On the Nonexcellence of Field Extensions $F(\pi)/F$

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Abstract. For any $n \geq 3$, we construct a field F and an n-fold Pfister form φ such that the field extension $F(\varphi)/F$ is not excellent. We prove that $F(\varphi)/F$ is universally excellent if and only if φ is a Pfister neighbor of dimension ≤ 4.

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Let F be a field of characteristic different from 2 and φ be a non-degenerate quadratic form on an F-vector space V, by which V gets the structure of a non-degenerate quadratic space. Choosing an orthogonal basis of V we can write φ in the form $a_1x_1^2 + \cdots + a_dx_d^2$. In this case we use the notation $\varphi = \langle a_1, \ldots, a_d\rangle$.

A quadratic form or space φ is called *isotropic* if $\varphi(v) = 0$ for some nonzero vector $v \in V$. We say that φ is *anisotropic* otherwise. Up to isometry, there is exactly one non-degenerated isotropic 2-dimensional quadratic space, namely the *hyperbolic plane* $\mathbb{H}$ equipped with the form $\langle 1, -1\rangle$. A non-degenerate quadratic space is called *hyperbolic* if it is isometric to the orthogonal sum of hyperbolic planes $m\mathbb{H} = \mathbb{H} \perp \cdots \perp \mathbb{H}$.

According to Witt's main theorem any non-degenerate quadratic space V can be decomposed in the orthogonal sum $V = V_{an} \perp V_h$, where V_{an} is anisotropic and $V_h \cong m\mathbb{H}$ is a hyperbolic space. (We will use $\cong$ to denote isometry of quadratic forms or spaces.) Moreover the quadratic space V_{an} is uniquely determined up to isometry. The restriction $\varphi|_{V_{an}}$ is called the *anisotropic part* (or anisotropic kernel) of φ and is denoted by φ_{an}. The number $m = \frac{1}{2}\dim V_h$ is called the Witt index of φ.

For any quadratic space V and any field extension L/F one can provide $V_L = V \otimes_F L$ with a structure of a quadratic space. The corresponding quadratic form we shall denote by φ_L. We say that a quadratic form φ over L is defined over F if there is a quadratic form ξ over F such that $\varphi \cong \xi_L$.

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It is an important problem to study the behavior of the anisotropic part of forms over F under a field extension L/F. It occurs sometimes that any anisotropic form over F is still anisotropic over L (for example if L/F is of odd degree). In this case for any quadratic form φ over F the anisotropic part $(\varphi_L)_{an}$ of φ over L coincides with $(\varphi_{an})_L$ and hence is defined over F.

However, very often φ becomes isotropic over L. In this case we do not know if the anisotropic part of φ over L is defined over F.

A field extension L/F is called *excellent* if for any quadratic form φ over F the anisotropic part $(\varphi_L)_{an}$ of φ over L is defined over F (i.e., there is a form ξ over F such that $(\varphi_L)_{an} \cong \xi_L$).

It is well known that any quadratic extension is excellent. Since any anisotropic quadratic form ψ over F is still anisotropic over the field of rational functions $F(t)$, every purely transcendental field extension is excellent.

Among all field extensions the fields $F(\varphi)$ of rational functions on the quadric hyper-surface defined by the equation $\varphi = 0$ are of special interest in the theory of quadratic forms. One of the important problems is to find a condition on φ so that the field extension $F(\varphi)/F$ is excellent.

We say that $F(\varphi)/F$ is *universally excellent* if for any extension K/F the extension $K(\varphi)/K$ is excellent.

If φ is isotropic then $F(\varphi)/F$ is purely transcendental, and it follows from Springer's theorem that $F(\varphi)/F$ is excellent and moreover is universally excellent. Thus it is sufficient to consider only the case of anisotropic forms φ.

In [Kn] Knebusch has proved that if φ is an anisotropic form such that $F(\varphi)/F$ is excellent then φ is a *Pfister neighbor*. This means that there is a quadratic form $\pi = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ (called *n-fold Pfister form*) such that φ is similar to a subform of π and $\dim(\varphi) > \frac{1}{2}\dim(\pi)$. This result gives rise to the natural question whether the field extension $F(\varphi)/F$ is excellent for any Pfister neighbor φ. This problem can be easily reduced to the case of an n-fold Pfister forms φ.

If $n = 1$ then $F(\varphi)/F$ is obviously excellent since $F(\varphi)/F$ is a quadratic extension. Arason [ELW1, Appendix II] has proved that, for $n = 2$, $F(\varphi)/F$ is always excellent (see also [R], [LVG]). Thus the answer to our question is yes for n-fold Pfister forms with $n \leq 2$. It was an open problem whether $F(\varphi)/F$ is excellent for any field F and any n-fold Pfister form φ over F (with $n \geq 3$).

In [ELW2] some special cases of this problem were considered: for an n-fold Pfister form φ with $n \geq 3$, the excellence of the field extension $F(\varphi)/F$ was proved for all fields with $\tilde{u}(F) \leq 4$. In [H2] Hoffmann considered another special case of the problem. An extension L/F is called *d-excellent* if for any quadratic form ψ of dimension $\leq d$ the anisotropic part $(\psi_L)_{an}$ of ψ over L is defined over F. Hoffmann has proved that the extension $F(\varphi)/F$ is 6-excellent for any Pfister neighbor φ.

In this paper we prove that for any $n \geq 3$ there is a field F and an n-fold Pfister form φ such that the field extension $F(\varphi)/F$ is not excellent. Moreover Theorem 1.1 of our paper says that $F(\varphi)/F$ is universally excellent if and only if φ is a Pfister neighbor of an n-fold Pfister form with $n \leq 2$, (i.e., either $\dim\varphi \leq 3$ or φ is a 4-dimensional form with $\det(\varphi) = 1$). In §3 we use the main construction of the paper to study "splitting pairs" φ, ψ of quadratic forms. More precisely, we construct a "non standard pair" φ, ψ such that φ is isotropic over the function field $F(\psi)$ of the quadric ψ.

Remark. Some results of this paper were developed further by D. Hoffmann in [H4].

1. Main Theorem

We will use the following notation throughout the paper: by $\varphi \perp \psi$, $\varphi \cong \psi$, and $[\varphi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of φ in the Witt ring $W(F)$ of the field F. The maximal ideal of $W(F)$ generated by the classes of even dimensional forms is denoted by $I(F)$. We write $\varphi \sim \psi$ if φ is similar to ψ, i.e., $k\varphi = \psi$ for some $k \in F^*$. The anisotropic part of φ is denoted by φ_{an} and $i_W(\varphi)$ denotes the Witt index of φ. We denote by $\langle\langle a_1, \ldots, a_n\rangle\rangle$ the n-fold Pfister form

$$\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms we denote by $GP_n(F)$. For any field extension L/F we put $\varphi_L = \varphi \otimes L$, $W(L/F) = \ker(W(F) \to W(L))$.

Main Theorem 1.1. *Let φ be an anisotropic form over F. Then the following conditions are equivalent.*

- (i) *The field extension $F(\varphi)/F$ is universally excellent, i.e., for any field extension E/F the extension $E(\varphi)/E$ is excellent.*
- (ii) *Either* $\dim(\varphi) \leq 3$ *or* $\varphi \in GP_2(F)$.

Proof of (ii) $\Rightarrow$ (i). The case $\dim(\varphi) = 2$ is obvious. If $\dim(\varphi) = 3$ or $\varphi \in GP_2(F)$ the excellence of the extension $E(\varphi)/E$ was proved by Arason (see the introduction).

Proof of (i) $\Rightarrow$ (ii). Since $E(\varphi)/E$ is excellent for any extension E/F, we see that $F(\varphi)/F$ is excellent. It was shown in [Kn, 7.13] that for $F(\varphi)/F$ to be excellent it is necessary that φ is a Pfister neighbor. Let φ be a Pfister neighbor of the n-fold Pfister form π. Since $F(\varphi)$ and $F(\pi)$ are F-equivalent, we can replace φ by π, i.e., we can suppose that $\varphi = \pi$ is an n-fold Pfister form. Thus it is sufficient to prove the following proposition.

Proposition 1.2. *Let π be anisotropic n-fold Pfister form over the field F. If $n \geq 3$ then there is a field extension E/F such that $E(\pi)/E$ is not excellent.*

2. Proof of Proposition 1.2

Lemma 2.1. *Let π and τ be anisotropic n-fold Pfister forms over the field F. Then there is a field extension K/F such that the following conditions hold.*

- a) $\pi_K = \tau_K$,
- b) *π_K and τ_K are anisotropic.*

Proof. Let φ be a Pfister neighbor of τ of dimension $2^{n-1}+1$. It follows from [H3, Theorem 4] that there exists a field extension K/F such that π_K is anisotropic and $\varphi_K \subset \pi_K$. Hence φ_K is a Pfister neighbor of π_K. Since φ_K is a Pfister neighbor of τ_K, we have $\pi_K = \tau_K$. $\square$

Lemma 2.2. *Let τ and π be anisotropic n-fold Pfister forms over F. Suppose that there is $a \in F^*$ such that $\tau_{F(\sqrt{a})}$ and $\pi_{F(\sqrt{a})}$ are isotropic. Then there is an extension E/F and $x \in E^*$ such that the following conditions hold.*

1) $\pi_{E(\sqrt{x})} = \tau_{E(\sqrt{x})}$,
2) $\pi_{E(\sqrt{x})}$ *and* $\tau_{E(\sqrt{x})}$ *are anisotropic,*
3) E/F *is unirational.*

Remark: We say that E/F is unirational, if there is a purely transcendental finitely generated field extension K/F such that $F \subset E \subset K$.

Proof. Since τ is an n-fold Pfister form and $\tau_{F(\sqrt{a})}$ is isotropic, we can write τ in the form $\tau = \langle\langle a, b_1, \dots, b_{n-1}\rangle\rangle$. Similarly, we can write π in the form $\pi = \langle\langle a, c_1, \dots, c_{n-1}\rangle\rangle$. Let $\widetilde{F} = F(A, B_1, \dots, B_{n-1}, C_1, \dots, C_{n-1})$ be the rational function field in $2n-1$ variables over $\widetilde{F}$.

Put $\widetilde{\tau} = \langle\langle A, B_1, \dots, B_{n-1}\rangle\rangle$ and $\widetilde{\pi} = \langle\langle A, C_1, \dots, C_{n-1}\rangle\rangle$. Let $\gamma = \tau \perp -\pi$ and $\widetilde{\gamma} = \widetilde{\tau} \perp -\widetilde{\pi}$. Let $E/\widetilde{F}$ be the universal field extension such that $\gamma_E = \widetilde{\gamma}_E$, i.e., $E = \widetilde{F}_h$, where $\widetilde{F} = \widetilde{F}_0, \widetilde{F}_1, \dots, \widetilde{F}_h$ is a generic splitting tower of the quadratic form $\gamma \perp -\widetilde{\gamma}$.

It is well known that the following universal property of E holds: For any field extension $K/\widetilde{F}$ the condition $\gamma_K = \widetilde{\gamma}_K$ implies that EK/K is purely transcendental.

Now we prove that conditions 1)–3) of the lemma hold for $x = A$.

1) We have $[\tau_{E(\sqrt{A})}] - [\pi_{E(\sqrt{A})}] = [\gamma_{E(\sqrt{A})}] = [\widetilde{\gamma}_{E(\sqrt{A})}] = [\widetilde{\tau}_{E(\sqrt{A})}] - [\widetilde{\pi}_{E(\sqrt{A})}] = 0$. Hence $[\tau_{E(\sqrt{A})}] = [\pi_{E(\sqrt{A})}]$.

2) Let K/F be as in Lemma 2.1, i.e., τ_K, π_K are anisotropic and $\tau_K = \pi_K$. We have $[\gamma_K] = [\tau_K] - [\pi_K] = 0$

Let $\widetilde{K} = K(A, B_1, \dots, B_{n-1}, C_1, \dots, C_{n-1})$ be the rational function field in $2n-1$ variables over K. We have $[\gamma_{\widetilde{K}(\sqrt{A})}] = [\tau_{\widetilde{K}(\sqrt{A})}] - [\pi_{\widetilde{K}(\sqrt{A})}] = 0$ and $[\widetilde{\gamma}_{\widetilde{K}(\sqrt{A})}] = [\widetilde{\tau}_{\widetilde{K}(\sqrt{A})}] - [\widetilde{\pi}_{\widetilde{K}(\sqrt{A})}] = 0$. Therefore $[\gamma_{\widetilde{K}(\sqrt{A})}] = [\widetilde{\gamma}_{\widetilde{K}(\sqrt{A})}]$. Using the universal property of $E/\widetilde{F}$ we see that $E\widetilde{K}(\sqrt{A})/\widetilde{K}(\sqrt{A})$ is purely transcendental.

It is clear that $\widetilde{K}(\sqrt{A})/K$ is purely transcendental. Therefore $E\widetilde{K}(\sqrt{A})/K$ is purely transcendental. Hence $\tau_{E\widetilde{K}(\sqrt{A})}$ and $\pi_{E\widetilde{K}(\sqrt{A})}$ are anisotropic. Therefore $\tau_{E(\sqrt{A})}$ and $\pi_{E(\sqrt{A})}$ are anisotropic.

3) Let $L = \widetilde{F}(\sqrt{A/a}, \sqrt{B_1/b_1}, \dots, \sqrt{B_{n-1}/b_{n-1}}, \sqrt{C_1/c_1}, \dots, \sqrt{C_{n-1}/c_{n-1}})$. It is clear that $\pi_L = \widetilde{\pi}_L$ and $\tau_L = \widetilde{\tau}_L$. Therefore $\gamma_L = \widetilde{\gamma}_L$. Using the universal property of $E/\widetilde{F}$ we see that EL/L is purely transcendental. It is clear that L/F is purely transcendental. Hence EL/F is purely transcendental. Since $E \subset EL$ we see that E/F is unirational. $\square$

Lemma 2.3. *Let F be a field and π be anisotropic n-fold Pfister form over F. Then there are a unirational extension E/F, an n-fold Pfister form τ over E, and $x \in E^*$ such that the following conditions hold.*

1) $\pi_{E(\sqrt{x})} = \tau_{E(\sqrt{x})}$,
2) $\pi_{E(\sqrt{x})}$ *and* $\tau_{E(\sqrt{x})}$ *are anisotropic,*
3) $\dim(\pi_E \perp -\tau_E)_{an} = 2^{n+1} - 4$.

Proof. Write π in the form $\pi = \langle\langle a, b_1, b_2, \ldots, b_{n-1}\rangle\rangle$. Let $\widetilde{F} = F(T_1, \ldots, T_{n-1})$ be the rational function field in $n-1$ variables over F. Let $\tau = \langle\langle a, T_1, \ldots, T_{n-1}\rangle\rangle$. Obviously

$$(\pi_{\widetilde{F}} \perp -\tau)_{an} = \langle\langle a\rangle\rangle\langle\langle b_1, \ldots, b_{n-1}\rangle\rangle'_{\widetilde{F}} \perp -\langle\langle a\rangle\rangle\langle\langle T_1, \ldots, T_{n-1}\rangle\rangle'.$$

Therefore $\dim(\pi_{\widetilde{F}} \perp -\tau)_{an} = 2^{n+1} - 4$.

The quadratic forms $\pi_{\widetilde{F}(\sqrt{a})}$ and $\tau_{\widetilde{F}(\sqrt{a})}$ are hyperbolic, i.e., all the conditions of Lemma 2.2 hold for $\widetilde{F}$, π, τ. Hence there is a unirational extension $E/\widetilde{F}$ such that

1) $\pi_{E(\sqrt{x})} = \tau_{E(\sqrt{x})}$,
2) $\pi_{E(\sqrt{x})}$ and $\tau_{E(\sqrt{x})}$ are anisotropic,

Since $E/\widetilde{F}$ is unirational, we have $\dim(\pi_E \perp -\tau_E)_{an} = \dim(\pi_{\widetilde{F}} \perp -\tau)_{an} = 2^{n+1} - 4$. Finally E/F is unirational since $E/\widetilde{F}$ is unirational and $\widetilde{F}/F$ is purely transcendental. $\square$

LEMMA 2.4. *Let E be a field, $n \geq 3$, $x \in E^*$. Let $\pi, \tau \in P_n(E)$ be such that*

1) $\pi_{E(\sqrt{x})} = \tau_{E(\sqrt{x})}$.
2) $\pi_{E(\sqrt{x})}$ *and* $\tau_{E(\sqrt{x})}$ *are anisotropic.*
3) $\dim(\pi \perp -\tau)_{an} = 2^{n+1} - 4$.

Let $\psi = \tau' \perp \langle x\rangle$ where τ' is such that $\tau = \tau' \perp \langle 1\rangle$.

Then

a) *ψ is anisotropic.*
b) *$\psi_{E(\pi)}$ is isotropic.*
c) *There is no quadratic form γ over E such that $(\psi_{E(\pi)})_{an} = \gamma_{E(\pi)}$.*
d) *For any subform $\xi \subsetneq \psi$ the form $\xi_{F(\pi)}$ is anisotropic, i.e., ψ is a minimal $F(\pi)$-form.*

Proof. a) Obviously $\psi_{E(\sqrt{x})} = \tau_{E(\sqrt{x})}$. By assumption we see that $\tau_{E(\sqrt{x})}$ is anisotropic. Hence $\psi_{E(\sqrt{x})}$ is anisotropic. Therefore ψ is anisotropic too.

b) Suppose that $\psi_{E(\pi)}$ is anisotropic. Since $\psi_{E(\sqrt{x})} = \tau_{E(\sqrt{x})} = \pi_{E(\sqrt{x})}$ we have $[\psi_{E(\pi)(\sqrt{x})}] = [\pi_{E(\pi)(\sqrt{x})}] = 0$. Since $\psi_{E(\pi)}$ is anisotropic and $\psi_{E(\pi)(\sqrt{x})}$ is hyperbolic, we conclude that $\psi_{E(\pi)} = \langle\langle x\rangle\rangle\xi$ where ξ is a quadratic form over $E(\pi)$. Since $\dim(\xi) = 2^{n-1}$ is even, we have $\xi \in I(E(\pi))$. Therefore $\psi_{E(\pi)} = \langle\langle x\rangle\rangle\xi \in I^2(E(\pi))$. Hence $\psi \in I^2(E)$. Therefore $[\langle\langle x\rangle\rangle] = [\tau] - [\psi] \in I^2(E)$, a contradiction.

c) Suppose that $(\psi_{E(\pi)})_{an} = \gamma_{E(\pi)}$ where γ is a quadratic form over E. It is clear that $\dim(\gamma) \leq 2^n - 2$. We have $(\psi \perp -\gamma)_{an} \in W(E(\pi)/E)$. Since π is a Pfister form we conclude that $(\psi \perp -\gamma)_{an} = \pi\mu$, with μ a quadratic form over E.

Since $2 = 2^n - (2^n - 2) \leq \dim(\psi \perp -\gamma)_{an} = 2^n + (2^n - 2) = 2^{n+1} - 2$ and $\dim(\pi) = 2^n$ divides $\dim(\pi\mu)$ we conclude that $\dim(\mu) = 1$. Writing μ in the form $\mu = \langle k\rangle$ we have $(\psi \perp -\gamma)_{an} = k\pi$. Hence $[k\pi] = [\psi] - [\gamma]$. Therefore

$$[\tau \perp -k\pi] = [\tau] - [k\pi] = ([\psi] + [\langle\langle x\rangle\rangle]) - ([\psi] - [\gamma]) = [\langle\langle x\rangle\rangle \perp \gamma].$$

Hence τ and $k\pi$ contain a common subform of dimension

$$\frac{1}{2}(\dim(\tau) + \dim(k\pi) - \dim(\langle\langle x\rangle\rangle \perp \gamma)) \geq \frac{1}{2}(2^n + 2^n - 2^n) = 2^{n-1} \geq 2^{3-1} = 4 > 3.$$

Therefore there is a 3-dimensional form ρ such that $\rho \subset \tau$, $\rho \subset k\pi$. Let $a, b \in E$ be such that $\rho \sim \langle 1, -a, -b\rangle$. Let $\varepsilon = \langle\langle a, b\rangle\rangle$. Obviously $\tau_{E(\varepsilon)}$ and $\pi_{E(\varepsilon)}$ are isotropic. Since τ, π, and ε are anisotropic Pfister forms, we conclude that $\varepsilon \subset \tau$ and $\varepsilon \subset \pi$. Therefore $\dim(\pi \perp -\tau)_{an} \leq \dim(\pi) + \dim(\tau) - 2\dim(\varepsilon) = 2^n + 2^n - 2\cdot 4 = 2^{n+1} - 8$, a contradiction.

d) We can suppose that ξ is a (2^n-1)-dimensional subform of ψ. let $k \in E^*$ be such that $\xi \perp \langle -k\rangle = \psi$. Set $\widetilde{\xi} = \xi \perp \langle -xk\rangle$. We have

$$[\tau] - [\widetilde{\xi}] = [\tau] - ([\xi] - [\langle xk\rangle]) = ([\psi] + [\langle\langle x\rangle\rangle]) - ([\psi] + [\langle k\rangle] - [\langle xk\rangle]) = [\langle\langle x, k\rangle\rangle].$$

Let $\rho = \langle\langle x, k\rangle\rangle$. We have $[\tau_{E(\rho)}] = [\widetilde{\xi}_{E(\rho)}]$. Comparing dimensions we see that $\tau_{E(\rho)} = \widetilde{\xi}_{E(\rho)}$. Therefore $\tau_{E(\rho,\pi)} = \widetilde{\xi}_{E(\rho,\pi)}$.

Our goal is to prove that $\xi_{E(\pi)}$ is anisotropic. Let us suppose that $\xi_{E(\pi)}$ is isotropic. Then $\widetilde{\xi}_{E(\rho,\pi)}$ is isotropic too. Therefore $\tau_{E(\rho,\pi)}$ is isotropic. Hence the Pfister form $\tau_{E(\rho)}$ becomes isotropic over the function field of the Pfister form $\pi_{E(\rho)}$. Therefore either $\tau_{E(\rho)}$ or $\tau_{E(\rho)} = \pi_{E(\rho)}$ is hyperbolic.

Suppose first that $\tau_{E(\rho)}$ is hyperbolic. Since $\rho_{E(\sqrt{x})} = \langle\langle x, k\rangle\rangle_{E(\sqrt{x})}$ is isotropic we conclude that $\tau_{E(\sqrt{x})}$ is isotropic. This contradicts the assumption in this lemma.

Let now $\tau_{E(\rho)} = \pi_{E(\rho)}$. Then $(\tau \perp -\pi)_{an} \in W(E(\rho)/E)$. Hence $(\tau \perp -\pi)_{an} = \rho\lambda$ with λ a quadratic form over E ([S, Ch.4,5.6]). Since $\dim(\tau \perp -\pi)_{an} = 2^n - 4$ and $\dim(\rho) = 4$ we conclude that $\dim(\lambda) = (2^n-4)/4 = 2^{n-2} - 1$. Since $n \geq 3$ we see that $\dim(\lambda)$ is odd and hence $[\lambda] \equiv [\langle 1\rangle] \pmod{I(E)}$. Since $\rho \in I^2(E)$ we have $[\rho\lambda] \equiv [\rho] \pmod{I^3(E)}$. Since $\tau, \pi \in P_n(E)$ and $n \geq 3$, we see that $[(\tau \perp -\pi)_{an}] \equiv 0 \pmod{I^3(E)}$. We have

$$[\rho] \equiv [\rho\lambda] = [(\tau \perp -\pi)_{an}] \equiv 0 \pmod{I^3(E)}.$$

Since $\dim(\rho) = 4 < 8$ we conclude that ρ is hyperbolic. Therefore $(\tau \perp -\pi)_{an} = \rho\lambda$ is hyperbolic. However $\dim(\tau \perp -\pi)_{an} = 2^n - 4 > 0$, a contradiction. □

Corollary 2.5. *Let π be an anisotropic n-fold Pfister form over the field F. If $n \geq 3$ then there is a unirational extension E/F such that $E(\pi)/E$ is not excellent.* □

This corollary completes the proof of Proposition 1.2 and Theorem 1.1.

Corollary 2.6. *Let $n \geq 3$. Then there are a field E, an n-fold Pfister form π over E, and a 2^n-dimensional form ψ over E such that ψ is an $E(\pi)$-minimal form.* □

Corollary 2.7. *Let $n \geq 3$. Then there are a field E and 2^n-dimensional forms ψ and π over E such that ψ is an $E(\pi)$-minimal form and ψ is not similar to π.* □

3. Nonstandard Splitting

An important problem in the theory of quadratic forms is to determine when an anisotropic quadratic form φ over F becomes isotropic over the function field $F(\psi)$ of another form ψ. There are some well-known situations when this occurs and we list some of them in the following two definitions.

Definition 3.1. Let φ and ψ be anisotropic quadratic forms. We say that the ordered pair φ, ψ is *elementary splitting* (or *elementary*) if one of the following conditions holds.

1) There is a $k \in F^*$ such that $k\psi \subset \varphi$;
2) There is a $k \in F^*$, such that $k\varphi \subset \psi$ and $\dim(\varphi) > \dim(\psi) - i_1(\psi)$;
3) There is a $\rho \in W(F(\psi)/F)$ such that $\dim(\rho) < 2\dim(\varphi)$ and $k\varphi \subset \rho$ for some $k \in F^*$.

Definition 3.2. Let φ and ψ be anisotropic quadratic forms. We say that the ordered pair φ, ψ is standard if there is a collection

$$\varphi_0 = \varphi, \varphi_1, \ldots, \varphi_{n-1}, \varphi_n = \psi$$

such that the pair φ_{i-1}, φ_i is elementary for each $i = 1, 2, \ldots, n$.

It is clear that if the pair (φ, ψ) is elementary splitting or standard, then $\varphi_{F(\psi)}$ is isotropic.

Examples 3.3. *Let φ and ψ be anisotropic quadratic forms such that $\varphi_{F(\psi)}$ is isotropic. Suppose that at least one of the following conditions holds*

a) *φ is a Pfister neighbor;*
b) *$\dim(\psi) \leq 3$, or $\psi \in GP_2(F)$;*
c) *$\dim(\varphi) \leq 5$;*

Then the pair φ, ψ is elementary.

Proof. a) Let φ be a Pfister neighbor of ρ. Then condition 3) of Definition 3.1 is fulfilled.

b) By the excellence property of the field extension $F(\psi)/F$ there exists an anisotropic form ξ over F such that $(\varphi_{F(\psi)})_{an} = \xi_{F(\psi)}$. Setting $\rho = \varphi \perp -\xi$ one can see that condition 3) of Definition 3.1 holds.

c) Let $\dim(\varphi) \leq 5$. We can suppose that φ is not a Pfister neighbor and $\psi \notin GP_2(F)$ (see a), b)). Then $\varphi_{F(\psi)}$ is isotropic if and only if φ contains a subform similar to ψ (see [H1, Th. 1, Main Theorem]). Therefore condition 1) of Definition 3.1 holds. □

Example 3.4. *Let $F = \mathbb{R}(T)$, $\varphi = \langle T, T, T, 1, 1, 1, 1, 1\rangle$, $\psi = \langle T, T, 1, 1, 1, 1, 1, 1\rangle$. Then the pair φ, ψ is standard but not elementary.*

Proof. Let $\rho = \langle T, T, 1, 1, 1, 1, 1\rangle$. Since $\rho \subset \varphi$, the pair (φ, ρ) is elementary. Since $\rho \subset \psi$ and $\dim(\rho) = 7 > 8 - 2 = \dim(\psi) - i_1(\psi)$, we see that the pair (ρ, ψ) is elementary. Since the pairs (φ, ρ) and (ρ, ψ) are elementary, we see that the pair (φ, ψ) is standard. It follows from Lemma 3.7 below that the pair (φ, ψ) is not elementary. □

In this section we construct a pair of anisotropic forms φ and ψ with $\varphi_{F(\psi)}$ isotropic which is not standard.

Lemma 3.5. *Let F be a field, $n \geq 3$, $x \in F^*$. Let $\pi, \tau \in P_n(F)$ be such that*

1) *$\pi \neq \tau$,*
2) *$\pi_{F(\sqrt{x})} = \tau_{F(\sqrt{x})}$,*
3) *$\pi_{F(\sqrt{x})}$ and $\tau_{F(\sqrt{x})}$ are anisotropic.*

Let $\varphi = \pi' \perp \langle x\rangle$ *and* $\psi = \tau' \perp \langle x\rangle$*. Then*

a) ψ *and* φ *are anisotropic,*

b) $\varphi_{F(\psi)}$ *and* $\psi_{F(\varphi)}$ *are isotropic,*

c) $\varphi \not\sim \psi$.

Proof. a) Obviously $\psi_{F(\sqrt{x})} = \pi_{F(\sqrt{x})}$ and $\psi_{F(\sqrt{x})} = \tau_{F(\sqrt{x})}$. It follows from condition 3) that φ and ψ are anisotropic.

b) Let us suppose that $\varphi_{F(\psi)}$ is anisotropic. Since $\varphi_{F(\sqrt{x})} = \pi_{F(\sqrt{x})}$ and $\psi_{F(\sqrt{x})} = \tau_{F(\sqrt{x})} = \pi_{F(\sqrt{x})}$ we see that $\varphi_{F(\psi,\sqrt{x})} = \pi_{F(\pi,\sqrt{x})}$. Since $\pi \in P_n(F)$ we conclude that $\varphi_{F(\psi,\sqrt{x})}$ is hyperbolic. Therefore $\varphi_{F(\psi)} = \langle\langle x\rangle\rangle\xi$ where ξ is a quadratic form over $F(\psi)$. Since $\dim(\xi) = 2^{n-1}$ is even, we have $\xi \in I(F(\psi))$. Therefore $\psi_{F(\psi)} = \langle\langle x\rangle\rangle\xi \in I^2(F(\psi))$. Hence $\psi \in I^2(F)$. Therefore $[\langle\langle x\rangle\rangle] = [\tau] - [\psi] \in I^2(F)$, a contradiction.

c) Suppose that $k\varphi = \psi$. Then $[k\pi] - [k\langle\langle x\rangle\rangle] = [k\varphi] = [\psi] = [\tau] - [\langle\langle x\rangle\rangle]$. Therefore $[\langle\langle x,k\rangle\rangle] = [\tau] - [k\pi] \in I^n(F) \subset I^3(F)$. Since $\dim(\langle\langle x,k\rangle\rangle) = 4 < 8$, we have $[\tau] - [k\pi] = [\langle\langle x,y\rangle\rangle] = 0$. Hence $\tau \sim \pi$. Since $\tau, \pi \in P_n(F)$ we see that $\tau = \pi$, a contradiction. □

Lemma 3.6. *Let* $\pi \in P_3(F)$ *and* $x \in F^*$ $(x \notin F^{*2})$ *be such that* $\pi_{F(\sqrt{x})}$ *is anisotropic. Let* $\varphi = \pi' \perp \langle x\rangle$*. Suppose that* ψ *is an anisotropic quadratic form such that* $\psi_{F(\varphi)}$ *and* $\varphi_{F(\psi)}$ *are isotropic. Then* $\dim(\psi) = 8$.

By $C(\varphi)$ (resp. $C_0(\varphi)$) we will denote the Clifford algebra (resp. even Clifford algebra) of the quadratic form φ. If they are central simple we denote their classes in the Brauer group of the underlying field by $[C(\varphi)]$ (resp. $[C_0(\varphi)]$).

Proof. Since $\dim(\varphi) = 8$ and $\varphi_{F(\psi)}$ is isotropic, it follows from Hoffmann's theorem [H3, §1, Theorem 1] that $\dim(\psi) \le 8$.

Suppose that $\dim(\psi) \le 6$. Since $\dim(\varphi) = 8$ and $\psi_{F(\varphi)}$ is isotropic, it follows from Hoffmann's theorems [H1], [H2] that $\varphi \in GP_3(F)$. Therefore $x = \det(\varphi) = 1$, a contradiction.

Consider now the case $\dim(\psi) = 7$. Since $\pi_{F(\psi,\sqrt{x})} = \varphi_{F(\psi,\sqrt{x})}$ is isotropic we see that $\psi_{F(\sqrt{x})}$ is a Pfister neighbor of $\pi_{F(\sqrt{x})}$. Therefore $[C_0(\psi)_{F(\sqrt{x})}] = 0$. Hence there is $y \in F^*$ such that $[C_0(\psi)] = [\left(\frac{x,y}{F}\right)]$. Let $\rho = \langle\langle x,y\rangle\rangle$.

We claim that $\psi_{F(\rho)}$ is an anisotropic Pfister neighbor. To prove this we consider the quadratic form $\widetilde{\psi} = \psi \perp \langle\det(\psi)\rangle$. Since $\dim(\widetilde{\psi}) = 8$ and $[C(\widetilde{\psi}_{F(\rho)})] = [\left(\frac{x,y}{F(\rho)}\right)] = 0$ we have $\widetilde{\psi}_{F(\rho)} \in GP_3(F(\rho))$. If $\psi_{F(\rho)}$ is isotropic then $\widetilde{\psi}_{F(\rho)}$ is isotropic too and hence hyperbolic. Therefore, $(\widetilde{\psi})_{an} = \rho\mu$. Since $\dim(\widetilde{\psi}) = 6$ or 8 we must have $\dim\mu = 2$ which implies $\widetilde{\psi}_{an} = \widetilde{\psi} \in GP_3(F)$. Therefore $[C(\rho)] = [C_0(\psi)] = [C(\widetilde{\psi})] = 0$. Hence, ρ is hyperbolic and ψ stays anisotropic over $F(\rho)$, a contradiction.

Since $\psi_{F(\varphi)}$ is isotropic, $\psi_{F(\rho)}$ becomes isotropic over the functional field of the form $\varphi_{F(\rho)}$. Since $\psi_{F(\rho)}$ is an anisotropic Pfister neighbor and $\dim(\varphi_{F(\rho)}) = 8$ we see that $\varphi_{F(\rho)} \in GP_3(F(\rho)) \subset I^2(F(\rho))$. Since $W(F)/I^2(F) \to W(F(\rho))/I^2(F(\rho))$ is injective we have $\varphi \in I^2(F)$. Hence $x = \det(\varphi) = 1$, a contradiction. □

Lemma 3.7. *Let* φ *and* ψ *be anisotropic 8-dimensional quadratic form such that* $\psi \notin GP_3(F)$ *and the pair* φ, ψ *is elementary. Then* $\varphi \sim \psi$.

Proof. Since the pair φ, ψ is elementary, one of conditions 1)–3) of Definition 3.1 holds. Since $\dim(\varphi) = \dim(\psi)$, both the conditions 1), 2) imply that $\varphi \sim \psi$. Now we suppose that condition 3) holds, i.e., there is $\rho \in W(F(\psi)/F)$ such that $\dim(\rho) < 2\dim(\varphi) = 16$ and $k\varphi \subset \rho$. Since $\dim(\psi) > 4$, the homomorphism $W(F)/I^3(F) \to W(F(\psi))/I^3(F(\psi))$ is injective. Hence $\rho \in I^3(F)$. Let $\sigma \in P_2(F)$ be such that ψ contains a Pfister neighbor of σ. Then $\rho \in W(F(\psi)/F) \subset W(F(\sigma)/F)$ and thus $\rho_{an} \cong \sigma\mu$ for some μ. If $\dim\mu$ is odd then $\sigma \equiv \sigma\mu = \rho \equiv 0 \pmod{I^3(F)}$, a contradiction. Thus $\dim\mu$ is even and $8|\dim(\rho_{an})$. Therefore $\dim(\rho_{an}) = 8$. Hence $\rho_{an} \in GP_3(F)$. Since $\rho_{F(\psi)}$ is hyperbolic, ψ is a Pfister neighbor in ρ_{an}. Since $\dim(\psi) = \dim(\rho_{an}) = 8$ we have $\psi \sim \rho_{an} \in GP_3(F)$, a contradiction. $\square$

Lemma 3.8. *Let $n = 3$, and let φ, ψ be as in Lemma 3.5. Then the pair φ, ψ is not standard.*

Proof. Assume that the pair φ, ψ is standard. Then there is a collection

$$\varphi_0 = \varphi, \varphi_1, \ldots, \varphi_{n-1}, \varphi_n = \psi$$

such that the pair φ_{i-1}, φ_i is elementary for each $i = 1, 2, \ldots, n$. Obviously, the quadratic forms $\varphi_{F(\varphi_i)}$ and $(\varphi_i)_{F(\psi)}$ are isotropic. Since $\psi_{F(\varphi)}$ is isotropic (see Lemma 3.5) and $(\varphi_i)_{F(\psi)}$ is isotropic, we see that $(\varphi_i)_{F(\varphi)}$ is isotropic too. Thus $\varphi_{F(\varphi_i)}$ and $(\varphi_i)_{F(\varphi)}$ are isotropic. It follows from Lemma 3.6 that $\dim(\varphi_i) = 8$.

Consider first the case $\psi_i \in GP_3(F)$. Since $(\varphi_i)_{F(\varphi)}$ and is isotropic, φ is a Pfister neighbor of ψ_i. Since $\dim(\varphi) = \dim(\psi_i) = 8$ we have $\varphi \sim \psi_i$. Hence $\varphi \in GP_3(F)$, a contradiction.

Thus we have proved that $\dim(\varphi_i) = 8$ and $\psi_i \notin GP_3(F)$ for each $i = 1, 2, \ldots, n$. It follows from Lemma 3.7 that $\varphi_{i-1} \sim \varphi_i$. We have

$$\varphi = \varphi_0 \sim \varphi_1 \sim \cdots \sim \varphi_n = \psi.$$

On the other hand, it follows from Lemma 3.5 that $\varphi \not\sim \psi$. The contradiction obtained proves the lemma. $\square$

Theorem 3.9. *For any field F there is a unirational field extension E/F and a pair of 8-dimensional anisotropic quadratic forms φ and ψ over E such that $\varphi_{E(\psi)}$ is isotropic, but the pair φ, ψ is not standard.*

Proof. Let $n = 3$. Let E, π and τ be such as in Lemma 2.3. Set $\varphi = \pi' \perp \langle x\rangle$, $\psi = \tau' \perp \langle x\rangle$. It is clear that all the conditions of Lemma 3.5 hold. Now the desired result follows immediately from Lemma 3.5 and Lemma 3.8. $\square$

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