On the Average Values of the Irreducible Characters of Finite Groups of Lie Type on Geometric Unipotent Classes

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Abstract. In 1980, Lusztig posed the problem of showing the existence of a unipotent support for the irreducible characters of a finite reductive group $G(\mathbb{F}_q)$. This is defined in terms of certain average values of the irreducible characters on unipotent classes. The problem was solved by Lusztig [16] for the case where q is a power of a sufficiently large prime. In this paper we show that, in general, these average values can be expressed in terms of the Green functions of G. In good characteristic, these Green functions are given by polynomials in q. Combining this with Lusztig's results, we can then establish the existence of unipotent supports whenever q is a power of a good prime.

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1 Introduction

Let G be a connected reductive group defined over the finite field with q elements, and let $F : G \to G$ be the corresponding Frobenius map. We are interested in the average values of the irreducible characters of the finite group of Lie type G^F on the F-fixed points of F-stable unipotent classes of G. In 1980, Lusztig [9] has stated the following problem.

Problem 1.1 *Let ρ be an irreducible character of G^F. Show that there exists a unique F-stable unipotent class C of maximal possible dimension such that the average value of ρ on C^F is non-zero, that is,*

$$\sum_{j=1}^{r}[G^F : C_G(u_j)^F]\,\rho(u_j) \neq 0,$$

where $u_1, \ldots, u_r \in G^F$ are representatives for the G^F-conjugacy classes contained in C^F and $C_G(u_j)$ denotes the centralizer of u_j. If this is the case, we call C the unipotent support *of ρ.*

In 1992, Lusztig [16] addressed this problem in the framework of his theory of character sheaves and its application to Kawanaka's theory [8] of generalized Gelfand-Graev representations. In this context, one is lead to consider the following related question.

PROBLEM 1.2 *Let ρ be an irreducible character of G^F. Show that there exists a unique F-stable unipotent class C of maximal possible dimension such that*

$$\sum_{j=1}^{r}[A(u_j) : A(u_j)^F]\,\rho(u_j) \neq 0,$$

where $A(u_j)$ denotes the group of components of $C_G(u_j)$.

Assuming that q is a sufficiently large power of a sufficiently large prime p, Lusztig proves in [16], (9.11), a formula which expresses a 'modified' average value as above in terms of the scalar products of the Alvis–Curtis–Kawanaka dual of ρ with the characters of the various generalized Gelfand-Graev representations corresponding to C. (The bound on p comes from the condition that, roughly speaking, one wants to operate with the Lie algebra of G as if it were in characteristic 0.) It is then an easy consequence of [16], Theorem 11.2, that Problem 1.2 has a positive solution.

Using the results in [16] and [6], we shall prove in Proposition 2.5 below a formula which expresses an average value as in Problem 1.1 in similar terms as above. Then the solution of Problem 1.1[1] also is an easy and formally completely analogous consequence of [16], Theorem 11.2. For this argument we have to assume, as in [*loc. cit.*], that q and p are large enough. It is one purpose of this paper to show that this condition on p can be relaxed so that Problem 1.1 and Problem 1.2 have a positive solution (and yield the same unipotent class) whenever p is a good prime for G. It may be true that, eventually, no condition on p will be needed but this seems to require some new arguments. (I have checked, using [19], that things go through for exceptional groups in characteristic $p \neq 2$. A more detailed discussion of the bad characteristic case appears in [7], where it is shown that Problem 1.2 always has a positive solution – Problem 1.1 in bad characteristic remains open.)

The idea of our argument is as follows. It is clear that an average value as in Problem 1.1 is given by the scalar product of ρ with the class function f_C on G^F such that

$$f_C(g) = \begin{cases} |G^F| & \text{if } g \in C^F, \\ 0 & \text{if } g \in G^F \setminus C^F. \end{cases}$$

A similar interpretation can also be given for the modified average value in Problem 1.2, using the class function f'_C on G^F with support on C^F and such that

$$f'_C(u_j) = [A(u_j) : A(u_j)^F]|C_G(u_j)^F| \quad \text{for } 1 \leq j \leq r,$$

where (as above) $u_1, \ldots, u_r \in C^F$ are representatives for the G^F-classes contained in C^F.

The statement concerning f_C in the following result was already conjectured by Lusztig in [9], (2.16). For large p, it follows easily from the results on Green functions in [17] (see also Kawanaka [8], (1.3.8)).

[1] Lusztig has informed me that this solution was known to him, but it was not included in [16]

PROPOSITION 1.3 *The functions f_C and f'_C are uniform, that is, they can be written as linear combinations of various Deligne-Lusztig generalized characters $R_{T,\theta}^G$.*

The proof of this result in (3.6) below will be based on Proposition 3.5, where we show that the known algorithm for computing the ordinary Green functions in [17] works without any restriction on p and q. This may also be of independent interest. It uses heavily the description of this algorithm in terms of Lusztig's character sheaves in [13], Section 24. The (mild) restrictions on p in [*loc. cit.*] can be removed by using Shoji's results [18] on cuspidal character sheaves in bad characteristic and the fact, also proved in [18], that the ordinary Green functions of G^F always coincide with those defined in terms of character sheaves.

It then follows that in order to compute our average values we only need to consider the uniform projection of ρ. We can also reduce to the case where G has a connected center and is simple modulo its center, see Lemmas 5.1 and 5.2. Then our average values can be expressed as linear combinations of Green functions of G^F where the coefficients are 'independent of q', by [11], Main Theorem 4.23. Up to this point we don't need any assumption on p or q.

Let now q be a power of a prime p which is good for G. Recall that this is the case if p is good for each simple factor involved in G, and that the conditions for the various simple types are as follows.

$$\begin{array}{rl} A_n: & \text{no condition,} \\ B_n, C_n, D_n: & p \neq 2, \\ G_2, F_4, E_6, E_7: & p \neq 2, 3, \\ E_8: & p \neq 2, 3, 5. \end{array}$$

Then the Green functions of G^F are given by evaluating certain well-defined polynomials at q (see [17]), and we obtain a similar statement for our average values. We can then replace a given q by a power of a larger prime p for which the results in [16] are applicable and thus deduce results about these average value polynomials being zero or not. Finally, we deduce from the formulae in Proposition 2.5 that our polynomials have the property that if one of them is non-zero then its evaluation at every prime power is non-zero. The details and the precise formulation of this argument can be found in Section 4, especially Proposition 4.4. Then the main result of this paper will be established in Section 5.

THEOREM 1.4 *Assume that q is a power of a good prime p for G. Let ρ be an irreducible character of G^F.*

(a) Both Problem 1.1 and Problem 1.2 have a positive solution for ρ, and they yield the same unipotent class, C say.

(b) The p-part in the degree of ρ is given by q^d where d is the dimension of the variety of Borel subgroups of G containing a fixed element in C.

The characterization of the p-part of ρ in terms of C was also conjectured in [9]. Lusztig [16] proves the following refinement (again assuming that q is a power of a large enough prime): Let $g \in G^F$ be any element such that $\rho(g) \neq 0$. Then the unipotent part of g lies in the unipotent support C of ρ or in a unipotent class of strictly smaller dimension than C. Note that it is not clear how to pass from results about the vanishing or non-vanishing of individual character values to results about the non-vanishing of average values.

We remark that, as far as this refinement is concerned, the situation definitely is different in the bad characteristic case. Consider, for example, the simple group G of type G_2 defined over a finite field of characteristic 3. Let C be the class of regular unipotent elements. Then there exist unipotent characters of G^F which are non-zero on some element in C^F but whose average value on C^F is zero (see the character table in [5]).

Completing earlier results of Lusztig's (see [14]), A.-M. Aubert [1] has shown that such a refinement holds for classical groups in good characteristic and with g unipotent. For that purpose, one has to use the full power of the theory of character sheaves and Shoji's proof of Lusztig's conjecture about almost characters and characteristic functions of character sheaves (see [18]). I have checked that this also works for exceptional groups in good characteristic. This will be discussed elsewhere.

I thank A.-M. Aubert for carefully reading earlier versions of this paper.

2 Generalized Gelfand-Graev representations and average values

Let G be a connected reductive group defined over $\mathbb{F}_q$, with corresponding Frobenius map F. All of our characters and class functions will have values in an algebraic closure of $\mathbb{Q}_l$, where l is prime not dividing q. If f, f' are two class functions on G^F we denote by

$$(f, f') := \frac{1}{|G^F|} \sum_{g \in G^F} f(g)\overline{f'(g)}$$

their usual hermitian product, where $x \mapsto \bar{x}$ is a field automorphism which maps roots of unity to their inverses. We denote by G_{uni} the set of unipotent elements in G. For each element $g \in G$ we let C_g denote the G-conjugacy class of g. There is a canonical partial order on the set of unipotent classes of G: if C, C' are two such classes we write $C \leq C'$ if C is contained in the Zariski closure of C'. We write $C < C'$ if $C \leq C'$ but $C \neq C'$.

2.1 Unipotently supported class functions on G^F

Let C be an F-stable unipotent class in G. Let $u \in C^F$ and $A(u)$ be the group of components of $C_G(u)$. If we twist u with any element $y \in A(u)$ we obtain an element $u_y \in C^F$, well-defined up to G^F-conjugacy. If we choose representatives for the F-conjugacy classes of $A(u)$ we obtain in this way a full set of representatives of the G^F-classes contained in C^F; denote such a set of representatives by $u_1, \ldots, u_r \in C^F$, where we let $u_1 = u$.

Let $I(C)^F$ be the set of pairs $i = (C, E)$ where E is an irreducible representation of $A(u)$ over $\bar{\mathbb{Q}}_l$ (given up to isomorphism) for which there exists an automorphism $\alpha_E : E \to E$ of finite order such that $\alpha_E \circ y = F(y) \circ \alpha_E$ for all $y \in A(u)$. We define a class function $Y_i : G^F \to \bar{\mathbb{Q}}_l$ by

$$Y_i(g) = \begin{cases} \text{Trace}(\alpha_E \circ y, E) & \text{if } g \text{ is } G^F\text{-conjugate to } u_y \text{ for some } y \in A(u), \\ 0 & \text{otherwise.} \end{cases}$$

These functions form a basis of the space of class functions of G^F with support on C^F. (Note that they are only well-defined up to non-zero scalar multiples.) For each j let $a_j := |A(u_j)^F|$. The order of $A(u_j)$ is independent of j; we denote it by a. With this notation we have the following orthogonality relations:

$$\sum_{j=1}^{r} \frac{a}{a_j} \overline{Y_i(u_j)} Y_{i'}(u_j) = a\delta_{ii'} \quad \text{and} \quad \sum_{i \in I_0(C)^F} \overline{Y_i(u_j)} Y_i(u_{j'}) = a_j \delta_{jj'},$$

for all $i, i' \in I(C)^F$, or all $1 \leq j, j' \leq r$, respectively.

The trivial module for $A(u)$ always satisfies the above condition. The corresponding pair will be denoted $i_0 = (C, \bar{\mathbb{Q}}_l)$, and the isomorphism α_E can be chosen so that the function Y_{i_0} is identically 1 on C^F. Thus, we have

$$f_C = |G^F| Y_{i_0} \quad \text{with} \quad i_0 = (C, \bar{\mathbb{Q}}_l).$$

On the other hand, using the definition of f'_C and the above orthogonality relations we compute that

$$(f'_C, Y_i) = a\delta_{i,i_0} \quad \text{for all } i \in I(C)^F.$$

Note that these relations determine f'_C uniquely.

2.2 GGGR's

Recall that if q is a power of a good prime for G then Kawanaka [8] has defined generalized Gelfand-Graev representations (GGGR's for short) for every unipotent class in G^F. (Usually, we will identify a GGGR with its character.) Very roughly, this is done as follows. Let C be an F-stable unipotent class in G. Using the corresponding weighted Dynkin diagram we can associate with C a pair of unipotent subgroups $U_2 \subseteq U_1$ where U_1 is the unipotent radical of an F-stable parabolic subgroup P of G and U_2 is an F-stable closed normal subgroup in P. Furthermore, $C \cap U_2$ is dense in U_2 and the centralizer in G of any element $u \in U_2 \cap C$ is already contained in P. (Note that Kawanaka [8] has checked that these statements indeed are true whenever the characteristic is good.) Hence the subgroup U_2^F contains representatives for all G^F-classes in C^F. Using a Killing type form on U_2 we can associate with each such representative $u \in C \cap U_2^F$ a certain linear character λ_u of U_2^F such that

$$\mathrm{Ind}_{U_2^F}^{G^F}(\lambda_u) = [U_1^F : U_2^F]^{1/2} \Gamma_u,$$

where Γ_u is the GGGR associated with u. With the notation in (2.1), we can assume that $u_j \in U_2$ for $1 \leq j \leq r$. As in [16], (7.5), we define the following 'twisted' version of GGGR's:

$$\Gamma_i = \sum_{j=1}^{r} \frac{a}{a_j} Y_i(u_j) \Gamma_{u_j} \quad \text{for } i \in I(C)^F.$$

2.3 Basic properties of GGGR's

We shall need two basic properties of GGGR which we now explain. Denote by D_G the Alvis–Curtis–Kawanaka duality operation on the character ring of G^F.

Assume that p and q are large enough so that the results in [16] are applicable.

(a) *For all $g \in G^F_{\text{uni}}$ we have*

$$D_G(\Gamma_i)(g) \neq 0 \quad \Rightarrow \quad C \leq C_g.$$

(b) *For all $i, i' \in I(C)^F$ we have*

$$(D_G(\Gamma_i), Y_{i'}) = a\zeta'_i q^{d_i} \delta_{i,i'}$$

where ζ'_i is a certain 4-th root of unity and d_i is half a certain integer.

Proofs of (a) and (b) can be obtained by combining [16], (8.6), with [16], (6.13)(i), and (6.13)(iii), respectively. Properties (a) and (b) are also contained in [6], Corollary 3.6(b) and Lemma 3.5. (Actually, the formula in the latter reference involves a certain function $X_{i'}$ instead of $Y_{i'}$, but $X_{i'}$ is zero on all elements $g \in G^F_{\text{uni}}$ unless $C_g \leq C$ and coincides with $Y_{i'}$ on C^F; using (a) we can therefore take $Y_{i'}$.) Note also that in [16] it is generally assumed that G is a split group, and the results in [6] referred to above are also proved under this assumption. However, by [16], (8.7), everything goes through for non-split groups as well, with only minor changes. Especially, properties (a) and (b) remain valid. Finally, we have the following special property of the numbers ζ', d_i appearing in (b).

(c) *If $i_0 = (C, \bar{\mathbb{Q}}_l)$ then $\zeta'_{i_0} = 1$ and $d_{i_0} = -d$ where d is the dimension of the variety of Borel subgroups of G containing u.*

For the proof see [6], Lemma 3.5, and the remarks concerning equation (a) in the proof of [16], Theorem 11.2. We also use the formula $\dim G - \dim C = \text{rank}(G) + 2d$ (see [3], Theorem 5.10.1).

LEMMA 2.4 *Assume that p and q are large enough so that the results in [16] are applicable. Let f_C and f'_C be the functions introduced in Section 1. Then the following hold.*

$$\begin{aligned} f_C(g) &= q^d \sum_{j=1}^{r} [G^F : C_G(u_j)^F]\, D_G(\Gamma_{u_j})(g) \qquad \text{for all } g \in C^F, \\ f'_C(g) &= q^d D_G(\Gamma_{i_0})(g) = q^d \sum_{j=1}^{r} \frac{a}{a_j} D_G(\Gamma_{u_j})(g) \quad \text{for all } g \in C^F. \end{aligned}$$

Proof. Let $i \in I(C)^F$ and Y_i the corresponding class function as in (2.1). Since the various functions Y_i form a basis of the space of class functions on G^F with support on C^F it will be sufficient to show that the scalar product of Y_i with the left and right hand sides of the above expressions are equal.

Consider at first f_C. The scalar product with the left hand side is just (f_C, Y_i). On the other hand, using the orthogonality relations in (2.1), we conclude that

$$\Gamma_{u_j} = \frac{1}{a} \sum_{i' \in I(C)^F} \overline{Y_{i'}(u_j)} \Gamma_{i'} \quad \text{for all } 1 \leq j \leq r.$$

Inserting this into the expression on the right hand side we obtain that

$$\begin{aligned}
(\text{r.h.s.}, Y_i) &= q^d \sum_{j=1}^{r} \frac{1}{a}[G^F : C_G(u_j)^F] \sum_{i' \in I(C)^F} \overline{Y_{i'}(u_j)}(D_G(\Gamma_{i'}), Y_i) \\
&= q^{d+d_i}\zeta_i' \sum_{j=1}^{r}[G^F : C_G(u_j)^F]\,\overline{Y_i(u_j)} \quad \text{by (2.3)(b)} \\
&= q^{d+d_i}\zeta_i'(f_C, Y_i) \quad \text{by definition of the scalar product.}
\end{aligned}$$

Hence it remains to prove that if $(f_C, Y_i) \neq 0$ then $\zeta_i' = 1$ and $d_i = -d$. This follows from the fact that the set $I(C)^F$ can be partitioned into 'blocks' according to the generalized Springer correspondence (see [16], (4.4)) and that the scalar product between $(Y_i, Y_{i'})$ is zero unless i, i' lie in the same block (see [16], (6.5)). Now remember that $f_C = |G^F| Y_{i_0}$. Hence, if $(f_C, Y_i) \neq 0$ then i lies in the same block as i_0. In this case, $\zeta_i' = \zeta_{i_0}'$ and $d_i = d_{i_0}$ by [6], Lemma 3.5. So we are done by (2.3)(c).

Now consider f_C'. By (2.1) we have $(f_C', Y_i) = a\delta_{i,i_0}$. The scalar product with the right hand side evaluates to the same expression using (2.3)(b) and (c). □

Proposition 2.5 (Cf. [16], (9.11)) *Assume that p and q are large enough so that the results in [16] are applicable. Let ρ be an irreducible character of G^F such that*

(*) $\rho(g) = 0$ for all $g \in G_{\text{uni}}^F$ with $C < C_g$.

Then we have

$$\begin{aligned}
(\rho, f_C) = \sum_{j=1}^{r}[G^F : C_G(u_j)^F]\,\rho(u_j) &= q^d \sum_{j=1}^{r}[G^F : C_G(u_j)^F]\,(\Gamma_{u_j}, D_G(\bar\rho)), \\
(\rho, f_C') = \sum_{j=1}^{r}[A(u) : A(u_j)^F]\,\rho(u_j) &= q^d \sum_{j=1}^{r}[A(u) : A(u_j)^F]\,(\Gamma_{u_j}, D_G(\bar\rho)).
\end{aligned}$$

Since these expressions are rational integers the above formulae are also valid with ρ instead of $\bar\rho$ on the right hand side.

Proof. It is clear that in order to evaluate the left hand sides of the above expressions we only need to know the values of ρ on C^F. Let us check that the same also holds for the expressions on the right hand side. We start by looking at the scalar product of $\bar\rho$ with $D_G(\Gamma_i)$, for $i \in I(C)^F$, that is, the expression

$$(D_G(\Gamma_i), \bar\rho) = \frac{1}{|G^F|} \sum_{g \in G^F} D_G(\Gamma_i)(g)\rho(g).$$

First, the sum need only be extended over $g \in G_{\text{uni}}^F$ since Γ_i, and hence also its dual, is zero outside G_{uni}^F. Now assume that $g \in G_{\text{uni}}^F$ gives a non-zero contribution to the above sum. On one hand, by (2.3)(a), we must have $C \leq C_g$. On the other hand, our assumption (*) then forces $C = C_g$. Hence, in order to evaluate the above scalar product we only need to look at the values of ρ and $D_G(\Gamma_i)$ on C^F. A similar remarks holds, of course, if we consider Γ_{u_j} instead of Γ_i. Using the self-adjointness of D_G

we can therefore conclude that the right hand sides of our desired equalities are also determined by the restriction of ρ to C^F.

To complete the proof, it remains to use the expressions for f_C and f'_C which are given in Lemma 2.4. □

COROLLARY 2.6 (LUSZTIG) *Assume that p and q are large enough so that the results in [16] are applicable. Let ρ be an irreducible character of G^F. Then both Problem 1.1 and Problem 1.2 have a positive solution for ρ, and the corresponding unipotent classes are equal.*

Proof. (Compare with the argument in the last part of the proof of [16], Theorem 11.2.) Let ρ' be the irreducible character such that $\rho' = \pm D_G(\rho)$. By [16], Theorem 11.2, there exists an F-stable unipotent class C such that the following two conditions hold (among others).

(1) There exists some $u \in C^F$ such that $(\Gamma_u, \rho') \neq 0$.

(2) If C' is an F-stable unipotent class such that $\rho(g) \neq 0$ for some $g \in {C'}^F$ then $\dim C' \leq \dim C$ with equality only if $C = C'$.

We show that C satisfies the requirements for both Problem 1.1 and Problem 1.2.

If C' is some F-stable unipotent class such that an average value on ${C'}^F$ as in Problem 1.1 or Problem 1.2 is non-zero then ρ has a non-zero value on some element in ${C'}^F$ and (2) implies that $\dim C' \leq \dim C$.

Recall that our average values are given by (ρ, f_C) and (ρ, f'_C), respectively. It remains to prove that these two scalar products are non-zero. By (2), assumption (*) in Proposition 2.5 is satisfied. So we have

$$\begin{aligned}(\rho, f_C) &= \pm q^d \sum_j [G^F : C_G(u_j)^F]\,(\Gamma_{u_j}, \rho'),\\ (\rho, f'_C) &= \pm q^d \sum_j [A(u_j) : A(u_j)^F]\,(\Gamma_{u_j}, \rho').\end{aligned}$$

In both cases all terms in the sums on the right hand sides are non-negative and at least one of them is non-zero by (1). Hence there are no cancellations and the left hand sides must be non-zero, too. This completes the proof. □

EXAMPLE 2.7 Assume that p and q are large enough so that the above results are applicable. Let ρ be an irreducible character of G^F and C its unipotent support. The assumption (*) in Proposition 2.5 is satisfied (see Property (2) in the proof of Corollary 2.6). Assume that the centralizer of an element in C is connected. In this case we have $r = 1$ in the formulae in Proposition 2.5 and the left and the right hand sides contain just one summand. So we find that

$$\rho(u) = q^d(\Gamma_u, D_G(\rho)).$$

In particular, the character value $\rho(u)$ is an integer divisible by q^d. In fact, using a similar argument as in [6], Proposition 5.4, one can show that the scalar product between $D_G(\rho)$ and Γ_u must be ± 1. Hence we have

$$\rho(u) = \pm q^d \qquad \text{where the sign is such that } \pm D_G(\rho)(1) > 0.$$

Theorem 1.4 and the results in Section 4 will imply that this last formula holds whenever q is a power of a good prime p. We omit further details.

3 Average values and uniform functions

The first aim of this section is to prove Proposition 1.3. We then derive in Corollary 3.8 a formula for the scalar products of an irreducible character of G^F with the functions f_C and f'_C for an F-stable unipotent class C. This will be in terms of Lusztig's parametrization of irreducible characters in [11].

We shall now introduce some notation and recall some facts from [13] which will be needed for the proof of Proposition 1.3. With each F-stable maximal torus T in G, we can associate two types of Green functions: one is the ordinary Green function Q_T^G defined as the restriction of a corresponding Deligne-Lusztig generalized character to G^F_{uni}; the other is a special case of a more general construction of generalized Green functions which are defined in terms of characteristic functions of F-stable character sheaves on G (see [13], (8.3.1)). Shoji has shown in [18], Theorem 5.5 (part II), that these two types of Green functions coincide (without any restriction on p or q).

We shall need some more detailed properties about the values of Green functions. For this purpose we take a closer look at Lusztig's algorithm in [13], Theorem 24.4, for the computation of all generalized Green functions. The properties that we need can be obtained from this algorithmic description. However, there is a mild restriction on p in [*loc. cit.*] which comes from the fact that certain properties of character sheaves on G are not yet established in complete generality. We will now go through [13], Section 24, and check that everything works without any restriction on p, if we only consider those generalized Green functions which correspond to the ordinary Green functions. This will use in an essential way Shoji's results in [18] on cuspidal character sheaves in bad characteristic.

3.1 The generalized Springer correspondence

Let I be the set of all pairs $(C,\mathcal{E})$ where C is a unipotent class in G and $\mathcal{E}$ is an irreducible G-equivariant $\bar{\mathbb{Q}}_l$-local system, given up to isomorphism. If $i=(C,\mathcal{E})$ and $i'=(C',\mathcal{E}')$ are elements in I we write $i \leq i'$ if $C \leq C'$, and $i \sim i'$ if $C = C'$. With each pair $i \in I$ there is associated a triple $(L, C_1, \mathcal{E}_1)$ consisting of a Levi subgroup L in some parabolic subgroup of G and $(C_1,\mathcal{E}_1)$ is a pair like i for L, but where $\mathcal{E}_1$ is 'cuspidal' (in the sense of [12]). The pairs in I associated with a fixed triple as above are parameterized by the irreducible characters of a group $W_G(L, C_1, \mathcal{E}_1)$ which is the inertia group of the pair $(C_1,\mathcal{E}_1)$ in the normalizer of L. This correspondence is the generalized Springer correspondence defined and studied in [12].

A pair $i \in I$ which corresponds to a triple where the Levi subgroup L is a maximal torus, the class C_1 is the trivial class and the local system $\mathcal{E}_1$ is trivial, will be called *uniform* (see the remark following [13], Theorem 24.4). In this case, the inertia group $W_G(T, \{1\}, \bar{\mathbb{Q}}_l)$ is nothing but the Weyl group of G with respect to T, and the above correspondence reduces to Springer's original correspondence. We will denote by I_0 the subset of I consisting of uniform pairs.

Remark 3.2 Let $i_0 = (C, \bar{\mathbb{Q}}_l) \in I$ where $\bar{\mathbb{Q}}_l$ denotes the trivial local system. Then i_0 is uniform.

Proof. This is a general property of the generalized Springer correspondence. Let $i = (C,\mathcal{E}) \in I$ and $\mathcal{B}_u^G$ be the variety of Borel subgroups containing a fixed element $u \in C$.

Recall from [12] that $\mathcal{E}$ corresponds to an irreducible representation of $A(u)$, and that i is uniform if and only if that representation appears with non-zero multiplicity in the permutation representation of $A(u)$ on the irreducible components of $\mathcal{B}_u^G$.

Now the trivial local system on C corresponds to the trivial representation of $A(u)$, and this certainly appears with non-zero multiplicity in any permutation representation of $A(u)$. Hence $i_0 = (C, \bar{\mathbb{Q}}_l)$ is uniform. $\square$

3.3 Basic relations

The Frobenius map F acts naturally on I. An F-stable pair $i = (C, \mathcal{E}) \in I^F$ gives rise to a pair in $I(C)^F$ as in (2.1) and hence to a function Y_i (cf. the proof of [13], (24.2.7).) This function can be extended to a function X_i on the Zariski closure of C by the construction in [10], (24.2.8), so that we have equations of the form

$$X_i = \sum_{i' \in I^F} P_{i',i} Y_{i'} \qquad \text{with } P_{i',i} \in \bar{\mathbb{Q}}_l \text{ for all } i, i' \in I^F,$$

and where $P_{i,i} = 1$ and $P_{i',i} = 0$ if $i' \not\leq i$ or if $i' \sim i$, $i' \neq i$. Now we also have 'contragredient' versions of these functions which will be denoted by $\tilde{X}_i$ and $\tilde{Y}_i$ (see [13], (24.2.12) and (24.2.13)). We have $\tilde{Y}_i = \overline{Y}_i$, see [13], (25.6,4). Correspondingly, we have similar equations as above with coefficients $\tilde{P}_{i',i}$. The various class functions introduced so far are only well-defined up to some scalar multiple, but [13], (24.2.1) and (24.2.2), singles out a certain 'good' normalization which we also assume chosen here. Finally, we define

$$\lambda_{i,i'} := (Y_i, Y_{i'}) \qquad \text{and} \qquad \omega_{i,i'} := \frac{1}{|G^F|} \sum_{g \in G^F_{\text{uni}}} X_i(g) \tilde{X}_{i'}(g) \qquad \text{for all } i, i' \in I^F.$$

As in [13], (24.3), we see that $\lambda_{i,i'} = 0$ unless $i \sim i'$, and that the matrix $(\lambda_{i,i'})_{i,i' \in I^F}$ is invertible. (The functions Y_i form a basis of the space of class functions on G^F_{uni}.) We obtain the following basic relations:

$$\sum_{i_1', i_2' \in I^F} P_{i_1', i_1} \tilde{P}_{i_2', i_2} \lambda_{i_1', i_2'} = \omega_{i_1, i_2} \qquad \text{for all } i_1, i_2 \in I^F.$$

Theorem 24.4 in [13] states that the coefficients $P_{i',i}$, $\tilde{P}_{i',i}$ and $\lambda_{i',i}$ are determined by this system of equations once the right hand side coefficients $\omega_{i',i}$ are known. Now, under some mild restriction on p, the coefficient $\omega_{i',i}$ is given by the equation [13], (24.3.4) (arising from a scalar product formula for characteristic functions of character sheaves). In general, we can at least obtain the following information.

LEMMA 3.4 *Assume that the center of G is connected. Let $i \in I_0^F$ and $i' \in I^F$. Then the following hold.*

(i) If $i' \notin I_0^F$ then $\omega_{i',i} = \omega_{i,i'} = 0$.

(ii) If $i' \in I_0^F$ then $\omega_{i',i} = \omega_{i,i'}$ is a rational number.

Proof. Given any $i, i' \in I^F$ the relevant scalar product formula for the evaluation of $\sum_g X_i(g)\tilde{X}_{i'}(g)$ (sum over all $g \in G^F_{\text{uni}}$) can be found in [13], Theorem 10.9. Let $(L, C_1, \mathcal{E}_1)$ and $(L', C_1', \mathcal{E}_1')$ be the triples associated with i and i', and K_1, K_1' the corresponding cuspidal perverse sheaves on L, L', respectively. One of the assumptions for the validity of [13], Theorem 10.9, is that K_1 and K_1' must be 'strongly cuspidal' (see the description of these assumptions in [13], (10.7)).

We claim that a cuspidal perverse sheaf on any group G with a connected center is always strongly cuspidal (hence in particular K_1 and K_1'; note that L, L' also have a connected center). This can be seen as follows. By [13], (7.1.6), it is sufficient to show that a cuspidal perverse sheaf on G is a character sheaf. By the reduction arguments in [13], (17.10) and (17.11), we can reduce to the case where G is simple of adjoint type. If p is an almost good prime the result is already covered by [13], Theorem 23.1(b). For G of type E_6 or E_7, see [13], Proposition 20.3. It remains to consider G of type G_2, F_4, E_8. The result in this case is contained in [18], Theorem 7.3(a) in part I and Proposition 5.3 in part II. So our claim is established.

Another assumption for the validity of [13], Theorem 10.9, is that if L, L' are conjugate in G then K_1, K_1' must be 'clean' (see again [13], (7.7)). Now if $i, i' \in I_0$ then both L and L' are maximal tori and the 'cleanness' is clear (we have to consider the trivial local system on the trivial class). If one of i, i' is uniform and the other is not, then one of the Levi subgroups L, L' is a maximal torus and the other is not, hence the above condition is vacuous. In combination with the 'good' normalization of $X_i, X_{i'}$ mentioned in (3.3), this proves both (i) and (ii) (cf. [13], (24.3.5)). $\square$

Now we can state the analogue of [13], Theorem 24.4, for uniform pairs $i \in I_0$.

PROPOSITION 3.5 *Assume that the center of G is connected. Let $i \in I_0^F$ and $i' \in I^F$. Then the following hold.*

(i) $P_{i',i} = \tilde{P}_{i',i}$ *and* $\lambda_{i',i} = \lambda_{i,i'}$ *are rational numbers.*

(ii) $P_{i',i}$ *and* $\lambda_{i',i}$ *are zero if* $i' \notin I_0^F$.

Moreover, the coefficients $P_{i',i}$ and $\lambda_{i',i}$ (for $i, i' \in I_0^F$) are determined from the basic relations in (3.3) by an algorithm as described in [13], Theorem 24.4 or [17], Remark 5.4.

Proof. This is almost completely analogous to the proof of [13], Theorem 24.4, with some minor changes concerning the ordering of the arguments. We will go through that proof and check that things go through as desired for uniform pairs in I_0^F. For any integer δ consider the following two statements.

(A_δ) If $i' = (C', \mathcal{E}') \in I^F$ with $\dim C' \leq \delta$ and if $i \in I^F$, then $P_{i',i} = \tilde{P}_{i',i}$ is a rational number if i or i' is uniform, and it is zero if one of i, i' is uniform and the other is not.

(B_δ) If $i' = (C', \mathcal{E}') \in I^F$ with $\dim C' \leq \delta$ and if $i \in I^F$, then $\lambda_{i',i} = \lambda_{i,i'}$ is a rational number if i or i' is uniform, and it is zero if one of i, i' is uniform and the other is not.

It is clear that these statements are true if $\delta < 0$. As in [*loc. cit.*] we first show that *if $\delta \geq 0$ and $(A_{\delta-1})$, (B_δ) are true then (A_δ) is true.*

Let us just describe this in more detail. Let $i \in I^F$ and $i' \in I^F$ such that $\dim C' = \delta$. If $i' \not\leq i$ or if $i \sim i'$, $i \neq i'$ then $P_{i',i} = \tilde{P}_{i',i} = 0$. So we may assume that $i' < i$. From the basic relations in (3.3) we derive, as in [*loc. cit.*], the following equations for any $a \in I^F$ with $a \sim i'$.

$$\begin{aligned} \sum_{i_2' \sim i'} \tilde{P}_{i_2',i} \lambda_{a,i_2'} &= \omega_{a,i} - \sum_{i_1' < i', i_2' \sim i_1'} P_{i_1',a} \tilde{P}_{i_2',i} \lambda_{i_1',i_2'}, \\ \sum_{i_2' \sim i'} P_{i_2',i} \lambda_{i_2',a} &= \omega_{i,a} - \sum_{i_1' < i', i_2' \sim i_1'} P_{i_2',i} \tilde{P}_{i_1',a} \lambda_{i_2',i_1'}. \end{aligned}$$

We denote the right hand sides of these two equations by $\tilde{r}(a)$ and $r(a)$, respectively. We claim that

(1) if a and i are uniform then $r(a) = \tilde{r}(a)$ is a rational number, and

(2) if one of a, i is uniform and the other is not then $\tilde{r}(a) = r(a) = 0$.

This is proved as follows. Lemma 3.4 shows that it is sufficient to consider the sum over i_1', i_2' in each of the defining equations for $r(a)$ and $\tilde{r}(a)$. At first let us consider $r(a)$, and assume that there exists some i_1', i_2' such that the corresponding term is non-zero. Then $P_{i_2',i} \neq 0$, $\tilde{P}_{i_1',a} \neq 0$, and $\lambda_{i_2',i_1'} \neq 0$. For each of these terms we can apply $(A_{\delta-1})$ or $(B_{\delta-1})$. If one of a, i is uniform and the other is not we obtain a contradiction; while if both of a, i are uniform we obtain a summand which is a rational number. We can argue similarly for $\tilde{r}(a)$. Moreover, if both a and i are uniform this analysis shows that $r(a) = \tilde{r}(a)$ is a rational number. Our claim is proved.

We have already mentioned above that the matrix of coefficients $(\lambda_{a,a'})$ (where $a, a' \in I^F$, $a \sim a' \sim i'$) is invertible. Let $(\lambda'_{a,a'})$ be the coefficients in the inverse of this matrix. Then we obtain that

$$\begin{aligned} \tilde{P}_{i',i} &= \sum_{i_2' \sim i'} \tilde{P}_{i_2',i} \left(\sum_{a \sim i'} \lambda'_{i',a} \lambda_{a,i_2'} \right) = \sum_{a \sim i'} \tilde{r}(a) \lambda'_{i',a}, \\ P_{i',i} &= \sum_{i_2' \sim i'} P_{i_2',i} \left(\sum_{a \sim i'} \lambda_{i_2',a} \lambda'_{a,i'} \right) = \sum_{a \sim i'} r(a) \lambda'_{a,i'}. \end{aligned}$$

By (B_δ) we know that $\lambda_{a,a'}$ is zero if one of a, a' is uniform and the other is not; moreover, $\lambda_{a,a'} = \lambda_{a',a}$ is a rational number if both a, a' are uniform. It follows that the matrix of coefficients $(\lambda'_{a,a'})$ has the analogous properties. Hence, if i' is uniform (respectively, not uniform) we can restrict the above sums to those a which are also uniform (respectively, not uniform).

Now assume that both i, i' are uniform. As we have just seen, we can assume that a in the above sums is uniform, and then $r(a) = \tilde{r}(a)$ by (1). Moreover, $\lambda'_{i',a} = \lambda'_{a,i'}$ is a rational number. Hence also $P_{i',i} = \tilde{P}_{i',i}$ is a rational number.

Next assume that i is uniform and i' is not uniform. We can now assume that a in the above sums is not uniform. By (2), we know that then both $r(a)$ and $\tilde{r}(a)$ are zero. Hence $P_{i',i} = \tilde{P}_{i',i} = 0$. A similar argument shows that this is also the case if i' is uniform and i is not uniform. This completes the proof of (A_δ).

In a completely similar way, we can also prove that

$$\textit{if } \delta \geq 0 \textit{ and } (A_{\delta-1}),\ (B_{\delta-1}) \textit{ are true then } (B_\delta) \textit{ is true.}$$

We can then proceed as in [*loc. cit.*] to complete the proof. □

3.6 Uniform pairs and uniform functions

We claim that (without any assumptions on the center of G or on p, q)

(a) *the pair $i \in I^F$ is uniform (cf. (3.1)) if and only if Y_i is a uniform function, and*

(b) *we have $\lambda_{i,i'} = (Y_i, Y_{i'}) = 0$ if one of $i, i' \in I^F$ is uniform and the other is not.*

Before we prove this let us check that this implies Proposition 1.3.

Let C be an F-stable unipotent class and $i_0 = (C, \bar{\mathbb{Q}}_l) \in I^F$. By Remark 3.2 we know that i_0 is uniform. By (2.1) we have $f_C = |G^F|Y_{i_0}$ (for a suitable normalization) hence (a) implies that this is a uniform function and we are done. Now consider f'_C. We can write $f'_C = \sum_i b_i Y_i$ where the sum is over all $i \in I(C)^F$ and $b_i \in \bar{\mathbb{Q}}_l$. By the orthogonality relations in (2.1) we have

$$a\delta_{i',i_0} = (f'_C, Y_{i'}) = \sum_{i \in I(C)^F} b_i (Y_i, Y_{i'}) = \sum_{i \in I(C)^F} b_i \lambda_{i,i'} \quad \text{for all } i' \in I(C)^F.$$

The matrix $(\lambda_{i,i'})$ (where $i', i \in I(C)^F$) is invertible. Let $(\lambda'_{i,i'})$ denote its inverse. Then the above equations imply that $b_i = a\lambda'_{i_0,i}$. Now (b) shows that $\lambda_{i,i'} = 0$ if one of i', i is uniform and the other is not. The coefficients $\lambda'_{i,i'}$ in the inverse matrix then have the analogous property. Since i_0 is uniform we conclude that $b_i = 0$ unless i is uniform. Hence f'_C is uniform. This completes the proof of Proposition 1.3.

We now prove (a). Recall that a class function on G^F_{uni} is uniform if and only it is a linear combination of the Green functions of G^F. Since the functions $\{Y_i \mid i \in I^F\}$ form a basis of the space of class functions on G^F_{uni} it will therefore be sufficient to show that the Green functions can be expressed as linear combinations of the functions $\{Y_i \mid i \in I_0^F\}$ and vice versa.

Assume at first that G has a connected center. By Proposition 3.5, we can write

$$X_i = \sum_{i' \in I_0^F} P_{i',i} Y_{i'} \quad \text{for all } i \in I_0^F.$$

If we choose a total order on I_0 which refines the order relation $i' \leq i$, we see that the matrix of coefficients $P_{i',i}$ has a triangular shape with 1's along the diagonal. Hence these equations can be inverted, and every Y_i (for $i \in I_0^F$) can be expressed as a linear combination of the functions $X_{i'}$, for various $i' \in I_0^F$.

By [13], (10.4.5), and the character formula in [13], Theorem 8.5, a function X_i for which $i \in I^F$ is uniform can be expressed as a linear combination of generalized Green functions corresponding to various F-stable maximal tori in G. (This is because i is uniform; otherwise, one would have to use generalized Green functions corresponding to Levi subgroups in G which are not maximal tori.) But now [18], Theorem 5.5 (part II), states that these generalized Green functions (corresponding to maximal tori) coincide with the ordinary Green functions of G^F. Moreover, this can be reversed and hence every Green function is a linear combination of the functions $\{X_i \mid i \in I_0^F\}$. Combining this with the above relations among the X_i and Y_i we see that, indeed, the Green functions can be expressed in terms of the functions $\{Y_i \mid i \in I_0^F\}$ and vive versa.

If the center of G is not connected let $\iota\colon G \to G'$ be a regular embedding. Recall from [15] that this means that ι is a homomorphism of connected reductive groups over $\mathbb{F}_q$ such that G' has a connected center, ι is an isomorphism onto a closed subgroup of G', and $\iota(G), G'$ have the same derived subgroup. To simplify notation, we identify G and its image $\iota(G)$.

The embedding $G \subseteq G'$ defines a bijection between the F-stable unipotent classes in G and in G'. Let $u \in C^F$ and consider the canonical quotient $C_{G'}(u) \to A_{G'}(u)$. Since $G' = GZ(G')$ the restriction of this map to $C_G(u)$ defines a surjective map $A_G(u) \to A_{G'}(u)$ whose kernel is given by the image of $Z(G)$ in $A_G(u)$. Via this surjection (which is compatible with the action of F) we also obtain a canonical injective map $I_{G'}(C)^F \to I_G(C)^F$. Since this holds for all F-stable unipotent classes C we obtain an injective map $I_{G'}^F \to I_G^F$. The characterization of uniform pairs in terms of multiplicities in permutation representations as in the proof of Remark 3.2 immediately shows that $i \in I_G^F$ certainly is uniform if i is the image of a uniform pair in $I_{G'}^F$ under this map. On the other hand the number of uniform pairs in I_G^F is always given (via the Springer correspondence) by the number of irreducible characters of the Weyl group W which are invariant under the action of F. Since the latter number is the same for G and G' we conclude that the uniform pairs in I_G^F are precisely the images of the uniform pairs in $I_{G'}^F$.

It follows from the definitions that for all $i \in I_{G'}^F$ we have

$$\mathrm{Res}_G^{G'}(Y_i^{G'}) = Y_i^G \text{ where we also regard } i \text{ as an element in } I_G^F.$$

Now it is also known (see [17]) that the Green functions for G^F are the restrictions of the Green functions for G'^F. Hence we can use the results from the connected center case to conclude that the Green functions of G^F are linear combinations of the functions $\{Y_i^G \mid i \in I_G^F \text{ uniform}\}$ and vice versa. This completes the proof of (a).

Finally, let us consider (b). If the center of G is connected then this is already contained in Proposition 3.5(ii). If the center of G is not connected we use a regular embedding $G \subseteq G'$ as above. Recall that we then have a surjective map $A_G(u) \to A_{G'}(u)$ with kernel given by the image of $Z(G)$ in $A_G(u)$. Using the definitions this easily implies that $(Y_i, Y_{i'}) = 0$ if one of $i, i' \in I_G^F$ lies in the image of the map $I_{G'}^F \to I_G^F$ and the other does not. This implies (b), and the proof is complete.

3.7 Series of irreducible characters

We assume for the rest of this section that the center of G is connected. (We will see in Section 5 that this is no loss of generality as far as Problem 1.1 and Problem 1.2 are concerned.) Let $T \subseteq G$ be an F-stable maximal torus contained in some F-stable Borel subgroup of G, and W be the Weyl group of G with respect to T. Let G^* be a group dual to G (see [11], (8.4)). Then G^* is also defined over $\mathbb{F}_q$ and we denote again by F the corresponding Frobenius map. We can identify W with the Weyl group of an F-stable maximal torus $T' \subseteq G^*$ dual to T; note that the actions of the Frobenius maps of G and G^* on W are inverse to each other.

(a) Let $s \in T'$ be a semisimple element such that the G^*-conjugacy class of s is F-stable. Let W_s be the stabilizer of s in W. Then W_s is a reflection subgroup of W. Let $w_1 \in W$ be the unique element of minimal length in the coset $Z_s = \{w \in W \mid F(s) = w(s)\}$. Then we have an induced automorphism $\gamma \colon W_s \to W_s$ defined by $\gamma^{-1}(w) = F(w_1 w w_1^{-1})$ for all $w \in W_s$ (see [11], (2.15) and the remarks in [11], p.258). Let $\bar{X}(W_s, \gamma)$ be the parameter set defined in [11], (4.21.12); this set only depends on W_s and γ.

(b) If $s \in T'$ is as in (a), we let $\tilde{W}_s = W_s\langle\sigma\rangle$ be the semidirect product of W and the cyclic group $\langle\sigma\rangle$ with generator σ such that $\sigma w \sigma^{-1} = \gamma(w)$ for all $w \in W_s$. Let ψ be an irreducible character of W_s which can be extended to $\tilde{W}_s$; we fix one possible extension of ψ and denote it by $\tilde{\psi}$. As in [11], (3.7), we define

$$R^s[\tilde{\psi}] := \frac{1}{|W_s|} \sum_{w \in W_s} \tilde{\psi}(\sigma w) R^G_{T_{w_1 w}, \theta_s},$$

where $T_{w_1 w} \subseteq G$ is an F-stable maximal torus obtained from T by twisting with $w_1 w$ and θ_s is an irreducible character of $T^F_{w_1 w}$ in 'duality' with s. (This 'duality' is described in [11], proof of Lemma 6.2 and the remarks on p.257.)

(c) The irreducible characters of G^F are divided into series corresponding to conjugacy classes of F-stable semisimple elements in G^*. If $s \in T'$ is as in (a), we denote by $\mathcal{E}_s$ the corresponding series. By [11], Main Theorem 4.23, there exists a bijection

$$\mathcal{E}_s \leftrightarrow \bar{X}(W_s, \gamma), \quad \rho \leftrightarrow \bar{x}_\rho,$$

such that the scalar product

$$(\rho, R^s[\tilde{\psi}])$$

is a rational number depending only on $w_1 W_s$, ψ, and $\bar{x}_\rho \in \bar{X}(W_s, \gamma)$. Let us denote this number by $a(w_1 W_s, \psi, \bar{x}_\rho)$.

(d) Consider the special case where $s = 1$. Then $W_s = W$, $w_1 = 1$ and σ is given by the action of F. We denote by $\mathrm{Irr}(W)^F$ the set of irreducible characters of W which can be extended to $\tilde{W}$, and we assume chosen once and for all a fixed extension for such a character. The corresponding functions $R^s[\tilde{\phi}]$ will be denoted by Q_ϕ, where $\phi \in \mathrm{Irr}(W)^F$. (These are the same as the functions in [17], Remark 5.5(i).)

With this notation we can now state the following result, which expresses our average values as linear combinations of Green functions with coefficients 'independent of q'.

Corollary 3.8 *Assume that the center of G is connected. Let $s \in T'$ be as in (3.7a) and $\rho \in \mathcal{E}_s$. Let C be an F-stable unipotent class in G and $u_1, \dots, u_r$ be representatives for the G^F-classes contained in C^F. Then there exists constants*

$b(w_1W_s, \phi, \bar{x}_\rho) \in \bar{\mathbb{Q}}_l$ *(depending only on* $w_1W_s, \phi, \bar{x}_\rho$*) such that*

$$\begin{aligned}(\rho, f_C) &= \sum_{j=1}^{r}\sum_{\phi}[G^F : C_G(u_j)^F]\, b(w_1W_s, \phi, \bar{x}_\rho)\, Q_\phi(u_j),\\ (\rho, f'_C) &= \sum_{j=1}^{r}\sum_{\phi}[A(u_j) : A(u_j)^F]\, b(w_1W_s, \phi, \bar{x}_\rho)\, Q_\phi(u_j),\end{aligned}$$

where in both formulae the second sum is over all $\phi \in \mathrm{Irr}(W)^F$.

Proof. Let ρ_{unif} denote the uniform projection of ρ. By Proposition 1.3 we know that f_C and f'_C are uniform. Hence we can replace ρ by ρ_{unif} in order to evaluate the scalar products with f_C and f'_C.

The various functions $R^s[\tilde{\psi}]$ have norm 1 and are mutually orthogonal. The uniform projection of ρ is given by projecting ρ on the space generated by the various $R^s[\tilde{\psi}]$. Hence we have

$$\rho_{\mathrm{unif}} = \sum_{\psi} a(w_1W_s, \psi, \bar{x}_\rho) R^s[\tilde{\psi}]$$

where the sum is over all irreducible characters ψ of W_s which can be extended to $\tilde{W}_s$. We insert the defining equation for $R^s[\tilde{\psi}]$ and note that the value of a Deligne-Lusztig generalized character at a unipotent element is the value of the corresponding Green function. Now the Green functions for G^F can be re-written in terms of the functions Q_ϕ, where $\phi \in \mathrm{Irr}(W)^F$ and where the coefficients are given by the entries in the inverse of the matrix of values $(\tilde{\phi}(Fw))$. This yields the above expressions for the average values.

Finally note that the coefficients in these linear combinations involve the constants $a(w_1W_s, \psi, \bar{x}_\rho)$, the character values $\tilde{\psi}(\sigma w)$, and the entries in the inverse of the matrix of values $(\tilde{\phi}(Fw))$. Having chosen fixed extensions of the various characters involved we see that these coefficients only depend on w_1W_s, ϕ and $\bar{x}_\rho$. This completes the proof. □

4 CONSIDERING q AS A VARIABLE

We continue to assume that G has a connected center. We have seen in Corollary 3.8 that average values of irreducible characters of G^F as in Problem 1.1 and Problem 1.2 can be expressed in terms of certain combinatorial objects associated with various reflection subgroups of the Weyl group of G and the values of the Green functions of G^F. There is a sense in which the latter are given by 'polynomials in q', and hence the same holds for our average value. In this section we will give a precise formulation for this statement, and this will eventually allow us to remove the assumption on p and q in Corollary 2.6. It will be technically simpler if our group G is simple modulo its center. (In Section 5 below we will see that this is no loss of generality as far as Problem 1.1 and Problem 1.2. are concerned.)

For the remainder of this section, our group G has a connected center and is simple modulo its center. As remarked above we will want to say that certain quantities or objects associated with G^F are given by 'polynomials in q' or are classified 'independently of q'. In order to make this precise, we let Ψ be the root datum of G with respect to a fixed F-stable maximal torus T contained in some F-stable Borel subgroup of G. We denote by W the Weyl group of G with respect to T; this only depends on Ψ. Let $X = X(T)$ be the character group of T. Then F acts as q times an automorphism F_0 of finite order on X, and the pair (G, T) together with the Frobenius map F is determined by (Ψ, F_0) and the choice of the prime power q. We now assume given, once and for all, the root datum Ψ, the corresponding Weyl group W, and the automorphism F_0. Then each choice of a prime power q_1 determines a pair (G_1, T_1) and a Frobenius map F_1 such that G_1 has root datum Ψ and F_1 acts as q_1 times F_0 on the character group of T_1.

4.1 Classification of unipotent classes

We summarize the known results on the classification of unipotent classes in good characteristic, as follows. There exists a finite index set A and a map $A \to \mathbb{N}_0$, $\alpha \mapsto d_\alpha$, depending only on (Ψ, F_0) and having the following properties. If q is a power of a good prime and G is the corresponding group over $\mathbb{F}_q$, there is a map

$$A \to G^F_{\text{uni}}, \quad \alpha \mapsto u_\alpha,$$

such that $\{u_\alpha \mid \alpha \in A\}$ is a set of representatives for the F-stable unipotent classes in G and $d_\alpha = \dim C_\alpha$ where C_α is the class of G containing u_α. (This is contained, for example, in [3], Chapter 5).

Moreover, there is a collection of finite groups $(A_\alpha)_{\alpha \in A}$ such that the map $A \to G^F_{\text{uni}}$ can be chosen to have the following additional properties.

(i) For each α, the group of components of the centralizer of u_α is isomorphic to A_α, and the action of F on this group is trivial.

(ii) For each α, the element u_α is split in the sense of [17], Remark 5.1, except possibly when G is of type E_8, $q \equiv -1$ mod 3, and u_α lies in the class $D_8(a_3)$ (notation of the table in [3], pp.405).

Each u_α is uniquely determined up to G^F-conjugacy by (i) and (ii). This follows in all cases where split elements exist, see Shoji [17] and the references there. For type E_8, see Kawanaka [8], (1.2.1); the uniqueness of u_α in this case is mentioned in [2], p.590.

For each $\alpha \in A$ we let $\mathrm{Cl}(A_\alpha)$ be a set of representatives of the conjugacy classes of A_α. By property (i), the set $\mathrm{Cl}(A_\alpha)$ parametrizes the various G^F-classes contained in C_α^F (for q and G as above). If $j \in \mathrm{Cl}(A_\alpha)$ we denote by $u_{\alpha,j}$ an element in C_α^F which is obtained from the representative u_α by twisting with j.

4.2 Values of Green functions

We summarize the known results about the values of Green functions in good characteristic as follows. For $\delta = 0, \pm 1$ there exist maps

$$Q^\delta \colon \mathrm{Irr}(W)^F \times \coprod_{\alpha \in A} A_\alpha \to \mathbb{Z}[t] \quad \text{and} \quad h^\delta \colon \coprod_{\alpha \in A} \mathrm{Cl}(A_\alpha) \to \mathbb{Q}[t]$$

depending only on (Ψ, F_0) and having the following properties. If q is a power of a good prime such that $q \equiv \delta \bmod 3$ and G is the corresponding group over $\mathbb{F}_q$ then

$$Q_\phi(u_{\alpha,j}) = Q^\delta(\phi, \alpha, j)(q) \quad \text{for all } w \in W,\ \alpha \in A \text{ and } j \in \mathrm{Cl}(A_\alpha),$$

where $\phi \in \mathrm{Irr}(W)^F$ and $u_{\alpha,j}$ is an element in C_α^F obtained by twisting the representative u_α with j. Moreover, $h^\delta(\alpha, j)(q)$ is the size of the G^F-conjugacy class of $u_{\alpha,j}$.

The results concerning the Green functions are contained in [17]. The existence of the polynomials $h^\delta(\alpha, j)$ follows, for example, from the algorithm for the computation of generalized Green functions in [13], Theorem 24.4. These polynomials (for fixed α) all have the same degree which is the integer $d_\alpha = \dim C_\alpha$. Note that the parameter δ makes a difference only for G of type E_8.

4.3 The average value polynomials

Fix $\delta = 0, \pm 1$. Let q be any power of a good prime with $q \equiv \delta \bmod 3$ and G the corresponding group over $\mathbb{F}_q$ with dual group G^*. Let $s \in T'$, $W_s \subseteq W$ and $w_1 \in W$ as in (3.7a). Then w_1 has minimal length in the coset $w_1 W_s$ and we have $F(w_1 W_s w_1^{-1}) = W_s$. The cosets $w_1 W_s$ arising in this way (for various choices of q and elements $s \in T'$) will be called the *δ-admissible cosets* of W.

Let $w_1 W'$ be a δ-admissible coset. We define the automorphism $\gamma : W' \to W'$ and the corresponding semidirect product $\tilde{W}'$ analogously as in (3.7a). The constructions in [11], Chapter 4, yield a parameter set $\bar{X}(W', \gamma)$ and rational numbers $a(w_1 W', \phi, \bar{x})$ (as in (3.7b)) for all irreducible characters ϕ of W' which can be extended to $\tilde{W}'$. Moreover, we obtain constants $b(w_1 W_s, \phi, \bar{x})$ (for $\phi \in \mathrm{Irr}(W)^F$) by the rewriting process as in the proof of Corollary 3.8. We now define two polynomial functions $A \times \bar{X}(W', \gamma) \to \mathbb{Q}[t]$ by

$$\begin{aligned}
\mathrm{AV}^\delta_{(1.1)}(\alpha, \bar{x}) &:= \sum_{j \in \mathrm{Cl}(A_\alpha)} \sum_{\phi \in \mathrm{Irr}(W)^F} h^\delta(\alpha, j)\, b(w_1 W', \phi, \bar{x})\, Q^\delta(\phi, \alpha, j), \\
\mathrm{AV}^\delta_{(1.2)}(\alpha, \bar{x}) &:= \sum_{j \in \mathrm{Cl}(A_\alpha)} \sum_{\phi \in \mathrm{Irr}(W)^F} [A_\alpha : C_{A_\alpha}(j)]\, b(w_1 W', \phi, \bar{x})\, Q^\delta(\phi, \alpha, j).
\end{aligned}$$

Given α and $\bar{x}$ we call the corresponding polynomials the *average value polynomials* of type (1.1) and (1.2), respectively.

The relevance of this definition is as follows. Let q be a power of a good prime with $q \equiv \delta \bmod 3$, and G the corresponding group over $\mathbb{F}_q$. Let $s \in T'$ and W_s, w_1, γ be as in (3.7a). Then $w_1 W_s$ is a δ-admissible coset, hence $\mathrm{AV}^\delta_{(1.1)}(\alpha, \bar{x})$ and $\mathrm{AV}^\delta_{(1.2)}(\alpha, \bar{x})$ are defined for all $\alpha \in A$ and $\bar{x} \in \bar{X}(W_s, \gamma)$. Corollary 3.8 can now be rephrased by saying that if $\rho \in \mathcal{E}_s$ we have

$$(\rho, f_C) = \mathrm{AV}^\delta_{(1.1)}(\alpha, \bar{x}_\rho)(q) \quad \text{and} \quad (\rho, f'_C) = \mathrm{AV}^\delta_{(1.2)}(\alpha, \bar{x}_\rho)(q).$$

PROPOSITION 4.4 *Let $w_1 W'$ be a δ-admissible coset and $\bar{x}$ a fixed element in the corresponding parameter set $\bar{X}(W', \gamma)$.*

(i) There exists a unique $\alpha \in A$ with maximal possible value d_α such that the polynomial $\mathrm{AV}^\delta_{(1.1)}(\alpha, \bar{x})$ is non-zero.

(ii) There exists a unique $\tilde\alpha \in A$ *with maximal possible value* $d_{\tilde\alpha}$ *such that the polynomial* $\mathrm{AV}^{\delta}_{(1.2)}(\tilde\alpha, \bar x)$ *is non-zero.*

(iii) We have $\alpha = \tilde\alpha$.

(iv) $\mathrm{AV}^{\delta}_{(1.1)}(\alpha, \bar x)(q) \neq 0$ *and* $\mathrm{AV}^{\delta}_{(1.2)}(\alpha, \bar x)(q) \neq 0$ *for all good prime powers* q *such that* $q \equiv \delta$ *mod 3.*

(Recall from (4.1) that $d_\alpha = \dim C_\alpha$.)

Proof. Let M be the set of integers q which are powers of various good primes and such that the following conditions are satisfied.

(a) All elements in M are congruent to δ modulo 3.

(b) If an average value polynomial is non-zero then it is non-zero when evaluated at every $q \in M$.

(c) If $q \in M$ and G is the corresponding group over $\mathbb{F}_q$ with Frobenius map F the results in Section 2 are applicable.

(d) If $q \in M$ and G is the corresponding group over $\mathbb{F}_q$ with Frobenius map F then the coset $w_1 W'$ arises from an F-stable semisimple class in G as in (3.7a).

The set M contains infinitely many elements. Indeed, condition (b) holds for all but finitely many good prime powers since we only have a finite number of average value polynomials; condition (c) holds for all large enough powers of large enough primes. Using Dirichlet's Theorem on primes in an arithmetic progression, the set M_1 of good prime powers satisfying (a), (b), (c) is infinite. Finally, Deriziotis has shown in [4], Theorem 3.3, that condition (d) either holds for none of for all but finitely many good prime powers q in a fixed congruence class modulo a certain integer depending only on (Ψ, F_0). The definition of δ-admissibility therefore implies that the set of elements in M_1 which also satisfy (d) is still infinite.

Let us prove (i). Let $q \in M$ and G the corresponding group over $\mathbb{F}_q$. By condition (d), the coset $w_1 W'$ arises from some F-stable semisimple class (s) in G^* as in (3.7a). Let ρ be an irreducible character of G^F in the corresponding series $\mathcal{E}_s$ such that $\bar x_\rho$ is the given element $\bar x \in \bar X(W', \gamma)$. By (c) we can apply Corollary 2.6 and conclude that there exists a unique $\alpha \in A$ with maximal possible value d_α such that

$$(\rho, f_C) = \mathrm{AV}^{\delta}_{(1.1)}(\alpha, \bar x)(q) \neq 0.$$

But then property (b) implies that (i) holds. The proof of (ii) is completely analogous, and yields the same class C_α by Corollary 2.6. This also proves (iii).

Now we prove (iv). For this purpose note that the class C_α has the properties (1) and (2) in the proof of Corollary 2.6. By [16], Theorem 11.2, we also have the following additional property.

(1') For all $u \in C_\alpha^F$, the absolute value of $(\Gamma_u, D_G(\rho))$ is $\leq |A_\alpha||W|$.

Recall that property (2) implies that assumption (*) in Proposition 2.5 is satisfied for the character ρ and the class C_α, and this holds for each choice of $q \in M$. The formulae in Proposition 2.5 yield that

$$\begin{aligned} \mathrm{AV}^\delta_{(1.1)}(\alpha,\bar{x})(q) &= q^d \sum_{j\in \mathrm{Cl}(A_\alpha)} h^\delta(\alpha,j)(q)N_j(q) \quad \text{and} \\ \mathrm{AV}^\delta_{(1.2)}(\alpha,\bar{x})(q) &= q^d \sum_{j\in \mathrm{Cl}(A_\alpha)} [A_\alpha : C_{A_\alpha}(j)]N_j(q), \end{aligned}$$

where $N_j(q)$ denotes the multiplicity of $D_G(\rho)$ in the GGGR associated with the representative in C_α corresponding to j. Property (1') gives a bound on the absolute value of $N_j(q)$ 'independently of q'. So there exists an infinite subset $M' \subseteq M$ such that $(N_j(q))_{j\in \mathrm{Cl}(A_\alpha)}$ is constant for all $q \in M'$. Let N_j denote this constant for $j \in \mathrm{Cl}(A_\alpha)$. We conclude that

$$\begin{aligned} \mathrm{AV}^\delta_{(1.1)}(\alpha,\bar{x})(q) &= q^d \sum_{j\in \mathrm{Cl}(A_\alpha)} h^\delta(\alpha,j)(q)N_j \quad \text{for all } q \in M', \\ \mathrm{AV}^\delta_{(1.2)}(\alpha,\bar{x})(q) &= q^d \sum_{j\in \mathrm{Cl}(A_\alpha)} [A_\alpha : C_{A_\alpha}(j)]N_j \quad \text{for all } q \in M'. \end{aligned}$$

So we actually obtain identities of polynomials in $\mathbb{Q}[t]$:

$$\begin{aligned} \mathrm{AV}^\delta_{(1.1)}(\alpha,\bar{x}) &= t^d \sum_{j\in \mathrm{Cl}(A_\alpha)} h^\delta(\alpha,j)N_j, \\ \mathrm{AV}^\delta_{(1.2)}(\alpha,\bar{x}) &= t^d \sum_{j\in \mathrm{Cl}(A_\alpha)} [A_\alpha : C_{A_\alpha}(j)]\, N_j. \end{aligned}$$

Since $\pm D_G(\bar{\rho})$ is an irreducible character, either all numbers N_j are non-negative or all numbers $-N_j$ are non-negative, and by (i) at least one N_j must be non-zero. So the expression

$$t^d \sum_{j\in \mathrm{Cl}(A_\alpha)} [A_\alpha : C_{A_\alpha}(j)]\, N_j$$

is a non-zero constant times t^d. Hence, in particular, its value at any q as in (iv) is non-zero. A slight modification of this argument also works for the other expression. Indeed, since $h^\delta(\alpha,j)$ gives a strictly positive integer when evaluated at any good prime power (namely the size of a conjugacy class), we conclude that the expression

$$t^d \sum_{j\in \mathrm{Cl}(A_\alpha)} h^\delta(\alpha,j)N_j$$

is a polynomial with the property that if we evaluate it at any good prime power then we obtain a strictly positive or a strictly negative number as a result. Again we are done. □

5 Proof of Theorem 1.4

Let q be a power of a prime p and G be a connected reductive group defined over $\mathbb{F}_q$, with corresponding Frobenius map F. For the moment we make no assumption on p or on the center of G.

LEMMA 5.1 *Let $G \subseteq G'$ be a regular embedding of G into a connected reductive group G' over $\mathbb{F}_q$ with a connected center and such that G, G' have the same derived subgroup. Then Problem 1.1 (respectively, Problem 1.2) has a positive solution for G if and only if it has a positive solution for G'.*

Proof. At first note that the embedding $G \subseteq G'$ defines a bijection between the F-stable unipotent classes of G and those of G'. Let C be any F-stable unipotent class of G, let ρ' be an irreducible character of ${G'}^F$, and let ρ be an irreducible component of the restriction of ρ' to G^F. By Clifford's Theorem the restriction of ρ' to G^F is a sum of irreducible characters of G^F which are of the form $\rho^x := \rho \circ c_x$, where c_x denotes the automorphism of G^F induced by conjugation with an element $x \in {G'}^F$.

It is clear that the function f_C defined with respect to G is $[{G'}^F : G^F]$ times the restriction of the corresponding function defined with respect to G'. Hence f_C is invariant under ${G'}^F$ and we have $f_C^x = f_C$ for all $x \in {G'}^F$.

Using the methods in (3.6) it can be easily seen that a similar statement also holds for the function f'_C on G^F. Hence it is also invariant under ${G'}^F$ and we have $(f'_C)^x = f'_C$ for all $x \in {G'}^F$.

We conclude that the scalar product of ρ^x with f_C (respectively, with f'_C) is the same as the scalar product of ρ with f_C (respectively, with f'_C). Using Clifford's Theorem in the above form, we see that the scalar product of ρ' with f_C (respectively, with f'_C) is a non-zero multiple of the scalar product of ρ with f_C (respectively, with f'_C). This implies the desired equivalence. $\square$

So from now on, we can assume that the center of G is connected. The next result shows that we can reduce to the case where G is simple modulo its center.

LEMMA 5.2 *Let p be a fixed prime. Assume that Problem 1.1 (respectively, Problem 1.2) has a positive solution for all groups G which are defined over a finite field of characteristic p, which have a connected center and which are simple modulo their center. Then Problem 1.1 (respectively, Problem 1.2) has a positive solution for all groups defined over a finite field of characteristic p.*

Proof. Let G be any group defined over $\mathbb{F}_q$, where q is a power of p. By Lemma 5.1 we may assume that the center of G is connected. The following reasoning is almost entirely analogous to that in [11], (8.8).

We can find a surjective homomorphism $f\colon G' \to G$ of algebraic groups over $\mathbb{F}_q$ such that the center of G' is connected, the kernel of f is a central torus, and the derived subgroup of G' is semisimple and simply-connected. We claim that if Problem 1.1 (respectively, Problem 1.2) has a positive solution for G' then it also has a positive solution for G. Indeed, the map f induces a bijection between the unipotent classes of G' and G. Since the kernel of f is connected this bijection also works on the level of the finite groups, and we have $f({G'}^F) = G^F$. So, if ρ is an irreducible character of G^F then $\rho' := \rho \circ f$ is an irreducible character of ${G'}^F$. Furthermore, the function f_C (respectively, f'_C) lifts to the analogously defined function of ${G'}^F$. This implies the claim.

Hence we may now also assume that the derived group G_{der} of G is simply-connected.

Let us write $G_{\mathrm{der}} = R_{f_1}(G_1) \times \ldots \times R_{f_n}(G_n)$ where each G_i is a closed simple simply-connected subgroup and R_f denotes restriction of scalars from $\mathbb{F}_{q^f}$ to $\mathbb{F}_q$ (for some $f \geq 1$). We can embed each G_i regularly (over $\mathbb{F}_{q^{f_i}}$) into a connected reductive group G_i'' with a connected center and which is simple modulo its center. Let $G'' := R_{f_1}(G_1'') \times \ldots \times R_{f_n}(G_n'')$. Then we also have a regular embedding $G_{\mathrm{der}} \to G''$ (over $\mathbb{F}_q$).

Finally, as in [*loc. cit.*], there exists a connected reductive group G''' with connected center and defined over $\mathbb{F}_q$ and there exist regular embeddings $G \to G'''$, $G'' \to G'''$ (over $\mathbb{F}_q$) which are compatible with the regular embedding $G_{\mathrm{der}} \to G''$.

Now we can argue as follows. Using Lemma 5.1 twice we see that Problem 1.1 (respectively, Problem 1.2) has a positive solution for G if and only if this is the case for G''' if and only if this is the case for G''. Now G'' has a decomposition into a direct product of various factors of the form $R_{f_i}(G_i)$, and this leads to a similar decomposition on the level of the finite groups. Correspondingly, the irreducible characters of G''^F are exterior tensor products of irreducible characters for the various factors, and it follows easily that Problem 1.1 (respectively, Problem 1.2) has a positive solution for G'' if this is the case for each factor G_i. Hence we are reduced to groups which have a connected center and are simple modulo their center. This completes the proof. $\square$

We are now ready for the proof of Theorem 1.4.

5.3 EXISTENCE OF UNIPOTENT SUPPORTS

Let G be as in the first sentence of this section, and assume that p is good.

Let us first show that Problem 1.1 has a positive solution (that is, the unipotent support of an irreducible character exists). By Lemmas 5.1 and 5.2 we may assume that G has a connected center and is simple modulo its center. Then we can apply the formalism of Section 4. Let ρ be an irreducible character of G^F contained in the series $\mathcal{E}_s$, say. Let $w_1 W_s$ and $\bar{X}(W_s, \gamma)$ as in (3.7a). By Proposition 4.4(i), there exists a unique $\alpha \in A$ with maximal possible value for d_α such that the average value polynomial $\mathrm{AV}^\delta_{(1.1)}(\alpha, \bar{x}_\rho)$ is non-zero (where $q \equiv \delta \bmod 3$). By Proposition 4.4(iv), we also have

$$\sum_{g \in C_\alpha^F} \rho(g) = \mathrm{AV}^\delta_{(1.1)}(\alpha, \bar{x}_\rho)(q) \neq 0.$$

Now let $\beta \in A$ be any element such that the average value of ρ on C_β^F is non-zero. Then, clearly, the corresponding average value polynomial itself is non-zero hence Proposition 4.4(i) implies that $\dim C_\beta = d_\beta \leq d_\alpha = \dim C_\alpha$ with equality only for $\alpha = \beta$. Hence the class C_α is the unipotent support of ρ.

A completely analogous argument shows that also Problem 1.2 has a positive solution, and Proposition 4.4(iii) proves that we obtain the same class as before. This proves part (a) in Theorem 1.4.

5.4 THE p-PARTS OF CHARACTER DEGREES

Let again G be as in the first sentence of this section, with p good. Now we turn to Theorem 1.4(b), that is, the problem concerning the p-part in the degree of an irreducible character ρ of G^F. We know already by (5.3) that ρ has a unipotent support, C say. Now we must show that the p-part in the degree of ρ is q^d where d is the dimension of the variety of Borel subgroups containing a fixed element in C. By the dimension formula in [3], Theorem 5.10.1, we have $d = (2N - \dim C)/2$ where N is the number of positive roots in the root system of G.

We can use a similar reasoning as before to reduce to the case where G has a connected center and is simple modulo its center. Indeed, let $G \subseteq G'$ be a regular embedding and ρ' an irreducible character of ${G'}^F$ whose restriction to G^F contains ρ as a constituent. Then ρ' also has unipotent support C (see Lemma 5.1). Since the index of G^F in ${G'}^F$ is certainly prime to p, Clifford's Theorem implies that the degree of ρ' is a multiple (coprime to p) of the degree of ρ. So the characters ρ and ρ' have the same p-part in their degrees and the dimensions of the unipotent supports are equal. Hence it is sufficient to consider groups G with a connected center. It is then also straightforward to check that the constructions in the proof of Lemma 5.2 behave well with respect to p-parts in character degrees and dimensions of unipotent supports. (This is certainly the case for the first reduction to groups G with a connected center and such that the derived group G_{der} is simply-connected; note that the remaining constructions just involve taking regular embeddings and direct products.)

Let us now assume that G has a connected center and is simple modulo its center. We use again the formalism of Section 4. Let $q \equiv \delta \bmod 3$ and ρ be an irreducible character of G^F contained in the series $\mathcal{E}_s$, say. Let $\bar{X}(W_s, \gamma)$ be the associated parameter set as in (3.7a), and $\bar{x} = \bar{x}_\rho$ be the element in this set corresponding to ρ. Let A be as in (4.1) and $\alpha_0 \in A$ be the unique element such that $d_{\alpha_0} = 0$ (so that C_{α_0} is the class of the trivial element in G). We define

$$\deg(\rho) := \mathrm{AV}^{\delta}_{(1.1)}(\alpha_0, \bar{x}_\rho) \in \mathbb{Q}[t].$$

Then, by the formula (4.4), the value of $\deg(\rho)$ at q is the degree of ρ. Let $a = a(\bar{x}) \geq 0$ such that t^a is the maximal power of t dividing $\deg(\rho)$. We claim that q^a is the p-part in the degree of ρ. Indeed, from the explicit description of the Fourier coefficients in [13], Chapter 4, and the formulae [13], (4.26.1) and (4.26.3), we deduce that there exists a positive integer d which is divisible by bad primes only and a monic polynomial $f \in \mathbb{Z}[t]$ such that $\deg(\rho) = (1/d)t^a f$ and $f \equiv \pm 1 \bmod t$. This implies our claim since p is good.

Thus, we have described the p-part in the degree of ρ purely in terms of our average value polynomials. On the other hand, we know by Proposition 4.4 and the argument in (5.3) that the unipotent support of ρ is also characterized purely in terms of the average value polynomials corresponding to our fixed $\bar{x} = \bar{x}_\rho$. Therefore, it will be sufficient to prove the following statement.

Given a δ-admissible coset $w_1 W'$ and $\bar{x} \in \bar{X}(W', \gamma)$ let $\alpha \in A$ be as in Proposition 4.4(i). Show that $a(\bar{x}) = (2N - d_\alpha)/2$.

Since this statement only concerns properties of the average value polynomials and dimensions of unipotent classes we can assume, without loss of generality, that q and p are large enough so that the results in [16] are applicable. Then the unipotent support $C = C_\alpha$ of our character ρ can also be characterized in terms of the map

$$\xi\colon \{\text{irreducible characters of } G^F\} \to \{F\text{-stable unipotent classes in } G\}$$

defined in [11], (13.4), or [16], (11.1). Indeed, by [16], Theorem 11.2, we have

$$C_\alpha = \xi(\rho') \quad \text{where} \quad \rho' = \pm D_G(\rho).$$

The above statement is an immediate consequence of the properties of the map ξ, as we will now check.

The first step in defining ξ is to associate with ρ' a so-called special conjugacy class in G^* (see [11], (13.2), for the precise definition). This is done as follows. With the character ρ' there is associated a family $\mathcal{F}$ of representations of W_s, and we let E_1 be the unique special representation in the family sign $\otimes\, \mathcal{F}$ (cf. [11], (13.1.3)). By the Springer correspondence, we can associate with E_1 the class of a unipotent element $v \in C_{G^*}(s)$. Then the G^*-conjugacy class C' of the element sv is the desired special class in G^*. Next, Lusztig [11], (13.3), defines a map Φ from special classes in G^* to unipotent classes in G, and we have $\xi(\rho') = \Phi(C')$. The main property of the map Φ that we need is that it preserves the dimensions of classes. So we can conclude that

$$d_\alpha = \dim C_\alpha = \dim \xi(\rho') = \dim \Phi(C') = \dim C',$$

and it remains to check that $a(\bar{x}) = (2N - \dim C')/2$. Translating this back using the dimension formula in [3], Theorem 5.10.1, we see that we must show that $a(\bar{x})$ equals the dimension of the variety of Borel subgroups of $C_{G^*}(s)$ containing the unipotent element v. By [11], (13.1.1), the latter dimension is equal to the integer a_{E_1} associated with the special representation E_1 as in [11], (4.1). So, eventually, we see that we must show that

$$a_{E_1} = a(\bar{x}).$$

Now since $\rho' = \pm D_G(\rho)$ and $\mathcal{F}$ is the family associated with ρ', the results in [11], (8.6), imply that the family associated with ρ is sign $\otimes\, \mathcal{F}$. But then the formula in [11], (4.26.3), just says that $a(\bar{x}) = a_{E_1}$, and we are done.

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