Boolean Localization, in Practice

J. F. Jardine

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Abstract. A new proof of the existence of the standard closed model structure for the category of simplicial presheaves on an arbitrary Grothendieck site is given. This proof uses the principle of Boolean localization.

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Introduction

This paper is an exposition on the use of the topos theoretic principle of Boolean localization in demonstrating the existence of closed model structures for the categories of simplicial sheaves and presheaves on a Grothendieck site $\mathcal{C}$.

Explicitly, a closed model category is a category $\mathcal{M}$ equipped with three classes of maps, called cofibrations, fibrations and weak equivalences, such that the following list of axioms is satisfied:

CM1: $\mathcal{M}$ is closed under all finite limits and colimits.

CM2: Suppose that the following diagram commutes in $\mathcal{M}$:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z. & \end{tikzcd}$$

If any two of f, g and h are weak equivalences, then so is the third.

CM3: If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4: Suppose that we are given a commutative solid arrow diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists, making the diagram commute, if either i or p is also a weak equivalence.

CM5: Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where p is a fibration and i is a trivial cofibration, and

(b) $f = q \cdot j$ where q is a trivial fibration and j is a cofibration.

Here, and as usual, one says that a map is a trivial cofibration (respectively trivial fibration) if it is both a cofibration (respectively fibration) and a weak equivalence.

The fundamental example of a closed model category is the category **S** of simplicial sets [11], [12], [2]: the cofibrations of **S** are the monomorphisms, the weak equivalences are the maps which induce isomorphisms in all possible homotopy groups of associated realizations, and the fibrations are the Kan fibrations. Recall that a Kan fibration is a map $q : X \to Y$ of simplicial sets which has the "right lifting property" with respect to all inclusions $\Lambda^n_k \subset \Delta^n$ of horns in simplices. Here, the k^{th} horn Λ^n_k is the subcomplex obtained from the boundary $\partial\Delta^n$ of the standard n-simplex by deleting the k^{th} face from its list of generators.

This paper addresses the various flavours of homotopy theory that arise from contravariant simplicial set-valued diagrams, or presheaves of simplicial sets, defined on small categories equipped with Grothedieck topologies. The list of all possible Grothendieck topologies includes the option of having no topology at all, so the theory includes that of ordinary small diagrams of simplicial sets.

There are both local and global homotopy theories for simplicial presheaves. The local theory is a theory of local weak equivalences and local fibrations. In particular, if one is working in a context so civilized as the category of simplicial presheaves on the category of open subsets of a topological space X, then a map (ie. natural transformation) $f : Y \to Z$ is a local fibration if each of the induced maps $f_x : Y_x \to Z_x$, $x \in X$, in stalks is a Kan fibration of simplicial sets. Similarly, a local weak equivalence in this case is a map which induces weak equivalences in all stalks. One uses the same notion of local weak equivalence in the global theory (so that the two theories induce equivalent homotopy categories), along with cofibrations, or monomorphisms of simplicial presheaves, and then global fibrations are defined by a lifting property. There is a difference between the two theories: the Eilenberg-Mac Lane objects $K(A, n)$ associated to sheaves of abelian groups A are certainly locally fibrant, but almost never globally fibrant. A globally fibrant model of $K(A, n)$ is most properly thought of as a type of injective resolution of the abelian sheaf A, up to a degree shift.

The main results of this paper (Theorems 18, 27) together assert that the cofibrations, local weak equivalences and global fibrations determine closed model structures on the categories of simplicial presheaves and simplicial sheaves on an arbitrary Grothendieck site, and that the homotopy categories associated to simplicial presheaves and sheaves on any such site are equivalent. In all of this, one of the main technical difficulties is to arrange for a definition of local weak equivalence which specializes to the stalkwise notion in cases where the underlying topos has enough points. Historically, this was done for simplicial presheaves in a somewhat ad hoc way [4], by using sheaves of homotopy groups for associated presheaves of Kan complexes. Here, one finds an alternative definition of local weak equivalence and proofs of the main results which are based on the method of Boolean localization. The proof in the simplicial sheaf case is roughly what Joyal had in mind in his letter to Grothendieck [7] of 1984, except that it's been somewhat reverse engineered so that the relationship between sheaves of homotopy groups and weak equivalences comes out only after the fact.

Stated bluntly, the Boolean localization principle asserts that every Grothendieck topos can be faithfully imbedded in a topos that satisfies the axiom of choice. The applicability of Boolean localization in homotopy theory was first noticed by Van Osdol [14] in the 1970's, in his proof of what was then called the Illusie conjecture [3], but the descriptions of the underlying topos theory in the literature remained fragmentary until the appearance of the Mac Lane-Moerdijk book [9] in 1992. Even so, the principle as stated in [9] has to be reinterpreted somewhat to achieve the form that is used in this paper. This is done in the first section below. This reinterpretation is trivial for a topos theorist, but quite opaque to almost everybody else.

The reader who is familiar with the "Simplicial presheaves" paper [4] will notice minor technical improvements here and there, particularly in the statement and proof of Lemma 12, and in the proof of Lemma 14, along with a more aggressive use of Kan's Ex^{∞} functor throughout. The basic thrust of using a transfinite small object argument to prove the factorization axiom CM5 survives, and the local fibration concept continues to be an essential building block of the theory.

The idea appearing in the third section, that homotopy groups should really be fibred group objects, is due to Joyal as far as I can tell. Such objects, combinatorially defined, are exactly the right kind of thing to feed to a Boolean localization functor. They also have other uses: in particular, fibred homotopy group objects appear implicitly (the π_*-Kan condition) in the proof of the Bousfield-Friedlander theorem [1], [2] that recognizes homotopy cartesian diagrams of bisimplicial sets. One can also express the theory of long exact sequences for fibrations in these terms.

The writeup that follows assumes that the reader knows the basic exactness properties of a topos, and is familiar with the nuts and bolts of the associated sheaf construction. In this connection, there is one notational oddity: I use the notation L^2F to denote the associated sheaf of a presheaf F. There is some precedent for this in the literature – see [13], for example. The notation is used in order to avoid the repeated appearance of some rather ugly very wide tildes. It is also assumed that the reader is familiar with the ordinary homotopy theory of simplicial sets [10], [2].

1. BOOLEAN LOCALIZATION.

Suppose that $\mathcal{C}$ is an arbitrary small Grothendieck site, and let $\mathcal{E}$ denote the sheaf category $Shv(\mathcal{C})$ on the site $\mathcal{C}$. A *Boolean localization* of $\mathcal{E}$ is a complete Boolean algebra $\mathcal{B}$ and a geometric topos morphism $\wp : Shv(\mathcal{B}) \to \mathcal{E}$, such that the inverse image functor $\wp^* : \mathcal{E} \to Shv(\mathcal{B})$ is faithful.

The definition is a bit of a mouthful. A complete Boolean algebra $\mathcal{B}$ can be characterized as a poset having at least a terminal object 1 and an initial object 0 such that $0 \neq 1$. Furthermore, $\mathcal{B}$ is required to have all limits (meets) and all colimits (joins), such that

(1) $\mathcal{B}$ is complemented in the sense that every element x has a complement $\neg x$ satisfying

$$x \vee \neg x = 1 \quad \text{and} \quad x \wedge \neg x = 0,$$

and

(2) $\mathcal{B}$ satisfies the distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

The word "complete" refers to the fact that $\mathcal{B}$ is required to have *all* meets as opposed to all finite meets. Complete Boolean algebras also satisfy the infinite distributive law:

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$$

(see [9, p.51,114]). Finally, in $\mathcal{B}$ a family of subobjects $y_i \leq x$ of x is said to be covering if $\bigvee_{i \in I} y_i = x$. The infinite distributive law guarantees that the covering families of $\mathcal{B}$ satisfy the axioms for a pretopology, and hence give rise to a category of sheaves $Shv(\mathcal{B})$.

Boolean localizations exist for all Grothendieck toposes $\mathcal{E}$: this is a major theorem of topos theory (Mac Lane and Moerdijk call it Barr's Theorem [9, p.513], but a result of Diaconescu plays a major part – see [9, p.511]). It's also important to know, so we don't leave the realm of small sites, that the construction doesn't blow up: if the cardinality of the set of morphisms of the underlying site $\mathcal{C}$ is bounded by some infinite cardinal β, then $|\mathcal{B}| < \beta$.

Boolean localization is a vast generalization of what it means for a topos to have enough points. Specifically, the topos $\mathcal{E}$ has enough points if there is a collection $x_i : \mathbf{Sets} \to \mathcal{E}$ of geometric morphisms such that two maps $f, g : F \to G$ of $\mathcal{E}$ coincide if and only if $x_i^* f = x_i^* g$ for all $i \in I$. The set category $\mathbf{Sets}$ is equivalent to the sheaf category $Shv(\{0,1\})$ on the Boolean algebra $\{0,1\}$; more generally, the product category $\prod_{i \in I} \mathbf{Sets}$ is equivalent to $Shv(\mathcal{P}(I))$ where $\mathcal{P}(I)$ is the complete Boolean algebra determined by the set of all subsets of the set I. Finally, any collection of points $x_i : \mathbf{Sets} \to \mathcal{E}$ determines a geometric morphism $x : Shv(\mathcal{P}(I)) \to \mathcal{E}$ which is a Boolean localization for $\mathcal{E}$ if the collection of points is big enough. In other words, the topos $\mathcal{E}$ has enough points if and only if there is a Boolean localization of the form $Shv(\mathcal{P}(I)) \to \mathcal{E}$ for some set I.

We shall discuss the homotopy theoretic consequences of the existence of Boolean localizations here, and defer to the Mac Lane-Moerdijk text for its proof. The applications depend explicitly on the fact that the topos $Shv(\mathcal{B})$ satisfies the axiom of choice in the sense that every epimorphism in $Shv(\mathcal{B})$ has a section; we begin by giving an explicit proof of this result (Proposition 2).

Lemma 1. *Suppose that F is a sheaf (of sets) on a complete Boolean algebra $\mathcal{B}$. Then the category $Sub(F)$ of subobjects of F is a complete Boolean algebra.*

Proof: The category $Sub(F)$ has all meets and joins, and satisfies the infinite distributive law, by an argument on the presheaf level. Given $G \in Sub(F)$, define

$$\neg G = \bigvee_{H \wedge G = \emptyset} H.$$

It's clear that $G \wedge \neg G = \emptyset$; the interesting bit is to show that $G \vee \neg G = F$.

First of all, we show that every subobject $G \leq hom(\ ,B)$ of a representable sheaf is representable. In effect,

$$G = \varinjlim_{\phi: hom(\ ,A) \to G} hom(\ ,A),$$

and the category of morphisms $\phi : hom(\ ,A) \to G$ is small, since it can be identified with a subcategory of subobjects of B in the Boolean algebra $\mathcal{B}$. There is an isomorphism

$$\bigvee_{\phi: hom(\ ,B) \to G} B \cong \varinjlim_{\phi: hom(\ ,B) \to G} B$$

in $\mathcal{B}$, and so G is represented by the object

$$\bigvee_{\phi: hom(\ ,B) \to G} B.$$

It follows that $Sub(hom(\ ,B))$ is a complete Boolean algebra. Every subobject $F \leq hom(\ ,B)$ is represented by a subobject $A \leq B$ of $\mathcal{B}$, and $\neg A$ in $Sub(B)$ is the subobject $(\neg A) \wedge B$. Observe that $(\neg A) \wedge B$ is terminal among all subobjects of B which miss A, so that $hom(\ ,(\neg A) \wedge B) = \neg hom(\ ,B)$ in the category of subobjects of $hom(\ ,B)$.

It's certainly the case, in general, that $G \vee \neg G \leq F$ in the category of subobjects of the sheaf F. Take a sheaf morphism $\phi : hom(\ ,A) \to F$, and form the pullback diagram

$$\begin{tikzcd} \phi^{-1}(G) \arrow[r] \arrow[d] & G \arrow[d] \\ hom(\ ,A) \arrow[r, "\phi"'] & F \end{tikzcd}$$

Then there is an induced diagram

$$\begin{tikzcd} \phi^{-1}(G) \vee \neg\phi^{-1}(G) \arrow[r] \arrow[d, "\cong"'] & G \vee \neg G \arrow[d] \\ hom(\ ,A) \arrow[r, "\phi"'] & F \end{tikzcd}$$

Such diagrams exist for all such maps ϕ, and F is a colimit of representables, so that the morphism $G \vee \neg G \leq F$ has a section, and is therefore an isomorphism. □

Proposition 2. *Suppose that $\mathcal{B}$ is a complete Boolean algebra. Then every epimorphism in the sheaf category $Shv(\mathcal{B})$ has a section.*

Proof: Suppose that $\pi : F \to G$ is an epimorphism of $Shv(\mathcal{B})$. Sheaf epimorphisms are defined by the existence of partial lifts along covering families, so by looking at the terminal object, one finds an object $A \in \mathcal{B}$ such that $A \neq 0$ and there is a lifting diagram

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ hom(\ ,A) \arrow[ur] \arrow[r] & G. \end{tikzcd}$$

Observe that the map $hom(\ ,A) \to G$ defines $hom(\ ,A)$ as a subobject of G, since $hom(\ ,A)$ is a subobject of the terminal sheaf $*$. It follows that the set of all partial lifts

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ N \arrow[ur, "s"] \arrow[r, hook] & G \end{tikzcd}$$

defined on subobjects N of G is non-empty. This set has maximal elements, by Zorn's Lemma.

Suppose that

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ M \arrow[ur, "s"] \arrow[r, hook] & G \end{tikzcd}$$

is such a maximal element, and suppose that $M \neq G$. Then M has a non-empty complement $\neg M$ in G, and we can form the pullback diagram

$$\begin{tikzcd} \pi^{-1}(\neg M) \arrow[r] \arrow[d, "\pi_*"'] & F \arrow[d, "\pi"] \\ \neg M \arrow[r] & G \end{tikzcd}$$

Then the map π_* is an epimorphism, and so there is a diagram

$$\begin{tikzcd} & \pi^{-1}(\neg M) \arrow[d, "\pi_*"] \\ hom(\ ,C) \arrow[ur, "s'"] \arrow[r, hook] & \neg M \end{tikzcd}$$

for some representable subobject $hom(\ ,C)$ of $\neg M$ with $C \neq 0$. Finally, $hom(\ ,C) \wedge M = \phi$, so that $M \neq hom(\ ,C) \vee M$, and there is a lift

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ hom(\ ,C) \vee M \arrow[ur, "s' \vee s"] \arrow[r, hook] & G, \end{tikzcd}$$

contradicting the maximality of the lifting s. □

Generally, a map $f : X \to Y$ of presheaves on a Grothendieck site $\mathcal{C}$ is said to be a *local epimorphism* if for all sections $y \in Y(U)$, $U \in \mathcal{C}$, there is a covering sieve $R \subset hom(\ ,U)$ and elements $x_\phi \in X(V)$ for each morphism $\phi : V \to U$ in R, such that y lifts to X along R in the sense that $\phi^*(y) = f(x_\phi)$ in $Y(V)$ for all $\phi \in R$, as in the picture

$$\begin{tikzcd}
X(U) \arrow[r, "\phi^*"] \arrow[d, "f"'] & X(V) \arrow[d, "f"] & x_\phi \arrow[dd] \\
Y(U) \arrow[r, "\phi^*"'] & Y(V) & \\
y \arrow[rr] & & \phi^*(y)
\end{tikzcd}$$

In cases where there is an adequate notion of stalk, local epimorphisms are stalkwise epimorphisms: the point is that all sections should be "liftable" up to local refinement.

Examples of local epimorphisms of presheaves include all sheaf epimorphisms and the associated sheaf map $\eta : X \to L^2X$. It's easy to show that local epimorphisms are closed under composition and that a map $f : X \to Y$ is a local epimorphism if and only if the induced map $f_* : L^2X \to L^2Y$ is an epimorphism of sheaves.

There is a dual notion of local monomorphism: a map $g : A \to B$ of presheaves is a *local monomorphism* if for all $x, y \in A(U)$, $U \in \mathcal{C}$, $g(x) = g(y)$ implies that there is a covering sieve $R \subset hom(\ ,U)$ such that $\phi^*(x) = \phi^*(y) \in A(V)$ for all maps $\phi : V \to U$ in R. Again, the associated sheaf map $\eta : X \to L^2X$ is a local monomorphism, local monomorphisms are closed under composition, and a map g is a local monomorphism if and only if the induced map g_* of associated sheaves is a monomorphism of sheaves.

Now suppose that $\wp : Shv(\mathcal{B}) \to \mathcal{E}$ is a fixed Boolean localization, where $\mathcal{E} = Shv(\mathcal{C})$. This means, in particular, that the inverse image functor $\wp^* : \mathcal{E} \to Shv(\mathcal{B})$ is faithful. The functor $\wp^*$ also preserves finite limits and all colimits – this is part of the definition of a geometric morphism. The combination of these properties for $\wp^*$, together with basic exactness properties of Grothendieck topoi, has the following rather powerful consequence:

Lemma 3. *Suppose that $\wp : \mathcal{F} \to \mathcal{E}$ is a geometric morphism of Grothendieck topoi. Then the following are equivalent:*

(1) *The inverse image functor $\wp^* : \mathcal{E} \to \mathcal{F}$ is faithful.*

(2) *The functor $\wp^*$ reflects isomorphisms.*

(3) *The functor $\wp^*$ reflects epimorphisms.*

(4) *The functor $\wp^*$ reflects monomorphisms.*

Proof: Suppose that $\wp^*$ is faithful. This means that $\wp^*(f_1) = \wp^*(f_2)$ for $f_1, f_2 : A \to B$ implies that $f_1 = f_2$. Then $\wp^*$ reflects monics. In effect, suppose that $m : B \to C$ is a morphism of $\mathcal{E}$ such that $\wp^*(m)$ is a monomorphism of $\mathcal{F}$. Suppose that $m \circ f_1 = m \circ f_2$. Then $\wp^*(m)\wp^*(f_1) = \wp^*(m)\wp^*(f_2)$ implies $\wp^*(f_1) = \wp^*(f_2)$, so that $f_1 = f_2$ in $\mathcal{E}$. Similarly, $\wp^*$ reflects epimorphisms. A morphism of $\mathcal{E}$ is an isomorphism if and only if it is both a monomorphism and an epimorphism, so it follows that $\wp^*$ reflects isomorphisms.

To see that (3) implies (1), observe that the maps $f_1, f_2 : A \to B$ coincide if and only if their equalizer $m : C \to A$ is an isomorphism. Suppose that $\wp^*(f_1) = \wp^*(f_2)$. Then $\wp^*(m)$ is the equalizer of $\wp^*(f_1)$ and $\wp^*(f_2)$, by exactness of $\wp^*$, so that $\wp^*(m)$ is an isomorphism. Thus, by assumption, m is an epimorphism and hence an isomorphism, so that $f_1 = f_2$. Statement (4) implies statement (1) by a dual argument. □

2. Closed model structures.

In this section, we show that any fixed Boolean localization $\wp : Shv(\mathcal{B}) \to Shv(\mathcal{C})$ determines a class of local weak equivalences of simplicial presheaves on the site $\mathcal{C}$. We further show that this class, along with the cofibrations (or monomorphisms) of simplicial presheaves, creates closed model structures for both simplicial presheaves and simplicial sheaves on $\mathcal{C}$, in such a way that the associated homotopy theories are equivalent (Theorem 18). These closed model structures are seen to be independent of the choice of Boolean localization $\wp$ in the next section.

The definition of local weak equivalence is based on universally defined notions of local fibration and trivial local fibration for simplicial presheaves on arbitrary sites, which specialize to Kan fibrations (respectively trivial Kan fibrations) in all sections in the case of morphisms of simplicial sheaves on a complete Boolean algebra $\mathcal{B}$, via the axiom of choice. With cofibrations and local weak equivalences in hand, one defines global fibrations by a right lifting property with respect to all maps which are both cofibrations and local weak equivalences, thus effectively forcing the factorization axiom CM5 to be the non-trivial part of Theorem 18. To prove it, one shows that a map is a global fibration if and only if it has the right lifting property with respect to some *set* of trivial cofibrations (Lemma 15). These are the α-bounded trivial cofibrations, defined with respect to a cardinal number α which is sufficiently large (and in particular larger than the cardinality of the set of morphisms of $\mathcal{C}$). The most interesting part, technically, is the proof of Lemma 12.

Suppose that K is a finite simplicial set, and that Y is a simplicial presheaf on the Grothendieck site $\mathcal{C}$. Write Y^K for the presheaf defined by simplicial set morphisms in sections via the formula

$$Y^K(U) = hom_{\mathbf{S}}(K, Y(U))$$

Observe that Y^K is a sheaf if Y is a simplicial sheaf, and that any exact functor preserves this definition, so that, for example, the sheaf associated to Y^K is canonically isomorphic to $(L^2Y)^K$. Also, any geometric topos morphism preserves this construction.

One says that a map $p : X \to Y$ of simplicial presheaves is a *local fibration* if the induced maps

(1) $$X^{\Delta^n} \xrightarrow{(i^*, p_*)} X^{\Lambda^n_k} \times_{Y^{\Lambda^n_k}} Y^{\Delta^n}$$

are local epimorphisms of presheaves for $n > 0$. Implicitly, a map $p : Z \to W$ of simplicial sheaves is a local fibration if and only if the maps (1) are sheaf epimorphisms. More than this is true in the Boolean topos setting:

Lemma 4. *A map $p : Z \to W$ of simplicial sheaves on a complete Boolean algebra $\mathcal{B}$ is a local fibration if and only if the induced maps in sections*

$$p : Z(b) \to W(b)$$

are Kan fibrations of simplicial sets for all $b \in \mathcal{B}$.

Proof: The sheaf epimorphisms

$$Z^{\Delta^n} \xrightarrow{(i^*,p_*)} Z^{\Lambda^n_k} \times_{W^{\Lambda^n_k}} W^{\Delta^n}$$

have sections, by Proposition 2, so that the maps

$$Z^{\Delta^n}(b) \xrightarrow{(i^*,p_*)} Z^{\Lambda^n_k}(b) \times_{W^{\Lambda^n_k}(b)} W^{\Delta^n}(b)$$

in sections are surjective, for all $b \in \mathcal{B}$. □

One says that a map $f : X \to Y$ of simplicial presheaves has the *local right lifting property with respect to the simplicial set inclusions* $\partial\Delta^n \subset \Delta^n$ if all of the maps

$$X^{\Delta^n} \xrightarrow{(i^*,f_*)} X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

are local epimorphisms. One can speak, as well, about local right lifting properties with respect to more general collections of inclusions $K \subset L$ of finite simplicial sets. In particular, a local fibration is a map which has the local right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$.

Suppose that X is a simplicial presheaf. The simplicial presheaf $Ex^m X$ has n-simplices defined by

$$(Ex^m X)_n = hom(sd^m\Delta^n, X).$$

with simplicial structure maps induced by precomposition with the induced simplicial set maps $sd^m\Delta^k \to sd^m\Delta^n$. The subdivision functor that we use here is the classical one: the subdivision $sd\Delta^n$ is the nerve of the poset of non-degenerate simplices of Δ^n, and the subdivision sdK of a simplicial set K is a colimit of simplicial sets $sd\Delta^m$, indexed over the simplices $\Delta^m \to K$ of K. The collection of "last vertex" maps $sd\Delta^m \to \Delta^m$, $m \geq 0$, induce a natural map $X \to ExX$, and iteration of the construction produces a sequence of simplicial presheaf maps

$$X \to ExX \to Ex^2X \to Ex^3X \to \dots$$

The simplicial presheaf $Ex^\infty X$ is defined to be the filtered colimit of these maps in the simplicial presheaf category. Write $\nu : X \to Ex^\infty X$ for the canonical map.

To put it a different way, Kan's original Ex^∞-construction [8], [2] is natural, so that it certainly applies to simplicial presheaves, and that's all we're doing here. In particular, the map $\nu : X \to Ex^\infty X$ consists of weak equivalences $\nu : X(U) \to Ex^\infty X(U)$, $U \in \mathcal{C}$, in all sections.

Now, fix a Boolean localization $\wp : Shv(\mathcal{B}) \to \mathcal{E}$, and consider the functors

$$\mathbf{S}Pre(\mathcal{C}) \xrightarrow{L^2} \mathbf{S}\mathcal{E} \xrightarrow{\wp^*} \mathbf{S}Shv(\mathcal{B})$$

relating the categories of simplicial objects in the categories $Pre(\mathcal{C})$ of presheaves on $\mathcal{C}$ and the toposes $\mathcal{E}$ and $Shv(\mathcal{B})$, where L^2 is the associated sheaf functor. We shall say that a map $f : X \to Y$ of $\mathbf{S}Shv(\mathcal{B})$ is a *pointwise weak equivalence* if it induces weak equivalences

$$f : X(b) \to Y(b)$$

of simplicial sets for all $b \in \mathcal{B}$. A map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is said to be a *local weak equivalence* if the induced map $\wp^*L^2(f) : \wp^*L^2Ex^\infty X \to \wp^*L^2Ex^\infty Y$ is a pointwise weak equivalence.

Remark 5. All of the decorations that appear in the definition of local weak equivalence are necessary. The categories of simplicial presheaves and sheaves on the site defined by the power set $\mathcal{P}(I)$ of an infinite set I are very good examples to keep in mind. The power set $\mathcal{P}(I)$ is, of course, a complete Boolean algebra, so that the Boolean localization $\wp$ can be taken to be the identity in this case. A simplicial presheaf X on $\mathcal{P}(I)$ is nothing more than a contravariant functor defined on the category of subsets of I, and taking values in simplicial sets. The stalks of the simplicial presheaf X are the simplicial sets $X_i = X(\{i\})$ corresponding to sections in the various singleton subsets of I, and the associated sheaf L^2X is defined in sections at a subset U of I by

$$L^2X(U) = \prod_{i \in U} X_i.$$

One says that a map $f : X \to Y$ of simplicial sheaves on $\mathcal{P}(I)$ is a stalkwise weak equivalence if all induced maps $f_i : X_i \to Y_i$, $i \in I$ are weak equivalences of simplicial sets. Observe that all induced maps in sections for the simplicial sheaf map f have the form

$$\prod f_i : \prod_{i \in U} X_i \to \prod_{i \in U} Y_i$$

for $U \subset I$. It is known that infinite products do not necessarily preserve weak equivalences (see the next paragraph), so that a stalkwise weak equivalence f of simplicial sheaves may not induce weak equivalences of simplicial sets in all sections. Infinite products do, however, preserve weak equivalences when all of the spaces X_i and Y_i are Kan complexes. The assertion that all of the X_i, $i \in I$, are Kan complexes is exactly what it means for the simplicial sheaf (or presheaf) X on $\mathcal{P}(I)$ to be locally fibrant. Thus, local weak equivalences as defined above coincide with stalkwise weak equivalences for simplicial sheaves and presheaves defined on $\mathcal{P}(I)$, and the implicit passage to locally fibrant models is fundamental.

Example 6. Here's an example of a countable collection of contractible simplicial sets X_n, $n \geq 0$, such that the product $\prod_{i \geq 0} X_i$ is not contractible. Let X_n be the subcomplex of Δ^n which is the union of the 1-simplices $\Delta^1 \subset \Delta^n$ defined by pairs of vertices $(i, i+1)$. The sequence of simplicial sets can therefore be identified with the graphs

$$0,\ 0 \to 1,\ 0 \to 1 \to 2, \dots$$

with no compositions allowed. Then the vertices $(0,0,0,0,\dots)$ and $(0,1,2,3,\dots)$ cannot be in the same path component of the product $\prod_{n \geq 0} X_n$. This observation can be expanded to a calculation of the homotopy type of the product: its path components are contractible, and two vertices $x = (x_n)$, $y = (y_n)$ of $\prod_{n \geq 0} X_n$ are in the same path component if and only if the list of combinatorial distances $d(x_n, y_n) = |y_n - x_n|$ (ie. number of 1-simplices between them in X_n) has a finite uniform bound.

Lemma 7. *Suppose, for a map $f : X \to Y$ of $\mathbf{S}Pre(\mathcal{C})$, the preheaf maps*

$$X^{\Delta^n} \xrightarrow{(i^*, f_*)} X^{\partial \Delta^n} \times_{Y^{\partial \Delta^n}} Y^{\Delta^n}$$

are local epimorphisms for $n \geq 0$. Then f is a local weak equivalence and a local fibration.

Proof: If f has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, then f has the local right lifting property with respect to all inclusions of finite simplicial sets $K \subset L$. In effect, the morphisms

$$\wp^* L^2 X^{\Delta^n} \xrightarrow{(i^*, f_*)} \wp^* L^2 X^{\partial\Delta^n} \times_{\wp^* L^2 Y^{\partial\Delta^n}} \wp^* L^2 Y^{\Delta^n}$$

are sheaf epimorphisms in $Shv(\mathcal{B})$, and hence pointwise epimorphisms, so that all maps $\wp^* L^2 X(b) \to \wp^* L^2 Y(b)$ in sections are trivial Kan fibrations. But this means that the sheaf maps

$$\wp^* L^2 X^L \xrightarrow{(i^*, f_*)} \wp^* L^2 X^K \times_{\wp^* L^2 Y^K} \wp^* L^2 Y^L$$

are pointwise epimorphisms by standard nonsense about trivial Kan fibrations, and are therefore sheaf epimorphisms. It follows that the maps

$$X^L \xrightarrow{(i^*, f_*)} X^K \times_{Y^K} Y^L$$

are local epimorphisms. In particular, the map f is a local fibration.

Also, if f has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, then f has the local right lifting property with respect to all induced inclusions $sd^m \partial\Delta^n \subset sd^m \Delta^n$, so that $Ex^m f : Ex^m X \to Ex^m Y$ has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$. But then $Ex^\infty f$ has the same local lifting property, and so does $\wp^* L^2 Ex^\infty f$. In particular, $\wp^* L^2 Ex^\infty f$ is a pointwise trivial fibration of simplicial sheaves on $\mathcal{B}$, and is therefore a weak equivalence. □

Corollary 8. *For any simplicial presheaf X, the canonical map $\eta : X \to L^2 X$ has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, and is therefore a local fibration and a local weak equivalence.* □

Lemma 9. *Suppose that a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is a pointwise weak equivalence in the sense that all maps in sections*

$$f : X(U) \to Y(U), \quad U \in \mathcal{C},$$

are weak equivalences of simplicial sets. Then f is a local weak equivalence.

Proof: The canonical map $\nu : X \to Ex^\infty X$ is a pointwise weak equivalence of simplicial presheaves, so it's enough to assume that $f : X \to Y$ is a pointwise weak equivalence of presheaves of Kan complexes, and then deduce that the map $\wp^* L^2 f : \wp^* L^2 X \to \wp^* L^2 Y$ is a pointwise weak equivalence of simplicial sheaves on $\mathcal{B}$.

Since X and Y are presheaves of Kan complexes, the classical method of replacing a map by a fibration gives a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & X \times_Y \mathbf{hom}(\Delta^1, Y) \arrow[d, "p"] \\ & Y \end{tikzcd}$$

of f in the simplicial presheaf category $\mathbf{S}Pre(\mathcal{C})$, where p is a map which is a pointwise Kan fibration and a pointwise weak equivalence, and the map i is right inverse to a map π which is a pointwise Kan fibration and a pointwise weak equivalence. The maps p and π have the local lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, so both maps are local fibrations and local weak equivalences by Lemma 7. In particular, the maps $\wp^* L^2 i$ and $\wp^* L^2 p$ are pointwise weak equivalences, so that $\wp^* L^2 f = (\wp^* L^2 p)(\wp^* L^2 i)$ is a pointwise weak equivalence. □

Corollary 10. *A map $f : X \to Y$ is a local weak equivalence of $\mathbf{S}Pre(\mathcal{C})$ if and only if $\wp^* L^2 f : \wp^* L^2 X \to \wp^* L^2 Y$ is a local weak equivalence of $\mathbf{S}Shv(\mathcal{B})$.*

Proof: Observe that (by definition, and with respect to the Boolean localization $1 : Shv(\mathcal{B}) \to Shv(\mathcal{B})$) a map $g : Z \to W$ of $\mathbf{S}Shv(\mathcal{B})$ is a weak equivalence if and only if the map $L^2 Ex^\infty g : L^2 Ex^\infty Z \to L^2 Ex^\infty W$ is a pointwise weak equivalence of $\mathbf{S}Shv(\mathcal{B})$

Also notice that there are natural isomorphisms

$$L^2 Ex^\infty \wp^* L^2 X \cong \wp^* L^2 Ex^\infty X$$

for $X \in \mathbf{S}Pre(\mathcal{C})$. Thus, $L^2 Ex^\infty \wp^* L^2 f$ is a pointwise weak equivalence if and only if $\wp^* L^2 Ex^\infty f$ is a pointwise weak equivalence. □

Generally, for a fixed property $\mathcal{P}$ of simplicial sets, one says that a map $f : X \to Y$ has the property $\mathcal{P}$ *pointwise* if each of the simplicial set maps $f : X(U) \to Y(U)$, $U \in \mathcal{C}$, in sections has the property $\mathcal{P}$. The class of pointwise weak equivalences appearing in the statement of Lemma 9 is a common example. Pointwise (Kan) fibrations and pointwise trivial fibrations also occur frequently: a map $f : X \to Y$ of simplicial presheaves is a pointwise fibration (respectively pointwise trivial fibration) if all of the maps $f : X(U) \to Y(U)$, $U \in \mathcal{C}$, are fibrations (respectively trivial fibrations) of simplicial sets. We have already met such maps in the context of simplicial presheaves on a complete Boolean algebra $\mathcal{B}$.

We shall also need the following partial converse to Lemma 7:

Lemma 11. *Suppose that X and Y are locally fibrant simplicial presheaves on $\mathcal{C}$, and that the map $q : X \to Y$ is a local fibration and a local weak equivalence. Then q has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$.*

Proof: It suffices to assume that X and Y are locally fibrant simplicial sheaves, since the associated sheaf functor L^2 preserves local fibrations and local weak equivalences, and reflects the desired local right lifting property.

The induced map

$$\wp^* L^2 Ex^\infty q : \wp^* L^2 Ex^\infty X \to \wp^* L^2 Ex^\infty Y$$

is a pointwise weak equivalence of simplicial sheaves on $\mathcal{B}$, since q is assumed to be a local weak equivalence. There is a natural isomorphism $\wp^* L^2 Ex^\infty \cong L^2 Ex^\infty \wp^*$, so the map

$$L^2 Ex^\infty \wp^* q : L^2 Ex^\infty \wp^* X \to L^2 Ex^\infty \wp^* Y$$

is a pointwise weak equivalence. The simplicial sheaf $\wp^* X$ is a presheaf of Kan complexes on $\mathcal{B}$, as is the object $Ex^\infty \wp^* X$. Furthermore, the natural map $L^2 \nu$:

$\wp^* X \to L^2 Ex^\infty \wp^* X$ can be identified with the effect of applying the associated sheaf functor L^2 to the canonical pointwise weak equivalence $\nu : \wp^* X \to Ex^\infty \wp^* X$. If we can prove that the associated sheaf functor on $\mathcal{B}$ preserves pointwise weak equivalences between presheaves of Kan complexes, then we'd be done, since then $L^2\nu$ would be a pointwise weak equivalence, and so the map $\wp^* q : \wp^* X \to \wp^* Y$ would be a pointwise Kan fibration and a pointwise weak equivalence, and would therefore have the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$. Finally, our faithful functor $\wp^*$ reflects this local right lifting property.

Suppose that $f : Z \to W$ is a pointwise weak equivalence between presheaves of Kan complexes on $\mathcal{B}$, and form a diagram of simplicial presheaf maps

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "f"'] & \overline{Z} \arrow[d, "\pi"] \\ & W \end{tikzcd}$$

such that π is a pointwise trivial fibration and i is right inverse to a pointwise trivial fibration $\pi' : \overline{Z} \to Z$. The associated sheaf functor L^2 preserves the local right lifting property with respect to the maps $\partial\Delta^n \subset \Delta^n$, and of course $Shv(\mathcal{B})$ satisfies the axiom of choice, so that the maps $L^2\pi$ and $L^2\pi'$ are pointwise trivial fibrations, and so $L^2 f$ is a pointwise weak equivalence. □

Pick some infinite cardinal α such that α is strictly larger than the cardinality of the set of morphisms of the site $\mathcal{C}$. A simplicial presheaf X on $\mathcal{C}$ is said to be *α-bounded* if

$$|X_n(U)| < \alpha$$

for all $n \geq 0$ and all objects U of $\mathcal{C}$. Standard cardinal arithmetic implies that if X is α-bounded, then so is its associated simplicial sheaf $L^2 X$.

Suppose that K is a simplicial set and U is an object of $\mathcal{C}$. Then the simplicial presheaf $L_U K$ is defined for $V \in \mathcal{C}$ by

$$L_U K(V) = \bigsqcup_{\phi : V \to U} K.$$

Observe that morphisms of simplicial presheaves $L_U K \to X$ are in one to one correspondence with simplicial set maps of the form $K \to X(U)$. If the simplicial set K is α-bounded in the sense that $|K_n| < \alpha$ for $n \geq 0$, then the simplicial presheaf $L_U K$ is α-bounded.

Lemma 12. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves on $\mathcal{C}$, and that pullback along f preserves α-bounded subcomplexes in the sense that if T is an α-bounded subcomplex of Y then $T \times_Y X$ is an α-bounded subcomplex of X. Suppose that there is a simplicial presheaf monomorphism $i : Z \hookrightarrow Y$ where Z is α-bounded. Then i has a factorization $Z \subset W \subset Y$ such that W is α-bounded and such that the projection map $f_* : W \times_Y X \to W$ is a local weak equivalence.*

Proof: First of all, one sees that any map of simplicial presheaves $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[dr, "i_*"] \arrow[dd, "f"'] & \\ & \overline{X} \arrow[dl, "p_*"] \\ Y & \end{tikzcd}$$

such that p_* is a pointwise Kan fibration and i_* is a pointwise weak equivalence, and that this factorization is natural and preserves filtered colimits in f. In effect, take the standard factorization

$$\begin{tikzcd} Ex^\infty X \arrow[dr, "i"] \arrow[dd, "Ex^\infty f"'] & \\ & Ex^\infty X \times_{Ex^\infty Y} \mathbf{hom}(\Delta^1, Ex^\infty Y) \arrow[dl, "p"] \\ Ex^\infty Y & \end{tikzcd}$$

and pull it back to Y using the diagram

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "f"'] & Ex^\infty X \arrow[d, "Ex^\infty f"] \\ Y \arrow[r, "\nu"'] & Ex^\infty Y, \end{tikzcd}$$

so that

$$\overline{X} = Y \times_{Ex^\infty Y} Ex^\infty X \times_{Ex^\infty Y} \mathbf{hom}(\Delta^1, Ex^\infty Y).$$

Note finally that if X and Y are α-bounded simplicial presheaves, then so is $\overline{X}$.

The pulled back map p_* has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, since Lemma 9 implies that p is a local weak equivalence as well as a pointwise fibration, so that Lemma 11 applies.

This means explicitly that given any diagram of simplicial set maps of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, hook] & \overline{X}(U) \arrow[d, "p_*"] \\ \Delta^n \arrow[r, "b"'] & Y(U), \end{tikzcd}$$

there is a covering sieve $R \subset hom(\ ,U)$ such that for each $\phi : V \to U$ in R, there is a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\phi^* a"] \arrow[d, hook] & \overline{X}(V) \arrow[d, "p_*"] \\ \Delta^n \arrow[r, "\phi^* b"'] \arrow[ur, "x_\phi"] & Y(V). \end{tikzcd}$$

Suppose given a diagram

$$
(2) \qquad
\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & \overline{Z \times_Y X}(U) \arrow[r] \arrow[d, "p_*"] & \overline{X}(U) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] & Z(U) \arrow[r] & Y(U).
\end{tikzcd}
$$

Then $\overline{X}$ is a filtered colimit of simplicial presheaves of the form $\overline{T \times_Y X}$, where T is an α-bounded subobject of Y containing Z. It follows that there is an α-bounded S containing Z such that all the liftings x_ϕ corresponding to the outer square live in $\overline{S \times_Y X}$. Taking the union of all such subcomplexes S over the α-bounded set of diagrams of the form (2) gives an α-bounded subcomplex Z_1 of Y such that $Z \subset Z_1 \subset Y$, and such that all local lifting problems (2) are solved in $\overline{Z_1 \times_Y X}$. Repeat the construction to obtain a sequence of α-bounded subobjects

$$Z = Z_0 \subset Z_1 \subset Z_2 \subset \dots$$

such that all local lifting problems

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & \overline{Z_i \times_Y X}(U) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] & Z_i(U)
\end{tikzcd}
$$

are solved over Z_{i+1}.

Let $W = \cup_i Z_i$. Then the map $p_* : \overline{W \times_Y X} \to W$ is a local weak equivalence by Lemma 7. Furthermore, $f_* : W \times_Y X \to W$ is a composite

$$
\begin{tikzcd}
W \times_Y X \arrow[r, "i_*"] \arrow[dr, "f_*"'] & \overline{W \times_Y X} \arrow[d, "p_*"] \\
& W,
\end{tikzcd}
$$

where i_* is a pointwise weak equivalence. The map i_* is therefore a local weak equivalence by Lemma 9, so that f_* is also a local weak equivalence. □

Corollary 13. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial sheaves which satisfies the boundedness condition of Lemma 12, and that there is a simplicial sheaf monomorphism $i : Z \hookrightarrow Y$ where Z is α-bounded. Then i has a factorization $Z \subset W \subset Y$ such that W is α-bounded and such that the projection map $f_* : W \times_Y X \to W$ is a local weak equivalence.*

Proof: Apply the associated sheaf functor to the output of Lemma 12. □

A *cofibration* of simplicial presheaves is a monomorphism $A \hookrightarrow B$ of simplicial presheaves. A map of simplicial presheaves which is both a cofibration and a local weak equivalence is called a *trivial cofibration*. A *global fibration* is a morphism $p : X \to Y$ of simplicial presheaves which has the right lifting property with respect to all trivial cofibrations. Finally a map which is simultaneously a global fibration and a local weak equivalence is said to be a *trivial global fibration*.

Lemma 14.

(1) *Trivial cofibrations of simplicial presheaves are closed under pushout.*

(2) *Suppose that γ is an limit ordinal, thought of as a poset, and that there is a functor $X : \gamma \to \mathbf{S}Pre(\mathcal{C})$ such that for each morphism $i \leq j$ of γ, the induced map $X(i) \to X(j)$ is a trivial cofibration. Then the canonical maps*

$$X(i) \xrightarrow{\tau_i} \varinjlim_{j \in \gamma} X(j)$$

are trivial cofibrations.

(3) *Suppose that the morphisms $f_i : X_i \to Y_i$ are local weak equivalences for $i \in I$. Then the morphism*

$$\bigsqcup_{i \in I} f_i : \bigsqcup_{i \in I} X_i \to \bigsqcup_{i \in I} Y_i$$

is a local weak equivalence.

Proof: It suffices, by Corollary 10 and Corollary 8, to prove all three statements for the category $\mathbf{S}Pre(\mathcal{B})$ of simplicial presheaves on the complete Boolean algebra $\mathcal{B}$.

For statement (1), suppose that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & B \cup_A C \end{tikzcd}$$

is a pushout of simplicial presheaves on $\mathcal{B}$, where i is a cofibration and a local weak equivalence. To show that i_* is a local weak equivalence, it suffices to show that the map i' in the pushout diagram of simplicial presheaves

$$\begin{tikzcd} L^2Ex^\infty A \arrow[r] \arrow[d, "L^2Ex^\infty i"'] & L^2Ex^\infty C \arrow[d, "i'"] \\ L^2Ex^\infty B \arrow[r] & L^2Ex^\infty B \cup_{L^2Ex^\infty A} L^2Ex^\infty C \end{tikzcd}$$

is a local weak equivalence. To see this, one invokes the ordinary patching lemma for simplicial sets and Corollary 8. But then the map i' is a pointwise weak equivalence since $L^2Ex^\infty i$ is a pointwise weak equivalence, so we're done.

For (2), let $X : \gamma \to \mathbf{S}Pre(\mathcal{B})$ be a functor as in the statement, and form a new functor $Ex^\infty X$ with $Ex^\infty X(i)$ defined in the obvious way for $i \in \gamma$, and consider the natural transformation $\nu : X \to Ex^\infty X$ arising from the pointwise weak equivalences

$$\nu : X(i) \to Ex^\infty X(i), \quad i \in \gamma.$$

Then each morphism $i \leq j$ of γ induces a trivial cofibration $Ex^\infty X(i) \to Ex^\infty X(j)$ by Lemma 9, and there is a commutative diagram

$$\begin{tikzcd} X(i) \arrow[r, "\tau_i"] \arrow[d, "\nu"'] & \varinjlim X(i) \arrow[d, "\nu_*"] \\ Ex^\infty X(i) \arrow[r, "\tau_i"'] & \varinjlim Ex^\infty X(i) \end{tikzcd}$$

where the filtered colimits are formed in the presheaf category, so that ν_* is a pointwise weak equivalence. It follows from Lemma 9 that one instance of τ_i in the diagram is a local weak equivalence if and only if the other is, so it suffices to assume that each of the simplicial presheaves $X(i)$ is a presheaf of Kan complexes.

Now suppose that X is a diagram of presheaves of Kan complexes, and form the diagram

$$\begin{tikzcd} X(i) \arrow[r, "\tau_i"] \arrow[d, "\eta"'] & \varinjlim X(i) \arrow[r, "\eta"] \arrow[d, "\eta_*"] & L^2(\varinjlim X(i)) \arrow[d, "\cong"] \\ L^2X(i) \arrow[r, "\tau_i"'] & \varinjlim L^2X(i) \arrow[r, "\eta"'] & L^2(\varinjlim L^2X(i)) \end{tikzcd}$$

which is induced the comparison transformation $\eta : X \to L^2X$ induced by the associated sheaf construction. The induced morphisms $L^2X(i) \to L^2X(j)$ are local weak equivalences of locally fibrant simplicial sheaves on the complete Boolean algebra $\mathcal{B}$, and are therefore pointwise weak equivalences, so that the simplicial presheaf maps $\tau_i : L^2X(i) \to \varinjlim L^2X(i)$ are pointwise weak equivalences and therefore local weak equivalences, by Lemma 9. The associated sheaf maps η are local weak equivalences by Corollary 8, so that the original maps $\tau_i : X(i) \to \varinjlim X(i)$ are local weak equivalences as well.

In the case of statement (3), the Ex^∞ construction preserves disjoint unions of simplicial sets, so it suffices to assume that the simplicial presheaves X_i and Y_i are presheaves of Kan complexes. In that case, the induced morphisms $L^2f_i : L^2X_i \to L^2Y_i$ are local weak equivalences of locally fibrant simplicial sheaves on $\mathcal{B}$, so that they are all pointwise weak equivalences. It follows that the induced morphism

$$\bigsqcup_{i\in I} L^2X_i \xrightarrow{\bigsqcup L^2f_i} \bigsqcup_{i\in I} L^2Y_i$$

are pointwise and hence local weak equivalences. One finishes by observing that there is a natural commutative diagram

$$\begin{tikzcd} \bigsqcup_{i\in I} X_i \arrow[r, "\bigsqcup_i \eta"] \arrow[d, "\eta"'] & \bigsqcup_{i\in I} L^2X_i \arrow[d, "\eta"] \\ L^2(\bigsqcup_{i\in I} X_i) \arrow[r, "\cong"'] & L^2(\bigsqcup_{i\in I} L^2X_i) \end{tikzcd}$$

in the category of simplicial presheaves on $\mathcal{B}$, so that the morphism $\bigsqcup_i \eta$ is a natural local weak equivalence by Corollary 8. ◻

A cofibration $A \hookrightarrow B$ of simplicial presheaves is said to be an *α-bounded cofibration* if the target simplicial presheaf B is α-bounded.

Lemma 15. *A map $p : X \to Y$ of simplicial presheaves is a global fibration if and only if it has the right lifting property with respect to all α-bounded trivial cofibrations.*

Proof: Suppose that p has the right lifting property with respect to all α-bounded trivial cofibrations, and consider the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & Y, \end{tikzcd}$$

where i is a trivial cofibration. We shall assume that $U \neq V$. Consider the set of all partial lifts

$$\begin{tikzcd} U \arrow[rr] \arrow[d, hook, "j"'] & & X \arrow[dd, "p"] \\ U' \arrow[urr] \arrow[d, hook, "i'"'] & & \\ V \arrow[rr] & & Y \end{tikzcd} \tag{3}$$

where $i'j = i$, $U' \neq U$, and j is a trivial cofibration. This set is non-trivial: take $x \in V(W) - U(W)$ for some $W \in \mathcal{C}$, and observe that x sits inside some α-bounded subcomplex C of V, namely the image of the map $L_W \Delta^m \to V$ which classifies x. By Lemma 12, there is an α-bounded subcomplex $B \subset V$ such that $C \subset B$ and such that the induced cofibration $j : B \cap U \hookrightarrow B$ is a local weak equivalence. Form the diagram

$$\begin{tikzcd} B \cap U \arrow[r] \arrow[d, "j"'] & U \arrow[r] \arrow[d, hook, "j_*"'] & X \arrow[dd, "p"] \\ B \arrow[r] & B \cup U \arrow[ur, dashed] \arrow[d, hook, "i'"'] & \\ & V \arrow[r] & Y \end{tikzcd}$$

and observe that the indicated lift exists because j_* is a pushout of the α-bounded trivial cofibration j. Then x is a section of $B \cup U$, so that $B \cup U \neq U$. Furthermore, j_* is a trivial cofibration: this is a consequence of Lemma 14.

The set of all partial lifts has maximal elements, by Zorn's lemma and part (2) of Lemma 14. Any maximal element must be a lift

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

by the argument (applied to maps of the form i' in (3)) that is used to demonstrate the existence of partial lifts. ◻

LEMMA 16. *Every simplicial presheaf map $f : X \to Y$ has a factorization*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

where p is a global fibration and i is a trivial cofibration.

PROOF: The proof is a transfinite small object argument.

Take a cardinal $\beta > 2^{\alpha}$, where α is the cardinality of the set of morphisms of the site $\mathcal{C}$. We define a functor $F : \beta \to \mathbf{S}Pre(\mathcal{C}) \downarrow Y$ by first setting $F(0) = f : X \to Y$. We let

$$X(\zeta) = \varinjlim_{\gamma<\zeta} X(\gamma)$$

for limit ordinals ζ. Finally, the map $X(\gamma) \to X(\gamma+1)$ is defined by taking the set of all diagrams

$$D : \quad \begin{tikzcd} U_D \arrow[r] \arrow[d, "i_D"'] & X(\gamma) \arrow[d, "F(\gamma)"] \\ V_D \arrow[r] & Y \end{tikzcd}$$

such that i_D is an α-bounded trivial cofibration, and then forming the pushout

$$\begin{tikzcd} \bigsqcup_D U_D \arrow[r] \arrow[d, "\bigsqcup_D i_D"'] & X(\gamma) \arrow[d, "i_*"] \\ \bigsqcup_D V_D \arrow[r] & X(\gamma+1) \end{tikzcd}$$

Then i_* is a trivial cofibration, by Lemma 14, as is the map i_β in the resulting diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i_\beta"'] & & Y \\ & X(\beta), \arrow[ur, "F(\beta)"'] & \end{tikzcd}$$

where $X(\beta) = \varinjlim_{\gamma<\beta} X(\gamma)$, and $F(\beta)$ is induced by all maps $F(\gamma)$. In any diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "i"'] & X(\beta) \arrow[d, "F(\beta)"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a trivial α-bounded cofibration, the simplicial presheaf U is α-bounded, so that g must factor through some subcomplex $X(\gamma) \subset X(\beta)$ with $\gamma < \beta$. It follows that the dotted arrow exists, making the diagram commute. □

Lemma 17. *Any simplicial presheaf map $f : X \to Y$ has a factorization*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where q is a trivial global fibration and j is a cofibration.

Proof: First of all, if a map $f : X \to Y$ has the right lifting property with respect to all morphisms of the form $A \subset L_U\Delta^n$, then f is a global fibration and a local weak equivalence. In effect, f has the right lifting property with respect to all cofibrations by an argument similar to that of Lemma 15, so that f is a global fibration, and f has the right lifting property with respect to all cofibrations of the form $L_U\partial\Delta^n \subset L_U\Delta^n$, so that f is a pointwise weak equivalence and hence a local weak equivalence by Lemma 9.

The existence of the required factorization is now a consequence of a transfinite small object argument similar to that given for Lemma 16. □

Theorem 18. *With respect to the definitions of cofibration, weak equivalence and global fibration given above,*

(1) *the category $\mathbf{S}Pre(\mathcal{C})$ of simplicial presheaves is a closed model category,*
(2) *the category $\mathbf{S}Shv(\mathcal{C})$ is a closed model category,*
(3) *the inclusion $\mathbf{S}Shv(\mathcal{C}) \subset \mathbf{S}Pre(\mathcal{C})$ induces an equivalence*

$$Ho(\mathbf{S}Shv(\mathcal{C})) \simeq Ho(\mathbf{S}Pre(\mathcal{C}))$$

of the associated homotopy categories.

Proof: The only non-trivial parts of the respective demonstrations are the factorization axiom and CM4, for both simplicial presheaves and simplicial sheaves. But the factorization axioms follow from Lemma 16 and Lemma 17, and their simplicial sheaf counterparts (which have the same arguments), and CM4 is a consequence of the assertion that every trivial global fibration has the right lifting property with respect to all cofibrations, for both categories.

For the latter, observe that if $p : X \to Y$ is a global fibration and a local weak equivalence, then the proof of Lemma 17 shows that p has a factorization

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "j"'] & & Y \\ & W, \arrow[ur, "q"'] & \end{tikzcd}$$

where j is a cofibration and q has the right lifting property with respect to all cofibrations and is a local weak equivalence. But then j is a trivial cofibration, so that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ W \arrow[r, "q"'] \arrow[ur] & Y, \end{tikzcd}$$

and so p is a retract of q.

The equivalence of categories

$$Ho(\mathbf{S}Shv(\mathcal{C})) \simeq Ho(\mathbf{S}Pre(\mathcal{C}))$$

is induced by the inclusion $\mathbf{S}Shv(\mathcal{C}) \subset \mathbf{S}Pre(\mathcal{C})$ and its left adjoint, the associated sheaf functor $L^2 : \mathbf{S}Pre(\mathcal{C}) \to \mathbf{S}Shv(\mathcal{C})$. Both of these functors preserve local weak equivalences, and the canonical simplicial presheaf map $X \to L^2X$ is a weak equivalence, by Corollary 8. $\square$

Suppose that X is a simplicial presheaf and that K is a simplicial set. There is a simplicial presheaf $\mathbf{hom}(K, X)$, which is defined in sections by

$$\mathbf{hom}(K, X)(U) = \mathbf{hom}(K, X(U)), \quad U \in \mathcal{C},$$

where $\mathbf{hom}(K, X(U))$, denotes the ordinary function space object in the category of simplicial sets. The simplicial presheaf $\mathbf{hom}(\Delta^1, X)$ is the path object that was used in the proof of Lemma 9.

The ordinary exponential law for simplicial sets induces a natural isomorphism of the form

$$hom(X, \mathbf{hom}(K, Y)) \cong hom(X \times K, Y),$$

where the indicated morphisms are in the category of simplicial presheaves, and $X \times K$ is the simplicial presheaf defined in sections by

$$(X \times K)(U) = X(U) \times K, \quad U \in \mathcal{C}.$$

The main homotopical result about function spaces of this type is the following:

Lemma 19. *Suppose that $q : X \to Y$ is a local fibration of simplicial presheaves on $\mathcal{C}$, and that $i : K \hookrightarrow L$ is a cofibration of simplicial sets where L is finite in the sense that it has only finitely many non-degenerate simplices. Then the induced simplicial presheaf map*

$$\mathbf{hom}(L, X) \xrightarrow{(i^*, q_*)} \mathbf{hom}(K, X) \times_{\mathbf{hom}(K,Y)} \mathbf{hom}(L, Y)$$

is a local fibration, and this map is a local weak equivalence if q is a local weak equivalence or if i is a trivial cofibration of simplicial sets.

Proof: There is a natural isomorphism

$$\wp^* L^2 \mathbf{hom}(K, X) \cong \mathbf{hom}(K, \wp^* L^2 X)$$

for all finite simplicial sets K and simplicial presheaves X, since the associated sheaf functor L^2 and the Boolean localization functor $\wp^*$ both preserve finite limits. The map $\wp^* L^2 q : \wp^* L^2 X \to \wp^* L^2 Y$ is a pointwise Kan fibration, so that the map

$$\mathbf{hom}(L, \wp^* L^2 X) \xrightarrow{(i^*, \wp^* L^2 q_*)} \mathbf{hom}(K, \wp^* L^2 X) \times_{\mathbf{hom}(K, \wp^* L^2 Y)} \mathbf{hom}(L, \wp^* L^2 Y)$$

is a pointwise Kan fibration, which is a pointwise weak equivalence if i is a trivial cofibration or if $\wp^* L^2 q$ is pointwise trivial. $\square$

Corollary 20. *Suppose that X is a locally fibrant simplicial presheaf, and that $i : K \hookrightarrow L$ is a cofibration of finite simplicial sets. Then the induced map*

$$i^* : \mathbf{hom}(L, X) \to \mathbf{hom}(K, X)$$

is a local fibration. The map i^* is a local weak equivalence if i is a trivial cofibration. □

Remark 21. Corollary 20 is the central device behind the path object and associated local fibration constructions that appear in the proof of Lemma 9.

Suppose that X and Y are simplicial presheaves. The *function space* $\mathbf{hom}(X, Y)$ is the simplicial set defined by having n-simplices

$$\mathbf{hom}(X, Y)_n = hom(X \times \Delta^n, Y),$$

where the morphism set on the right is in the category of simplicial presheaves. The standard exponential law for the simplicial set category also induces a natural isomorphism

$$hom(X \times K, Y) \cong hom_{\mathbf{S}}(K, \mathbf{hom}(X, Y)),$$

so that the category of simplicial presheaves acquires the structure of a simplicial category in the sense of Quillen.

Similar observations obtain for the category of simplicial sheaves on $\mathcal{C}$. In that case, one writes $X \otimes K = L^2(X \times K)$ for $X \in \mathbf{S}Shv(\mathcal{C})$. Then, if Y is a simplicial sheaf, there is an isomorphism

$$hom(X \otimes K, Y) \cong hom_{\mathbf{S}}(K, \mathbf{hom}(X, Y)),$$

so that the category of simplicial sheaves on $\mathcal{C}$ also has the structure of a simplicial category.

Lemma 22. *Suppose that $i : A \to B$ is a cofibration and $q : X \to Y$ is a global fibration of simplicial presheaves. Then the induced simplicial set map*

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, q_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

is a Kan fibration which is trivial if either i or q is a local weak equivalence.

Proof: The map

$$(B \times \Lambda^n_k) \cup_{(A \times \Lambda^n_k)} (A \times \Delta^n) \subset B \times \Delta^n$$

is a cofibration and a pointwise weak equivalence; it is therefore a local weak equivalence by Lemma 9. Finish the argument that (i^*, q_*) is a Kan fibration with the standard adjointness trick [11], [2].

It remains to show that the cofibration

$$(B \times \partial\Delta^n) \cup_{(A \times \partial\Delta^n)} (A \times \Delta^n) \subset B \times \Delta^n$$

is a local weak equivalence in the case where the cofibration $i : A \to B$ is a local weak equivalence.

One should know first that the functor $X \mapsto X \times K$ preserves local weak equivalences in simplicial presheaves X, for all simplicial sets K. For this, there are natural local equivalences

$$\wp^* L^2(X \times K) \xrightarrow{\wp^* L^2(\nu \times K)} \wp^* L^2(Ex^\infty X \times K) \cong L^2(\wp^* L^2 Ex^\infty X \times K).$$

The functor $X \mapsto \wp^* L^2 Ex^\infty X \times K$ takes local weak equivalences to pointwise weak equivalences, and so the desired result follows from Corollary 10.

It follows that, in the diagram

$$\begin{tikzcd} A \times \Delta^n \arrow[d, "i_*"'] \arrow[dr, "i \times \Delta^n"] & \\ (B \times \partial\Delta^n) \cup_{(A \times \partial\Delta^n)} (A \times \Delta^n) \arrow[r, hook] & B \times \Delta^n, \end{tikzcd}$$

the maps $i \times \Delta^n$ and i_* are trivial cofibrations. □

There is a corresponding statement about simplicial sheaves, which is an immediate corollary of Lemma 22.

Corollary 23. *The simplicial presheaf category* $\mathbf{S}Pre(\mathcal{C})$ *and the simplicial sheaf category* $\mathbf{S}Shv(\mathcal{C})$ *are both closed simplicial model categories.*

One says that a closed model category is *proper* if weak equivalences are preserved by pullback along fibrations and by pushout along cofibrations.

Theorem 24. *The simplicial presheaf category* $\mathbf{S}Pre(\mathcal{C})$ *and the simplicial sheaf category* $\mathbf{S}Shv(\mathcal{C})$ *are proper closed simplicial model categories.*

Properness is very commonly used: it is fundamental to all patching lemmas [2, 2.8], and is essential for constructing stable homotopy theories for simplicial presheaves [1], [5], [6].

Proof of Theorem 24: Suppose that the diagram of simplicial presheaf morphisms

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g_*"] \arrow[d] & X \arrow[d, "q"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback with q a global fibration and g a local weak equivalence. To show that g_* is a local weak equivalence, it suffices, by Corollary 8 and Corollary 10, to assume that X, Y, and Z are simplicial sheaves on the complete Boolean algebra $\mathcal{B}$. By Corollary 8 and exactness, applying the composite functor $L^2 Ex^\infty$ doesn't change the problem, so it suffices to assume that X, Y and Z are locally fibrant simplicial sheaves on $\mathcal{B}$. But then g is a pointwise weak equivalence, so that g_* is a pointwise, hence local, weak equivalence.

Suppose given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ B \arrow[r, "f_*"'] & B \cup_A X \end{tikzcd}$$

with i a cofibration and f a local weak equivalence. By the patching lemma for simplicial sets, Corollary 8 and Corollary 10, it suffices to assume that A, B and X are locally fibrant simplicial sheaves on $\mathcal{B}$ and that the pushout is formed in the category of simplicial presheaves on $\mathcal{B}$. In that case, f is a pointwise weak equivalence, so that f_* is a pointwise hence local weak equivalence, by Lemma 9. □

3. Homotopy groups.

Traditionally, weak equivalences of simplicial sheaves and presheaves have been defined via sheaves of homotopy groups, which we haven't even mentioned yet. We have so far used a definition of weak equivalence that appears to depend on a fixed Boolean localization $\wp : Shv(\mathcal{B}) \to \mathcal{E} = Shv(\mathcal{C})$. In this section we will show that this apparent dependence on $\wp$ can be removed by introducing a notion of fibred homotopy group objects which is preserved by the inverse image functor $\wp^*$ and specializes to the standard homotopy groups for ordinary simplicial sets over all vertices (but see also Remark 28 below). These homotopy group objects are made up of sheaves of homotopy groups in the usual sense, and our definition of weak equivalence is seen to coincide with the familiar one.

Suppose that X is a Kan complex, with base point x. The set underlying the homotopy group $\pi_n(X,x)$ can be identified with the set of path components $\pi_0 F_{n,x}X$, where the $F_{n,x}X$ is defined by the pullback diagram

$$\begin{tikzcd} F_{n,x}X \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\ * \arrow[r, "x"'] & \mathbf{hom}(\partial\Delta^n, X) \end{tikzcd}$$

and i^* is the fibration induced by the inclusion $i : \partial\Delta^n \hookrightarrow \Delta^n$. Note, in particular, that $F_{n,x}X$ is a Kan complex, so that $\pi_0 F_{n,x}X$ can be identified with a set of homotopy classes of vertices.

To collect all such definitions together, use the notation X_0 for the discrete simplicial set $\bigsqcup_{x\in X_0} *$ on the set of vertices of X as well as for the set of vertices itself, and form the pullback

$$\begin{tikzcd} F_nX \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\ X_0 \arrow[r] & \mathbf{hom}(\partial\Delta^n, X), \end{tikzcd}$$

where the map $X_0 \to \mathbf{hom}(\partial\Delta^n, X)$ takes the vertex x to the map $x : \partial\Delta^n \to X$ which factors through x. The simplicial set X_0 is a Kan complex, so that

$$F_nX \cong \bigsqcup_{x\in X_0} F_{n,x}X$$

is a Kan complex fibred over X_0, and we write

$$\pi_n X = \pi_0 F_n X = \bigsqcup_{x\in X_0} \pi_0 F_{n,x}X = \bigsqcup_{x\in X_0} \pi_n(X,x).$$

There is a canonical function

$$\pi_n X = \pi_0 F_n X \to X_0$$

which gives $\pi_n X$ a fibred structure over the set of vertices X_0.

To see the group multiplication, let $\Lambda^{[0,n-2]} \subset \Delta^{n+1}$ be the subcomplex which is generated by the simplices $d^i : \mathbf{n} \to \mathbf{n+1}$, $0 \leq i \leq n-2$, and write $K_n = \Lambda^{[0,n-2]} \cup sk_{n-1}\Delta^{n+1}$ let j denote the inclusion $K_n \subset \Delta^{n+1}$. Form the pullback diagram

$$\begin{tikzcd} G_n X \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^{n+1}, X) \arrow[d, "j^*"] \\ X_0 \arrow[r] & \mathbf{hom}(K_n, X) \end{tikzcd}$$

in the category of simplicial sets. The maps $d^i : \Delta^n \to \Delta^{n+1}$ induce morphisms $d_i : G_n X \to F_n X$ of spaces fibred over X_0 for $n-1 \leq i \leq n+1$. Furthermore, the induced map $(d_{n-1}, d_{n+1}) : G_n X \to F_n X \times_{X_0} F_n X$ is surjective, since it is induced by pulling back a trivial fibration $\mathbf{hom}(\Delta^{n+1}, X) \to \mathbf{hom}(\Delta^n \times_{\Delta^{n-1}} \Delta^n, X)$. By looking at vertices and taking path components one sees, via the standard constructions, that there is a unique map $m : \pi_0 F_n X \times_{X_0} \pi_0 F_n X \to \pi_0 F_n X$ of objects fibred over X_0 making the following diagram commute:

$$\begin{tikzcd} G_n X_0 \arrow[r, "d_{n*}"] \arrow[d, "(d_{n-1}, d_{n+1})_*"'] & F_n X_0 \arrow[r, two heads] & \pi_0 F_n X \\ (F_n X \times_{X_0} F_n X)_0 \arrow[d] & & \\ \pi_0 F_n X \times_{X_0} \pi_0 F_n X \arrow[uurr, "m"'] & & \end{tikzcd}$$

Observe that the map m can be identified with the map

$$\bigsqcup_{x \in X_0} \pi_n(X,x) \times \pi_n(X,x) \to \bigsqcup_{x \in X_0} \pi_n(X,x)$$

that one obtains by collecting all of the ordinary homotopy group multiplication maps together.

The group inverse $\sigma : \pi_n X \to \pi_n X$ is defined as a fibred map over X_0 by letting $\Lambda^{n+1}_{n-1,n+1}$ be the subcomplex of Δ^{n+1} generated by the simplices d^i for $i \neq n-1, n+1$, and forming the pullback

$$\begin{tikzcd} H_n X \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^{n+1}, X) \arrow[d] \\ X_0 \arrow[r] & \mathbf{hom}(\Lambda^{n+1}_{n-1,n+1}, X) \end{tikzcd}$$

The maps d_{n-1} and d_{n+1} induce functions $d_{n-1*}, d_{n+1*} : H_n X \to F_n X$ of spaces fibred over X_0, and both of these maps are surjective because they are induced by trivial fibrations of the form $(d^j)^* : \mathbf{hom}(\Delta^{n+1}, X) \to \mathbf{hom}(\Delta^n, X)$. Then σ is the unique map of sets fibred over X_0 which makes the following diagram commute:

$$\begin{tikzcd} H_nX_0 \arrow[r, two heads, "d_{n+1*}"] \arrow[d, two heads, "d_{n-1*}"'] & F_nX_0 \arrow[r, two heads] & \pi_0F_nX \\ F_nX_0 \arrow[d, two heads] & & \\ \pi_0F_nX \arrow[uurr, "\sigma"'] & & \end{tikzcd}$$

Again, the map σ can be identified with the map

$$\bigsqcup_{x\in X_0} \pi_n(X,x) \to \bigsqcup_{x\in X_0} \pi_n(X,x)$$

which consists of the group inverses for the regular homotopy groups.

The identity $e : X_0 \to \pi_nX$ is the section of the structure map $\pi_nX \to X_0$ which is induced by the canonical section of the simplicial set map $F_nX \to X_0$. Of course e specializes to the map $* \to \pi_n(X,x)$ which picks out the identity map of the group $\pi_n(X,x)$ over each summand of X_0.

The defining axioms for the group structures on the various $\pi_n(X,x)$ can now be used to show that the fibred objects $\pi_nX \to X_0$, together with the multiplication map m, the inverse map σ and the identity section e, give π_nX the structure of a group object in the category of sets fibred over X_0. This group object is abelian if $n \geq 2$. The existence of the group object isn't news by itself, but the descriptions of the maps m, σ and e are combinatorial and functorial, and are therefore more broadly applicable.

Observe that a map $f : X \to Y$ of Kan complexes is a weak equivalence if and only if

(1) the induced map $f_* : \pi_0X \to \pi_0Y$ of path components is a bijection, and

(2) the induced diagrams

$$\begin{tikzcd} \pi_nX \arrow[r, "f_*"] \arrow[d] & \pi_nY \arrow[d] \\ X_0 \arrow[r, "f"'] & Y_0 \end{tikzcd}$$

are pullbacks for $n \geq 1$.

This is easily verified, given that the displayed group objects consist of ordinary homotopy groups.

Suppose that Y is a simplicial presheaf, and define a presheaf π_0^pY by forming the coequalizer diagram

$$Y_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y_0 \xrightarrow{c} \pi_0^pY$$

in the presheaf category. Let π_0Y denote the associated sheaf for π_0^p; one oftens says that π_0Y is the *sheaf of path components* of Y. Observe that the canonical map $Y \to L^2Y$ from Y to its associated sheaf induces an isomorphism $\pi_0Y \cong \pi_0L^2Y$.

Lemma 25. *Suppose that X is a locally fibrant simplicial sheaf on a complete Boolean algebra $\mathcal{B}$. Then the associated sheaf map*

$$\eta : \pi_0^p X \to \pi_0 X$$

is an isomorphism of presheaves.

Proof: The locally fibrant simplicial sheaf X is a presheaf of Kan complexes, by Lemma 4. It follows that the canonical presheaf map

$$X_1 \xrightarrow{(d_1,d_0)} X_0 \times_{\pi_0^p X} X_0$$

is a pointwise epimorphism.

Form the comparison diagram

$$\begin{tikzcd}
X_1 \arrow[r, two heads, "{(d_1,d_0)}"] \arrow[dr, dashed] & X_0 \times_{\pi_0^p X} X_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\eta"'] & X_0 \arrow[r, "c"] \arrow[d, "1_{X_0}"'] & \pi_0^p X \arrow[d, "\eta"'] \\
& X_0 \times_{\pi_0 X} X_0 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r, "L^2 c"'] & \pi_0 X
\end{tikzcd}$$

The bottom sequence is a coequalizer in the sheaf category, while the top sequence is a coequalizer in the presheaf category.

The sheaf epimorphism L^2c is a pointwise epimorphism, by the axiom of choice (Proposition 2), so that the canonical presheaf map $\eta : \pi_0^p X \to \pi_0 X$ is also a pointwise epi. The composite map displayed by the dotted arrow can be identified with the sheaf map associated to the presheaf epimorphism (d_0, d_1), so it's a sheaf epi and hence a pointwise epi, again by the axiom of choice. If $L^2c(x) = L^2c(y)$ in $\pi_0 X$, then (x, y) defines an element of $X_0 \times_{\pi_0 X} X_0$, and so there is a section z of X_1 which maps to (x, y) under the dotted composite. But then $x = d_1 z$ and $y = d_0 z$, so that x and y represent the same element of $\pi_0^p X$. The associated sheaf map $\eta : \pi_0^p X \to \pi_0 X$ is therefore pointwise monic as well as pointwise epi. □

Suppose that X is a locally fibrant simplicial sheaf on the site $\mathcal{C}$. The homotopy group sheaves $\pi_n X \to X_0$ are defined as sheaves fibred over the sheaf of vertices X_0 by letting $F_n X$ be the locally fibrant simplicial sheaf defined by the pullback diagram

$$\begin{tikzcd}
F_n X \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\
X_0 \arrow[r] & \mathbf{hom}(\partial\Delta^n, X),
\end{tikzcd}$$

and then by defining $\pi_n X = \pi_0 F_n X$, where the latter denotes the *sheaf* of path components of the simplicial sheaf $F_n X$, as above. The group object multiplication $m : \pi_n X \times_{X_0} \pi_n X \to \pi_n X$ is defined by analogy with the group object multiplication for Kan complexes: define a locally fibrant simplicial sheaf $G_n X$ by requiring that the diagram

$$\begin{tikzcd}
G_n X \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^{n+1}, X) \arrow[d, "j^*"] \\
X_0 \arrow[r] & \mathbf{hom}(K_n, X)
\end{tikzcd}$$

is a pullback, and then consider the resulting diagram

$$\begin{tikzcd}
R \arrow[d, shift left] \arrow[d, shift right] & & \\
G_nX_0 \arrow[r, "d_{n*}"] \arrow[d, two heads, "(d_{n-1},d_{n+1})_*"'] & F_nX_0 \arrow[r, two heads] & \pi_0F_nX \\
(F_nX \times_{X_0} F_nX)_0 \arrow[d, two heads] & & \\
\pi_0F_nX \times_{X_0} \pi_0F_nX \arrow[uurr, dashed, "m"'] & &
\end{tikzcd}$$

We haven't exactly shown that the morphism m exists yet, but the indicated morphisms $R \rightrightarrows G_nX_0$ are supposed to denote the kernel pair of the composite sheaf epimorphism

$$G_nX_0 \to \pi_0F_nX \times_{X_0} \pi_0F_nX.$$

A unique morphism $m : \pi_0F_nX \times_{X_0} \pi_0F_nX \to \pi_0F_nX$ exists and makes the diagram commute if it can be shown that the horizontal composite $G_nX_0 \to \pi_0F_nX$ equalizes the arrows $R \rightrightarrows G_nX_0$ in the sense that it gives the same result when composed with each of them. This is shown by applying the Boolean localization functor $\wp^*$. This functor commutes with the constructions π_0, F_n and G_n, and $\wp^*X$ is a presheaf of Kan complexes. There is an isomorphism $\pi_0F_n\wp^*X \cong \pi_0^pF_n\wp^*X$ by Lemma 25, so that the ordinary group object structure on the presheaves of homotopy group objects for the presheaf of Kan complexes $\wp^*X$ determines a map

$$m : \pi_0F_n\wp^*X \times_{\wp^*X_0} \pi_0F_n\wp^*X \to \pi_0F_n\wp^*X.$$

In other words, applying the functor $\wp^*$ to the map $G_nX_0 \to \pi_0F_nX$ gives a morphism which equalizes the induced maps $\wp^*R \rightrightarrows \wp^*G_nX_0$. The functor $\wp^*$ is faithful, so that $G_nX_0 \to \pi_0F_nX$ equalizes the morphisms $R \rightrightarrows G_nX_0$, and the multiplication map m is defined uniquely.

The inverse map $\sigma : \pi_0F_nX \to \pi_0F_nX$ of sheaves over X_0 exists by a completely analogous argument, and the identity $e : X_0 \to \pi_0F_nX$ is a canonical section. Finally, the maps m, σ and e define a group object structure on $\pi_0F_nX = \pi_nX \to X_0$: just use the fact that $\wp^*$ is faithful (and exact) again, together with the observation that the corresponding group object structure for $\pi_n\wp^*X$ already exists, since $\wp^*X$ is a presheaf of Kan complexes, and the sheaves of homotopy groups for $\wp^*X$ coincide with their underlying presheaves.

LEMMA 26. *A map $f : X \to Y$ of simplicial presheaves on a site $\mathcal{C}$ is a weak equivalence if and only if*

(1) *the induced map*

$$f_* : \pi_0L^2Ex^\infty X \to \pi_0L^2Ex^\infty Y$$

is an isomorphism of sheaves, and

(2) *the diagrams*

$$\begin{tikzcd} \pi_n L^2 Ex^\infty X \arrow[r, "f_*"] \arrow[d] & \pi_n L^2 Ex^\infty Y \arrow[d] \\ (L^2 Ex^\infty X)_0 \arrow[r, "f_*"'] & (L^2 Ex^\infty Y)_0 \end{tikzcd}$$

are pullbacks for $n \geq 1$.

Proof: The map f is a local weak equivalence if and only if the induced map $f_* : L^2Ex^\infty X \to L^2Ex^\infty Y$ is a local weak equivalence, so it's enough to show that a map $f : X \to Y$ of locally fibrant simplicial sheaves on $\mathcal{C}$ is a weak equivalence if and only if the map

$$f_* : \pi_0 X \to \pi_0 Y$$

is a sheaf isomorphism, and all of the diagrams

$$\begin{tikzcd} \pi_n X \arrow[r, "f_*"] \arrow[d] & \pi_n Y \arrow[d] \\ X_0 \arrow[r, "f_*"'] & Y_0 \end{tikzcd}$$

are pullbacks. The Boolean localization functor $\wp^*$ reflects isomorphisms and pullbacks, so that these conditions are equivalent to the assertions that

$$\wp^* f_* : \pi_0 \wp^* X \to \pi_0 \wp^* Y$$

is a sheaf isomorphism, and all diagrams

$$\begin{tikzcd} \pi_n \wp^* X \arrow[r, "\wp^* f_*"] \arrow[d] & \pi_n \wp^* Y \arrow[d] \\ \wp^* X_0 \arrow[r, "\wp^* f_*"'] & \wp^* Y_0 \end{tikzcd}$$

are pullbacks. The simplicial sheaves $\wp^* X$ and $\wp^* Y$ are presheaves of Kan complexes, and their associated presheaves of homotopy group objects coincide with the respective sheaves of homotopy group objects, so these last conditions are jointly equivalent to the assertion that $\wp^* f : \wp^* X \to \wp^* Y$ is a pointwise weak equivalence. □

We can now give our independence result:

Theorem 27. *Suppose that $\mathcal{C}$ is an arbitrary Grothendieck site. Say that a cofibration of simplicial presheaves on $\mathcal{C}$ is a pointwise monomorphism, a local weak equivalence is a map satisfying the conditions of Lemma 26, and a global fibration is a map which has the right lifting property with respect to all maps which are simultaneously cofibrations and local weak equivalences. Then, with these definitions, the categories $\mathbf{S}Pre(\mathcal{C})$ and $\mathbf{S}Shv(\mathcal{C})$, respectively, of simplicial presheaves and simplicial sheaves on the site $\mathcal{C}$ satisfy the axioms for a proper closed simplicial model category. Furthermore, the associated sheaf functor induces an equivalence*

$$Ho(\mathbf{S}Pre(\mathcal{C})) \simeq Ho(\mathbf{S}Shv(\mathcal{C}))$$

between the associated homotopy categories. □

Suppose that U is an object of the site $\mathcal{C}$ and that x is a vertex of the simplicial set $X(U)$, where X is a locally fibrant simplicial sheaf. Then there is a pullback diagram

$$\begin{tikzcd}
\pi_n(X|_U, x)(U) \arrow[r] \arrow[d] & \pi_n X(U) \arrow[d] \\
* \arrow[r, "x"'] & X_0(U)
\end{tikzcd}$$

where $\pi_n(X|_U, x)$ is the n^{th} ordinary sheaf of homotopy groups for the restricted simplicial sheaf $X|_U$ on the site $\mathcal{C} \downarrow U$ of objects over U, based at the global section x. Also if $f : X \to Y$ is a map of locally fibrant simplicial sheaves, then the diagram

$$\begin{tikzcd}
\pi_n X \arrow[r, "f_*"] \arrow[d] & \pi_n Y \arrow[d] \\
X_0 \arrow[r, "f_*"'] & Y_0
\end{tikzcd}$$

is a pullback if and only if all of the induced maps

$$\pi_n(X|_U, x) \xrightarrow{f_*} \pi_n(Y|_U, f(x))$$

are isomorphisms for $U \in \mathcal{C}$, $x \in X_0(U)$, so the definition of local weak equivalence given here coincides with the standard form.

Remark 28. There is another, much easier, way to see the independence result for the closed model structure on $\mathbf{S}Pre(\mathcal{C})$. The key point is a combination of Lemma 7, Lemma 9, and Lemma 11: if $f : X \to Y$ is a map of $\mathbf{S}Pre(\mathcal{C})$, then there is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "\nu"] \arrow[d, "f"'] & Ex^\infty X \arrow[d, "Ex^\infty f"'] \arrow[dr, "i"] & \\
& & Z \arrow[dl, "q"] \\
Y \arrow[r, "\nu"'] & Ex^\infty Y &
\end{tikzcd}$$

where i is a pointwise weak equivalence and a cofibration, and q is a pointwise Kan fibration. The maps ν are pointwise weak equivalences, so Lemma 9 says that f is a local weak equivalence if and only if the pointwise Kan fibration q is a local weak equivalence. The objects Z and $Ex^\infty Y$ are presheaves of Kan complexes, so that one infers from Lemma 7 and Lemma 11 that q is a local weak equivalence if and only if it has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$. This local right lifting property is an internal criterion for simplicial presheaves on the site $\mathcal{C}$, and is independent of any Boolean localization.

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Mathematics Department
University of Western Ontario
London, Ont., N6A 5B7
CANADA
jardine@jardine.math.uwo.ca