$\phi = \sum_{i=0}^{d+1} z_i \otimes v_i, \qquad \phi \in \Gamma(X, \mathcal{O}_X(1) \otimes V).$$

The complex

$$\cdots \xrightarrow{\phi\cdot} \mathcal{O}_X(-n)\otimes C_{n+d+1} \xrightarrow{\phi\cdot} \mathcal{O}_X(1-n)\otimes C_{n+d} \xrightarrow{\phi\cdot} \mathcal{O}_X(2-n)\otimes C_{n+d-1} \xrightarrow{\phi\cdot} \cdots \tag{3.3.1}$$

is exact and locally splits ([11], Prop. 8.2.(a)).

Definition 3.3.1.

$$\mathcal{U}_n = \mathrm{Coker}(\mathcal{O}_X(-n-2)\otimes C_{n+d+3} \xrightarrow{\phi\cdot} \mathcal{O}_X(-n-1)\otimes C_{n+d+2}),$$
$$\mathcal{U} = \mathcal{U}_{d-1}.$$

Since the complex 3.3.1 is - up to a twist - periodical with period two, we have

$$\mathcal{U}_{n+2} = \mathcal{U}_n(-2).$$

Consider the exact sequences

$$\mathcal{O}_X(-n-2)\otimes C_{n+d+3} \xrightarrow{\phi\cdot} \mathcal{O}_X(-n-1)\otimes C_{n+d+2} \to \mathcal{U}_n \to 0$$

for two consecutive values n; twist the first one by 1. For any anisotropic vector $w \in V$ the isomorphism given by right multiplication by $1\otimes w$ fits into the commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{O}_X(-n-2)\otimes C_{n+d+4} & \xrightarrow{\phi\cdot} & \mathcal{O}_X(-n-1)\otimes C_{n+d+3} & \longrightarrow & \mathcal{U}_{n+1}(1) & \longrightarrow & 0 \\
\cong\downarrow{\cdot 1\otimes w} & & \cong\downarrow{\cdot 1\otimes w} & & & & \\
\mathcal{O}_X(-n-2)\otimes C_{n+d+3} & \xrightarrow{\phi\cdot} & \mathcal{O}_X(-n-1)\otimes C_{n+d+2} & \longrightarrow & \mathcal{U}_n & \longrightarrow & 0.
\end{array}$$

Thus we have proved the following lemma:

Lemma 3.3.2.

$$\mathcal{U}_{n+1} \cong \mathcal{U}_n(-1) \quad \textit{and} \quad \mathcal{U}_n \cong \mathcal{U}_0(-n)$$

for arbitrary integer n.

□

There is an exact sequence

$$0 \to \mathcal{U}_0 \xrightarrow{\phi} \mathcal{O}_X \otimes C_0 \to \mathcal{U}_{-1} \to 0 \tag{3.3.2}$$

where an isomorphism $\cdot(1\otimes w)$ was used to replace $\mathcal{O}_X\otimes C_1$ by $\mathcal{O}_X\otimes C_0$ for even d.

Lemma 3.3.3. $\mathrm{End}_X(\mathcal{U}_n) \cong C_0$ *acts on* $\mathcal{U}_n$ *from the right.*

Proof. [11], Lemma 8.7. □

3.4

We are now ready to compute $\mathcal{U}_n{}^\wedge$.

Lemma 3.4.1. *$\mathcal{U}_n{}^\wedge \cong \mathcal{U}_n(2n+1)$, in particular $\mathcal{U}^\wedge \cong \mathcal{U}(2d-1)$.*

Proof. We have chosen a basis $v_0,\ v_1,\ \dots,\ v_{d+1}$ of V in 3.1 above. The set of naturally ordered products of several v_i's in an even number forms a basis of C_0. Define a quadratic form Q on C_0 as follows: let the distinct basis products be orthogonal to each other and

$$Q(v_{i_1}\cdot v_{i_2}\cdot\dots\cdot v_{i_k}) = q(v_{i_1})\cdot q(v_{i_2})\cdot\dots\cdot q(v_{i_k}).$$

The form Q is nonsingular and defines - by scalar extension - a nonsingular symmetric bilinear form Δ on $\mathcal{O}_X\otimes C_0$. Since $(q(v_i))^2 = 1$, a direct computation shows that $\mathrm{Im}(\mathcal{O}_X(-1)\otimes C_1 \xrightarrow{\phi} \mathcal{O}_X\otimes C_0) = \phi\cdot\mathcal{U}_0 \cong \mathcal{U}_0$ is a totally isotropic subspace of $\mathcal{O}_X\otimes C_0$. Therefore

$$\mathcal{U}_0 \cong \phi\cdot\mathcal{U}_0 = (\phi\cdot\mathcal{U}_0)^\perp \cong ((\mathcal{O}_X\otimes C_0)/(\phi\cdot\mathcal{U}_0))^\wedge \cong \mathcal{U}_{-1}{}^\wedge$$

follows quickly from sect. 1.1 above and the exactness of 3.3.2. Thus

$$\mathcal{U}_0{}^\wedge \cong \mathcal{U}_{-1} \cong \mathcal{U}_0(1)$$

and, in general

$$\mathcal{U}_n{}^\wedge \cong (\mathcal{U}_0(-n))^\wedge \cong \mathcal{U}_0{}^\wedge(n) \cong \mathcal{U}_0(n+1) \cong \mathcal{U}_n(2n+1).$$

□

Remark 3.4.2. This argument was pointed out to the author by the referee.

Corollary 3.4.3. *i) $[\mathcal{U}^\wedge] = [\mathcal{U}(2d-1)]$ and $[\mathcal{U}(d-1)] + [\mathcal{U}(d-1)]^\wedge = 2d+1$ in $K_0(X)$;*

ii) $\mathrm{rank}\,\mathcal{U} = \frac{1}{2}\dim C_0 = 2^d$.

□

In case of an even $d = 2m$ the algebra $\mathrm{End}_X(\mathcal{U}) = C_0$ splits into the direct product of subalgebras defined in 3.2 above: $C_0 = P_0 \times P_1$.

Definition 3.4.4. In case of an even d:

$$\mathcal{U}'_n = \mathcal{U}_n\otimes_{C_0}P_0,\ \mathcal{U}''_n = \mathcal{U}_n\otimes_{C_0}P_1,$$
$$\mathcal{U}' = \mathcal{U}\otimes_{C_0}P_0,\ \mathcal{U}'' = \mathcal{U}\otimes_{C_0}P_1.$$

Note that $\mathcal{U}_n = \mathcal{U}'_n \oplus \mathcal{U}''_n$, $\mathcal{U} = \mathcal{U}'\oplus\mathcal{U}''$. $\mathcal{U}'_0$ and $\mathcal{U}''_0$ correspond to spinor representation and we shall copy here the standard argument on dualization (compare [4], sect. 4.3).

In case of an even $d = 2m$ another property of ϕ and the quadratic form Q introduced in the proof of Lemma 3.4.1 may be verified by direct computation:

LEMMA 3.4.5. *In case of an even $d = 2m$*

i) *if m is even, then $P_i = (1 \pm \delta)C_0$ are orthogonal to each other, hence self-dual;*

ii) *if m is odd, then $P_i = (1 \pm \delta)C_0$ are totally isotropic, hence dual to each other;*

iii) $\phi(1 \pm \delta) = (1 \mp \delta)\phi$.

□

COROLLARY 3.4.6. *In case of an even $d = 2m$*

i) $\mathcal{U}'^{\wedge} \cong \mathcal{U}'(2d-1)$ *and* $\mathcal{U}''^{\wedge} \cong \mathcal{U}''(2d-1)$ *for even m;*

ii) $\mathcal{U}'^{\wedge} \cong \mathcal{U}''(2d-1)$ *and* $\mathcal{U}''^{\wedge} \cong \mathcal{U}'(2d-1)$ *for odd m;*

iii) $\mathrm{End}_X(\mathcal{U}') \cong \mathrm{End}_X(\mathcal{U}'') \cong M_{2^m}(F)$;

iv) *the exact sequence 3.3.2 splits into two exact parts*

$$0 \to \mathcal{U}'_0 \xrightarrow{\phi\cdot} \mathcal{O}_X \otimes P_0 \to \mathcal{U}''_0(1) \to 0$$
$$0 \to \mathcal{U}''_0 \xrightarrow{\phi\cdot} \mathcal{O}_X \otimes P_1 \to \mathcal{U}'_0(1) \to 0$$

□

The standard way to determine indecomposable components is tensoring with the simple left module over an appropriate endomorphism algebra. We will use (from here onwards) superscript for the direct sum of identical objects.

DEFINITION 3.4.7.

i) in case of an odd $d = 2m+1$ $\quad \mathcal{V} = \mathcal{U} \otimes_{C_0} F^{2^{m+1}}$;

ii) in case of an even $d = 2m$ $\quad \mathcal{V}_0 = \mathcal{U}' \otimes_{M_{2^m}(F)} F^{2^m}$,
$\quad \mathcal{V}_1 = \mathcal{U}'' \otimes_{M_{2^m}(F)} F^{2^m}$.

For convenience we will use mod 2 subscripts in $\mathcal{V}_i$. Since $M_n(F) = (F^n)^n$ as a left $M_n(F)$-module, indecomposable components inherit properties of the Swan bundle: we have

PROPOSITION 3.4.8. *a) In case of an odd $d = 2m+1$:*

i) $\mathcal{U} \cong \mathcal{V}^{2^{m+1}}$;

ii) $\mathcal{V}^{\wedge} = \mathcal{V}(2d-1)$;

iii) $\mathrm{End}_X(\mathcal{V}) \cong F$ *and* $\mathrm{rank}\, \mathcal{V} = 2^m$;

iv) $[\mathcal{V}(d-1)] + [\mathcal{V}(d)] = 2^m$ *in* $K_0(X)$.

b) In case of an even $d = 2m$:

i) $\mathcal{U}' = \mathcal{V}_0^{2^m}$ *and* $\mathcal{U}'' = \mathcal{V}_1^{2^m}$;

ii) $\mathcal{V}_i{}^{\wedge} = \mathcal{V}_{i+m}(2d-1)$;

iii) $\mathrm{End}_X(\mathcal{V}_i) \cong F$ *and* $\mathrm{rank}\, \mathcal{V}_i = 2^{m-1}$

iv) $[\mathcal{V}_i(d-1)] + [\mathcal{V}_{i+1}(d)] = 2^m$ *in* $K_0(X)$.

□

In particular there is no global morphism $\mathcal{V}_i \to \mathcal{V}_{i+1}$.

COROLLARY 3.4.9. *In case of an even $d = 2m$ following identities hold in $K_0(X)$:*

i) $([\mathcal{V}_0] - [\mathcal{V}_1]) \cdot H = 0$;

ii) $([\mathcal{V}_0] - [\mathcal{V}_1]) \cdot [\mathcal{O}_X(n)] = [\mathcal{V}_0] - [\mathcal{V}_1]$;

iii) $([\mathcal{V}_0] - [\mathcal{V}_1])^\wedge = (-1)^m([\mathcal{V}_0] - [\mathcal{V}_1])$.

Proof. Proposition 3.4.8.b) *iv*) yields

$$[\mathcal{V}_0(d-1)] + [\mathcal{V}_1(d)] = [\mathcal{V}_1(d-1)] + [\mathcal{V}_0(d)].$$

Tensoring with $\mathcal{O}_X(-d)$ one obtains

$$[\mathcal{V}_0] - [\mathcal{V}_1] = ([\mathcal{V}_0] - [\mathcal{V}_1]) \cdot [\mathcal{O}_X(-1)],$$

hence i) and ii). Thus iii) results from 3.4.8. b) ii). □

PROPOSITION 3.4.10. *$K_*(X)$ is a free $K_*(F)$-module of the rank $2m+2$; moreover*

i) in case of an odd $d = 2m+1$ the classes $[\mathcal{O}_X]$, $[\mathcal{O}_X(-1)]$, $\ldots$, $[\mathcal{O}_X(1-d)]$, $[\mathcal{V}]$ form a basis of $K_*(X)$;

ii) in case of an even $d = 2m$ the classes $[\mathcal{O}_X]$, $[\mathcal{O}_X(-1)]$, $\ldots$, $[\mathcal{O}_X(1-d)]$, $[\mathcal{V}_0]$, $[\mathcal{V}_1]$ form a basis of K_*X.

Proof. Apply Theorem 9.1 of [11]. □

We have expressed the action of $^\wedge$ on $K_0(X)$ in terms of a twist. We need a plain expression in order to determine EX and E^-X.

3.5

We recall here several facts known from section 6 of [11] needed for establishing plain formulas for the action of $^\wedge$.

Every regular sheaf $\mathcal{F}$ on X has a *canonical resolution* (infinite in general):

$$\cdots \to \mathcal{O}_X(-p)^{k_p} \to \cdots \to \mathcal{O}_X(-1)^{k_1} \to {\mathcal{O}_X}^{k_0} \to \mathcal{F} \to 0$$

where superscript k_p means a direct sum of k_p copies. One may compute the coefficients k_p and the differentials recursively as follows: put $\mathcal{Z}_{-1} = \mathcal{F}$. Since a regular sheaf is generated by its global sections, put $k_p = \dim \Gamma(X, \mathcal{Z}_{p-1}(p))$ and define $\mathcal{Z}_p$ as the twisted kernel in

$$0 \to \mathcal{Z}_p(p) \to \mathcal{O}_X^{k_p} \to \mathcal{Z}_{p-1}(p) \to 0.$$

Then $\mathcal{Z}_p(p+1)$ is a regular sheaf. Therefore the sequence

$$0 \to \mathcal{Z}_p(p) \to {\mathcal{O}_X}^{k_p} \to \cdots \mathcal{O}_X(p-1)^{k_1} \to \mathcal{O}_X(p)^{k_0} \to \mathcal{F}(p) \to 0$$

is exact. Twisting it by 1 one obtains an exact sequence of regular sheaves

$$0 \to \mathcal{Z}_p(p+1) \to \mathcal{O}_X(1)^{k_p} \to \cdots \to \mathcal{O}_X(p)^{k_1} \to \mathcal{O}_X(p+1)^{k_0} \to \mathcal{F}(p+1) \to 0.$$

Since the functor of global sections is exact on regular sheaves, there is following recurrence for $k_{p+1} = \dim \Gamma(X, \mathcal{Z}_p(p+1))$:

$$\dim \Gamma(X, \mathcal{F}(p+1)) - k_0 \cdot \dim \Gamma(X, \mathcal{O}_X(p+1)) + \cdots \\ + (-1)^{p-1} k_p \cdot \dim \Gamma(X, \mathcal{O}_X(1)) + (-1)^p k_{p+1} = 0. \quad (3.5.1)$$

In case of a -1- regular $\mathcal{F}$ to obtain an expression for $[\mathcal{F}] \in K_0(X)$ in terms of the basis from Proposition 3.4.10 one truncates the canonical resolution of $\mathcal{F}$:

$$0 \to \mathcal{Z}_{d-1} \to \mathcal{O}_X(1-d)^{k_{d-1}} \to \cdots \to \mathcal{O}_X(-1)^{k_1} \to \mathcal{O}_X{}^{k_0} \to \mathcal{F} \to 0$$

and replaces $\mathcal{Z}_{d-1}$ by $\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1}) \cong \mathcal{Z}_{d-1}$. Then in $K_0(X)$

$$[\mathcal{F}] = \sum_{i=1}^{d-1} [\mathcal{O}_X(-i)] + [\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})].$$

Depending on the parity of d we have there

$$[\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})] = a[\mathcal{V}]$$

or

$$[\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})] = a[\mathcal{V}_0] + b[\mathcal{V}_1],$$

where the integers a, b in turn depend on the decomposition of $\operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})$ into a direct sum of simple left C_0 - modules. Conversely, for a given $\mathcal{F}$ the equality

$$[\mathcal{F}] = \sum_{i=1}^{d-1} [\mathcal{O}_X(-i)] + W$$

holds, where W is either $a[\mathcal{V}]$ or $a[\mathcal{V}_0] + b[\mathcal{V}_1]$, then k_0 is the Euler characteristic $\sum (-1)^i \dim \mathrm{H}^i(X, \mathcal{F})$ of $\mathcal{F}$. So if $\mathcal{F}$ is regular, then $k_0 = \dim \Gamma(X, \mathcal{F})$. Next, $\mathcal{Z}_0(1) = \operatorname{Ker}(\mathcal{O}_X(1)^{k_0} \to \mathcal{F}(1))$ is regular, and iterating yields that for a regular $\mathcal{F}$ the congruence

$$[\mathcal{F}] \equiv [\mathcal{O}_X(-i)] \mod \operatorname{Im}(K_0(C_0) \to K_0(X))$$

holds if and only if integers k_i satisfy 3.5.1. In case of an odd $d = 2m+1$, in order to express a class $[\mathcal{F}]$ of a regular sheaf $\mathcal{F}$ in terms of the basis of Proposition 3.4.10, it is enough to know the dimensions of $\Gamma(X, \mathcal{F}(i))$ for $i = 0,\ 1,\ 2,\ \ldots,\ d-1$ to determine the k_i's and the rank $\mathcal{F}$ to determine the coefficient a of $[\mathcal{V}]$. An analogous statement remains valid for an arbitrary sheaf $\mathcal{F}$ with Euler characteristic of $\mathcal{F}(i)$ in place of $\dim \Gamma(X, \mathcal{F}(i))$. In case of an even $d = 2m$, in view of Corollary 3.4.9 ii) and Proposition 3.4.8 ii), the bundles $\mathcal{V}_0$ and $\mathcal{V}_1$ have the same Euler characteristic, rank and even the highest exterior power. Thus, without special considerations, one can express a class $[\mathcal{F}]$ in terms of basis of the Proposition 3.4.10 only up to a multiple of $[\mathcal{V}_0] - [\mathcal{V}_1]$.

4 The group EX for a split projective quadric

4.1 A Poincaré series of a sheaf

We introduce here a method for the determination of the coefficients of the canonical resolution of a large enough class of regular sheaves. A *Poincaré series* $\Pi_{\mathcal{F}}(t)$ of a sheaf $\mathcal{F}$ is the formal power series

$$\Pi_{\mathcal{F}}(t) \stackrel{def}{=} \sum_{i=0}^{\infty} \dim\Gamma(X, \mathcal{F}(i)) \cdot t^i \in \mathbb{Z}[[t]].$$

The Poincaré series $\Pi_S(t)$ of a variety S is the Poincaré series of its structural sheaf:

$$\Pi_S(t) \stackrel{def}{=} \Pi_{\mathcal{O}_S}(t).$$

In particular if $S = \operatorname{Proj} A$ for a graded algebra A, then $\Pi_S(t)$ is the usual Poincaré series of A.

Example 4.1.1. If S is the projective space, $S = \mathbb{P}^n_F$, then $\dim\Gamma(S, \mathcal{O}_S(i)) = \binom{n+i}{i}$, so

$$P_n(t) \stackrel{def}{=} \Pi_S(t) = \sum_{i=0}^{\infty} \binom{n+i}{i} \cdot t^i = (1-t)^{-n-1}.$$

Example 4.1.2. Let f be a homogeneous polynomial of degree k in homogeneous coordinates in $\mathbb{P}^{d+1}_F = \operatorname{Proj} B$, $B = F[x_0, x_1, \ldots, x_{d+1}]$, $A = B/(f)$, $S = \operatorname{Proj} A$ - a hypersurface $f = 0$ in $\mathbb{P}^{d+1}_F$. Since the exact sequence

$$0 \to B_n \xrightarrow{f\cdot} B_{n+k} \to A \to 0$$

splits for every n, the following equality holds:

$$\Pi_S(t) = P_{d+1}(t) - t^k P_{d+1}(t).$$

Thus $\Pi_S(t) = \dfrac{1-t^k}{(1-t)^{d+2}} = \dfrac{1+t+\ldots+t^{k-1}}{(1-t)^{d+1}}$.

Lemma 4.1.3. *For a projective quadric X of dimension d*

$$Q_d(t) \stackrel{def}{=} \Pi_X(t) = \frac{1+t}{(1-t)^{d+1}}.$$

□

Proposition 4.1.4. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$ - modules and either $\mathcal{F}'$, $\mathcal{F}''$ are regular or $\mathcal{F}$, $\mathcal{F}'(1)$ are regular, then*

$$\Pi_{\mathcal{F}}(t) = \Pi_{\mathcal{F}'}(t) + \Pi_{\mathcal{F}''}(t).$$

Proof. By [7], Sect. 8, Lemma 1.2 either $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are regular or $\mathcal{F}'(1)$, $\mathcal{F}$, $\mathcal{F}''$ are regular. Hence each exact sequence of sheaves

$$0 \to \mathcal{F}'(i) \to \mathcal{F}(i) \to \mathcal{F}''(i) \to 0$$

induces an exact sequence of global sections. □

4.2 THE GENERATING FUNCTION FOR THE CANONICAL RESOLUTION

The recursive method of finding a canonical resolution

$$\cdots \to \mathcal{O}_X(-p)^{k_p} \to \cdots \to \mathcal{O}_X(-1)^{k_1} \to {\mathcal{O}_X}^{k_0} \to \mathcal{F} \to 0$$

of a regular sheaf $\mathcal{F}$, described in 3.5 above, namely the identity 3.5.1, yields following identities for the generating function $G_{\mathcal{F}}(t) \overset{def}{=} \sum_{i=0}^{\infty} k_i t^i$:

$$\Pi_{\mathcal{F}}(t) = G_{\mathcal{F}}(-t) \cdot \Pi_X(t) \qquad \text{and} \qquad G_{\mathcal{F}}(t) = \frac{\Pi_{\mathcal{F}}(-t)}{Q_d(-t)}.$$

EXAMPLE 4.2.1. The generating function for the canonical resolution of the sheaf $\mathcal{O}_X(1)$:

$$\Pi_{\mathcal{O}_X(1)}(t) = \frac{\Pi_{\mathcal{O}_X}(t) - 1}{t},$$

so

$$G_{\mathcal{O}_X(1)}(t) = \frac{\Pi_{\mathcal{O}_X}(-t)}{Q_d(-t)} = \frac{Q_d(-t) - 1}{-tQ_d(-t)} = \frac{\frac{1-t}{(1+t)^{d+1}} - 1}{-t\frac{1-t}{(1+t)^{d+1}}} = \frac{(1+t)^{d+1} - (1-t)}{t(1-t)}.$$

EXAMPLE 4.2.2. For a linear section $H^l = (1 - [\mathcal{O}_X(-1)])^l$ of codimension l in X

$$G_{H^l} = (1+t)^l.$$

EXAMPLE 4.2.3. Continue the notation of 3.1. Since X splits, it contains linear subvarieties $S_k = \operatorname{Proj} F[x_0, \ldots, x_k]$ given by the following equations:
a) in case of an even $d = 2m$:

$$y_0 = \ldots = y_m = x_{k+1} = \ldots = x_m = 0 \text{ for } k < m \text{ and}$$
$$y_0 = \ldots = y_m = 0 \text{ for } k = m;$$

b) in case of an odd $d = 2m + 1$:

$$y_0 = \ldots = y_m = z_d = x_{k+1} = \ldots = x_m = 0 \text{ for } k < m \text{ and}$$
$$y_0 = \ldots = y_m = z_d = 0 \text{ for } k = m.$$

S_k is isomorphic to $\mathbb{P}^k_F$, in particular its structural sheaf $\mathcal{L}_k$ is regular. Therefore

$$G_{\mathcal{L}_k}(t) = \frac{P_k(-t)}{Q_d(-t)} = \frac{(1+t)^{-k-1}}{\frac{1-t}{(1+t)^{d+1}}} = \frac{(1+t)^{d-k}}{1-t}.$$

LEMMA 4.2.4.

$$2G_{\mathcal{L}_k} - G_{\mathcal{L}_{k-1}} = (1+t)^{d-k}.$$

□

4.3 THE GENERATING FUNCTION FOR A TRUNCATED CANONICAL RESOLUTION

Truncating a generating function $G_{\mathcal{F}}$ one obtains a polynomial $T_{\mathcal{F}}$. For $l < d$ the canonical resolution for H^l is itself truncated:

$$T_{H^l} = (1+t)^l \qquad \text{for} \qquad l < d.$$

The sequence (c_i) of coefficients of the canonical resolution of the sheaf $\mathcal{L}_k$ stabilizes from the degree $d-k$ onwards:

$$G_{\mathcal{L}_k} = \frac{(1+t)^{d-k}}{1-t} = (1+t)^{d-k} \cdot \sum_{i=0}^{\infty} t^i = \sum_{i=0}^{\infty} c_i t^i$$

so

$$c_{d-k} = c_{d-k+1} = \ldots = 2^{d-k}.$$

Thus

$$T_{\mathcal{L}_k}(t) = \frac{(1-t)^{d-k} - 2t^d}{1-t}.$$

PROPOSITION 4.3.1. *If, for fixed k, $\mathcal{L}_k$ is a structural sheaf of a linear subvariety S_k of dimension k in X, then in $K_0(X)$:*
a) in case of an odd $d = 2m+1$

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + 2^{m-k}[\mathcal{V}];$$

b) in case of an even $d = 2m$ for a suitable integer a

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + (2^{m-k} - a)[\mathcal{V}_1].$$

Proof. Substituting $t = -[\mathcal{O}_X(-1)]$ into the expansion for $T_{\mathcal{L}_k}(t)$ yields, depending on the parity of d, the expressions

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}];$$

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + b[\mathcal{V}_1].$$

for suitable integers a, b. Thus

$$0 = \operatorname{rank}[\mathcal{L}_k] = \begin{cases} T_{\mathcal{L}_k}(-1) + (-1)^d a \cdot 2^m = & \\ \quad = (-1)^d (2^m a - 2^{d-k-1}) & \text{for } d = 2m+1 \\ T_{\mathcal{L}_k}(-1) + (-1)^d (a+b) \cdot 2^{m-1} = & \\ \quad = (-1)^d (2^{m-1}(a+b) - 2^{d-k-1}) & \text{for } d = 2m. \end{cases}$$

□

4.4 THE TOPOLOGICAL FILTRATION

Now we shall find a basis of $K_0(X)$ which is convenient for computations. Since the quadric X is regular, $K_0'(X) = K_0(X)$ and one may transfer the topological filtration

$$\mathrm{F}^p K_0'(X) = \text{ subgroup generated by } \left\{ [\mathcal{F}] : \begin{array}{c} \text{the stalk } \mathcal{F}_x = 0 \text{ for all generic points} \\ x \text{ of subvarieties of codimension} < p \end{array} \right\}$$

of $K_0'(X)$ to $K_0(X)$. We omit the standard proof of following

PROPOSITION 4.4.1. *For a split projective quadric X the Chow groups $A^p(X)$ are isomorphic to the corresponding factors of the topological filtration:*

$$A^p(X) \cong \mathrm{F}^p K_0(X) / \mathrm{F}^{p+1} K_0(X).$$

□

Continue the notation of 3.1. Recall the classical computation of the Chow ring of a split projective quadric.

PROPOSITION 4.4.2. *For a split projective quadric X of dimension d*
a) in case of an even $d = 2m$

$$A^p(X) \cong \mathbb{Z} \text{ for } p \neq m,\ 0 \leq p \leq 2m \text{ and } A^m(X) \cong \mathbb{Z} \oplus \mathbb{Z};$$

b) in case of an odd $d = 2m + 1$

$$A^p(X) \cong \mathbb{Z} \text{ for all } p, 0 \leq p \leq 2m.$$

Explicit generators are given as follows:
Case $d = 2m$:
i) for $p > m$, a class of any linear subvariety of dimension $d - p$, e.g., $S_{d-p} : y_0 = \ldots = y_m = x_{d-p+1} = \ldots = x_m = 0$;
ii) for $p < m$, a class H^p of a linear section of codimension p;
iii) for $p = m$, $A^m(X)$ is generated by two classes of linear subvarieties $S_m' :\ x_0 = \ldots = x_m = 0$ and $S_m'' :\ y_0 = x_1 = \ldots = x_m = 0$; the classes in $A^m(X)$ remain unchanged if an even number of x_i, y_i are exchanged in these equations.
Case $d = 2m + 1$:
i) for $p > m$, a class of any linear subvariety of dimension $d - p$, e.g. $S_{d-p} : y_0 = \ldots = y_m = z_{d+1} = x_{d-p+1} = \ldots = x_m = 0$;
ii) for $p \leq m$, a class H^p of a linear section of codimension p.

For a sketch of proof and references see [10], Thm. 13.3.

Now we can give an explicit description of the ring structure and the action of the involution $\wedge$ on $K_0(X)$. To do this denote $L_p = [\mathcal{L}_p]$ the class of the structural sheaf of the linear subvariety S_p of dimension p. Moreover, in case of an even $d = 2m$ denote by L_m' and L_m'' the class of the structural sheaf of S_m' and S_m'' respectively.

THEOREM 4.1. *Let X be a split projective quadric of dimension d. Then*

i) in case of an odd $d = 2m+1$ classes $1, H, H^2, \dots, H^m, L_m, \dots, L_0$ form a basis of the free Abelian group $K_0(X)$;

ii) in case of an even $d = 2m$ classes $1, H, H^2, \dots, H^{m-1}, L'_m, L''_m, L_{m-1}, \dots, L_0$ form a basis of the free Abelian group $K_0(X)$;

iii) in case of an even $d = 2m$ classes may be chosen as follows:

$$L'_m = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{m}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + [\mathcal{V}_0],$$

$$L''_m = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{m}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + [\mathcal{V}_1]$$

and for dimensions $k < m$

$$L_k = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + 2^{m-k-1}([\mathcal{V}_0] + [\mathcal{V}_1]);$$

iv) if $d = 2m$, then $H^m = L'_m + L''_m - L_{m-1}$;

v) $H \cdot L_p = L_{p-1}$, $H \cdot L'_m = H \cdot L''_m = L_{m-1}$;

vi) $H^{d-k} = 2L_k - L_{k-1}$ for $k \le \frac{d-1}{2}$, $H^d = 2L_0$, $H^{d+1} = 0$;

vii) $L_p \cdot L_q = L_p \cdot L'_m = L_p \cdot L''_m = 0$;

viii) if $d = 2m$ and m is even, then ${L'_m}^2 = {L''_m}^2 = L_0$, $L'_m \cdot L''_m = 0$, if $d = 2m$ and m is odd, then ${L'_m}^2 = {L''_m}^2 = 0$, $L'_m \cdot L''_m = L_0$.

Proof. First of all note that the classes H^k, L_k for $k \le \frac{d-1}{2}$ and the pair $\{L'_m, L''_m\}$ are determined uniquely by the conditions of irreducibility of the underlying subvariety and to form a basis of some appropriate $A^p(X)$. In fact, by Proposition 4.4.1 this is clear for $\mathrm{F}^d K_0(X) \cong A^d(X)$. Thus, the general case results by induction. Statements i) and ii) follow from Proposition 4.4.1 and 4.4.2. To verify iii), note that the reflection ρ_{v_1} fixes $v_0, v_2, \dots, v_{d+1}$ and changes v_1 into the opposite (3.2 above). Thus, this reflection induces an automorphism of the symmetric algebra $S(\mathcal{V}^\wedge)$, which interchanges x_0 with y_0 and fixes other coordinates and q. Therefore it induces an automorphism of $S(\mathcal{V}^\wedge)/(q)$, $X = \operatorname{Proj} S(\mathcal{V}^\wedge)/(q)$, a semilinear automorphism of $\mathcal{O}_X(n)$ for all n, and an automorphism of $K_0(X)$. By Lemma 3.2.1 ii), the reflection ρ_{v_1} interchanges the P_i's with each other. So the induced automorphism of $\mathcal{U}$ interchanges direct summands $\mathcal{U}' = U \otimes P_0$ and $\mathcal{U}'' = U \otimes P_1$ of $\mathcal{U}$ and their indecomposable components $\mathcal{V}_0$, $\mathcal{V}_1$. Therefore, the induced automorphism of $K_0(X)$ fixes the basic elements $[\mathcal{O}_X]$, $[\mathcal{O}_X(-1)]$, $\dots$, $[\mathcal{O}_X(1-d)]$ and interchanges $[\mathcal{V}_0]$ with $[\mathcal{V}_1]$. By the uniqueness statement this automorphism fixes $L_0, \dots, L_{m-1}$. The explicit description given in Proposition 4.4.2 ii) shows that this automorphism interchanges L'_m with L''_m. Hence, by the explicit formula of Proposition 4.3.1 ii) for $k < m$

$$L_k = [\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + (2^{m-k} - a)[\mathcal{V}_1]$$

the integer a must be equal to 2^{m-k-1}. This same argument for $k = m$ yields

$$L'_m = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + (1-a)[\mathcal{V}_1]$$

and

$$L''_m = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + (1-a)[\mathcal{V}_0] + a[\mathcal{V}_1].$$

Since the statement ii) of the theorem holds, the integer a must be 0 or 1 (this follows from the regularity of the structural sheaves of S'_m and S''_m , too). Statements iv) - vii) are obvious consequences of the uniqueness and the explicit equations of Proposition 4.4.2. For to prove *viii)* assume, without loss of generality, that L'_m is the class of S'_m and L''_m is the class of S''_m. Consider the class L_m of the subvariety $S_m : \ y_0 = \ldots = y_m = 0$. In case of an even m the classes L''_m and L_m coincide, and S_m meets S'_m transversally at the empty set of points, so $L'_m \cdot L''_m = 0$. Moreover, S_m meets S''_m transversally at the rational point S_0, so ${L''_m}^2 = L_0$. Analogously, ${L'_m}^2 = L_0$.
In case of an odd m $L'_m = L_m$, so ${L'_m}^2 = {L''_m}^2 = 0$, $L'_m \cdot L''_m = L_0$. □

Theorem 4.2. *For a split projective quadric X of dimension d, the involution $^\wedge$ acts as follows:*

i) $L_k{}^\wedge = (-1)^{d-k} \cdot \sum_{i=0}^{k} \binom{d-k-2+i}{i} L_{k-i}$ *for* $k \leq \frac{d-1}{2}$;

ii) *in case of an even $d = 2m$:*

$$L'_m{}^\wedge = (-1)^m \cdot \left(L'_m + \sum_{i=1}^{m} \binom{m-2+i}{i} L_{m-i} \right),$$

$$L''_m{}^\wedge = (-1)^m \cdot \left(L''_m + \sum_{i=1}^{m} \binom{m-2+i}{i} L_{m-i} \right),$$

$$\begin{aligned} H^{k\wedge} =& (-1)^k \cdot \left(\sum_{j=k}^{m-1} \binom{j-1}{k-1} H^j + \binom{m-1}{k-1} (L'_m + L''_m) \right) \\ &+ (-1)^k \cdot \left(\sum_{j=m+1}^{d} \left(\binom{j-1}{k-1} - \binom{j-1}{k-2} \right) \cdot L_{d-j} \right); \end{aligned}$$

iii) in case of an odd $d = 2m+1$

$$H^{k\wedge} = (-1)^k \cdot \left(\sum_{j=k}^{m-1} \binom{j-1}{k-1} H^j + 2\binom{m-1}{k-1} L_m \right) + (-1)^k \left(\sum_{j=m+1}^{d} \left(\binom{j-1}{k-1} - \binom{j-1}{k-2} \right) \cdot L_{d-j} \right).$$

Proof. i) Since $H^{d-k} = 2L_k - L_{k-1} = 2L_k - H \cdot L_k$ by the Theorem 4.1 iv), vi),

$$L_k = \frac{H^{d-k}}{2-H} \quad \text{and} \quad H^\wedge = \frac{-H}{1-H}$$

by Lemma 2.1.2 iii),

$$L_k{}^\wedge = \frac{\left(\frac{-H}{1-H}\right)^{d-k}}{2 + \frac{H}{1-H}} = \frac{(-H)^{d-k}}{(2-H)(1-H)^{d-k-1}} = (-1)^{d-k} L_k \frac{1}{(1-H)^{d-k-1}}$$
$$= (-1)^{d-k} L_k \sum_{i=0}^{d} \binom{d-k-2+i}{i} H^i = (-1)^{d-k} \sum_{i=0}^{d} \binom{d-k-2+i}{i} L_{k-i}.$$

ii) To obtain the formula for $H^{k\wedge}$ substitute $H^{d-k} = 2L_k - L_{k-1}$ and $H^m = L'_m + L''_m - L_{m-1}$ into the formula of Lemma 2.1.2 iv). Analogously one proves iii). In case $d = 2m$

$$L'_m + L''_m = H^m + L_{m-1} = H^m + \frac{H^{m+1}}{2-H} = \frac{2H^m}{2-H},$$

so

$$(L'_m + L''_m)^\wedge = 2\left(\frac{-H}{1-H}\right)^m \frac{1}{2 + \frac{H}{1-H}} = (-1)^m \cdot 2 \cdot \frac{H^m}{2-H} \cdot \frac{1}{(1-H)^{m-1}}$$
$$= (-1)^m \cdot (L'_m + L''_m) \frac{1}{(1-H)^{m-1}} = (-1)^m \cdot (L'_m + L''_m) \sum_{i=0}^{d} \binom{m-2+i}{i} H^i,$$

and thus

$$(L'_m + L''_m)^\wedge = (-1)^m \cdot \left(L'_m + L''_m + 2\sum_{i=1}^{m} \binom{m-2+i}{i} L_{m-i} \right).$$

On the other hand, by Theorem 4.1 iii)

$$L'_m - L''_m = [\mathcal{V}_0] - [\mathcal{V}_1]$$

and by Corollary 3.4.9 iii)

$$(L'_m - L''_m)^\wedge = (-1)^m \cdot ([\mathcal{V}_0] - [\mathcal{V}_1]) = (-1)^m \cdot (L'_m - L''_m).$$

The formula for $L'_m{}^\wedge$ and $L''_m{}^\wedge$ follows directly, since we know their sum and difference. $\square$

Consider the matrix A of the involution $^\wedge$ in the free Abelian group $K_0(X)$ with respect to the basis given in Theorem 4.1 i), ii). We shall write it in a slightly unusual way:

$$A = [a_{i,j}] \qquad , \qquad 0 \le i,j \le 2m+1.$$

In case of an even $d = 2m$ we regard A as a block matrix B, arranging two central rows and two central columns into separate blocks:

$$b_{m,m} = \begin{bmatrix} a_{m,m} & a_{m,m+1} \\ a_{m+1,m} & a_{m+1,m+1} \end{bmatrix} \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2),$$

$$b_{m,i} = \begin{bmatrix} a_{m,i} \\ a_{m+1,i} \end{bmatrix} \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^2),$$

$$b_{i,m} = [a_{i,m}\ a_{i,m+1}] \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}) \quad \text{for } i \neq m,$$

$$b_{i,j} = \begin{cases} a_{i,j} & \text{for } i,j < m \\ a_{i,j+1} & \text{for } i < m, j > m \\ a_{i+1,j} & \text{for } i > m, j < m \\ a_{i+1,j+1} & \text{for } i,j > m. \end{cases}$$

As one may expect in view of Proposition 1.3.2 iv), the matrix A is triangular in the odd dimensional case and the matrix B is triangular in the even dimensional case. We summarize the most important results of Theorem 4.2 as follows:

COROLLARY 4.4.3. *a) In case of an odd $d = 2m+1$ the matrix A is triangular with*

$$a_{i,i} = (-1)^i \quad \textit{for } i = 0,1,\ldots,2m+1,$$

$$a_{i,0} = 0 \quad \textit{for } i > 0,$$

$$a_{i+1,i} = \begin{cases} (-1)^i i & \textit{for } i = 0,1,\ldots,m-1 \\ (-1)^m \cdot 2m & \textit{for } i = m \\ (-1)^i (i-1) & \textit{for } i = m+1,\ldots,2m. \end{cases}$$

b) In case of an even $d = 2m$ the matrix B is block triangular with

$$b_{i,i} = (-1)^i \textit{for } i \neq m,$$

$$b_{m,m} = (-1)^m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b_{i,0} = 0 \textit{for } i > 0,$$

$$b_{i+1,i} = \begin{cases} (-1)^i i & \textit{for } i = 0,1,\ldots,m-2 \\ (-1)^i (i-1) & \textit{for } i = m+1,\ldots,2m, \end{cases}$$

$$b_{m,m-1} = (-1)^{m-1}(m-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$b_{m+1,m} = (-1)^m (m-1)\,[1\ 1]\,,$$

$$b_{2m,m} = (-1)^m \binom{2m-2}{m} [1\ 1]\,.$$

□

THEOREM 4.3. *Let X be a split projective quadric of dimension d.*
a) If $d = 2m+1$ is odd, then

$$EX = \mathbb{Z}/2\mathbb{Z} \cdot [\mathcal{O}_X],$$

$$E^- X = \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot [L_m] & \text{for even } m \\ \mathbb{Z}/2\mathbb{Z} \cdot [H^m] & \text{for odd } m; \end{cases}$$

b) If $d = 2m$ is even, then

$$EX = \mathbb{Z}/2\mathbb{Z} \cdot [\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [L_0],$$

$$E^- X = \begin{cases} 0 & \text{for even } m \\ \mathbb{Z}/2\mathbb{Z} \cdot [L'_m] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [L''_m] & \text{for odd } m. \end{cases}$$

Proof. Consider the complex 1.3.1:

$$\cdots \to K_0(X) \xrightarrow{1+\wedge} K_0 X \xrightarrow{1-\wedge} K_0(X) \xrightarrow{1+\wedge} K_0(X) \to \cdots$$

with the topological filtration

$$K_0(X) = \mathrm{F}^0\, K_0(X) \supset \mathrm{F}^1\, K_0(X) \supset \cdots \supset \mathrm{F}^d\, K_0(X) \supset \mathrm{F}^{d+1}\, K_0(X) = 0,$$

and the corresponding spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{Ker}(1 - (-1)^{p+q} \cdot \wedge) / \operatorname{Im}(1 + (-1)^{p+q} \cdot \wedge) \Longrightarrow E^{(-1)^{p+q}} X$$

where $E^1 X = EX$, $E^{-1} X = E^- X$. The E_1 - term has period 2 with respect to q.
a) In case of an odd $d = 2m+1$ the term E_1 looks like

$$\begin{array}{cccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
{\scriptstyle q=1} & 0 & & 0 & & \cdots & & 0 & & 0 \\
{\scriptstyle q=0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\partial_0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\partial_1} & \cdots & \to & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\partial_{d-1}} & \mathbb{Z}/2\mathbb{Z}
\end{array}$$

The differential ∂_i is induced by the multiplication by the entry $a_{i+1,i}$ of the matrix A of $\wedge$. Thus, for each even q, we have complex $\mathrm{E}_1^{\cdot,q}$:

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2\cdot} \cdots$$
$$\xrightarrow{(m-1)\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2m\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{m\cdot} \cdots$$
$$\to \mathbb{Z}/2\mathbb{Z} \xrightarrow{(2m-1)\cdot} \mathbb{Z}/2\mathbb{Z}$$

Therefore for even m we have $\mathrm{E}_2\, 0, q = E_2^{m+1,q} = \mathbb{Z}/2\mathbb{Z}$ and $E_2^{i,q} = 0$ for other values of i. Since the left (the zeroth) column of A has zero entries except $a_{0,0} = 1$, all the differentials starting from $\mathrm{E}_r^{0,q}$ are trivial. So $EX = \mathbb{Z}/2\mathbb{Z} \cdot [\mathcal{O}_X]$, $E^- X = \mathbb{Z}/2\mathbb{Z} \cdot [L_m]$. Analogously, for an odd m, we have $E_2^{0,q} = E_2^{m,q} = \mathbb{Z}/2\mathbb{Z}$ and $E_2^{i,q} = 0$ for other values of i, so $EX = \mathbb{Z}/2\mathbb{Z} \cdot [\mathcal{O}_X]$, $E^- X = \mathbb{Z}/2\mathbb{Z} \cdot [H^m]$.

b) In case of an even $d = 2m$, the term E_1 looks like

$$\begin{array}{ccccccccccc}
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & & \cdots & 0 & & 0 & & 0 & \cdots & & 0 \\
\mathbb{Z}/2\mathbb{Z} & \xrightarrow{\partial_0} & \cdots & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\partial_{m-1}} & (\mathbb{Z}/2\mathbb{Z})^2 & \xrightarrow{\partial_m} & \mathbb{Z}/2\mathbb{Z} & \cdots & \xrightarrow{\partial_{d-1}} & \mathbb{Z}/2\mathbb{Z}.
\end{array}$$

The differential ∂_i is induced by the corresponding block of the matrix B of $^\wedge$. For each even q we have a complex $E_1^{\cdot,q}$:

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2\cdot} \cdots$$
$$\xrightarrow{(m-2)} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(m-1)\begin{bmatrix}1\\1\end{bmatrix}} (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{(m-1)[1\ \ 1]\cdot} \mathbb{Z}/2\mathbb{Z} \xrightarrow{m\cdot} \cdots$$
$$\to \mathbb{Z}/2\mathbb{Z} \xrightarrow{(2m-2)\cdot} \mathbb{Z}/2\mathbb{Z}$$

Thus for even m and even q only $\mathrm{E}_2^{0,q} = \mathrm{E}_2^{2m,q} = \mathbb{Z}/2\mathbb{Z}$ are nonzero. By the dimension argument the sequence degenerates from E_2 onwards. Hence

$$EX = \mathbb{Z}/2\mathbb{Z}\cdot[\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z}\cdot[L_0],\ E-X = 0 \text{ for even } m.$$

For odd m and even q only $\mathrm{E}_2^{0,q} = \mathrm{E}_2^{2m,q} = \mathbb{Z}/2\mathbb{Z}$ and $E_2^{m,q} = (\mathbb{Z}/2\mathbb{Z})^2$ are nonzero. There is no nonzero differential starting from $\mathrm{E}_r^{0,q}$ since the entries of the left (zeroth) column of B are 0 except $b_{0,0} = 1$. All the differentials but $\mathrm{E}_m^{m,q} \to \mathrm{E}_m^{2m,q-m+1}$ must be zero. This exceptional one is zero too, since it is induced by $b_{2m,m} = (-1)^m \cdot \binom{2m-2}{m}[1\ \ 1]$, and $\binom{2m-2}{m}$ is even for odd m. Therefore the spectral sequence degenerates, and finally

$$EX = \mathbb{Z}/2\mathbb{Z}\cdot[\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z}\cdot[L_0],\ E^-X = \mathbb{Z}/2\mathbb{Z}\cdot[L'_m] \oplus \mathbb{Z}/2\mathbb{Z}\cdot[L''_m] \text{ for odd } m.$$

The theorem is proved. $\square$

5 Non-extended Witt classes on certain split projective quadrics

We shall show here that if the dimension d of a split projective quadric X is even and greater than two, then the invariant $e^0 : W(X) \to EX \cong (\mathbb{Z}/2\mathbb{Z})^2$ is surjective.

5.1 For an arbitrary locally free coherent sheaf $\mathcal{M}$ the sheaf $\mathcal{E} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}) = \mathcal{M} \otimes \mathcal{M}^\wedge$ is self-dual and supports a canonical symmetric bilinear form θ, which reduces to the trace of a product on stalks:

$$\theta(\alpha)(\beta) = \mathrm{tr}(\alpha\cdot\beta) \ \text{ for } \alpha, \beta \in \mathcal{E}_x, x \in X$$

or if $\mu : \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{E} \to E$ is the multiplication map, then $\theta : E \to \mathcal{E}^\wedge$ is adjoint of $\mathrm{tr}\circ\mu : \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{E} \to \mathcal{O}_X$.

Theorem 5.1. *If X is a split projective quadric of dimension $d = 2m$, $m > 1$, then, for an indecomposable component $\mathcal{V}_0$ of the Swan sheaf $\mathcal{U}$,*

$$e^0(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V}_0), \theta) = [L_0].$$

Thus $(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V}_0), \theta)$ represents a non-extended Witt class in $W(X)$.

Proof. The case $m = 0$ is special, so assume $m > 0$. We shall compute the class of $[\mathcal{V}_0] \cdot [\mathcal{V}_0]^\wedge = [\mathcal{V}_0(d)] \cdot [\mathcal{V}_0(d)]^\wedge$ in EX. We know from Proposition 3.4.8 b) iv) that, for $d = 2m$,

$$[\mathcal{V}_0(d)] + [\mathcal{V}_1(d-1)] = 2^m.$$

On the other hand by Corollary 3.4.9 ii) and Theorem 4.1 iii)

$$[\mathcal{V}_0(d-1)] - [\mathcal{V}_1(d-1)] = [\mathcal{V}_0] - [\mathcal{V}_1] = L'_m - L''_m.$$

Thus

$$[\mathcal{V}_0(d)](1 + [\mathcal{O}_X(-1)]) = [\mathcal{V}_0(d)] + [\mathcal{V}_1(d-1)] = 2^m + L'_m - L''_m.$$

or

$$[\mathcal{V}_0(d)](2 + H) = 2^m + L'_m - L''_m.$$

The rules of multiplication in $K_0(X)$, given in Theorem 4.1 and Lemma 2.1.2 yield that multiplying both sides of this equality by

$$\begin{aligned}
\sum_{i=0}^{d} 2^{d-i} H^i &= \sum_{i=0}^{m-1} 2^{d-i} H^i + 2^m \cdot (L'_m + L''_m - L_{m-1}) + \sum_{j=1}^{m} 2^{m-j} H^{m+j} \\
&= \sum_{i=0}^{m-1} 2^{d-i} H^i + 2^m \cdot (L'_m + L''_m - L_{m-1}) + \sum_{j=1}^{m} 2^{m-j} (2L_{m-j} - L_{m-j-1}) \\
&= \sum_{i=0}^{m-1} 2^{d-i} H^i + 2^m \cdot (L'_m + L''_m)
\end{aligned}$$

we obtain

$$\begin{aligned}
2^{d+1}[\mathcal{V}_0(d)] &= [\mathcal{V}_0(d)](2^{d+1} + H^{d+1}) \\
&= \left(\sum_{i=0}^{m-1} 2^{d-i} H^i + 2^m \cdot (L'_m + L''_m) \right) \cdot (L'_m + L''_m - L_{m-1}) \\
&= \left(2^{m+1} \sum_{i=0}^{m-1} 2^{m-i-1} H^i + 2^m \cdot (L'_m + L''_m) \right) \cdot (L'_m + L''_m - L_{m-1}) \\
&= 2^{d+1} \sum_{i=0}^{m-1} 2^{m-i-1} H^i + 2^d \cdot (L'_m + L''_m) + 2^d \cdot (L'_m - L''_m) \\
&= 2^{d+1} \cdot \sum_{i=0}^{m-1} 2^{m-i-1} H^i + 2^{d+1} \cdot L'_m.
\end{aligned}$$

Since $K_0(X)$ is torsion free, $[\mathcal{V}_0(d)] = \sum_{i=0}^{m-1} 2^{m-i-1} H^i + L'_m$. Thus

$$[\mathcal{V}_0(d)]^\wedge = \sum_{i=0}^{m-1} 2^{m-i-1} H^{i\wedge} + {L'_m}^\wedge.$$

Note that $(\alpha+\beta)\cdot(\alpha+\beta)^\wedge \equiv \alpha\cdot\alpha^\wedge+\beta\cdot\beta^\wedge \mod \mathrm{Im}(1+{}^\wedge)$, since $\alpha^\wedge\cdot\beta+\alpha\cdot\beta^\wedge$ is a member of $\mathrm{Im}(1+{}^\wedge)$. Also $2\alpha\cdot\alpha^\wedge\equiv 0 \mod \mathrm{Im}(1+{}^\wedge)$. Therefore

$$\begin{aligned}[\mathcal{E}] &= [\mathcal{V}_0]\otimes[\mathcal{V}_0{}^\wedge] = [\mathcal{V}_0(d)]\cdot[\mathcal{V}_0(d)]^\wedge \\ &\equiv \sum_{i=0}^{m-1} 2^{2(m-i-1)}H^iH^{i\wedge} + L'_mL'_m{}^\wedge \mod \mathrm{Im}(1+{}^\wedge).\end{aligned}$$

If $m=1$, then the first summand equals 1 while the second is 0. For $m>1$

$$\begin{aligned}[\mathcal{E}] &\equiv \sum_{i=0}^{m-1} 2^{d-2i-2}H^iH^{i\wedge} + L'_mL'_m{}^\wedge \\ &= \sum_{i=0}^{m-2} 2^{d-2i-2}H^iH^{i\wedge} + H^{m-1}H^{m-1\wedge} + L'_mL'_m{}^\wedge \\ &\equiv H^{m-1}H^{m-1\wedge} + L'_mL'_m{}^\wedge && \text{since } 2\alpha\alpha^\wedge\equiv 0 \\ &\equiv H^{m-1}H^{m-1}\left(1\binom{m-1}{1}H+\binom{m}{2}H^2\right) + L'_mL'_m{}^\wedge && \text{by Lemma 2.1.2;} \\ &\equiv H^{d-2}+(m-1)H^{d-1}+\frac{m(m-1)}{2}H^d + L'_mL'_m{}^\wedge \\ &\equiv 2L_2-L_1+2(m-1)L_1-(m-1)L_0+m(m-1)L_0 \\ &\qquad + L'_mL'_m{}^\wedge && \text{by Theorem 4.1 vi);} \\ &\equiv 2L_2+(d-3)L_1-(m-1)^2L_0+L'_mL'_m{}^\wedge \\ &\equiv L_2+L_2{}^\wedge-(m-2)L_0 \\ &\qquad +(-1)^mL'_m(L'_m+\text{terms of higher codim}) && \text{by Theorem 4.2 i);} \\ &\equiv \begin{cases}(2-m)L_0+L_0 \text{ for even } m\\ (2-m)L_0 \text{ for odd } m\end{cases} && \text{by Theorem 4.1 viii);} \\ &\equiv L_0 \mod \mathrm{Im}(1+{}^\wedge).\end{aligned}$$

Anyway, $e^0((\mathcal{V}_0),\theta)=[L_0]$ for $m>1$. □

5.2 In the particular case $d=4$ there exists another symmetric bilinear form ϑ on $\mathcal{E}=\mathcal{V}_0\otimes_{\mathcal{O}_X}\mathcal{V}_0{}^\wedge$: the tensor product of exterior multiplications

$$\mathcal{V}_0\otimes_{\mathcal{O}_X}\mathcal{V}_0\to\bigwedge^2\mathcal{V}_0\cong\mathcal{O}_X(-7) \text{ and } \mathcal{V}_0{}^\wedge\otimes_{\mathcal{O}_X}\mathcal{V}_0{}^\wedge\to\bigwedge^2\mathcal{V}_0{}^\wedge\cong\mathcal{O}_X(7).$$

On the stalks the associated quadratic form is the determinant map. Since the value of e^0 depends only on supporting bundle, $(\mathcal{E},\vartheta)$ is non-extended as well as $(\mathcal{E},\theta)$.

The symmetric bilinear space $(\mathcal{E},\vartheta)$ has the following interesting property: it is not metabolic (since it has a nontrivial e^0) but is hyperbolic on stalks, i.e., *locally hyperbolic*. In fact, any stalk $\mathcal{V}_{0,x}$ at $x\in X$ is a free rank two $\mathcal{O}_{X,x}$ - module, so any stalk of $(\mathcal{E},\vartheta)$ is $(M_2(\mathcal{O}_{X,x}),det)$, which is hyperbolic. Thus, there is no local invariant to detect the symmetric bilinear space $(\mathcal{E},\vartheta)$ and such a global invariant as e^0 is useful. If -1 is a sum of two squares in F, then $(\mathcal{E},\theta)$ is locally hyperbolic, too.

Note that the case $d = 4$ is of particular interest, since the split four-dimensional quadric X is the smallest non-trivial Graßmann variety $G_2(4)$. Thus on general Graßmann varieties there may exist non-extended Witt classes contrary to the case of projective spaces, i.e., Graßmann varieties $G_1(n)$.

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