Two Interesting Oriented Matroids

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Abstract. Oriented matroids are a combinatorial model for configurations in real vector spaces. A central role in the theory is played by the realizability problem: Given an oriented matroid, find an associated vector configuration. In this paper we present two closely related oriented matroids Ω_{14}^{+} and Ω_{14}^{-} of rank 3 with 14 elements that have interesting properties with respect to realizability. Ω_{14}^{+} and Ω_{14}^{-} differ in exactly one basis orientation.

The realizable oriented matroid Ω_{14}^{+} has at least two interesting properties: First it has a combinatorial symmetry that has no metric realization, and second it has a disconnected realization space. In other words, there are different realizations of Ω_{14}^{+} that cannot be continuously deformed into each other while staying in the same isotopy class. The oriented matroid Ω_{14}^{-} is non-realizable but it has no bi-quadratic final polynomial. In other words, the only known effective algorithmic method fails to prove the non-realizability of Ω_{14}^{-}.

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1 Introduction

Oriented matroids are combinatorial models for vector configurations in vector spaces over ordered fields. They form a basic combinatorial concept for treating many different objects on the borderline of combinatorics and geometry — such as convex polytopes, simplicial complexes, hyperplane-arrangements, quasi-crystals, etc. The *realizability question* is of fundamental importance in this theory: *When does a discrete structure have a geometric representation? What does the space of all representations look like?* Questions of this type occur in many different mathematical contexts (e.g. embedding of polyhedral manifolds, the theory of moduli spaces, Cairns' smoothing theory, etc.). The basic effects that arise here are often due to the properties of the

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underlying oriented matroids, and they can be profitably studied in this model. A systematic study of "small" oriented matroids that have interesting behavior with respect to realizability is a fruitful source for producing examples and counterexamples in many different mathematical disciplines. Here we present two new such oriented matroids.

Every vector configuration has an associated oriented matroid, but the converse is not true: there are oriented matroids that have no corresponding vector configuration. An oriented matroids is *realizable* if it corresponds to a vector configuration, and *non-realizable* otherwise. In this paper we present two closely related oriented matroids Ω_{14}^{+} and Ω_{14}^{-} of rank 3 with 14 elements that are interesting because of their properties with respect to realizability.

The oriented matroid Ω_{14}^{+} is realizable, but its realization space is not connected. The *realization space* of an oriented matroid χ is the set of all vector configurations X that have the associated oriented matroid χ, modulo linear equivalence. (For a more formal definition of realization spaces see Section 2). For a long time it was an outstanding open question whether oriented matroids with disconnected realization space exist. This problem was solved by N.E. Mnëv in a surprising way [6, 7]. He proved that for any basic semi-algebraic set V (defined over the rationals) there is an oriented matroid whose realization space is stably equivalent (in the sense of [9]) to V. Thus realization spaces can be homotopy equivalent to any finite simplicial complex (in particular they may have an arbitrary number of connected components). The examples produced by Mnëv's method in general involve a large number of points. At the same time P.Y. Suvorov [12] constructed an example of rank 3 with disconnected realization space that contains only 14 elements.

The oriented matroid Ω_{14}^{+} shares these properties with Suvorov's example, but it has the following additional nice properties:

- Ω_{14}^{+} is constructible. (After fixing the position of the points $x_1, \ldots, x_4$ that form a projective basis and choosing a point $x_5 = (t+1)x_3 + (t-1)x_4$ each point x_i for $i = 6, \ldots, 14$ is of the form $(x_a \vee x_b) \wedge (x_c \vee x_d)$ where "$\vee$" is the *join* operator and "$\wedge$" is the *meet* operator and a, b, c, d are indices that are smaller than i.)
- up to stable equivalence (see [9]) the realization space of Ω_{14}^{+} is an open interval from which one point has been deleted.
- Ω_{14}^{+} has rational realizations.
- Ω_{14}^{+} has a combinatorial symmetry of order two that has no metric realization. (The smallest example with this property, known so far, with 90 points, was constructed by P. Shor [11].)

It is still an open question whether there exists an oriented matroid with disconnected realization space and less than 14 points.

If we switch the orientation of one particular basis in Ω_{14}^{+} we obtain the *non-realizable* oriented matroid Ω_{14}^{-}. This oriented matroid has a remarkable property. It is the first known example of a non-realizable oriented matroid for which non-realizability cannot be proved by a bi-quadratic final polynomial.

Final polynomials [3, 5] are certificates for the non-realizability of matroids and oriented matroids. However, no algorithmic method for computing final polynomials is known to be both generally applicable and effective. Indeed, this is not surprising since the realizability problem is known to be NP-hard [11]. Bi-quadratic final polynomials (as introduced in [2] and [8]) are special kinds of final polynomials which can be computed very efficiently. The method of bi-quadratic final polynomials for the oriented matroid case was originally inspired by J. Bokowski [5], who suggested that one consider only inequalities of the form $[\ldots][\ldots] < [\ldots][\ldots]$ which are consequences of three-term Graßmann-Plücker polynomials and the signature of the oriented matroid. These inequalities have to be satisfied in the realizable case. If this system of these inequalities is inconsistent one has a bi-quadratic final polynomial. Deciding whether an oriented matroid has a bi-quadratic final polynomial can be translated into an LP-feasibility-problem and therefore solved in polynomial time. This is the first example of a non-realizable oriented matroid which cannot be certified to be non-realizable by a bi-quadratic final polynomial.

2 Realization spaces

Oriented matroids are combinatorial models for vector configurations in linear vector spaces over ordered fields. For an extensive introduction into oriented matroid theory we recommend [1] and [10]. Throughout the paper we will restrict ourselves to the case of vector configurations in $\mathbb{R}^3$, the case of oriented matroids of rank 3. Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^{3n}$ be a configuration consisting of n vectors in $\mathbb{R}^3$. We set $E = \{1, \ldots, n\}$. To every triple of indices $(i, j, k) \in E^3$ we assign a sign

$$\chi_X(i,j,k) = \text{SIGN DET}(x_i, x_j, x_k).$$

The map $\chi_X : E^3 \to \{-1, 0, +1\}$ is called the oriented matroid of X. We omit the general definition of an oriented matroid (it can be found in [1] and [10]).

For us it is sufficient to know that an oriented matroid $\chi : E^3 \to \{-1, 0, +1\}$ is a sign map that models the combinatorial behavior of signs of determinants. In particular χ always satisfies the alternating determinant rules:

$$\chi(i,j,k) = \chi(k,i,j) = \chi(j,k,i) = -\chi(j,i,k) = -\chi(k,j,i) = -\chi(i,k,j).$$

Since χ is alternating it is sufficient to specify χ on the set

$$\Lambda(E,3) = \{(i,j,k) \in E^3 \mid i < j < k\}.$$

An oriented matroid χ is *realizable* if there is a vector configuration X with $\chi_X = \chi$. If there is no such vector configuration, then χ is called *non-realizable*. Deciding the question whether an oriented matroid is realizable or not algorithmically is known to be an NP-hard problem [11].

For a realizable oriented matroid one is often interested not only in a particular realization, but also in the space of all realizations. There are various ways of describing this space, depending on how much of the actions on $\mathbb{R}^{3n}$ that preserve the oriented matroid of X are factored out. If at least a linear basis is fixed all these descriptions turn out to be isomorphic up to stable equivalence (compare [9]). We here use the version where a projective basis is fixed. Reorientation of a point i (i.e. reversing all signs $\chi(a,b,c)$ with $i \in \{a,b,c\}$) does not change the behavior of χ with respect to realizability: if $X = (x_1,\ldots,x_n)$ is a realization of χ then we get a realization of the reversed situation if we replace x_i by $-x_i$. Hence, we may (up to relabeling, reorientation of points 1, 2, 3 or 4 and the assumption that χ has at least four points in general position) assume that we have $\chi(1,2,3) = \chi(1,2,4) = \chi(1,3,4) = \chi(2,3,4) = 1$.

Definition 2.1. Let $\chi\colon E^3 \to \{-1,0,+1\}$ be a rank 3 oriented matroid that satisfies $\chi(1,2,3) = \chi(1,2,4) = \chi(1,3,4) = \chi(2,3,4) = 1$. Let $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (1,0,1)$, and $x_4 = (0,1,1)$. The *realization space* of χ is the set of all $(x_5,\ldots,x_n) \in \mathbb{R}^{3(n-4)}$ with $\chi_X = \chi$ for $X = (x_1,\ldots,x_n)$.

3 Ω_{14}^+ has disconnected realization space

The configuration that we will study here is defined by the following construction sequence. The oriented matroid Ω_{14}^+ is the underlying oriented matroid for choices of the parameter t in $(-3+\sqrt{8},\ 0) \cup (0,\ 3-\sqrt{8})$.

$$
\begin{array}{rll}
x_1 & = (1,0,0), & \\
x_2 & = (0,1,0), & \\
x_3 & = (1,0,1), & \\
x_4 & = (0,1,1), & \\
x_5 & = (1-t)x_3 + (1+t)x_4, & \\
x_6 & = x_5x_2 \wedge x_1x_4 & = (1-t,2,2), \\
x_7 & = x_5x_1 \wedge x_2x_3 & = (-2,-1-t,-2), \\
x_8 & = x_6x_3 \wedge x_5x_1 & = (3-2t-t^2, 2+2t, 4), \\
x_9 & = x_7x_4 \wedge x_5x_2 & = (2-2t, 3+2t-t^2, 4), \\
x_{10} & = x_3x_4 \wedge x_8x_2 & = (-3+2t+t^2, -1-2t-t^2, -4), \\
x_{11} & = x_3x_4 \wedge x_9x_1 & = (-1+2t-t^2, -3-2t+t^2, -4), \\
x_{12} & = x_7x_{10} \wedge x_{11}x_2 & = (1-2t^2+t^4, -1+4t+10t^2+4t^3-t^4, 4+8t+4t^2), \\
x_{13} & = x_6x_{11} \wedge x_{10}x_1 & = (-1-4t+10t^2-4t^3-t^4, 1-2t^2+t^4, 4-8t+4t^2), \\
x_{14} & = x_1x_3 \wedge x_2x_4 & = (0,0,1)
\end{array}
$$

Here $x_\alpha x_\beta$ denotes the "join" of x_α and x_β, and $a \wedge b$ denotes the "meet". Both operations can be computed in terms of the standard cross-product in $\mathbb{R}^3$.

After fixing a projective basis consisting of the points $x_1,\ldots,x_4$ the whole construction only depends on the choice of the parameter t. The following matrix gives coordinates for the situation $t = 0$ (the situation where x_5 is in the middle of x_3 and x_4).

$$
X_0 = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 1 & 2 & 3 & 2 & 3 & 1 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 3 & 1 & 3 & -1 & 1 & 0 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 1
\end{pmatrix}
$$

We can visualize the situation if we normalize the last coordinate for $x_3,\ldots,x_{14}$ to 1 by multiplying each vector with a suitable positive scalar. The situation in the plane $\{(x,y,1) \mid x,y \in \mathbb{R}\}$ gives an affine image of our vector configuration in $\mathbb{R}^3$. Figure 1 shows the affine situation for a value t slightly smaller than zero. The points x_1 and x_2 are the points at infinity that lie on the x-axis and y-axis. The little displacement of x_5 away from the symmetric position forces that the lines $(1,3)$, $(2,4)$ and $(12,13)$ not to be concurrent (as in the case $t = 0$).

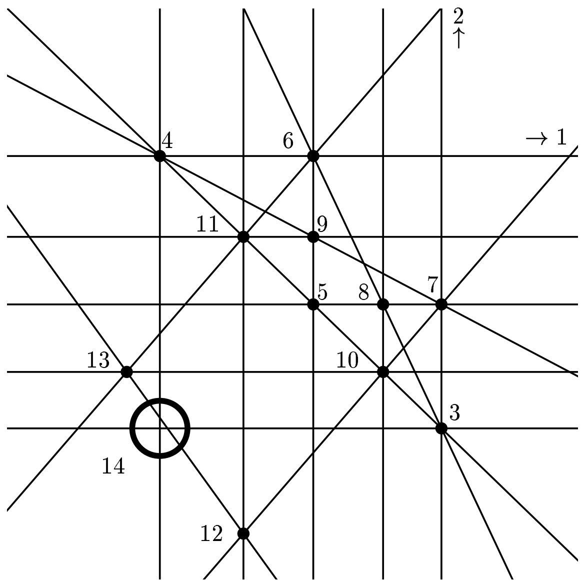

Figure 1

The whole construction sequence has a combinatorial symmetry that is induced by the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 1 & 4 & 3 & 5 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 14 \end{pmatrix}.$$

Evaluating the determinant $\mathrm{DET}(x_{12}, x_{13}, x_{14})$ we get

$$\mathrm{DET}(x_{12}, x_{13}, x_{14}) = 32\,t^2 - 64\,t^4 + 32\,t^6 = 32t^2(t^2-1)^2,$$

a polynomial that has a root which is actually a minimum at $t = 0$.

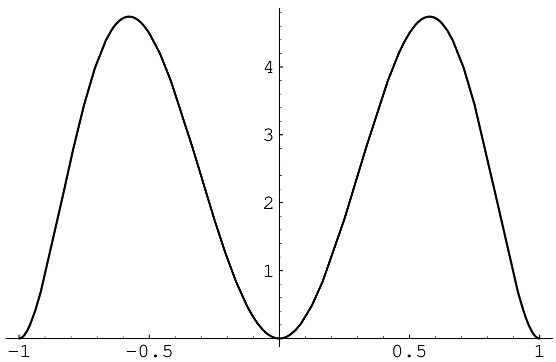

Figure 2

The fact that this polynomial is symmetric in t is already a consequence of the symmetry of the underlying construction of the configuration and of the symmetric choice of our basis $x_1, \ldots, x_4$. A graph of this polynomial is given in Figure 2.

We now define for all $(i,j,k) \in \Lambda(\{1,\ldots,14\},3)$ and $\sigma \in \{-1,0,+1\}$

$$\Omega_{14}^{\sigma}(i,j,k) := \begin{cases} \sigma & \text{if } (i,j,k) = (12,13,14), \\ \chi_{X_0}(i,j,k) & \text{otherwise.} \end{cases}$$

The oriented matroids Ω_{14}^{σ} have a combinatorial symmetry which is induced by π. For all $(i,j,k) \in \Lambda(\{1,\ldots,14\},3)$ and $\sigma \in \{-1,0,+1\}$ we have

$$\Omega_{14}^{\sigma}(\pi(i),\pi(j),\pi(k)) = -\Omega_{14}^{\sigma}(i,j,k).$$

A realization X of Ω_{14}^{σ} is *symmetric* if there is a linear involution $R: \mathbb{R}^3 \to \mathbb{R}^3$ with $R(x_i) = x_{\pi(i)}$ for $i \in \{1,\ldots,14\}$.

Theorem 3.1. *The oriented matroids Ω_{14}^{σ} have the following properties:*

(i) *There is a polynomial function f from $((0,1) \setminus \{\frac{1}{2}\}) \times (0,\infty)^{10}$ to the realization space of Ω_{14}^{+} that is an isomorphism of semi-algebraic sets.*

(ii) *Ω_{14}^{+} has no symmetric realization.*

(iii) *Ω_{14}^{+} has rational realizations.*

(iv) *Ω_{14}^{-} is not realizable.*

Proof. The construction sequence at the beginning of this section shows that after the choice of the parameter t all points are determined up to multiplication by a positive number. The signs that are identical in Ω_{14}^{+}, Ω_{14}^{0}, and Ω_{14}^{-} are exactly taken for values of t in the open interval $(-3+\sqrt{8}, 3-\sqrt{8})$. (The basis that collapse at the end points of this open interval are (x_1,x_3,x_{12}) and (x_2,x_4,x_{13}).) We get realizations of Ω_{14}^{+} exactly for all choices of t in $I = (-3+\sqrt{8},0) \cup (0,3-\sqrt{8})$. For $t=0$ we get a realization of Ω_{14}^{0}. The factor $(0,\infty)^{10}$ in (i) is due to the fact that multiplication of any of the points $x_5,\ldots,x_{14}$ by a positive scalar does not change the underlying oriented matroid. This proves (i).

Assume that there was a symmetric realization X of Ω_{14}^{+}. After a suitable projective transformation we may assume that $x_1,\ldots,x_4$ are located at $(1,0,0)$, $(0,1,0)$, $(1,0,1)$, $(0,1,1)$, respectively, and that the reflection R is given by $R(x,y,z) = (y,x,z)$. Since x_5 is a fix-point of R it must be of the form $(x,x,z) \neq (0,0,0)$. Up to a positive multiple the only possible choice for x_5 is induced by $t=0$ in our construction sequence. For $t=0$ the determinant $\mathrm{DET}(x_{12},x_{13},x_{14})$ evaluates to zero. Hence, there is no symmetric realization. This proves (ii).

If we choose t as a rational number in $(-3+\sqrt{8},0) \cup (0,3-\sqrt{8})$ we get a rational realization, as stated in (iii). Fact (iv) is a direct consequence of the fact that for $t \in (-3+\sqrt{8}, 3-\sqrt{8})$ the determinant $\mathrm{DET}(x_{12},x_{13},x_{14})$ is always positive or zero. $\square$

4 Final polynomials

Bi-quadratic final polynomials [2, 8] are special final polynomials that can be found by linear programming. They provide an effective tool to prove non-realizability for a large class of oriented matroids. Here we restrict ourselves to the case of realizability over $\mathbb{R}$ and to the case of oriented matroids in rank 3 on a ground set $E = \{1, \ldots, n\}$. Our starting point is the structure of three-term Graßmann-Plücker polynomials. For this the *brackets* $[i, j, k]$ with $i, j, k \in E$ are considered as formal variables. We identify brackets according to the alternating determinant rules:

$$[i, j, k] = [k, i, j] = [j, k, i] = -[j, i, k] = -[k, j, i] = -[i, k, j].$$

The polynomial ring in all brackets $\mathbb{R}[\{[\lambda] \mid \lambda \in E^3\}]$ modulo these identifications is abbreviated $B_{3,n}$. (This is a polynomial ring in $\binom{n}{3}$ generators.) For an oriented matroid χ and a bracket monomial $[\lambda^1] \cdot [\lambda^2] \cdot \ldots \cdot [\lambda^k]$ we write

$$\chi([\lambda^1] \cdot [\lambda^2] \cdot \ldots \cdot [\lambda^k]) := \chi(\lambda^1) \cdot \chi(\lambda^2) \cdot \ldots \cdot \chi(\lambda^k).$$

For a vector configuration $X = (x_1, \ldots, x_n) \in \mathbb{R}^{3n}$ and $(i, j, k) \in E^3$ we write

$$[i, j, k]_X = \mathrm{DET}(x_i, x_j, x_k).$$

Definition 4.1. Let χ be a rank 3 oriented matroid on a finite set E of cardinality $n > 3$, let $\tau \in E, \lambda = (a, b, c, d) \in E^4$ with $|\{\tau, a, b, c, d\}| = 5$ and let

$$\begin{array}{ll} A := (\tau, a, b), & B := (\tau, c, d), \\ C := (\tau, a, c), & D := (\tau, b, d), \\ E := (\tau, a, d), & F := (\tau, b, c). \end{array}$$

(1) The pair (τ, λ) is called χ-*normalized* if

$$\chi([A][B]) \geq 0, \quad \chi([C][D]) \geq 0, \quad \chi([E][F]) \geq 0.$$

(2) A χ-normalized pair (τ, λ) is called χ-*non-degenerate* if $\chi([C][D]) > 0$.

(3) For a χ-non-degenerate pair (τ, λ) we call

$$\begin{array}{lll} [A][B] < [C][D] & \text{a } \textit{bi-quadratic inequality} & \text{if } \chi([E][F]) > 0, \\ [A][B] = [C][D] & \text{a } \textit{bi-quadratic equation} & \text{if } \chi([E][F]) = 0, \\ [E][F] < [C][D] & \text{a } \textit{bi-quadratic inequality} & \text{if } \chi([A][B]) > 0, \\ [E][F] = [C][D] & \text{a } \textit{bi-quadratic equation} & \text{if } \chi([A][B]) = 0. \end{array}$$

In fact (as a consequence of the oriented matroid axioms) for any $\tau \in E$ and $\lambda \in E^4$ there is always a suitable permutation $\pi \in S_4$ of the elements in λ such that $(\tau, \pi(\lambda))$ is χ-normalized. Furthermore, if $[A][B] = [C][D]$ is a bi-quadratic equation, $[C][D] = [A][B]$ is a bi-quadratic equation as well.

The set of all bi-quadratic inequalities of χ will be denoted by $\mathcal{B}_\chi$ and the set of all its bi-quadratic equations will be denoted by $\mathcal{A}_\chi$. Each element in $\mathcal{B}_\chi \cup \mathcal{A}_\chi$ is called a bi-quadratic expression. The bi-quadratic expressions can be considered as natural consequences of Graßmann-Plücker relations in the realizable case, as we will see now.

LEMMA 4.2. *For a vector configuration $X \in (\mathbb{R}^d)^n$ and its corresponding oriented matroid χ_X we have*

(i) $[A]_X[B]_X < [C]_X[D]_X$ *for all* $[A][B] < [C][D] \in \mathcal{B}_{\chi_X}$.

(ii) $[A]_X[B]_X = [C]_X[D]_X$ *for all* $[A][B] = [C][D] \in \mathcal{A}_{\chi_X}$

Proof.
<u>(i):</u> Assume that $[A][B] < [C][D]$ is a bi-quadratic inequality and let (τ, λ) be the corresponding χ-non-degenerate pair. Let $A, \dots, F$ be defined as in Definition 4.1. We have $\chi([E][F]) = 1$. The polynomial $[A][B] - [C][D] + [E][F]$ is a Graßmann-Plücker-polynomial. Hence its evaluation is identical to zero for every configuration $X \in (\mathbb{R}^d)^n$:

$$[A]_X[B]_X - [C]_X[D]_X + [E]_X[F]_X = 0.$$

Since $\chi([E][F]) = 1$, in any realization X of χ we have $[A]_X[B]_X - [C]_X[D]_X < 0$. This proves the first part of the lemma.
<u>(ii):</u> Let $[A][B] = [C][D]$ be a bi-quadratic equation and let $(\tau, \lambda), E, F$ be defined as above. Then we have $\chi([E][F]) = 0$. Therefore in any realization X of χ we have $[A]_X[B]_X - [C]_X[D]_X = 0$. □

The following definition of *bi-quadratic final polynomials* is more general than the one given in [2], where only the uniform case (no zero determinants) was considered.

DEFINITION 4.3. For an oriented matroid χ a non-empty collection of bi-quadratic inequalities

$$[A_i][B_i] < [C_i][D_i] \in \mathcal{B}_\chi; 1 \leq i \leq k$$

together with a (possibly empty) collection of bi-quadratic equations

$$[A_i][B_i] = [C_i][D_i] \in \mathcal{A}_\chi; k+1 \leq i \leq l$$

is called a *bi-quadratic final polynomial* if the following equality holds within the ring $B_{3,n}$ (where brackets are identified according to the alternating determinant rule):

$$\prod_{i=1}^{l} [A_i][B_i] = \prod_{i=1}^{l} [C_i][D_i].$$

LEMMA 4.4. [2, Lemma 4.1] *If χ admits a bi-quadratic final polynomial, then χ is not realizable over* $\mathbb{R}$.

Proof. Assume on the contrary that χ admits a bi-quadratic final polynomial as defined above, and χ is realizable, i.e $\chi = \chi_X$ for a suitable vector configuration X. By Lemma 4.2 we have

$$\begin{aligned} [A_i]_X[B_i]_X &< [C_i]_X[D_i]_X \quad \text{for all } 1 \leq i \leq k, \text{ and} \\ [A_i]_X[B_i]_X &= [C_i]_X[D_i]_X \quad \text{for all } k+1 \leq i \leq l. \end{aligned}$$

At least one proper inequality appears. By definition the products on the left side are all positive and the products on the right side are positive as well. If we multiply all right and all left sides we obtain:

$$\prod_{i=1}^{l}[A_i]_X[B_i]_X < \prod_{i=1}^{l}[C_i]_X[D_i]_X.$$

On the other hand the fact that we have a bi-quadratic final polynomial implies

$$\prod_{i=1}^{l}[A_i]_X[B_i]_X = \prod_{i=1}^{l}[C_i]_X[D_i]_X.$$

This contradicts the assumption that χ was realizable. □

5 Ω_{14}^{-} has no bi-quadratic final polynomial

The main result of this section is:

Theorem 5.1. *Let χ^0, χ^+, χ^- be three oriented matroids that differ in exactly one basis $\mu \in \Lambda(E,3)$ with $\chi^\sigma(\mu) = \sigma$. If χ^0 and χ^- are realizable and χ^+ is not, then χ^+ cannot have a bi-quadratic final polynomial.*

Proof. Assume that a bi-quadratic final polynomial for χ^+ exists. Let

$$\{[A_i][B_i] < [C_i][D_i] \mid 1 \le i \le k\} \subseteq \mathcal{B}_{\chi^+}$$

together with

$$\{[A_i][B_i] = [C_i][D_i] \mid k+1 \le i \le l\} \subseteq \mathcal{A}_{\chi^+}$$

be a bi-quadratic final polynomial for χ^+ consisting of $k > 0$ bi-quadratic inequalities and $l - k \ge 0$ bi-quadratic equations. Since $[\tau, b, c][\tau, e, f] = [\tau, c, b][\tau, f, e]$ holds, we may assume that every bracket in the bi-quadratic final polynomial has positive signature. In each bi-quadratic expression the bracket $[\mu]$ can be contained at most once (since each three-term Graßmann-Plücker-polynomial contains each bracket at most once). Since we have a bi-quadratic final polynomial the overall number r of occurrences of $[\mu]$ on the right sides of the expressions equals the number of overall occurrences of $[\mu]$ on the left sides. Thus we may assume that the bi-quadratic expressions are sorted in a way that each expression of the form $[A_i][B_i] \le [C_i][D_i]$ with $\mu \in \{A_i, B_i\}$ is directly followed by an expression $[A_{i+1}][B_{i+1}] \le [C_{i+1}][D_{i+1}]$ with $\mu \in \{C_{i+1}, D_{i+1}\}$ (indices taken modulo r).

With suitable $\tau_i \in E$ and $\lambda_i := (\lambda_{i1}, \ldots, \lambda_{i4}) \in E^4$ we have

$$\begin{aligned} A_i &:= (\tau_i, \lambda_{i1}, \lambda_{i2}), \quad B_i := (\tau_i, \lambda_{i3}, \lambda_{i4}), \\ C_i &:= (\tau_i, \lambda_{i1}, \lambda_{i3}), \quad D_i := (\tau_i, \lambda_{i_2}, \lambda_{i_4}). \end{aligned}$$

With this choice the Graßmann-Plücker polynomials

$$\{\tau_i|\lambda_i\} := [A_i][B_i] - [C_i][D_i] + [E_i][F_i]$$

are χ-normalized and χ-non-degenerate. By Definition 4.1 we know that $\chi([E_i][F_i])$ is $+1$ for $1 \leq i \leq k$ and 0 for $k+1 \leq i \leq l$. Furthermore $\chi([A_i][B_i]) = 1$ and $\chi([C_i][D_i]) = 1$ for all $1 \leq i \leq l$. We define monomials

$$m_i := \prod_{j=1}^{i-1}([A_i][B_i]) \cdot \prod_{j=i+1}^{l}([C_i][D_i])$$

and consider the polynomial

$$p := \sum_{i=1}^{l}\Big(m_i \cdot \{\tau_i|\lambda_i\}\Big).$$

We have

$$m_i \cdot [A_i][B_i] = m_{i+1} \cdot [C_{i+1}][D_{i+1}].$$

Furthermore, since all bi-quadratic expressions together form a bi-quadratic final polynomial, we also have

$$m_l \cdot [A_l][B_l] = \prod_{i=1}^{l}([A_i][B_i]) = \prod_{i=1}^{l}([C_i][D_i]) = m_1 \cdot [C_1][D_1].$$

Thus, canceling pairwise vanishing summands in p yields:

$$p = \sum_{i=1}^{l}\Big(m_i \cdot [E_i][F_i]\Big).$$

(In fact p is an ordinary final polynomial for χ^+ in the sense of Bokowski & Sturmfels [1, 5].) Since all Graßmann-Plücker-polynomials that are involved were χ-normalized we get:

$$\chi(m_i \cdot [E_i][F_i]) = 1 \text{ for } i = 1, \ldots, k$$

and

$$\chi(m_i \cdot [E_i][F_i]) = 0 \text{ for } i = k+1, \ldots, l.$$

By our assumption on the order of the bi-quadratic expressions in each of the monomials $m_i = [\mu]^r \cdot m_i'$ the bracket $[\mu]$ occurs with degree r (the total number of occurrences of $[\mu]$ on the right side of bi-quadratic expressions). Thus if we consider the polynomial

$$p' := \sum_{i=1}^{l}\Big(m_i' \cdot \{\tau_i|\lambda_i\}\Big) = \sum_{i=1}^{l}\Big(m_i' \cdot [E_i][F_i]\Big).$$

each summand $m_i' \cdot [E_i][F_i]$ is either linear in $[\mu]$ (in case that $\mu \in \{E_i, F_i\}$) or does not contain $[\mu]$ at all. Furthermore (since $\chi^+(\mu) = 1$) we have $\chi(m_i' \cdot [E_i][F_i]) = 1$ for $i = 1, \ldots, k$ and $\chi(m_i' \cdot [E_i][F_i]) = 0$ for $i = k+1, \ldots, l$. Thus we have

$$p' = [\mu] \cdot \sum_{i=1}^{s} p_i + \sum_{i=1}^{l-s} q_i$$

with $\chi(p_i)$ and $\chi(q_i)$ all either zero or positive and at least one of these monomials positive. Observe that the p_i and q_i are independent on $[\mu]$ thus the corresponding signs $\chi(p_i)$ and $\chi(q_i)$ are identical for χ^+, χ^0 and χ^-.

We now replace the brackets of p' by the values of the actual determinants of a realization of χ^0 (we know that such a realization does exist). The polynomial p' is a linear combination of Graßmann-Plücker-polynomials, hence this expression must evaluate to zero. Since $\chi^0([\mu]) = 0$ and the monomials q_i evaluate to a non-negative number we can conclude that $\chi(q_i) = 0$ for all $i = 1, \dots, l - s$.

Using this information we now consider the case where we replace the brackets of p' by the values of the actual determinants of a realization of χ^- (we know that such a realization does also exist). The summands q_i for all $i = 1, \dots, l - s$ evaluate to zero. Each of the summands $[\mu] \cdot p_i$ for $i = 1, \dots, s$ evaluates either to zero or to a number with sign since $\chi^-([\mu]) = -1$. At least one non-zero summand occurs. Thus we have a non-empty collection of negative numbers summing up to zero. □

Corollary 5.2. *The oriented matroid Ω_{14}^- is not realizable and does not admit a bi-quadratic final polynomial.*

Proof. The non-realizability of Ω_{14}^- was proved in Theorem 3.1. Since Ω_{14}^+ and Ω_{14}^0 are realizable Theorem 5.1 applies and the corollary follows. □

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