The Polytope of All Triangulations of a Point Configuration

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Abstract. We study the convex hull $P_{\mathcal{A}}$ of the 0-1 incidence vectors of all triangulations of a point configuration $\mathcal{A}$. This was called the universal polytope in [4]. The affine span of $P_{\mathcal{A}}$ is described in terms of the cocircuits of the oriented matroid of $\mathcal{A}$. Its intersection with the positive orthant is a quasi-integral polytope $Q_{\mathcal{A}}$ whose integral hull equals $P_{\mathcal{A}}$. We present the smallest example where $Q_{\mathcal{A}}$ and $P_{\mathcal{A}}$ differ. The duality theory for regular triangulations in [5] is extended to cover all triangulations. We discuss potential applications to enumeration and optimization problems regarding all triangulations.

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1 Introduction

We are interested in the set of all triangulations of a configuration $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbf{R}^d$. The subset of *regular* triangulations is well-understood thanks to its bijection with the vertices of the *secondary polytope* (see [7, Chapter 7] and [16, Lecture 9]). But non-regular triangulations remain a mystery: for instance, it is still unknown whether any two triangulations of $\mathcal{A}$ can be connected by a sequence of bistellar flips [12]. Non-regular triangulations are abundant: if $\mathcal{A}$ is the vertex set of the cyclic polytope $C_{4n-4}(4n)$, there are at least 2^n triangulations (Proposition 5.10 in this article), while the number of regular triangulations is $O(n^4)$.

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One approach in understanding non-regular triangulations is to replace the secondary polytope by a larger polytope $P_{\mathcal{A}}$ whose vertices are in bijection with all triangulations of $\mathcal{A}$. The polytope $P_{\mathcal{A}}$ is isomorphic to the *universal polytope* introduced by Billera, Filliman and Sturmfels [4]. They expressed the secondary polytope as a projection of $P_{\mathcal{A}}$ and they showed $dim(P_{\mathcal{A}}) = \binom{n-1}{d+1}$ when $\mathcal{A}$ is in general position. We shall now fix some notation, and define the polytope $P_{\mathcal{A}}$.

Throughout this paper $\mathcal{A} \subset \mathbf{R}^d$ will denote a d-dimensional configuration of n possibly repeated points. By a *k-simplex* we mean a sub-configuration of $\mathcal{A}$ consisting of $k+1$ affinely independent points. A *triangulation* of $\mathcal{A}$ is a collection T of d-simplices whose convex hulls cover $conv(\mathcal{A})$ and intersect properly: for any σ and τ in T we have $conv(\sigma \cap \tau) = conv(\sigma) \cap conv(\tau)$. Let $\Delta(\mathcal{A})$ denote the collection of d-simplices in $\mathcal{A}$. We define $P_{\mathcal{A}}$ as the convex hull in $\mathbf{R}^{\Delta(\mathcal{A})}$ of the set of incidence vectors of all triangulations of $\mathcal{A}$. For a triangulation T the *incidence vector* v_T has coordinates $(v_T)_\sigma = 1$ if $\sigma \in T$ and $(v_T)_\sigma = 0$ if $\sigma \notin T$. We also consider the polytope $Q_{\mathcal{A}} = \mathit{aff}(P_{\mathcal{A}}) \cap \mathbf{R}_+^{\Delta(\mathcal{A})}$, which is the linear programming relaxation of $P_{\mathcal{A}}$. We denote by $M(\mathcal{A})$ the oriented matroid of affine dependencies of the point configuration $\mathcal{A}$.

We first present linear equations defining the affine hull $\mathit{aff}(P_{\mathcal{A}})$ of $P_{\mathcal{A}}$. These equations involve the cocircuits (see [6, Chapter 1] or [16, Lecture 6]) of $M(\mathcal{A})$: for any τ which is a $(d-1)$-simplex of $\mathcal{A}$, let H_τ be the hyperplane that contains τ and let H_τ^+ and H_τ^- denote the two open half-spaces defined by H_τ. We recall that the cocircuits of the oriented matroid $M(\mathcal{A})$ are the resulting partitions $(\mathcal{A} \cap H_\tau^+, \mathcal{A} \cap H_\tau, \mathcal{A} \cap H_\tau^-)$ of $\mathcal{A}$. Consider the following linear form:

$$Co_\tau \quad := \sum_{\sigma=\tau\cup\{a\},\ a\in\mathcal{A}\cap H_\tau^+} x_\sigma \quad - \sum_{\sigma=\tau\cup\{a\},\ a\in\mathcal{A}\cap H_\tau^-} x_\sigma \tag{1}$$

We call Co_τ the *cocircuit form* associated with the $(d-1)$-simplex τ. If $conv(\tau) \cap int(conv(\mathcal{A})) \neq \emptyset$, we say that τ is an *interior* $(d-1)$-simplex. In this case neither of the two sums in (1) is void. Moreover, every triangulation T of $\mathcal{A}$ contains either no d-simplex containing τ or exactly two, one in the first sum and one in the second. Thus Co_τ vanishes at the incidence vector v_T of every triangulation of $\mathcal{A}$, and hence, on $\mathit{aff}(P_{\mathcal{A}})$. We call the equations $Co_\tau = 0$, for interior $(d-1)$-simplices τ, the *interior cocircuit equations.* We summarize our main results:

Theorem 1.1 *Let $\mathcal{A}$ be a point configuration with the above conventions.*

(i) *The affine span of $P_{\mathcal{A}}$ in $\mathbf{R}^{\Delta(\mathcal{A})}$ is defined by the linear equations $Co_\tau = 0$ for every interior $(d-1)$-simplex τ, together with one non-homogeneous linear equation valid on $P_{\mathcal{A}}$.*

(ii) *$P_{\mathcal{A}}$ coincides with the integral hull of $Q_{\mathcal{A}}$; i.e., the lattice points in $Q_{\mathcal{A}}$ are precisely the incidence vectors of triangulations of $\mathcal{A}$.*

(iii) *Two triangulations T_1 and T_2 of $\mathcal{A}$ are neighbors in the edge graph of $P_{\mathcal{A}}$ if and only if they are neighbors in the edge graph of $Q_{\mathcal{A}}$.*

(iv) *For the case of the n-gon and configurations with at most $d+3$ points, we have $Q_{\mathcal{A}} = P_{\mathcal{A}}$. This is not true in general for $n \geq d+4 \geq 6$.*

The following three types of non-homogeneous equations may be used to complete the description of *aff*$(P_{\mathcal{A}})$ in part (i) of Theorem 1.1: If τ is a non-interior $(d-1)$-simplex and $conv(\tau)$ is a facet of $conv(\mathcal{A})$, then the cocircuit form Co_τ has constant value equal to ± 1 on $P_{\mathcal{A}}$. This produces new valid equations for *aff*$(P_{\mathcal{A}})$ which we call *boundary cocircuit equations.* They can be expressed in the form

$$\sum_{\sigma=\tau\cup\{a\},\ a\in\mathcal{A}\setminus\tau} x_\sigma \quad = \quad 1. \tag{2}$$

Another set of valid equations for *aff*$(P_{\mathcal{A}})$ can be obtained as follows: let $p \in conv(\mathcal{A})$ be a point not lying in the convex hull of any $(d-1)$-simplex of $\mathcal{A}$. Every triangulation of $\mathcal{A}$ satisfies the equation:

$$\sum_{\sigma\in\Delta(\mathcal{A}),\ p\in conv(\sigma)} x_\sigma \ = \ 1. \tag{3}$$

Recall that the *chamber complex* of $\mathcal{A}$ is the common refinement of all triangulations of $\mathcal{A}$ (see [1],[5]). We call the equations of type (3) *chamber equations*, because the simplices in the sum only depend on the chamber in which p lies. Note that the boundary cocircuit equations (2) are a particular case of chamber equations.

Finally, if we denote by $vol(\cdot)$ the standard volume form on $\mathbf{R}^d$, the following *volume equation* is satisfied by every triangulation of $\mathcal{A}$:

$$\sum_{\sigma\in\Delta(\mathcal{A})} vol(conv(\sigma))\, x_\sigma \quad = \quad vol(conv(\mathcal{A})). \tag{4}$$

REMARK 1.2 The (interior and boundary) cocircuit equations depend only on the oriented matroid $M(\mathcal{A})$ of affine dependencies of $\mathcal{A}$. This holds neither for the volume equation nor for chamber equations: for example, all configurations consisting on the six vertices of a convex planar hexagon have the same oriented matroid, while the number of chambers can be 24 or 25, depending on the coordinates of the vertices.

Clearly, if $\mathcal{A}$ has no simplicial facets, then there are no boundary cocircuit equations. However, every configuration $\mathcal{A}$ has *some* chamber equations which can be obtained from the oriented matroid $M(\mathcal{A})$. (Such chambers arise from *lexicographic extensions*; see [6, Figure 7.2.2, page 296].) Any of them, together with the interior cocircuit equations, will provide a description of *aff*$(P_{\mathcal{A}})$ in terms of $M(\mathcal{A})$. Part (ii) of Theorem 1.1 implies that this yields a description of $P_{\mathcal{A}}$ itself in terms of $M(\mathcal{A})$.

In Section 2 we examine the affine span of $P_{\mathcal{A}}$ and we prove part (i) of Theorem 1.1. A surprising consequence (Corollary 2.3) is that *aff*$(P_{\mathcal{A}})$ is spanned by the *regular* triangulations only. This implies the formula $dim(P_{\mathcal{A}}) = \binom{n-1}{d+1}$ when $\mathcal{A}$ is in general position. Section 3 contains the proof of parts (ii) and (iii) in Theorem 1.1. As a consequence of part (iii) we obtain a combinatorial characterization of the edges of $P_{\mathcal{A}}$ (Theorem 3.3). Section 4 contains the proof of part (iv). We also discuss computational issues regarding the enumeration of triangulations and the optimization of linear cost functions over $P_{\mathcal{A}}$. In Section 5 we present a duality theory relating (non-regular) triangulations of $\mathcal{A}$ with (virtual) chambers in the Gale transform of $\mathcal{A}$.

2 Equations defining the affine span of $P_{\mathcal{A}}$

We introduce now some basic definitions and properties concerning regular triangulations. For a more detailed description and the relevant background the reader may consult [4],[5],[7, Chapter 7],[9] and [16].

A regular triangulation of $\mathcal{A}$ is a triangulation which is obtained by projecting the lower envelope of a $(d+1)$-dimensional simplicial polytope onto $conv(\mathcal{A})$. In other words, a triangulation is regular if it supports a piecewise convex linear functional.

In [7, Chapter 7] the collection of regular triangulations of a point configuration is identified with the vertex set of a polytope $\Sigma(\mathcal{A})$ of dimension $n-d-1$ embedded in $\mathbf{R}^n$. This polytope, called the *secondary polytope*, is a projection of $P_{\mathcal{A}}$ [4]. The projection map $\pi : \mathbf{R}^{\Delta(\mathcal{A})} \to \mathbf{R}^{\mathcal{A}}$ is given by $\pi(e_\sigma) = vol(\sigma)\sum_{a\in\sigma} e_a$, where e_σ and e_a denote the standard basis vectors.

A characterization of the edges of $\Sigma(\mathcal{A})$ is given in [7, Chapter 7]. This uses the notions of *circuits* and *bistellar flips*. We only define bistellar flips in the general position case: a *circuit* of the point configuration $\mathcal{A}$ is a minimal affinely dependent set. If $\mathcal{A}$ is in general position circuits are subsets of cardinality $d+2$. The unique (up to scaling factor) affine dependency equation satisfied by a circuit Z splits it into two subsets Z^+ and Z^- consisting of the points which have positive and negative coefficients respectively. Any circuit Z has exactly two triangulations $t(Z)^+ = \{Z \setminus \{a\}, \quad a \in Z^+\}$ and $t(Z)^- = \{Z \setminus \{a\}, \quad a \in Z^-\}$. If a triangulation T of $\mathcal{A}$ contains one of the two triangulations of a circuit Z (say $t(Z)^+$), then $T' = T \setminus t(Z)^+ \cup\ t(Z)^-$ is again a triangulation of $\mathcal{A}$. The operation that passes from T to T' (or vice versa) is called a *bistellar flip*. Two regular triangulations are neighbors in the 1-skeleton of the secondary polytope $\Sigma(\mathcal{A})$ if and only if they differ by a bistellar flip. This implies that any two regular triangulations can be transformed to one another by a finite sequence of bistellar flips. It is unknown whether this property is true for non-regular triangulations.

Our next goal is to prove part (i) of Theorem 1.1. We first state a lemma about the behavior of triangulations under the matroidal operations of *deletion* $\mathcal{A} \mapsto \mathcal{A}\backslash a_i$ and *contraction* $\mathcal{A} \mapsto \mathcal{A}/a_i$, where a_i is a vertex of $\mathcal{A}$. We regard $\mathcal{A}/a_i$ as a configuration of typically $n-1$ points (maybe less, if a_i was a repeated point) in affine $(d-1)$-space. The convex hull of $\mathcal{A}/a_i$ is the *vertex figure* of $conv(\mathcal{A})$ at a_i.

Lemma 2.1 *Let $a_1 \in \mathcal{A}$ be a vertex of $conv(\mathcal{A})$. Every triangulation of $\mathcal{A}\backslash a_1$ can be extended to a triangulation of $\mathcal{A}$. Every* REGULAR *triangulation of $\mathcal{A}/a_1$ can be extended to a triangulation of $\mathcal{A}$. The latter fails for non-regular triangulations.*

Proof: Every (regular or non-regular) triangulation T' of $\mathcal{A}\backslash a_1$ can be extended to a triangulation T of $\mathcal{A}$ by the *placing* operation described on page 444 in [9]. In this situation T is regular if and only if T' is regular.

The extension property of regular triangulations for contractions follows from the identification of the secondary fan of $\mathcal{A}$ with the chamber complex of $\mathcal{B}$, where $\mathcal{B}$ is a Gale transform of $\mathcal{A}$ (see [5], in particular Lemma 3.2 and the paragraph after Lemma 3.4). The reasoning is this: let T' be a regular triangulation of $\mathcal{A}/a_1$. It corresponds to a chamber $C_{T'}$ of $\mathcal{B}\backslash b_1$, where $b_1 \in \mathcal{B}$ is the point corresponding to a_1 in the Gale transform. The chamber $C_{T'}$ may split into smaller chambers when passing to the chamber complex of $\mathcal{B}$ and any such smaller chamber C_T corresponds to a triangulation T of $\mathcal{A}$ with $T/a_1 = T'$.

To see that regularity is necessary in the previous paragraph, let $\mathcal{A} \subset \mathbf{R}^3$ be the configuration $a_1 = (20,0,1)$, $a_2 = (1,20,0)$, $a_3 = (0,1,20)$, $a_4 = (10,0,0)$, $a_5 = (0,10,0)$, $a_6 = (0,0,10)$ and $a_7 = (-10,-10,-10)$. Let T' be the triangulation of $\mathcal{A}/a_7$ given by the triangles $\{1,2,5\}$, $\{1,3,4\}$, $\{1,4,5\}$, $\{2,3,6\}$, $\{2,5,6\}$, $\{3,4,6\}$ and $\{4,5,6\}$. There is no triangulation T of $\mathcal{A}$ such that $T/a_7 = T'$. The non-convex polyhedron with vertices $a_1, \ldots, a_6$ and triangular faces determined by the above list, together with the triangle $\{1,2,3\}$, is the *Schönhardt polyhedron* [13] which cannot be triangulated without adding a new point. □

Part (i) of Theorem 1.1 is a consequence of the following theorem and the existence of non-homogeneous forms vanishing on $P_{\mathcal{A}}$.

THEOREM 2.2 *Let $h = \sum_{\sigma\in\Delta(\mathcal{A})} c_\sigma x_\sigma$ ($c_\sigma \in \mathbf{R}$) be any homogeneous linear form on $\mathbf{R}^{\Delta(\mathcal{A})}$. The following properties are equivalent:*

(i) *h is a linear combination of the interior cocircuit forms Co_τ* (1).

(ii) *h vanishes on (the incidence vector of) every triangulation of $\mathcal{A}$.*

(iii) *h vanishes on (the incidence vector of) every regular triangulation of $\mathcal{A}$.*

Proof: (i)⇒(ii)⇒(iii) are obvious. We prove (iii)⇒(i).

Let $P_{\mathcal{A}}^{reg}$ denote the convex hull of all points v_T where T is a regular triangulation of $\mathcal{A}$. Thus $P_{\mathcal{A}}^{reg} \subset P_{\mathcal{A}}$. Let $h = \sum c_\sigma x_\sigma$ be any linear form which vanishes on $P_{\mathcal{A}}^{reg}$. We shall prove that h is a linear combination of the interior cocircuit forms using a double induction on $n = |\mathcal{A}|$ and $d = dim(\mathcal{A})$. Assume that the statement is true for any configuration of smaller cardinality or smaller dimension.

Let a_1 be a vertex of $conv(\mathcal{A})$. Let us suppose that $\mathcal{A}\backslash a_1$ still spans $\mathbf{R}^d$. Otherwise $P_{\mathcal{A}}^{reg}$ and $P_{\mathcal{A}\backslash a_1}^{reg}$ are affinely isomorphic and the theorem follows by induction. The interior cocircuit forms Co_τ vanish on $P_{\mathcal{A}}^{reg}$. If $a_1 \notin \tau$ then Co_τ involves at most one d-simplex of the form $\sigma = \{a_1\} \cup \tau$. Subtracting appropriate multiples of those Co_τ from h we get another linear form h_1 in which the variables x_σ corresponding to these simplices do not appear. That is,

$$h_1 \quad = \sum_{\substack{\sigma:a_1\in\sigma \\ conv(\sigma\backslash a_1)\subset\, boundary(conv(\mathcal{A}))}} c_\sigma x_\sigma \quad + \sum_{\sigma:a_1\notin\sigma} c'_\sigma x_\sigma .$$

The second sum $h_2 = \sum_{\sigma:a_1\notin\sigma} c'_\sigma x_\sigma$ is a linear form vanishing on $P_{\mathcal{A}\backslash a_1}^{reg}$. Indeed, let T' be any regular triangulation of $\mathcal{A}\backslash a_1$. Pick a regular triangulation T of $\mathcal{A}$ that extends T' as in Lemma 2.1. Since a_1 is a vertex of $conv(\mathcal{A})$, the triangulation T cannot contain a simplex σ of the form $\{a_1\} \cup \tau$ where $conv(\tau)$ is in the boundary of $conv(\mathcal{A})$. This fact together with $h_1(v_T) = 0$ implies $h_2(v_T) = 0$, and consequently $h_2(v_{T'}) = 0$.

Every cocircuit form Co_τ of $\mathcal{A}\backslash a_1$ is either a cocircuit form of $\mathcal{A}$ as well or can be extended to a cocircuit form of $\mathcal{A}$ by adding a single variable $x_{\{a_1\}\cup\tau}$ with the appropriate sign. By induction hypothesis, h_2 is a linear combination of the cocircuit forms of $\mathcal{A}\backslash a_1$. We extend this presentation to a linear combination of cocircuit forms of $\mathcal{A}$, which vanishes on $P_{\mathcal{A}}^{reg}$. We subtract it from h_1 to get a new form h_3 which vanishes on $P_{\mathcal{A}}^{reg}$ and involves only d-simplices σ of the form $a_1 \cup \tau$:

$$h_3 \quad = \sum_{\sigma : a_1 \in \sigma} c''_\sigma x_\sigma .$$

The assignment $\tau \mapsto \{a_1\} \cup \tau$ defines a bijection between the simplices of $\mathcal{A}/a_1$ and the simplices of $\mathcal{A}$ containing a_1. Therefore we can interpret h_3 as a linear form on $P_{\mathcal{A}/a_1}^{reg}$. Lemma 2.1 guarantees that h_3 vanishes on $P_{\mathcal{A}/a_1}^{reg}$. By the induction hypothesis, h_3 is a linear combination of cocircuit forms Co_τ of $\mathcal{A}/a_1$. We replace each variable x_τ in this linear combination by the corresponding variable $x_{\{a_1\}\cup\tau}$. This transforms cocircuit forms of $\mathcal{A}/a_1$ into cocircuit forms $Co_{\{a_1\}\cup\tau}$ of $\mathcal{A}$. Therefore h_3 is a linear combination of cocircuit forms of $\mathcal{A}$. This proves Theorem 2.2 and Theorem 1.1 (i). $\square$

Corollary 2.3 *The linear subspace of $\mathbf{R}^{\Delta(\mathcal{A})}$ parallel to aff($P_{\mathcal{A}}$) is spanned by all vectors $v_T - v_{T'}$ where T and T' are regular triangulations of $\mathcal{A}$ differing by a bistellar flip.*

Proof: It follows from Theorem 2.2 that the linear subspace in question is spanned by all vectors $v_T - v_{T'}$ where T and T' are regular triangulations. Since every pair of regular triangulations is connected by a sequence of bistellar flips, the corollary follows. $\square$

Theorem 2.2 implies $dim(P_{\mathcal{A}}) = |\Delta(\mathcal{A})| - R - 1$, where R is the rank of the interior cocircuit forms. If the points of $\mathcal{A}$ are in general position then $|\Delta(\mathcal{A})| = \binom{n}{d+1}$. In this case the vector $v_T - v_{T'}$, where T and T' are regular triangulations of $\mathcal{A}$ differing by a bistellar flip, equals $v_{t(Z)^+} - v_{t(Z)^-}$, where $t(Z)^+$ and $t(Z)^-$ are the two triangulations of the circuit Z on which the bistellar flip is supported.

Theorem 2.4 *Let $\mathcal{A} \subset \mathbf{R}^d$ be a configuration of n points in general position. Let $a_1 \in \mathcal{A}$ and R be the rank of the interior cocircuit forms.*

(i) *The (interior and boundary) cocircuit forms Co_τ for which $a_1 \notin \tau$ form a basis for the space of linear forms vanishing on the linear space parallel to $P_{\mathcal{A}}$. Thus, $R + 1 = \binom{n-1}{d}$.*

(ii) *The vectors $v_{t(Z)^+} - v_{t(Z)^-}$, for the circuits Z containing a_1, form a basis of the linear space parallel to $P_{\mathcal{A}}$. Thus, $dim(P_{\mathcal{A}}) = \binom{n-1}{d+1}$.*

Proof: The cocircuit forms Co_τ for the $(d-1)$-simplices τ not containing a_1 are linearly independent, because each simplex σ containing a_1 appears in exactly one of them. Thus, $R + 1 \geq \binom{n-1}{d}$. Likewise the vectors $v_{t(Z)^+} - v_{t(Z)^-}$ for the circuits Z containing a_1 are linearly independent, because each simplex σ not containing a_1 appears in exactly one of them. Thus $dim(P_{\mathcal{A}}) \geq \binom{n-1}{d+1}$. This together with the formula $dim(P_{\mathcal{A}}) = \binom{n}{d+1} - R - 1$ finishes the proof. $\square$

PROPOSITION 2.5 *If $\mathcal{A}$ is a configuration in general position, then aff($P_{\mathcal{A}}$) is defined by the chamber equations* (3).

Proposition 2.5 is generally false for configurations in special position, because a collection of d-simplices may uniquely cover all open chambers without being a triangulation. For example, the configuration consisting of the four vertices of a quadrilateral plus the intersection of its diagonals has 3 triangulations ($dim(P_{\mathcal{A}}) = 2$) while the chamber equations have 7 non-negative integer solutions whose convex hull is a 4-dimensional polytope. For the vertex set of the 3-cube we have calculated that $dim(P_{\mathcal{A}}) = 29$ but the chamber equations define an affine space of dimension 35. Proposition 2.5 is implied by Theorem 2.2 and the following lemma, which expresses interior cocircuit forms as differences of chambers.

LEMMA 2.6 *Let $\mathcal{A}$ be a configuration in general position in $\mathbf{R}^d$. Let C_1 and C_2 be two neighboring maximal chambers and τ the unique $(d-1)$-simplex containing their common facet. Then*

$$\sum_{\sigma : C_1 \subset conv(\sigma)} x_\sigma \quad - \sum_{\sigma : C_2 \subset conv(\sigma)} x_\sigma \quad = \quad Co_\tau .$$

Proof: Let H be the hyperplane defined by τ, with half-spaces $H^+ \supset C_1$ and $H^- \supset C_2$. If σ is any d-simplex which contains C_1, then either σ contains C_2 as well or $\sigma = a \cup \tau$ where $a \in H^+$ (similarly for C_2). □

If $\mathcal{A}$ is in special position then more than one $(d-1)$-simplex may contain the common facet of C_1 and C_2. Call Ω the collection of them. In this case, with similar arguments one can prove that the formula in Lemma 2.6 has to be corrected by substituting $\sum_{\tau \in \Omega} Co_\tau$ for Co_τ.

REMARK 2.7 Let M be the incidence matrix of the chambers and the d-simplices of $\mathcal{A}$. If $\mathcal{A}$ is in general position then Proposition 2.5 implies that

$$\mathit{aff}(P_{\mathcal{A}}) \quad = \quad \{x \in \mathbf{R}^{\Delta(\mathcal{A})} : M \cdot x = \mathbf{1}\}.$$

Row and column bases of M have been studied in [1] and [2] and a formula is given for *rank*(M) in [2]. The formulae in Theorem 2.4 are a special case of that formula.

3 THE RELATION BETWEEN $P_{\mathcal{A}}$ AND $Q_{\mathcal{A}}$

Part (i) of Theorem 1.1 implies that the linear programming relaxation $Q_{\mathcal{A}}$ of $P_{\mathcal{A}}$ is defined by the interior cocircuit equations $Co_\tau = 0$ plus an extra non-homogeneous equation satisfied on $P_{\mathcal{A}}$, and the inequalities $x_\sigma \geq 0$ for each simplex σ of $\mathcal{A}$. Clearly $P_{\mathcal{A}} \subset Q_{\mathcal{A}}$. We shall examine the relationship between these two polytopes.

We call *support* of a point $v \in \mathbf{R}^{\Delta(\mathcal{A})}$ (and denote it *supp*(v)) the collection of d-simplices σ for which $v_\sigma \neq 0$.

LEMMA 3.1 (i) *$Q_{\mathcal{A}}$ is a subpolytope of the unit cube $conv(x : x \in \{0,1\}^{\Delta(\mathcal{A})})$.*

(ii) *Every vertex of $P_{\mathcal{A}}$ is also a vertex of $Q_{\mathcal{A}}$.*

(iii) *If v is any point in $Q_{\mathcal{A}}$ then $supp(v)$ covers $conv(\mathcal{A})$, i.e.,*

$$\bigcup_{\sigma:\sigma\in supp(v)} conv(\sigma) \quad = \quad conv(\mathcal{A}).$$

(iv) *A vertex v of $Q_{\mathcal{A}}$ is a vertex of $P_{\mathcal{A}}$ if and only if its support contains a triangulation.*

Proof: The chamber equations (3) are valid for $Q_{\mathcal{A}}$, and they imply part (iii). Also, since every x_σ appears in at least one of them, the non-negativity constraints imply $x_\sigma \leq 1$ for all σ. This proves part (i) which, in turn, implies part (ii). The only-if-direction of (iv) is obvious. For the if-direction, suppose first that $supp(v)$ is the support of a triangulation T. Then the chamber equations imply that v is the incidence vector of T, hence is a vertex of $P_{\mathcal{A}}$. If $supp(v)$ strictly contains a triangulation T then it cannot be a vertex of $Q_{\mathcal{A}}$ because $v_\varepsilon := \frac{v-\varepsilon v_T}{1-\varepsilon}$ is still a point in $Q_{\mathcal{A}}$, for a sufficiently small positive ε. But then $v = (1-\varepsilon)v_\varepsilon + \varepsilon v_T$ is not a vertex of $Q_{\mathcal{A}}$. □

Theorem 3.2 *Every integral point of $Q_{\mathcal{A}}$ is the incidence vector of a triangulation of $\mathcal{A}$; i.e., $P_{\mathcal{A}}$ is the integral hull of $Q_{\mathcal{A}}$.*

Proof: Let v be an integral point of $Q_{\mathcal{A}}$. By Lemma 3.1 (iii) we only need to prove that any two simplices in $supp(v)$ intersect properly. Suppose this is not the case for two simplices σ_1 and σ_2 in $supp(v)$, i.e.:

$$conv(\sigma_1 \cap \sigma_2) \neq conv(\sigma_1) \cap conv(\sigma_2).$$

Take a point a in $(conv(\sigma_1) \cap conv(\sigma_2)) \backslash conv(\sigma_1 \cap \sigma_2)$. Then the minimal face (subset) F of σ_1 with $a \in conv(F)$ is not a face of σ_2. For each simplex σ of $supp(v)$ having F as a face, consider the convex polyhedral cone

$$c(\sigma) \;:=\; a + pos(conv(\sigma) - a) \;=\; \{\, \lambda p + (1-\lambda)a \;:\; p \in conv(\sigma),\, \lambda \geq 0 \,\}.$$

Note that the facets of $c(\sigma)$ are in 1-to-1 correspondence with the facets of σ which contain F. We claim that $conv(\mathcal{A})$ is contained in the union of such cones. Suppose a point b of $conv(\mathcal{A})$ lies outside their union. Then b "sees" a facet of some cone $c(\sigma)$, where $\sigma \in supp(v)$. Let τ be the corresponding facet of σ, which contains F. By the choice of τ, there is no d-simplex in $supp(v)$ having τ as a facet and lying in the half-space containing b. This violates the interior cocircuit equation $Co_\tau(v) = 0$, since $v \geq 0$. Therefore an open neighborhood of a in $conv(\mathcal{A})$ is covered by those simplices in $supp(v)$ which have F as a face. The interior of one of these simplices intersects the interior of $conv(\sigma_2)$. This violates the chamber equations for v. □

We next prove that the edges of $P_{\mathcal{A}}$ are also edges of $Q_{\mathcal{A}}$. A different proof for this theorem can be derived from more general results about $0-1$ polytopes due to Matsui and Tamura [10]. Here we present a self-contained proof in the context of triangulations.

Theorem 3.3 *Let T_1 and T_2 be two distinct triangulations of $\mathcal{A}$. The following statements are equivalent:*

(i) *v_{T_1} and v_{T_2} are not neighbors in $Q_{\mathcal{A}}$.*

(ii) *v_{T_1} and v_{T_2} are not neighbors in $P_{\mathcal{A}}$.*

(iii) *There exist two triangulations T_3 and T_4 of $\mathcal{A}$ (different from T_1 and T_2) such that $v_{T_1} + v_{T_2} = v_{T_3} + v_{T_4}$.*

(iv) *There exist partitions $T_1 = R_1 \cup L_1$ and $T_2 = R_2 \cup L_2$ such that $L_1 \cup R_2$ and $R_1 \cup L_2$ are two other triangulations of $\mathcal{A}$, different from T_1 and T_2.*

Proof: (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) are obvious. We only need to show (i)$\Rightarrow$(iv). We define a graph G whose nodes are the d-simplices of T_1. Two d-simplices of T_1 are adjacent in G if and only if they share a common facet and this facet is not a facet of any d-simplex of T_2. The graph G has the following property: If H is a connected component of G then the (topological) boundary of $\cup_{\sigma\in H} conv(\sigma)$ is the union of $(d-1)$-simplices which are faces of both T_1 and T_2.

Next we will construct L_1, L_2, R_1 and R_2. Let σ_0 be a simplex of T_1 which is not in T_2. Let L_1 be the collection of d-simplices in the same connected component of G as σ_0 and let $R_1 = T_1 \backslash L_1$. Moreover, let $|R| = \cup_{\sigma\in R_1} conv(\sigma)$ and $|L| = \cup_{\sigma\in L_1} conv(\sigma)$. Finally, let L_2 (resp. R_2) be the collection of d-simplices of T_2 whose convex hull intersect the interior of $|L|$ (resp. of $|R|$). By the property of G mentioned above, no simplex of T_2 can intersect the interiors of both $|L|$ and $|R|$. Thus T_2 is the disjoint union of R_2 and L_2. Also, by the same property, the simplices of $L_1 \cup R_2$ (same for $R_1 \cup L_2$) intersect properly. Moreover, they cover $conv(\mathcal{A})$, because their union covers $conv(\mathcal{A})$ twice. We conclude that the disjoint unions $L_1 \cup R_2$ and $L_2 \cup R_1$ are triangulations of $\mathcal{A}$. Clearly $L_1 \neq L_2$, because $\sigma_0 \in L_1 \backslash L_2$. Let us assume that $R_1 = R_2$ and prove that then v_{T_1} and v_{T_2} are neighbors in $Q_{\mathcal{A}}$. This will finish the proof of the theorem.

Let v be a point in the minimal face F of $Q_{\mathcal{A}}$ containing v_{T_1} and v_{T_2}. This face is defined setting all coordinates not appearing in v_{T_1} or v_{T_2} equal to zero. Thus, $supp(v) \subset T_1 \cup T_2$. The entry of v corresponding to any d-simplex in $R_1 = R_2$ equals 1, because of the chamber equations. On the other hand, for any two d-simplices σ_1 and σ_2 adjacent in G, the interior cocircuit equations imply $v_{\sigma_1} = v_{\sigma_2}$, since these two simplices are the only ones in $T_1 \cup T_2$ having $\tau = \sigma_1 \cap \sigma_2$ as a facet. Thus, the entries of v corresponding to simplices in L_1 have a constant value ε. With this, the chamber equations imply that the entries corresponding to simplices in L_2 have a constant value $1-\varepsilon$. Thus, $v = \varepsilon v_{T_1} + (1-\varepsilon)v_{T_2}$. This implies that F is a segment, i.e., that v_{T_1} and v_{T_2} are neighbors. □

The above theorem implies that any two integral vertices of $Q_{\mathcal{A}}$ (triangulations) are connected by a path of integral vertices. Note that it is still conceivable that we could have two triangulations of $\mathcal{A}$ which are not connected by bistellar flips.

4 EXAMPLES AND APPLICATIONS

Our linear programming relaxation $Q_{\mathcal{A}}$ is generally not a lattice polytope. Therefore $P_{\mathcal{A}}$ is strictly contained in $Q_{\mathcal{A}}$. In this section we will exhibit some cases where $P_{\mathcal{A}} = Q_{\mathcal{A}}$ and the smallest configuration for which $P_{\mathcal{A}} \neq Q_{\mathcal{A}}$.

Theorem 4.1 *Let $\mathcal{A} \subset \mathbf{R}^d$ be a configuration of n points. The equality $P_{\mathcal{A}} = Q_{\mathcal{A}}$ holds in the following cases:*

(i) *$d = 2$ and all points lie on the boundary of a convex polygon.*

(ii) *$d = 1$.*

(iii) *$n \leq d + 3$.*

Proof: (i) Let v be a vertex of $Q_{\mathcal{A}}$. Let S be a subset of $supp(v)$ where all triangles in S intersect properly and cover a convex sub-polygon of $conv(\mathcal{A})$. Suppose that S is maximal with these two properties. Let e be an edge of the sub-polygon covered by S. Then S is contained in one of the two half-planes defined by e. By maximality of S and the interior cocircuit equations, e must be a segment on the boundary of $conv(\mathcal{A})$. This proves that S covers $conv(\mathcal{A})$ and hence is a triangulation. Lemma 3.1 (iv) implies that v is a vertex of $P_{\mathcal{A}}$.
(ii) The proof is a minor variation of case (i).
(iii) Let $\mathcal{S} = \{T_1, \ldots, T_k\}$ be the collection of all triangulations of $\mathcal{A}$. By Corollary 5.9 below, every triangulation T_i of $\mathcal{A}$ contains a simplex σ_i which is not contained in any other triangulation. Therefore, setting the coordinate of σ_i equal to zero defines a facet of $P_{\mathcal{A}}$ that contains every triangulation but T_i. Thus, $P_{\mathcal{A}}$ is a $(k-1)$-simplex. The fact that all facets of $P_{\mathcal{A}}$ are defined by setting coordinates equal to zero implies that $P_{\mathcal{A}} = Q_{\mathcal{A}}$. □

Example 4.2 *A fractional vertex of $Q_{\mathcal{A}}$.*
For any $\mathcal{A}$ with $P_{\mathcal{A}} \neq Q_{\mathcal{A}}$ we must have $n \geq d+4 \geq 6$. A minimal example is provided by the vertices $1, \ldots, 5$ of a regular pentagon plus its center 0. This configuration is in general position and has 20 triangles and 16 triangulations. Consider the vector $v \in \mathbf{R}^{20}$ with coordinates $v_{\{123\}} = v_{\{234\}} = v_{\{345\}} = v_{\{145\}} = v_{\{125\}} = v_{\{013\}} = v_{\{024\}} = v_{\{035\}} = v_{\{014\}} = v_{\{025\}} = 1/2$ and all other coordinates zero. It satisfies the interior and boundary cocircuit equations. Therefore v lies in $Q_{\mathcal{A}}$. Since $supp(v)$ does not contain any triangulation, $P_{\mathcal{A}} \neq Q_{\mathcal{A}}$. This fractional point is the only vertex of $Q_{\mathcal{A}}$ which is not in $P_{\mathcal{A}}$.

Remark 4.3 The property $P_{\mathcal{A}} = Q_{\mathcal{A}}$ is neither sufficient nor necessary for a configuration $\mathcal{A}$ to have all triangulations regular. In Example 4.2 all triangulations are regular but $P_{\mathcal{A}} \neq Q_{\mathcal{A}}$. For the canonical example of the planar configuration which has non-regular triangulations (six points which form two triangles with parallel edges) we still have $P_{\mathcal{A}} = Q_{\mathcal{A}}$.

Let C_n be the vertex set of a planar n-gon. The following proposition gives an irredundant inequality presentation of the $\binom{n-1}{3}$-dimensional polytope $P_{C_n} = Q_{C_n}$.

Proposition 4.4 *For $n \geq 5$ the facets of P_{C_n} are defined by $x_\sigma = 0$ where σ is a triangle with at most one of its edges lying on the boundary of C_n.*

Proof: We call a triangle *external* if it has two edges on the boundary of C_n. We first show that, for any external triangle σ, $x_\sigma = 0$ does not define a facet of P_{C_n}. Without loss of generality we can assume $\sigma = \{1, 2, n\}$. Suppose that $x_\sigma = 0$ defines a facet of

P_{C_n}. Then $\binom{n-1}{3}$ affinely independent triangulations not involving σ together with a triangulation T that contains σ will affinely span $\mathit{aff}(P_{C_n})$. The triangulation T contains a triangle $\{i, 2, n\}$ for some $i \neq 1$. Since $n \geq 5$ there exists another triangulation S which contains $\{j, 2, n\}$ where $j \neq 1, i$. But then S cannot be expressed as an affine combination of the above set of triangulations since $\{j, 2, n\}$ cannot appear in any of them. This contradiction shows that an external triangle does not define a facet of Q_{C_n}.

For $n = 5$ a direct investigation shows that the five non-external triangles correspond to facets. For $n > 5$, suppose σ is a non-external triangle. Then there exists a vertex i such that $C_{n-1} := C_n \backslash i$ contains σ as a non-external triangle. Without loss of generality we can assume $i = 1$. Let τ be the external triangle $\{1, 2, n\}$. By induction, $x_\sigma = 0$ defines a facet of C_{n-1}. In other words, there are $\binom{n-2}{3}$ affinely independent triangulations of C_{n-1} which do not contain σ. This set of affinely independent triangulations can be extended to triangulations of C_n by adding τ. Now we need to produce $\binom{n-2}{2}$ additional affinely independent triangulations which do not contain σ: for each j and k such that $2 < j < k < n$, we can construct a triangulation which contains the triangles $\{1, 2, j\}, \{1, j, k\}$ and $\{1, k, n\}$ (and hence not τ), and which does not contain σ. Similarly, for each $2 < l < n$ we can construct triangulations which contain $\{1, 2, l\}$ and $\{1, l, n\}$. These additional $\binom{n-2}{2}$ triangulations together with the previous ones form a set of $\binom{n-1}{3}$ affinely independent points in P_{C_n}, none of them containing σ. □

Remark 4.5 The argument that $x_\sigma = 0$ does not define a facet of Q_{C_n} whenever σ is an external triangle can be generalized. The "external" simplices of a point configuration $\mathcal{A}$ in general position are the following: if any vertex a_i of $\mathit{conv}(\mathcal{A})$ is deleted, then the point configuration $\mathcal{A} \backslash a_i$ will again be in general position. Each facet of $\mathit{conv}(\mathcal{A} \backslash a_i)$ that is visible from a_i together with a_i will form a simplex σ which will not define a facet for $Q_\mathcal{A}$. The argument is identical to the one above.

We close this section with remarks about using $Q_\mathcal{A}$ to enumerate the triangulations of $\mathcal{A}$ and to solve optimization problems over $P_\mathcal{A}$. If $\mathcal{A}$ is in general position then, by Remark 2.7

$$Q_\mathcal{A} = \{x \in \mathbf{R}_+^{\Delta(\mathcal{A})} : M \cdot x = \mathbf{1}\}\ ,$$
$$P_\mathcal{A} = \mathit{conv}\{x \in \{0,1\}^{\Delta(\mathcal{A})} : M \cdot x = \mathbf{1}\}.$$

This means that $P_\mathcal{A}$ is a *set partitioning polytope.* Even if $\mathcal{A}$ is not in general position, by introducing extra variables for the interior $(d-1)$-simplices of $\mathcal{A}$, $P_\mathcal{A}$ can be realized as a set partitioning polytope. This has important implications for enumeration and optimization purposes. For example, Trubin [15] shows that if P is a set partitioning polytope and Q its linear relaxation then P is quasi-integral, i.e., every edge of P is also an edge of Q. Balas and Padberg [3] and Matsui and Tamura [10] have given a characterization of adjacency between vertices of P. This leads to an algorithm for optimizing linear functionals over P using Q. Starting from an integral vertex of Q the algorithm finds the optimal solution visiting only integral vertices of Q in a fashion similar to the simplex method in linear programming. The same adjacency characterization can be used to enumerate the vertices of P as well.

Unfortunately, no implementation of the Balas-Padberg procedure is known to us. Moreover enumerating the triangulations of $\mathcal{A}$ using the existing vertex enumeration packages has two major drawbacks. First of all, an inequality presentation of $P_{\mathcal{A}}$ is either hard to get or it might involve too many constraints. Secondly if one uses $Q_{\mathcal{A}}$ instead of $P_{\mathcal{A}}$ then there is an excessive number of fractional vertices which have to be enumerated too. Our experiments using several vertex enumeration packages allowed us only to compute small examples. However the situation is more promising for optimization problems over $P_{\mathcal{A}}$ thanks to the efficient implementations of *branch-and-bound* and *branch-and-cut* algorithms for integer programming. Finding triangulations using minimum or maximum number of simplices, or finding a minimum/maximum cost triangulation fall into the category of these optimization problems. In order to illustrate the sizes of the problems one can attempt to solve and the efficiency of using linear and integer programming techniques over $P_{\mathcal{A}}$ we give the following example. Triangulating the d-dimensional cube with minimal number of simplices is important for their use in algorithms for the computation of fixed points of continuous maps [14]. This is equivalent to minimizing the functional $\sum x_\sigma$ over $P_{\mathcal{A}}$ where $\mathcal{A}$ is the vertex set of the d-cube. In the case of the 4-cube ($d = 4, n = 16$) the system defining $Q_{\mathcal{A}}$ has 1257 equations and 3008 variables. Using CPLEX 3.0 on a SPARC10 workstation, it takes 150 seconds to verify that the 4-cube's minimal triangulation has 16 simplices. The size of the corresponding linear programs for the d-cube with $d \geq 5$ gets too big. In this case one should formulate a smaller linear program which exploits the symmetries of the d-cube. This was done successfully for $d = 5, 6$ in [8].

5 Duality

In this section we extend the duality theory in [5] to cover all triangulations. Two applications are included: a short proof of Carl Lee's result that every triangulation of a configuration of $d+3$ points is regular [9] and an exponential lower bound for the number of triangulations of the cyclic polytope $C_{4n-4}(4n)$.

Following [5], we now consider a configuration $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ of *vectors* spanning $\mathbf{R}^{d+1}$. Thus $|\Delta(\mathcal{A})| \leq \binom{n}{d+1}$ and equality holds if $\mathcal{A}$ is in general position. If there exists an affine hyperplane H ($0 \notin H$) which intersects $pos(a_i)$ for every $a_i \in \mathcal{A}$, then we can consider $\mathcal{A}$ as a d-dimensional point configuration in H (identified with $\mathbf{R}^d$). If this is the case $\mathcal{A}$ is said to be *acyclic*, and we are in the setting of the previous sections. For the non-acyclic case see Remark 5.2 below.

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a spanning subset of $\mathbf{R}^{n-d-1}$ which is a *Gale transform* of $\mathcal{A}$. This means that $\sum_{i=1}^n a_i \otimes b_i = 0$ in $\mathbf{R}^{d+1} \otimes \mathbf{R}^{n-d-1}$. In particular, $M(\mathcal{B})$ is the oriented matroid dual to $M(\mathcal{A})$.

Let $\mathcal{T}(\mathcal{A}) \subset \{0,1\}^{\Delta(\mathcal{A})}$ be the set of incidence vectors of all triangulations of $\mathcal{A}$ and $\mathcal{T}_{reg}(\mathcal{A})$ the subset corresponding to regular triangulations. Similarly let $\Gamma(\mathcal{A}) \subset \{0,1\}^{\Delta(\mathcal{A})}$ be the set of all chambers of $\mathcal{A}$. By the results in [5],

$$\mathcal{T}_{reg}(\mathcal{A}) = \Gamma(\mathcal{B}) \qquad \text{and} \qquad \mathcal{T}_{reg}(\mathcal{B}) = \Gamma(\mathcal{A}). \tag{5}$$

We identify each cocircuit form Co_τ of $\mathcal{A}$ with its coefficient vector in $\{0,+1,-1\}^{\Delta(\mathcal{A})}$. Let $Co(\mathcal{A})$ denote the collection of all *cocircuit vectors* Co_τ, where τ runs over all linearly independent d-subsets of $\mathcal{A}$, and let $Co_{int}(\mathcal{A})$ be the subset of interior cocircuit vectors. Recall that Co_τ is *interior* if and only if both $+1$ and -1 appear among the coordinates of Co_τ. Dually, let ρ be any spanning $(d+2)$-subset of $\mathcal{A}$. Then ρ

contains a unique signed circuit $Z = (Z_+, Z_-)$ of $\mathcal{A}$. We define the *circuit vector*

$$Ci_\rho \quad := \quad \sum_{i \in Z_-} e_{\rho \backslash i} \quad - \quad \sum_{j \in Z_+} e_{\rho \backslash j}.$$

(The $e_{\rho\backslash i}$ are standard basis vectors in $\mathbf{R}^{\Delta(\mathcal{A})}$.) We say that Ci_ρ is an *interior* circuit vector if $Z_+ \neq \emptyset$ and $Z_- \neq \emptyset$. Let $Ci(\mathcal{A})$ denote the set of all circuit vectors and $Ci_{int}(\mathcal{A})$ the subset of interior circuit vectors. $\mathcal{A}$ is acyclic if and only if $Ci(\mathcal{A}) = Ci_{int}(\mathcal{A})$. We fix the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^{\Delta(\mathcal{A})}$. Here is our first duality theorem.

Theorem 5.1 *Let $\mathcal{A} \subset \mathbf{R}^{d+1}$ and $\mathcal{B} \subset \mathbf{R}^{n-d-1}$ be Gale transforms of each other.*

(i) *Circuit and cocircuit vectors satisfy $Ci(\mathcal{A}) = Co(\mathcal{B})$ and $Ci(\mathcal{B}) = Co(\mathcal{A})$.*

(ii) *If $\mathcal{A}$ is in general position then the subspaces spanned by $Ci(\mathcal{A})$ and $Co(\mathcal{A})$ are orthogonal complements in $\mathbf{R}^{\Delta(\mathcal{A})}$.*

Proof: Recall the following two facts from oriented matroid duality: A $(d+2)$-subset of $\mathcal{A}$ is spanning if and only if the complementary $(n-d-2)$-subset of $\mathcal{B}$ is linearly independent. The signed circuits of $\mathcal{A}$ are the signed cocircuits of $\mathcal{B}$ and vice versa. These two facts imply assertion (i).
(ii) Let Ci_ρ be a circuit vector and let Co_τ be a cocircuit vector. If $supp(Ci_\rho) \cap supp(Co_\tau) = \emptyset$ then $\langle Co_\tau, Ci_\rho \rangle = 0$. Otherwise $supp(Ci_\rho) \cap supp(Co_\tau) = \{\sigma_1, \sigma_2\}$ where $\sigma_1 = \{\tau, i\} = \rho\backslash j$ and $\sigma_2 = \{\tau, j\} = \rho \backslash i$. If a_i and a_j are in different half-spaces defined by τ then $\{\tau, i\}$ and $\{\tau, j\}$ have the same sign in Ci_ρ since they appear in the same triangulation of ρ. If a_i and a_j are in the same half-space then $\{\tau, i\}$ and $\{\tau, j\}$ have opposite signs in Ci_ρ since they do not belong to the same triangulation of ρ. This shows

$$\langle Co_\tau, Ci_\rho \rangle \; = \; 0 \qquad \text{for all} \quad Co_\tau \in Co(\mathcal{A})\,, \;\; Ci_\rho \in Ci(\mathcal{A}). \tag{6}$$

If $\mathcal{A}$ is acyclic, by the proof of Theorem 2.4 we have $dim(Co(\mathcal{A})) = \binom{n-1}{d}$ and $dim(Ci(\mathcal{A})) = \binom{n-1}{d+1}$. We conclude that $Ci(\mathcal{A})$ and $Co(\mathcal{A})$ span orthogonal complements. If $\mathcal{A}$ is in general position but not acyclic, then $\mathcal{B}$ is acyclic and the result follows from (i). ◻

The orthogonality relation (6) need not hold for configurations $\mathcal{A}$ in special position. For example, let $\mathcal{A} = \{a_1, a_2, \ldots, a_6\}$ be the vertex set of the regular octahedron where a_5 and a_6 are not connected by an edge. If we choose $\tau = \{1,2,3\}$ and $\rho = \{1,2,3,4,5\}$, then $\langle Co_\tau, Ci_\rho \rangle \neq 0$.

Remark 5.2 The main results of Section 2 and 3 can be extended to vector configurations. The dimension formula given for point configurations in general position should be corrected to $dim(P_\mathcal{A}) = \binom{n-1}{d+1} - 1$ whenever $\mathcal{A}$ is a non-acyclic configuration in general position. This can be proved using duality.

Proposition 5.3 *For every vector configuration $\mathcal{A}$, the orthogonal complement of $Ci(\mathcal{A})$ is contained in $Co(\mathcal{A})$ and vice versa.*

Proposition 5.3 can be deduced from the following theorem.

Theorem 5.4 *For a vector* $X \in \{0,1\}^{\Delta(\mathcal{A})}$ *the following are equivalent:*

(a) $\forall\, T \in \mathcal{T}_{reg}(\mathcal{A}) \quad \langle X, T\rangle = 1$ *and* $\forall\, Ci_\rho \in Ci_{int}(\mathcal{A}) \quad \langle X, Ci_\rho\rangle = 0.$

(b) $\forall\, T \in \mathcal{T}(\mathcal{A}) \quad \langle X, T\rangle = 1$ *and* $\forall\, Ci_\rho \in Ci_{int}(\mathcal{A}) \quad \langle X, Ci_\rho\rangle = 0$

(c) X *is the incidence vector of a triangulation of* $\mathcal{B}$*, that is, the set of all* $(n-d-1)$*-subsets* σ *satisfying* $X_{\{1,\ldots,n\}\setminus\sigma} = 1$ *defines a triangulation of* $\mathcal{B}$*.*

Proof: The equivalence of (a) and (b) follows from $\mathit{aff}(\mathcal{T}_{reg}(\mathcal{A})) = \mathit{aff}(\mathcal{T}(\mathcal{A}))$ which is a consequence of Corollary 2.4. Using Theorem 5.1 (i) and equation (5) we see that (a) is equivalent to

$\forall\, C \in \Gamma(\mathcal{B}) \quad \langle X, C\rangle = 1$ and $\forall\, Co_\tau \in Co_{int}(\mathcal{B}) \quad \langle X, Co_\tau\rangle = 0.$

The 0-1-solutions X to this are the triangulations of $\mathcal{B}$ by Theorem 1.1. □

Corollary 5.5 *If* $\mathcal{A}$ *is in general position then for a vector* $X \in \{0,1\}^{\Delta(\mathcal{A})}$ *the following are equivalent:*

(a) $\langle X, T\rangle = 1$ *for all* $T \in \mathcal{T}_{reg}(\mathcal{A})$.

(b) $\langle X, T\rangle = 1$ *for all* $T \in \mathcal{T}(\mathcal{A})$.

(c) X *is the incidence vector of a triangulation of* $\mathcal{B}$.

The chambers of $\mathcal{A}$ constitute a (generally proper) subset of the vectors $X \subset \{0,1\}^{\Delta(\mathcal{A})}$ characterized by the three equivalent conditions in Theorem 5.4. We propose the following interpretation for the remaining solutions:

Definition 5.6 *The solutions* $X \in \{0,1\}^{\Delta(\mathcal{A})}$ *to the system* (a) *in Theorem 5.4, viewed as collections of* $(d+1)$*-subsets in* $\mathcal{A}$*, are called the* virtual chambers *of* $\mathcal{A}$*. Writing* $\Gamma_{virt}(\cdot)$ *for the set of virtual chambers, we have*

$$\mathcal{T}(\mathcal{A}) = \Gamma_{virt}(\mathcal{B}) \qquad \text{and} \qquad \mathcal{T}(\mathcal{B}) = \Gamma_{virt}(\mathcal{A}). \tag{7}$$

Remark 5.7 There are two kinds of virtual chambers in $\Gamma_{virt}(\mathcal{B}) \setminus \Gamma(\mathcal{B})$: the first kind of these can become real chambers in a different realization of $M(\mathcal{B})$. These correspond to the triangulations of $\mathcal{A}$ which are regular in some other realization of $M(\mathcal{A})$. The second kind, the "truly" virtual chambers, will never show up as real chambers and thus they correspond to non-regular triangulations which never become regular. An example of the second kind can be found in Proposition 9.6.4 in [6].

We now present two applications of our duality results.

Proposition 5.8 (Carl Lee, [9]) *If* $\mathcal{A}$ *is a vector configuration in* $\mathbf{R}^{d+1}$ *with* $|\mathcal{A}| \le d+3$ *then every triangulation of* $\mathcal{A}$ *is regular.*

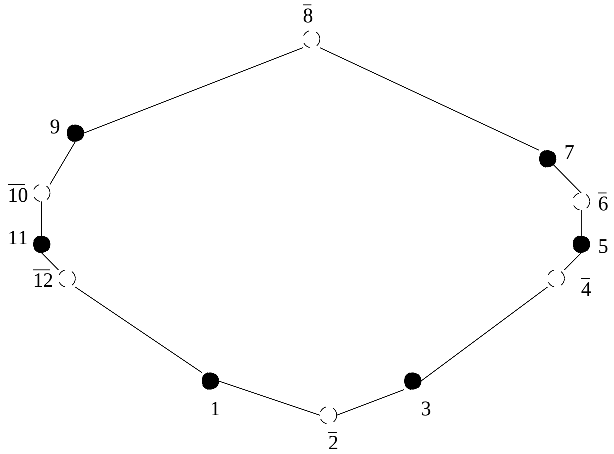

Figure 1: A Gale diagram of $C_{12}(8)$

Proof: If $|\mathcal{A}| = d+1$, there is a trivial triangulation which is regular. The case $|\mathcal{A}| = d+2$ is still very easy, since the linear transform $\mathcal{B}$ of $\mathcal{A}$ is one-dimensional. Let us assume that $|\mathcal{A}| = d+3$. Then $\mathcal{B}$ is two-dimensional and its simplices are pairs of independent vectors. Let C be a virtual chamber of $\mathcal{B}$ and let $\{b_1, b_2\}$ be a simplex in the support of C. We have the following:
(a) if $b' \in \mathcal{B}$ is such that $pos(b_1, b_2) \subset pos(b_1, b')$ then $\{b_1, b'\}$ is in the support of C.
(b) if $b' \in \mathcal{B}$ lies in $pos(b_1, b_2)$, then exactly one of $\{b_1, b'\}$ and $\{b', b_2\}$ is in the support of C.

Both properties follow from $\forall Ci_\rho \in Ci_{int}(\mathcal{B})$, $\langle C, Ci_\rho\rangle = 0$. Moreover, since $\forall T \in \mathcal{T}(\mathcal{B})$, $\langle C, T\rangle = 1$, a simplex $\{b_i, b_j\}$ where $pos(b_i, b_j) \not\subseteq pos(b_1, b_2)$ cannot be in C. From these we conclude that there is a unique minimal simplex lying in C. This implies that C is a real chamber. □

COROLLARY 5.9 *If $\mathcal{A}$ is a vector configuration in $\mathbf{R}^{d+1}$ with $|\mathcal{A}| \leq d+3$, then every triangulation of $\mathcal{A}$ contains a simplex which is not used in any other triangulation of $\mathcal{A}$.*

Proof: For $|\mathcal{A}| = d+1$ and for $|\mathcal{A}| = d+2$ the statement is again trivial. For the case $|\mathcal{A}| = d+3$, the minimal simplex of C in the proof of the Theorem 5.8 is such a simplex. □

The triangulations of all cyclic polytopes $C_d(n)$ are known to be connected by bistellar flips [11]. We close the paper with a result about the abundance of non-regular triangulations of $C_d(n)$.

PROPOSITION 5.10 *If $\mathcal{A}$ is the vertex set of $C_{4n-4}(4n)$, then $\mathcal{A}$ has $O(n^4)$ regular triangulations and has at least 2^n triangulations.*

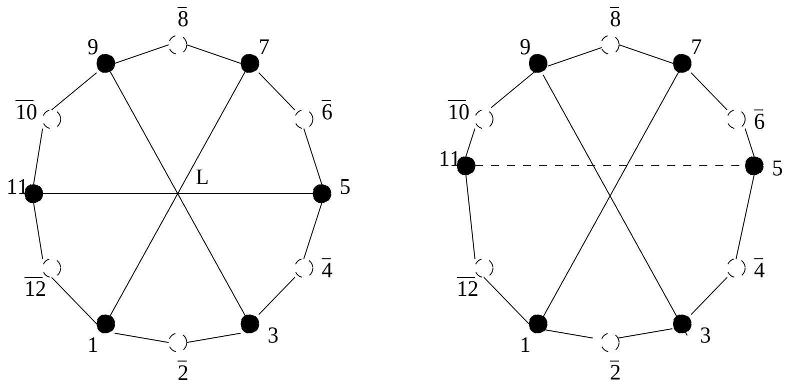

Figure 2: A perturbation that creates a chamber

Proof: The fact that $\mathcal{A}$ has $O(n^4)$ regular triangulations was established in [4], Theorem 5.7 (note that their "d" corresponds to the "$d+1$" in our notation). A Gale transform $\mathcal{B} = \{b_1, b_2, \ldots, b_{4n}\}$ of $\mathcal{A}$ can be depicted as a $(4n)$-gon under taking the antipodals of $\{b_2, b_4, \ldots, b_{4n}\}$ (see Figure 1 and also compare with Figure 1 in [5]). That configuration can be assumed to be a regular $4n$-gon, in particular, the sub-configuration $\{b_1, b_3, \ldots, b_{4n-1}\}$ is a regular $2n$-gon. Hence, the cones $pos(b_i, b_{i+2n}), 1 \leq i \leq 2n-1$ and i odd, intersect in a half-line L. Now by perturbing the vertices of this $2n$-gon, we can create 2^n different chambers which correspond to 2^n distinct triangulations of $\mathcal{A}$ (see Figure 2). These chambers are virtual chambers of $\mathcal{A}$, because the small perturbation does not change the oriented matroid. This shows that $C_{4n-4}(4n)$ has at least 2^n non-regular triangulations. All of them become regular for some point configuration combinatorially equivalent to $\mathcal{A}$. □

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