Manis Valuations and Prüfer Extensions I

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Abstract. We call a commutative ring extension $A \subset R$ Prüfer, if A is an R-Prüfer ring in the sense of Griffin (Can. J. Math. 26 (1974)). These extensions relate to Manis valuations in much the same way as Prüfer domains to Krull valuations. We develop a basic theory of Prüfer extensions and give some examples. In the introduction we try to explain why Prüfer extensions deserve interest from a geometric viewpoint.

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Contents

Introduction

If F is a formally real field then it is well known that the intersection of the real valuation rings of F is a Prüfer domain H, and that H has the quotient field F. {A valuation ring is called real if its residue class field is formally real.} H is the so called *real holomorphy ring* of F, cf. [B, §2], [S], [KS, Chap.III §12]. If F is the function field $k(V)$ of an algebraic variety V over a real closed field k (e.g. $k = \mathbb{R}$), suitable overrings of H in R can tell us a lot about the algebraic and the semi-algebraic geometry of $V(k)$.

These rings, of course, are again Prüfer domains. A very interesting and – to our opinion – still mysterious role is played by some of these rings which are related to the orderings of higher level of F, cf. e.g. [B_2], [B_3]. Here we meet a remarkable phenomenon. For orderings of level 1 (i.e. orderings in the classical sense) the usual procedure is to observe first that the convex subrings of ordered fields are valuation rings, and then to go on to Prüfer domains as intersections of such valuation rings, cf. e.g. [B], [S], [KS]. But for higher levels, up to now, the best method is, to construct directly a Prüfer domain A in F from a "torsion reordering" of F, and then to obtain the valuation rings necessary for analyzing the reordering as localizations $A_{\mathfrak{p}}$ of A, cf. [B_2, p.1956 f], [B_3]. Thus there is a two way traffic between valuations and Prüfer domains.

Less is done up to now for F the function field $k(V)$ of an algebraic variety V over a p-adically closed field k (e.g. $k = \mathbb{Q}_p$). But work of Kochen and Roquette (cf. §6 and §7 in the book [PR] by Prestel and Roquette) gives ample evidence, that also here Prüfer domains play a prominent role. In particular, every formally p-adic field F contains a "*p-adic holomorphy ring*", called the *Kochen ring*, in complete analogy to the formally real case [PR, §6]. Actually the Kochen ring has been found and studied much earlier than the real holomorphy ring ([Ko], [R_1]).

If R is a commutative ring (with 1) and k is a subring of R then we can still define a real holomorphy ring $H(R/k)$ consisting of those elements a of R which on the real spectrum of R (cf. [BCR], [B_1], [KS]) can be bounded by elements of k. {If R is a formally real field F and k the prime ring of F this coincides with the real holomorphy ring H from above}. These rings $H(R/k)$ have proved to be very useful in real semi-algebraic geometry. In particular, N. Schwartz and M. Prechtel have used them in order to complete a real closed space and, more generally, to turn a morphism between real closed spaces into a proper one in a universal way ([Sch, Chap V, §7], [Pt]).

The algebra of these holomorphy rings turns out to be particularly good natured if we assume that $1 + \Sigma R^2 \subset R^*$, i.e. that all elements $1 + a_1^2 + \cdots + a_n^2$ $(n \in \mathbb{N}, a_i \in R)$ are units in R. This is a natural condition in real algebra. The rings used by Schwartz and Prechtel, consisting of abstract semi-algebraic functions, fulfill the condition automatically. More generally, if A is any commutative ring (always with 1) then the localization $S^{-1}A$ with respect to the multiplicative set $S = 1 + \Sigma A^2$ is a ring R fulfilling the condition, and R has the same real spectrum as A. Thus for many problems in real geometry we may replace A by R.

Recently V. Powers has proved that, if $1 + \Sigma R^2 \subset R^*$, the real holomorphy ring

$H(R/k)$ with respect to any subring k is an R-Prüfer ring, as defined by Griffin in 1973 [G_2].*) More generally V. Powers proved that, if $1 + \Sigma R^{2d} \subset R^*$ for some even number $2d$, every subring A of R containing the elements $\frac{1}{1+q}$ with $q \in \Sigma R^{2d}$ is R-Prüfer ([P, Th.1.7], cf. also [BP]).

An R-Prüfer ring is related to Manis valuations on R in much the same way as a Prüfer domain is related to valuations of its quotient field. Why shouldn't we try to repeat the success story of Prüfer domains and real valuations on the level of relative Prüfer rings and Manis valuations? Already Marshall in his important paper [Mar] has followed such a program. He has worked there with "Manis places" in a ring R with $1 + \Sigma R^2 \subset R^*$, and has related them to the points of the real spectrum Sper$R$.

We mention that Marshall's notion of Manis places is slightly misleading. By his definition these places do not correspond to Manis valuations but to a broader class of valuations which we call "special valuations", cf. §1 of the present paper. But then V. Powers (and independently one of us, D.Z.) observed that, in the case $1+\Sigma R^2 \subset R^*$, the places of Marshall in fact do correspond to the Manis valuations of R [P]. {In §1 of the present paper we prove that every special valuation of R is Manis under a much weaker condition on R, cf. Theorem 1.1.}

The program to study Manis valuations and relative Prüfer rings in rings of real functions has gained new impetus and urgency from the fact, that the theory of orderings of higher level has recently been pushed from fields to rings leading to *real spectra of higher level.* These spectra in turn have already proved to be useful for ordinary real semi-algebraic geometry. We mention an opus magnum by Ralph Berr [Be], where spectra of higher level are used in a fascinating way to classify the singularities of real semi-algebraic functions.

p-adic semi-algebraic geometry seems to be accessible as well. L. Bröcker and H.-J. Schinke have brought the theory of p-adic spectra to a rather satisfactory level by studying the "L-spectrum" L-spec A of a commutative ring A with respect to a given non-Archimedean local field L (e.g. $L = \mathbb{Q}_p$). There seems to be no major obstacle in sight which prevents us from defining and studying rings of semialgebraic functions on a constructible (or even pro-constructible) subset X of L-spec A. Here "semi-algebraic" means definability in a model theoretic sense plus a suitable continuity condition. Relative Prüfer subrings of such rings should be quite interesting.

The present paper is the first version of Chapter I of a book in preparation, devoted to a study of relative Prüfer rings and Manis valuations, with an eye to applications in real and p-adic geometry. In this chapter we present the basic theory and some examples.

Now, there exists already a rich theory of "*Prüfer rings with zero divisors*" also started by Griffin [G_1], cf. the books [LM], [Huc], and the literature cited there. But this theory seems not to be tailored to geometric needs. A Prüfer ring with zero divisors A is the same as an R-Prüfer ring with $R = \operatorname{Quot} A$, the total quotient ring of A. While this is a reasonable notion from the viewpoint of ring theory it may be artificial from a geometric viewpoint. A typical situation in real geometry is the following. R

*) The definition by Griffin needs a slight modification, cf. Def.1 in §5 below.

is the ring of (continuous) semialgebraic functions on a semialgebraic set M over a real closed field k or, more generally, the set of abstract semialgebraic functions on a pro-constructible subset X of a real spectrum (cf. [Sch], [Sch_1]). Although the ring R has very many zero divisors we have experience that in some sense R behaves nearly as well as a field, cf. e.g. our notion of "convenient ring extensions" in §6 of the present paper. Now, if A is a subring of R, then it is natural and interesting from a geometric viewpoint to study the R-Prüfer rings $B \supset A$, while the total quotient rings Quot$A$ and Quot$B$ seem to bear little geometric relevance.

Except in a paper by P.L. Rhodes from 1991 [Rh] very little seems to be done on relative Prüfer rings in general, and in the original paper of Griffin the proofs of important facts [G_2, Prop.6, Th.7] are omitted. Moreover the paper by Rhodes has a gap in the proof of his main theorem. {[Rh, Th.2.1], condition (5b) there is apparently not a characterization of Prüfer extensions. Any algebraic field extension is a counterexample.} Thus we have been careful about a foundation of this theory.

In §1 and §2 we gather what we need about Manis valuations. Then in §3 and §4 we develop an auxiliary theory of "weakly surjective" ring homomorphisms. These form a class of epimorphisms in the category of commutative rings close to the flat epimorphisms studied by D. Lazard and others in the sixties, cf. [L], [Sa_1], [A]. In §5 the up to then independent theories of Manis valuations and weakly surjective homomorphisms are brought together to study Prüfer extensions. {We call a ring extension $A \subset R$ Prüfer, if A is R-Prüfer in the sense of Griffin.} It is remarkable that, although Prüfer extensions are defined in terms of Manis valuations (cf. §5, Def.1 below), they can be characterized entirely in terms of weak surjectivity. Namely, a ring extension $A \subset R$ is Prüfer iff every subextension $A \subset B$ is weakly surjective (cf. Th.5.2 below). A third way to characterize Prüfer extensions is by multiplicative ideal theory, as we will explicate in Chapter II of our planned book.

Our first major result on Prüfer extensions is Theorem 5.2 giving various characterizations of these extensions which sometimes make it easy to recognize a given ring extension as Prüfer, cf. the examples in §6. We then establish various permanence properties of the class of Prüfer extensions. For example we prove for Prüfer extensions $A \subset B$ and $B \subset C$ that $A \subset C$ is again Prüfer (Th.5.6).

At the end of §5 we prove that any commutative ring A has a universal Prüfer extension $A \subset P(A)$ which we call the *Prüfer hull* of A. Every other Prüfer extension $A \hookrightarrow R$ can be embedded into $A \hookrightarrow P(A)$ in a unique way. The Prüfer rings with zero divisors are just the rings A with $P(A)$ containing the total quotient ring Quot$A$. Prüfer hulls mean new territory leading to many new open questions. We will pursue some of them in later chapters of our planned book.

In §6 we prove theorems which give us various examples of Manis valuations and Prüfer extensions. We illustrate how naturally they come up in algebraic geometry over a field k which is not algebraically closed (§6, Example 5, Th.6.5, Th.6.9), and in real algebraic and semialgebraic geometry (§6, Examples 3 and 10). Perhaps our best result here is Theorem 6.8 giving a far-reaching generalization of an old lemma by A. Dress (cf. [D, Satz 2′]). This lemma states for F a field, in which -1 is not a square, that the subring of F generated by the elements $1/(1+a^2)$, $a \in F$, is Prüfer in F.

Dress's innocent looking lemma seems to have inspired generations of real algebraists (cf. e.g. [La, p.86], [KS, p.163]) and also ring theorists, cf. [Gi_1].

We finally prove in §7 for various Prüfer extensions $A \subset R$ that, if $\mathfrak{a}$ is a finitely generated A-submodule of R with $R\mathfrak{a} = R$, then some power $\mathfrak{a}^d$ (with d specified) is principal. Our main result here (Theorem 7.8) is a generalization of a theorem by P. Roquette [R, Th.1] which states this for R a field (cf. also [Gi_1]). Roquette used his theorem to prove by general principles that the Kochen ring of a formally p-adic field is Bézout [loc.cit]. Similar applications should be possible in p-adic semialgebraic geometry. Roquette's paper has been an inspiration for our whole work since it indicates well the ubiquity of Prüfer domains in algebraic geometry over a non algebraically closed field.

Important topics missing in the present paper are multiplicative ideal theory, the characterization of a given Prüfer extension $A \subset R$ by a suitable lattice of ideals of A, approximation theory for Manis valuations and, finally, the construction of a "Manis valuation spectrum", i.e. a suitable space whose points are the Manis valuations of a given ring R. (One needs a condition on the ring R to establish this spectrum, otherwise one has to be content with the valuation spectrum $\text{Spev}R$, cf. [HK].) We will deal with these topics in later chapters of our planned book. A good deal of multiplicative ideal theory and the characterization business has already been done by Rhodes [Rh].

We have been forced to change some of the terminology used by ring theorists, say in the books of Larsen-McCarthy [LM] and of Huckaba [Huc]. While these authors mean by valuation on a ring a Manis valuation we use the word "valuation" in the much broader sense of Bourbaki [Bo, Chap.VI, §3]. It is true that Manis valuations are the really good ones for computations. But the central notion is the Bourbaki valuation, since only with these valuations one can build an honest spectral space, the valuation spectrum [HK]. Valuation spectra have already proved to be immensely useful both in algebraic geometry (cf. [HK]) and rigid analytic geometry (e.g. [Hu_1], [Hu_2]). The closely related real valuation spectra (cf. [Hu_3, §1]) seem to be the natural basic spaces for endeavors in real algebra concerning valuations and Prüfer extensions.

Some notations. In this paper all rings are commutative with 1. For A a ring we denote the group of units of A by A^*. We denote the total quotient ring of A by $\text{Quot}A$. For $\mathfrak{p}$ a prime ideal of A we denote the field $\text{Quot}(A/\mathfrak{p})$ by $k(\mathfrak{p})$.

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If A and B are sets then $A \subset B$ means that A is a subset of B and $A \subsetneqq B$ means that A is a proper subset of B. If two subsets M and N of some set X are given then $M \setminus N$ denotes the complement of $M \cap N$ in M.

§1 Valuations on rings

Let R be a ring and Γ an (additive) totally ordered Abelian group. We extend Γ to an ordered monoid $\Gamma \cup \infty := \Gamma \cup \{\infty\}$ by the rules $\infty + x = x + \infty = \infty$ for all $x \in \Gamma \cup \infty$ and $x < \infty$ for all $x \in \Gamma$.

DEFINITION 1 (Bourbaki [Bo, VI. 3.1]).
A *valuation on* R with values in Γ is a map $v: R \to \Gamma \cup \infty$ such that:

(1) $v(xy) = v(x) + v(y)$ for all $x, y \in R$.
(2) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in R$.
(3) $v(1) = 0$ and $v(0) = \infty$.

If $v(R) = \{0, \infty\}$ then v is said to be *trivial*, otherwise v is called *non-trivial*.

We recall some very basic facts[1) about valuations on rings and fix notations. Let $v: R \to \Gamma \cup \infty$ be a valuation on R.

The subgroup of Γ generated by $v(R) \setminus \{\infty\}$ is called the *value group* of v and is denoted by Γ_v. The set $v^{-1}(\infty)$ is a prime ideal of R. It is called the *support* of v and is denoted by $\operatorname{supp} v$. v induces a valuation $\hat{v}: k(\operatorname{supp} v) \to \Gamma \cup \infty$ on the quotient field $k(\operatorname{supp} v)$ of $R/\operatorname{supp} v$. We denote by $\mathfrak{o}_v$ the valuation ring of $k(\operatorname{supp} v)$ corresponding to $\hat{v}$, by $\mathfrak{m}_v$ its maximal ideal, and by $\kappa(v)$ its residue class field, $\kappa(v) := \mathfrak{o}_v/\mathfrak{m}_v$.

Notice that $\hat{v}(\mathfrak{o}_v) = (\Gamma_v)_+ \cup \{\infty\}$, where $(\Gamma_v)_+$ denotes the set of nonnegative elements in Γ_v. (We use such a notation for any ordered Abelian group.)

We further denote by A_v the set $\{x \in R \mid v(x) \geq 0\}$ and by $\mathfrak{p}_v$ the set $\{x \in R \mid v(x) > 0\}$. Clearly A_v is a subring of R and $\mathfrak{p}_v$ is a prime ideal of A_v. We call A_v the *valuation ring* of v and $\mathfrak{p}_v$ the *center* of v.

DEFINITION 2. Two valuations v, w on R are said to be *equivalent*, in short, $v \sim w$, if the following equivalent conditions are satisfied:

(1) There is an isomorphism $f: \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ of ordered monoids with $w(x) = f(v(x))$ for all $x \in R$.
(2) $v(a) \geq v(b) \Longleftrightarrow w(a) \geq w(b)$ for all $a, b \in R$.
(3) $\operatorname{supp} v = \operatorname{supp} w$ and $\mathfrak{o}_v = \mathfrak{o}_w$.

By abuse of language we will often regard equivalent valuations as "equal".

DEFINITION 3. a) The *characteristic subgroup* $c_v(\Gamma)$ of Γ with respect to v is the smallest convex subgroup of Γ (convex with respect to the total ordering of Γ) which contains all elements $v(x)$ with $x \in R$, $v(x) \leq 0$. Clearly $c_v(\Gamma)$ is the set of all $\gamma \in \Gamma$ such that $v(x) \leq \gamma \leq -v(x)$ for some $x \in R$ with $v(x) \leq 0$.
b) v is called *special*,[2) if $c_v(\Gamma_v) = \Gamma_v$. (We replaced Γ by Γ_v.)

If H is any convex subgroup of Γ containing $c_v(\Gamma)$ then we obtain from v a new valuation $v|H: R \to \Gamma_\infty$ putting $(v|H)(x) = v(x)$ if $v(x) \in H$ and $v(x) = \infty$ else. Taking $H = c_v\Gamma$ we obtain from v a special valuation $w = v|c_v\Gamma$. Notice that $A_w = A_v$, $\mathfrak{p}_w = \mathfrak{p}_v$.

DEFINITION 4 (cf. [M]). v is called a *Manis valuation* on R, if $v(R) = \Gamma_v \cup \infty$.[3)

1) For this we refer to [Bo, VI.3.1] and [HK, §1]

2) The word "special" alludes to the fact that such a valuation has no proper primary specialization in the valuation spectrum of R, cf. [HK, §1].

3) Since we often identify equivalent valuations we have slightly altered the definition in [M]. Manis demands that $v(R) = \Gamma \cup \infty$.

Manis valuation will be in the focus of the present paper. Notice that every Manis valuation is special, but that the converse is widely false.

Example. Let R be the polynomial ring $k[x]$ in one variable x over some field k. Consider the valuation $v: R \to \mathbb{Z} \cup \infty$ with $v(f) = -\deg f$ for any $f \in R \setminus \{0\}$. This valuation is special but definitely not Manis. □

One of our primary observations is that nevertheless there are many interesting rings, on which every special valuation is Manis. For example this holds if for every $x \in R$ the element $1 + x^2$ is a unit in R. More generally we have the following theorem.

Theorem 1.1. *Let k be a subring of R. Assume that for every $x \in R \setminus k$ there exists some monic polynomial $F(T) \in k[T]$ (one variable T) with $F(x) \in R^*$. Then every special valuation v on R with $A_v \supset k$ is Manis.*

Proof. We may assume that v is non trivial. Let $x \in R$ be given with $v(x) \neq 0, \infty$. We have to find some $y \in R$ with $v(y) = -v(x)$. Since v is special there exists some $a \in R$ with $v(ax) < 0$. Let $F(T) = T^d + c_1 T^{d-1} + \cdots + c_d$ be a polynomial with $c_1, \ldots, c_d \in k$ and $F(ax) \in R^*$. Since $v(ax) < 0$, but $v(c_i) \geq 0$ for $i = 1, \ldots, d$, we have $v(F(ax)) = dv(ax)$. The element $y := \frac{a^d x^{d-1}}{F(ax)}$ does the job.[4] q.e.d.

We return to valuations in general. Up to the end of this section we will keep the following

Notations. *$v: R \to \Gamma \cup \infty$ is a valuation on some ring R, $A := A_v$, $\mathfrak{p} := \mathfrak{p}_v$, $\mathfrak{q} :=$ $\operatorname{supp} v$, $\bar{R} := R/\mathfrak{q}$, $\bar{A} := A/\mathfrak{q}$, $\bar{\mathfrak{p}} := \mathfrak{p}/\mathfrak{q}$. $\pi: R \to \bar{R}$ is the evident epimorphism from R to $\bar{R}$. We have a unique valuation $\bar{v}: \bar{R} \to \Gamma \cup \infty$ on $\bar{R}$ such that $\bar{v} \circ \pi = v$.*

We have $A_{\bar{v}} = \bar{A}$, $\mathfrak{p}_{\bar{v}} = \bar{\mathfrak{p}}$, $\operatorname{supp} \bar{v} = \{0\}$, $\Gamma_{\bar{v}} = \Gamma_v$, $\mathfrak{o}_v = \mathfrak{o}_{\bar{v}}$. It is evident that v is special iff $\bar{v}$ is special, and that v is Manis iff $\bar{v}$ is Manis. Looking at the valuation $\hat{v}$ on the quotient field $k(\mathfrak{q})$ of $\bar{R}$ (which extends $\bar{v}$) one now obtains by an easy exercise

Proposition 1.2.

a) *v is Manis iff $k(\mathfrak{q}) = \bar{R} \cdot \mathfrak{o}_v^*$.*
b) *v is special iff $k(\mathfrak{q}) = \bar{R} \cdot \mathfrak{o}_v$.*

Here $\bar{R} \cdot \mathfrak{o}_v^*$ (resp. $\bar{R} \cdot \mathfrak{o}_v$) denotes the set of products xy with $x \in \bar{R}$, $y \in \mathfrak{o}_v^*$ (resp. $\mathfrak{o}_v$). The set $\bar{R} \cdot \mathfrak{o}_v$ is also the subring of $k(\mathfrak{q})$ generated by $\bar{R}$ and $\mathfrak{o}_v$. □

Definition 5. v is called *local* if the pair $(A, \mathfrak{p})$ is local, i.e. $\mathfrak{p}$ is the unique maximal ideal of A.

Proposition 1.3 (cf. [G$_2$, Prop. 5]). *The following are equivalent.*

i) *v is Manis and local.*
ii) *The pair $(R, \mathfrak{q})$ is local.*
iii) *v is local and $\mathfrak{q}$ is a maximal ideal of R.*

[4] We are indebted to Roland Huber for this simple argument.

Proof. i) $\Rightarrow$ ii): Let $x \in R \setminus \mathfrak{q}$ be given. Since v is Manis there exists some $y \in R$ with $v(xy) = 0$. Since v is local this implies that xy is a unit of A, hence also a unit of R. Thus x is a unit of R.
ii) $\Rightarrow$ i): $\bar{v}$ is a valuation of the field $\bar{R}$. Thus $\bar{v}$ is Manis, which implies that v is Manis. Let $x \in A \setminus \mathfrak{p}$ be given. Then x is a unit in R. We have $v(x^{-1}) = -v(x) = 0$. Thus $x^{-1} \in A$, $x \in A^*$.
i), ii) $\Rightarrow$ iii): trivial.
iii) $\Rightarrow$ i): $\bar{v}$ is a valuation of the field $\bar{R}$. From this we conclude again that v is Manis.
$\square$

If S is any multiplicative subset of R with $S \cap \mathfrak{q} = \emptyset$ then we denote by v_S the unique "extension" of v to a valuation on $S^{-1}R$, defined by

$$v_S\left(\frac{a}{s}\right) = v(a) - v(s) \qquad (a \in R, s \in S).$$

For $w = v_S$ we have $\Gamma_w = \Gamma_v$ and $c_w(\Gamma) \supset c_v(\Gamma)$. Thus if v is Manis then v_S is Manis and if v is special then v_S is special. v_S has the support $S^{-1}\mathfrak{q}$.

We now consider the special case $S = A \setminus \mathfrak{p}$. Then

$$v_S\left(\frac{a}{s}\right) = v(a) \qquad (a \in R, s \in S).$$

Thus for $w = v_S$ we now have $A_w = S^{-1}A = A_\mathfrak{p}$ and $\mathfrak{p}_w = S^{-1}\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$, and we see that v_S is a local valuation. Moreover $A \setminus \mathfrak{p}$ is the smallest saturated multiplicative subset S of R such that v_S is local. We write $S^{-1}R = R_\mathfrak{p}$.

Definition 6. The valuation v_S with $S = A \setminus \mathfrak{p}$ is called *the localization of* v, and is denoted by $\tilde{v}$.

We have $\tilde{v}(R_\mathfrak{p}) = v(R)$, $\Gamma_{\tilde{v}} = \Gamma_v$, $c_v\Gamma = c_{\tilde{v}}\Gamma$. Thus v is Manis iff $\tilde{v}$ is Manis and v is special iff $\tilde{v}$ is special. Applying Proposition 3 [5] to $\tilde{v}$ we obtain

Proposition 1.4. *The following are equivalent.*

i) *v is Manis.*
ii) *$\mathfrak{q}$ is the unique ideal of R which is maximal among all ideals of R which do not meet $A \setminus \mathfrak{p}$.*
iii) *$\mathfrak{q}$ is maximal among all ideals of R which do not meet $A \setminus \mathfrak{p}$.* $\square$

If S is a (non empty) multiplicative subset of R then we denote by $\mathrm{Sat}_R(S)$ the set of all elements of R which divide some element of S ("saturum of S in R"). Recall from basic commutative algebra that, if T is a second multiplicative subset of R, then $S^{-1}R = T^{-1}R$ iff $\mathrm{Sat}_R(S) = \mathrm{Sat}_R(T)$.

The following characterization of Manis valuations can be deduced from Proposition 4, but we will give an independent proof.

[5] Reference to Prop.1.3 in this section. In later sections we will refer to this proposition as "Prop.1.3", instead of "Prop.3".

Proposition 1.5. *The following are equivalent.*

i) *v is Manis.*
ii) $Sat_R(A \setminus \mathfrak{p}) = R \setminus \mathfrak{q}$.
iii) $R_\mathfrak{p} = R_\mathfrak{q}$.

Proof. The multiplicative set $R \setminus \mathfrak{q}$ is saturated. Thus the equivalence ii) $\Longleftrightarrow$ iii) is evident from what has been said above.
i) $\Longleftrightarrow$ ii): v is Manis $\Longleftrightarrow$ For every $x \in R \setminus \mathfrak{q}$ there exists some $y \in R$ with $v(x) + v(y) = 0$, i.e. with $xy \in A \setminus \mathfrak{p} \Longleftrightarrow R \setminus \mathfrak{q} = \mathrm{Sat}_R(A \setminus \mathfrak{p})$. □

Proposition 1.6. *If v is Manis then $\mathfrak{o}_v = \bar{A}_{\bar{\mathfrak{p}}}$.*

Proof. We may pass from v to $\bar{v}$. Thus we assume without loss of generality that $\mathfrak{q} = 0$. We have $\mathfrak{o}_v = \mathfrak{o}_{\tilde{v}}$ and v is Manis iff $\tilde{v}$ is Manis. Thus we may assume without loss of generality that v is also local. Now R is a field (cf. Prop. 3), and $\mathfrak{o}_v = A = A_\mathfrak{p}$. □

Definition 7. We say that v *has maximal support* if $\mathfrak{q}$ is a maximal ideal of R.

Proposition 1.7. *v has maximal support iff $\bar{v}$ is local and Manis. Then v is also a Manis valuation on R.*

Proof. If v has maximal support, then $\bar{v}$ is a valuation on the field $\bar{R}$. Thus $\bar{v}$ is certainly Manis and local. Since $\bar{v}$ is Manis, also v is Manis.

If $\bar{v}$ is local and Manis then, applying Proposition 3 to $\bar{v}$, we learn that the pair $(\bar{R}, \{0\})$ is local. This means that $\mathfrak{q}$ is a maximal ideal of R. □

Definition 8. An additive subgroup M of R is called *v-convex*, if for any elements $x \in M$, $y \in R$ with $v(x) \leq v(y) (\leq v(0) = \infty)$ it follows that $y \in M$.

If M is a v-convex additive subgroup of R, then certainly $ax \in M$ for any $a \in A$, $x \in M$, i.e. M is an A-submodule of R. We now have a closer look at the v-convex ideals of A.

Clearly $\mathfrak{q}$ is a v-convex ideal of A and is contained in any other v-convex ideal of A. Also $\mathfrak{p}$ is v-convex and $I \subset \mathfrak{p}$ for every v-convex ideal $I \neq A$.

Proposition 1.8. *If v has maximal support then every A-submodule of R containing $\mathfrak{q}$ is v-convex.*

Proof. Let I be an A-submodule of R containing $\mathfrak{q}$, and $\bar{I} := I/\mathfrak{q}$. It is easy to see that I is v-convex iff $\bar{I}$ is $\bar{v}$-convex. Since v has maximal support, $\bar{v}$ is a valuation on the field $\bar{R} := R/\mathfrak{q}$. From classical valuation theory we conclude that $\bar{I}$ is $\bar{v}$-convex. □

Corollary 1.9. *If v is a local Manis valuation then every A-submodule of R containing $\mathfrak{q}$ is v-convex.*

Proof. By Proposition 3 we know that v has maximal support. □

PROPOSITION 1.10. [M, Prop. 3]. *Assume that the valuation v is Manis. Then a prime ideal $\mathfrak{r}$ of A is v-convex iff $\mathfrak{q} \subset \mathfrak{r} \subset \mathfrak{p}$.*

PROOF. Replacing v by $\bar{v}$ we assume without loss of generality that $\mathfrak{q} = 0$. Since $v(A \setminus \mathfrak{p}) = \{0\}$ it is evident that the v-convex prime ideals $\mathfrak{r}$ of A correspond uniquely with the $\tilde{v}$-convex prime ideals $\mathfrak{r}'$ of $A_{\mathfrak{p}}$ via $\mathfrak{r}' = \mathfrak{r}_{\mathfrak{p}}$. Thus we may pass from v to $\tilde{v}$ and assume without loss of generality that v is local. All prime ideals (in fact, all ideals) of A are v-convex (Cor. 9). q.e.d.

PROPOSITION 1.11. *Assume that v is a non trivial Manis valuation. The following are equivalent.*

i) *Every ideal I of A with $\mathfrak{q} \subset I \subset \mathfrak{p}$ is v-convex.*
ii) *Any two ideals I, J of A with $\mathfrak{q} \subset I \subset \mathfrak{p}$ and $\mathfrak{q} \subset J \subset \mathfrak{p}$ are comparable by inclusion.*
iii) *$\bar{A}$ is a (Krull)valuation domain.*
iv) *$\mathfrak{p}$ is the unique maximal ideal of A which contains $\mathfrak{q}$.*
v) *v has maximal support.*
vi) *Every ideal I of A containing $\mathfrak{q}$ is v-convex.*

PROOF. We assume without loss of generality that $\mathfrak{q} = \{0\}$. Now R is an integral domain.

i) $\Rightarrow$ ii) is evident, since for any two v-convex ideals I and J of A we have $I \subset J$ or $J \subset I$. (This holds more generally for v-convex additive subgroups I, J of R.)

ii) $\Rightarrow$ iii): We verify: If $x \in A$, $y \in A$ then $Ax \subset Ay$ or $Ay \subset Ax$. This will imply that A is a valuation domain. We assume without loss of generality that $v(x) \le v(y)$. If $x \in \mathfrak{p}$ then also $y \in \mathfrak{p}$. The ideals Ax and Ay are comparable by our assumption ii). There remains the case that $x \notin \mathfrak{p}$. We choose an element $c \neq 0$ in $\mathfrak{p}$. Then $xc \in \mathfrak{p}$ and $v(xc) \le v(yc)$. As we have proved this implies $Ayc \subset Axc$ or $Axc \subset Ayc$. Since R is a domain we conclude that $Ay \subset Ax$ or $Ax \subset Ay$.

iii) $\Longrightarrow$ iv): trivial. iv) $\Longrightarrow$ v) is evident by Proposition 7, and v) $\Longrightarrow$ vi) is evident by Proposition 8. Clearly vi) $\Rightarrow$ i). □

DEFINITION 9. A valuation $w: R \to \Gamma' \cup \infty$ is called *coarser* than v (or a *coarsening* of v) if there exists an order preserving homomorphism[6] $f: \Gamma_v \to \Gamma_w$ such that, for all $x \in R$, $w(x) = f(v(x))$ (put $f(\infty) = \infty$).

If H is a convex subgroup of Γ then the quotient Γ/H is a totally ordered Abelian group in such a way that the natural projection from Γ to Γ/H is an order preserving homomorphism. We have $(\Gamma/H)_+ = (\Gamma_+ + H)/H$. From v we obtain a coarsening $w: R \to (\Gamma/H) \cup \infty$ putting $w(x) := x + H$ for all $x \in R$. (Read $\infty + H = \infty$.) This valuation w is denoted by v/H.

REMARKS 1.12. a) v/H has the center $\mathfrak{p}_H := \{x \in R \mid v(x) > H\}$, and this is a v-convex prime ideal of A. $\{v(x) > H$ means $v(x) > \gamma$ for every $\gamma \in H\}$. If $\Gamma_+ \subset v(R)$

[6] This means f is a homomorphism of Abelian groups with $f(\alpha) \ge f(\beta)$ if $\alpha \ge \beta$. The homomorphism f is necessarily surjective.

(e.g. v is Manis and $\Gamma = \Gamma_v$) then the v-convex prime ideals $\mathfrak{r}$ of A correspond uniquely with the convex subgroups H of Γ via $\mathfrak{r} = \mathfrak{p}_H$.

b) Assume (without loss of generality) that $\Gamma = \Gamma_v$. The coarsenings w of v correspond, up to equivalence, uniquely with the convex subgroups H of Γ via $w = v/H$. We have $A \subset A_w$, $\mathfrak{p} \supset \mathfrak{p}_w$, $\operatorname{supp} w = \mathfrak{q}$, $\hat{w} = \hat{v}/H$, $\bar{w} = \bar{v}/H$, $\tilde{w} = (\tilde{v}/H)^\sim$. If S is a multiplicative subset of R with $S \cap \mathfrak{q} = \emptyset$ then $v_S/H = (v/H)_S$. If v is special then v/H is special. If v is Manis then v/H is Manis.

All this is either trivial or can be verified in a straightforward way.

How do we obtain the ring A_w from $A_v = A$ if $w = v/H$? In order to give a satisfactory answer, at least in special cases, we need a definition which will be widely used also later on.

Definition 10. Let B be a subring of R, let S be a multiplicative subset of B and let $j_S\colon R \to S^{-1}R$ denote the localization map $x \mapsto \frac{x}{1}$ of R with respect to S. For any B-submodule M of R we define

$$M_{[S]} := j_S^{-1}(S^{-1}M).$$

Clearly $M_{[S]}$ is the set of all $x \in R$ such that $sx \in M$ for some $s \in S$. We call $M_{[S]}$ the *saturation of* M *(in* R*) by* S.[7] In the case $S = B \setminus \mathfrak{r}$ with $\mathfrak{r}$ a prime ideal of B we usually write $j_\mathfrak{r}$ and $M_{[\mathfrak{r}]}$ instead of j_S, $M_{[S]}$.

Notice that $B_{[S]}$ is a subring of R and $M_{[S]}$ is a $B_{[S]}$-submodule of R. If M is an ideal of B then $M_{[S]}$ is an ideal of $B_{[S]}$. If M is a prime ideal of B with $M \cap S = \emptyset$ then $M_{[S]}$ is a prime ideal of $B_{[S]}$.

Proposition 1.13. *Let S be a multiplicative subset of $A \setminus \mathfrak{q}$, and let H denote the convex subgroup of Γ generated by $v(S)$, i.e. the smallest convex subgroup of Γ containing $v(S)$. Let $w := v/H$ and $\mathfrak{r} := \mathfrak{p}_H$. Then*

$$A_w = A_{[S]} = A_{[\mathfrak{r}]},$$
$$\mathfrak{p}_w = \mathfrak{r} = \{x \in R \mid v(x) > v(S)\}.$$

Proof. We already stated above that $\mathfrak{p}_w = \mathfrak{p}_H = \mathfrak{r}$. This ideal coincides with the set of all $x \in R$ with $v(x) > v(S)$. It is evident that $A_{[S]} \subset A_w$. Let now $x \in A_w$ be given. There exists some element $\gamma \in H_+$ with $v(x) \geq -\gamma$, and some element $s \in S$ with $\gamma \leq v(s)$. We obtain $v(xs) \geq 0$, i.e. $xs \in A$. This proves that $A_w = A_{[S]}$. We have $S \subset A \setminus \mathfrak{r}$, thus $A_{[S]} \subset A_{[\mathfrak{r}]}$. Let $x \in A_{[\mathfrak{r}]}$ be given. We choose $y \in A \setminus \mathfrak{r}$ with $xy \in A$. There exists some $\gamma \in H_+$ with $v(y) \leq \gamma$ and some $s \in S$ with $\gamma \leq v(s)$. We have

$$0 \leq v(x) + v(y) \leq v(x) + v(s) = v(sx).$$

Thus $sx \in A$, $x \in A_{[S]}$. This proves $A_{[S]} = A_{[\mathfrak{r}]}$. q.e.d.

[7] $M_{[S]}$ is called the "S-component of M" in [LM].

REMARK. The converse of Proposition 13 for the case of non-trivial Manis valuations is also true (Th.2.6.ii).

COROLLARY 1.14. *Assume that* $\Gamma_+ \subset v(R)$ *(e.g.* v *Manis and* $\Gamma_v = \Gamma$*). Let* H *be a convex subgroup of* Γ, $w := v/H$ *and* $\mathfrak{r} := \mathfrak{p}_H$. *We have* $A_w = A_{[\mathfrak{r}]}$ *and* $\mathfrak{p}_w = \mathfrak{r}$.

PROOF. Apply Prop. 13 to the set $S := \{x \in R \mid v(x) \in H_+\}$. $\square$

PROPOSITION 1.15. *Let* I *be an* A*-submodule of* R *with* $\mathfrak{q} \subset I$. *Assume that* v *is Manis. Then* I *is* v*-convex iff* $I = I_{[\mathfrak{p}]}$.

PROOF. Assume first that I is v-convex. We have $I \subset I_{[\mathfrak{p}]}$. Let $x \in I_{[\mathfrak{p}]}$ be given. We choose $d \in A \setminus \mathfrak{p}$ with $dx \in I$. We have $v(x) = v(dx)$. Since I is v-convex this implies $x \in I$. Thus $I = I_{[\mathfrak{p}]}$.

Assume now that $I = I_{[\mathfrak{p}]}$. This means $I = j_{\mathfrak{p}}^{-1}(I_{\mathfrak{p}})$ with $j_{\mathfrak{p}}$ the localization map from R to $R_{\mathfrak{p}}$. As always let $\tilde{v}: R_{\mathfrak{p}} \to \Gamma \cup \infty$ denote the localization of v. We have $A_{\tilde{v}} = A_{\mathfrak{p}}$, $\operatorname{supp} \tilde{v} = \mathfrak{q}_{\mathfrak{p}}$. Since $\tilde{v}$ is local, every $A_{\mathfrak{p}}$-submodule of $R_{\mathfrak{p}}$ containing $\mathfrak{q}_{\mathfrak{p}}$ is $\tilde{v}$-convex (Cor. 1.9). In particular $I_{\mathfrak{p}}$ is $\tilde{v}$-convex. Since $I = j_{\mathfrak{p}}^{-1}(I_{\mathfrak{p}})$ and $v = \tilde{v} \circ j_{\mathfrak{p}}$ we conclude that I is v-convex. $\square$

We briefly discuss a process of restriction which gives us special valuations on subrings of R.

Let B be a subring of R. The restriction $u = v|B: B \to \Gamma \cup \infty$ of the map $v: R \to \Gamma \cup \infty$ is a valuation on B. Let $\Delta := c_u(\Gamma)$ and $w := u|\Delta$. Then $w: B \to \Delta \cup \infty$ is a special valuation on B.

DEFINITION 11. We call w the *special restriction* of v to B, and denote this valuation by $v|_B$.

For $w = v|_B$ we have $A_w = A \cap B$, $\mathfrak{p}_w = \mathfrak{p} \cap B$, $\operatorname{supp} w \supset \mathfrak{q} \cap B$. Notice also that $v|_B = (v|c_v\Gamma)|_B$. Thus in essence our restriction process deals with special valuations.

In the case that v is Manis the question arises, under which conditions on B the special restriction $v|_B$ is again Manis. We need an easy lemma.

LEMMA 1.16. *If* $v: R \to \Gamma \cup \infty$ *is special and* $(\Gamma_v)_+ \subset v(R)$, *then* v *is Manis.*

PROOF. This is a consequence of Proposition 2. By that proposition $k(\mathfrak{q}) = \bar{R}\mathfrak{o}_v$. From $(\Gamma_v)_+ \subset v(R) = \bar{v}(\bar{R})$ we conclude that $\mathfrak{o}_v \subset \bar{R}\mathfrak{o}_v^*$, hence $k(\mathfrak{q}) = \bar{R}\mathfrak{o}_v^*$, and this means that v is Manis. $\square$

PROPOSITION 1.17. *Assume that* v *is Manis and that* B *is a subring of* R *containing* $\mathfrak{p} = \mathfrak{p}_v$. *Then the special restriction* $v|_B: B \to \Delta \cup \infty$ *of* v *is again Manis. If* v *is surjective (i.e.* $\Gamma = \Gamma_v$*) then* $v|_B$ *is surjective.*

PROOF. We assume without loss of generality that v is surjective. Let $u := v|B$ and $w := v|_B$. Let $\gamma \in \Delta$ be given with $\gamma > 0$. There exists some $a \in \mathfrak{p}_v$ with $v(a) = \gamma$. Since $\mathfrak{p}_v \subset B$ we have $a \in B$, hence $v(a) = u(a) = w(a)$. {Recall that for any $x \in B$

with $u(x) \in \Delta$ we have $w(x) = u(x)$.} This proves that $\Delta_+ \subset w(B)$. By the lemma w is Manis. □

Scholium 1.18. Let $v: R \to \Gamma \cup \infty$ be a Manis valuation and H a convex subgroup of Γ. Let $w := v/H$ and $B := A_w$. We have

$$\begin{aligned} A_w &= \{x \in R \mid v(x) \geq h \quad \text{for some} \quad h \in H\} =: A_H \\ \mathfrak{p}_w &= \{x \in R \mid v(x) > h \quad \text{for all} \quad h \in H\} =: \mathfrak{p}_H. \end{aligned}$$

Let $v_H: B \to \Delta \cup \infty$ denote the special restriction $v|_B$ of v. Here $\Delta = c_{v|B}(\Gamma) \subset H$. v_H has support $\mathfrak{p}_H$, hence gives us a Manis valuation $\overline{v_H}: A_H/\mathfrak{p}_H \to \Delta \cup \infty$ of support zero. If v is surjective then $\Delta = H$.

The proof of all this is a straightforward exercise. Later we will prove a converse to these statements (Prop. 2.8).

Using Lemma 16 from above we can prove a converse to Proposition 6.

Proposition 1.19. *Assume that the valuation v on R is special and that $\mathfrak{o}_v = \bar{A}_{\bar{\mathfrak{p}}}$ (cf. notations above). Then v is Manis.*

Proof. Replacing A by $\bar{A} = A/\mathfrak{q}$ and v by $\bar{v}$ we assume without loss of generality that $\mathfrak{q} = 0$. Now R is an integral domain, and $A \subset R \subset K$ with K the quotient field of R. We also assume without loss of generality that $\Gamma = \Gamma_v$. The valuation $v: R \to \Gamma \cup \infty$ extends to the valuation $\hat{v}: K \twoheadrightarrow \Gamma \cup \infty$, and $\hat{v}$ has the valuation ring $\mathfrak{o}_v$. We have $v(A \setminus \mathfrak{p}) = \{0\}$, hence $v(A) = \hat{v}(A_\mathfrak{p}) = \hat{v}(\mathfrak{o}_v) = \Gamma_+$. By Lemma 16 we conclude that v is Manis. □

§2 Valuation subrings and Manis pairs

As before let R be a ring (commutative, with 1).

Definition 1. a) A *valuation subring of* R is a subring A of R such that there exists some valuation $v: R \to \Gamma \cup \infty$ with $A = A_v$. A *valuation pair in* R (also called "R-valuation pair") is a pair $(A, \mathfrak{p})$ consisting of a subring A of R and a prime ideal $\mathfrak{p}$ of A such that $A = A_v$, $\mathfrak{p} = \mathfrak{p}_v$ for some valuation v of R.
b) We speak of a *Manis subring* A of R and a *Manis pair* $(A, \mathfrak{p})$ in R respectively if here v can be chosen as a Manis valuation of R.

Two bunches of questions come to mind immediately. 1) How can a valuation subring or a Manis subring of R be characterized ring theoretically? Ditto for pairs.
2) How far is a valuation v determined by the associated ring A_v or pair $(A_v, \mathfrak{p}_v)$?

As stated in §1 the pair $(A_v, \mathfrak{p}_v)$ does not change if we pass from v to the associated special valuation $v|c_v\Gamma$. Thus, starting from now, we will concentrate on special valuations.

If $A = R$ then a special valuation v with $A_v = A$ must be trivial, and any prime ideal $\mathfrak{p}$ of R occurs as the center (= support) of such a valuation v. The valuation v is completely determined by $(R, \mathfrak{p})$ and is Manis. These pairs $(R, \mathfrak{p})$ are called the *trivial Manis pairs* in R.

If $A \neq R$ and A is a valuation subring of R then clearly $R \setminus A$ is a multiplicatively closed subset of R. P. Samuel started an investigation of such subrings of R. We quote one of his very remarkable results.

DEFINITION 2. Let A be a subring of R with $A \neq R$ and $S := R \setminus A$ multiplicatively closed. We define the following subsets $\mathfrak{p}_A$ and $\mathfrak{q}_A$ of A. $\mathfrak{p}_A$ is the set of all $x \in A$ such that there exists some $s \in S$ with $sx \in A$, and $\mathfrak{q}_A$ is the set of all $x \in A$ with $sx \in A$ for all $s \in R \setminus A$.

Clearly $\mathfrak{q}_A \subset \mathfrak{p}_A$. Also $\mathfrak{q}_A = \{x \in R \mid rx \in A \text{ for all } r \in R\}$. Thus $\mathfrak{q}_A$ is the biggest ideal of R contained in A, called the *conductor* of A in R.

THEOREM 2.1. [Sa, Th.1 and Th.2]. *Let A be a proper subring of R with $R \setminus A$ multiplicatively closed.*

i) *$\mathfrak{p}_A$ is a prime ideal of A and $\mathfrak{q}_A$ is a prime ideal both of A and R.*
ii) *A is integrally closed in R.*
iii) *If R is a field then A is a valuation domain, and R is the quotient field of A.* □

If v is a special nontrivial valuation then the support of v is determined by the ring A_v alone. More precisely we have the following proposition, whose proof is an easy exercise.

PROPOSITION 2.2. *Let v be a non trivial valuation on R and $A := A_v$. Then $\mathfrak{q}_A \supset$ $\operatorname{supp} v$. The valuation v is special iff $\mathfrak{q}_A = \operatorname{supp} v$.* □

We cannot expect that a special valuation v is determined up to equivalence by the pair $(A, \mathfrak{p}) := (A_v, \mathfrak{p}_v)$, as is already clear from the example in §1. But this holds if v is Manis. Indeed, if v is also non trivial, then we see from Prop. 2 and Prop.1.6 that $\mathfrak{o}_v = \bar{A}_{\bar{\mathfrak{p}}}$ with $\bar{A} = A/\mathfrak{q}_A$, $\bar{\mathfrak{p}} = \mathfrak{p}/\mathfrak{q}_A$. Even more is true. The following proposition implies that v is determined up to equivalence by A alone. The proof is again an easy exercise.

PROPOSITION 2.3. *Let v be a non trivial valuation on R and $A := A_v$. Then $\mathfrak{p}_A \subset \mathfrak{p}_v$. If v is Manis then $\mathfrak{p}_A = \mathfrak{p}_v$.* □

We have the following important characterization of Manis pairs.

THEOREM 2.4 ([M, Prop. 1], or [Huc, Th. 5.1]). *Let A be a subring of R and $\mathfrak{p}$ a prime ideal of A. The following are equivalent.*

i) *$(A, \mathfrak{p})$ is a Manis pair in R.*
ii) *If B is a subring of R and $\mathfrak{q}$ a prime ideal of B with $A \subset B$ and $\mathfrak{q} \cap A = \mathfrak{p}$ then $A = B$.* [1)]

[1)] In [M] and [Huc] it is not assumed that $\mathfrak{q}$ is a prime ideal. It can be proved easily that their condition can be changed to our condition (ii).

iii) *For every $x \in R \setminus A$ there exists some $y \in A$ with $xy \in A \setminus \mathfrak{p}$.* $\square$

There also exists a satisfying characterization of the valuation subrings of R in ring theoretic terms, due to Samuel and Griffin [e.g.Huc, Th.5.5], but we do not need this here.

We give a characterization of local Manis pairs in a classical style.

Theorem 2.5. *Let $A \subset R$ be a ring extension, $A \neq R$.*
i) The following are equivalent
(1) *Every $x \in R \setminus A$ is a unit in R and $x^{-1} \in A$.*
(2) *A has a unique maximal ideal $\mathfrak{p}$ (hence is local) and $(A, \mathfrak{p})$ is Manis in R.*
ii) If (1), (2) hold, then R is a local ring with maximal ideal $\mathfrak{q} := \mathfrak{q}_A$ and $A_\mathfrak{q} = R_\mathfrak{p} = R$. Moreover, $\mathfrak{p} = \mathfrak{q} \cup \{x^{-1} | x \in R \setminus A\}$.

Proof. Assume that (1) holds. Then $R \setminus A$ is closed under multiplication. Indeed, let $x, y \in R \setminus A$ be given. Then $(xy)y^{-1} \in R \setminus A$, but $y^{-1} \in A$, hence $xy \in R \setminus A$. We introduce the prime ideals $\mathfrak{p} := \mathfrak{p}_A$ and $\mathfrak{q} := \mathfrak{q}_A$ (cf. Def. 2). If $\mathfrak{M}$ is any maximal ideal of R then $\mathfrak{M} \cap (R \setminus A) = \emptyset$, since $R \setminus A \subset R^*$, and $\mathfrak{M} \subset A$. Thus $\mathfrak{M}$ is contained in the conductor $\mathfrak{q}$ of A in R, and we conclude that $\mathfrak{M} = \mathfrak{q}$. Thus $\mathfrak{q}$ is the only maximal ideal of R. Let K denote the field $R/\mathfrak{q}$ and $\overline{A}$ the subring $A/\mathfrak{q}$ of K. For every $z \in K \setminus \overline{A}$ the inverse z^{-1} is contained in $\overline{A}$. Thus $\overline{A}$ is a valuation domain with quotient field K. We conclude that A is Manis in R, and then, that $(A, \mathfrak{p})$ is a Manis pair in R (cf. Prop. 3). Since $(R, \mathfrak{q})$ is local we learn from Proposition 1.3 that $(A, \mathfrak{p})$ is local.

Now assume that (2) holds. We know from Proposition 1.3 that R is local with maximal ideal $\mathfrak{q} := \mathfrak{q}_A$. Thus $R \setminus A \subset R \setminus \mathfrak{q} = R^*$. Since $(A, \mathfrak{p})$ is Manis in R we have $x^{-1} \in \mathfrak{p} \subset A$ for every $x \in R \setminus A$, and it is also clear that $\mathfrak{p} = \mathfrak{q} \cup \{x^{-1} | x \in R \setminus A\}$.

We have $A \setminus \mathfrak{q} \subset R^*$, hence $A_\mathfrak{q} \subset R$. If $x \in R \setminus A$ then $x = \frac{1}{y}$ with $y \in A \setminus \mathfrak{q}$. Thus $x \in A_\mathfrak{q}$. This proves that $A_\mathfrak{q} = R$. Since $A \setminus \mathfrak{p} \subset R^*$ also $R_\mathfrak{p} = R$.

Assume now that (2) holds. We know from Proposition 1.3 that R is local with maximal ideal $\mathfrak{q} := \mathfrak{q}_A$. Thus $R \setminus A \subset R \setminus \mathfrak{q} = R^*$. Since $(A, \mathfrak{p})$ is Manis in R we have $x^{-1} \in \mathfrak{p}$ for every $x \in R \setminus A$, a fortiori $x^{-1} \in A$. $\square$

Let $v : R \longrightarrow \Gamma \cup \infty$ and w be valuations on R. We have called w coarser than v if w is equivalent to v/H for some convex subgroup H of v (§1, Def. 9 and Remark 1.12). How can the coarsening relation be expressed in terms of the pairs $(A_v, \mathfrak{p}_v)$, $(A_w, \mathfrak{p}_w)$ if both v and w are Manis?

Theorem 2.6 (cf. [M, Prop.4] for a weaker statement). *Assume that $v : R \longrightarrow \Gamma \cup \infty$ and w are two non-trivial Manis valuations of R.*

i) *The following are equivalent:*
(1) *w is coarser than v.*
(2) *$supp(v) = supp(w)$ and $\mathfrak{o}_v \subset \mathfrak{o}_w$.*
(3) *$A_v \subset A_w$ and $\mathfrak{p}_w \subset \mathfrak{p}_v$.*
(4) *$\mathfrak{p}_w$ is an ideal of A_v contained in $\mathfrak{p}_v$.*

ii) *Let $A := A_v$, $\mathfrak{p} := \mathfrak{p}_v$, and let $\mathfrak{r}$ be a prime ideal of A with* $\operatorname{supp} v \subset \mathfrak{r} \subset \mathfrak{p}$. *Let H denote the convex subgroup of Γ generated by $v(A \setminus \mathfrak{r})$ and $w := v/H$. Then $\mathfrak{r} = \mathfrak{p}_H = \mathfrak{p}_w$ and $A_{[\mathfrak{r}]} = A_w = A_H$.*[2)]

PROOF: (1) $\iff$ (2): We may assume in advance that $\operatorname{supp} v = \operatorname{supp} w$. It is now evident that w is coarser than v iff $\hat{w}$ is coarser than $\hat{v}$. By classical valuation theory this holds iff the valuation ring $\mathfrak{o}_v$ of $\hat{v}$ is contained in $\mathfrak{o}_w$.

(2) $\Longrightarrow$ (3): Replacing R by $R/\operatorname{supp} v$ we assume without loss of generality that $\operatorname{supp} v = \operatorname{supp} w = \{0\}$. In the quotient field K of R we have $\mathfrak{o}_v \cap R = A_v$, $\mathfrak{o}_w \cap R = A_w$, $\mathfrak{m}_v \cap R = \mathfrak{p}_v$ and $\mathfrak{m}_w \cap R = \mathfrak{p}_w$. By assumption $\mathfrak{o}_v \subset \mathfrak{o}_w$. This implies $\mathfrak{m}_v \supset \mathfrak{m}_w$. We conclude that $A_v \subset A_w$ and $\mathfrak{p}_v \supset \mathfrak{p}_w$.

(3) $\Longrightarrow$ (2): We verify first that $\operatorname{supp}(v) = \operatorname{supp}(w)$. We know that $\operatorname{supp}(v) = \{x \in R \mid xR \subset A_v\}$ and $\operatorname{supp}(w) = \{x \in R \mid xR \subset A_w\}$ (cf. Proposition 2). Using the assumption $A_v \subset A_w$ we conclude $\operatorname{supp} v \subset \operatorname{supp} w$. Since v, w are Manis valuations, it is also evident that $\operatorname{supp}(v) = \{x \in R \mid xR \subset \mathfrak{p}_v\}$ and $\operatorname{supp}(w) := \{x \in R \mid xR \subset \mathfrak{p}_w\}$. Using the assumption $\mathfrak{p}_v \supset \mathfrak{p}_w$ we conclude that $\operatorname{supp} v \supset \operatorname{supp} w$. Thus indeed $\operatorname{supp}(v) = \operatorname{supp}(w)$.

In order to prove that $\mathfrak{o}_v \subset \mathfrak{o}_w$ we may replace R by $R/\operatorname{supp} v$. Thus we may assume that $\operatorname{supp} v = \operatorname{supp} w = \{0\}$. Now we know from Proposition 1.6 that $\mathfrak{o}_v = (A_v)_{\mathfrak{p}_v}$ and $\mathfrak{o}_w = (A_w)_{\mathfrak{p}_w}$. The inclusions $A_v \subset A_w$ and $\mathfrak{p}_v \supset \mathfrak{p}_w$ imply that $\mathfrak{o}_v \subset \mathfrak{o}_w$.

(3) $\Longrightarrow$ (4): trivial.

(4) $\Longrightarrow$ (3): Since w is Manis we have $A_w = \{x \in R \mid x\mathfrak{p}_w \subset \mathfrak{p}_w\}$. Now $\mathfrak{p}_w$ is an ideal of A_v. Thus $A_v \subset A_w$.

ii): We know from Prop.1.10 that the ideal $\mathfrak{r}$ is v-convex, and from Remark 1.12.a that $\mathfrak{r} = \mathfrak{p}_H$. Let $w := v/H$ and $B := A_w$. We have $B = A_H$ (cf. 1.18) and $\mathfrak{p}_w = \mathfrak{p}_H = \mathfrak{r}$.

It remains to prove that $B = A_{[\mathfrak{r}]}$. Let $x \in A_{[\mathfrak{r}]}$ be given. We choose some $d \in A \setminus \mathfrak{r}$ with $dx \in A$. Since $A \subset A_w$, $\mathfrak{r} = \mathfrak{p}_w$, we have $w(dx) \geq 0$, $w(d) = 0$, hence $w(x) \geq 0$, i.e. $x \in B$. This proves that $A_{[\mathfrak{r}]} \subset B$. Let now $x \in B$ be given. Suppose that $x \notin A_{[\mathfrak{r}]}$. Since $x \notin A$ there exists some $x' \in \mathfrak{p}$ with $xx' \in A \setminus \mathfrak{p} \subset A \setminus \mathfrak{r} \subset A$. Since $x \notin A_{[\mathfrak{r}]}$ we have $x' \in \mathfrak{r}$. Thus $x\mathfrak{r} \not\subset \mathfrak{r}$. This is a contradiction, since $\mathfrak{r}$ is an ideal of B and $x \in B$. Thus $x \in A_{[\mathfrak{r}]}$. We have proved $B = A_{[\mathfrak{r}]}$. q.e.d.

COROLLARY 2.7. *Let $v : R \to \Gamma \cup \infty$ be a Manis valuation and $A := A_v$, $\mathfrak{p} = \mathfrak{p}_v$. The coarsenings w of v correspond uniquely, up to equivalence, with the prime ideals $\mathfrak{r}$ of A between $\operatorname{supp} v$ and $\mathfrak{p}$ via $\mathfrak{r} = \mathfrak{p}_w$. Also $A_{[\mathfrak{r}]} = A_w$.*

PROOF. If v is trivial then $\operatorname{supp} v = \mathfrak{p}$, and all assertions are evident. Assume now that v is not trivial. For the trivial coarsening t of v we have $\mathfrak{p}_t = \operatorname{supp} t = \operatorname{supp} v$ and $A_{[\mathfrak{p}_t]} = R$. If w is a non trivial coarsening of v then $\mathfrak{p}_w$ is an ideal of A with $\operatorname{supp} v \subsetneqq \mathfrak{p}_w \subset \mathfrak{p}$ (cf. Th.6.i). This ideal is prime in A since it is prime in the ring $A_w \supset A$. Conversely, if $\mathfrak{r}$ is a prime ideal of A with $\operatorname{supp} v \subsetneqq \mathfrak{r} \subset \mathfrak{p}$ then, by Theorem 6.ii, there exists a coarsening w of v with $\mathfrak{p}_w = \mathfrak{r}$, $A_w = A_{[\mathfrak{r}]}$, and w is not trivial.

[2)] Recall the notations from 1.12 and 1.18.

Finally, if w and w' are two nontrivial coarsenings of v with $\mathfrak{p}_w = \mathfrak{p}_{w'} = \mathfrak{r}$, then $A_w = \{x \in R \mid x\mathfrak{r} \subset \mathfrak{r}\} = A_{w'}$, and we learn from (3) in Theorem 6.i (or by a direct argument), that $w \sim w'$. $\square$

We establish a converse to the construction 1.18.

Proposition 2.8. *Let w be a non-trivial Manis valuation on R and u a Manis valuation on $A_w/\mathfrak{p}_w$. Let A and $\mathfrak{p}$ denote the pre-images of A_u and $\mathfrak{p}_u$ in A_w under the natural homomorphism $\varphi: A_w \to A_w/\mathfrak{p}_w$.*

i) *$(A, \mathfrak{p})$ is a Manis pair in R iff $\operatorname{supp} u = \{0\}$.*

ii) *If this holds, let $v: R \twoheadrightarrow \Gamma \cup \infty$ be a surjective valuation with $A_v = A$, $\mathfrak{p}_v = \mathfrak{p}$. Then Γ has a convex subgroup H, uniquely determined by w and u, such that w is equivalent to v/H and u is equivalent to $\overline{v_H}$ (cf. 1.18).*

Proof. We have $\mathfrak{p}_w \subset \mathfrak{p} \subset A \subset A_w \subset R$.

a) We assume that $\operatorname{supp} u = \{0\}$ and prove that the pair $(A, \mathfrak{p})$ is Manis in R. Let $x \in R \setminus A$ be given. By Theorem 4 we are done if we find some $y \in \mathfrak{p}$ with $xy \in A \setminus \mathfrak{p}$.

Case 1: $x \in A_w$. Since $\varphi(x) \notin A_u$ there exists some $y \in \mathfrak{p}$ with $\varphi(x)\varphi(y) \in A_u \setminus \mathfrak{p}_u$, hence $xy \in A \setminus \mathfrak{p}$.

Case 2: $x \in R \setminus A_w$. Since w is Manis there exists some $y \in \mathfrak{p}_w$ with $xy \in A_w \setminus \mathfrak{p}_w$. We have $\varphi(xy) \neq 0$. Since u has support zero there exists some $z \in A_w$ with $\varphi(xy)\varphi(z) \in A_u \setminus \mathfrak{p}_u$, hence $xyz \in A \setminus \mathfrak{p}$. Clearly $yz \in \mathfrak{p}_w \subset \mathfrak{p}$.

b) Assume now that $(A, \mathfrak{p})$ is Manis in R, and that $v: R \twoheadrightarrow \Gamma \cup \infty$ is a surjective valuation with $A_v = A$, $\mathfrak{p}_v = \mathfrak{p}$. We verify that u has support zero and prove the second part of the proposition. Since w is not trivial, we know from Theorem 6 that w is a coarsening of v. There is a unique convex subgroup H of Γ with $w \sim v/H$, and $A_w = A_H$, $\mathfrak{p}_w = \mathfrak{p}_H$ (notations from 1.18). We obtain from v and H a Manis valuation $v_H: A_w \twoheadrightarrow H \cup \infty$ with support $\mathfrak{p}_w$, as explained in 1.18. The pair associated to v_H is $(A, \mathfrak{p})$. Thus $v_H \sim u \circ \varphi$ and $\overline{v_H} \sim u$. In particular $\operatorname{supp} u = \operatorname{supp} \overline{v_H} = \{0\}$. $\square$

We now consider the following situation: A is a subring of R and $\mathfrak{p}$ is a prime ideal of A. We are looking for criteria that the pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ (cf. §1, Def. 10) is Manis.

We need an easy lemma.

Lemma 2.9. a) $R_{\mathfrak{p}} = R_{(\mathfrak{p}_{[\mathfrak{p}]})}$.

b) *If M is an A-submodule of R then $M_{\mathfrak{p}} = (M_{[\mathfrak{p}]})_{\mathfrak{p}_{[\mathfrak{p}]}}$.*

c) *If M is an A-submodule of R and $\mathfrak{r}$ is a prime ideal of A contained in $\mathfrak{p}$, then*

$$M_{[\mathfrak{r}]} = (M_{[\mathfrak{p}]})_{[\mathfrak{r}_{[\mathfrak{p}]}]}.$$

Proof. We have $R_{\mathfrak{p}} = S^{-1}R$ and $R_{(\mathfrak{p}_{[\mathfrak{p}]})} = T^{-1}R$ with $S = A \setminus \mathfrak{p}$, $T = A_{[\mathfrak{p}]} \setminus \mathfrak{p}_{[\mathfrak{p}]}$. Notice that $S \subset T$. Let $x \in T$ be given. Choose some $d \in S$ with $dx \in A$. Then $dx \in A \setminus \mathfrak{p} = S$. This proves that $\operatorname{Sat}_R(S) = \operatorname{Sat}_R(T)$, and we conclude that $S^{-1}R = T^{-1}R$.

If M is an A-submodule of R, then $M_{[\mathfrak{p}]}$ is an $A_{[\mathfrak{p}]}$-submodule of R, and $M_{\mathfrak{p}} = S^{-1}M$, $(M_{[\mathfrak{p}]})_{\mathfrak{p}_{[\mathfrak{p}]}} = T^{-1}M_{[\mathfrak{p}]}$. Clearly $S^{-1}M \subset T^{-1}M_{[\mathfrak{p}]}$. (N.B. Both are subsets of $S^{-1}R = T^{-1}R$.) Also $T^{-1}M_{[\mathfrak{p}]} = S^{-1}M_{[\mathfrak{p}]}$. Let $z \in S^{-1}M_{[\mathfrak{p}]}$ be given. Write $z = \frac{x}{s}$ with $x \in M_{[\mathfrak{p}]}$, $s \in S$. We choose some $d \in S$ with $dx = m \in M$. We have $z = \frac{m}{sd} \in M_{\mathfrak{p}}$. This proves part b) of the lemma. The last statement c) follows from the obvious equality $M_{\mathfrak{r}} = (M_{\mathfrak{p}})_{\mathfrak{r}_{\mathfrak{p}}}$ by taking pre-images under the various localization maps. $\square$

Proposition 2.10. *$(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R iff $(A_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$ is a Manis pair in $R_{\mathfrak{p}}$. In this case, if $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ comes from the Manis valuation v on R, then $(A_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$ comes from the localization $\tilde{v}$ of v defined in §1 (Def. 6). {Recall from the lemma that $A_{\mathfrak{p}} = (A_{[\mathfrak{p}]})_{\mathfrak{p}_{[\mathfrak{p}]}}, \mathfrak{p}_{\mathfrak{p}} = (\mathfrak{p}_{[\mathfrak{p}]})_{\mathfrak{p}_{[\mathfrak{p}]}}$.} With $\mathfrak{q} := A \cap \operatorname{supp} v$ we have $\operatorname{supp} v = \mathfrak{q}_{[\mathfrak{p}]}, \operatorname{supp} \tilde{v} = \mathfrak{q}_{\mathfrak{p}}$.*

Proof. a) Assume first that there exists a Manis valuation $v: R \to \Gamma \cup \infty$ with $A_v = A_{[\mathfrak{p}]}, \mathfrak{p}_v = \mathfrak{p}_{[\mathfrak{p}]}$. Let $\tilde{v}: R_{\mathfrak{p}_v} \to \Gamma \cup \infty$ denote the localization of v. Then $\tilde{v}$ is again Manis and $A_{\tilde{v}} = (A_v)_{\mathfrak{p}_v}, \mathfrak{p}_{\tilde{v}} = (\mathfrak{p}_v)_{\mathfrak{p}_v}$, $\operatorname{supp} \tilde{v} = (\operatorname{supp} v)_{\mathfrak{p}_v}$ (cf. §1). By part a) of the lemma above we have $R_{\mathfrak{p}_v} = R_{\mathfrak{p}}$, $A_{\tilde{v}} = A_{\mathfrak{p}}, \mathfrak{p}_{\tilde{v}} = \mathfrak{p}_{\mathfrak{p}}$. Let $\mathfrak{q} := A \cap \operatorname{supp} v$. Certainly $\mathfrak{q}_{[\mathfrak{p}]} \subset \operatorname{supp} v$. Let $x \in \operatorname{supp} v$ be given. We have $x \in A_v = A_{[\mathfrak{p}]}$. We choose some $d \in A \setminus \mathfrak{p}$ with $dx \in A$. Then $v(dx) = \infty$, thus $dx \in A \cap \operatorname{supp} v = \mathfrak{q}$, $x \in \mathfrak{q}_{[\mathfrak{p}]}$. This proves $\operatorname{supp} v = \mathfrak{q}_{[\mathfrak{p}]}$. Using part b) of the lemma we obtain $\operatorname{supp} \tilde{v} = \mathfrak{q}_{\mathfrak{p}}$.
b) Assume finally that $w: R_{\mathfrak{p}} \to \Gamma \cup \infty$ is a Manis valuation with $A_w = A_{\mathfrak{p}}$, $\mathfrak{p}_w = \mathfrak{p}_{\mathfrak{p}}$. Let $j_T: R \to R_{\mathfrak{p}}$ denote the localization map of R with respect to $T := A_{[\mathfrak{p}]} \setminus \mathfrak{p}_{[\mathfrak{p}]}$. Let v denote the valuation $w \circ j_T$ on R. We have $v(T) = \{0\}$. Thus $v(R) = w(R_{\mathfrak{p}}) = \Gamma_w$, and we conclude that v is Manis. Also $A_v = j_T^{-1}(A_w) = A_{[\mathfrak{p}]}$, $\mathfrak{p}_v = j_T^{-1}(\mathfrak{p}_w) = \mathfrak{p}_{[\mathfrak{p}]}$, and w coincides with the localization $\tilde{v}$ of v. q.e.d.

Proposition 2.11. *Let $\mathfrak{r}$ be a prime ideal of A contained in $\mathfrak{p}$. Assume that $v: R \to \Gamma \cup \infty$ is a valuation with $A_v = A_{[\mathfrak{p}]}$, $\mathfrak{p}_v = \mathfrak{p}_{[\mathfrak{p}]}$, $A \cap \operatorname{supp} v \subset \mathfrak{r}$. Let H denote the convex subgroup of Γ generated by $v(A \setminus \mathfrak{r})$ and let $w := v/H$. Then $A_w = A_{[\mathfrak{r}]}$, $\mathfrak{p}_w = \mathfrak{r}_{[\mathfrak{p}]} = \mathfrak{r}_{[\mathfrak{r}]}$. Thus, if $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R the same holds for $(A_{[\mathfrak{r}]}, \mathfrak{r}_{[\mathfrak{r}]})$.*

Proof. By the last statement in Prop. 10 we have $\operatorname{supp} v \subset \mathfrak{r}_{[\mathfrak{p}]}$. It follows from Proposition 1.13 and part c) of lemma 9 above that $A_w = A_{[\mathfrak{r}]}$, $\mathfrak{p}_w = \mathfrak{r}_{[\mathfrak{p}]}$. It is evident that $\mathfrak{r}_{[\mathfrak{p}]} \subset \mathfrak{r}_{[\mathfrak{r}]} \subset \mathfrak{p}_w$. Thus $\mathfrak{r}_{[\mathfrak{p}]} = \mathfrak{r}_{[\mathfrak{r}]}$. $\square$

We now state a criterion which will play a key role for the theory of relative Prüfer rings in §5.

Theorem 2.12. *Assume that A is integrally closed in R. The following are equivalent.*

i) *$(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R.*
ii) *For each $x \in R$ there exists some polynomial $F[T] \in A[T] \setminus \mathfrak{p}[T]$ with $F(x) = 0$.*

Proof. i) $\Rightarrow$ ii): We first consider the case that $x \in A_{[\mathfrak{p}]}$. We choose some $s \in A \setminus \mathfrak{p}$ with $sx = a \in A$. The polynomial $F(T) := sT - a$ fulfills the requirements. Let now $x \in R \setminus A_{[\mathfrak{p}]}$. Since $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair there exists some $y \in \mathfrak{p}_{[\mathfrak{p}]}$ with $xy \in A_{[\mathfrak{p}]} \setminus \mathfrak{p}_{[\mathfrak{p}]}$. We choose elements s and t in $A \setminus \mathfrak{p}$ with $ty \in \mathfrak{p}$, $sxy \in A$. We have $sxy \in A \setminus \mathfrak{p}$. Put $a_0 := sty \in \mathfrak{p}$, $a_1 := -stxy \in A \setminus \mathfrak{p}$. The polynomial $F(T) := a_0 T + a_1$ fulfills the requirements.

ii) $\Rightarrow$ i): We verify the property (iii) in Theorem 4. Let $x \in R \setminus A_{[\mathfrak{p}]}$ be given. We look for an element $y \in \mathfrak{p}_{[\mathfrak{p}]}$ with $xy \in A_{[\mathfrak{p}]} \setminus \mathfrak{p}_{[\mathfrak{p}]}$. Let

$$F(T) := a_0 T^n + a_1 T^{n-1} + \cdots + a_n$$

be a polynomial of minimal degree $n \geq 1$ in $A[T] \setminus \mathfrak{p}[T]$ with $F(x) = 0$. From $F(x) = 0$ we deduce that $b := a_0 x$ is integral over A. Thus $b \in A$. Since $x \notin A_{[\mathfrak{p}]}$ we conclude that $a_0 \in \mathfrak{p}$. Suppose that $n > 1$. We put

$$G(T) := a_0 T - b \quad \text{in the case} \quad b \notin \mathfrak{p},$$

and

$$G(T) := (b + a_1) T^{n-1} + a_2 T^{n-2} + \cdots + a_n$$

in the case $b \in \mathfrak{p}$. In both cases

$$G(T) \in A[T] \setminus \mathfrak{p}[T] \quad \text{and} \quad G(x) = 0.$$

This contradicts the minimality of n. Thus $n = 1$, $F(T) = a_0 T + a_1$. Since $a_0 \in \mathfrak{p}$, certainly $a_1 \in A \setminus \mathfrak{p}$. For $y := a_0$ we have $y \in \mathfrak{p}_{[\mathfrak{p}]}$, $xy \in A_{[\mathfrak{p}]} \setminus \mathfrak{p}_{[\mathfrak{p}]}$. q.e.d.

Essentially as a consequence of Theorems 4 and 12 we derive still another criterion for a pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ to be Manis in R. In the case of Krull valuation rings (i.e. R a field) such a criterion had been observed by Gilmer [Gi, Th. 19.15]. We need (a special case of) an easy lemma.

Lemma 2.13. *Let $(B, \mathfrak{q})$ be a Manis pair in R. Let I be a B-submodule of R with $I \cap B \subset \mathfrak{q}$. Then $I \subset \mathfrak{q}$.*

Proof. Suppose there exists an $x \in I$ with $x \notin \mathfrak{q}$, hence $x \notin B$. Since $(B, \mathfrak{q})$ is Manis there exists some $y \in B$ with $xy \in B \setminus \mathfrak{q}$. Then $xy \notin I$. On the other hand $x \in I$ and $y \in B$, a contradiction. □

Theorem 2.14 (cf. [Gi, Th. 19.15] for R a field). *Assume that A is integrally closed in R, and let $\mathfrak{p}$ be a prime ideal of A. The following are equivalent.*

i) *$(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R.*

ii) *If B is a subring of R containing $A_{[\mathfrak{p}]}$ and $\mathfrak{q}, \mathfrak{q}'$ are prime ideals of B with $\mathfrak{q} \subset \mathfrak{q}'$ and $\mathfrak{q} \cap A_{[\mathfrak{p}]} = \mathfrak{q}' \cap A_{[\mathfrak{p}]} \subset \mathfrak{p}_{[\mathfrak{p}]}$, then $\mathfrak{q} = \mathfrak{q}'$.*

ii′) *If B is a subring of R containing $A_{[\mathfrak{p}]}$ and $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals of B lying over $\mathfrak{p}_{[\mathfrak{p}]}$, then $\mathfrak{q} = \mathfrak{q}'$.*

iii) *If B is a subring of R containing A and $\mathfrak{q}, \mathfrak{q}'$ are prime ideals of B with $\mathfrak{q} \subset \mathfrak{q}'$ and $\mathfrak{q} \cap A = \mathfrak{q}' \cap A \subset \mathfrak{p}$ then $\mathfrak{q} = \mathfrak{q}'$.*

iii′) *If B is a subring of R containing A and $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals of B lying over $\mathfrak{p}$ then $\mathfrak{q} = \mathfrak{q}'$.*

iv) *There exists only one Manis pair $(B, \mathfrak{q})$ in R over $(A, \mathfrak{p})$, i.e. with $A \subset B$ and $\mathfrak{q} \cap A = \mathfrak{p}$.*

v) *For every subring B of R containing A there exists at most one prime ideal $\mathfrak{q}$ of B over $\mathfrak{p}$.*

vi) *For every Manis pair $(B,\mathfrak{q})$ in R over $(A,\mathfrak{p})$ the field extension $k(\mathfrak{p}) \subset k(\mathfrak{q})$ is algebraic.*

PROOF. The implication i) $\Rightarrow$ ii) is evident by the preceding lemma. The implications ii) $\Rightarrow$ ii$'$) and iii) $\Rightarrow$ iii$'$) are trivial.
ii$'$) $\Rightarrow$ iii$'$): If $\mathfrak{q}$ and $\mathfrak{q}'$ are prime ideals of B over $\mathfrak{p}$ with $\mathfrak{q} \subset \mathfrak{q}'$, then $\mathfrak{q}_{[\mathfrak{p}]}$ and $\mathfrak{q}'_{[\mathfrak{p}]}$ are prime ideals of $B_{[\mathfrak{p}]}$ over $\mathfrak{p}_{[\mathfrak{p}]}$ with $\mathfrak{q}_{[\mathfrak{p}]} \subset \mathfrak{q}'_{[\mathfrak{p}]}$. Thus $\mathfrak{q}_{[\mathfrak{p}]} = \mathfrak{q}'_{[\mathfrak{p}]}$. Intersecting with B we obtain $\mathfrak{q} = \mathfrak{q}'$. ii) $\Rightarrow$ iii): The proof is similar.
iii$'$) $\Rightarrow$ i): Suppose that $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is not Manis in R. By Theorem 12 there exists some $x \in R$ such that $F(x) \neq 0$ for every polynomial $F(T) \in A[T] \setminus \mathfrak{p}[T]$. We introduce the subring $B := A[x]$ of R and the surjective ring homomorphism $\varphi: A[T] \longrightarrow B$ over A with $\varphi(T) = x$. The kernel of φ is contained in $\mathfrak{p}[T]$. This implies that the ideals $\mathfrak{q}$ and $\mathfrak{q}'$ of B defined by

$$\mathfrak{q} := \varphi(\mathfrak{p}[T]) = \mathfrak{p}[x] = \mathfrak{p}B, \quad \mathfrak{q}' := \varphi(\mathfrak{p} + TA[T]) = \mathfrak{p} + xB = \mathfrak{q} + xB,$$

both are prime and lie over $\mathfrak{p}$. Since $\mathfrak{q} \neq \mathfrak{q}'$ this contradicts the assumption iii$'$). Thus $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is Manis in R.
i) $\Rightarrow$ iv): Let $(B,\mathfrak{q})$ be a Manis pair in R over $(A,\mathfrak{p})$. It is easily verified that $(B,\mathfrak{q})$ is a pair over $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$. Since $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is Manis in R we conclude by Theorem 4 that $(B,\mathfrak{q}) = (A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$.
iv) $\Rightarrow$ v): Assume that B is a subring of R containing A and $\mathfrak{q}_1$, $\mathfrak{q}_2$ are prime ideals of B over $\mathfrak{p}$. We extend the pairs $(B,\mathfrak{q}_1)$ and $(B,\mathfrak{q}_2)$ to maximal pairs $(C,\mathfrak{q}'_1)$ and $(D,\mathfrak{q}'_2)$ in R. These pairs are Manis in R by Theorem 4. They both lie over $(A,\mathfrak{p})$, hence $(C,\mathfrak{q}'_1) = (D,\mathfrak{q}'_2)$. Intersecting with B we obtain $\mathfrak{q}_1 = \mathfrak{q}_2$.
v) $\Rightarrow$ iii$'$): trivial.
i) $\Rightarrow$ vi): Since (i) and (iv) hold we know that $(B,\mathfrak{q}) := (A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is the only Manis pair in R over $(A,\mathfrak{p})$. We have $k(\mathfrak{p}) = k(\mathfrak{q})$.
vi) $\Rightarrow$ iii$'$): Suppose that $(B,\mathfrak{q}_1)$ and $(B,\mathfrak{q}_2)$ are pairs in R over $(A,\mathfrak{p})$ with $\mathfrak{q}_1 \subsetneqq \mathfrak{q}_2$. We choose a maximal pair $(C,\mathfrak{r})$ in R over $(B,\mathfrak{q}_1)$. Then $(C,\mathfrak{r})$ is Manis, hence $k(\mathfrak{r})$ is algebraic over $k(\mathfrak{p})$. It follows that $k(\mathfrak{q}_1)$ is algebraic over $k(\mathfrak{p})$. We choose an element $x \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$. Since $k(\mathfrak{q}_1)$ is algebraic over $k(\mathfrak{p})$ we have a relation

$$(*) \qquad \sum_{i=0}^{n} a_i x^i = b$$

with $a_0, a_1, \ldots, a_n \in A$, $a_n \notin \mathfrak{p}$, $b \in \mathfrak{q}_1$. Let B' denote the subring $A[b, a_n x]$ of B, and $\mathfrak{q}'_1 := \mathfrak{q}_1 \cap B'$, $\mathfrak{q}'_2 := \mathfrak{q}_2 \cap B'$. We have $\mathfrak{q}'_1 \subsetneqq \mathfrak{q}'_2$, since $a_n x \in \mathfrak{q}'_2 \setminus \mathfrak{q}'_1$. But $\mathfrak{q}'_1 \cap A = \mathfrak{q}'_2 \cap A = \mathfrak{p}$. We learn from the relation $(*)$ that $B'/\mathfrak{q}'_1$ is integral over $A/\mathfrak{p}$. But the ring $B'/\mathfrak{q}'_1$ contains the prime ideal $\mathfrak{q}'_2/\mathfrak{q}'_1 \neq \{0\}$ with $(\mathfrak{q}'_2/\mathfrak{q}'_1) \cap A/\mathfrak{p} = \{0\}$. Such a situation is impossible in an integral ring extension (cf. [Bo, V §2, n^o 1]). Thus (iii$'$) is valid. $\square$

§3 Weakly surjective homomorphisms

In section §5 we will start our theory of "Prüfer extensions". In the terminology developed there the Prüfer rings (with zero divisors) of the classical literature (e.g. [LM], [Huc]) are those commutative rings A which are Prüfer in their total quotient rings Quot$A$. In the present section and the following one we develop an auxiliary theory of "weakly surjective" ring extensions. The inclusions $A \subset \mathrm{Quot}A$ are (very special) examples of such extensions.

Definition 1. i) Let $\varphi: A \to B$ be a ring homomorphism. We call φ *locally surjective* (abbreviated: ls) if for every prime ideal $\mathfrak{q}$ of B the induced homomorphism $\varphi_{\mathfrak{q}}: A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is surjective. We call φ *weakly surjective* (abbreviated: ws) if for every prime ideal $\mathfrak{p}$ of A with $\mathfrak{p}B \neq B$ the induced homomorphism $\varphi_{\mathfrak{p}}: A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is surjective.
ii) If A is a subring of a ring B, then we say that A is *locally surjective in* B (resp. *weakly surjective in* B) if the inclusion mapping $A \hookrightarrow B$ is ls (resp. ws).

At first glance "locally surjective" seems to be a more natural notion than "weakly surjective", but it is the latter notion which will be needed below.

Of course, a surjective homomorphism is both weakly surjective and locally surjective. We now prove that weak surjectivity is a stronger property than local surjectivity.

Proposition 3.1. *If $\varphi: A \to B$ is weakly surjective then φ is locally surjective.*

This follows from

Lemma 3.2. *Let $\varphi: A \to B$ be a ring homomorphism. Let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. Assume that $\varphi_{\mathfrak{p}}: A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is surjective. Then the natural map $B_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an isomorphism, in short, $B_{\mathfrak{p}} = B_{\mathfrak{q}}$. Furthermore $\mathfrak{p}B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$.*

Proof of the lemma. One easily retreats to the case that A is a subring of B and φ is the inclusion $A \hookrightarrow B$. Now $\mathfrak{p} = \mathfrak{q} \cap A$ and $A_{\mathfrak{p}} = B_{\mathfrak{p}}$. We have $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}} \subset \mathfrak{q}B_{\mathfrak{p}}$. Since $\mathfrak{p}A_{\mathfrak{p}}$ is the maximal ideal of $A_{\mathfrak{p}}$ and $(\mathfrak{q}B_{\mathfrak{p}}) \cap B = \mathfrak{q}$, hence $\mathfrak{q}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$, we have $\mathfrak{p}B_{\mathfrak{p}} = \mathfrak{q}B_{\mathfrak{p}}$. The natural homomorphism $B \to B_{\mathfrak{p}}$ maps $B \setminus \mathfrak{q}$ into the group of units of $B_{\mathfrak{p}}$, hence factors through a homomorphism from $B_{\mathfrak{q}}$ to $B_{\mathfrak{p}}$. This homomorphism is inverse to the natural map from $B_{\mathfrak{p}}$ to $B_{\mathfrak{q}}$. □

Example 3.3. If S is a multiplicative subset of a ring A then the localization map $A \to S^{-1}A$ is weakly surjective. □

Example 3.4. Let K be a field. The diagonal homomorphism $K \to K \times K$, $x \mapsto (x, x)$, is locally surjective but not weakly surjective, as is easily verified.

Proposition 3.5. *If $\varphi: A \to B$ is locally surjective and B is an integral domain then φ is weakly surjective.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p}B \neq B$. We choose a prime ideal $\mathfrak{q}$ of B containing $\mathfrak{p}B$. Let $\mathfrak{r} := \varphi^{-1}(\mathfrak{q})$. We have a natural commuting triangle

$$\begin{array}{ccc} A_{\mathfrak{r}} & \xrightarrow{\varphi_{\mathfrak{r}}} & B_{\mathfrak{r}} \\ {\scriptstyle \varphi_{\mathfrak{q}}} \searrow & & \swarrow {\scriptstyle \psi} \\ & B_{\mathfrak{q}} & . \end{array}$$

$\varphi_{\mathfrak{q}}$ is surjective since φ is ls. On the other hand ψ is injective since B is a domain. Thus ψ is bijective and $\varphi_{\mathfrak{r}}$ is surjective. (We have $B_{\mathfrak{r}} = B_{\mathfrak{q}}$, $\varphi_{\mathfrak{r}} = \varphi_{\mathfrak{q}}$.) Since $\mathfrak{p} \subset \mathfrak{r}$ also $\varphi_{\mathfrak{p}}$ is surjective. □

PROPOSITION 3.6. *Every locally surjective homomorphism is an epimorphism in the category $\mathcal{R}$ of rings (commutative, with 1).*

PROOF. Assume that $\varphi: A \to B$ is locally surjective, and that $\psi_1: B \to C$, $\psi_2: B \to C$ are two ring homomorphisms with $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. For every prime ideal $\mathfrak{q}$ of B the map $\varphi_{\mathfrak{q}}: A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is surjective, thus $\psi_{1\mathfrak{q}} = \psi_{2\mathfrak{q}}$. We conclude that $\psi_1 = \psi_2$ (cf. [Bo, Chap II, §3]). □

A fortiori every ws map is an epimorphism in $\mathcal{R}$. We now verify that this class of epimorphisms has pleasant formal properties.

PROPOSITION 3.7. *Let $\varphi: A \to B$ and $\psi: B \to C$ be ring homomorphisms.*
a) If both φ and ψ are weakly surjective then $\psi \circ \varphi$ is weakly surjective.
b) If $\psi \circ \varphi$ is weakly surjective then ψ is weakly surjective.

PROOF. a): Let $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p}C \neq C$. We choose a prime ideal $\mathfrak{r}$ of C containing $\mathfrak{p}C$. Let $\mathfrak{q} := \psi^{-1}(\mathfrak{r})$ and $\tilde{\mathfrak{p}} := \varphi^{-1}(\mathfrak{q})$. The map $\varphi_{\tilde{\mathfrak{p}}}: A_{\tilde{\mathfrak{p}}} \to B_{\tilde{\mathfrak{p}}}$ is surjective. By lemma 3.2 we know that $B_{\mathfrak{q}} = B_{\tilde{\mathfrak{p}}}$. Thus also $C_{\tilde{\mathfrak{p}}} = C \otimes_A A_{\tilde{\mathfrak{p}}} = C \otimes_B (B \otimes_A A_{\tilde{\mathfrak{p}}}) = C \otimes_B B_{\tilde{\mathfrak{p}}} = C \otimes_B B_{\mathfrak{q}} = C_{\mathfrak{q}}$, and $\psi_{\tilde{\mathfrak{p}}} = \psi_{\mathfrak{q}}$, which is surjective. We conclude that $(\psi \circ \varphi)_{\tilde{\mathfrak{p}}} = \psi_{\tilde{\mathfrak{p}}} \circ \varphi_{\tilde{\mathfrak{p}}}$ is surjective.
b): Let $\mathfrak{q}$ be a prime ideal of B with $\mathfrak{q}C \neq C$. Let $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. The map $\psi_{\mathfrak{p}} \circ \varphi_{\mathfrak{p}} = (\psi \circ \varphi)_{\mathfrak{p}}$ is surjective. Thus $\psi_{\mathfrak{p}}$ is surjective. Since $\varphi(A \setminus \mathfrak{p}) \subset B \setminus \mathfrak{q}$ also $\psi_{\mathfrak{q}}$ is surjective.
□

PROPOSITION 3.8. *If $\varphi: A \to B$ and $\psi: B \to C$ are ring homomorphisms and φ is ws then $\psi\varphi(A)$ is ws in $\psi(B)$.*

PROOF. We have a commuting square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \psi\varphi(A) & \underset{i}{\hookrightarrow} & \psi(B) \end{array}$$

with i an inclusion mapping and surjections p and q. Since φ and q are ws, the composite $q \circ \varphi = i \circ p$ is ws. Thus also i is ws. □

COROLLARY 3.9. *Let $\varphi: A \to B$ a ring homomorphism. φ is ws iff $\varphi(A)$ is ws in B.*

PROOF. Applying Proposition 8 with $\psi = id_B$ we see that weak surjectivity of φ implies weak surjectivity of the inclusion mapping $i: \varphi(A) \hookrightarrow B$. Conversely, if i is ws, then φ is ws, since $\varphi = i \circ p$ with p a surjection. □

It is also easy to verify the corollary directly by using Definition 1.

Proposition 3.10. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \alpha\downarrow & & \downarrow\beta \\ C & \underset{\psi}{\longrightarrow} & D \end{array}$$

be a commuting square of ring homomorphisms. Assume that φ is ws and $D = \beta(B)\cdot\psi(C)$. Then ψ is ws.

Proof. Let $\mathfrak{q} \in \operatorname{Spec} C$ be given with $\psi(\mathfrak{q})D \neq D$, and let $\mathfrak{p} := \alpha^{-1}(\mathfrak{q})$. The commuting square above "extends" to a commuting square

$$\begin{array}{ccc} A_{\mathfrak{p}} & \xrightarrow{\tilde{\varphi}} & B_{\mathfrak{p}} \\ \tilde{\alpha}\downarrow & & \downarrow\tilde{\beta} \\ C_{\mathfrak{q}} & \underset{\tilde{\psi}}{\longrightarrow} & D_{\mathfrak{q}} \end{array}$$

with $\tilde{\varphi} = \varphi_{\mathfrak{p}}$, $\tilde{\psi} = \psi_{\mathfrak{q}}$. We have $\mathfrak{p}B \neq B$. The map $\tilde{\varphi}$ is surjective. We are done, if we verify that $\tilde{\psi}$ is surjective.

Let $\xi \in D_{\mathfrak{q}}$ be given. Write $\xi = \frac{x}{s}$ with $x \in D$, $s \in C \setminus \mathfrak{q}$. Since $D = \beta(B)\psi(C)$ we have an equation

$$x = \sum_{i\in I} \beta(b_i)\psi(c_i)$$

with finite index set I, $b_i \in B$, $c_i \in C$. This equation gives us

$$\xi = \sum_{i\in I} \tilde{\beta}(\frac{b_i}{1})\tilde{\psi}(\frac{c_i}{s}).$$

Since $\tilde{\varphi}$ is surjective we have elements $a_i \in A \quad (i \in I)$ and an element $t \in A \setminus \mathfrak{p}$ with $\frac{b_i}{1} = \tilde{\varphi}(\frac{a_i}{t})$ for every $i \in I$. Then

$$\xi = \tilde{\psi}(\frac{y}{s\alpha(t)})$$

with $y := \sum\limits_{\alpha\in I} \alpha(a_i)c_i$. This proves that $\tilde{\psi}$ is surjective. □

In order to understand weakly surjective homomorphisms it suffices by Cor. 9 to analyze weakly surjective ring extensions.

In the following R is a ring and A is a subring of R.

Definition 2. An *R-overring of A* is a subring B of R with $A \subset B$.

Proposition 3.11.

a) *Let B_1 and B_2 be R-overrings of A. If A is ws both in B_1 and B_2 then A is ws in B_1B_2.*

b) *There exists a unique R-overring $M(A,R)$ of A such that A is ws in $M(A,R)$ and $M(A,R)$ contains every R-overring of A in which A is ws.*

Proof. a) Since $A \hookrightarrow B_1$ is ws, the inclusion $B_2 \hookrightarrow B_1B_2$ is ws, as follows from Proposition 10. Since also $A \hookrightarrow B_2$ is ws, the composite $A \hookrightarrow B_2 \hookrightarrow B_1B_2$ is ws (Prop. 7).
b) Let $\mathfrak{A}$ denote the set of all R-overrings of A in which A is ws. Then $\mathfrak{A}$ is an upward directed system of subrings of R. Let $M(A,R)$ denote the union of all these subrings, which is again a subring of R. A is ws in $M(A,R)$ by the following general remark, which is immediate from Definition 1.

Remark 3.12. Let $(B_i|i \in I)$ be an upward directed system of R-overrings of A. If A is ws in each B_i then A is ws in $\bigcup\limits_{i\in I} B_i$.

Definition 3. We call $M(A,R)$ the *weakly surjective hull* of A in R.

We now derive criteria for a homomorphism to be weakly surjective. Without essential loss of generality we concentrate on ring extensions. Let R be a ring and A a subring of R. Recall from §2 that for $\mathfrak{p}$ a prime ideal of A we denote by $A_{[\mathfrak{p}]}$ the pre-image of $A_\mathfrak{p}$ under the localization map $R \to R_\mathfrak{p}$.

Notation. If $x \in R$ then $(A{:}\,x)$ denotes the ideal of A consisting of all $a \in A$ with $ax \in A$.

Theorem 3.13 (cf. [G$_1$, Prop. 10] in the case $R = \mathrm{Quot}A$). *Let B be an R-overring of A. The following are equivalent.*

(1) *A is weakly surjective in B.*
(2) *$B_{[\mathfrak{q}]} = A_{[\mathfrak{q}\cap A]}$ for every prime ideal $\mathfrak{q}$ of B.*
(2′) *$B_{[\mathfrak{q}]} = A_{[\mathfrak{q}\cap A]}$ for every maximal ideal $\mathfrak{q}$ of B.*
(3) *$B \subset A_{[\mathfrak{p}]}$ for every prime ideal $\mathfrak{p}$ of A with $\mathfrak{p}B \neq B$.*
(4) *$(A{:}\,x)B = B$ for every $x \in B$.*

Proof. (1) $\Longleftrightarrow$ (3): We verify the following: For any $\mathfrak{p} \in \mathrm{Spec}A$

$$B \subset A_{[\mathfrak{p}]} \Longleftrightarrow B_\mathfrak{p} = A_\mathfrak{p}.$$

Then we will be done according to Def. 1.

$\Rightarrow$: If $B \subset A_{[\mathfrak{p}]}$, then $B_\mathfrak{p} \subset (A_{[\mathfrak{p}]})_\mathfrak{p} = A_\mathfrak{p}$.
$\Leftarrow$: If $B_\mathfrak{p} = A_\mathfrak{p}$ then the pre-image $A_{[\mathfrak{p}]}$ of $A_\mathfrak{p}$ under the localization map $R \to R_\mathfrak{p}$ contains B.

(3) $\Rightarrow$ (2): Let $\mathfrak{q} \in \mathrm{Spec}B$ and $\mathfrak{p}{:}= \mathfrak{q} \cap A$. Of course, $A_{[\mathfrak{p}]} \subset B_{[\mathfrak{q}]}$. In order to prove the converse inclusion we first remark that $\mathfrak{p}B \subset \mathfrak{q}$, hence $\mathfrak{p}B \neq B$. By hypothesis $B \subset A_{[\mathfrak{p}]}$. Let $x \in B_{[\mathfrak{q}]}$ be given. Choose $b \in B \setminus \mathfrak{q}$ with $bx =: b_1 \in B$. We then have elements a, a_1 in $A \setminus \mathfrak{p}$ with $ab \in A$, $a_1b_1 \in A$. Since $a \in B \setminus \mathfrak{q}$, also $ab \in B \setminus \mathfrak{q}$, hence $ab \in A \cap (B \setminus \mathfrak{q}) = A \setminus \mathfrak{p}$. Also $a_1ab \in A \setminus \mathfrak{p}$. From $(a_1ab)x = a(a_1bx) = a(a_1b_1) \in A$ we see that $x \in A_{[\mathfrak{p}]}$.
(2) $\Rightarrow$ (2′): trivial.

$(2') \Rightarrow (4)$: Let $x \in B$ be given. Suppose that $(A{:}\,x)B \neq B$. We choose a maximal ideal $\mathfrak{q}$ of B containing $(A{:}\,x)B$. Let $\mathfrak{p} := \mathfrak{q} \cap A$. Then $(A{:}\,x) \subset \mathfrak{p}$. But it follows from $(2')$ that $x \in A_{[\mathfrak{p}]}$, i.e. $(A{:}\,x) \not\subset \mathfrak{p}$. This contradiction proves that $(A{:}\,x)B = B$.
$(4) \Rightarrow (3)$: Let $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p}B \neq B$. Suppose there exists some $x \in B$ with $x \notin A_{[\mathfrak{p}]}$. Then $(A{:}\,x) \subset \mathfrak{p}$. Thus $(A{:}\,x)B \subset \mathfrak{p}B \subsetneqq B$. This contradicts the assumption (4). We conclude that $B \subset A_{[\mathfrak{p}]}$. □

Remarks. In the case of domains Richman [Ri, §2] has studied the properties (3), (4) under the name "good extensions". If $A \subset B$ and B is a domain then good means the same as weakly surjective and as locally surjective. Theorem 13 has a close relation to work of Lazard [L, Chap. IV] and Akiba [A], cf. Theorem 4.4 in the next section.

Definition 4. [Lb, §2.3].

a) An ideal $\mathfrak{a}$ of a ring C is called *dense in* C if its annulator ideal $\mathrm{Ann}_C(\mathfrak{a})$ is zero.
b) A *ring of quotients of* A is a ring $B \supset A$ such that $(A{:}\,x)B$ is dense in B for every $x \in B$.

We recall the following important fact from Lambek's book [Lb, §2.3]. For any ring A there exists a ring of quotients $Q(A)$ of A, explicitly constructed in [Lb], such that for any other ring of quotients B of A there exists a unique homomorphism from B to $Q(A)$ over A. Every such homomorphism is injective. $Q(A)$ is called the *complete ring of quotients* of A. Of course $Q(A)$ contains the total quotient ring $\mathrm{Quot}(A)$ {also called the "classical" quotient ring of A}. For A Noetherian it is known that $\mathrm{Quot}A = Q(A)$, cf. [A, Prop. 1], but in general these two extensions of A may be different.

From condition (4) in Theorem 13 it is clear that, if $A \subset B$ is a weakly surjective ring extension, then B is a ring of quotients of A. Thus every weakly surjective ring extension of A embeds into $Q(A)$ in a unique way.

Definition 5. The *weakly surjective hull* $M(A)$ of A is defined as the ws hull $M(A, Q(A))$ of A in $Q(A)$.

From our discussion of the hulls $M(A, R)$ above the following is evident.

Proposition 3.14. *For every weakly surjective ring extension $A \subset B$ there exists a unique homomorphism $B \to M(A)$ over A, and this is a monomorphism.* □

Thus, without serious abuse, we may regard any ws extension $A \subset B$ as a subextension of $A \subset M(A)$. In particular, $A \subset \mathrm{Quot}A \subset M(A)$.

Remark 3.15. If C is any subring of $M(A)$ containing A then $M(C) = M(A)$. In particular, $MM(A) = M(A)$.

Proof. Since C is ws in $M(A)$ we have embeddings $C \subset M(A) \subset M(C)$. Now A is ws in $M(A)$ and $M(A)$ is ws in $M(C)$, hence A is ws in $M(C)$. Due to the maximality of $M(A)$ we have $M(C) = M(A)$. □

Caution. In general, if C is a subring of $M(A)$ containing A, then A is not necessarily ws in C (cf. §5).

Corollary 3.16. *Let $A \subset B_1$ and $A \subset B_2$ be weakly surjective extensions. Then there exists at most one homomorphism $\lambda: B_1 \to B_2$ over A, and λ is injective.*

Proof. We have unique homomorphisms $\mu_i: B_i \to M(A)$ over $A \quad (i = 1, 2)$, and they both are injective. If $\lambda: B_1 \to B_2$ is a homomorphism over A, this implies that $\mu_2 \circ \lambda = \mu_1$. Thus λ is injective and is uniquely determined by μ_1 and μ_2. □

Of course, the uniqueness of λ is a priori clear, since $A \hookrightarrow B_1$ is an epimorphism (Prop. 6).

§4 More on weakly surjective extensions

Having set the stage we discuss some properties of weakly surjective ring extensions. We are mainly interested in functorial properties and the behavior of ideals.

In the following *we assume that $A \subset B$ is a weakly surjective ring extension.*

Proposition 4.1. *Every weakly surjective ring extension $A \subset B$ is flat (i.e., B is a flat A-module).*

Proof. Let $\alpha: M' \to M$ be an injective homomorphism of A-modules.
We verify that $\alpha \otimes_A B: M' \otimes_A B \to M \otimes_A B$ is again injective. Let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} := \mathfrak{q} \cap A$. Then $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, thus

$$(\alpha \otimes_A B)_{\mathfrak{q}} = (\alpha \otimes_A B) \otimes_B B_{\mathfrak{q}} = \alpha \otimes_A B_{\mathfrak{q}} = \alpha \otimes_A A_{\mathfrak{p}}.$$

Since $A \to A_{\mathfrak{p}}$ is flat the homomorphism $(\alpha \otimes_A B)_{\mathfrak{q}}$ is injective. Since this holds for every $\mathfrak{q} \in \mathrm{Spec} B$ we conclude that $\alpha \otimes_A B$ is injective. □

Proposition 4.2. *Let $A \subset B_1$ and $A \subset B_2$ be weakly surjective ring extensions.*

a) *Then the natural map $A \to B_1 \otimes_A B_2$ is injective and weakly surjective, hence may be regarded as a ws extension.*

b) *If both $A \subset B_1$ and $A \subset B_2$ are subextensions of a ring extension $A \subset R$, then the natural map $B_1 \otimes_A B_2 \to B_1 B_2$ is an isomorphism, in short, $B_1 \otimes_A B_2 = B_1 B_2$.*

Proof. a) Since B_1 is flat over A the natural map $B_1 \to B_1 \otimes_A B_2$ is injective. Also $B_2 \to B_1 \otimes_A B_2$ and $A \to B_1 \otimes B_2$ are injective. We regard A, B_1, B_2 as subrings of $B_1 \otimes_A B_2$ and conclude from Propositions 3.7.a and 3.8. that A is ws in $B_1 \otimes B_2$.
b) In the situation $B_1 \subset R$, $B_2 \subset R$ the ring A is also ws in $B_1 B_2$. The natural map $\lambda: B_1 \otimes_A B_2 \to B_1 B_2$ is a surjective homomorphism over A. By Cor.3.16 λ is also injective, hence is an isomorphism. □

Example 4.3. If $\varphi: A \to B$ is a weakly surjective homomorphism then the natural map $B \otimes_A B \longrightarrow B, x \otimes y \longmapsto xy$, is an isomorphism.

This follows from the proposition since $B \otimes_A B = B \otimes_{\varphi(A)} B$. The statement is just a reformulation of the fact, already known to us (Prop. 3.6), that φ is an epimorphism, cf. e.g. [St, p. 380].

We now invoke the important work of Lazard in his thesis [L] and of Akiba [A]. We have seen that every injective weakly surjective homomorphism is a flat epimorphism (in the category $\mathcal{R}$ of rings). By [L, IV. Prop. 2.4] or [A, Th.1] the converse also holds.

THEOREM 4.4 (Lazard, Akiba). *An injective homomorphism φ is weakly surjective iff φ is a flat epimorphism.* □

PROPOSITION 4.5. *Let $A \subset B$ be a weakly surjective extension and C a subring of B containing A. Then $A \subset C$ is weakly surjective iff C is flat over A.*

PROOF. We know already that weak surjectivity of $A \hookrightarrow C$ implies flatness. Conversely, if $A \hookrightarrow C$ is flat then $A \hookrightarrow C$ is epimorphic by the theory of Lazard [L, IV Cor. 3.2], hence is ws. □

Up to very minor points also the results to follow, up to Proposition 10, are contained in Lazard's thesis [L], and many more. For the convenience of the reader we give short proofs in the present frame work. Our focus is different from Lazard's, since we only strive for the understanding of a special class of flat epimorphic extensions, the Prüfer extensions to be defined in §5.

As before we are given a ws extension $A \subset B$.

PROPOSITION 4.6. *Let $\mathfrak{b}$ be an ideal of B and $\mathfrak{a} := \mathfrak{b} \cap A$. Then $\mathfrak{b} = \mathfrak{a}B$.*

PROOF. Let $\mathfrak{c} := \mathfrak{a}B$. Then $\mathfrak{c} \subset \mathfrak{b}$ and $\mathfrak{c} \cap A = \mathfrak{a}$. We have a commuting triangle of natural homomorphisms

$$\begin{tikzcd} & B/\mathfrak{c} \arrow[dd, "\lambda"] \\ A/\mathfrak{a} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & B/\mathfrak{b} \end{tikzcd}$$

with α and β injective (and λ surjective). Both α and β are ws. Thus λ is injective (hence an isomorphism) by Cor. 3.16. This means that $\mathfrak{c} = \mathfrak{b}$. □

The nil radical of a ring C will be denoted by $\operatorname{Nil} C$.

EXAMPLE 4.7. $\operatorname{Nil} B = (\operatorname{Nil} A)B$.

Indeed, we have $(\operatorname{Nil} B) \cap A = \operatorname{Nil} A$. □

THEOREM 4.8. *Let $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p}B \neq B$. Then $\mathfrak{q} := \mathfrak{p}B$ is a prime ideal of B. This is the unique prime ideal of B lying over $\mathfrak{p}$. If B is given as a subextension of an extension $A \subset R$ then $\mathfrak{q} = \mathfrak{p}_{[\mathfrak{p}]} \cap B$.*

Proof. We have $A_{\mathfrak{p}} = B_{\mathfrak{p}}$. Thus $\mathfrak{p}B_{\mathfrak{p}}$ is the unique maximal ideal of $B_{\mathfrak{p}}$. Let $\mathfrak{q}$ denote the pre-image of $\mathfrak{p}B_{\mathfrak{p}}$ under the localization map $B \to B_{\mathfrak{p}}$. From the natural commuting triangle

$$\begin{array}{ccc} A & \hookrightarrow & B \\ & \searrow \quad \swarrow & \\ & A_{\mathfrak{p}} = B_{\mathfrak{p}} & \end{array}$$

we read off that $\mathfrak{q} \cap A = \mathfrak{p}$. By Prop. 6 we have $\mathfrak{p}B = \mathfrak{q}$. Thus $\mathfrak{p}B$ is a prime ideal. Now assume that $A \subset B \subset R$. Then $B \subset A_{[\mathfrak{p}]}$ by Theorem 3.13. $\mathfrak{q}' := \mathfrak{p}_{[\mathfrak{p}]} \cap B$ is a prime ideal of B with $\mathfrak{q}' \cap A = \mathfrak{p}_{[\mathfrak{p}]} \cap A = \mathfrak{p}$. Thus $\mathfrak{q}' = \mathfrak{q}$. $\square$

Remark 4.9. If $\mathfrak{p}B = B$ then certainly $\mathfrak{p}B \neq \mathfrak{p}_{[\mathfrak{p}]} \cap B$.

Let $X(B/A)$ denote the image of the restriction map $\mathfrak{q} \mapsto \mathfrak{q} \cap A$ from Spec$B$ to Spec$A$. We endow $X(B/A)$ with the subspace topology in Spec$A$. It follows from Theorem 8 that $X(B/A)$ is the set of all $\mathfrak{p} \in$ Spec$A$ with $\mathfrak{p}B \neq B$.

Proposition 4.10. *The restriction map SpecB $\to$ SpecA is a homeomorphism from SpecB to $X(B/A)$. The set $X(B/A)$ is pro-constructible and dense in SpecA. It is closed under generalizations in SpecA.*

Proof. We use the framework of spectral spaces, cf. [Ho] or e.g. [KS, Chap. III]. The restriction map Spec$B \to$ Spec$A$ is spectral. Thus $X(B/A)$ is pro-constructible in Spec$A$, hence is itself a spectral space. Again by Theorem 8 the restriction map r: Spec$B \to X(B/A)$ is bijective. If $x, y \in$ Spec$B$ and $r(y)$ is a specialization of $r(x)$ then y is a specialization of x. Since r is spectral this implies that r is a homeomorphism.

Since A is a subring of B, the image $X(B/A)$ of the restriction map contains all minimal prime ideals of A and is dense in Spec$A$. If $\mathfrak{p} \in$ Spec$A$ and $\mathfrak{p}B \neq B$, then $\mathfrak{r}B \neq B$ for the prime ideals $\mathfrak{r}$ of A contained in $\mathfrak{p}$. Thus $X(B/A)$ is closed under generalizations. {This already follows from the fact that $A \hookrightarrow B$ is flat, hence the "going down theorem" holds for prime ideals.} $\square$

We briefly discuss relations between weakly surjective extensions and integral extensions.

Proposition 4.11(cf. [G$_1$, Prop. 11]). *If a ring homomorphism $\varphi: A \to B$ is both weakly surjective and integral then φ is surjective.*

Proof. Replacing A by $\varphi(A)$ we assume without loss of generality that $A \subset B$ and φ is the inclusion mapping. We have to prove that $A = B$.

Suppose there exists an element $x \in B \setminus A$. Then $(A{:}x)$ is a proper ideal of A. Since B is integral over A, this implies that $(A{:}x)B \neq B$. This contradicts property (4) in Theorem 3.13. Thus $A = B$. $\square$

Proposition 4.12. ([Ri, §4] for R a field, [G$_1$, Prop. 11] for $R =$ Quot$A$). *Assume that $A \subset B \subset R$ are ring extensions, and that A is weakly surjective in B. For the integral closures $\tilde{A}$ and $\tilde{B}$ of A and B in R the following holds.*

i) $\tilde{B} = \tilde{A} \cdot B$.
ii) *$\tilde{A}$ is weakly surjective in $\tilde{B}$.*

Proof. The argument in [Ri] (p.797, proof of Prop.1) extends to our more general situation.

$\square$

§5 Basic theory of relative Prüfer rings

Let R be a ring and A a subring of R.

Definition 1 [G_2, §4] *) A is called an *R-Prüfer ring*, or a *Prüfer subring of R*, if $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R for every maximal ideal $\mathfrak{p}$ of A. We then also say that A is *Prüfer in R*, or that R is a *Prüfer extension* of A.

N.B. According to Prop. 2.10 this holds iff $(A_\mathfrak{p}, \mathfrak{p}_\mathfrak{p})$ is a Manis pair in $R_\mathfrak{p}$ for every maximal ideal $\mathfrak{p}$ of A.

In particular, if R is a field, we arrive at the classical notion of a Prüfer domain.

Proposition 5.1. *Assume that A is Prüfer in R.*

i) *For every prime ideal $\mathfrak{p}$ of A the pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is Manis in R.*
ii) *The following are equivalent.*

(1) *A is a Manis subring of R.*
(2) *A is a valuation subring of R.*
(3) *$R \setminus A$ is multiplicatively closed, i.e. $(R \setminus A)(R \setminus A) \subset R \setminus A$.*

Moreover, if $A \neq R$ and (1) – (3) hold then $(A, \mathfrak{p}_A)$ is a Manis pair of R. $\{\mathfrak{p}_A$ had been defined in §2, Def.2.$\}$

Proof. i) Let $\mathfrak{p}$ be a prime ideal of A. We choose a maximal ideal $\mathfrak{m} \supset \mathfrak{p}$. There exists a Manis valuation v on R with $A_v = A_{[\mathfrak{m}]}$, $\mathfrak{p}_v = \mathfrak{m}_{[\mathfrak{m}]}$. If $A \cap \operatorname{supp} v \not\subset \mathfrak{p}$, then we choose some $s \in (\operatorname{supp} v) \cap (A \setminus \mathfrak{p})$. We have $sR \subset A_{[\mathfrak{m}]} \subset A_{[\mathfrak{p}]}$, and we conclude that $A_{[\mathfrak{p}]} = R$. Thus $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is certainly Manis in R in this case. Assume now that $A \cap \operatorname{supp} v \subset \mathfrak{p}$. Then it follows from Prop.2.11 that $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is Manis in R.

In assertion (ii) the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. We prove (3) $\Rightarrow$ (1). We may assume $A \neq R$. Let $\mathfrak{p} := \mathfrak{p}_A$. Let $x \in A_{[\mathfrak{p}]}$ be given. There exists some $d \in A \setminus \mathfrak{p}$ with $dx \in A$. If $x \notin A$ this would imply $d \in \mathfrak{p}$ by definition of $\mathfrak{p} = \mathfrak{p}_A$. Thus $x \in A$. This proves $A_{[\mathfrak{p}]} \subset A$, i.e. $A_{[\mathfrak{p}]} = A$. Then $\mathfrak{p}_{[\mathfrak{p}]} \subset A$, hence $\mathfrak{p} = \mathfrak{p}_{[\mathfrak{p}]} \cap A = \mathfrak{p}_{[\mathfrak{p}]}$. Since A is Prüfer in R we conclude that the pair $(A, \mathfrak{p})$ is Manis in R. $\square$

*) It turned out that Griffin's definition is not quite "correct". He only demands that the $A_{[\mathfrak{p}]}$ are Manis subrings of R. For a reasonable theory it is necessary to include a condition on the $\mathfrak{p}_{[\mathfrak{p}]}$, cf. also [Gr, p.285].

The following theorem gives a bunch of criteria for a given ring extension $A \subset R$ to be Prüfer. It is here that the theory of Manis valuations and the theory of weakly surjective ring extensions, displayed in §1, §2 and in §3, §4 respectively, come together.

THEOREM 5.2. *The following are equivalent.*

(1) *A is an R-Prüfer ring.*
(2) *A is weakly surjective in every R-overring.*
(2′) *A is weakly surjective in $A[x]$ for every $x \in R$.*
(3) *If B is any R-overring of A then $(A\colon x)B = B$ for every $x \in B$.*
(4) *Every R-overring of A is integrally closed in R.*
(5) *A is integrally closed in R, and $A[x] = A[x^n]$ for every $x \in R$ and $n \in \mathbb{N}$.*
(5′) *A is integrally closed in R, and $A[x] = A[x^2]$ for every $x \in R$.*
(6) *A is integrally closed in R. For every $x \in R$ there exists a polynomial $F[T] = \sum_{i=0}^{d} a_i T^i$ with all $a_i \in A$, $a_j = 1$ for at least one index j, such that $F(x) = 0$.*
(7) *A is integrally closed in R. For every $x \in R$ and every maximal ideal $\mathfrak{p}$ of A there exists a polynomial $F_{x,\mathfrak{p}}(T) \in A[T] \setminus \mathfrak{p}[T]$ such that $F_{x,\mathfrak{p}}(x) = 0$.*
(8) *$(A\colon x) + x(A\colon x) = A$ for every $x \in R$.*
(9) *A is integrally closed in R. For every overring B of R the restriction map $SpecB \to SpecA$ is injective.*
(9′) *A is integrally closed in R. If B is an R-overring of A and $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals of B with $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$ then $\mathfrak{q} = \mathfrak{q}'$.*
(10) *A is integrally closed in R. For every prime ideal $\mathfrak{p}$ of A there exists a unique Manis pair $(B, \mathfrak{q})$ in R over $(A, \mathfrak{p})$, i.e. with $A \subset B$, $\mathfrak{q} \cap A = \mathfrak{p}$.*
(11) *For every R-overring B of A the inclusion map $A \hookrightarrow B$ is an epimorphism (in the category of rings).*
(11′) *For every $x \in R$ the inclusion map $A \hookrightarrow A[x]$ is an epimorphism.* □

REMARKS. The equivalence of (1), (2), (3), (4) had already been stated by Griffin [G_2, Prop.6, Th.7], but he made additional assumptions and did not present the proofs. On the other hand, Griffin weakened the hypothesis that our rings have unit elements. The equivalence of (1), (4), (8) has been proved by Eggert for $R = Q(A)$, the complete ring of quotients of A [Eg, Th.2]. The equivalence of (1) and any of the conditions (4) - (7) is a generalization of classical results for R a field (cf. e.g. [E, Th.11.10]). The equivalence of (1) and (11) for R a field has been proved by Storrer [St_1]. The equivalence of (1), (2), (4), (8) has been stated in full generality by Rhodes [Rh, Th.2.1]. Unfortunately his proof contains a gap (cf. Introduction to the present paper). E.D. Davis studied extensions $A \subset R$ with property (4) under the name "normal pairs". In the case of domains some of our results in this section can be read off from his paper [Da].

PROOF. (1) $\Rightarrow$ (2): Let B be an R-overring of A and $\mathfrak{q}$ a prime ideal of B. Let $\mathfrak{p} := \mathfrak{q} \cap A$. We verify that $A_{[\mathfrak{p}]} = B_{[\mathfrak{q}]}$ and then will be done by Theorem 3.13. Of course, $A_{[\mathfrak{p}]} \subset B_{[\mathfrak{q}]}$. Let $x \in R \setminus A_{[\mathfrak{p}]}$ be given. We prove that $x \notin B_{[\mathfrak{q}]}$, and then will be done.

Since $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R there exists an element y of $\mathfrak{p}_{[\mathfrak{p}]}$ with $xy \in A_{[\mathfrak{p}]} \setminus \mathfrak{p}_{[\mathfrak{p}]}$. We choose elements a and c in $A \setminus \mathfrak{p}$ with $a(xy) \in A$ and $cy \in \mathfrak{p}$. We have $a(xy) \in A \setminus \mathfrak{p}$. Suppose that $x \in B_{[\mathfrak{q}]}$. Then there exists some $b \in B \setminus \mathfrak{q}$ with $bx \in B$. We have $a(bx)(cy) \in \mathfrak{q}$. On the other hand, $a(bx)(cy) = bc(axy) \in B \setminus \mathfrak{q}$. This contradiction proves that $x \notin B_{[\mathfrak{q}]}$.

(2) $\Rightarrow$ (2′): trivial.

(2) $\Leftrightarrow$ (3): Clear from Th. 3.13.

(2′) $\Rightarrow$ (3): Let $x \in B$. Then $(A{:}\,x)A[x] = A[x]$. A fortiori $(A{:}\,x)B = B$.

(2) $\Rightarrow$ (4): Let B be an R-overring of A, and let $C = \tilde{B}$ denote the integral closure of B in R. By (2) A is ws in C. Thus B is ws in C (Prop. 3.7.b). Prop. 4.11 tells us that $C = B$, i.e. B is integrally closed in R.

(4) $\Rightarrow$ (5): x is integral over $A[x^n]$. By assumption (4) the subring $A[x^n]$ is integrally closed in R. Thus $x \in A[x^n]$.

(5) $\Rightarrow$ (5′): trivial.

(5′) $\Rightarrow$ (6): For every $x \in R$ we have a relation $x = \sum\limits_{i=0}^{m} a_i x^{2i}$ with $m \in \mathbb{N}_0$, $a_i \in A$.

(6) $\Rightarrow$ (7): trivial.

(7) $\Rightarrow$ (1): Theorem 2.12 tells us that $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in R for every $\mathfrak{p} \in \operatorname{Spec} A$.

(1) $\Rightarrow$ (8): Suppose there exists some $x \in R$ with $I{:}= (A{:}\,x) + x(A{:}\,x) \neq A$. We choose a maximal ideal $\mathfrak{m}$ of A containing I. Then $x \in R \setminus A_{[\mathfrak{m}]}$ since $(A{:}\,x) \subset \mathfrak{m}$. By (1) and Theorem 2.4 (iii) there exists some $x' \in \mathfrak{m}_{[\mathfrak{m}]}$ with $xx' \in A_{[\mathfrak{m}]} \setminus \mathfrak{m}_{[\mathfrak{m}]}$. We then choose some $d \in A \setminus \mathfrak{m}$ with $dx' \in \mathfrak{m}$ and $dxx' \in A \setminus \mathfrak{m}$. It follows that $dx' \in (A{:}\,x)$ and $dxx' \in x(A{:}\,x) \subset \mathfrak{m}$, a contradiction. Thus (8) holds.

(8) $\Rightarrow$ (1): We prove for a given prime ideal $\mathfrak{p}$ of A that the pair $(A_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is Manis in R by verifying condition (iii) in Theorem 2.4. Let $x \in R \setminus A_{[\mathfrak{p}]}$. Then $(A{:}\,x) \subset \mathfrak{p}$. By (8) we know that $x(A{:}\,x) \not\subset \mathfrak{p}$. Thus there exists some $x' \in (A{:}\,x) \subset \mathfrak{p}$ with $xx' \in A \setminus \mathfrak{p}$.

The equivalence of (1), (9), (9′), (10) is evident from Theorem 2.14. The implication (2′) $\Rightarrow$ (11′) follows from the fact that every weakly surjective map is an epimorphism (cf. Prop.3.6).

(11′) $\Rightarrow$ (11): Suppose there exists an R-overring B of A such that the inclusion map $A \hookrightarrow B$ is not an epimorphism. Then there exist two ring homomorphisms φ_1, φ_2 from B to some ring C with $\varphi_1|A = \varphi_2|A$ but $\varphi_1 \neq \varphi_2$. We choose some $x \in B$ with $\varphi_1(x) \neq \varphi_2(x)$. The restrictions $\varphi_1|A[x]$ and $\varphi_2|A[x]$ are different, but $\varphi_1|A = \varphi_2|A$. This contradicts the assumption (11′).

(11) $\Rightarrow$ (4): Let B be an R-overring of A, and let $x \in R$ be integral over B. We want to prove that $x \in B$. The inclusion $A \hookrightarrow B[x]$ is an epimorphism. Thus (for purely categorial reasons) also the inclusion $B \hookrightarrow B[x]$ is an epimorphism. By an easy proposition of Lazard [L, Chap. IV, Prop.1.7], $B[x] = B$. □

From condition (4) in this theorem one obtains immediately

Corollary 5.3. *Let B be an R-overring of A. If A is Prüfer in R then B is Prüfer in R and A is Prüfer in B.* □

From condition (8) in the theorem we obtain

COROLLARY 5.4. *If A is Prüfer in R then for any $x \in R$ the ideal $(A{:}\, x)$ is generated by two elements.*

Indeed, we have elements a and b in $(A{:}\, x)$ with $1 = a + xb$. If $u \in (A{:}\, x)$ then $u = ua + (ux)b$. Thus $(A{:}\, x) = Aa + Ab$. $\square$

Theorem 2 contains the fact that every R-Prüfer ring is integrally closed in R. The reader might ask for a more direct proof of this statement. Indeed this follows from the definition of R-Prüfer rings and an elementary fact which holds without any assumption about our subring A of R.

REMARK 5.5. If M is an A-submodule of R, then

$$M = \bigcap_{\mathfrak{p}\in\Omega} A_{[\mathfrak{p}]} \cdot M = \bigcap_{\mathfrak{p}\in\Omega} M_{[\mathfrak{p}]},$$

with Ω denoting the set of maximal ideals of A. In particular $A = \bigcap\limits_{\mathfrak{p}\in\Omega} A_{[\mathfrak{p}]}$.

PROOF. Of course, $M \subset A_{[\mathfrak{p}]}M \subset M_{[\mathfrak{p}]}$ for every $\mathfrak{p} \in \Omega$. Let $x \in \bigcap\limits_{\mathfrak{p}\in\Omega} M_{[\mathfrak{p}]}$ be given. Consider the ideal $\mathfrak{a}{:}= \{a \in A \mid ax \in M\}$. For every $\mathfrak{p} \in \Omega$ the intersection $\mathfrak{a} \cap (A \setminus \mathfrak{p})$ is not empty, i.e. $\mathfrak{a} \not\subset \mathfrak{p}$. Thus $\mathfrak{a} = A$, i.e. $x \in M$. $\square$

We now look for permanence properties of relative Prüfer rings.

THEOREM 5.6 [Rh, Prop.3.1.3]. *Assume that A is a Prüfer subring of B and B is a Prüfer subring of C. Then A is Prüfer in C.*

PROOF (cf. [Rh, loc.cit]). We verify for a given prime ideal $\mathfrak{p}$ of A that the pair $(A_\mathfrak{p}, \mathfrak{p}_\mathfrak{p})$ is Manis in $C_\mathfrak{p}$. Replacing A, B, C by $A_\mathfrak{p}$, $B_\mathfrak{p}$, $C_\mathfrak{p}$ we assume without loss of generality that A is local and $\mathfrak{p}$ is the maximal ideal of A. We will apply Theorem 2.5. By this theorem (or Prop.1.3) B is local, and the maximal ideal $\mathfrak{q}$ of B is contained in $\mathfrak{p}$. Let $x \in C \setminus A$ be given. If $x \in B$ then, by Theorem 2.5, $x \in B^*$ and $x^{-1} \in \mathfrak{p}$. If $x \notin B$ then, by the same theorem, $x \in C^*$ and $x^{-1} \in \mathfrak{q} \subset \mathfrak{p}$. Thus in both cases x is a unit in C and $x^{-1} \in A$. We conclude, again by Theorem 2.5, that $(A, \mathfrak{p})$ is Manis in C. $\square$

PROPOSITION 5.7. *Assume that A is a Prüfer subring of B. Then, for any ring homomorphism $\psi{:}\, B \to D$ the ring $\psi(A)$ is Prüfer in $\psi(B)$.*

PROOF. Let C' be a subring of $\psi(B)$ containing $\psi(A)$. We verify that $\psi(A)$ is weakly surjective in C', and then will be done by condition (2) in Theorem 2. Indeed, $C{:}= \psi^{-1}(C')$ is a subring of B containing A. Thus A is weakly surjective in C. By Proposition 3.8 $\psi(A)$ is weakly surjective in $\psi(C) = C'$. $\square$

PROPOSITION 5.8 [Rh, Prop.3.1.1]. *Let $A \subset R$ be a ring extension and I an ideal of R contained in A. Then A is Prüfer in R iff A/I is Prüfer in R/I.*

Proof. If A is Prüfer in R then the preceding proposition tells us that A/I is Prüfer in R/I. Assume now that the latter holds. We verify condition (4) in Theorem 2 and then will be done.

Let B be an R-overring of A. Then B/I is an R/I-overring of A/I. Thus B/I is integrally closed in R/I. Let $x \in R$ be integral over B. Then $x + I \in B/I$. Since $I \subset B$ we conclude that $x \in B$. Thus B is integrally closed in R. $\square$

Theorem 5.9. *Let $\varphi: R \to R'$ be an integral ring homomorphism. Let A be a Prüfer subring of R, and let A' denote the integral closure of $\varphi(A)$ in R'. Then A' is a Prüfer subring of R'.*

Proof. We verify condition (7) in Theorem 2. Let an element x of R' and a prime ideal $\mathfrak{q}$ of R' be given. Let $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. We look for a polynomial $G(T) \in A[T] \setminus \mathfrak{p}[T]$ with $G^{\varphi}(x) = 0$, where $G^{\varphi}(T)$ denotes the polynomial obtained from $G(T)$ by applying φ to the coefficients.

We start with a polynomial

$$F(T) = T^n + a_1 T^{n-1} + \cdots + a_n \in R[T]$$

such that $F^{\varphi}(x) = 0$. Such a polynomial exists since φ is integral. Let $v: R \twoheadrightarrow \Gamma \cup \infty$ denote the Manis valuation on R with $A_v = A_{[\mathfrak{p}]}$, $\mathfrak{p}_v = \mathfrak{p}_{[\mathfrak{p}]}$. We choose an index $r \in \{1, \ldots, n\}$ with $v(a_r) = \mathrm{Min}\{v(a_i) \mid 1 \le i \le n\}$. We distinguish two cases.

Case 1: $v(a_r) = \infty$. Now certainly $a_i \in A_{[\mathfrak{p}]}$ for $i = 1, 2, \ldots, n$. We choose some $d \in A \setminus \mathfrak{p}$ with $da_i \in A$ for all i. The polynomial $G(T) := dF(T)$ does the job.

Case 2: $v(a_r) < \infty$. We choose some $b \in R$ with $v(ba_r) = 0$. This is possible since v is Manis. We have

$$ba_i \in A_{[\mathfrak{p}]} \quad \text{for every} \quad i \in \{1, \ldots, n\}$$

and $ba_r \notin \mathfrak{p}_{[\mathfrak{p}]}$. We choose some $c \in A \setminus \mathfrak{p}$ with $cba_i \in A$ for $i = 1, \ldots, n$. The polynomial $G(T) := cbF(T)$ does the job. $\square$

Remark. Since $\varphi(A)$ is weakly surjective in $\varphi(R)$ we conclude from Prop.4.12 that $R' = A' \cdot \varphi(R)$.

Theorem 5.10. *Let A be a subring of R and B, C be two R-overrings of A. Assume that A is Prüfer in B and weakly surjective in C. Then C is Prüfer in BC.*

Proof. We pick a prime ideal $\mathfrak{q}$ of C and verify that $(C_{\mathfrak{q}}, \mathfrak{q}_{\mathfrak{q}})$ is a Manis pair in $(BC)_{\mathfrak{q}}$.

Let $\mathfrak{p} := \mathfrak{q} \cap A$. Then $A_{\mathfrak{p}} = C_{\mathfrak{p}} = C_{\mathfrak{q}}$ and $\mathfrak{q}_{\mathfrak{q}} = \mathfrak{q}_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$ (cf. Lemma 3.2). Thus $C \setminus \mathfrak{q}$ is the saturum of the multiplicative set $A \setminus \mathfrak{p}$ in C. Notice also that $BC = B \otimes_A C$ (Prop. 4.2). Thus $(BC)_{\mathfrak{p}} = B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} C_{\mathfrak{p}} = B_{\mathfrak{p}}$, more precisely, the subrings $(BC)_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ are equal. We conclude that $B_{\mathfrak{q}} = B_{\mathfrak{p}}$, $(BC)_{\mathfrak{q}} = (BC)_{\mathfrak{p}} = B_{\mathfrak{p}}$, $(C_{\mathfrak{q}}, \mathfrak{q}_{\mathfrak{q}}) = (A_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$. Since A is Prüfer in B, the pair $(C_{\mathfrak{q}}, \mathfrak{q}_{\mathfrak{q}})$ is Manis in $(BC)_{\mathfrak{q}}$. $\square$

COROLLARY 5.11. *Let A be a subring of R and B, C be two R-overrings of A. If A is Prüfer in B and in C, then A is Prüfer in BC.*

This follows from theorems 10 and 6. $\square$

COUNTEREXAMPLE 5.12. If $A \subset B$ is a Prüfer extension and $A \subset C$ is a flat ring extension then C is not necessarily Prüfer in $B \otimes_A C$. Here is a simple example: Let A be a non trivial valuation ring of a field K. Then A is Prüfer in K, but the polynomial ring $A[T]$ in one variable T is not Prüfer in $K[T]$.

Indeed, let $\mathfrak{m}$ be the maximal ideal of A and let $M := \mathfrak{m} + TA[T]$, which is a maximal ideal of $C := A[T]$. In the extension $K[T]$ of C we have $C_{[M]} = C$, $M_{[M]} = M$, as is easily verified. The pair (C, M) is not Manis in $K[T]$. $\square$

REMARK 5.13. Let $A \subset R$ be a ring extension and $(B_i | i \in I)$ an upward directed family of R-overrings of A. Assume that A is Prüfer in every B_i. Then A is Prüfer in $B := \bigcup_{i \in I} B_i$.

PROOF. Let C be an R-overring of A contained in B. We verify that A is weakly surjective in C and then will be done by Theorem 2. Now C is the union of the upward directed family of subrings $(C \cap B_i | i \in I)$. A is weakly surjective in $C \cap B_i$ for every $i \in I$. Thus A is weakly surjective in C (Remark 3.12). q.e.d.

We now have the means to establish a theory of "Prüfer hulls" analogous to the theory of weakly surjective hulls in §3.

THEOREM 5.14. *Let $A \subset R$ be a ring extension. Then there exists a unique R-overring $\mathrm{Pr}(A, R)$ of A such that A is Prüfer in $\mathrm{Pr}(A, R)$, and $\mathrm{Pr}(A, R)$ contains every R-overring of A in which A is Prüfer.*

This follows from Corollary 11 and Remark 13 (cf. the proof of Prop. 3.11). $\square$

DEFINITION 2. We call $\mathrm{Pr}(A, R)$ the *Prüfer hull of A in R.*

Of course, $\mathrm{Pr}(A, R)$ is contained in the weakly surjective hull $M(A, R)$ of A in R, and $\mathrm{Pr}(A, R) = \mathrm{Pr}(A, C)$ for every R-overring C with $C \supset \mathrm{Pr}(A, R)$. Also $\mathrm{Pr}(A, R) = \mathrm{Pr}(B, R)$, if B is any R-overring of A contained in $\mathrm{Pr}(A, R)$.

DEFINITION 3. For any ring A the *Prüfer hull $P(A)$ of A* is defined as the Prüfer hull of A in the complete quotient ring $Q(A)$ (cf. §3), $P(A) := \mathrm{Pr}(A, Q(A))$.

Of course, $P(A)$ is contained in the weakly surjective hull $M(A)$. The classical Prüfer rings (with zero divisors) are precisely the rings A with $\mathrm{Quot} A \subset P(A)$. If A' is a weakly surjective ring extension of A, contained in $M(A)$ without loss of generality, then
$A' \cdot P(A) \subset P(A')$ by Theorem 10 above.

EXAMPLE 5.15. Let V be an affine algebraic variety over some real closed field k. The ring R of (k-valued, continuous) semialgebraic functions on $V(k)$ is "Prüfer closed", i.e. $P(R) = R$. This has been proved recently by Niels Schwartz [Sch_2] within the framework of his theory of real closed rings. His proof would take us here too far afield.

Let d be a natural number. In §6 we will see that R is Prüfer over the subring $A = k\left[\frac{1}{1+x^{2d}} | x \in R\right]$ generated by k and the elements $\frac{1}{1+x^{2d}}$, $x \in R$, cf. Th.6.8. Thus $R = P(A)$.

$\square$

§6 Examples of convenient ring extensions and relative Prüfer rings

In this section R is a ring and A is a subring of R. We are looking for handy criteria which guarantee that A is Manis or Prüfer in R, and we will discuss examples emanating from some of these criteria.

Theorem 6.1. *Assume that A is integrally closed in R. Assume further that for every $x \in R \setminus A$ there exists a monic polynomial $F(T) \in A[T]$ and a unimodular polynomial $G(T) \in A[T]$ (i.e. the ideal of A generated by all coefficients of $G(T)$ is A), such that $F(x) \in R^*$, $\deg G < \deg F$ and $G(x)/F(x) \in A$. Then A is Prüfer in R.*

Proof. We verify that for a given element x of R and a given maximal ideal $\mathfrak{m}$ of A there exists a polynomial $H(T) \in A[T] \setminus \mathfrak{m}[T]$ with $H(x) = 0$, and then will be done by Theorem 5.2.

If $x \in A$ we take $H(T) = T - x$. Now let $x \in R \setminus A$. We choose polynomials $F(T)$, $G(T)$ as indicated in the theorem. We put $b := G(x)/F(x) \in A$ and take $H(T) := bF(T) - G(T)$. Then $H(x) = 0$. If $b \in \mathfrak{m}$ then $H(T) \notin \mathfrak{m}[T]$, since $G(T)$ is unimodular. If $b \notin \mathfrak{m}$ then again $H(T) \notin \mathfrak{m}[T]$, since $\deg G < \deg F$. q.e.d.

Definition 1. We call a valuation v on R a *Prüfer-Manis valuation* (or *PM-valuation*, for short), if v is Manis and A_v is Prüfer in R. We call a subring B of R a *Prüfer-Manis subring* of R if $B = A_v$ for some Prüfer-Manis valuation v on R. We then also say that the ring B is *Prüfer-Manis* (or *PM*, for short) *in R.*

If A is Prüfer in R and B is an R-overring of A which is Manis in R, then, of course, B is PM in R. Thus the valuations which really matter in the theory of relative Prüfer rings are the PM-valuations and not just the Manis valuations. We defer a systematic theory of PM-subrings of R to later chapters, but now look for examples of such rings.

Theorem 6.2. *Assume that $A \neq R$ and the set $S := R \setminus A$ is multiplicatively closed. Assume further that for every $x \in R \setminus A$ there exists a monic polynomial $F(T) \in A[T]$ of degree ≥ 1 with $F(x) \in R^*$. Then A is PM in R.*

Proof. We verify that A is Prüfer in R and then will be done by Prop. 5.1.ii. We know from Theorem 2.1 that A is integrally closed in R. Let $x \in R \setminus A$ be given. We choose a polynomial $F(T) \in A[T]$ as indicated in the theorem. Certainly $F(x) \in R \setminus A$, since A is integrally closed in R. We conclude from the equation $1 = F(x) \cdot F(x)^{-1}$ that $1/F(x) \in A$, since otherwise we would get the contradiction $1 \in R \setminus A$. Now Theorem 1 tells us that A is Prüfer. $\square$

DEFINITION 2. a) Let k be a subring of R. We say that R is *convenient over* k, if every R-overring A of k which has a multiplicatively closed complement $R \setminus A$ is PM in R.
b) We call the *ring* R *convenient*, if R is convenient over its prime ring $\mathbb{Z} \cdot 1$.

EXAMPLE 1. Every field is a convenient ring. $\square$

The idea behind Definition 2 is that, as far as valuations are concerned, a convenient ring is nearly as "convenient" as a field. If R is only convenient over some subring k then at least this should be true for the (special) valuations v with $A_v \supset k$. In particular we expect that for a convenient ring extension $k \subset R$ we have a theory of R-Prüfer rings $A \supset k$ nearly as good as in the field case.

From Theorem 2 we extract

SCHOLIUM 6.3. Let k be a subring of R with the following property.
($*$) For every $x \in R \setminus k$ there exists some monic polynomial $F_x(T) \in k[T]$, $F_x \neq 1$, with $F_x(x) \in R^*$.
Then R is convenient over k. $\square$

We give some examples of ring extensions which are convenient and, up to the first and the last one, even fulfill condition ($*$).

EXAMPLE 2 (Generalization of Example 1). If R has Krull dimension zero then R is convenient.

PROOF. Let A be a subring of R with $A \neq R$ and $R \setminus A$ multiplicatively closed. We prove that A is Prüfer in R. Then it will follow from Prop. 5.1.ii that A is also Manis in R.

The ring A is integrally closed in R by Theorem 2.1.ii. Given an element $x \in R$ we prove that there exists a unimodular polynomial $F(T) \in A[T]$ with $F(x) = 0$. Then we will be done by Theorem 5.2.

If $x \in A$ take $F(T) = T - x$. Now let $x \in R \setminus A$. There exists some $n \in \mathbb{N}$ and $y \in R$ with $x^{n+1}y = x^n$, cf. [Huc, Th.3.5]. Then $(xy)^{n+1} = (xy)^n$. Since A is integrally closed in R, this implies $xy \in A$. Since $R \setminus A$ is closed under multiplication we conclude that $y \in A$. The polynomial $F(T) = yT^{n+1} - T^n$ fits our needs. $\square$

EXAMPLE 3. Every ring R with $1 + \Sigma R^2 \subset R^*$ is convenient. Indeed, it suffices to know that $1 + R^2 \subset R^*$ in order to conclude that R is convenient. $\square$

COMMENT. This is the most important class of rings we have in mind for use in real algebra. Recall that for every ring A the localization $\Sigma^{-1}A$ with respect to the multiplicative set $\Sigma := 1 + \Sigma A^2$ is such a ring, and that A and $\Sigma^{-1}A$ have the same real spectrum. For many problems in real algebra we may replace A by $\Sigma^{-1}A$ and thus arrive at a convenient ring. {If A is not real, i.e. $-1 \in \Sigma A^2$, then $\Sigma^{-1}A$ is the null ring, but this does not bother us.}

Subexample 3 bis. If A is any ring and X is a pro-constructible subset of the real spectrum Sper$A$ then the ring $C(X,A)$ of abstract semialgebraic functions on X (cf. [Sch]) is convenient, since in this ring R we have $1+\Sigma R^2 \subset R^*$. In general $C(X,A)$ has very many zero divisors.

Example 4. If more generally R is a ring such that, for every $x \in R$, there exists a natural number d with $1+x^d \in R^*$, then R is convenient.

Such rings (with d even) seem to be important in the theory of orderings of higher level and higher real spectra (cf. e.g. [B_2], [B_3], [P], [BP], [Be]).

Example 5. Let A be an affine algebra over a field k which is not algebraically closed. Let $V(k)$ denote the set of rational points of the associated k-variety V. {We may identify $V(k) = \mathrm{Hom}_k(A,k)$.} Let U be a k-Zariski-open subset of $V(k)$. {In other words, U is open in the subspace topology of $V(k)$ in Spec$A$.} Let finally S be the multiplicative set consisting of all $a \in A$ with $a(p) \neq 0$ for every $p \in U$. Then $S^{-1}A$ is convenient over k.

Proof. We choose a monic polynomial $F(T) \in k[T]$, $F \neq 1$, in one variable T which has no zeros in k. Let $x \in S^{-1}A$ be given. Write $x = \frac{a}{s}$ with $a \in A$, $s \in S$, and $F(T) = T^d + c_1T^{d-1} + \cdots + c_d$. We have $F(x) = \frac{b}{s^d}$ with $b = a^d + c_1a^{d-1}s + \cdots + c_ds^d$. For every point $p \in U$ we have $\frac{b(p)}{s(p)^d} = F\big(\frac{a(p)}{s(p)}\big) \neq 0$, hence $b(p) \neq 0$. Thus $b \in S$ and $F(x)$ is a unit in $S^{-1}A$. $\square$

Definition 3. We call this ring $S^{-1}A$ the *ring of regular functions on* U.

If the field k is real closed and $U = V(k)$ then S is the set of divisors of the elements in $\Sigma := 1 + \Sigma A^2$, as is well known (e.g. [BCR, Cor. 4.4.5.], [KS, p.142]). Thus $S^{-1}A = \Sigma^{-1}A$, and we are back to Example 3.

Example 6. If R is a semi-local ring containing an infinite field k then R is convenient over k. Indeed, if R has the maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$, then for a given $x \in R$ we find some $\lambda \in k$ with $x - \lambda \notin \mathfrak{m}_i$ for $i = 1, \dots, r$, hence $x - \lambda \in R^*$. $\square$

If k is any ring and $(R_\alpha | \alpha \in I)$ is a direct system of k-algebras fulfilling condition $(*)$ from above then the same holds for the inductive limit $\varinjlim R_\alpha$. Thus Example 6 can be amplified to

Example 7. An inductive limit of semi-local k-algebras over some infinite field k is convenient over k. In particular, if R is a semi-local algebra over some infinite field k, the infinite Galois extensions of R (cf. e.g. [K], there called "coverings") are convenient over k. $\square$

Example 8. Assume that R is the total quotient ring of A, $R = \mathrm{Quot}A$. The ring A is called *additively regular* [Huc, p.32], if for every $x \in R$ there exists some $a \in A$

such that $x + a$ is a "regular element", i.e. a unit in R. Of course then condition $(*)$ is satisfied for $k := A$, and thus R is convenient over A. As Huckaba observes [Huc, p.32 f], if A is Noetherian or, more generally, if the set of zero divisors of A is a union of finitely many prime ideals, then A is additively regular [Huc, p.32 f]. □

Example 9. Assume again that $R = \mathrm{Quot} A$. The ring A is called a *Marot ring* [Huc, p.31], if each ideal of A which contains a non zero divisor is generated by a set of non zero divisors. Marot rings form a very broad class of rings. In particular, every additively regular ring is Marot [Huc, p.33 f]. If A is Marot then $R = \mathrm{Quot} A$ is convenient over A, cf. [Huc, Th.7.7 and Cor.7.8]. But now condition $(*)$ may be violated, as we can show by examples. □

As before R denotes a ring and A a subring of R. We return to the search for Prüfer subrings of R which are not necessarily Manis in R.

If R is a field then the intersection of finitely many valuation subrings of R is Prüfer in R, as is well known. Does the same hold if $k \subset R$ is a convenient extension and if all the valuation rings contain k? Or does this at least hold if the extension $k \subset R$ fulfills the stronger condition $(*)$ in 6.3? We can only prove the following result.

Theorem 6.4. *Let k be a subring of R with the following property.*
$(**)$ *For every $x \in R \setminus k$ there exists a monic polynomial $F_x(T) \in k[T]$, $F_x \neq 1$, with $F_x(x) \in R^*$ and constant term $F_x(0) \in k^*$.*

Let $v_1, \dots, v_n$ be valuations on R with $A_{v_i} \supset k$ for all i. Then the intersection A of the rings A_{v_i} is Prüfer in R.

Proof. A is integrally closed in R. Let $x \in R \setminus A$ be given. We prove that there exists a monic polynomial $H(T) \in k[T]$ of degree ≥ 1 with $H(x) \in R^*$ and $1/H(x) \in A$, and then will be done by Theorem 1.

For every index i with $1 \leq i \leq n$ we choose a monic polynomial $F_i(T) \in k[T]$ with $v_i(F_i(x)) > 0$, if such a polynomial exists. Otherwise we put $F_i(T) := 1$.
Let $G(T) := T F_1(T) \cdots F_n(T)$ and $y := G(x)$. Certainly $y \notin A$, since $x \notin A$ and A is integrally closed in R. A fortiori $y \notin k$. We claim that the polynomial $H(T) := F_y(G(T))$ fits our needs (with F_y as indicated in the theorem).

Certainly $H(T)$ is monic and $H(x) = F_y(y) \in R^*$. Given $i \in \{1, \dots, n\}$ we verify that $v_i(1/H(x)) \geq 0$, and then will be done.

Case 1. $v_i(x) < 0$. Now $v_i(H(x)) = (\deg H) \cdot v_i(x) < 0$, since $H(T)$ is monic and has coefficients in A_{v_i}. Thus $v_i(1/H(x)) > 0$.

Case 2. $v_i(x) \geq 0$. Then $v_i(H(x)) \geq 0$. Suppose that $v_i(H(x)) > 0$. Then $v_i(F_i(x)) > 0$, hence $v_i(y) > 0$. But $F_x(T) = T^d + c_1 T^{d-1} + \cdots + c_d$ has constant term $c_d \in k^*$. Thus $H(x) = y^d + c_1 y^{d-1} + \cdots + c_d$ has value $v_i(H(x)) = 0$. This is a contradiction. We conclude that $v_i(H(x)) = 0$, hence $v_i(1/H(x)) = 0$. □

Notice that, for k a subfield of a ring R, the previous condition $(*)$ (cf. 6.3) implies $(**)$. In particular $(**)$ holds in the examples 5 – 7 above. $(**)$ holds also in the examples 1, 3, 4 for k the prime ring in R.

Definition 4. Let $F(T) \in R[T]$ be a non-constant monic polynomial. Let v be a valuation on R. We call v an *F-valuation*, if $v(c) \geq 0$ for every coefficient c of F and $F(T)$ has no zero in the residue class field $\kappa(v) = \mathfrak{o}_v/\mathfrak{m}_v$. {Of course, this means that the image polynomial $\bar{F}(T) \in \kappa(v)[T]$ has no zero in $\kappa(v)$.}

Theorem 6.5. *Let $(v_i|i \in I)$ be a family of valuations on R. Assume that A is the intersection of the valuation rings A_{v_i} $(i \in I)$. Assume also that for each $x \in R \setminus A$ there exists a monic polynomial $F_x(T) \in A[T]$ of degree $d_x \geq 1$, such that $F_x(x) \in R^*$ and every v_i is an F_x-valuation. Then A is Prüfer in R.*

Proof. Each A_{v_i} is integrally closed in R. Thus A is integrally closed in R. By Theorem 1 we are done if we verify that $1/F_x(x) \in A$ for each $x \in R \setminus A$, i.e. $v_i(F_x(x)) \leq 0$ for each $x \in R \setminus A$ and $i \in I$. If $v_i(x) < 0$ then $v_i(F_x(x)) = d_x \cdot v_i(x) < 0$. If $v(x_i) \geq 0$ then $x \in A_{v_i}$, and $v_i(F_x(x)) = 0$ since v_i is an F_x-valuation. □

Here we quote the seminal paper [R] by Peter Roquette, which in the case, that R is a field, bears close connection to Theorem 5. Roquette also obtained results on class groups which allow to conclude in important cases that A has trivial class group, hence is a Bezout ring. Our Theorem 5 generalizes the first part of [R, Theorem 1]. The second part, dealing with the class group of A, will be generalized in §7.

We now aim to criteria that A is Prüfer in R, which do not assume in advance that A is integrally closed in R. A prototype of the criteria to follow is a lemma of A. Dress, which states for R a field of characteristic not 2, that the subring of R generated by the elements $1/(1+a^2)$ with $a \in F$, $a^2 \neq -1$, is Prüfer in R, cf. [D, Satz 2′], [KS, Chap III §12], [La, p.86].*)

Theorem 6.6. *Assume that for every $x \in R \setminus A$ there exists some monic polynomial $F(T) \in A[T]$ of degree ≥ 1 with $F(x) \in R^*$, $\frac{1}{F(x)} \in A$, $\frac{x}{F(x)} \in A$. Then A is Prüfer in R.*

Proof. Let B be an R-overring of A and $S := A \cap B^*$. We verify that $B = S^{-1}A$. Then we know that A is weakly surjective in every R-overring, and will be done by Theorem 5.2.
Of course, $S^{-1}A \subset B$. Let $x \in B \setminus A$ be given. We choose a polynomial $F(T)$ as indicated in the theorem. $s := \frac{1}{F(x)} \in A \subset B$. Also $F(x) \in B$, hence $s \in S$. By assumption $a := \frac{x}{F(x)} \in A$. Thus $x = \frac{a}{s} \in S^{-1}A$. □

The following remark sheds additional light both on Theorem 6 and Theorem 1.

Remark 6.7. Assume that A is integrally closed in R (e.g. A is Prüfer in R). Let $x \in R$ and let $F(T) \in A[T]$ be a monic polynomial of degree $n \geq 1$ with $F(x) \in R^*$ and $\frac{1}{F(x)} \in A$. Then $\frac{x^r}{F(x)} \in A$ for $0 \leq r \leq n$.

Proof (cf. [Gi, p.154]). We proceed by induction on r. For $r = 0$ the assertion is trivial. Assume that $1 \leq r \leq n$ and that $\frac{x^s}{F(x)} \in A$ for $0 \leq s < r$.

*) Actually Dress made the slightly stronger assumption that -1 is not a square in F.

We write

$$F(T)^r = T^{nr} + \sum_{j=1}^{n} h_j(T) T^{(n-j)r}$$

with polynomials $h_j(T) \in A[T]$ of degree $< r$. The relation

$$\frac{1}{F(x)^{n-r}} = \frac{F(x)^r}{F(x)^n} = \left(\frac{x^r}{F(x)}\right)^n + \sum_{j=1}^{n} \frac{1}{F(x)^{j-1}} \frac{h_j(x)}{F(x)} \cdot \left(\frac{x^r}{F(x)}\right)^{n-j}$$

proves that $\frac{x^r}{F(x)}$ is integral over A, since by induction hypothesis $\frac{h_j(x)}{F(x)} \in A$ for every $j \in \{1, \dots, n\}$. Thus $\frac{x^r}{F(x)} \in A$. $\square$

THEOREM 6.8. *Let k be a subring of R. (We will often take for k the prime ring in R.) Let $F(T) \in k[T]$ be a monic polynomial of degree $d \geq 1$. Assume that $d! \in R^*$ and that $F(x) \in R^*$ for every $x \in R$ with $F(x) \notin k$. The subring A of R generated by k, the element $1/d!$ and the set $\{1/F(x) | x \in R,\, F(x) \notin k\}$ is Prüfer in R.*

PROOF. a) Let $B := \tilde{A}$, the integral closure of A in R. By Theorem 1 B is Prüfer in R. We now verify that for a given prime ideal $\mathfrak{q}$ of B and $\mathfrak{p} := \mathfrak{q} \cap A$ we have $A_{[\mathfrak{p}]} = B_{[\mathfrak{q}]}$. Since over every prime ideal $\mathfrak{p}$ of A there lies a prime ideal $\mathfrak{q}$ of B we then may conclude (Remark 5.5) that

$$B = \bigcap_{\mathfrak{q} \in \operatorname{Spec} B} B_{[\mathfrak{q}]} \subset \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} A_{[\mathfrak{p}]} = A,$$

hence $A = B$, and we will be done.

b) We first prove that for any $x \in B_{[\mathfrak{q}]}$ we have $F(x) \in A_{[\mathfrak{p}]}$. Put $y := F(x) - 1$. Suppose $F(x) \notin A_{[\mathfrak{p}]}$, hence $y \notin A_{[\mathfrak{p}]}$. Clearly $F(x) \notin k$. By hypothesis $1 + y = F(x) \in R^*$ and $\frac{1}{1+y} \in A$. Also $\frac{y}{1+y} = 1 - \frac{1}{1+y} \in A$. Since $y \notin A_{[\mathfrak{p}]}$ we conclude that $\frac{1}{1+y} \in \mathfrak{p}$. On the other hand $\frac{y}{1+y} = y \cdot (\frac{1}{1+y}) \in (B_{[\mathfrak{q}]} \cdot \mathfrak{p}) \cap A \subset \mathfrak{q}_{[\mathfrak{q}]} \cap A = \mathfrak{p}$. We arrive at the contradiction $1 = \frac{1}{1+y} + \frac{y}{1+y} \in \mathfrak{p}$. Thus indeed $F(x) \in A_{[\mathfrak{p}]}$.

c) For $\ell = 0, 1, 2, \dots$ we successively define polynomials $\Delta^\ell F(T)$ by

$$\Delta^0 F(T) := F(T), \quad \Delta^{\ell+1} F(T) := \Delta^\ell F(T+1) - \Delta^\ell F(T).$$

For every $x \in B_{[\mathfrak{q}]}$ we have $F(x) \in A_{[\mathfrak{p}]}$, thus also $\Delta^\ell F(x) \in A_{[\mathfrak{p}]}$ for any $\ell \in \mathbb{N}$. But $\Delta^{d-1} F(T) = d!T + c$ with $c \in k$. Thus $(d!)x \in A_{[\mathfrak{p}]}$ for every $x \in B_{[\mathfrak{q}]}$. Since $1/d! \in A \subset A_{[\mathfrak{p}]}$, we conclude that $A_{[\mathfrak{p}]} = B_{[\mathfrak{q}]}$, q.e.d.

EXAMPLE 10. We denote the prime ring in R by $\mathbb{Z} \cdot 1$. Let $d \in \mathbb{N}$. Assume that $d! \in R^*$ and $1 + x^d \in R^*$ for all $x \in R$ with $x^d \notin \mathbb{Z} \cdot 1$. The subring A of R generated by $1/d!$ and the elements $1/(1 + x^d)$ with $x \in R$, $x^d \notin \mathbb{Z} \cdot 1$ is Prüfer in R. $\square$

N.B. For $d = 2$ and R a field this example states a slight improvement of Dress's lemma cited above.

Remark. The condition $d! \in R^*$ cannot be omitted. For example, let $R := \mathbb{F}_2[T]/(1+T^2)$ with $\mathbb{F}_2$ the field consisting of 2 elements. Let A be the subring of R generated by the elements $1/(1+x^2)$ for all $x \in R$ with $x^2 \neq 1$. Then $A = \mathbb{F}_2$, and this not Prüfer in R, since $\mathbb{F}_2$ is not integrally closed in R.

As an illustration what has been done so far we return to Example 5. Thus let V be an affine variety over some field k which is not algebraically closed. Let U be a k-Zariski-open subset of $V(k)$, and let R be the ring of regular functions on U. We choose a monic polynomial $F(T) \in k[T]$, $F \neq 1$, which has no zeros in k.

Let B be any subring of R containing k (e.g. $B = k$). Let H_0 denote the subring $B[\frac{1}{F(x)} \mid x \in R]$ of R generated by B and the functions $\frac{1}{F(x)}$ for all $x \in R$. Let H denote the integral closure of H_0 in R.

Theorem 6.9. i) *H is an R-Prüferring.*

ii) *H is the set of all $x \in R$ such that $v(x) \geq 0$ for every Manis F-valuation v on R with $v(b) \geq 0$ for all $b \in B$.*

iii) $H = B[\frac{x^i}{F(x)} \mid x \in R,\ 0 \leq i \leq 1]$.

iv) *If the characteristic of k is zero or exceeds d, then $H = H_0$.*

Proof. H is an R-Prüferring by Theorem 1. Thus H is the intersection of the valuation rings A_v with v running through the set Ω of all Manis valuations on R with $A_v \supset H$.

Let v be a Manis valuation on R. Then $v \in \Omega$ iff $A_v \supset H_0$. This means that $A_v \supset B$ and $v\big(\frac{1}{F(x)}\big) \geq 0$ for every $x \in R$. If $x \notin A_v$ then $v(F(x)) < 0$, hence $v\big(\frac{1}{F(x)}\big) > 0$ automatically. Let $x \in A_v$. Then $v\big(\frac{1}{F(x)}\big) \geq 0$ iff $v(F(x)) = 0$ iff $\bar{F}(\bar{x}) \neq 0$ for $\bar{F}(T)$ the image of $F(T)$ in $\kappa(v)[T]$ and $\bar{x}$ the image of x in $\kappa(v)$. Thus Ω is the set of all Manis F-valuations v on R with $A_v \supset B$.

The ring $H' := B\big[\frac{x^i}{F(x)} \mid x \in R,\ 0 \leq i \leq 1\big]$ is Prüfer in R by Theorem 6. Every valuation $v \in \Omega$ has nonnegative values on H'. Thus $H_0 \subset H' \subset H$. Since H' is integrally closed in R, we have $H' = H$. If $d! \in k^*$, then we know from Theorem 8 that H_0 is Prüfer in R and conclude that $H_0 = H$. □

§7 Principal ideal results

We start out for a generalization of the second half of Roquette's theorem 1 in [R] mentioned in §6. We will rely on techniques developed by Alan Loper in the case of subrings of fields [Lo_1], [Lo_2].

In the following we fix a ring A and a monic polynomial $F(T) \in A[T]$ of degree $d \geq 1$.

Definition 1 (cf. [Lo_1]). Let $\varphi: A \to B$ be a ring extension of A. We call the polynomial F *unit valued in B* (abbreviated: uv in B), if $F(b) \in B^*$ for every $b \in B$. {Of course, $F(b) := F^\varphi(b)$ with $F^\varphi(T)$ the image polynomial of $F(T)$ in $B[T]$.}

More precisely we then should call F "uv with respect to φ", but in the following it will be always clear which homomorphism φ from A to B is taken.

N.B. If F is uv in some extension B of A different from the null ring then certainly $d \geq 2$.

PROPOSITION 7.1 (cf. [Lo$_1$, Prop.1.14]). *Let $\mathfrak{m}$ be a maximal ideal of A. Then $F(T)$ is uv in $A_\mathfrak{m}$ iff $F(A) \subset A \setminus \mathfrak{m}$.*

PROOF. If there exists some $a \in A$ with $F(a) \in \mathfrak{m}$, then certainly $F(T)$ is not uv in $A_\mathfrak{m}$. Assume now that $F(A) \subset A \setminus \mathfrak{m}$. Suppose that $F(T)$ is not uv in $A_\mathfrak{m}$. We have some $a \in A$, $s \in A \setminus \mathfrak{m}$ with $F\left(\frac{a}{s}\right) \in \mathfrak{m}A_\mathfrak{m}$. Since the ideal $\mathfrak{m}$ is maximal there exists some $t \in A$ with $st \equiv 1 \mod \mathfrak{m}$. Then in $A_\mathfrak{m}$

$$\frac{F(at)}{1} \equiv F\big(\frac{a}{s}\big) \equiv 0 \qquad \mod \mathfrak{m}A_\mathfrak{m},$$

hence $F(at) \in \mathfrak{m}$. This contradiction proves that $F(T)$ is uv in $A_\mathfrak{m}$. □

COROLLARY 7.2. *$F(T)$ is uv in A iff $F(T)$ is uv in $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A.* □

We write $F(T) = T^d + c_1T^{d-1} + \cdots + c_d$ with $a_i \in A$, and introduce the homogenization $G(X,Y) \in A[X,Y]$ of F,

$$G(X,Y) := Y^d F\big(\frac{X}{Y}\big) = X^d + c_1X^{d-1}Y + \cdots + c_dY^d.$$

PROPOSITION 7.3. *Let $\mathfrak{p}$ be a prime ideal of A. The following are equivalent.*

i) *F is uv in $A_\mathfrak{p}$.*
ii) *F is uv in $k(\mathfrak{p}) = Quot(A/\mathfrak{p})$, i.e. F has no zero in $k(\mathfrak{p})$.*
iii) *If $x, y \in A$ and $G(x,y) \in \mathfrak{p}$, then $y \in \mathfrak{p}$.*
iv) *If $x, y \in A$ and $G(x,y) \in \mathfrak{p}$, then $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$.*

PROOF. i) $\Leftrightarrow$ ii) is evident. iv) $\Rightarrow$ iii) is trivial, and iii) $\Rightarrow$ iv) is evident, since the form $G(X,Y)$ contains the term X^d.
i) $\Rightarrow$ iii): Let $x, y \in A$ and $G(x,y) \in \mathfrak{p}$. Suppose $y \notin \mathfrak{p}$. Then we have in $A_\mathfrak{p}$

$$F\big(\frac{x}{y}\big) = \frac{G(x,y)}{y^d} \in \mathfrak{p}A_\mathfrak{p}.$$

This contradicts the assumption that F is uv in $A_\mathfrak{p}$.
iii) $\Rightarrow$ i): Let $a \in A$, $s \in A \setminus \mathfrak{p}$ be given. Then $G(a,s) \in A \setminus \mathfrak{p}$. Thus

$$F\big(\frac{a}{s}\big) = \frac{G(a,s)}{s^d} \in A_\mathfrak{p}^*.$$ □

PROPOSITION 7.4 (cf. [Lo$_2$, Cor.2.3]). *Assume that $(A, \mathfrak{p})$ is a Manis pair in some ring R. Let v denote a Manis valuation on R with $A_v = A$, $\mathfrak{p}_v = \mathfrak{p}$. The following are equivalent.*

i) *F is uv in $A_{\mathfrak{p}}$.*
ii) *v is an F-valuation.*
iii) *$v(G(x,y)) = d\min(v(x), v(y))$ for all $x, y \in R$.*

Proof. The equivalence i) $\Leftrightarrow$ ii) is clear from i) $\Leftrightarrow$ ii) in Proposition 3.
i) $\Rightarrow$ iii): Let $x, y \in R$ be given. The formula is a priori valid if $v(x) < v(y)$, since $G(X,Y)$ contains the term X^d. It is also valid if $v(x) = v(y) = \infty$. Assume now that $v(x) \geq v(y) \neq \infty$. We choose some $z \in R$ with $v(yz) = 0$. This is possible since v is Manis. Then $v(xz) \geq 0$. Thus $xz \in A$ and $yz \in A \setminus \mathfrak{p}$. We know from Prop. 3 that $G(xz, yz) = z^d G(x,y) \in A \setminus \mathfrak{p}$. Thus $v(G(x,y)) = -dv(z) = dv(y)$.
iii) $\Rightarrow$ i): Let $x, y \in A$ and $G(x,y) \in \mathfrak{p}$. Then the formula in iii) tells us that $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$. Thus F is uv in $A_{\mathfrak{p}}$ by Proposition 3. □

We now study finitely generated A-submodules $\mathfrak{a}$ of R with $R\mathfrak{a} = R$. These submodules should be viewed as analogues of the finitely generated fractional ideals in the classical case that A is a domain and R its quotient field. We are looking for criteria that some power $\mathfrak{a}^d$ is a *principal module*, i.e. $\mathfrak{a}^d = Rb$ with some $b \in R^*$.

Definition 2. Let $(a_1, \dots, a_n)$ be a finite sequence in R. The *F-transform* of this sequence is the sequence $(b_1, \dots, b_n)$ in R defined inductively by

$$b_1 := a_1, \quad b_i := G(b_{i-1}, a_i^{d^{i-2}}) \qquad (i > 1).$$

In the following lemmas $(a_1, \dots, a_n)$ is a sequence in R and $(b_1, \dots, b_n)$ is its F-transform.

Lemma 7.5. *Assume that all $a_i \in A$. Let $\mathfrak{p}$ be a prime ideal of A such that F is uv in $A_{\mathfrak{p}}$. Then $Aa_1 + \cdots + Aa_n \subset \mathfrak{p}$ iff $b_n \in \mathfrak{p}$.*

Proof. If $x, y \in A$ and $t \in \mathbb{N}$, then $Ax + Ay \subset \mathfrak{p}$ iff $Ax + Ay^t \subset \mathfrak{p}$. By Proposition 3 the latter is equivalent to $G(x, y^t) \in \mathfrak{p}$. The lemma follows from this by induction on n. □

Lemma 7.6 (cf. [Lo$_2$, Cor.2.4]). *Assume that A is the valuation ring A_v of a Manis valuation v on some ring R which is also an F-valuation. Then*

$$v(b_n) = d^{n-1} \min\{v(a_1), \dots, v(a_n)\}.$$

The proof goes by induction on n using the formula in Proposition 4.iii. □

Lemma 7.7. *Let $\mathfrak{a} := Aa_1 + \cdots + Aa_n$. Assume that F is uv in R. Then*

$$R\mathfrak{a} = R \Longleftrightarrow b_n \in R^*.$$

Proof. $\Leftarrow$: This is evident since $b_n \in \mathfrak{a}$.
$\Rightarrow$: Suppose $b_n \notin R^*$. We choose a maximal ideal $\mathfrak{M}$ of R containing b_n. Our polynomial F is uv in R hence uv in $R_{\mathfrak{M}}$ by Corollary 2. Now Lemma 5, applied to

F as a polynomial over R, tells us that $Ra_1 + \cdots + Ra_n \subset \mathfrak{M}$. This contradicts the assumption $R\mathfrak{a} = R$. Thus $b_n \in R^*$. $\square$

Now we are prepared to prove a generalization of the theorem by Roquette mentioned in §6.

Theorem 7.8 (cf. [R, Th.1] for R a field). *Assume that S is a set of Manis valuations on a ring R and that $A = \bigcap_{v \in S} A_v$. Assume further that there exists a monic polynomial $F(T) \in A[T]$ of degree $d \geq 1$ with the following two properties:*
(i) *$F(T)$ is uv in R.*
(ii) *Every $v \in S$ is an F-valuation.*

Then A is Prüfer in R. If $\mathfrak{a}$ is any finitely generated A-submodule of R with $R\mathfrak{a} = R$ then there exists some $t \in \mathbb{N}$ such that $\mathfrak{a}^{d^t}$ is principal. More precisely, if $a_1, \ldots, a_n$ is a system of generators of $\mathfrak{a}$ and $(b_1, \ldots, b_n)$ is the F-transform of the sequence $(a_1, \ldots, a_n)$, then

$$\mathfrak{a}^{d^{n-1}} = Ab_n.$$

Proof. Theorem 6.5 tells us that A is Prüfer in R. Let $a_1, \ldots, a_n$ be a system of generators of $\mathfrak{a}$ and $(b_1, \ldots, b_n)$ the F-transform of $(a_1, \ldots, a_n)$. Lemma 7 tells us that $b_n \in R^*$.

It is evident that $b_n \in \mathfrak{a}^{d^{n-1}}$. The module $\mathfrak{a}^{d^{n-1}}$ is generated over A by the monomials $a_1^{e_1} \ldots a_n^{e_n}$ with $e_i \geq 0$, $e_1 + \cdots + e_n = d^{n-1}$. We now verify that

$$v(a_1^{e_1} \ldots a_n^{e_n}) \geq v(b_n) \tag{$*$}$$

for every such monomial and every $v \in S$. It then follows that $a_1^{e_1} \ldots a_n^{e_n}/b_n$ is an element of A_v for every $v \in S$, hence of A, and we conclude that $\mathfrak{a}^{d^{n-1}} = Ab_n$. The verification of $(*)$ is immediate by use of Lemma 6. Let $\gamma := \min\{v(a_1), \ldots, v(a_n)\}$. Then $v(a_1^{e_1} \ldots a_n^{e_n}) \geq (e_1 + \cdots + e_n)\gamma = d^{n-1}\gamma = v(b_n)$. $\square$

In part II of the paper we will see that for A a Prüfer subring of a ring R the finitely generated A-submodules $\mathfrak{a}$ of R with $R\mathfrak{a} = R$ form an Abelian group. The quotient of this group by the subgroup of principal modules should be called the *class group of A in R*. Starting with Theorem 8 it is possible to get bounds on the torsion of the class group in good cases in much the same way as Roquette has explicated for R a field [R]. Here we only quote the following theorem which is an immediate consequence of Theorem 8.

Theorem 7.9 (cf.[R, Th.2]). *Assume again that $A = \bigcap_{v \in S} A_v$ for a set S of Manis valuations on some ring R. Assume further that there exist non-constant monic polynomials $F_1(T), \ldots, F_r(T)$ with coefficients in A $(r \geq 1)$, such that for every $j \in \{1, \ldots, r\}$ the following holds*
(1) *F_j is uv in R.*
(2) *Every $v \in S$ is an F_j-valuation.*

Let d denote the greatest common divisor of the degrees of $F_1, \dots, F_r$. Then A is Prüfer in R, and for each finitely generated A-submodule $\mathfrak{a}$ of R with $R\mathfrak{a} = R$ there exists some $t \in \mathbb{N}$ such that $\mathfrak{a}^{d^t}$ is principal. $\square$

EXAMPLE 7.10. Let R be a ring such that $X^d + 1$ is uv in R for some (even) $d \in \mathbb{N}$ and $d!$ is a unit in R. Let A be a subring of R which contains $1/d!$ and the elements $1/(1 + x^d)$ for all $x \in R$. Then A is Prüfer in R by Example 10 in §6. For every finitely generated A-submodule $\mathfrak{a}$ of R with $\mathfrak{a}R = R$ there exists some $t \in \mathbb{N}$ with $\mathfrak{a}^{d^t}$ principal.

PROOF. A is the intersection of the rings $A_{[\mathfrak{m}]}$ with $\mathfrak{m}$ running through the maximal ideals of A (Remark 5.5). These rings are Manis in R. The polynomial $X^d + 1$ is uv in $A_\mathfrak{m}$ for every $\mathfrak{m}$ (Cor.2), and thus the Manis valuations giving the rings $A_{[\mathfrak{m}]}$ are $(X^d + 1)$-valuations. Theorem 8 applies. $\square$

In an important more special situation this result can be improved. Assume that $1 + \Sigma R^d \subset R^*$. A subring A of R containing the elements $1/(1 + q)$ with $q \in \Sigma R^d$ is Prüfer in R. If $\mathfrak{a} = Ax_1 + \cdots + Ax_n$ is a finitely generated submodule of R with $R\mathfrak{a} = R$, then $\mathfrak{a}^d = A(x_1^d + \cdots + x_n^d)$. This has been proved recently by E. Becker and V. Powers [BP, Cor. 5.11, Cor.5.13].

A slight expansion of the techniques used so far will give us a theorem containing this result of Becker and Powers as a special case, together with a proof which is rather different from the one in [BP].

DEFINITION 3. Let $H(X_1, \dots, X_n) \in A[X_1, \dots, X_n]$ be a form, i.e. a homogeneous polynomial over A in $n \geq 2$ variables. Let $\varphi: A \to K$ be a homomorphism into a field K. We call H *isotropic over* K, if the image form $H^\varphi(X_1, \dots, X_n) \in K[X_1, \dots, X_n]$ is isotropic, i.e. has a non trivial zero in K^n, and we call H *anisotropic over* K otherwise.

In the following it will be always clear which homomorphism φ is under consideration. Thus the impreciseness in this definition will do no harm.

THEOREM 7.11. *Let S be a set of Manis valuations on a ring R and $A := \bigcap_{v \in S} A_v$. Assume there is given a form $H(X_1, \dots, X_n)$ over A in $n \geq 2$ variables of degree $d \geq 1$ with the following properties:*

i) *For every maximal ideal $\mathfrak{M}$ of R the form H is anisotropic over $R/\mathfrak{M}$.*
ii) *For every $v \in S$ the form H is anisotropic over $\kappa(v)$.*

Then A is Prüfer in R. If $\mathfrak{a}$ is an A-submodule of R generated by n elements $x_1, \dots, x_n$ and $R\mathfrak{a} = R$ then $\mathfrak{a}^d = H(x_1, \dots, x_n)A$.

PROOF. a) We start with a proof of the second claim. Suppose that $H(x_1, \dots, x_n)$ is not a unit in R. Then there exists a maximal ideal $\mathfrak{M}$ of R with $H(x_1, \dots, x_n) \in \mathfrak{M}$. Since $Rx_1 + \cdots + Rx_n = R$ we conclude that H is isotropic over $R/\mathfrak{M}$, in contradiction to assumption (i) above. Thus $H(x_1, \dots, x_n) \in R^*$.

b) Let $v \in S$ be given. We verify that

$$(*) \qquad v(H(x_1,\dots,x_n)) = d\min\{v(x_1),\dots,v(x_n)\}.$$

This is obvious if $v(x_i)=\infty$ for all $i \in \{1,\dots,n\}$. Assume now that $\gamma := \min\{v(x_1),\dots,v(x_n)\} < \infty$. We choose some $z \in R$ with $v(z) = -\gamma$, which is possible, since v is Manis. Then $v(zx_i) \geq 0$ for all $i \in \{1,\dots,n\}$ and $v(zx_i)=0$ for at least one i. Since H is anisotropic over $\kappa(v)$ we conclude that $v(H(zx_1,\dots,zx_n))=0$, hence $v(H(x_1,\dots,x_n)) = -dv(z) = d\gamma$, as desired.
c) Now we see, as in the proof of Theorem 8, that

$$v(x_1^{e_1}\dots x_n^{e_n}) \geq v(H(x_1,\dots,x_n))$$

for any integers $e_i \geq 0$ with $e_1+\dots+e_n = d$ and any $v \in S$, and we conclude that $x_1^{e_1}\dots x_n^{e_n}/H(x_1,\dots,x_n) \in A$. This proves that $\mathfrak{a}^d = H(x_1,\dots,x_n)A$.
d) Let

$$G(X,Y) := H(X,\dots,X,Y) = c_0X^d + c_1X^{d-1}Y + \dots + c_dY^d.$$

$c_0 = H(1,\dots,1,0)$ is a unit in A, since the elements $1,\dots,1,0$ generate the ideal $\mathfrak{a} = A$ and $\mathfrak{a}^d = H(1,\dots,1,0)A$. We consider the monic polynomial

$$F(T) := c_0^{-1}G(T,1) \in A[T].$$

F is uv in R, since $H(x,\dots,x,1) \in R^*$ for every $x \in R$. If $v(x) \geq 0$ for some $v \in S$, then $v(H(x,\dots,x,1)) = v(1) = 0$. Thus every $v \in S$ is an F-valuation. We conclude by Theorem 6.5 that A is Prüfer in R. □

REMARK. The multiplicative ideal theory to be developed in part II of this paper will give a more natural proof that A is Prüfer in R.

In order to exploit Theorem 11 in the real algebraic setting, we need an easy lemma.

LEMMA 7.12. *Let $H(X_1,\dots,X_n)$ be a form over a ring A of degree $d \geq 1$ in $n \geq 2$ variables. For each $i \in \{1,\dots,n\}$ we define*

$$F_i(T_1,\dots,T_{n-1}) := H(T_1,\dots,T_{i-1},1,T_i,\dots,T_{n-1}).$$

The following are equivalent

(1) *H is anisotropic over $A/\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A.*
(2) *$F_i(x_1,\dots,x_{n-1}) \in A^*$ for all $x_1,\dots,x_{n-1} \in A$ and $1 \leq i \leq n$.*

PROOF. (1) $\Longrightarrow$ (2): Let $x_1,\dots,x_{n-1} \in A$ and $i \in \{1,\dots,n\}$. Then $H(x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1}) \notin \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. Thus $F_i(x_1,\dots,x_{n-1}) \in A^*$.
(2) $\Longrightarrow$ (1): Suppose there exists a maximal ideal $\mathfrak{m}$ of A such that H is isotropic over $A/\mathfrak{m}$. Then there exist elements $a_1,\dots,a_n \in A$ with $H(a_1,\dots,a_n) \in \mathfrak{m}$ but $a_i \notin \mathfrak{m}$ for some i. We choose an element $b_i \in A$ with $a_ib_i \equiv 1 \mod \mathfrak{m}$. We have $b_i^dH(a_1,\dots,a_n) = H(a_1b_i,\dots,a_nb_i) \equiv F_i(a_1b_i,\dots,a_{i-1}b_i,a_{i+1}b_i,\dots,a_nb_i) \mod \mathfrak{m}$. Thus, $F_i(a_1b_i,\dots,a_{i-1}b_i,a_{i+1}b_i,\dots,a_nb_i) \in \mathfrak{m}$, a contradiction. □

Corollary 7.13 (cf. [BP]). *Let $d \in \mathbb{N}$ and let R be a ring with $1 + \Sigma R^{2d} \subset R^*$. Then the subring*

$$H := H_d(R) = \mathbb{Z}\left[\frac{1}{1+q} \middle| q \in \Sigma R^{2d}\right]$$

is Prüfer in R. For each finitely generated H-submodule $\mathfrak{a} = Hx_1 + \cdots + Hx_n$ of R with $\mathfrak{a}R = R$ we have $\mathfrak{a}^{2d} = (x_1^{2d} + \cdots + x_n^{2d})H$.

Proof. Applying Theorem 6.8 with $F(T) = 1 + T^{2d}$ we see that H is Prüfer in R (cf. §6, Example 10). For every maximal ideal $\mathfrak{m}$ of H we choose a Manis valuation v on R with $A_v = H_{[\mathfrak{m}]}$, $\mathfrak{p}_v = \mathfrak{m}_{[\mathfrak{m}]}$. Let S denote the set of these valuations. Then $H = \bigcap_{v \in S} A_v$ (cf. 5.5). Now, if $v \in S$, $A_v = H_{[\mathfrak{m}]}$, then $H/\mathfrak{m} = H_{[\mathfrak{m}]}/\mathfrak{m}_{[\mathfrak{m}]}$, as is easily checked, and we learn from Proposition 1.7 that $\kappa(v)$ is the quotient field of $H/\mathfrak{m}$. Since $H/\mathfrak{m}$ is already a field, we have $\kappa(v) = H/\mathfrak{m}$. Let $n \geq 2$. Using Lemma 12 we see that the form $X_1^{2d} + \cdots + X_n^{2d}$ is anisotropic in $R/\mathfrak{M}$ for every maximal ideal $\mathfrak{M}$ of R, and also anisotropic in $H/\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of H. Now Theorem 11 gives the second claim above. $\square$

Becker and Powers have proved that $1 + \Sigma R^{2d} \subset R^*$ implies $1 + \Sigma R^2 \subset R^*$, and that then $H := H_d(R)$ coincides with $H_1(R)$ and the "real holomorphy ring" of R [BP, Prop.5.1 and Prop.5.7]. Thus, if $\mathfrak{a}$ is a finitely generated H-submodule of R with $R\mathfrak{a} = R$, then already $\mathfrak{a}^2$ is a principal submodule.

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