Étale Cohomology of Rigid Analytic Spaces

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Abstract. The paper serves as an introduction to étale cohomology of rigid analytic spaces. A number of basic results are proved, e.g. concerning cohomological dimension, base change, invariance for change of base fields, the homotopy axiom and comparison for étale cohomology of algebraic varieties. The methods are those of classical rigid analytic geometry and along the way a number of known results on rigid cohomology are re-established.

Key Phrases: "étale cohomology", "rigid analytic spaces", "rigid cohomology", "overconvergent sheaves"

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1 Introduction

The origin of this paper lies in the questions on étale cohomology for rigid analytic spaces posed in [S-S]. In that paper an étale site and a corresponding cohomology theory for analytic varieties are defined. We prove here that the axioms for an 'abstract cohomology' (as stated in [S-S]) hold for this cohomology theory. In addition, we prove a (quasi-compact) base change theorem for rigid étale cohomology and a comparison theorem comparing rigid and algebraic étale cohomology of algebraic varieties.

The main tools in this paper are analytic (resp. étale) points and rigid (resp. étale) overconvergent sheaves. The rigid overconvergent sheaves on affinoids were first introduced in [P82] and were called constructible in that paper. They were further studied in [S93] and were called conservative there. The term 'overconvergent', also used by P. Berthelot in recent work, seemed more appropriate this time.

In Section 2 we (re)introduce some basic notations concerning analytic points and rigid overconvergent sheaves, which are needed later on. We (re)prove a number of folklore results, most importantly: 1) Rigid cohomology agrees with Čech cohomology on quasi-compact spaces. 2) The cohomological dimension of a paracompact space

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is at most its dimension. 3) A base change theorem for rigid spaces which is more general than the results of [P82] or [S93].

The rest of the paper deals with étale sites and étale cohomology. Étale points and étale overconvergent sheaves are introduced. A key point is the introduction of special étale morphisms of affinoids $U \to X$, analogous to rational subdomains in the rigid case. Included in the paper is the proof by R. Huber that any étale morphism of affinoids is special étale. This simplifies the original exposition somewhat. A structure theorem for étale morphisms (3.1.2) allows us to give a proof of the étale base change theorem following closely the proof in the rigid case. We calculate the cohomology groups of one dimensional spaces in Section 4. This allows us to prove the basic results mentioned at the beginning of this introduction (Sections 5, 6 and 7).

We have tried to be complete in the proofs of various statements. We hope that this paper may serve as an introduction to rigid and étale cohomology of rigid analytic spaces.

Berkovich, in the paper [B93], develops an étale cohomology theory for analytic spaces. The category of analytic spaces used there was introduced in [B90] and extended in [B93]. It is different from the category of rigid analytic spaces. For this reason we have not borrowed from his work. However, we have to mention that the approach taken here, in some sense, does not differ from his (although in this paper we have to deal with non-overconvergent sheaves also, which do not correspond to sheaves on the Berkovich analytic spaces). For example, Lemma 2.1.1, which controls the étale stalk functors, is more or less equivalent to Theorems 2.1.5 & 2.3.3 of [B93]. Furthermore, using the equality of Berkovich cohomology with ours in the case of paracompact varieties (see [Hu, Section 8.3]), all our results on cohomology of overconvergent sheaves are in principle deducible from the references [B93, B94a, B94b, B94c].

Étale cohomology theories for rigid analytic spaces were developed by O. Gabber (unpublished) and K. Fujiwara, who proved Deligne's conjecture using his theory. As mentioned above R. Huber constructed an étale cohomology theory for his adic spaces, this specializes to give a theory for rigid analytic spaces also.

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1.1 Notations and conventions

- Unless stated otherwise k will be a complete non Archimedean valued field.
- As general reference for the basic facts and definitions concerning rigid analytic varieties we take [BGR].
- All rigid analytic varieties occurring in this work will be quasi-separated analytic varieties. This means that the diagonal morphism $X \to X \times X$ is quasi-compact, or equivalently that the intersection of any two affinoid subvarieties of X is a finite union of affinoid subvarieties of X. It is clear that fibre products of such are still quasi-separated.
- We work frequently with sites and associated topoi as in [SGA 4]. We recall that a morphism of sites $f : S_1 \to S_2$ is a continuous functor $u : S_2 \to S_1$ (remark that u goes in the opposite direction!), which induces a morphism of

associated topoi $S_1^\sim \to S_2^\sim$ (see [SGA 4, IV 4.9]). We remark that if S_2 allows finite projective limits then it suffices that u is continuous and preserves fibred products.

- A sheaf $\mathcal{F}$ on a site S is said to be flabby if for any object U in S we have $\mathrm{H}^q(U, \mathcal{F}) = 0$ for all $q > 0$. It is said to be flasque if for any morphism $U \to V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. A flasque sheaf is flabby since Čech cohomology may be used to determine whether a sheaf is flabby ([M80, III 2.12]).

2 Analytic points and rigid overconvergent sheaves

In this section we will review the base change theorem for rigid analytic spaces (see [P82, S93]). We will introduce our basic notations and reprove the statements of [P82] (whose proofs are perhaps somewhat sketchy). We try to avoid using results from [B90] except for the basic fact that the space $\mathcal{M}(X)$ (see below) is Hausdorff and compact (this is not hard to prove). Finally, we prove a slightly stronger version of the base change theorem, namely that it holds for arbitrary sheaves.

2.1 Sites, sheaves and analytic points on affinoids

Let X be an affinoid space over some complete non Archimedean valued field k. On X we consider the *special Grothendieck topology* given by the collection of finite unions of open affinoid subspaces and the admissible coverings. (See [FP, GP], this is a G-topology slightly stronger than the weak G-topology of X in [BGR, 9.1.4].) We will write X_{rigid} for the following site:

1. The objects are the admissible open subsets of X. We choose here as admissible opens the finite unions of open affinoid subsets. These will also be called the special subsets of X.

2. A morphism between to objects is an inclusion between the admissible subsets.

3. For an object U the elements of $\mathrm{Cov}(U)$ are those set-theoretical coverings of U by admissible opens which can be refined to finite coverings.

We use the special G-topology rather than the strong G-topology since it behaves better with respect to base change and change of base field. We remark that this gives the same category of sheaves.

It is sometimes easier to work with a subcategory X_{rigid}^{rat} of X_{rigid}. The objects of X_{rigid}^{rat} are the rational subsets of X. A rational subset of X is a set of the form

$$\{x \in X | \; |f_1(x)| \geq |f_i(x)| \text{ for all } i \text{ with } 1 \leq i \leq n\}$$

where $f_1, \ldots, f_n$ are elements of $O(X)$ generating the unit ideal. We note that a small change of the $f_1, \ldots, f_n$ does not affect the subset above. It is known that every open affinoid subset of X is a finite union of rational subsets ([GG]). A *rational covering* of a rational $U \subset X$ is a covering of the form $U = \cup_{i=1}^m U_i$ given by elements $f_1, \ldots f_m \in O(U)$ generating the unit ideal such that the U_i are the rational subsets (of U and also of X) $U_i := \{x \in U | \; |f_i(x)| \geq |f_j(x)| \text{ for all } j\}$. This defines for every object the collection of coverings. The morphism of sites $X_{rigid} \to X^{rat}_{rigid}$ (given by the inclusion functor $X^{rat}_{rigid} \to X_{rigid}$, see our conventions) defines an isomorphism of associated topoi, this follows from the fact that any special subset of X is a finite union of rational subsets and any finite affinoid covering of an affinoid variety can be refined to a rational covering (see for example [BGR, 8.2.2/2]).

It is well known that the set of ordinary points of X is too small to "separate" the sheaves on X_{rigid}. For this purpose one introduces new points, called *analytic points.* (See [P82, S93]). We will adopt here the terminology of [S93].

An *analytic point* a of X is a semi-norm $|\ |_a : O(X) \to \mathbf{R}_{\geq 0}$ on the affinoid algebra $O(X)$ of X satisfying:

1. $|f+g|_a \leq \max(|f|_a, |g|_a)$ for all $f, g \in O(X)$.
2. $|fg|_a = |f|_a |g|_a$ for all $f, g \in O(X)$.
3. For $\lambda \in k$ the value $|\lambda|_a$ is the absolute value of λ.
4. $|\ |_a : O(X) \to \mathbf{R}_{\geq 0}$ is continuous with respect to the norm topology on $O(X)$.

The *filter of the analytic point* a consist of the affinoid subdomains U of X for which there exists a rational covering given by $f_1, ..., f_n$ and an i such that $U \supset U_i$ and $|f_i|_a \geq |f_j|_a$ for all j. This is equivalent with the property that $|\,|_a$ extends to a $|\,|_a : O(U) \to \mathbf{R}_{\geq 0}$, i.e., that a is also an analytic point of U. We write $a \in U$ to denote that U belongs to the filter of a. We will also need the concept of a *wide neighborhood* of an analytic point a of X (see [S93, p. 131]). An element U of the filter of a is a wide neighborhood of a if there exists an affinoid generating system $f_1, \ldots, f_n$ of $O(U)$ over $O(X)$ such that $|f_i|_a < 1$ for all i.

Let $\mathcal{M}(X)$ denote the set of analytic points of X. We give $\mathcal{M}(X)$ the coarsest topology such that for every $g \in O(X)$ the map $\mathcal{M}(Z) \to \mathbf{R}$ given by $a \mapsto |g|_a$ is continuous. For an analytic point a a fundamental system of neighborhoods is given by the subsets $\mathcal{M}(U)$ where U runs through the (affinoid) wide neighborhoods of a. The space $\mathcal{M}(Z)$ is Hausdorff and compact for this topology. These results are not hard to prove, they follow from 1.2.2 and 1.3.3 of [P82], but see [B90, §1], [S93, §1] for more details. We will repeatedly make use of the following corollary of the above: *Suppose that $\{X_i\}_{i \in I}$ are affinoid subdomains of X such that for any analytic point a of X some X_i is a wide neighborhood of a, then the covering $X = \bigcup X_i$ is admissible, i.e., finitely many of the X_i cover X.*

The stalk of a sheaf S on X_{rigid} at an analytic point a is defined as $S_a = \lim_{\to} S(U)$ where the direct limit is taken over all U in the filter of a. The modified stalk of S at a is $S^{mod}_a = \lim_{\to} S(U)$ where the limit is over the wide open neighborhoods of a in X.

For every U in the filter of a the semi-norm $|\ |_a$ extends to a semi-norm on $O(U)$. Hence we get a semi-norm $|\ |_a$ on O_a the stalk of $O = O_X$ at a. A fundamental fact that we will use is (see [P82, 1.3.1]) that for $f \in O(X)$:

$$|f|_a = \inf\{||f||_U\}$$

where U runs through the filter of a. In fact it suffices to consider only wide open neighborhoods of a (use that for $U \subset X$ rational we have $||f||_U = \inf_{r>1} ||f||_{U(r)}$

where $U(r)$ is defined as in 2.3 below). It follows from these considerations that the ideal m_a of elements $f \in O_a$ satisfying $|f|_a = 0$ is the unique maximal ideal of O_a (and similar for O_a^{mod}). The field O_a/m_a will be denoted by k_a. The semi-norm $|\ |_a$ induces a valuation on k_a. This valuation extends the valuation of the subfield k of k_a. In general the field k_a is not complete and its completion is denoted by F_a. (The same constructions give k_a^{mod} and F_a^{mod}.)

Let $\phi : O(X) \to F_a$ denote the continuous homomorphism of k-algebras obtained above from $|\ |_a$. Then one sees that $|f|_a = |\phi(f)|$. This remark shows that our definition of analytic point coincides with the equivalence classes of analytic points as defined in [S93]. Every ordinary point of X is also an analytic point (with $F_a = k_a$ a finite extension of k). The following lemma will be useful in our study of the étale site of X.

Lemma 2.1.1 *Notations are as above.*

1. *O_a and O_a^{mod} are Henselian local rings.*
2. *k_a and k_a^{mod} are Henselian valued fields.*
3. *F_a is finite over a complete subfield K which has a dense subfield $k(t_1, ..., t_d)$ with $d \leq$ the dimension of X.*
4. *The homomorphism $O_a^{mod} \to O_a$ is local, flat and induces an isomorphism $F_a^{mod} \cong F_a$.*

Proof. Let $O_a \subset A$ be a finite free extension of rings. We claim the following: the ring $A\hat{\otimes}F_a$ has a nontrivial idempotent if and only if A has one. (We also claim a similar result for O_a^{mod}.)

This immediately implies (1) (see [R70, I Proposition 5]). Statement (2) means that the valuation ring of k_a (resp. k_a^{mod}) is an Henselian ring. Our claim implies that a finite separable ring extension $k_a \subset k'$ contains a copy of k_a if and only if the tensor product $k' \otimes F_a$ contains a copy of F_a (use a lift $O_a \to A$ of the finite extension $k_a \to k'$). This gives that any scheme étale over the valuation ring of k_a has a k_a-valued point if and only if it has a F_a-valued point. This assertion combined with the fact that the valuation ring of F_a is Henselian implies that k_a is a Henselian valued field (use the criterium of [R70, Proposition 3 page 76]).

To prove our claim, note that the ring extension $O_a \subset A$ comes from a finite free ring extension $O(U) \subset A_U$ for some U in the filter of a. Clearly, A_U is an affinoid algebra and hence determines a finite flat morphism $\phi : V = \mathrm{Spm}(A_U) \to U$. The fact that $A\hat{\otimes}F_a \cong A_U\hat{\otimes}F_a$ has a nontrivial idempotent is equivalent to the fact that $\phi^{-1}(a) = b_1, \ldots, b_s$ has at least two elements. Let us take disjoint wide neighbourhoods V_i of the b_i in V. There exists a smaller U' in the filter of a such that $\phi^{-1}(U')$ is contained in $\cup V_i$ (see Lemma 3.1.6 below; the reader may check that this lemma is not used before that lemma). Therefore the algebra $A_{U'} = A_U \otimes O(U')$ decomposes and hence so does A. The proof for O_a^{mod} is the same.

(3) After dividing $O(X)$ by a prime ideal we may suppose that $|\ |_a$ is a norm on $O(X)$. The field of quotients of $O(X)$ is a dense subfield of F_a. The algebra $O(X)$ is finite over some $A := k\langle T_1, \ldots, T_d\rangle$ with d equal to the dimension of X. Let $K \subset F_a$ denote the completion of the field of quotients of A with respect to $|\ |_a$. The field F_a is finite over K and K has $k(T_1, \ldots, T_d)$ as dense subfield with respect to $|\ |_a$.

(4) It is clear that the homomorphism $O_a^{mod} \to O_a$ is local and flat. Suppose that $\wp$ is the kernel of the seminorm $|\ |_a$ on $O(X)$. It is clear that the fraction field of $O(X)/\wp$ is dense in both F_a and F_a^{mod}. The result follows. □

REMARK 2.1.2 It follows from this lemma and its proof that there are equivalences between the following categories: the category of finite separable extensions of F_a, of finite separable extensions of k_a, of finite separable extensions of k_a^{mod}, of finite étale extensions of local rings $O_a \subset A$, and of finite étale extensions of local rings $O_a^{mod} \subset A$. Furthermore, any such extension comes from a finite étale (see paragraph 4) morphism $V \to U$ where U is a wide neighbourhood of a.

It is clear that the above constructions are functorial in the following sense. If $f : Y \to X$ is a morphism of affinoids over k, then we get a morphism of sites $Y_{rigid} \to X_{rigid}$ (resp. $Y_{rigid}^{rat} \to X_{rigid}^{rat}$). Indeed, if $U \subset X$ is an affinoid subdomain (resp. rational subset) then so is $f^{-1}(U) \subset Y$. Hence a functor $X_{rigid} \to Y_{rigid}, U \mapsto f^{-1}(U)$, it is easy to see that this is continuous and compatible with fibre products (i.e., intersections). The associated adjoint functors on sheaves are denoted f^*, f_* as usual.

The morphism f also induces a continuous map: $\mathcal{M}(Y) \to \mathcal{M}(X)$. The seminorm $O(Y) \to \mathbf{R}_{\geq 0}$ is mapped to the composition $O(X) \to O(Y) \to \mathbf{R}_{\geq 0}$. We remark that if f identifies Y with an affinoid subdomain of X then 1) $Y_{rigid} \cong X_{rigid}/Y$ and 2) the analytic points of Y are identified with those analytic points a of X such that Y is in the filter of a, i.e., $a \in Y$.

2.2 SITES, SHEAVES AND ANALYTIC POINTS FOR GENERAL X

To the analytic variety X we associate the site X_{rigid} by exactly the same definition as for affinoid X's. The objects are the finite unions of affinoid open subvarieties and the coverings are coverings which can be refined to finite coverings. (Since X is quasi-separated, the intersection of two affinoid open subvarieties is an object of the category X_{rigid}, so that X_{rigid} is indeed a site.) We remark that the the associated topos $X_{rigid}^{\sim}$ is again naturally isomorphic to the category of sheaves on X (as defined in [BGR, 9.2]). A morphism $f : Y \to X$ induces a morphism of topoi $f_{rigid} : Y_{rigid}^{\sim} \to X_{rigid}^{\sim}$ but not in general a morphism of sites $Y_{rigid} \to X_{rigid}$. Indeed, this morphism of sites exists if and only if f is quasi-compact.

The space X has some admissible covering $\{X_i\}$ by affinoids subsets. The analytic points of X are just the analytic points of the X_i, subject to the usual equivalence relation. (For a more precise definition see [S93, §2].) We remark that our $f : Y \to X$ induces a map on analytic points.

Finally, suppose $f : Y \to X$ is an open immersion (in the sense of [BGR, p. 354]). It is easy to prove (using the above) that: 1) f induces an injection between the sets of analytic points and 2) f induces an isomorphism $Y_{rigid}^{\sim} \to X_{rigid}^{\sim}/Y$ (where Y denotes the sheaf $V \mapsto \mathrm{Mor}_X(V, Y)$ on X_{rigid}). However, it is not true that any f satisfying 1) and 2) is an open immersion.

2.3 OVERCONVERGENT SHEAVES ON AFFINOIDS

Let X be an affinoid variety over k. The collection of analytic points of X is still not large enough to "separate" the Abelian sheaves on X_{rigid}. We can introduce a larger collection of points as in [P82] to remedy this fact. However, this larger collection of points seems not to be of much use for questions like base change theorems et cetera. We choose to work with a restricted collection of sheaves, namely the overconvergent sheaves on X_{rigid}.

Suppose that $V \subset U$ are special subsets of X. We will say that V is inner in U (w.r.t. X), or that U is a *wide neighborhood* of V in X, if for any analytic point a of V there is an affinoid wide neighborhood U_a of a in X with $U_a \subset U$. Notation: $V \subset\subset_X U$. It is proved in [S93, §1 Proposition 23] that this agrees with the notion V is relatively compact in U over X (see [BGR, 9.6.2]) if V and U are affinoid subdomains of X: $V \subset\subset_X U \Leftrightarrow$ there is an affinoid generating system $f_1, \ldots, f_r$ of $O(U)$ over $O(X)$ such that

$$V \subset \{x \in U; |f_1(x)| < 1, \ldots, |f_r(x)| < 1\}.$$

Suppose $V \subset X$ is rational in X given by the inequalities $|g_0| \geq |g_1|, ..., |g_m|$. For $r > 1$ and $r \in \sqrt{|k^*|}$ we define the rational set $V(r)$ by the inequalities $r|g_0| \geq |g_1|, ..., |g_m|$. It is easy to see that $V \subset\subset_X V(r)$. (The notation $V(r)$ will be used even if no explicit system $g_0, \ldots, g_m$ defining V and $V(r)$ is indicated.)

Lemma 2.3.1 *With notations as above.*

1. *The $V(r)$ form a co-final system of (special) wide neighborhoods in X of the rational set V.*

2. *If $V_1, \ldots, V_n$ are rational in X then*

$$V_1 \cap \ldots \cap V_m \subset\subset_X V_1(r) \cap \ldots \cap V_m(r)$$

($r > 1$ and $r \in \sqrt{|k^*|}$) and this forms a co-final system of wide neighborhoods of $V_1 \cap \ldots \cap V_m$. Similarly for $V_1 \cup \ldots \cup V_m \subset\subset_X V_1(r) \cup \ldots \cup V_m(r)$.

Proof. Suppose that $V \subset\subset_X U$ (with U a special subset of X). We claim the covering $X = U \cup (X \setminus V)$ is admissible. This is proved in [P92, Lemma 1.1], but let us indicate another proof: For any analytic point a of X, $a \notin V$ choose an affinoid wide neighborhood W_a of a with $W_a \cap V = \emptyset$ (just define W_a by suitable inequalities). For an analytic point $a \in V$ we choose the affinoid wide neighborhood W_a of a in X which is contained in U. Since $\mathcal{M}(X)$ is compact the covering $X = \bigcup W_a$ is admissible (see 3.1), hence so is $X = U \cup (X \setminus V)$. This proves our claim. In particular there is a special $W \subset X \setminus V$ such that $X = U \cup W$.

Next, put $W_i = \{w \in W;\ |g_i(x)| \geq |g_j(x)|\ j = 0, \ldots, n\}$ for $i = 1, \ldots, n$. Of course $W = \bigcup W_i$ since $W \cap V = \emptyset$. On W_i the function g_i is invertible hence we can put

$$\epsilon_i = ||g_0/g_i||_{W_i} \quad \text{and} \quad \epsilon = \max_i \epsilon_i.$$

By the maximum modulus principle on W_i and since $W \cap V = \emptyset$ we get $\epsilon_i < 1$ and $\epsilon < 1$. It is now clear that for any $r \in \sqrt{|k^*|}$, $\epsilon^{-1} > r > 1$ we have $V(r) \cap W = \emptyset$ and hence $V(r) \subset U$.

We prove 2) only in the case $m = 2$. Suppose that V_1 is given by the inequalities $|g_0| \geq |g_1|, \ldots, |g_n|$ and that V_2 is given by the inequalities $|f_0| \geq |f_1|, \ldots, |f_{n'}|$. The intersection $V_1(r) \cap V_2(r)$ is given by the inequalities $r^2|g_0 f_0| \geq |g_i f_j|$, $i = 0, \ldots, n$, $j = 0, \ldots, n'$. The result follows. The statement for unions is trivial from 1). □

At this point we are able to define the rigid overconvergent sheaves on our affinoid variety X. A (pre)sheaf S (on X_{rigid}) is called *(rigid) overconvergent* if for every admissible open $V \subset X$ we have

$$S(V) = \varinjlim_{V \subset\subset_X U} S(U).$$

It follows from the lemma above that if S is a sheaf then S is overconvergent if and only if $S(V) = \lim S(V(r))$ for any rational $V \subset X$. These sheaves were called the constructible sheaves in [P82]; they agree with the conservative sheaves of [S93] by [S93, §1 Lemma 25]. In [S93, §1] it is shown that these overconvergent sheaves correspond to sheaves on the topological space $\mathcal{M}(X)$.

LEMMA 2.3.2 *(Properties of overconvergent sheaves.) In this lemma all (pre)sheaves are (pre)sheaves of Abelian groups on the affinoid variety X.*

1. *The sheaf associated to a overconvergent presheaf is overconvergent.*
2. *For any overconvergent sheaf S the presheaves $U \mapsto H^i(U, S)$ are overconvergent.*
3. *The category of overconvergent sheaves is an exact subcategory of the category of all sheaves.*
4. *If $f : Y \to X$ is a morphism of affinoids then f^* and f_* preserve overconvergent sheaves. The same holds for $R^q f_*$.*
5. *If $X = \bigcup X_i$ is written as the finite union of affinoid subdomains then a sheaf S on X is overconvergent if and only if the restriction of S to any of the X_i is overconvergent.*
6. *A overconvergent sheaf S is zero if and only if all of its stalks S_a at analytic points of X are zero.*

Proof. Let S be a overconvergent presheaf. Suppose $V \subset X$ is the union of rational subsets $V_1, \ldots, V_m$ of X. Denote by $\mathcal{V} = \{V_i\}$ the covering of V and by $\mathcal{V}(r) = \{V_i(r)\}$ the covering of $V(r) := \bigcup_i V_i(r)$. It is immediate from Lemma 2.3.1 that

$$\mathcal{C}^{\cdot}(\mathcal{V}, S) = \varinjlim{}_{r>1} \mathcal{C}^{\cdot}(\mathcal{V}(r), S).$$

(These symbols denote Čech complexes.) It is therefore clear that the map

$$\varinjlim_{V \subset\subset_X U} \check{H}^p(U, S) \longrightarrow \check{H}^p(V, S)$$

is surjective.

Let us prove that it is also injective. Take a special $U \subset X$ with $V \subset\subset_X U$, an admissible covering $\mathcal{U} = \{U_i\}$ of U, a co-cycle $\xi \in \mathcal{C}^p(\mathcal{U}, S)$ whose Čech cohomology class maps to zero in $\check{H}^p(V, S)$. This means there is a covering $\mathcal{V} = \{V_j\}$ of V which refines $\mathcal{U} \cap V$, i.e., there is a function α such that $V_j \subset U_{\alpha(j)} \cap V$, and a chain $\eta \in \mathcal{C}^{p-1}(\mathcal{V}, S)$ with $\alpha(\xi) - \mathrm{d}\eta = 0 \in \mathcal{C}^p(\mathcal{V}, S)$. Here $\alpha(\xi)$ is the image of ξ under the map $\mathcal{C}^p(\mathcal{U}, S) \to \mathcal{C}^p(\mathcal{V}, S)$ determined by α. By refining $\mathcal{U}$ and $\mathcal{V}$ we may assume that $\mathcal{U}$ and $\mathcal{V}$ are finite and that all U_i and V_j are rational subdomains of X.

By the above, the co-cycle ξ lifts to a co-cycle $\xi' \in \mathcal{C}^p(\mathcal{U}(r), S)$ for some $r > 1$. Lemma 2.3.1 implies that there exists an $r^* > 1$ such that $V_j(r^*) \subset U_{\alpha(j)}(r)\forall j$. For an even smaller r^*, we may also assume η lifts to a chain $\eta' \in \mathcal{C}^{p-1}(\mathcal{V}(r^*), S)$. The co-cycle $\alpha(\xi') - \mathrm{d}\eta' \in \mathcal{C}^p(\mathcal{V}(r^*), S)$ maps to zero as a chain in $\mathcal{C}^p(\mathcal{V}, S)$, thus it is already zero in some $\mathcal{C}^p(\mathcal{V}(r^{**}), S)$, $r^* > r^{**} > 1$. We conclude that the cohomology class of ξ in $\check{H}^p(V(r^{**}), S)$ is zero, which was what we wanted to show.

The isomorphism of Čech cohomologies above proves that the presheaf $\check{\mathcal{H}}^0(S)$ is overconvergent if S is overconvergent. Hence also the sheaf associated to S is overconvergent. It proves (2) since Čech cohomology agrees with usual cohomology for any special $U \subset X$. (See [P82, 1.4.4] or our Proposition 2.5.4.)

The third statement of our lemma means that the kernels and co-kernels of overconvergent sheaves are overconvergent and that if a short exact sequence of sheaves $0 \to S_1 \to S_2 \to S_3 \to 0$ is given, S_1 and S_3 are overconvergent then so is S_2. These statements follow easily from (1) and (2).

(4) If $V \subset X$ is a rational subset, then $f^{-1}(V)$ is a rational subdomain of Y and we have: $f^{-1}\big(V(r)\big) = \big(f^{-1}(V)\big)(r)$. Thus it is clear from Lemma 2.3.1 that for special $V \subset\subset_X U$ in X we have $f^{-1}(V) \subset\subset_Y f^{-1}(U)$ and that these $f^{-1}(U)$ form a co-final system of wide neighborhoods of $f^{-1}(V)$.

Take an overconvergent sheaf S on Y. The sheaf $R^q f_* S$ is the sheaf associated to the presheaf $U \mapsto H^q(f^{-1}(U), S)$. It is immediate from the remarks above and (2) that this presheaf is overconvergent.

If S is a sheaf on X then f^*S is the sheaf associated to the presheaf P defined as follows on $V \in X_{rigid}$:

$$P(V) = \varinjlim_{U \in X_{rigid}, f^{-1}(U) \supset V} S(U)$$

Suppose S is overconvergent. If $t \in P(V)$, i.e., t comes from $s \in S(U)$ for some $U \subset X$ as in the limit, then s comes from $s' \in S(U')$ for some $U' \in X_{rigid}$ with $U \subset\subset_X U'$. By the above we see that $V \subset\subset_Y f^{-1}(U')$. We conclude that the map

$$\varinjlim_{V \subset\subset_Y V'} P(V') \to P(V)$$

is surjective. Let us prove that it is injective: Suppose $t' \in P(V')$ comes from some $s' \in S(U')$ with $f^{-1}(U') \supset V'$ and maps to zero in some $S(U)$ with $U \subset U'$ and $f^{-1}(U) \supset V$. There exists a wide neighborhood U'' of U' and $s'' \in S(U'')$ mapping to s'. Since S is overconvergent there is a special U''' with $U''' \subset U''$, $U''' \supset\supset_X U$ such that s'' maps to zero in U'''. It is clear that $V'' := V' \cap f^{-1}(U''')$ is a wide neighborhood of V in Y such that t' maps to zero in $P(V'')$. We have proved that P, hence f^*S, is overconvergent.

(5) This follows from (3) and (4) since any sheaf S on X fits into an exact sequence

$$0 \longrightarrow S \longrightarrow \bigoplus_i S|_{X_i} \longrightarrow \bigoplus_{i,j} S|_{X_i \cap X_j}.$$

Here $S|_{X_i} := j_* j^* S$ where $j : X_i \to X$ is the inclusion.

(6) Take a section $s \in \Gamma(X, S)$. By assumption any analytic point a in X has an affinoid wide neighborhood $V_a \subset X$ such that $s|_{V_a} = 0$. By compactness of $\mathcal{M}(X)$ we get that the covering $X = \bigcup V_a$ is admissible, hence $s = 0$. The same proof gives that $\Gamma(V, S) = 0$ for arbitrary special $V \subset X$. $\square$

2.4 Overconvergent sheaves on general X

Let X be an arbitrary analytic variety over k. We will say that a sheaf S on X is *overconvergent* if for any affinoid open subvariety $V \subset X$, the restriction $S|_V$ is overconvergent on V. Suppose $X = \bigcup X_i$ is an admissible affinoid covering. It follows from Lemma 2.3.2 that S is overconvergent if and only if $S|_{X_i}$ is overconvergent for all i.

Suppose $f : Y \to X$ is a morphism of rigid varieties. It is clear from Lemma 2.3.2 that f^* preserves overconvergent sheaves. This is not true in general for f_* or $R^q f_*$. But it is true if f is quasi-compact.

Proposition 2.4.1 *If $f : Y \to X$ is a quasi-compact morphism then f_* and $R^q f_*$ preserve overconvergent presheaves.*

Proof. Take an overconvergent sheaf S on Y. The question is local on X, hence we may assume X affinoid. Thus Y is quasi-compact and hence by Lemma 2.5.3 we can find a finite admissible affinoid covering $Y = \bigcup Y_i$ such that all intersections $Y_{i_0 \ldots i_q} := Y_{i_0} \cap \ldots \cap Y_{i_q}$ are affinoid. At this point we use the spectral sequence (deduced from the Cartan-Leray spectral sequence [SGA 4, V 3.3]) $\{E_n^{pq}\}$ abutting to $R^n f_* S$ and with E_2-term:

$$E_2^{pq} = \bigoplus_{i_0 \ldots i_q} R^p \big(f|_{Y_{i_0 \ldots i_q}}\big)_* S|_{Y_{i_0 \ldots i_q}}$$

By Lemma 2.3.2 all its terms are overconvergent sheaves. Hence by the same lemma we see that $R^n f_* S$ is overconvergent too. $\square$

2.5 Cohomology and Čech cohomology

In this subsection we prove that cohomology agrees with Čech cohomology on quasi-compact varieties. Further we prove that the cohomological dimension of such an analytic variety is at most its dimension.

Lemma 2.5.1 *Let X be an affinoid variety, $V \subset X$ special and a an analytic point of X. There exists a wide neighborhood $W = W_a$ of a such that $W \cap V$ is a finite union of Weierstrass domains, each defined by invertible functions.*

Proof. Since V is a finite union of rational subsets of X we may assume that V is rational itself. Say it is defined by the inequalities $|g_0| \geq |g_1|, \ldots, |g_n|$, where the g_i generate the unit ideal of $O(X)$. If $a \notin V$, then we can find a wide neighborhood W of a disjoint with V. If $a \in V$ then $|g_0|_a \geq |g_i|_a$ and since the g_i generate the unit ideal we get $|g_0|_a > 0$. Thus we may replace X by a wide neighborhood of a, so that g_0 becomes invertible. In this situation V is defined by $1 \geq |f_i|$ with $f_i = g_i / g_0$, i.e., V is a Weierstrass domain in X. For those i such that $\epsilon_i := |f_i|_a < 1$, we may replace

X by the wide neighborhood of a defined by $|f_i| \leq 1/2(1+\epsilon_1)$ and drop f_i. At this point $V \subset X$ is defined as $1 \geq |f_i|$ with $|f_i|_a = 1$ for all i. Hence the subset $|f_i| \geq |\pi|$, $\pi \in k, 0 < |\pi| < 1$ defines a wide neighborhood of a such that f_i is invertible on it. □

Lemma 2.5.2 *Suppose X is affinoid, $V \subset X$ special. There exists a finite covering $X = \bigcup X_i$ by affinoids of X such that $X_i \cap V$ is affinoid for all i.*

Proof. By compactness of $\mathcal{M}(X)$ and the lemma above we may assume $V \subset X$ is a finite union of Weierstrass domains, each given by invertible functions. Say $V = \bigcup_{i=1}^n V_i$ and V_i is defined by $1 \geq |f_1^i|, \ldots, |f_{n_i}^i|$ and each f_j^i invertible.

Consider combinatorial data of the form $A = (i, (j_1, \ldots, \widehat{j_i}, \ldots, j_n))$ where $i \in \{1, \ldots, n\}$ and $j_l \in \{1, \ldots, n_l\}$ for each $l \neq i, l \in \{1, \ldots, n\}$. We put

$$V_A = \left\{ x \in X ; |f_j^i(x)| \leq |f_{j_l}^l(x)|,\, l = 1, \ldots, \hat{\imath}, \ldots, n,\, j = 1, \ldots, n_i \right\}$$

Remark that $X = \bigcup_A V_A$ since for any $x \in X$ there is some $i \in \{1, \ldots, n\}$ such that $\max_j |f_j^i(x)| \leq \max_j |f_j^l(x)|$ for all $l \neq i$. On the other hand, if $A = (i, (j_1, \ldots, \widehat{j_i}, \ldots, j_n))$ as above then

$$V_A \cap V \subset V_i,$$

and hence $V_A \cap V = V_A \cap V_i$ is affinoid. This is immediate from the definitions. □

We remark that in proving the lemmata above we proved something slightly stronger: Suppose we had started with an admissible affinoid covering $V = \bigcup V_i$. This we can refine to a finite covering $V = \bigcup V_i$ with $V_i \subset X$ rational. The proof of Lemma 2.5.1 shows that we can cover X by finitely many affinoids X_j such that each $X_j \cap V_i$ is a Weierstrass domain in X_j defined by invertible functions. The proof of Lemma 2.5.2 shows that we can cover each X_j by finitely many $X_{j,A}$'s such that $X_{j,A} \cap (V \cap X_j) = X_{j,A} \cap V$ is contained in some V_i. Thus we have proved the first statement of the following lemma in the case that X is affinoid.

Lemma 2.5.3 *Let X be a quasi-compact variety over k.*

1. *Given an admissible covering $\mathcal{V} : V = \bigcup V_i$ of the special subset V of X, there exists a finite affinoid covering $\mathcal{U} : X = \bigcup X_j$ such that the covering $\mathcal{U} \cap V$ refines $\mathcal{V}$. In addition we may assume $X_j \cap V_i$ affinoid for all j.*

2. *There exists a finite affinoid covering $X = \bigcup X_j$ such that $X_i \cap X_j$ is affinoid for all i, j.*

Proof. (1) This assertion follows immediately from the case X affinoid (proved above) by writing X as the finite admissible union of affinoids (use that X is quasi-separated by our conventions).

(2) Take first an arbitrary finite affinoid covering $X = \bigcup X_i$. By (1) we can find finite affinoid coverings $\mathcal{U}_{ij} : X_i = \bigcup_k X_{ijk}$ such that $X_{ijk} \cap (X_i \cap X_j)$ is affinoid for all k. Next we take a finite affinoid covering $\mathcal{U}_i : X_i = \bigcup_l X_{il}$ refining $\mathcal{U}_{ij}$ for all j. It is clear that $X_{il} \cap X_{jm} = X_{il} \cap (X_i \cap X_j) \cap X_{jm}$ is affinoid (all intersections are taken in X). Thus the covering $X = \bigcup X_{il}$ works. □

PROPOSITION 2.5.4 *Suppose X is a quasi-compact (and quasi-separated) variety. Čech cohomology agrees with cohomology on X.*

Proof. The Leray spectral sequence relating Čech cohomology with cohomology [SGA 4, V 3.4] shows that it suffices to prove: $\check{H}^p(X,S) = 0$ if S is a presheaf whose associated sheaf is zero. Suppose $\mathcal{V}$ is some finite admissible covering of X and $\xi = \prod \xi_{i_o \ldots i_p} \in \mathcal{C}^p(\mathcal{V}, S) = \prod_{i_o \ldots i_p} S(V_{i_o \ldots i_p})$. We can find a covering $\mathcal{V}_{i_o \ldots i_p}$ of $V_{i_o \ldots i_p}$ such that $\xi_{i_o \ldots i_p}$ restricts to zero on each member of $\mathcal{V}_{i_o \ldots i_p}$. By Lemma 2.5.3 we can find a covering $\mathcal{U}_i : V_i = \bigcup U_{ij}$ of V_i such that $\mathcal{U}_i \cap V_{i_o \ldots i_p}$ (some $i_l = i$) refines $\mathcal{V}_{i_o \ldots i_p}$ for all choices of the i_l. Put $\mathcal{U} = \bigcup \mathcal{U}_i$, it is an admissible covering of X and the map

$$\alpha : \mathcal{C}^*(\mathcal{V}, S) \longrightarrow \mathcal{C}^*(\mathcal{U}, S)$$

is defined using $U_{ij} \subset V_i$. It is clear that the chain ξ maps to zero under α. □

REMARK 2.5.5 By Lemma 2.5.3 this is a special case of [P82, 1.4.4]. The argument in the proof of [P82, 1.4.5] together with Lemma 2.5.3 shows that Čech cohomology agrees with cohomology on any (quasi-separated, see conventions) X which is of countable type (see Definition 2.5.6 below).

We introduce some convenient topological notions for the Grothendieck topology on our analytic varieties X.

DEFINITION 2.5.6 *Let X be an analytic variety over k.*

1. *We say that X is of countable type if there exists a countable admissible affinoid covering of X.*

2. *Suppose that $X = \bigcup X_i$ is an admissible affinoid covering of X. We say that the covering is locally finite if each X_i meets finitely many X_j.*

3. *The variety X will be called paracompact if there exists an admissible locally finite affinoid covering.*

LEMMA 2.5.7 *A paracompact space X is the admissible disjoint union of paracompact varieties of countable type. A connected paracompact variety X can be written as the admissible union $X = \bigcup_{n \in \mathbb{N}} X_n$, with X_n quasi-compact and $X_i \cap X_j = \emptyset$ when $|i - j| \geq 2$.*

Proof. Since any rigid analytic space is the admissible disjoint union of its connected components, it suffices to prove the second statement. Therefore we assume that X is connected and has a locally finite affinoid admissible covering $X = \bigcup X_\alpha$. Let us choose a fixed index α_0. For any α we define the distance $d(\alpha)$ of α to α_0 to be the minimal length d of a sequence of indices $\alpha_0, \alpha_1, \ldots, \alpha_d = \alpha$ such that $X_{\alpha_i} \cap X_{\alpha_{i+1}} \neq \emptyset$ for all $i = 0, \ldots, d-1$. Since X is connected all distances are finite. We put $X_n = \bigcup_{d(\alpha)=n} X_\alpha$. Since the covering was locally finite the spaces X_n are quasi-compact. The last condition of the lemma follows immediately from our definition of distance. □

In the proof of the next proposition we need the relation of rigid analytic geometry with formal geometry (see [R70] and [BL]). We recall that if $\mathfrak{X}$ is a formal scheme of finite type and flat over $\mathrm{Spf}(k^\circ)$ then there is canonically associated a quasi-compact rigid analytic variety $X = \mathfrak{X}^{rig}$. If $\mathfrak{U} \subset \mathfrak{X}$ is a formal open subscheme then $\mathfrak{U}^{rig} \subset \mathfrak{X}^{rig}$ is an open subvariety. If $\mathfrak{X} = \bigcup \mathfrak{U}_i$ then $\mathfrak{X}^{rig} = \bigcup \mathfrak{U}_i^{rig}$ is an admissible covering (see [BL, §4]). Thus we get a morphism of sites $X_{rigid} = \mathfrak{X}^{rig}_{rigid} \to \mathfrak{X}_{Zar}$.

It is also possible to perform the construction $\mathfrak{X} \mapsto \mathfrak{X}^{rig}$ for formal schemes $\mathfrak{X}$ which are only locally of finite type over $\mathrm{Spf}(k^\circ)$. It is not true that any rigid variety X comes from such a formal scheme. A counterexample can be constructed by gluing a countable number of closed discs to a fixed closed disc along mutually disjoint closed sub-discs. (This is also an example of a variety of countable type which is not paracompact.) It can be proved using the lemma above and [BL] that any paracompact X comes from a (paracompact) formal scheme $\mathfrak{X}$.

Proposition 2.5.8 *(See [P82, 1.4.13]). If X is a quasi-compact rigid analytic variety of dimension d then $H^p(X, S) = 0$ for all $p > d$ and all sheaves S on X.*

Proof. Let us choose a formal scheme $\mathfrak{X}$ with $\mathfrak{X}_{rig} \cong X$ (see [R70] or [BL, Theorem 4.1]). Let us denote by $\{\mathfrak{X}_\alpha\}$ the directed system of admissible blowing ups of $\mathfrak{X}$. These all satisfy $\mathfrak{X}_\alpha^{rig} \cong X$. Hence we get the morphism of sites $\pi_\alpha : X_{rigid} \to \mathfrak{X}_{\alpha,Zar}$. Let us write $S_\alpha := \pi_{\alpha,*}S$. There is a map $H^p_\alpha := H^p(\mathfrak{X}_{\alpha,Zar}, S_\alpha) \to H^p(X, S)$ deduced from the map $\pi^*\pi_{\alpha,*}S \to S$. It is proved in [BL, 4.4] that any finite covering of X comes from a covering of some $\mathfrak{X}_\alpha$. Therefore, by our result that Čech cohomology agrees with cohomology on X, we see that any cohomology class in $H^p(X, S)$ comes from some H^p_α. At this point we just remark that the underlying Zariski topological space associated to $\mathfrak{X}_\alpha$ is the underlying topological space of a scheme of finite type over the field $\bar{k}$ of dimension at most n. The result follows. □

Remark 2.5.9 If we allow in X_{rigid} only finite coverings then it is true that $\lim \mathfrak{X}_{\alpha,Zar} \cong X_{rigid}$ as sites (see letter of Deligne to Raynaud of 23 august 1992). In this way it becomes clear that in fact $\lim H^p_\alpha = H^p(X, S)$. This follows from the following general fact: Suppose the site $\mathcal{S}$ is the direct limit of a directed system of sites $\mathcal{S}_\alpha$. Then for any sheaf $\mathcal{F}$ on $\mathcal{S}$ there is a canonical isomorphism

$$\varinjlim_{\alpha} H^q(\mathcal{S}_\alpha, \mathcal{F}|_{\mathcal{S}_\alpha}) \cong H^q(\mathcal{S}, \mathcal{F}).$$

This isomorphism is in fact easy to prove by induction on q, using the Cartan-Leray spectral sequence and the fact that any cohomology class can be killed by some covering.

Corollary 2.5.10 *If X is paracompact and of dimension $\leq d$ then cohomology of sheaves on X is zero in degrees $\geq d + 1$.*

Proof. It suffices to do the case where X is connected. Choose a covering $X = \bigcup X_n$ as in Lemma 2.5.7. Put $V_1 = \bigcup_{n \text{ odd}} X_n$ and $V_2 = \bigcup_{n \text{ even}} X_n$. The spaces V_1, V_2 and $V_1 \cap V_2$ are admissible disjoint unions of quasi-compact varieties. Note that for any sheaf S on X the maps $H^d(X_n, S) \oplus H^d(X_{n+1}, S) \to H^d(X_n \cap X_{n+1}, S)$ is surjective, otherwise the sheaf S on $X_n \cup X_{n+1}$ would have a nontrivial $d+1^{th}$-cohomology group, a contradiction with the proposition. With these remarks the result of the corollary follows from a consideration of the Cartan-Leray spectral sequence associated to the covering $X = V_1 \cup V_2$. $\square$

Remark 2.5.11 Any separated variety of dimension 1 is paracompact. See [LP]. Similarly, the analytic space associated to a scheme of finite type over $\mathrm{Spec}(k)$ is paracompact.

2.6 General morphisms

Consider an extension of complete valued fields $k \subset K$. In [BGR, 9.3.6] there is constructed a base change functor $X \mapsto X\hat{\otimes}K$ of analytic varieties over k to analytic varieties over K. If X is affinoid then $X\hat{\otimes}K$ is affinoid with algebra $O(X)\hat{\otimes}_k K$. In general, if $X = \bigcup X_i$ is an admissible affinoid covering then $X\hat{\otimes}K$ is defined as the gluing of the $X_i\hat{\otimes}K$. If V is an affinoid open subvariety of X then so is $V\hat{\otimes}K \subset X\hat{\otimes}K$. In this way (use [BGR, 9.3.6/1& 2]) we see that there is a morphism of sites

$$\varphi = \varphi_{K/k} : (X\hat{\otimes}K)_{rigid} \to X_{rigid}.$$

Lemma 2.6.1 *The functors φ^*, φ_* and $R^q\varphi_*$ preserve overconvergent sheaves.*

Proof. There is a trivial reduction to the case that X is affinoid. Let V be a rational subdomain of X. It is clear that $V(r)\hat{\otimes}K = \big(V\hat{\otimes}K\big)(r)$ for $r > 1, r \in \sqrt{|k^*|}$ (see [BGR, 9.3.6/1]). These form a co-final system of wide neighborhoods of $V\hat{\otimes}K$ since $\sqrt{|k^*|}$ is dense in $\mathbf{R}_{\geq 0}$. Thus it is clear from Lemma 2.3.1 that for special $V \subset\subset_X U$ in X we have $V\hat{\otimes}K \subset\subset_{X\hat{\otimes}K} U\hat{\otimes}K$ and that these $U\hat{\otimes}K$ form a co-final system of wide neighborhoods of $V\hat{\otimes}K$. The rest of the proof is exactly the same as the proof of Lemma 2.3.2 part 4. $\square$

Let $k \subset K$ denote an extension of complete valued fields. Let X (resp. Y) denote an arbitrary analytic variety over the field k (resp. K). The most convenient way to define a *general morphism* $f : Y \to X$ is to say that f is a morphism of the K-analytic spaces $Y \to X\hat{\otimes}K$. If both X and Y are affinoid then this is simply a continuous k-algebra homomorphism $O(X) \to O(Y)$, since any such factors as $O(X) \to O(X)\hat{\otimes}_k K \to O(Y)$. By the above, a general morphism $f : Y \to X$ gives rise to a morphism of topoi $f_{rigid} : Y^{\sim}_{rigid} \to X^{\sim}_{rigid}$. The pullback functor, written f^*, preserves overconvergent sheaves. We say that the morphism f is *quasi-compact* if $Y \to X\hat{\otimes}K$ is quasi-compact. In this case f induces a morphism of sites $Y_{rigid} \to X_{rigid}$ and R^qf_* preserves overconvergent sheaves for all q. (Use the lemma above and Proposition 2.4.1.)

If, in addition, we are given a morphism $Z \to X$ of analytic varieties over k, then we can form the fibre product:

$$Y \times_X Z := Y \times_{X\hat{\otimes}K} X\hat{\otimes}K$$

It is an analytic variety over K which satisfies a certain universal property regarding general morphisms; we leave it to the reader to describe this property explicitly.

2.7 Base change

The aim of the base change theorem is to compare $H^q(Y_a, S|_{Y_a})$ with $\left(R^q f_* S\right)_a$ for sheaves S on Y. Here Y_a is the fibre of a morphism f over the analytic point a. Let us first define this fibre.

Consider a morphism $f : Y \to X$ of analytic varieties over k and let an analytic point a of X be given. The *fibre* Y_a of f over a is defined as the fibre product of the general morphism $\mathrm{Spm}(F_a) \to X$ with f. It can also be defined as the fibre of $f\hat{\otimes}F_a : Y\hat{\otimes}F_a \to X\hat{\otimes}F_a$ over the usual point $a \in X\hat{\otimes}F_a$. There results a general morphism $\alpha : Y_a \to Y$. We remark that α is quasi-compact; the morphism of sites $(Y_a)_{rigid} \to Y_{rigid}$ comes from the functor $V \mapsto V_a$ on special subsets of Y. For a sheaf S on Y we write $S|Y_a$ instead of $\alpha^*(S)$. Finally, we remark that if both X and Y are affinoid then Y_a is affinoid with algebra $O(Y)\hat{\otimes}_{O(X)}F_a$.

Lemma 2.7.1 *(Key lemma for the rigid case.) Let a morphism $f : Y \to X$ of affinoid spaces over k be given together with an analytic point a of X. Write $\alpha : Y_a \to X$ for the resulting general morphism.*

1. *For every admissible open $V \subset Y_a$ (i.e., $V \in (Y_a)_{rigid}$) there is an admissible open $W \subset Y$ such that $V = W_a$.*

2. *Suppose W, Z are admissible open in Y and $W_a \subset Z_a$. There is a U in the filter of a such that $W \cap f^{-1}(U) \subset Z$.*

Proof. (1) We may assume that V is a rational subset of Y_a. Thus V is given by inequalities $|g_1| \geq |g_1|, ..., |g_m|$ with elements $g_1, ..., g_m \in O(Y_a) = O(Y)\hat{\otimes}_{O(X)}F_a$ generating the unit ideal. Say that $f_1g_1 + \ldots + f_mg_m = 1$. We may suppose that the g_i come from elements $g_i \in O(Y) \otimes_{O(X)} k_a$. So there is some U in the filter of a and elements $G_i \in O(Y)\hat{\otimes}_{O(X)}O(U) = O(f^{-1}(U))$ mapping to the g_i. If we take $F_i \in O(Y)\hat{\otimes}_{O(X)}O(U) = O(f^{-1}(U))$ mapping to elements close to the f_i then we see that $F_1G_1 + \ldots + F_mG_m = 1 + \delta$ where δ maps to an element of $O(Y_a) = O(Y)\hat{\otimes}_{O(X)}F_a$ with small norm, say with spectral norm < 1. By Lemma 2.7.2 this implies that δ gets spectral norm < 1 in $O(Y)\hat{\otimes}_{O(X)}O(U) = O(f^{-1}(U))$ for some smaller U in the filter of a. Hence we see that $G_1, \ldots, G_m$ generate the unit ideal in $O(f^{-1}(U))$. Thus $W \subset f^{-1}U$ given by the inequalities $|G_1| \geq |G_1|, ..., |G_m|$ works.

(2) We may assume that W is a rational subdomain of Y. Next we write Z as a finite union $Z = \bigcup Z_i$ of rational subdomains Z_i of Y. The finite covering $W_a = \bigcup_i W_a \cap (Z_i)_a$ can be refined by a rational covering $W_a = \bigcup_j V_j$ given by a number of elements $g_1, \ldots, g_m$ in $O(W_a)$ generating the unit ideal. Arguing as above, we may suppose that the g_i come from $G_i \in O(W)$ generating the unit ideal, after replacing X by some U in the filter of a. The rational subsets W_j of W defined by $|G_j| \geq |G_1|, ..., |G_m|$ cover W and each $(W_j)_a$ is contained in some $(Z_i)_a$. If we solve the problem for all the pairs (W_j, Z_i) with $(W_j)_a \subset (Z_i)_a$ then we solve the problem for (W, Z). Thus we have reduced to the case that both W and Z are rational subdomains of Y.

At this point we replace Z by $Z \cap W$, then we are in the situation that $Z \subset W$ is a rational subdomain, $Z_a = W_a$ and we want to show that there is some U such that $W \cap f^{-1}(U) = Z \cap f^{-1}(U) \subset Z$. Suppose that Z is given by inequalities $|h_0| \geq |h_1|, ..., |h_n|$ where $h_0, ..., h_n$ generate the unit ideal in $O(W)$. In particular, h_0

is an invertible function on Z, hence on $Z_a = W_a$. Arguing as in (1), we may shrink X and assume that h_0 is invertible on W. Dividing by h_0 we see that we may suppose that Z is given by the inequalities $|h_1| \leq 1, \ldots, |h_m| \leq 1$. The h_i have norms ≤ 1 on W_a. Hence, by Lemma 2.7.2, we can find a U in the filter of a such that the h_i have norm ≤ 1 on $W \cap f^{-1}(U)$, i.e., such that $W \cap f^{-1}(U) = Z \cap f^{-1}(U)$. □

LEMMA 2.7.2 *Let $f : Y \to X$ be a morphism of afinoid spaces over k, let a be an analytic point of X. Let $g \in O(Y)$ whose image $\alpha(g) \in O(Y_a)$ has spectral norm ≤ 1 (resp. < 1). There is a U in the filter of a such that the spectral norm of g on $f^{-1}(U)$ is ≤ 1 (resp. < 1).*

Proof. Let us write $O(Y) = O(X)\langle T_1, ..., T_n\rangle/(G_1, ..., G_m)$. With obvious notations we have $O(Y_a) = F_a\langle T_1, ..., T_n\rangle/(G_1(a), ..., G_m(a))$. If the spectral norm of $\alpha(g)$ is ≤ 1 it follows that $\alpha(g)$ is integral over the ring $F_a\langle T_1, ...T_n\rangle^o$. Let such an equation be

$$\alpha(g)^e + c_{e-1}\alpha(g)^{e-1} + ... + c_0 = 0$$

Write $c_i = \sum c_{i,\beta}T^\beta$ with all $c_{i,\beta} \in F_a$ satisfying $|c_{i,\beta}|_a \leq 1$.

Choose some $\pi \in k$ with $0 < |\pi| < 1$. For the $c_{i,\beta}$ with $|c_{i,\beta}|_a \geq |\pi|$ (there are only finitely many of these!) we take a suitable U in the filter of a and elements $C_{i,\beta} \in O(U)$ with images $\alpha(C_{i,\beta}) \in F_a$ such that $|\alpha(C_{i,\beta}) - c_{i,\beta}|_a < |\pi|$. (This is possible, the image of O_a is dense in F_a.) It follows that $|\alpha(C_{i,\beta})|_a \leq 1$. Thus the inequalities $|C_{i,\beta}| \leq 1$ define a smaller U in the filter of a where the elements $C_{i,\beta}$ have spectral norm ≤ 1. For convenience we replace X by U and Y by $f^{-1}U$. The $C_{i,\beta} \in O(X)$ are elements with spectral norm ≤ 1. We consider the expression

$$R := g^e + \gamma(\sum C_{e-1,\beta}T^\beta)g^{e-1} + ... + \gamma(\sum C_{0,\beta}T^\beta)$$

where γ denotes the map $O(X)\langle T_1, ..., T_n\rangle \to O(Y)$. This element $R \in O(Y)$ has an image $\alpha(R) \in O(Y_a)$ with spectral norm $< |\pi|$. If we can find a U in the filter of a such that the spectral norm of R on $f^{-1}U$ is < 1 then we replace again X by U and Y by $f^{-1}U$. After this is done the spectral norm of R on Y is < 1 and the spectral norms of the $\gamma(\sum C_{i,\beta}T^\beta)$ are ≤ 1. It follows at once that the spectral norm of g on Y is ≤ 1.

In this way we have reduced the case ≤ 1 of the lemma to the case < 1. Let us therefore assume that the spectral norm of $\alpha(g)$ is < 1. For some $N \geq 1$ the element $\alpha(g^N) \in O(Y_a)$ has a pre-image $g_1 \in F_a\langle T_1, ..., T_n\rangle$ with norm < 1. Take also a $g_2 \in O(X)\langle T_1, ..., T_n\rangle$ with image $g^N \in O(Y)$. Then $\alpha(g_2) - g_1 \in F_a\langle T_1, ..., T_n\rangle$ lies in the ideal generated by the $\{G_1(a), ..., G_m(a)\}$ and we can write

$$\alpha(g_2) - g_1 = \sum_i G_i(a)(\sum_\beta a_{i,\beta}T^\beta)$$

where the coefficients $a_{i,\beta} \in F_a$ have limit 0. For the $a_{i,\beta}$ with $|a_{i,\beta}| \geq |\pi|$ we choose a U in the filter of a and elements $A_{i,\beta} \in O(U)$ such that the difference of the image of $A_{i,\beta}$ and $a_{i,\beta}$ in F_a has absolute value $< |\pi|$. We may suppose again that $U = X$. We suppose that π is chosen such that all coefficients of $\pi G_i(a)$ have norm < 1 (in F_a). After changing g_2 into $g_2 - \sum_i G_i(\sum_\beta A_{i,\beta}T^\beta) \in O(X)\langle T_1, ..., T_n\rangle$ we have the situation that $\alpha(g_2) - g_1 \in F_a\langle T_1, ..., T_n\rangle$ and $\alpha(g_2) \in F_a\langle T_1, ..., T_n\rangle$ are power series with coefficients having absolute values < 1. For the finitely many coefficients in $O(X)$ of g_2 with absolute value ≥ 1 we can find a U in the filter of a such that their spectral norms on $f^{-1}U$ are < 1. After shrinking X to U and Y to $f^{-1}U$ we arrive at the situation where all the coefficients of g_2 are < 1. Hence the spectral norm of g^N on Y is < 1 and so the spectral norm of g on Y is < 1. $\square$

Lemma 2.7.3 *In the situation of the Key Lemma.*

1. *The functor $S \mapsto S|_{Y_a}$ preserves flasque sheaves.*
2. *For any sheaf S on Y the sheaf $\alpha^*_{rigid}(S) = S|_{Y_a}$ can be described as follows:*

$$\Gamma(W_a, S|_{Y_a}) = \varinjlim_{a \in U \subset X} S\big(W \cap f^{-1}(U)\big).$$

Here W is any special subset of Y.

Proof. (1) This is clear from the first assertion of our Lemma 2.7.1. (2) From Lemma 2.7.1 it follows that for any S we have

$$\Gamma(W_a, S|_{Y_a}) = \varinjlim_{Z \subset Y, Z_a = W_a} S\big(Z\big).$$

The limit is over all admissible open $Z \subset Y$ such that $Z_a = W_a$. From Lemma 2.7.1 part 2 it follows that the $Z = W \cap f^{-1}(U)$ are co-final in this system. $\square$

Finally, we come to the base change theorem. To give a natural statement recall that a δ-functor between to Abelian categories $\mathcal{A}$ and $\mathcal{B}$ is a sequence of functors $\{T_n\}_{n \geq 0}$ (equipped with certain boundary operators) such that any short exact sequence in $\mathcal{A}$ is transformed into a long exact sequence in $\mathcal{B}$. (For a more precise definition see for example [H77]).

Theorem 2.7.4 *(Base change for rigid spaces.) Let $f : Y \to X$ be a quasi-compact morphism of rigid analytic varieties over k. Take any analytic point a of X and denote by Y_a the fibre of f over a. The functors $S \mapsto H^n(Y_a, S|_{Y_a})$ (resp. $S \mapsto \big(R^n f_* S\big)_a$) on the category of Abelian sheaves on Y_{rigid} form a δ-functor. These δ-functors are isomorphic: $\big(R^n f_* S\big)_a \cong H^n(Y_a, S|_{Y_a})$ for any Abelian sheaf S on Y.*

Proof. The functor $S \mapsto S|_{Y_a}$ is exact and so is the functor $\mathcal{F} \mapsto \mathcal{F}_a$ on sheaves on X. From this follows immediately that the functors under consideration form δ-functors.

Let us define the canonical morphisms:

$$(*) \qquad \big(R^n f_* S\big)_a \longrightarrow H^n(Y_a, S|_{Y_a})$$

Since $S|_{Y_a} = \alpha^*(S)$ there are canonical homomorphisms $H^n(X, S) \to H^n(Y_a, S|_{Y_a})$ and these form a transformation of δ-functors. For any open subvariety $U \subset X$, with $a \in U$, we have $f^{-1}(U)_a = Y_a$ and $\big(S|_{f^{-1}(U)}\big)|_{Y_a} = S|_{Y_a}$. Hence the same argument gives

$$H^n(f^{-1}(U), S) \longrightarrow H^n(Y_a, S|_{Y_a}).$$

Since $\big(R^n f_* S\big)_a = \lim H^n(f^{-1}(U), S)$ (the limit is taken over U as above) we get the desired map of δ-functors.

To prove that (*) is an isomorphism we may assume that X is affinoid. Let us first do the case that Y is affinoid. The result for $n=0$ is Lemma 2.7.3 with $W=Y$. For a flasque sheaf on Y both sides of (*) are zero for $n \geq 1$ (use 2.7.3), hence the standard argument gives the result for general n. (Inject S into a flasque sheaf and argue by induction on n.)

There are two ways to get the result for general quasi-compact Y.
(1) Choose a finite affinoid covering $Y = \bigcup Y_i$ such that all $Y_{i_0 \ldots i_q} = Y_{i_0} \cap \ldots \cap Y_{i_q}$ are affinoid (see 2.5.3). The maps (*) for Y and all $Y_{i_0 \ldots i_q}$ induce a morphism of spectral sequences $\{{}_1E_n^{pq}\} \to \{{}_2E_n^{pq}\}$ abutting to the maps $\left(R^{p+q}f_*S\right)_a \longrightarrow H^{p+q}(Y_a, S|_{Y_a})$ and with as E_2-terms the maps (*):

$$\bigoplus_{i_0 \ldots i_q} \left(R^p(f|_{Y_{i_0 \ldots i_q}})_* S\right)_a \longrightarrow \bigoplus_{i_0 \ldots i_q} H^p((Y_{i_0 \ldots i_q})_a, S|_{Y_{i_0 \ldots i_q}})$$

These maps are isomorphisms by the above hence we get the result.
(2) Here we just remark that the Key Lemma holds for $f : Y \to X$ with X affinoid and Y quasi-compact. This follows immediately from the Key Lemma as it stands now. The base change theorem now follows from the same argument as for the case Y affinoid. □

REMARK 2.7.5 (1) The result is of course most useful for overconvergent sheaves S since in that case the sheaves $R^n f_* S$ are overconvergent too and hence "determined" by their stalks at analytic points.

(2) The proof given above is the one of [P82]. In [S93] the translation of rigid overconvergent sheaves on Z to sheaves on $\mathcal{M}(Z)$ is used to translate the statement into the topological base change theorem for the continuous map $\mathcal{M}(f) : \mathcal{M}(Y) \to \mathcal{M}(X)$.

(3) One aim of this paper is to develop a theory of étale points and étale overconvergent sheaves such that the base change theorem and related theorems are valid.

3 ÉTALE POINTS AND ÉTALE OVERCONVERGENT SHEAVES

A morphism $f : Y \to X$ of analytic spaces over k is called *étale* if for every $y \in Y$ the induced homomorphism of the local rings $O_{X,f(y)} \to O_{Y,y}$ is flat and un-ramified. The term un-ramified means that $O_{Y,y}/mO_{Y,y}$ is a (finite) separable field extension of the field $O_{X,f(y)}/m$ where m denotes the maximal ideal of $O_{X,f(y)}$.

This notion of étale morphism is somewhat complicated. First of all the image of an étale morphism is in general not an admissible open subset. For affinoids Y, X however, it has been shown in [M81] that $f(Y)$ is a finite union of affinoid subdomains of X. We will give a proof of this fact below (see Proposition 3.1.7).

We define the étale site in 3.2 (see [S-S]) and we compare the étale topology with the rigid topology. We define étale points and étale stalks in 3.3. In order to be able to work with étale overconvergent sheaves we construct étale wide neighborhoods in the affinoid case. The proof of the étale base change theorem is then similar to the proof in the rigid case.

3.1 Étale morphisms of affinoids

Let an extension of rings $A \to B$ be given. Let $d : B \to \Omega^f_{B/A}$ denote the universal *finite* differential module of B over A. By definition $\Omega^f_{B/A}$ is a finitely generated B-module and every derivation of B/A into a finitely generated B-module factors uniquely over $d : B \to \Omega^f_{B/A}$. This module exists in many cases where the usual universal differential module of B over A is not a finitely generated module. For affinoid algebras A, B over the same field k one can give B a presentation $B = A\langle T_1, ..., T_n\rangle/(G_1, ..., G_m)$. In this case $\Omega^f_{B/A}$ exists and is the quotient of the free B-module generated by $dT_1, ..., dT_n$ by its submodule generated by $dG_1, ..., dG_m$. Let m be a maximal ideal of B and n the corresponding maximal ideal of A. The completions of the local rings are denoted by $\hat{B}_m$ and $\hat{A}_n$. One can show that $\Omega^f_{\hat{B}_m/\hat{A}_n}$ coincides with $\Omega^f_{B/A} \otimes \hat{B}_m$. In the situation $Y = Spm(B)$ and $X = Spm(A)$ and y, x corresponding to m, n one has that $\hat{B}_m, \hat{A}_n$ are the completions of the local rings $O_{Y,y}$ and $O_{X,f(y)}$. The map between the last two local rings is un-ramified if and only if the map between the completed rings is un-ramified. The last statement is equivalent with $\Omega^f_{\hat{B}_m/\hat{A}_n} = 0$. From this and the fact that flatness is a local property one finds the following:

Observation 3.1.1 *A morphism of affinoid spaces $Y \to X$ is étale if and only if $O(X) \to O(Y)$ is flat and $\Omega^f_{O(Y)/O(X)} = 0$. Further $\Omega^f_{O(Y)/O(X)} = 0$ if and only if the $n \times n$ minors of the matrix $(\frac{\partial G_k}{\partial T_l})$ generate the unit ideal in $O(Y)$.*

A *special étale morphism* of affinoid spaces $f : Y \to X$ is a morphism such that $O(Y)$ has a presentation

$$O(Y) = O(X)\langle T_1, ..., T_n\rangle/(G_1, ..., G_n)$$

such that the functional determinant $\Delta := \det(\partial G_i/\partial T_j)$ is an invertible function on Y. The morphism $Y \to X$ is indeed étale. We need only prove flatness. Let us check this in a point $y \in Y$ where $T_i = 0$. The completion of the local ring $O_{Y,y}$ is isomorphic to:

$$\widehat{O_{X,f(y)}}[[T_1, \ldots, T_n]]/(G_1, \ldots, G_n)$$

Our assumption on Δ gives that $(G_1, \ldots, G_n) = (T_1, \ldots, T_n)$ in this ring. Hence the map $O_{X,f(y)} \to O_{Y,y}$ is flat and un-ramified since it induces an isomorphism on completions (this is not true for general points $y \in Y$!).

We now present the proof by Huber of the fact that any étale morphism of affinoids is special étale.

Let $Y \to X$ be an étale morphism of affinoids. Choose a surjection $O(X)\langle T_1, \ldots, T_n\rangle \to O(Y)$ with kernel I. Since the module of differentials of $O(Y)$ over $O(X)$ is zero, there is an isomorphism

$$I/I^2 \longrightarrow O(Y)dT_1 \oplus \ldots \oplus O(Y)dT_n.$$

Thus we may choose $G_1, \ldots, G_n \in I$ whose classes mod I^2 are a basis of I/I^2. It follows that $\mathrm{Sp}\big(O(X)\langle T_1, \ldots, T_n\rangle/(G_1, \ldots, G_n)\big)$ is equal to $Y \amalg Z$ for some affinoid

Z. Let us choose an element $G \in O(X)\rangle T_1, \ldots, T_n\rangle$ which is 1 on Y and 0 on Z. It follows that $O(Y)$ has the presentation

$$O(Y) = O(X)\langle T, T_1, \ldots, T_n\rangle/(TG-1, G_1, \ldots, G_n).$$

It follows immediately that Y is special étale over X.

OBSERVATION 3.1.2 *[Hu, 1.7.1] Any étale morphism $Y \to X$ of affinoids is a special étale morphism.*

Let $f: Y \to X$ be an étale morphism of affinoids and choose a representation $O(Y) = O(X)\langle T_1, \ldots, T_n\rangle/(G_1, \ldots, G_n)$ such that $\{\Delta, G_1, \ldots, G_n\}$ generate the unit ideal of the algebra $O(X)\langle T_1, \ldots, T_n\rangle$. If $Z \to X$ is a general morphism, where Z is an affinoid variety over K, then the fibre product $Z \times_X Y$ is given by the affinoid algebra:

$$O(Z)\hat{\otimes}_{O(X)}O(Y) = O(Z)\langle T_1, ..., T_n\rangle/(G_1, ..., G_m)$$

Thus it is clear that $Z \times_X Y \to Z$ is again special étale.

OBSERVATION 3.1.3 *Étale morphisms of affinoids are preserved by general base change. It follows that arbitrary étale morphisms $f: Y \to X$ are preserved by general base change. In particular, if b is an analytic point with image a in X then F_b is a finite separable extension of F_a.*

The following proposition shows that any étale morphism of affinoids can locally be embedded in a finite étale morphism.

PROPOSITION 3.1.4 *Let $f: Y \to X$ be an étale morphism of affinoids. There exists a finite affinoid covering $X = \bigcup X_j$, finite étale morphisms $g_j : Z_j \to X_j$ and open immersions $h_j : f^{-1}(X_j) \to Z_j$ such that $f|_{f^{-1}(X_j)} = g_j \circ h_j$.*

Proof. Let us take an analytic point a of X. By compactness of $\mathcal{M}(X)$ we need only to find a wide neighborhood of a in X over which f can be factored as in the proposition. By Lemma 3.1.6 we may assume that $f^{-1}(\{a\}) = \{b\}$ for some analytic point b of Y. By the above the field extension $F_a \subset F_b$ is finite separable. Hence we can find a wide U in the filter of a and a finite étale morphism $\phi: V \to U$ such that $\phi^{-1}(\{a\})$ consists of one analytic point v with $F_v \cong F_b$ as F_a extensions. See Remark 2.1.2. Let us consider the fibre product $Y \times_X V$ and its projections. The projection to the first factor is a finite étale map onto $f^{-1}(U)$, the projection to the second factor is étale to V and there is an analytic point $c =$"$b \times_a v$" with $F_v = F_c = F_b$. Thus by the lemma below a wide neighborhood W of b is isomorphic to a wide neighborhood of c which is mapped isomorphically to an affinoid subdomain of a wide neighborhood of v. Lemma 3.1.6 shows that replacing U by a smaller wide U we get that $f^{-1}(U) \subset W$ is isomorphic to an affinoid open subdomain of V. □

LEMMA 3.1.5 *Let $f: Y \to X$ be an étale morphism of affinoids and b an analytic point of Y. Put $a = f(b)$. If $F_a \cong F_b$ then there exists a wide U in the filter of a such that $f^{-1}(U) = V \amalg W$ where V is a wide neighborhood of b in Y and the morphism $V \to U$ is an open immersion. If f is finite then $V \to U$ is an isomorphism.*

Proof. In the case that f is finite we may assume that $f^{-1}(\{a\}) = \{b\}$ by replacing X by U as in Lemma 3.1.6 and Y by a connected component of $f^{-1}(U)$. Now $O(Y)$ is a finite locally free $O(X)$-module which we may assume to have constant rank by replacing X by one of its connected components. Our assumptions imply that $F_a \cong O(Y)\hat{\otimes}_{O(X)}F_a \cong O(Y) \otimes_{O(X)} F_a$ hence this rank must be one. This proves the finite case.

The general case. We may replace X by a U as in the lemma below and Y by one of its connected components, hence we may assume that $f^{-1}(\{a\}) = \{b\}$. Let us consider the fibre product $Y \times_X Y$. Since f is étale, the diagonal $\triangle(Y)$ is a union of connected components of $Y \times_X Y$: $\triangle$ is a closed immersion and it is étale (look at local rings!), hence by the finite case above it is also open immersion. Put $Z = Y \times_X Y \setminus \triangle(Y)$, it is an affinoid variety. By assumption, b is not in the image of $\mathrm{pr}_1|_Z : Z \to Y$. Hence we can find a wide neighborhood V of b in Y such that $\mathrm{pr}_1^{-1}(V) \cap Z = \emptyset$. (Use that the spaces $\mathcal{M}(Z)$ and $\mathcal{M}(Y)$ are Hausdorff and compact.) Next we replace X by a wide neighborhood U of a such that $f^{-1}(U) \subset V$ (see lemma below) and Y by $f^{-1}(U)$. We see that $Y \times_X Y \cong Y$. Thus $Y \to X$ is an open immersion ([BGR, 7.3.3], look at complete local rings in ordinary points of Y) and we have won. □

Lemma 3.1.6 *Suppose $f : Y \to X$ is a morphism of affinoids and a is an analytic point of X.*

1. *If $Y_a = \bigcup Y_i$ is the decomposition of the fibre of f into connected components, then there is a wide U in the filter of a such that $f^{-1}(U) = \coprod V_i$ with $(V_i)_a = Y_i$.*

2. *If $Y_a = \{b_1, \ldots, b_s\}$ and we are given wide neighborhoods $W_i \subset Y$ of b_i then we may choose U such that $f^{-1}(U) \subset \bigcup W_i$.*

Proof. The first assertion is a direct consequence of the base change theorem combined with the fact that $f_*\mathbb{Z}$ is overconvergent. For 2) take neighborhoods W_i' of b_i in Y such that $W_i' \subset\subset_Y W_i$. By our Key Lemma we can find a neighborhood U' of a such that $f^{-1}(U') \subset \bigcup W_i'$. For some $U \subset X$ with $U' \subset\subset_X U$ we get $f^{-1}(U) \subset \bigcup W_i$. (Since $\bigcup W_i' \subset\subset_Y \bigcup W_i$, compare with proof of Lemma 2.3.2 part 4.) □

Proposition 3.1.7 *Let $f : Y \to X$ be an étale morphism with Y quasi-compact.*

1. *The image $f(Y)$ of f is a special subset of X, i.e., it is a finite union of open affinoid subvarieties of X.*

2. *An analytic point a of X comes from an analytic point of $f(Y)$ if and only if there exists an analytic point of Y mapping to a.*

3. *The formation of the image of f commutes with general base change: if $X' \to X$ is a general morphism then $f(Y \times_X X') = f(Y) \times_X X'$.*

Proof. We remark that the last assertion follows from the other two.

Let us take an admissible affinoid covering $X = \bigcup X_j$. The admissible covering $Y = \bigcup f^{-1}(X_j)$ has a finite affinoid refinement $Y = \bigcup_{i=1}^n Y_i$. It suffices to prove the proposition for the maps $Y_i \to X_{\alpha(i)}$. Thus we may assume that both X and Y are affinoid. At this point let us prove the assertion on analytic points assuming proven the result on the image. Take an analytic point a of X. If a is not an analytic point of $f(Y)$ then there exists a neighborhood $U \subset X$ of a such that $U \cap f(Y) = \emptyset$. Hence $f^{-1}(U) = \emptyset$ and so $Y_a = \emptyset$. On the other hand, if $a = f(b)$ for some analytic point b of Y then for any U in the filter of a, $f^{-1}(U) \neq \emptyset$. Hence $U \cap f(Y) \neq \emptyset$, hence a is an analytic point of $f(Y)$.

Let us prove the first assertion. Using our preceding proposition we may assume that f factors as $Y \to Z \to X$ where $Y \to Z$ is an open immersion and $Z \to X$ is finite étale. We may also assume that Y is a rational subdomain of Z. We have a morphism

$$\varphi : Z \longrightarrow (\mathbb{P}_n)^{an}$$

with $Y = \varphi^{-1}(R)$ where $R = \{(x_0, \ldots, x_n); |x_0| \geq |x_i|\}$.

Suppose the degree of $Z \to X$ is constant and equal to d. Consider the d-fold fibre product

$$Z^d := Z \times_X Z \times \ldots \times_X Z$$

The diagonals $\triangle_{ij} = \{(z_1, \ldots, z_d) \in Z^d | z_i = z_j\}$ are unions of connected components of Z^d since $Z \to X$ is étale. We put

$$W := Z^d \setminus \bigcup_{i,j} \triangle_{ij}$$

It is an affinoid variety, finite étale over X endowed with an action of S_d (the symmetric group on d letters). The quotient of W under this action is X in the sense that $\Gamma(W, O_W)^{S_d} = \Gamma(X, O_X)$. (Since $Z \to X$ is finite we are doing just algebraic geometry here.) There is a S_d-equivariant map

$$\varphi \times \ldots \times \varphi : W \longrightarrow \mathbb{P}_n^{an} \times \ldots \times \mathbb{P}_n^{an}$$

which descends to a morphism

$$S_d(\varphi) : X \to \big((\mathbb{P}_n)^d / S_d\big)^{an}.$$

It is clear that $f(Y) = S_d(\varphi)^{-1}\big(R(d)/S_d\big)$ with

$$R(d) = \bigcup_i \mathbb{P}_n^{an} \times \ldots \times R \times \ldots \mathbb{P}_n^{an}.$$

There is an obvious formal scheme $(\mathbb{P}^d_{n,k^\circ})^\wedge$ giving rise to $(\mathbb{P}^d_n)^{an}$ and $R(d)$ corresponds to a S_d-stable formal open subscheme of it, namely:

$$\mathcal{U} := \bigcup_i (\mathbb{P}_{n,k^\circ})^\wedge \times \ldots \times (\mathbb{A}_{n,k^\circ})^\wedge \times \ldots \times (\mathbb{P}_{n,k^\circ})^\wedge$$

It follows that $R(d)/S_d$ corresponds to the formal open subscheme $\mathcal{U}/S_d$ of $(\mathbb{P}^d_{n,k^\circ})^\wedge$. Thus $R(d)/S_d$ is a special subset of $\big((\mathbb{P}_n)^d/S_d\big)^{an}$ and hence so is $S_d(\varphi)^{-1}$ of it. □

3.2 The étale site

Let X be an analytic variety over k. In this subsection we recall the definition of the étale site of X (see [S-S, p. 58]). We give a criterium for a presheaf to be a sheaf and we give some examples of étale sheaves. Finally, we prove Hilbert 90 in our situation and we prove that étale cohomology of coherent O-modules agrees with rigid cohomology.

The underlying category of the site $X_{étale}$ will be the category of étale morphisms f of analytic varieties $f : Y \to X$. A morphism of f into f' is a morphism $g : Y \to Y'$ such that $f' \circ g = f$; the morphism g is automatically étale.

We say that a family of étale morphisms $\{g_i : Z_i \to Y\}_{i \in I}$ is an *étale covering* if it has the following property:

For any (some) choice of admissible affinoid coverings $Z_i = \bigcup_j Z_{i,j}$ we have $Y = \bigcup_{i,j} g_i(Z_{i,j})$ and this is an admissible covering in the G-topology of Y.

This makes sense since the subsets $g_i(Z_{i,j})$ are admissible (special) subsets (see Proposition 3.1.7). We remark that the property is local on Y in the following sense: if $Y = \bigcup Y_l$ is an admissible affinoid covering then $\{g_i : Z_i \to Y\}$ is an étale covering if and only if $\{g_i : g_i^{-1}(Y_l) \to Y_l\}$ is an étale covering for all l. This is so since both assertions are equivalent to the following assertion:

For each l there are finitely many (i_α, j_α), $\alpha = 1, \dots, n$ such that $Y_l \subset \bigcup_{\alpha=1}^n g_{i_\alpha}(Z_{i_\alpha, j_\alpha})$.

From this it also immediately follows that if $\{Z_i \to Y\}$ is an étale covering and $\{X_{i,j} \to Z_i\}$ are étale coverings then $\{X_{i,j} \to Y\}$ is an étale covering.

Lemma 3.2.1 *Suppose $\{Y_i \to X\}$ is an étale covering and $Z \to X$ is a general morphism. The fibre product $\{Z \times_X Y_i \to Z\}$ is an étale covering.*

Proof. This follows immediately from the definition, the remarks above and Proposition 3.1.7. □

It follows from the above that the category $X_{étale}$, equipped with the family of étale coverings as defined above is a site. It is also clear from the lemma that any (general) morphism $f : Z \to X$ defines a morphism of sites $Z_{étale} \to X_{étale}$ (given by the functor $(Y \to X) \mapsto (Z \times_X Y \to Z)$). The functors on étale sheaves will be denoted by f_* and f^* as usual.

For any object $Y \to X$ of $X_{étale}$ we get a morphism of sites

$$r_{Y/X} : X_{étale} \longrightarrow Y_{rigid},$$

comparing rigid and étale topologies. It is defined by the inclusion of categories $Y_{rigid} \subset X_{étale}$, if S is a sheaf on $X_{étale}$ then $\Gamma(V, (r_{Y/X})_* S) = \Gamma(V, S)$. Sometimes we will use the notation $S|_{Y_{rigid}}$ in stead of $(r_{Y/X})_* S$; we will also use this notation for presheaves S on $X_{étale}$. If $Y = X$ the morphism $r_{X/X}$ will be denoted $r : X_{étale} \to X_{rigid}$. If a is an analytic point of X then we put $S_a := (S|_{X_{rigid}})_a = r_*(S)_a$.

Proposition 3.2.2 *The presheaf S on $X_{étale}$ is a sheaf if and only if the following two conditions hold:*

1. *For any Y in $X_{étale}$ the presheaf $S|_{Y_{rigid}}$ is a sheaf.*

2. *For any surjective finite étale morphism $Y' \to Y$ of affinoids in $X_{étale}$ the sequence $\emptyset \to S(Y) \to S(Y') \rightrightarrows S(Y' \times_Y Y')$ is exact.*

Proof. Suppose S satisfies 1) and 2). We claim that S also satisfies 2) for any finite étale morphism $Y' \to Y$ in $X_{étale}$ with Y quasi-compact. Just cover Y by affinoids as in Lemma 2.5.3 and use 1) to show that it suffices to know 2) for all the resulting affinoid finite étale coverings.

Let us take a morphism $\varphi : Z \to U$ in $X_{étale}$ such that

1. φ is surjective,

2. Z and U are quasi-compact,

3. φ factors as $Z \to V \to U$ with $V \to U$ finite étale and $Z \to V$ an open immersion.

We claim that for any such φ the sequence

$$\emptyset \to S(U) \to S(Z) \rightrightarrows S(Z \times_U Z)$$

is exact. We prove this by induction on the degree of the morphism $V \to U$. (If it is 1 then φ is an isomorphism and our claim trivial.) Suppose therefore that the degree of $V \to U$ is d and that we have proved our claim in the cases where the corresponding degree is less than d.

Since $V \to U$ is finite étale we have that the diagonal $\triangle(V) \subset V \times_U V$ is a union of connected components of $V \times_U V$. Its complement $W \subset V \times_U V$ is thus a quasi-compact variety and the morphism $pr_2 : W \to V$ is finite étale of degree $< d$. Put $Z' = Z \times_U V \cap W$ and $U' = pr_2(Z')$, both are quasi-compact (see Proposition 3.1.7). The surjective étale morphism $\varphi' = pr_2 : Z' \to U'$ factors through $V' := W \cap pr_2^{-1}(U') \to U'$ which is finite étale of degree $< d$. Furthermore, it is clear that $V = U' \cap Z$.

We have the following commutative diagram:

$$\begin{array}{ccccccc} \emptyset & \longrightarrow & S(U) & \longrightarrow & S(Z) & \rightrightarrows & S(Z \times_U Z) \\ & & \downarrow & & \downarrow & & \downarrow \\ \emptyset & \longrightarrow & S(U') & \longrightarrow & S(Z') & \rightrightarrows & S(Z' \times_U' Z') \end{array}$$

The diagram shows that any element $s \in S(Z)$ such that $p_1^*(s) = p_2^*(s)$ gives a unique element (by induction) $s' \in S(U')$ such that $s'|_{Z'} = s|_{Z'}$. It is also true that $s'|_{U' \cap Z} = s|_{U' \cap Z}$ (use induction hypothesis for the morphism $(\varphi')^{-1}(U' \cap Z) \to U' \cap Z$). Hence by 1) for the covering $V = Z \cup U'$ we get a unique section $s_V \in S(V)$ with $s_V|_Z = s$ and $s_V|_{U'} = s'$. We want to show that $p_1^*(s_V) = p_2^*(s_V)$ on $V \times_U V$. Remark that $V \times_U V$ has the following admissible special covering

$$V \times_U V = Z \times_U Z \cup U' \times_U Z \cup Z \times_U U' \cup U' \times_U U'.$$

Hence by 1) we need only to prove $p_1^*(s_V) = p_2^*(s_V)$ on each of these. For the most difficult case, namely $U' \times_U U'$, we remark that the morphism $Z' \times_U Z' \to U' \times_U U'$

is the composition $Z' \times_U Z' \to U' \times_U Z' \to U' \times_U U'$ of morphisms to which our induction hypothesis applies. Hence the map $S(U' \times_U U') \to S(Z' \times_U Z')$ is injective. At this point the commutative diagram

$$\begin{array}{ccc} S(Z) & \rightrightarrows & S(Z \times_U Z) \\ \downarrow & & \downarrow \\ S(Z') & \rightrightarrows & S(Z' \times'_U Z') \end{array}$$

gives the desired result.

To prove that the presheaf S is a sheaf we have to show that any étale covering $\{g_i : Z_i \to Y\}$ in $X_{\acute{e}tale}$ gives an exact sequence

$$\emptyset \longrightarrow S(Y) \longrightarrow \prod S(Z_i) \rightrightarrows \prod S(Z_i \times_Y Z_j).$$

By choosing an admissible affinoid covering $Y = \bigcup Y_j$ and using 1) it is easy to reduce to the case Y affinoid. Similarly we may reduce to the case all Z_i affinoid also. Using propositions 3.1.4 and 3.1.7 we may assume that each $Z_i \to Y$ factors as $Z_i \to V_i \to U_i \subset Y$ as above. It is now easy to deduce the result from our claim above. Compare also with [M80, II 1.5]. $\square$

Examples of sheaves on the étale site. It follows easily from the criterium given above that the following presheaves are sheaves. A general object of $X_{\acute{e}tale}$ will be denoted by $f : Y \to X$.

1. The structure sheaf $\mathbb{G}_a$ defined by $Y \mapsto \Gamma(Y, O_Y)$.

2. The sheaf $\mathbb{G}_m$ defined by $Y \mapsto \Gamma(Y, O_Y^*)$.

3. For any real number r we can look at the subsheaf of $\mathbb{G}_a$ given by $Y \mapsto \{f \in \Gamma(Y, O_Y) : |f(y)| \le r \forall y \in Y\}$. We can also replace the $\le$-sign by the $<$ sign. If $r \le 1$ we can define a subsheaf of $\mathbb{G}_m$ by inequalities of the form $|1 - f(y)| \le r$.

4. Any representable sheaf $Y \mapsto \mathrm{Mor}_X(Y, Z)$ given by some variety Z over X.

5. For any Abelian group A we have the constant sheaf A_X with stalks A defined by: $Y \mapsto$ the set of maps $Y \to A$ constant on connected components of Y. (This is in fact a representable sheaf, namely represented by $\coprod_{a \in A} X$.)

6. If n is prime to the characteristic of k then we define μ_n as the kernel of the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by multiplication by n. If k contains a primitive n^{th} root of unity ζ then $\mu_n \cong \mathbb{Z}/n\mathbb{Z}_X \cdot \zeta$. There is a Kummer exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

7. Suppose that $\mathcal{F}$ is a coherent sheaf of O_X-modules on X. We define a sheaf $W(\mathcal{F})$ of $\mathbb{G}_a$-modules on $X_{\acute{e}tale}$ as follows: $Y \mapsto \Gamma(Y, f^*\mathcal{F})$, here f^* denotes pullback of coherent O-modules: $f^*\mathcal{F} := f^*(\mathcal{F}) \otimes_{f^*O_X} O_Y$. It is clear that $W(O_X) = \mathbb{G}_a$.

Suppose that we are given an étale covering $\{Y_i \to X\}$. We claim that this coverings allows effective descent of coherent O-modules. This means the following: suppose we are given for each i a coherent O_{Y_i}-module $\mathcal{F}_i$ and descent data. This means isomorphisms of coherent sheaves

$$\varphi_{ij} : pr_1^* \mathcal{F}_i \longrightarrow pr_2^* \mathcal{F}_j$$

on $Y_i \times_X Y_j$ satisfying the co-cycle relation $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ on $Y_i \times_X Y_j \times_X Y_k$. In this situation there exists a unique coherent sheaf of O_X-modules $\mathcal{F}$ giving rise to $\mathcal{F}_i$ on each Y_i and inducing the isomorphisms φ_{ij}. In addition, homomorphisms of systems $(\mathcal{F}_i, \varphi_{ij})$ as above should correspond to homomorphisms of the corresponding sheaves $\mathcal{F}$. The proof of this is gotten by paraphrasing the proof of Proposition 3.2.2 in this case. Indeed, the question we are considering is whether the association $Y \mapsto$ the category of coherent O_X-modules defines a sheaf of categories. It is clear that rigid coverings and finite étale coverings allow effective descent for coherent O-modules and hence the reasoning of the Proposition applies.

COROLLARY 3.2.3 *Any descent datum for coherent sheaves over an étale covering of X is effective.*

COROLLARY 3.2.4 *For any analytic variety X we have the following isomorphisms:*

$$H^1(X, \mathbb{G}_m) \cong H^1(X, O_X^*) \cong Pic(X)$$

Proof. Of course the group $Pic(X)$ is the group of isomorphism classes of line bundles on X. Since $H^1 = \check{H}^1$ any element in $H^1(X, \mathbb{G}_m)$ can be considered as descent data for invertible O-modules. By the above these are effective and hence come from an element of $H^1(X, O_X^*)$. □

PROPOSITION 3.2.5 *Suppose $\mathcal{F}$ is a coherent sheaf of O_X-modules. The natural maps $H^i(X_{rigid}, \mathcal{F}) \to H^i(X, W(\mathcal{F}))$ are isomorphisms.*

Proof. The maps arise from the identification $r_* W(\mathcal{F}) \cong \mathcal{F}$ and the adjunction map $r^* \mathcal{F} = r^* r_* W(\mathcal{F}) \to W(\mathcal{F})$. Hence the result for $i = 0$. We are going to prove the proposition by induction on i. Take n and suppose the proposition is proven for all $X, \mathcal{F}$ and $i \leq n-1$.

For any $f : Y \to X$ in $X_{étale}$ consider the map

$$H^n(Y, f^* \mathcal{F}) = H^n(Y, (r_{Y/X})_* W(\mathcal{F})) \longrightarrow H^n(Y, W(\mathcal{F})).$$

This map is injective: by induction hypothesis the sheaves $R^i(r_{Y/X})_* W(\mathcal{F})$ on Y_{rigid} are zero for $i = 1, \ldots, n-1$ (they are the sheaves associated to the presheaves $U \mapsto H^i(U, W(\mathcal{F}))$). Thus the spectral sequence $H^j(Y, R^i(r_{Y/X})_* W(\mathcal{F})) \Rightarrow H^{i+j}(Y, W(\mathcal{F}))$ gives the result. Consider the presheaf $\mathcal{H}^n$ on $X_{étale}$ defined by

$$Y \longmapsto \mathcal{H}^n := \mathrm{Coker}\big(H^n(Y, f^* \mathcal{F}) \to H^n(Y, W(\mathcal{F}))\big).$$

The sheaf associated to this presheaf is zero since any cohomology class in $H^n(Y, W(\mathcal{F}))$ can be killed by an étale covering. Therefore, if we show that $\mathcal{H}^n$ is a sheaf then we are done. To do this we use the criterium from Proposition 3.2.2.

Take any admissible covering $\mathcal{U} : Y = \bigcup Y_j$ of some étale $f : Y \to X$. We have the morphism of spectral sequences

$$\begin{array}{ccc} \check{H}^i(\mathcal{U}, \underline{H}^j(f^*\mathcal{F})) & \Rightarrow & H^{i+j}(Y, f^*\mathcal{F}) \\ \downarrow & & \downarrow \\ \check{H}^i(\mathcal{U}, \underline{H}^j(W(\mathcal{F}))) & \Rightarrow & H^{i+j}(Y, W(\mathcal{F})) \end{array}$$

(see for example [M80, III Proposition 2.7]). We leave it to the reader to verify that this and our induction hypothesis immediately imply that

$$0 \longrightarrow \mathcal{H}^n(Y) \longrightarrow \prod \mathcal{H}^n(Y_j) \longrightarrow \prod \mathcal{H}^n(Y_i \cap Y_j)$$

is exact.

Finally, let $Z \to Y$ be a finite étale morphism of affinoids in $X_{étale}$. Put $A = O(Y)$, $B = O(Z)$ and $M = \Gamma(Y, W(\mathcal{F}))$. We use the notation $Z^n = Z \times_X Z \times \ldots \times_X Z$. It is an affinoid variety. Thus we have that $H^i(Z^n_{rigid}, \mathcal{F} \otimes O^n_Z) = 0$ for all i, n. Furthermore, the complex

$$0 \longrightarrow M \longrightarrow M \hat{\otimes}_A B \longrightarrow M \hat{\otimes}_A B \hat{\otimes}_A B \longrightarrow \ldots$$

is exact. (As the ring extension $A \subset B$ is finite we may replace the completed tensor products by usual ones and then the result is classical.) Thus the spectral sequence $\check{H}^i(\mathcal{U}, \underline{H}^j(W(\mathcal{F}))) \Rightarrow H^{i+j}(Y, W(\mathcal{F}))$ for the covering $\mathcal{U} = \{Z \to Y\}$ and induction hypothesis gives that

$$0 \longrightarrow H^n\big(Y, W(\mathcal{F})\big) \longrightarrow H^n\big(Z, W(\mathcal{F})\big) \longrightarrow H^n\big(Z \times_Y Z, W(\mathcal{F})\big)$$

is exact. We have won. □

Corollary 3.2.6 *Suppose the homomorphism $A \to B$ of affinoid algebras defines a surjective étale morphism of affinoids. For any finite A-module M the complex*

$$0 \longrightarrow M \longrightarrow M \hat{\otimes}_A B \longrightarrow M \hat{\otimes}_A B \hat{\otimes}_A B \longrightarrow \ldots$$

is exact.

3.3 Étale points and stalks

Let us define an étale point of the analytic variety X. An *étale point* e above the analytic point a of X is a separable closure $F_a \subset \mathcal{H}_e$ of F_a. We will always denote by F_e the completion of $\mathcal{H}_e$. Note that the field F_e is algebraically closed (see [BGR, 3.4.1/6]). Therefore an étale point e over a also corresponds to an algebraically closed complete extension $F_a \subset F_e$ such that the algebraic closure of F_a lies dense in F_e. The group $\mathrm{Gal}(\mathcal{H}_e/F_a)$ is equal to the group of continuous F_a-isomorphisms $F_e \to F_e$; this pro-finite group will be denoted $\mathcal{G}_e$.

An *étale neighborhood* of e is a triple (Y, b, ϕ), where Y is a variety étale over X, the analytic point b of Y maps to a and $\phi : F_b \to F_e$ is an F_a-embedding. A morphism $(Y, b, \phi) \to (Y', b', \phi')$ is a morphism $g : Y \to Y'$ over X such that $g(b) = b'$ and $\phi' = \phi \circ g^*$. Two étale neighborhoods (Y_1, b_1, ϕ_1) and (Y_2, b_2, ϕ_2) are dominated by a third one: take $Y = Y_1 \times_X Y_2$, take the point b in Y corresponding to some factor of $F_{b_1} \otimes_{F_a} F_{b_2}$ and $\phi = \phi_1 \otimes \phi_2$. In this way we see that the category of all étale neighborhoods of e give a filtered system.

The *stalk* S_e of a sheaf S on $X_{\acute{e}tale}$ at the étale point e is defined by the formula:

$$S_e := \varinjlim_{(Y,b,\phi)} S(Y)$$

The limit is take over the category of étale neighborhoods of e. If e' (given by $F_a \subset F_{e'}$) is another étale point lying over a, we get by choosing a continuous isomorphism $\psi : F_e \to F_{e'}$ a functor $(Y, b, \phi) \mapsto (Y, b, \psi \circ \phi)$ of the category of étale neighborhoods of e to the category of étale neighborhoods of e'. This gives an isomorphism of stalks

$$\begin{array}{ccc} S_e := \varinjlim_{(Y,b,\phi)} S(Y) & \xrightarrow{\psi^*} & S_{e'} := \varinjlim_{(Y,b,\phi)} S(Y) \\ s \in S(Y) \text{ w.r.t. } (Y,b,\phi) & \longmapsto & s \in S(Y) \text{ w.r.t. } (Y,b,\psi\circ\phi) \end{array}$$

In particular we get an action of $\mathcal{G}_e$ on the stalk functor $S \mapsto S_e$. It is clear that this action is continuous (with the discrete topology on S_e), since any ϕ is stabilized by an open subgroup of $\mathcal{G}_e$.

Let us construct some étale neighborhoods of e. Take a finite Galois extension $F_a \subset L$, say with group G, contained in F_e. By Remark 2.1.2 we can find an affinoid neighborhood U of a in X and a finite étale morphism $g : V \to U$, such that G acts on V over U, $g^{-1}(\{a\}) = \{v\}$ and $F_v \cong L$ (G-equivariant). We may also assume that V and U are connected. It is clear that $(V, v, F_v \to L \subset F_e)$ is an étale neighborhood of e. We claim that these étale neighborhoods are cofinal in the system of all étale neighborhoods of e.

Indeed, given an arbitrary (Y, b, ϕ) take L such that it contains $\phi(F_b)$. The fibre product $Y \times_X V$ contains a point c with $pr_1(c) = b$, $pr_2(c) = v$ and $F_b \to F_c \cong F_v \cong L \subset F_e$ equals ϕ. Using Lemma 3.1.5 we see that there is a commutative diagram

$$\begin{array}{ccc} V' & \longrightarrow & Y \\ \downarrow & & \uparrow \\ V & \longleftarrow & Y \times_X V \end{array}$$

where $V' \subset V$ is an affinoid subdomain containing v. By the Key Lemma we can find a smaller affinoid neighborhood U' of a in X such that $g^{-1}(U') \subset V'$. It is clear that $(g^{-1}(U'), v, F_v \to L \subset F_e)$ is of the form described above and dominates (Y, b, ϕ).

Lemma 3.3.1 *In the situation above.*

1. *The association $S \mapsto S_e$ is an exact functor of the category of étale sheaves on X to the category of continuous $\mathcal{G}_e$-sets.*

2. *For any étale neighborhood (Y, b, ϕ) we have*

$$\left(S|_{Y_{rigid}}\right)_b = \left((r_{Y/X})_* S\right)_b = H^0(\mathrm{Gal}(\mathcal{H}_e/\phi(F_b)), S_e).$$

In particular $S_a = (r_* S)_a = H^0(\mathcal{G}_e, S_e)$.

3. *The cohomological dimension $\mathrm{cd}(\mathcal{G}_e)$ of the pro-finite group $\mathcal{G}_e = \mathrm{Gal}(\mathcal{H}_e/F_a)$ is less than or equal to $\dim X + \mathrm{cd}(\mathrm{Gal}(k^{sep}/k))$.*

Proof. (1) We have to show that a surjection of étale sheaves $S \to T$ induces a surjection $S_e \to T_e$. Take an element $t \in T(Y)$ for some étale neighborhood (Y, b, ϕ). There is some étale covering $\{Z_i \to Y\}$ of Y and elements $s_i \in S(Z_i)$ such that s_i maps to the element $t|_{Z_i}$ in $T(Z_i)$. By Lemma 3.2.1 the family $\{(Z_i)_b \to b\}$ is a covering, hence there is some i_0 and analytic point $b_{i_0} \in Z_{i_0}$ mapping to b. Thus (Z_{i_0}, b_{i_0}, χ) is a neighborhood of e (here χ is some extension of ϕ) and s_{i_0} gives an element of S_e lifting t. (Another proof follows from the result $S_e \leftrightarrow i_a^* S$ below.)

(2) We only do the case $X = Y$, $a = b$. Take an element $s \in S_e$, which is fixed by the group $\mathcal{G}_e$. By our results above we may assume that $s \in S(V)$ for some special neighborhood $(V, v, F_v \to L \subset F_e)$ constructed above. Our assumption is that s is stable under the action of G acting on $S(V)$ via its action on V. If we show that $V \times_U V = V \times_X V$ is isomorphic to $V \times G$ then the sheaf property of S will imply that s comes from a unique section of S over U and hence we will be done. However, this again is a consequence of Remark 2.1.2 at least after shrinking U a bit.

(3) We use the notations of Lemma 2.1.1. By [S64, II 4.1] we may replace F_a by K, since this can at most increase the cohomological dimension. The field K is the completion of $k(\underline{t}) = k(t_1, \ldots, t_d)$ for some valuation and some $d \leq \dim X$. But then the group $\mathrm{Gal}(K^{sep}/K)$ is a closed subgroup of the group $\mathrm{Gal}(k(\underline{t})^{sep}/k(\underline{t}))$. We conclude by [S64, I Proposition 14, II Proposition 11]. $\square$

There is a more canonical way to understand the étale stalks S_e. Consider the general morphism

$$i_a : a = \mathrm{Spm}(F_a) \longrightarrow X.$$

It is clear that the category of sheaves on $a_{étale}$ is equivalent to the category of discrete $\mathcal{G}_e$-sets. (Compare [M80, II 1.9].) Therefore i_a^* is a functor of sheaves on $X_{étale}$ to the category of discrete $\mathrm{Gal} - \mathrm{cont}(F_e/F_a)$-sets. This functor is precisely our functor $S \mapsto S_e$. The functor $(i_a)_*$ has the following description: if M is a set with a continuous $\mathcal{G}_e$-action, then

$$\Gamma(Y, (i_a)_* M) = \prod_{b \in Y_a} \left(\mathrm{Hom}_{F_a}(F_b, F_e) \times M\right)^{\mathcal{G}_e}$$

We leave it to the reader as a nice exercise that this functor is exact. It follows from the yoga of adjoint functors that $S \mapsto (i_a)^* S = S_e$ transforms injective sheaves into injective $\mathcal{G}_e$-modules.

COROLLARY 3.3.2 *There are canonical isomorphisms*

$$\left(R^q r_* S\right)_a \cong H^q(\mathcal{G}_e, S_e).$$

Proof. For $q = 0$ this is the lemma above. It follows for general q by the usual argument using that if S is injective then both sides are zero. (See above.) $\square$

As in the rigid case we do not have enough étale points to separate étale sheaves. To overcome this difficulty we introduce the *étale overconvergent* sheaves: A sheaf S on $X_{étale}$ is said to be *(étale) overconvergent* if $S|_{Y_{rigid}}$ is overconvergent for all Y in $X_{étale}$. Before we can prove interesting properties of these sheaves we need some technical preparations; these will be done in the next section.

3.4 Étale overconvergent sheaves on affinoids

In this section X will be an affinoid variety. Let $f : Y \to X$ be a morphism with Y affinoid and let b be an analytic point of Y. We will say that Y is a *wide neighborhood* of b over X if there exists an affinoid generating system $f_1, \ldots, f_n$ of $O(Y)$ over $O(X)$ such that $|f_i|_b < 1$ for all $i = 1, \ldots, n$. Note that this agrees with our definition in §3.1 in the case that f is an open immersion.

Next we define the notion of relative compactness over X. Let us take a quasi-compact analytic variety Z over X and a quasi-compact open subvariety $Y \subset Z$. We say that Y is *relatively compact* in Z over X, or that Z is a *wide neighborhood* of Y over X, if for any analytic point b of Y there is an affinoid neighborhood $V \subset Z$ of b which is a wide neighborhood of b over X. Notation: $Y \subset\subset_X Z$. Remark that if $Y \subset\subset_Z Y'$ in this situation then also $Y \subset\subset_X Y'$. We note that if both Y and Z are affinoid then this agrees with the definition of [BGR, p. 394] (proof same as proof of [S93, Proposition 23], see also [B90, §2.5]).

Suppose that $f : Y \to X$ is an étale morphism with Y quasi-compact. We want to construct wide neighborhoods of f. We only do this in the case that f is an étale morphism of affinoids. Thus Y is affinoid and $O(Y)$ has a presentation:

$$O(Y) = O(X)\langle T_1, \ldots, T_n\rangle/(G_1, \ldots, G_n)$$

such that $\Delta = \det(\partial G_i/\partial T_j)$ generates the unit ideal of $O(Y)$. A fundamental property of special étale morphisms is that we may always choose this presentation such that $G_1, \ldots, G_n \in O(X)[T_1, \ldots, T_n]$. This follows immediately from the proposition below; in it we use $|R|$ for the supremum norm of an element $R \in O(X)\langle T_1, \ldots, T_n\rangle$.

Lemma 3.4.1 *In the situation above there exists an $\epsilon > 0$ such that if we take any $R_1, \ldots, R_n \in O(X)\langle T_1, \ldots, T_n\rangle$ with $|R_i| < \epsilon$ then we have:*

1. *The affinoid algebra $O(X)\langle T_1, \ldots, T_n\rangle/(G_1 + R_1, \ldots, G_n + R_n)$ defines a special étale morphism $f' : Y' \to X$.*

2. *There exists an isomorphism $Y \cong Y'$ of analytic varieties over X.*

Proof. Let us write $\Delta + R$ for the determinant of the matrix $\big(\partial(G_i + R_i)/\partial T_j\big)$. It is clear that if the R_i have small norm then R has small norm. Since Δ, G_i generate the unit ideal of $O(X)\langle T_1, \ldots, T_n\rangle$ it follows that $\Delta + R, G_i + R_i$ also generate the unit ideal if $|R_i|$ is small enough. This proves (1).

We claim there exists for any positive $\delta < 1$ an $\epsilon > 0$ such that for any affinoid $O(X)$-algebra A the following holds: If there are $a_1, ..., a_n \in A$ with all $|a_i| \le 1$ and all $|G_i(a_1, ..., a_n)| < \epsilon$, then there are $b_1, \ldots, b_n \in A$ such that all $|a_i - b_i| < \delta$ and all $G_i(b_1, ..., b_n) = 0$.

We suppose given $a_1, ..., a_n \in A$ with all $|a_i| \le 1$ and all $|G_i(a_1, ..., a_n)| < \epsilon$, the size of ϵ will be determined later. For an element $b = (b_1, ..., b_n) \in A^n$ we write $\|b\| = \max|b_i|$. Further $G = (G_1, ..., G_n)$ is seen as a map from $\{b \in A^n; \|b\| \le 1\}$ to A^n. Let $\partial G/\partial T$ denote the Jacobian matrix of G. Note that $|(\partial G/\partial T)|$ is bounded from below away from zero on Y, hence is bounded from below by $\eta > 0$ in a neighborhood of the form $|G_i| \le \epsilon_0$, some $\epsilon_0 > 0$. We apply Newton's method; consider the map $Z : b \mapsto b - (\partial G/\partial T(b))^{-1}G(b)$. By the remark above, and by considering a power series expansion of the map G, we see that for δ small enough (so that the quadratic and higher order terms of the power series are negligible) and $\epsilon < \eta\delta$ (for the constant terms) this defines a selfmap of the set $S := \{b \in A^n; \|b-a\| \leq \delta\}$. Moreover it is then clear that the map $Z : S \to S$ is a contraction. The fixed point b of the contraction satisfies $G(b) = 0$.

We apply this claim to $A = O(X)\langle T_1, ..., T_n\rangle/(G_1 + R_1, ..., G_n + R_n)$ with the $|R_i| < \epsilon$ and where a_i, $i = 1, ..., n$ is the image of T_i in A. There results a morphism of affinoid $O(X)$-algebras $\alpha : O(X)\langle T_1, ..., T_n\rangle/(G_1, ..., G_n) \to O(X)\langle T_1, ..., T_n\rangle/(G_1 + R_1, ..., G_n + R_n)$, with $\alpha(T_i)$ close to T_i. We can do the same in the other direction to get $\beta : O(X)\langle T_1, ..., T_n\rangle/(G_1 + R_1, ..., G_n + R_n) \to O(X)\langle T_1, ..., T_n\rangle/(G_1, ..., G_n)$, with $\beta(T_i)$ close to T_i. The composition is an endomorphism of $O(Y) = O(X)\langle T_1, ..., T_n\rangle/(G_1, ..., G_n)$ as an $O(X)$-algebra which is close to the identity. It follows that this must be the identity by looking at the graph of it in the fibre product $Y \times_X Y$, where the diagonal is a union of connected components. □

Let us take an étale morphism of affinoids $f : Y \to X$ and take a presentation $O(Y) = O(X)\langle T_1, \ldots, T_n\rangle/(G_1, \ldots, G_n)$ with $G_i \in O(X)[T_1, \ldots, T_n]$. The functional determinant of this presentation $\Delta = \det(\partial G_i/\partial T_j)$ is viewed as a function on $X \times \mathbb{A}^{N,an}$. We define a morphism $f(r) : Y(r) \to X$ for $r \in \sqrt{|k^*|}, r > 1$ as follows:

$$Y(r) = \{(x, t_1, \ldots, t_n) \in X \times \mathbb{A}^{n,\mathrm{an}};\ |t_i| \leq r \text{ and } G_i(x, t_1, \ldots, t_n) = 0\}$$

We claim that if our r is close to 1 then $f(r)$ will again be special étale. To see this we note that there is a presentation:

$$O(Y(r)) = O(X)\langle S_1, \ldots, S_n, T'_1, \ldots, T'_n\rangle/\big((T'_i)^m - \pi^{-m+1}S_i,\ G_i(\pi T'_1, \ldots, \pi T'_n)\big)$$

Here π is an element of k with $r^m = |\pi|$ and the relation of the coordinates is that $S_i = \pi^{-1}T_i^m$ and $T'_i = \pi^{-1}T_i$. The functional determinant of this presentation is $\pi^{-mn}\Delta|_{Y(r)}$. It is therefore clear that $Y(r) \to X$ is special étale as soon as $\Delta \in \Gamma(Y(r), O_{Y(r)})$ is invertible; this will be the case for r sufficiently close to 1 (the zero locus of Δ lies a positive distance away from Y!). Finally it is clear that $Y \subset\subset_X Y(r)$. We will use the notation $Y(r)$ even if no explicit presentation of $O(Y)$ is given, the number r will always denote an element of $\sqrt{|k^*|}$ bigger than 1 and small enough.

At this point we want to prove the analog of Lemma 2.3.1 in this situation. However, we need to be careful since any étale $U \to X$ has many *non-separated* wide neighborhoods, so the wide neighborhoods $Y \subset Y(r)$ can only be cofinal in the system of separated wide neighborhoods. Although this is in fact true, we restrict ourselves to the case of affinoid varieties.

Lemma 3.4.2 *With notations as above.*

1. *Let W, $U \subset\subset_X V$ be affinoid varieties étale over X and $f : V \to W$ a morphism over X. If V' is a wide neighborhood of U in V, then $f(V')$ is a wide neighborhood of $f(U)$ in W. For varying V' these give a cofinal system of wide neighborhoods of $f(U)$. If $f|_U$ is an isomorphism $U \to f(U)$ then for some $U \subset\subset_V V'$ f induces an isomorphism $V' \to f(V')$.*

2. *If $U \subset\subset_X V$ are affinoid varieties étale over X and $\varphi : Y \to U$ is a morphism then for some $r > 1$ there exists an extension $\varphi(r) : Y(r) \to V$ of φ. This*

extension is unique if r is sufficiently close to 1. *In particular the $Y(r)$, $r > 1$ form a cofinal system of affinoid wide neighborhoods of Y over X.*

3. *Suppose that $Y_i \to X$, $i = 1, \ldots, n$ is étale and Y_i affinoid. We have:*

$$Y_1(r) \times_X \ldots \times_X Y_n(r) = \big(Y_1 \times_X \ldots \times_X Y_m\big)(r)$$

Proof. Suppose we show that if b is an analytic point of U (with image a in W), then there is a wide neighborhood V_b of b in V' such that $f(V_b)$ is a wide neighborhood of a in W. This immediately implies the first statement of (1). The statement on co-finality then follows immediately by letting V' run through the inverse images of such a system of wide neighborhoods of $f(U)$. Let Z denote the complement of the diagonal in $V \times_W V$; it is a union of connected components and hence affinoid. Under the last assumption of (1) we have $pr_1(Z) \cap U = \emptyset$. Thus for some wide neighborhood V' of V we have $V' \times_W V' \cong V'$ and hence it will map isomorphically onto $f(V')$ (compare with Lemma 3.1.5).

Let us construct the neighborhood V_b. By assumption there exists an affinoid generating system $f_1, \ldots, f_r$ of $O(V)$ over $O(X)$ such that $|f_i|_b < 1$. Take a wide neighborhood V_b of b such that $||f_i||_{V_b} < 1$. By Lemma 3.1.6 we can find a wide neighborhood W_a of a in W such that $f^{-1}(W_a) = \bigcup V_i$ as in 3.1.6.1 and $V_1 \subset V_b$ is a wide neighborhood of b. Thus we may replace W by W_a and V by V_1 and assume that $||f_i||_V < 1$ for a generating system $f_1, \ldots, f_r$ of $O(V)$ over $O(W)$. But then V is finite over W [BGR, 9.6.3/6], so we get the existence of V_b by Lemma 3.1.5.

To prove (2) we apply (1) to the projection $Y(r) \times_X V \to Y(r)$. We see that there exists a wide neighborhood of the graph $\Gamma_\varphi \subset Y \times_X U \subset Y(r) \times_X V$ which maps isomorphically onto a wide neighborhood Y' of Y in $Y(r)$. Hence we can find some r', $1 < r' < r$ such that $Y(r') \subset Y'$ (see Lemma 2.3.1). This r' works.

The proof of (3) is formal. □

The lemma above allows us to work with étale morphisms of affinoids only. Therefore we introduce the *special étale site* of X. (Recall that X is affinoid.) It is denoted $X^{sp}_{\acute{e}tale}$ and is defined as follows:

1. Objects are étale morphisms $Y \to X$ with Y affinoid, i.e. special étale ones.

2. Morphisms are morphisms of analytic spaces over X.

3. Coverings are those finite families of morphisms $\{f_i : Y_i \to Y\}$ such that $\bigcup f_i(Y_i) = Y$.

It follows from the remarks made after the definition of special étale morphisms and Lemma 3.2.1 that this is indeed a site. It is functorial with respect to (general) morphisms of affinoids: $Z \to X$ induces a morphism of sites $Z^{sp}_{\acute{e}tale} \to X^{sp}_{\acute{e}tale}$.

The morphism of site $X_{\acute{e}tale} \to X^{sp}_{\acute{e}tale}$, given by the inclusion functor, induces an equivalence of associated topoi. (Use 3.1.2.)

LEMMA 3.4.3 *The topos of sheaves on $X^{sp}_{\acute{e}tale}$ is coherent (see [SGA 4, Exposé VI]). In particular, étale cohomology of étale Abelian sheaves on X commutes with filtered direct limits, see [Ibid, 5.2].*

Proof. In the site $X^{sp}_{étale}$ finite fibered products are representable and any object is quasi-compact (see [Ibid, Definition 1.1]). Since it also has a final object we are done, see [Ibid, 2.4.1]. $\square$

The following lemma characterizes overconvergent étale sheaves on X in terms of the site $X^{sp}_{étale}$.

LEMMA 3.4.4 *A sheaf S on $X^{sp}_{étale}$ corresponds to an overconvergent sheaf on $X_{étale}$ if and only if the natural map*

$$\varinjlim_{r>1} S(Y(r)) \longrightarrow S(Y)$$

is an isomorphism for all $Y \to X$ affinoid étale.

Proof. Suppose that S is overconvergent. In this case $S|_{Y(r_0)}$ is overconvergent for some $r_0 > 1$. Hence 3.4.4 is an isomorphism since $Y \subset\subset_{Y(r_0)} Y(r)$, $1 < r < r_0$ forms a cofinal system of wide neighborhoods of Y in $Y(r_0)$.

Conversely suppose 3.4.4 is an isomorphism always. Let $Y \to X$ be an étale morphism of affinoids. We have to show that $S|_Y$ is rigid overconvergent. Let $U \subset Y$ be a rational subset of Y. Choose some $r_0 > 1$ such that $Y \subset\subset_X Y(r_0)$ and $Y(r_0)$ is étale over X. Denote for $r > 1$ by $U(r)$ the wide neighborhood of U in $Y(r_0)$ defined in §3.3. It follows easily from Lemma 3.4.2 that these wide neighborhoods $U \subset\subset_X U(r)$ form a cofinal system of affinoid étale wide neighborhoods of U over X. Hence our assumption gives the isomorphism $\lim S(U(r)) = S(U)$.

However, we want to show that the map $\lim S(U(r) \cap Y) \to S(U)$ is an isomorphism. It is clear from the above that this is a surjection. Using for all the rational subdomains $U(r) \cap Y$ of Y the bijectivity of the map $S\big((U(r) \cap Y)(r')\big) \to S(U(r) \cap Y)$, it also follows that the map is injective. This proves our lemma. $\square$

We will say that a presheaf on $X^{sp}_{étale}$ is overconvergent if the map 3.4.4 is always an isomorphism. At this point we introduce a useful method to produce overconvergent (pre)-sheaves on $X^{sp}_{étale}$. Let S be a presheaf on $X^{sp}_{étale}$. We define the presheaf cS on $X^{sp}_{étale}$ as follows:

$$\Gamma(Y, cS) = cS(Y) := \varinjlim_{r>1} \Gamma(Y(r), S)$$

for any Y in $X^{sp}_{étale}$. Note that Lemma 3.4.2 implies that this is independent of the chosen representation of $O(Y)$ over $O(X)$ and that cS is indeed a presheaf. The construction c is a functor, there is an obvious functorial arrow $cS \to S$ and the map $ccS \to cS$ is an isomorphism. Hence the presheaf cS is overconvergent. It is therefore clear that the functor c is a right adjoint of the inclusion functor: overconvergent presheaves on $X^{sp}_{étale} \to$ presheaves on $X^{sp}_{étale}$.

LEMMA 3.4.5 *With notations as above.*

1. *If S is a sheaf then cS is a (overconvergent) sheaf. The functor $S \mapsto cS$ is a right adjoint of the inclusion functor: overconvergent sheaves on $X \to$ sheaves on X. The functor $S \mapsto cS$ is left exact on the category of sheaves on $X^{sp}_{étale}$.*

2. *If $\mathcal{J}$ is an injective sheaf on $X^{sp}_{étale}$ and $\mathcal{U} = \{Y_i \to Y\}$ is a covering in $X^{sp}_{étale}$ then*
$$\check{H}^i(\mathcal{U}, c\mathcal{J}) = 0 \; \forall i > 0.$$
It follows that $c\mathcal{J}$ is a flabby sheaf on $X^{sp}_{étale}$.

3. *Any overconvergent sheaf can be embedded into a sheaf of the form $c\mathcal{J}$ with $\mathcal{J}$ injective.*

Proof. Let $\mathcal{U} = \{g_i : Y_i \to Y\}$ be a covering in $X^{sp}_{étale}$. We want to show the following: there exists a set of coverings $\mathcal{U}_\alpha$ such that for any (pre)sheaf S we have a canonical isomorphism:

$$(*) \qquad \varinjlim_{\alpha} \check{\mathcal{C}}(\mathcal{U}_\alpha, S) \cong \check{\mathcal{C}}(\mathcal{U}, cS)$$

(The symbols $\check{\mathcal{C}}$ denote Čech-complexes.) It is clear that this will prove that cS is a sheaf if S is a sheaf and it will prove the second assertion of the lemma. We leave the adjointness property to the reader, as well as the third part of the lemma.

We will only prove the above in the case that the covering $\mathcal{U} = \{g : Z \to Y\}$ is given by one map. Since for an arbitrary (and hence finite) covering in $X^{sp}_{étale}$ there exists a covering consisting of a single morphism giving an isomorphic Čech-Complex there is no loss of generality. To do this we fix $r_0 > 1$ small enough such that $Y(r_0)$ is étale over X and a $r_1 > 1$ small enough such that g extends to $\tilde{g} : Z(r_1) \to Y(r_0)$. Next, for any r_2, $1 < r_2 < r_1$, we choose a $r_3(r_2)$, $1 < r_3(r_2) < r_0$ such that $Y(r_3(r_2)) \subset \tilde{g}(Z(r_2))$. This is possible by Lemma 3.4.2, which also implies that we may choose $r_3(r_2)$ to be a decreasing function of r_2, decreasing to 1 in fact.

We put $Z_{r_2} = Z(r_2) \cap \tilde{g}^{-1}\big(Y(r_3(r_2))\big)$. The coverings we are looking for are $\mathcal{U}_{r_2} = \{Z_{r_2} \to Y(r_3(r_2))\}$. Note that there are commutative diagrams for $1 < r_2' < r_2$:

$$\begin{array}{ccccc} Z & \longrightarrow & Z_{r_2'} & \longrightarrow & Z_{r_2} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y(r_3(r_2')) & \longrightarrow & Y(r_3(r_2)) \end{array}$$

Hence we get the map (*). To show that (*) is an isomorphism we only need to prove that
$$Z \times_Y \ldots \times_Y Z \subset\subset_X Z_{r_2} \times_{Y(r_3(r_2))} \ldots \times_{Y(r_3(r_2))} Z_{r_2}$$
forms a cofinal system of wide neighborhoods of $Z \times_Y \ldots \times_Y Z$ as r_2 decreases to 1. This is clear from the following three facts: 1) $Z_{r_2} \times_X \ldots \times_X Z_{r_2}$ forms a cofinal system of wide neighborhoods of $Z \times_X \ldots \times_X Z$ (see Lemma 3.4.2), 2) $Z \times_Y \ldots \times_Y Z$ is a union of connected components of $Z \times_X \ldots \times_X Z$ and 3) the intersection of $Z_{r_2} \times_{Y(r_3(r_2))} \ldots \times_{Y(r_3(r_2))} Z_{r_2}$ with $Z \times_X \ldots \times_X Z$ is $Z \times_Y \ldots \times_Y Z$. □

LEMMA 3.4.6 *(Properties of overconvergent sheaves on $X^{sp}_{étale}$.) In this lemma all (pre-)sheaves are (pre-)sheaves of Abelian groups.*

1. *The sheaf associated to an overconvergent presheaf is overconvergent.*

2. *For any overconvergent sheaf S the presheaves $Y \mapsto H^q(Y, S)$ are overconvergent; for any q the rigid sheaf $R^q(r_{X/Y})_*S$ is overconvergent on Y, in particular the sheaves $R^q r_* S$ are overconvergent on X_{rigid}.*

3. *The category of overconvergent sheaves is an exact subcategory of the category of all sheaves on $X^{sp}_{étale}$.*

4. *If $f : Z \to X$ is a general morphism of affinoids then f^* and f_* preserve overconvergent sheaves. The same holds for $R^q f_*$ for any q.*

5. *If $\{f_i : X_i \to X\}$ is a special étale covering of X then a sheaf on $X^{sp}_{étale}$ is overconvergent if and only if each $f_i^*(S)$ is overconvergent.*

6. *An overconvergent sheaf S is zero if and only if its étale stalks S_e are zero for all étale points e of X.*

Proof. 1) Let $a(S)$ denote the sheaf associated to S. The map $S \to a(S)$ factors as $S \to ca(S) \to a(S)$ since S is overconvergent. By the universal property of aS we get a section $a(S) \to ca(S)$ (as $ca(S)$ is a sheaf). It follows that $a(S)$ is a direct summand of the overconvergent sheaf $ca(S)$ and hence overconvergent.

2) Embed S in an overconvergent flabby sheaf as in the preceding lemma: $0 \to S \to c\mathcal{J}$. The quotient presheaf is overconvergent hence so is the quotient sheaf Q by 1). For any affinoid Y étale over X we get the exact sequence

$$0 \longrightarrow H^0(Y,S) \longrightarrow H^0(Y,c\mathcal{J}) \longrightarrow H^0(Y,Q) \longrightarrow H^1(Y,S) \longrightarrow 0$$

and isomorphisms $H^q(Y,S) \cong H^{q-1}(Y,Q)$ for $q > 1$. It follows immediately that the presheaf $Y \mapsto H^1(Y,S)$ is overconvergent and the usual induction on q does the rest.

3) Follows from 1) and 2) and the results on rigid overconvergent sheaves.

4) Remark that if $Y \to X$ is affinoid étale then $Y(r) \times_X Z \cong \big(Y \times_X Z\big)(r)$. The rest of the argument is completely analogous to the proof of Lemma 2.3.2 part 4.

5) Same argument as in the rigid case.

6) This is immediate from Lemma 3.3.1 combined with the result for rigid overconvergent sheaves. □

3.5 Étale overconvergent sheaves on general X

Let X be an arbitrary analytic variety over k. Recall that a sheaf S on $X_{étale}$ is overconvergent if $S|_Y$ is rigid overconvergent for any Y étale over X. It is clear from Lemma 3.4.6 that this condition is local in the étale topology on X.

There are now a number of easy consequences of the above which we list here:

1. If $f : Z \to X$ is an arbitrary (general) morphism then f^* preserves overconvergent sheaves.

2. If $f : Z \to X$ is quasi-compact then $R^q f_*$ preserves overconvergent sheaves. (Compare proof of Proposition 2.4.1.)

3. For any overconvergent sheaf S on X the rigid sheaves $R^q(r_{Y/X})_* S$ (in particular $R^q r_* S$) are overconvergent.

Finally, we have the following result.

Proposition 3.5.1 *If X is paracompact and S is an overconvergent torsion sheaf on $X_{étale}$ then $H^q(X,S) = 0$ for all $q > 2\dim X + \mathrm{cd}(k)$, where $\mathrm{cd}(k)$ denotes the cohomological dimension of k.*

Proof. Consider the spectral sequence with E_2-term $H^i(X_{rigid}, R^j r_* S)$ converging to $H^{i+j}(X,S)$. By Corollary 3.3.2 and Lemma 3.4.6 we get that the sheaves $R^j r_* S$ are zero for $j > \dim X + \mathrm{cd}(k)$. Hence we get the result from Corollary 2.5.10. □

3.6 Galois action on cohomology

Let us take a separable closure k^{sep} of k and let us denote by K the completion of k^{sep} with respect to the absolute value $||$. Note that K is algebraically closed (see e.g. [BGR, 3.4]). We remark that the group $\mathcal{G} := \mathrm{Gal}(k^{sep}/k)$ can be identified with the group of continuous automorphisms of K over k.

Take an analytic variety X over k and an étale sheaf S on it. Consider the variety $X\hat{\otimes}K$ over K and the general morphism $\alpha : X\hat{\otimes}K \to X$. For any $\sigma \in \mathcal{G}$ there is an obvious general morphism $\varphi_\sigma : X\hat{\otimes}K \to X\hat{\otimes}K$. This is not a morphism of analytic varieties over K unless $\sigma = \mathrm{id}_K$; it lies over the continuous field homomorphism $\sigma : K \to K$. Since it is clear that $\alpha = \alpha \circ \varphi_\sigma$, we get an isomorphism $\alpha^*(S) \cong (\alpha \circ \varphi_\sigma)^*(S) \cong (\varphi_\sigma)^*\alpha^*(S)$. Thus we get

$$\varphi_\sigma^* : H^i(X\hat{\otimes}K, \alpha^* S) \longrightarrow H^i(X\hat{\otimes}K, \alpha^* S).$$

This defines an action of $\mathcal{G}$ on $H^i(X\hat{\otimes}K, \alpha^* S)$.

Another way to get a $\mathcal{G}$-module is to consider the morphism

$$p : X \to Sp(k)$$

As was noted above the sheaves $R^i p_* S$ correspond to $\mathcal{G}$-modules $(R^i p_* S)_e$. It will be shown below that these two Galois modules agree in the case that X is quasi-compact.

3.7 Étale base change

Let $f : Y \to X$ be a quasi-compact morphism of analytic varieties over k and S an étale sheaf on Y. The étale base change theorem compares the cohomology of S on the étale fibre Y_e with the étale stalks at e of the sheaves $R^q f_* S$. The étale fibre is just defined as $Y_a \hat{\otimes} F_e$, or as the fibre product of the general morphism $Sp(F_e) \to X$ with the morphism $Y \to X$. The result will be an isomorphism of $\mathcal{G}_e$-modules. As in the rigid case the theorem will follow formally from a lemma describing the étale site of the fibre Y_e in the affinoid case.

Therefore we suppose that $f : Y \to X$ is a morphism of affinoids over k and we fix an étale point e lying over the analytic point a of X. For any étale neighborhood (U, b, ϕ) of e with U affinoid we can consider the special étale site of $Y_U := Y \times_X U$. Using ϕ we can see e as an étale point of U lying over b and then it is clear that $(Y_U)_e = Sp(F_e) \times_U Y_U \cong Sp(F_e) \times_X Y = Y_e$. Thus a general morphism $Y_e \to Y_U$ which gives rise to the functor

$$\begin{array}{ccc} (Y_U)^{sp}_{\acute{e}tale} & \longrightarrow & (Y_e)^{sp}_{\acute{e}tale} \\ V \to Y_U & \longmapsto & V_e \to Y_e. \end{array}$$

On the other hand, if the affinoid étale neighborhood (U', b', ϕ') dominates (U, b, ϕ), there is clearly a functor $(Y_U)^{sp}_{\acute{e}tale} \to (Y_{U'})^{sp}_{\acute{e}tale}$ compatible with the functor described above.

Lemma 3.7.1 *The functors above define an equivalence of sites:*

$$\varinjlim_{(U,b,\phi)} \left(Y_U\right)^{sp}_{\acute{e}tale} \cong \left(Y_e\right)^{sp}_{\acute{e}tale}.$$

Proof. Note that all functors defined above underly morphisms of sites in the reverse directions. The statement follows from the following three assertions:

1. For any étale morphism $V \to Y_e$ with V affinoid there exists an affinoid étale neighborhood (U, b, ϕ) and an étale $W \to Y_U$ morphism of affinoids such that $W_e \cong V$ as varieties over Y_e.

2. Given two étale morphisms $W_i \to Y_U$, W_i affinoid ($i = 1, 2$ and U as above) and a morphism $\psi_e : W_{1,e} \to W_{2,e}$ there exists an affinoid étale neighborhood (U', b', ϕ') dominating (U, b, ϕ) and a morphism $\psi_{U'} : W_{1,U'} \to W_{2,U'}$ such that $\psi_{U',e} = \psi_e$. This $\psi_{U'}$ is unique if U' is small enough.

3. If $\{g_i : W_i \to W\}$ is a finite set of morphisms in $(Y_U)^{sp}_{\acute{e}tale}$ and $\{W_{i,e} \to W_e\}$ is an étale covering then $\{W_{i,U'} \to W_{U'}\}$ is an étale covering if U' is small enough.

Let us prove 1). By definition $O(V)$ has a presentation

$$O(V) \cong O(Y_e)\langle T_1, \ldots, T_n\rangle/(G_1, \ldots, G_n)$$

such that $\Delta := \det(\partial G_i/\partial T_j)$ is invertible. By Lemma 3.4.1 we may suppose that the G_i are polynomials. Since $O(Y_e) = O(Y)\hat{\otimes}_{O(X)}F_e$ we can approximate the G_i by polynomials with coefficients in $O(Y)\otimes_{O(X)}L$ for some finite separable field extension $F_a \subset L \subset F_e$. By Lemma 3.4.1 we may assume $G_i \in O(Y) \otimes_{O(X)} L[T_1, \ldots, T_n]$. We can construct (U, b, ϕ) such that $\phi(F_b) \supset L$ (see 3.3); for this U we can find polynomials $P_i \in O(Y)\hat{\otimes}_{O(X)}O(U)[T_1, \ldots, T_n]$ mapping to the G_i. The function $\Delta(P) := \det(\partial P_i/\partial T_j)$ on

$$W := Sp(O(Y)\hat{\otimes}_{O(X)}O(U)\langle T_1, \ldots, T_n\rangle/(P_1, \ldots, P_n)$$

is such that its restriction to W_e is invertible. Hence, $\Delta(P)$ is invertible on W_b, hence by Lemma 2.7.2 or 2.7.1 we get that $\Delta(P)$ is invertible on W after shrinking U. This gives that $W \to Y_U$ is special étale and $W_e \cong V$ by construction.

Next we do 2). Note that the morphism $\psi_e : W_{1,e} \to W_{2,e}$ gives rise to a graph morphism $\Gamma_e : W_{1,e} \to (W_1 \times_{Y_U} W_2)_e$ and that this morphism identifies $W_{1,e}$ with a union of connected components of $(W_1 \times_{Y_U} W_2)_e$. Hence Γ_e is an étale morphism of affinoids. By 1) there exists a smaller étale neighborhood (U', b', ϕ') and an étale morphism of affinoids $\Gamma_{U'} : W \to (W_1 \times_{Y_U} W_2) \times_U U' \cong W_{1,U'} \times_{Y_{U'}} W_{2,U'}$ with $W_e \cong W_{1,e}$ and $\Gamma_{U',e} = \Gamma_e$. We replace (U, b, ϕ) by (U', b', ϕ') and hence we have $\Gamma : W \to W_1 \times_{Y_U} W_2$. Consider $p_2 = pr_2 \circ \Gamma$. By the above we have that $(p_2)_e$ is an isomorphism. It follows that $(p_2)_b$ is a bijective (on analytic points) étale morphism of affinoids and hence an isomorphism. Thus for any analytic point $c \in W_{1,b}$ we have that $p_2^{-1}(c)$ consists of one analytic point $c' \in W$ with $F_c \cong F_{c'}$. Lemma 3.1.5 implies that p_2 is an open immersion in a wide open neighborhood $W_1(c)$ of c in W_1. Finitely many $W_1(c)_b$'s cover $W_{1,b}$ and $W_b = W_{1,b}$ hence by the key lemma for the rigid case we may shrink U and get that p_2 is an isomorphism (apply the key lemma to both W and W_1). Clearly, the morphism $pr_1 \circ \Gamma \circ (p_2)^{-1} : W_1 \to W_2$ does the job.

The uniqueness follows easily from the rigid key lemma by looking at graphs as above.

Finally, if the assumptions are as in 3) then $W_e = \bigcup_i g_{i,e}(W_{i,e})$ implies $W_b = \bigcup g_{i,b}(W_{i,b})$, since formation of image commutes with arbitrary base change, see Lemma 3.1.7. Thus the statement follows from the rigid case, i.e., if U small enough then $W = \bigcup g_i(W_i)$. □

This was the hard part of the proof of the base change theorem in the étale case. We can deduce the following analog of 2.7.3.

COROLLARY 3.7.2 *Consider the general morphism $\alpha : Y_e \to Y$.*

1. *The functor α^* preserves flabby sheaves.*
2. *For any sheaf S on $Y^{sp}_{\acute{e}tale}$, any (U, b, ϕ) and any $W \to Y_U$ as above we have:*

$$\Gamma(W_e, \alpha^* S) = \varinjlim_{(U',b',\phi') \geq (U,b,\phi)} \Gamma(W \times_U U', S)$$

Proof. For any S on $Y^{sp}_{\acute{e}tale}$ we have $H^q(Y_e, \alpha^* S) = \lim H^q(Y_U, S)$, by Remark 2.5.9 and the previous lemma. The same argument gives $H^q(W_e, \alpha^* S) = \lim H^q(W_U, S)$ for W as in 2). The results of the corollary follow directly from this. □

THEOREM 3.7.3 *Let $f : Y \to X$ be a quasi-compact morphism of analytic spaces over k. Take an étale point e of X and denote by Y_e the (étale) fibre of f at e. The functors*

$$S \mapsto H^q(Y_e, \alpha^* S) \ \ \textit{resp.}\ S \mapsto (R^q f_* S)_e$$

are δ-functors of the category of Abelian sheaves on $Y_{\acute{e}tale}$ to the category of continuous $\mathcal{G}_e$-modules. These δ-functors are isomorphic.

Proof. Remark that $Y_e = Y_a \hat{\otimes} F_e$ and that $\alpha^* S$ is the pullback of $S|_{Y_{a,\acute{e}tale}}$ via the general morphism $Y_e \to Y_a$. Thus we see by 3.6 that the groups $H^q(Y_e, \alpha^* S)$ indeed have a Galois module structure. In the same way as in the proof of the rigid base change theorem it is proved that the functors under consideration form δ-functors. The maps

$$\left(R^q f_* S\right)_e \longrightarrow H^q(Y_e, \alpha^* S)$$

are defined similarly as in the proof of Theorem 2.7.4. These maps commute with Galois action since the action on both sides is defined through the action of $\mathcal{G}_e$ on F_e.

Let us prove that these maps are isomorphisms only in the case that both X and Y are affinoid. The general case then follows as it did in the rigid case. The result for $q = 0$ is just Corollary 3.7.2 part 2) with $W = Y$. The general result follows by induction on q and the fact that α^* preserves flabby sheaves. □

COROLLARY 3.7.4 *If $f : Y \to X$ is quasi-compact and has finite fibres then $R^q f_* S$ is zero for $q \geq 1$ and any overconvergent sheaf S on $Y_{\acute{e}tale}$. In particular the cohomology of S on Y is equal to the cohomology of $f_* S$ on X.*

Corollary 3.7.5 *(Hochschild-Serre spectral sequence.) Let K be a completion of a separable closure k^{sep} of k. Let $\mathcal{G} = \mathrm{Gal}(k^{sep}/k)$ denote the continuous Galois group of K over k. For any quasi-compact variety X over k and any Abelian sheaf S on $X_{étale}$ there is a spectral sequence*

$$H^i(\mathcal{G}, H^j(X\hat{\otimes}K, \alpha^* S)) \Rightarrow H^{i+j}(X, S).$$

Here $\alpha : X\hat{\otimes}K \to X$ is as in 3.6 and $H^q(\mathcal{G}, -)$ denotes continuous cohomology.

Proof. Let us write $p : X \to \mathrm{Sp}(k)$ as in 3.6. Let a denote the unique analytic point of $\mathrm{Sp}(k)$ and let e be an étale point lying over a. First we note that (if S is overconvergent)

$$H^q(X\hat{\otimes}K, \alpha^* S) \cong (R^q p_* S)_e$$

as $\mathcal{G}$-modules by the theorem above. This shows that the $\mathcal{G}$-module on the left has a continuous $\mathcal{G}$-action if it is given the discrete topology. It also proves that

$$H^0(\mathcal{G}, H^0(X\hat{\otimes}K, \alpha^* S)) = H^0(\mathcal{G}, (p_*S)_e) = (p_*S)_a = H^0(\mathrm{Sp}(k), p_*S) = H^0(X, S).$$

Here we used Lemma 3.3.2. Hence we only need to show that the functor which maps S to the Galois module $H^0(X\hat{\otimes}K, \alpha^* S)$ transforms an injective sheaf S on X into an acyclic $\mathcal{G}$-module. Since we are taking continuous cohomology we have:

$$H^q(\mathcal{G}, H^0(X\hat{\otimes}K, \alpha^* S)) = \varinjlim_{k\subset k'} H^q(\mathrm{Gal}(k'/k), H^0(X\hat{\otimes}K, \alpha^* S)^{\mathcal{G}'}),$$

where the limit runs over all finite Galois extensions $k \subset k'$ contained in K. By an argument as above this is the limit over the groups

$$H^q(\mathrm{Gal}(k'/k), H^0(X \otimes k', S)).$$

But since S is injective these groups compute the cohomology groups $H^q(X, S)$ (compare [M80, Theorem 2.20]) and these are zero for $q \geq 1$. □

4 Cohomology of varieties of dimension at most 1

In this section we suppose that the field k is algebraically closed. Let $p \geq 1$ denote the characteristic of the residue field of k. We put $p = 1$ if the residue field of k contains the field of rational numbers. Further X will denote an analytic space over k of dimension ≤ 1.

4.1 Some general results

For $n > 1$ which is not divisible by the characteristic of k, we consider the exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 0$$

of sheaves on $X_{étale}$. This sequence induces the following distinguished triangle of complexes on X_{rigid}:

$$\longrightarrow Rr_*\mu_n \longrightarrow Rr_*\mathbb{G}_m \longrightarrow Rr_*\mathbb{G}_m \longrightarrow Rr_*\mu_n[1]$$

We already know quite a lot about the homology sheaves of these complexes: We know that $R^q r_* \mu_n = 0$ for all $q \geq 2$ by Lemma 3.3.1 and Corollary 3.3.2 combined with the fact that $R^q r_* \mu_n$ are overconvergent (Lemma 3.4.6). Further we know that $R^1 r_* \mathbb{G}_m = 0$ by Corollary 3.2.4. The following result is a formal consequence of this.

LEMMA 4.1.1 *In the derived category of Abelian sheaves on X_{rigid} we have the following isomorphism*

$$Rr_* \mu_n \cong \left(O_X^* \xrightarrow{n} O_X^*\right),$$

where the first term on the right is placed in degree 0.

LEMMA 4.1.2 *Let X be connected paracompact (and still have dimension ≤ 1). We denote by $\mathbb{Z}/n\mathbb{Z}_X$ the constant sheaf with fibre $\mathbb{Z}/n\mathbb{Z}$ on $X_{étale}$, where n is prime to the characteristic of k. We have $H^0(X, \mathbb{Z}/n\mathbb{Z}_X) = \mathbb{Z}/n\mathbb{Z}$ and $H^q(X, \mathbb{Z}/n\mathbb{Z}_X) = 0$ for $q \geq 3$.*

1. *There is an exact sequence*

$$0 \longrightarrow H^1(X_{rigid}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^1(X, \mathbb{Z}/n\mathbb{Z}_X) \longrightarrow H^0(X_{rigid}, R^1 r_* \mathbb{Z}/n\mathbb{Z}_X) \to 0$$

   and we have $H^2(X, \mathbb{Z}/n\mathbb{Z}_X) = H^1(X_{rigid}, R^1 r_* \mathbb{Z}/n\mathbb{Z}_X)$.

2. *A choice of a primitive n^{th}-root of unity determines an exact sequence*

$$0 \longrightarrow O(X)^*/O(X)^{*\,n} \longrightarrow H^1(X, \mathbb{Z}/n\mathbb{Z}_X) \longrightarrow \ker(n, Pic(X)) \longrightarrow 0$$

   and an isomorphism $H^2(X, \mathbb{Z}/n\mathbb{Z}_X) = Pic(X)/nPic(X)$.

Proof. The statement on H^0 is trivial. Consider the spectral sequence with E_2-terms $H^p(X_{rigid}, R^q r_* \mathbb{Z}/n\mathbb{Z}_X)$ abutting to $H^{p+q}(X, \mathbb{Z}/n\mathbb{Z}_X)$. Clearly 1) follows since $R^q r_* \mathbb{Z}/n\mathbb{Z}_X = (0)$ for $q \geq 2$ and cohomology of rigid sheaves is zero on X in dimensions ≥ 2 by Corollary 2.5.10.

A choice of a primitive n^{th}-root of unity determines an isomorphism of sheaves $\mathbb{Z}/n\mathbb{Z}_X \cong \mu_n$. Thus statement 2) follows from the lemma above and the vanishing of rigid cohomology in degrees ≥ 2 on X. □

4.2 THE COHOMOLOGY OF $\mathbb{Z}/n\mathbb{Z}$ WITH $(n,p) = 1$

In this subsection we will determine the cohomology of X in certain cases where X is smooth and irreducible. We will use the word *curve* to denote a separated analytic variety of pure dimension 1. Recall that we are working over an algebraically closed field.

PROPOSITION 4.2.1 *Let C be a nonsingular projective curve of genus g. We compute the cohomology with values in $\mathbb{Z}/n\mathbb{Z}$ for $(n,p) = 1$ of an open subvariety X of C as follows.*

1. *If $X = C$, then $H^1(X, \mathbb{Z}/n\mathbb{Z}_X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ and $H^2(X, \mathbb{Z}/n\mathbb{Z}_X) \cong \mathbb{Z}/n\mathbb{Z}$.*
2. *If X is the complement of finitely many points $c_1, \ldots, c_a$ $(a > 0)$ in C, then $H^1(X, \mathbb{Z}/n\mathbb{Z}_X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g+a-1}$ and $H^2(X, \mathbb{Z}/n\mathbb{Z}_X) = 0$.*

3. Suppose X is $C \setminus (D_1 \cup \ldots \cup D_a)$ where the D_i are disjoint open discs in C. In this case $H^1(X, \mathbb{Z}/n\mathbb{Z}_X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g+a-1}$ and $H^2(X, \mathbb{Z}/n\mathbb{Z}_X) = 0$.

In all of these cases, for any extension of algebraically closed complete valued fields $k \subset K$, the natural map $H^q(X, \mathbb{Z}/n\mathbb{Z}_X) \to H^q(X_K, \mathbb{Z}/n\mathbb{Z}_{X_K})$ is an isomorphism.

Proof. Part 1) follows from Lemma 4.1.2 part 2) and the fact that $Pic(C)$ corresponds to the algebraic Picard group of C by GAGA. Note that the isomorphism for H^2 is given by the isomorphism $Pic(C)/nPic(C) \cong \mathbb{Z}/n\mathbb{Z}$ induced by taking degrees of line bundles on C.

Note that in case 2) the space X is the admissible increasing union $X = \bigcup X_n$ of affinoid spaces X_n as in 3). Just take $D_{i,n}$ to be smaller and smaller open discs in C with center c_i. Thus if we prove 3) then 2) will follow by considering the Cartan-Leray spectral sequence associated to the covering $X = \bigcup X_n$. (Here we also need that the maps $H^i(X_{n+1}) \to H^i(X_n)$ are isomorphisms; this follows from the proof of 3) below.)

Let us assume X is as in 3). Any line bundle on X is the restriction of a line bundle of degree zero on C. In other terms, $Pic^0(C) \to Pic(X)$ is surjective. In particular $Pic(X)$ is a divisible group and by Lemma 4.1.2 we get $H^2(X, \mathbb{Z}/n\mathbb{Z}_X) = 0$. For the calculation of the group $H^1(X, \mathbb{Z}/n\mathbb{Z}_X)$ we start with the case where X is the closed unit disk $\mathbb{B} := \{z \in k; |z| \leq 1\}$.

Now $H^1(\mathbb{B}, \mathbb{G}_m) = Pic(\mathbb{B}) = 0$ and by 4.1.2 one has $H^1(\mathbb{B}, \mathbb{Z}/n\mathbb{Z}_{\mathbb{B}}) = O(\mathbb{B})^*/O(\mathbb{B})^{*n}$. The invertible functions on $\mathbb{B}$ are of the form $\lambda(1+f)$ with $\lambda \in k^*$ and $f \in O(\mathbb{B})$ has a norm < 1. The condition on n implies that such a function has an n-th root. Hence $H^1(\mathbb{B}, \mathbb{Z}/n\mathbb{Z}_{\mathbb{B}}) = 0$.

Next, we want to investigate a ring domain (or annulus) $\partial\mathbb{B} := \{z \in k; |z| = 1\}$. Again $Pic(\partial\mathbb{B}) = 0$. Further every invertible function on $\partial\mathbb{B}$ has uniquely the form $\lambda z^s(\sum_m a_m z^m)$ where $\lambda \in k^*$, $s \in \mathbb{Z}$ and where the Laurent series satisfies $a_0 = 1$, $|a_m| < 1$ for all $m \neq 0$ and $\lim |a_m| = 0$. It follows that $H^1(\partial\mathbb{B}, \mathbb{Z}/n\mathbb{Z}_{\partial\mathbb{B}}) = O(\partial\mathbb{B})^*/O(\partial\mathbb{B})^{*n} = \mathbb{Z}/n\mathbb{Z}$, a generator is given by the class of z. Clearly this is independent of the base field k.

Now we start proving the general statement. The pre-sheaves $U \mapsto H^i(U_{etale}, \mathbb{Z}/n\mathbb{Z}_C)$ are overconvergent on C_{rigid}. Hence it suffices to prove the statement for all wide neighborhoods X' of X in C. For such an X' we can find closed unit discs $B_i \subset D_i$ such that $X' \cap B_i$ is isomorphic to a ring domain $\partial\mathbb{B}$. If we have this then the covering

$$C = X' \cup \bigcup B_i$$

will be admissible. In particular it is also an étale covering of C. Therefore, we have the Mayer-Vietoris sequence exact sequence [M80, p. 110]

$$\begin{aligned}0 \longrightarrow H^1(C, \mathbb{Z}/n\mathbb{Z}_C) &\longrightarrow H^1(X', \mathbb{Z}/n\mathbb{Z}_C) \oplus \bigoplus H^1(B_i, \mathbb{Z}/n\mathbb{Z}_C)\\ &\longrightarrow \bigoplus H^1(X' \cap B_i, \mathbb{Z}/n\mathbb{Z}_C) \longrightarrow H^2(C, \mathbb{Z}/n\mathbb{Z}_C) \longrightarrow 0.\end{aligned}$$

The zero on the right follows by the vanishing of H^2 on affinoid curves proved above and the zero on the left is trivial to establish. The result follows by the computation of cohomology of $\mathbb{B}$ and $\partial\mathbb{B}$ given above.

For a precise definition of the map $H^q(X, \mathbb{Z}/n\mathbb{Z}_X) \to H^q(X_K, \mathbb{Z}/n\mathbb{Z}_{X_K})$ see 5.1 below. The invariance of cohomology under extension of base field for X follows from the invariance of cohomology for the algebraic curve C and the spaces $\mathbb{B}$, resp. $\partial\mathbb{B}$. □

REMARK 4.2.2 Let $k_0 \subset k$ denote a complete subfield of k such that k is the completion of the separable algebraic closure of k_0. Suppose that the g.c.d.$(n,p) = 1$. Let $\partial\mathbb{B}$ be the ring domain $\{z \in k_0; |z| = 1\}$ over k_0. The Galois action (see 3.6) on $H^1(((\partial\mathbb{B})\hat{\otimes}_{k_0} k), \mu_n) = \mathbb{Z}/n\mathbb{Z}$ is trivial. By the proof above this cohomology group is canonically isomorphic to $O(\partial\mathbb{B}\hat{\otimes}_{k_0} k)^*/O(\partial\mathbb{B}\hat{\otimes}_{k_0} k)^{*\,n}$. The generator of this group is the class of the invertible function z. This is clearly invariant under the Galois group.

REMARK 4.2.3 The open unit disc D is the increasing union of closed discs. Thus we see, by the argument that proved part 2 of the proposition, that $H^q(D, \mathbb{Z}/n\mathbb{Z}_D) = 0$ for $q \geq 1$. This result is partially generalized in the corollary below.

COROLLARY 4.2.4 *Let $\mathcal{L}$ be a compact subset of $\mathbb{P}^1_k$ and put $X = \mathbb{P}^1_k \setminus \mathcal{L}$. Then $H^1(X, \mathbb{Z}/n\mathbb{Z}_X)$ coincides with the group of $\mathbb{Z}/n\mathbb{Z}$-valued currents on the tree of X (or the tree of $\mathcal{L}$). More generally, for any connected open subspace X of $\mathbb{P}^1_k$, the group $H^1(X, \mathbb{Z}/n\mathbb{Z}_X)$ is equal to $O(X)^*/O(X)^{*n}$.*

Proof. The line bundles on any open subspace X of $\mathbb{P}^1_k$ are trivial (see [FP]) and hence $H^1(X, \mathbb{Z}/n\mathbb{Z}_X) \cong O(X)^*/O(X)^{*n}$. (Use Lemma 4.1.2.) The structure of the group $O(X)^*$ is well known if $X = \mathbb{P}^1_k - \mathcal{L}$. Namely, there is an exact sequence

$$0 \longrightarrow k^* \longrightarrow O(X)^* \longrightarrow C(T) \to 0,$$

where T denotes the tree of $\mathcal{L}$ and where $C(T)$ denotes the group of currents with values in $\mathbb{Z}$ on T. (See [FP].) It follows that $O(X)^*/O(X)^{*n} = C(T)/nC(T)$ is the group of currents on T with values in $\mathbb{Z}/n\mathbb{Z}$. □

PROPOSITION 4.2.5 *If X is a connected smooth affinoid curve then there is an embedding $X \subset C$ as in Proposition 4.2.1 part 3) above. We deduce from this the following results. Let A be an Abelian torsion group of exponent n, with $(n,p) = 1$.*

1. *There are natural isomorphisms $H^q(X, \mathbb{Z}/n\mathbb{Z}_X) \otimes A \cong H^q(X, A_X)$.*
2. *The cohomology groups $H^q(X, A_X)$ are invariant under algebraically closed extensions of base fields.*

Proof. The existence of such an embedding $X \to C$ is proved in [P80]. The group A is the direct limit of its finite subgroups. Taking cohomology commutes with direct limits (3.4.3), hence it suffices to do the case A is finite. Writing A as the direct sum of cyclic subgroups it follows that we may assume $A \cong \mathbb{Z}/n'\mathbb{Z}$ where $n'|n$. In this case both 1) and 2) follow easily from Proposition 4.2.1. □

REMARK 4.2.6 Other constant sheaves.

1. The cohomology of $\mathbb{Q}_X$. Since in this case the rigid sheaves $R^q r_* \mathbb{Q}_X$ for $q \geq 1$ are both torsion (by 3.3.2 and 3.4.6) and sheaves of $\mathbb{Q}$-vector spaces, they are zero. Hence we have
$$H^q(X, \mathbb{Q}_X) \cong H^q(X_{rigid}, \mathbb{Q})$$

for all q. If X is a separated quasi-compact smooth curve then we have

$$H^1(X, \mathbb{Q}_X) = H^1(X_{rigid}, \mathbb{Q}) = \mathbb{Q}^b,$$

where b is the Betti number of the graph of a semi-stable reduction of X. A semi-stable reduction of X is defined as follows: take a separated formal scheme $\mathfrak{X}$ of finite type over $\mathrm{Spf}(k^\circ)$, whose associated rigid space $\mathfrak{X}^{rig}$ is isomorphic to X. (See [R74] or [BL].) By blowing up $\mathfrak{X}$ a bit we may assume that the singularities of the special fibre are ordinary double points. This special fibre is a semi-stable reduction of X. Since any other such formal scheme $\mathfrak{X}'$ may be compared with $\mathfrak{X}$ by a sequence of blow ups and blow downs in points it follows that the associated graphs have the same homotopy type. The result now follows from Remark 2.5.9 and a computation of the Zariski cohomology of a constant sheaf on an algebraic semi-stable curve.

2. The constant sheaf $\mathbb{Z}/p\mathbb{Z}_X$.

   (a) Let the characteristic of k be $p > 1$. We will give a calculation of $H^1(\mathbb{B}, \mathbb{Z}/p\mathbb{Z}_X)$ where $\mathbb{B}$ is the closed unit disk. Consider the Artin-Schreier exact sequence

   $$0 \longrightarrow \mathbb{Z}/p\mathbb{Z}_{\mathbb{B}} \longrightarrow \mathbb{G}_a \xrightarrow{\phi} \mathbb{G}_a \longrightarrow 0$$

   on $X_{\acute{e}tale}$. Here of course $\phi(f) = f^p - f$. On cohomology we get an exact sequence

   $$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow O(\mathbb{B}) \xrightarrow{\phi} O(\mathbb{B}) \longrightarrow H^1(\mathbb{B}, \mathbb{Z}/p\mathbb{Z}_{\mathbb{B}}) \longrightarrow 0,$$

   since $H^1(\mathbb{B}, \mathbb{G}_a) = (0)$ by 3.2.5. The co-kernel of $\phi : O(\mathbb{B}) \to O(\mathbb{B})$ is a rather large group and is not invariant under algebraically closed extensions of base fields. This reflects the fact that the closed disk has many p-cyclic un-ramified coverings.

   (b) Here the characteristic of k is zero, but the characteristic of the residue field $\tilde{k}$ is $p > 1$. With the methods above it follows that $H^1(\mathbb{B}, \mathbb{Z}/p\mathbb{Z}_{\mathbb{B}}) = O(\mathbb{B})^* / O(\mathbb{B})^{*p}$. This is again a very large group not invariant under base field extensions. It can be shown that every (algebraic) finite étale covering of the affine line over $\tilde{k}$ lifts to a finite étale covering of $\mathbb{B}$. The conjecture of S.S.Abhyankar on the coverings of the affine line in characteristic p (proved by M.Raynaud) implies that the totality of nontrivial finite étale coverings of $\mathbb{B}$ is very large.

5 Base change revisited

In this section we prove a general base change theorem for quasi-compact morphisms and overconvergent étale sheaves. In order to be able to apply Theorem 3.7.3 we have to prove invariance of cohomology under extensions of algebraically closed base fields. This is done below for rigid and étale cohomologies and overconvergent sheaves.

5.1 A change of fields, étale case.

Let $k \subset K$ be an extension of complete and algebraically closed fields. For any analytic space X over k we denote by $p_K : X_K = X \hat{\otimes}_k K \to X$ the general morphism associated to the change of fields. For an étale sheaf S on X we have the étale sheaf $p_K^* S$ on X_K and comparison maps

$$H^q(X, S) \longrightarrow H^q(X_K, p_K^* S).$$

We would like to know when these are isomorphisms. As before we put $p \geq 1$ equal to the characteristic of the residue field of k (and $p = 1$ if $\mathrm{char}(\tilde{k}) = 0$).

Theorem 5.1.1 *The canonical maps $H^q(X, S) \longrightarrow H^q(X_K, p_K^* S)$ are isomorphisms if S satisfies the following conditions:*

1. *The sheaf S is overconvergent.*
2. *All étale stalks S_e of S are torsion groups, with torsion prime to p.*

Proof. By taking an admissible affinoid covering of X we see that it suffices to do the case that X is affinoid. Let us consider S as a sheaf on the site $X^{sp}_{\acute{e}tale}$. For any $n \in \mathbb{N}$ with $(n, p) = 1$ let $S_n \subset S$ be the subsheaf of S consisting of sections annihilated by n, i.e., $S_n := \mathrm{Ker}(S \stackrel{n}{\to} S)$ is also overconvergent. By our two conditions on S and Lemma 3.3.1 we see that any section $s \in S(Y)$ is torsion ($Y \to X$ is special étale, hence Y affinoid, hence quasi-compact). Thus we see that $S = \bigcup S_n$. By looking at stalks we see that $(p_K^* S)_n = p_K^* S_n$, hence also $p_K^* S = \bigcup p_K^* S_n$. Since cohomology commutes with direct limits (3.4.3) it suffices to do the case that S is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules.

Consider fields L with $k \subset L \subset K$, which are complete and algebraically closed. We say that L has topological transcendence degree $\leq r$ over k if there exist elements $t_1, \ldots, t_r \in L$ such that L is the completion of the algebraic closure of $k(t_1, \ldots, t_r)$. The reasoning of Lemma 3.7.1 shows that the site $(X_K)^{sp}_{\acute{e}tale}$ is the direct limit of the sites $(X_L)^{sp}_{\acute{e}tale}$, taken over all L of finite topological transcendence degree over k. Therefore it suffices to prove $H^q(X, S) = H^q(X_L, p_L^* S)$ for $k \subset L$ of topological transcendence degree $\leq r$. By induction on r it suffices to do the case: $k \subset K$ of topological transcendence degree 1.

Take an element $t \in K$ such that K is the completion of the algebraic closure of $k(t)$. We may assume that $|t| \leq 1$. Consider the continuous k-algebra homomorphism $k\langle T \rangle \to K$ mapping T to t. This determines an étale point e of the closed unit disc $\mathbb{B}$ over k with $F_e = K$.

The problem we are studying may now be formulated with the help of the following diagram of analytic spaces and general morphisms.

$$\begin{array}{ccccc} X_K & \longrightarrow & X \times \mathbb{B} & \xrightarrow{p_1} & X \\ \downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q_2} \\ Sp(K) & \xrightarrow{e} & \mathbb{B} & \xrightarrow{q_1} & Sp(k). \end{array}$$

There is a general base change morphism (see [SGA 4, Exp. XVII 4.1.5])

$$q_1^* R^q (q_2)_* S \longrightarrow R^q (p_2)_* p_1^* S \qquad (1).$$

The comparison map from the theorem is the base change map for the big rectangle of the diagram. Our étale base change theorem 3.7.3 asserts that the base change map for the left square with p_1^*S is an isomorphism. By the functoriality properties of the base change morphism it will suffice to prove that (1) is an isomorphism.

Let $V \to \mathbb{B}$ be étale and V affinoid. Put $p : X \times V \to X$ equal to the projection. We write $H^m(V) = H^m(V, \mathbb{Z}/n\mathbb{Z}_V)$ and $H^m(V)_X$ is the constant sheaf with fibre $H^m(V)$ on $X_{\acute{e}tale}$. There is a natural map

$$H^m(V)_X \otimes_{\mathbb{Z}/n\mathbb{Z}} S \longrightarrow R^m p_* p^* S. \tag{2}$$

It is the composition

$$H^m(V)_X \otimes S \longrightarrow R^m p_*(\mathbb{Z}/n\mathbb{Z}_{X\times V}) \otimes S \longrightarrow R^m p_* p^* S,$$

the first map given by base change, the second deduced from $S \to p_*p^*S$ by the cup-product $R^m p_*(\mathbb{Z}/n\mathbb{Z}_{X\times V}) \otimes R^0 p_* p^* S \to R^m p_* p^* S$ associated to $\mathbb{Z}/n\mathbb{Z}_{X\times V} \otimes p^*S \to p^*S$. By étale base change 3.7.3 the stalk of $R^m p_* p^* S$ in the étale point f of X is $H^m(V\hat{\otimes}F_f, (S_f)_{V\hat{\otimes}F_f})$. Hence, by Proposition 4.2.5 (2) is an isomorphism on étale stalks for all f. Since both sides of (2) are overconvergent we get that (2) is an isomorphism. Thus we get that $R^m p_* p^* S = (0)$ for $n \geq 2$. Finally, since $p : X \times V \to X$ has a section, the maps $H^m(X, S) \to H^m(X \times V, p^*S)$ have sections. We conclude that the spectral sequence $H^i(X, R^j p_* p^* S) \Rightarrow H^{i+j}(X \times V, S)$ degenerates and gives:

$$\begin{array}{rcccc} H^m(X \times V, S) & \cong & H^m(X, H^0(V)_X \otimes S) & \oplus & H^{n-1}(X, H^1(V)_X \otimes S) \\ & \cong & H^0(V) \otimes H^m(X, S) & \oplus & H^1(V) \otimes H^{n-1}(X, S) \end{array}$$

Therefore, the sheaf associated to the presheaf $V \mapsto H^m(X \times V, S)$ (on $\mathbb{B}^{sp}_{\acute{e}tale}$) is the constant sheaf with fibre $H^m(X, S)$. Clearly this means that the right side of (1) is constant and hence that (1) is an isomorphism (look at fibres over $0 \in \mathbb{B}$). □

5.2 A change of fields, rigid case.

We think it is quite amusing that a similar theorem also holds for the rigid case. Notations are as in 5.1.

Theorem 5.2.1 *Let S be an overconvergent sheaf on X_{rigid}. The canonical maps*

$$H^q(X_{rigid}, S) \longrightarrow H^q((X_K)_{rigid}, p_K^* S)$$

are isomorphisms.

Proof. As in the proof of the étale case we may assume that X is affinoid and $k \subset K$ of topological transcendence degree 1 (using X^{rat}_{rigid} in stead of $X^{sp}_{\acute{e}tale}$). We consider subfields $L \subset K$, which are complete and are the completion of a function field of transcendence degree 1 over k. In this case we remark that $(X_K)^{rat}_{rigid}$ is the direct limit of the sites $(X_L)^{rat}_{rigid}$ for such fields L. Again it suffices to do the case $K = L$. (The field K is no longer algebraically closed!)

Suppose Z is a nonsingular projective irreducible curve over k, whose function field $k(Z)$ is a dense subfield of K. The embedding $k(Z) \to K$ defines an analytic point a of Z with $F_a = K$.

OBSERVATION 5.2.2 *There is an affinoid subdomain $U \subset Z$ in the filter of a with the following property: For every affinoid $V \subset U$ and any constant sheaf T on U the cohomology groups $H^n(V, T)$ are zero for $n \geq 1$.*

This follows quite easily from the stable reduction of Z. As was noted in Remark 4.2.6 part 1) the cohomology of a rigid constant sheaf on an affinoid smooth curve depends only on the Betti number of the graph of its stable reduction. Thus we take for U the pre-image of a Zariski open W of the stable reduction of Z, such that W contains no cycles. The assertion of the observation then holds for $V = U$. But also for any such $V \subset U$ it holds, since this corresponds to a Zariski open part in a blow up of the stable model of Z. Blow ups do not introduce extra cycles.

The rest of the proof of the theorem is similar to the proof of Theorem 5.1.1: just replace $\mathbb{B}$ by U and étale by rigid cohomology. □

REMARK 5.2.3 Both theorems are false when k is not algebraically closed. Just take $X = \mathrm{Sp}(k')$ where $k \subset k'$ is a finite Galois extension and $S = \mathbb{Z}/n\mathbb{Z}_X$. In this case $H^0(X, S) = \mathbb{Z}/n\mathbb{Z}$ and $H^0(X_K, p_K^* S) = (\mathbb{Z}/n\mathbb{Z})^{[k':k]}$. Even if X is a geometrically connected smooth projective curve and S is a constant sheaf the result is false in general. (Both rigid and étale case.)

COROLLARY 5.2.4 *Suppose that S is an overconvergent sheaf of $\mathbb{Z}[1/p]$-modules on $X_{étale}$. The canonical comparison maps $H^q(X, S) \to H^q(X_K, p_K^* S)$ are isomorphisms.*

Proof. There is an exact sequence

$$0 \longrightarrow S_{tors} \longrightarrow S \longrightarrow S \otimes \mathbb{Q} \longrightarrow Q \longrightarrow 0$$

By Theorem 5.1.1 the result is true for both S_{tors} and Q. Since $H^q(X, S \otimes \mathbb{Q})$ agrees with $H^q(X_{rigid}, r_* S \otimes \mathbb{Q})$ (compare Remark 4.2.6) we see the result is true for $S \otimes \mathbb{Q}$ also by the theorem above. The snake lemma gives the result for the sheaf S. □

5.3 QUASI-COMPACT BASE CHANGE.

By a combination of our previous results we can now prove a general base change theorem for quasi-compact morphisms.

THEOREM 5.3.1 *(Quasi-compact base change.) Consider a diagram*

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X \end{array}$$

and an overconvergent sheaf of $\mathbb{Z}[1/p]$-modules S on $Y_{étale}$. Here f is a quasi-compact morphism of analytic varieties over k and g is a (arbitrary) general morphism of analytic varieties. The base change morphism [SGA 4, Exposé XVII]

$$g^* R f_* S \longrightarrow R f'_* (g')^* S$$

is a quasi-isomorphism.

Proof. We only have to show that this morphism induces an isomorphism on the étale stalks of the overconvergent sheaves $g^*R^qf_*S$ and $R^qf'_*(g')^*S$. Any étale point e' of Z lies over an étale point e of X, i.e., such that $F_e \subset F_{e'}$. By the étale base change theorem 3.7.3 the map on the stalks is the map 5.1 between

$$\left(g^*R^qf_*S\right)_{e'} = \left(R^qf_*S\right)_e = H^q(Y_e, S|_{Y_e})$$

and

$$\left(R^qf'_*(g')^*S\right)_{e'} = H^q(Y_{e'}, (g')^*S|_{Y_{e'}}) = H^q(Y_e\hat{\otimes}F_{e'}, p^*_{F_{e'}}(S|_{Y_e})).$$

The statement follows from Corollary 5.2.4. □

6 The axioms for cohomology

Let k be a complete valued field. An 'abstract' cohomology theory for rigid analytic spaces over k is defined in [S-S, section 2] to be a cohomology theory $X \mapsto H^*(X)$ satisfying four axioms. There is also given a candidate for such a cohomology theory. Let A be a finite ring of order prime to the residue field of k. Let K be the completion of the algebraic closure of k. We put

$$H^*(X) := H^*(X\hat{\otimes}K, A_{X\hat{\otimes}K}).$$

As remarked in [S-S, p. 58], the nontrivial axioms to check in this case are the 'homotopy axiom' and the axiom concerning the cohomology of the projective space. In this section we will prove those axioms.

The homotopy axiom states that $H^*(X \times D) \cong H^*(X)$ for an open disc D. This follows immediately from the following theorem.

Theorem 6.0.2 *(The homotopy axiom.) Let X be an analytic space over k. Let S be an overconvergent sheaf of $\mathbb{Z}[1/p]$-modules on $X_{étale}$. Suppose D is an open or closed disc over k; let $p : X \times D \to X$ denote the projection. The canonical maps $H^q(X, S) \to H^q(X \times D, p^*S)$ are isomorphisms.*

Proof. If the disc D is open then it is the admissible union $D = \bigcup B_n$ of closed discs B_n of radius $\rho_n \in \sqrt{|k^*|}$. The covering $X \times D = \bigcup X \times B_n$ is the also admissible. Therefore, it suffices to prove the theorem for a closed disc B.

In this case we prove that $p_*p^*S \cong S$ and that $R^qp_*p^*S = (0)$ for $q \geq 1$. By the étale base change theorem the étale stalk at e of these sheaves are equal to $H^q(B\hat{\otimes}F_e, (S_e)_{B\hat{\otimes}F_e})$. Note that $B\hat{\otimes}F_e \cong \mathbb{B}$, the closed unit disc of radius 1 over F_e. If we prove that $H^q(\mathbb{B}, A_{\mathbb{B}}) = (0)$ for $q \geq 1$ for any $\mathbb{Z}[1/p]$-module A then we are done. A standard argument, compare with 4.2.5, reduces to the cases $A = \mathbb{Q}$ or $A = \mathbb{Z}/n\mathbb{Z}$. These cases where done in Remark 4.2.6 and Proposition 4.2.1. □

For the formulation of the following theorem, we need to be more precise about the Galois action on the cohomology groups. Let K denote the completion of the separable closure k^{sep} of k. The symbol $\mathcal{G} = \mathrm{Gal}(k^{sep}/k)$ denotes the continuous Galois group of K over k. See 3.6. The finite ring A is given the trivial $\mathcal{G}$ action. For any $i \in \mathbb{Z}$ we define $A(i) := A \otimes \left(\mu_n(K)\right)^{\otimes i}$ as a $\mathcal{G}$-module, where $n = \#\mathcal{G}$. The following result also follows from the comparison theorem in the following section.

THEOREM 6.0.3 *(Cohomology of $\mathbb{P}^d$.) Let $\mathbb{P}^d$ denote the d-dimensional projective space over k. We have as Galois modules*

$$H^q(\mathbb{P}^d) = H^q(\mathbb{P}^d_K, A_{\mathbb{P}^d_K}) = \begin{cases} A(-q/2) & \text{for } q \text{ even, } 0 \le q \le 2d, \\ (0) & \text{otherwise.} \end{cases}$$

Proof. The calculation of the cohomology is done by applying the Mayer-Vietoris sequence to the covering $\{U_0, U_1\}$ of $\mathbb{P}^d$ given by $U_0 = \{(z_0; ...; z_d) | \ |z_j| \le |z_0| \text{ for all } j\}$ and $U_1 = \{(z_0; ...; z_d) | \ |z_0| \le max(|z_1|, ..., |z_d|)\}$. The space U_0 is a product of disks and has therefore trivial cohomology. The spaces U_1 and $U_0 \cap U_1$ respectively, admit a surjective morphism to $\mathbb{P}^{d-1}$ given by $(z_0; ...; z_d) \mapsto (z_1; ...; z_d)$. The fibres are disks or ring domains respectively and the fiberings are locally trivial. Base change, with respect to the first map, yields $H^i(U_1) \xrightarrow{\sim} H^i(\mathbb{P}^{d-1})$. Base change applied to the second map gives rise to an exact sequence

$$0 \longrightarrow H^i(U_1) \longrightarrow H^i(U_0 \cap U_1) \longrightarrow H^{i-1}(\mathbb{P}^{d-1})(-1) \longrightarrow 0.$$

The (-1) in the last cohomology group is a consequence of the Galois action on the cohomology of a ring domain. See 4.2.2. Induction on d and the Mayer-Vietoris sequence imply that $H^i(\mathbb{P}^d) \cong H^{i-2}(\mathbb{P}^{d-1})(-1)$ for $i \ge 2$ and the expected values of H^0 and H^1. □

7 PURITY AND COMPARISON

Let X be a scheme of finite type over the complete valued field k. We write $X_{ét}$ for the small étale site of the scheme X. Further, X^{an} denotes the rigid analytic variety associated to X. There is a morphism of sites

$$\epsilon : X^{an}_{étale} \longrightarrow X_{ét}$$

comparing the algebraic and rigid étale sites. It is given by the functor that associates to the scheme Y étale over X the analytic space Y^{an} étale over X^{an}. We want to compare sheaves on both sides and their cohomology. It will turn out that if the characteristic of k is zero then we get results as proved in [SGA 4] comparing étale cohomology and classical cohomology over $\mathbb{C}$. However, if the characteristic is $p > 1$, only a weaker version holds. We will give counterexamples for the full statement.

In order to prove the statements above we use a purity result for rigid étale cohomology. It tells us what the cohomology of the complement of a smooth divisor in a smooth rigid analytic variety is.

7.1 A PRELIMINARY RESULT

We start by proving that sheaves of the form ϵ^*S are overconvergent.

LEMMA 7.1.1 *For any sheaf S on $X_{ét}$ the sheaf ϵ^*S is overconvergent.*

Proof. Let $Y \to X$ be an algebraic étale morphism, with Y affine. We also denote by Y the sheaf on $X_{ét}$ it defines. We only need to show that the sheaf $\epsilon^*(Y)$ is overconvergent. (The sheaves Y generate the category of sheaves on $X_{ét}$ and the direct limit of overconvergent sheaves is overconvergent.) By Zariski's main theorem we can embed Y as a Zariski open set in a scheme $\overline{Y}$ finite over X. Suppose $U \subset X^{an}$ is an affinoid subdomain and $V \to U$ is an étale morphism of affinoids. The notation $V(r) \to U$ is as in 3.4. We have to show that any morphism $\varphi : V \to Y^{an}$ over X^{an} extends (uniquely) to some $V(r) \to Y^{an}$. See 3.4.4. By Lemma 3.4.2 it suffices to show that $\varphi(V) \subset\subset_U Y^{an} \times_{X^{an}} U$. Clearly, we have that $\varphi(V) \subset\subset_U \overline{Y}^{an} \times_{X^{an}} U$, since the last space is finite over U. The result follows since $Y^{an} \times_{X^{an}} U$ is Zariski open in $\overline{Y}^{an} \times_{X^{an}} U$. See for example [S93, §3 Proposition 3]. □

7.2 Purity for rigid étale cohomology

Let $i : Z \to X$ be a closed immersion of analytic varieties over k. Let $U = X \setminus Z$ denote the admissible open subvariety of X which is the complement of Z. As usual $j : U \to X$ denotes the open immersion of U into X. We want to prove that sheaves on $Z_{étale}$ correspond to sheaves S on $X_{étale}$ such that j^*S is a final object in the category of sheaves on U, i.e., a sheaf which has exactly one section over each object of $U_{étale}$. This means that the category of sheaves on Z can be viewed as the closed sub-topos of $X^{\sim}_{étale}$ complementary to the open sub-topos $U^{\sim}_{étale}$. Compare [SGA 4, Exposé IV 9.3.5]. Although this follows easily for overconvergent sheaves, we need the result in general for the proof of purity below. It implies in particular that $Ri_*\mathcal{F} \cong i_*\mathcal{F}$ for any Abelian sheaf $\mathcal{F}$ on $Z_{étale}$.

Lemma 7.2.1 *The functor i_* identifies the category of sheaves (of sets) on Z with the category of sheaves S on X such that j^*S is a final object of $U^{\sim}_{étale}$.*

Proof. Let us take an admissible affinoid covering $X = \bigcup X_i$ of X and admissible affinoid coverings $X_i \cap X_j = \bigcup X_{ijk}$. Any sheaf on X is given by sheaves on X_i glued on the X_{ijk}, whereas a sheaf on Z (resp. U) is given by sheaves on $Z \cap X_i$ (resp. $U \cap X_i$) glued on the $Z \cap X_{ijk}$ (resp. $U \cap X_{ijk}$). In this way one reduces to the case that X is affinoid.

In this case we work with the sites $X^{sp}_{étale}$ and $Z^{sp}_{étale}$. The functor $X^{sp}_{étale} \to Z^{sp}_{étale}$ is denoted $W \mapsto W_Z = Z \times_X W$. Consider the following statements:

1. For any étale $W_0 \to Z$, W_0 affinoid, there exists an étale $W \to X$ morphism of affinoids such that $W_0 \cong W_Z$.
2. If $V, W \in X^{sp}_{étale}$ and $\phi_0 : W_Z \to V_Z$ is a morphism over Z then there is a Weierstrass domain $W' \subset W$ with $W'_Z = W_Z$ and a morphism $\phi : W' \to V$ lifting ϕ_0.
3. If $W \in X^{sp}_{étale}$ then any special étale covering $\{W_{i,0} \to W_Z\}$ may be lifted to a special étale covering of W.

Let us first prove that these imply the lemma.

We denote by e a final object of $(U_{étale})^{\sim}$. Further for any sheaf S on X we denote by $P(S)$ the presheaf on $Z^{sp}_{étale}$ defined by the formula:

$$P(S)(W_0) = \varinjlim_{V, \phi_0 : W_0 \to V_Z} \Gamma(V, S)$$

By definition, i^*S is the sheaf associated to the presheaf $P(S)$. It is clear from 1) and 2) above that $P(S)$ may be described as follows

$$P(S)(W_Z) = \varinjlim_{W' \subset W \text{ as in } 2)} \Gamma(W', S).$$

Such a subdomain W' is automatically a wide neighborhood of W_Z in W, since W_Z is closed in W. Therefore there exists a special subset $V \subset W$, disjoint with W_Z such that $W = W' \cup V$. This means that if S has the property that $j^*S \cong e$ then $\Gamma(W, S) = \Gamma(W', S)$ since both $\Gamma(V, S)$ and $\Gamma(V \cap W', S)$ consist of one element. In particular we see that for such S we have $P(S)(W_Z) = \Gamma(W, S)$. Property 3) above implies that $P(S)$ is a sheaf in this case and hence $i^*(S) = P(S)$. It follows immediately that $i_*i^*S \cong S$ for such sheaves S.

Conversely, if $\mathcal{F}$ is a sheaf on Z, it is immediate that $j^*i_*\mathcal{F} \cong e$. Hence by the above we have that $i^*i_*\mathcal{F} = P(i_*\mathcal{F})$ and

$$\Gamma(W_Z, i^*i_*\mathcal{F}) = \Gamma(W, i_*\mathcal{F}) = \Gamma(W_Z, \mathcal{F}).$$

We have proved that i_* and i^* are mutually inverse functors defining the desired equivalence of categories.

Let us prove 1). Let $Y_0 \to Z$ be an étale morphism of affinoids. We can choose a presentation (see Lemma 3.4.1)

$$O(Y_0) = O(Z)\langle T_1, \ldots, T_n\rangle/(G_1, \ldots, G_n)$$

with $G_1, \ldots, G_n \in O(Z)[T_1, \ldots, T_n]$ such that the determinant $\Delta_0 = \det(\partial G_j/\partial T_i)$ is invertible in $O(Z)$. Let us lift the polynomials G_i to polynomials $F_i \in O(X)[T_1, \ldots, T_n]$. Put $\Delta = \det(\partial F_j/\partial T_i)$. Take $\pi \in k^*$ such that $|\pi|$ is smaller than the infimum of $|\Delta_0|$ on Z. We consider the algebra

$$O(X)\langle T_0, T_1, \ldots, T_n\rangle/(F_1, \ldots, F_n, \Delta T_0 - \pi).$$

This defines a special étale morphism $Y \to X$ since the corresponding functional determinant is Δ^2, which is invertible. The isomorphism $Y_0 \cong Z \times_X Y$ follows by construction.

The proof of 2) is similar to the proof of 2) in Lemma 3.7.1. We consider the product $V \times_X W$ and the graph morphism $\Gamma_0 : W_Z \to (V \times_X W)_Z$. This morphism is étale. By 1) (with $W \times_X V$ in stead of X) we can find $\Gamma : Y \to V \times_X W$ such that $Y_Z \cong W_Z$ and $\Gamma_Z = \Gamma_0$. Next argue as in the proof of 3.7.1 to see that there is some wide neighborhood $W' \subset W$ of W_Z such that $\mathrm{pr}_1 \circ \Gamma : \Gamma^{-1}(V \times_X W') \to W'$ is an isomorphism. Thus we get $W' \to V$. Finally, the Weierstrass domains in W are cofinal in the set of neighborhoods of W_Z in W. To see this apply the rigid key lemma to a morphism $f : W \to \mathbb{B}^n$ with $W_Z = f^{-1}(0)$.

For 3) we first note that by 1) we may lift each of the special étale $W_{i,0} \to W_Z$ to special étale $f_i : W_i \to W$. The special subset $\bigcup f_i(W_i)$ is a neighborhood of W_Z in W, hence a wide neighborhood, hence there exists some special $V \subset W$ such that $V \cap W_Z = \emptyset$ and $W = V \cup \bigcup f_i(W_i)$. Write $V = \bigcup V_j$ as a finite union of rational subdomains of W, then the special étale covering of W we are looking for is the covering $\{W_i \to W, V_j \to W\}$. $\square$

Next, we prove some kind of purity in the rigid étale case. Let X be a smooth rigid variety over k. let $i : H \to X$ be a closed immersion, with H smooth over k and everywhere in X of co-dimension 1. Thus it is a smooth divisor in X. Let U denote the admissible open subset $X \setminus H$ of X and let j denote the open immersion $j : U \to X$.

Theorem 7.2.2 *(Purity.) With the notations as above and with n prime to the characteristic of k we have*

$$R^q j_* \mathbb{Z}/n\mathbb{Z}_U = \begin{cases} \mathbb{Z}/n\mathbb{Z}_X & q = 0 \\ i_*(\mu_n^{\otimes -1}) & q = 1 \\ (0) & q \geq 2. \end{cases}$$

Proof. The statement is local on X, hence we may assume X affinoid. Locally on X (in the Zariski topology) the ideal of H is generated by a single function, hence we may assume that H is given as $f = 0$ for some $f \in O(X)$. By [K68] we can find an affinoid neighborhood of H in X which has an admissible covering by affinoids of the form $H_i \times \mathbb{B}$. Here $\mathbb{B}$ is the closed unit ball over k with coordinate z. Thus we may assume that $X = H \times \mathbb{B}$ and $U = H \times \mathbb{B}^*$ where $\mathbb{B}^*$ is the punctured unit disc. Let us write $\bar{f} : X \to H$ for the projection and $f = \bar{f}|_U$ so that we have the following commutative diagram:

$$\begin{array}{ccccc} U & \xrightarrow{j} & X & \xleftarrow{i} & H \\ & f \searrow & \downarrow \bar{f} & \swarrow \mathrm{id} & \\ & & H & & \end{array}$$

We note that the sheaves $R^q j_* \mathbb{Z}/n\mathbb{Z}_U$ for $q \geq 1$ have are zero restricted to U, hence are of the form $i_* F_q$ for certain sheaves F_q on H (use lemma above). Further, it is clear that $j_* \mathbb{Z}/n\mathbb{Z}_U = \mathbb{Z}/n\mathbb{Z}_X$ on X. We study the spectral sequence associated to the isomorphism $Rf_* \cong R\bar{f}_* \circ Rj_*$. For the sheaf $\mathbb{Z}/n\mathbb{Z}_U$ its E_2-terms are $E_2^{ab} = R^a \bar{f}_* R^b j_* \mathbb{Z}/n\mathbb{Z}_U$ and it abuts to $R^{a+b} f_* \mathbb{Z}/n\mathbb{Z}_U$. In view of the fact that $R^b j_* \mathbb{Z}/n\mathbb{Z}_U = i_* F_b \cong Ri_* F_b$ for $b \geq 1$ (by 7.2.1), we see that $E_2^{ab} = 0$ for $a, b \geq 1$ and $E_2^{0b} = F_b$ for $b \geq 1$. Also we have $E_2^{a0} = R^a \bar{f}_* \mathbb{Z}/n\mathbb{Z}_X$. This is an overconvergent sheaf, whose étale stalks are $H^a(\mathbb{B}, \mathbb{Z}/n\mathbb{Z}_{\mathbb{B}})$, over various algebraically closed base fields. Hence, by Lemma 4.1.2 and since $Pic(\mathbb{B}) = (0)$, we see that $R^a \bar{f}_* \mathbb{Z}/n\mathbb{Z}_X = (0)$ for $a \geq 2$. The upshot of all of this is: 1) we have $R^0 f_* \mathbb{Z}/n\mathbb{Z}_U = \mathbb{Z}/n\mathbb{Z}_H$, 2) there is an exact sequence

$$0 \longrightarrow R^1 \bar{f}_* \mathbb{Z}/n\mathbb{Z}_X \longrightarrow R^1 f_* \mathbb{Z}/n\mathbb{Z}_U \longrightarrow F_1 \longrightarrow 0$$

and 3) there are isomorphisms $R^q f_* \mathbb{Z}/n\mathbb{Z}_U \cong F_q$ for $q \geq 2$.

We have already used the morphism

$$R\bar{f}_* \mathbb{Z}/n\mathbb{Z}_X \longrightarrow Rf_* \mathbb{Z}/n\mathbb{Z}_U.$$

In addition, there is a map

$$\mathbb{Z}/n\mathbb{Z}_H[-1] \longrightarrow Rf_* \mu_n,$$

which associates to $1 \in \mathbb{Z}/n\mathbb{Z}$ the section of $R^1 f_* \mu_n$ corresponding to the μ_n-torsor of $U = H \times \mathbb{B}^*$ given by the equation $y^n = z$. We claim that together these induce a quasi-isomorphism

$$R\bar{f}_* \mathbb{Z}/n\mathbb{Z}_X \oplus \mu_n^{\otimes -1}[-1] \longrightarrow Rf_* \mathbb{Z}/n\mathbb{Z}_U. \tag{1}$$

From the considerations above it follows that this implies the theorem.

To prove the claim we may assume that n is a prime power $n = p^r$. Let us treat the case that p equals the characteristic of k. In this case k is a p-adic field. The other cases are easier and similar arguments work.

Let us take $c \in \mathbb{N}$ large enough. For $m \in \mathbb{N}$ we put $R_m = \{x \in \mathbb{B}; |x| \geq |p^{cm}|\}$, a ring domain. Put $U_m = H \times R_m$, note that the covering $U = \bigcup U_m$ is admissible. Let us write $f_m : U_m \to H$ for the projection. We will study the overconvergent sheaf $R^q f_{m,*}\mu_n$. Its étale fibres are $H^q(R_m \hat{\otimes} F_e, \mu_n)$. These are zero for $q \geq 2$ and equal to μ_n for $q = 0$. Note that

$$\mathbb{P}^1 = \{|z| \geq |p^{cm}|\} \cup \{|z| \leq 1\},$$

hence that we have an exact sequence

$$0 \longrightarrow F_e^* \longrightarrow O^*(|z| \geq |p^{cm}|) \oplus O^*(|z| \leq 1) \longrightarrow O^*(R_m) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

This follows by the computation of cohomology of $\mathbb{P}^1$, use for example Lemma 4.1.2. We see immediately that

$$H^1(R_m \hat{\otimes} F_e) = \mathbb{Z}/n\mathbb{Z} \oplus H^1(|z| \leq 1) \oplus H^1(|z| \geq |p^{cm}|).$$

This implies a corresponding decomposition of the overconvergent sheaf

$$R^1 f_{m,*}\mu_n = \mathbb{Z}/n\mathbb{Z}_H \oplus R^1 \bar{f}_* \mu_n \oplus Rest_m.$$

The maps $Rest_{m+1} \to Rest_m$ are zero, since by [L93, Theorem 2.1] the maps on the étale fibres $H^1(|z| \geq |p^{c(m+1)}|) \to H^1(|z| \geq |p^{cm}) $ are zero if c is large enough.

This means that for any $V \to H$ affinoid étale we have a decomposition

$$H^q(V \times R_m) = H^{q-1}(V, \mu_n^{\otimes -1}) \oplus H^q(V \times \mathbb{B}) \oplus \mathrm{Rest}_m,$$

the transition maps $H^q(V \times R_{m+1}) \to H^q(V \times R_m)$ are the identity on the first two summands, zero on the last one. This proves that

$$\varprojlim H^q(V \times R_m) = H^{q-1}(V, \mu_n^{\otimes -1}) \oplus H^q(V \times \mathbb{B})$$

and the derived limit

$$\varprojlim{}^{(1)} H^q(V \times R_m) = (0).$$

We conclude that $H^q(V \times \mathbb{B}^*) = H^{q-1}(V, \mu_n^{\otimes -1}) \oplus H^q(V \times \mathbb{B})$, hence (1) is an isomorphism. $\square$

7.3 COMPARISON

In this section X will denote a variety of finite type over over the complete valued field k. We state the results corresponding to [SGA 4, Exposé XI Theorem 4.4] in our case. Further, we will indicate the necessary changes in the proof given there so that it will work in our case also.

THEOREM 7.3.1 *Suppose the characteristic of k is zero. There is an equivalence between the category of locally constant sheaves on $X_{ét}$ with finite stalks and the category of locally constant sheaves on $X^{an}_{étale}$ with finite stalks. The equivalence is given by the functors ϵ^* and ϵ_*.*

Proof. Since sheaves of this kind are representable by finite étale coverings we see that it suffices to prove the following statement: If $Y \to X^{an}$ is finite étale then there exists a (unique) finite étale morphism of schemes $Z \to X$ such that $Y \cong Z^{an}$. This was recently proved by Lütkebohmert, see [L93]. □

In the next theorem k is no longer of characteristic zero.

Theorem 7.3.2 *Let X be smooth over* Spec(k) *and let k be algebraically closed. Suppose S is an Abelian locally constant sheaf on $X_{ét}$ with finite stalks where all orders of torsion are prime to the characteristic of k. In this case we have $R^q\epsilon_*\epsilon^*S = (0)$ for $q \geq 1$. The canonical morphisms $H^q(X_{ét}, S) \to H^q(X^{an}_{étale}, \epsilon^*S)$ are isomorphisms. In particular we have*

$$H^q(X_{ét}, \mathbb{Z}/n\mathbb{Z}) \cong H^q(X^{an}, \mathbb{Z}/n\mathbb{Z}_{X^{an}}).$$

Proof. With the results proved above, we can use the proof of [SGA 4, Exposé XI Theorem 4.4 part (ii)]. In stead of the 'calcul direct' of line 1 on page 13 we use Theorem 7.2.2. The only other fact used in the proof which is not immediately clear is the following: Suppose $\bar{f} : \bar{X} \to S$ is a family of smooth projective curves over the scheme S, which is of finite type over k, suppose n is relatively prime to the characteristic of k. In this case $R^1\bar{f}^{an}_*\mathbb{Z}/n\mathbb{Z}_{\bar{X}^{an}}$ is a locally constant sheaf on $S^{an}_{étale}$. However, this is immediately clear from: 1) The corresponding fact in the algebraic case. 2) The base change map $\epsilon^*R^1f_*\mathbb{Z}/n\mathbb{Z} \to R^1\bar{f}^{an}_*\mathbb{Z}/n\mathbb{Z}_{\bar{X}^{an}}$ is an isomorphism (look at étale fibres). □

Remark 7.3.3 The more general results proved in [SGA 4, Exposé XVI §4] should hold true for the rigid analytic case also. At least if the characteristic of k is zero then it should be possible with some effort to follow the reasoning of locus citatus in this case.

7.4 Counterexamples in characteristic $p > 0$.

Take k algebraically closed of characteristic $p > 0$. Riemann's existence theorem is no longer valid in this case. We give an example of this.

Lemma 7.4.1 *Consider the covering $\psi : Y \to \mathbb{A}^{1\ an}$, given by the equation $T^p - T = F := \sum_{i\geq 0} a_i z^{p^i}$, where the series F converges on $\mathbb{A}^{1\ an}$. We suppose that there are infinitely many non zero a_i and that for every $k \geq 0$ one has*

$$|a_k + a_{k-1}^p + a_{k-2}^{p^2} + ... + a_0^{p^k}| = \max{}_{0\leq i\leq k}(|a_i|^{p^{k-i}}).$$

Then Y is not isomorphic to Z^{an} for any covering $Z \to \mathbb{A}^1$.

Proof. Any p-cyclic (un-ramified) covering of the unit disk D is given by an equation $T^p - T = f$ with $f \in O(D)$. Two functions $f_1, f_2 \in O(D)$ define isomorphic coverings if and only if $\lambda_1 f_1 + \lambda_2 f_2 = h^p - h$ holds with $\lambda_1, \lambda_2 \in \mathbb{F}_p^*$ and $h \in O(D)$. Using that the structure sheaf O on the analytic space $\mathbb{A}^{1,an}$ has trivial cohomology, one finds for $\mathbb{A}^{1,an}$ similar results. Namely: Any p-cyclic analytic covering of $\mathbb{A}^{1\ an}$ is given by an equation $T^p - T = f$ with f a holomorphic function on $\mathbb{A}^{1\ an}$. Two holomorphic functions f_1, f_2 define the same p-cyclic extension if and only if there is a $\lambda \in \mathbb{F}_p$ such that the equation $T^p - T = \lambda_1 f_1 + \lambda_2 f_2$ has a holomorphic solution.

If $Y = Z^{an}$ then Z is a p-cyclic covering of $\mathbb{A}^1$ given by an equation of the form $U^p - U = g$ with $g \in zk[z]$. Let the equation $T^p - T = -G := F - \lambda g$ have a holomorphic solution $h = \sum_{i\geq 1} h_i z^i$. In some disk around 0 the spectral norm of G is less than 1. Therefore $\sum_{i\geq 0} G^{p^i}$ converges and is on this disk a solution of $T^p - T = -G$. So h coincides with $\sum_{i\geq 0} G^{p^i}$ on this disk and the power series expansion of h is equal to the power series expansion of $\sum_{i\geq 0} G^{p^i}$. One takes a disc $D(0,R)$ around 0 such that $1 < B := \|G\|_R = \|F\|_R > \|g\|_R$. After replacing z by $z\lambda$ for a suitable $\lambda \in k^*$, we may suppose that $R = 1$. First we look at $\sum_{i\geq 0} F^{p^i} = \sum_{k\geq 0} A_k z^{p^k}$ with $A_k = (a_k + a_{k-1}^p + a_{k-2}^{p^2} + \ldots + a_0^{p^k})$. A calculation shows that for $N >> 0$ one has $|A_N| = |A_{N-1}|^p$ and $|A_N| \geq B$. Then we look at $\sum_{i\geq 0} g^{p^i} = \sum_{k\geq 1} b_k z^k$. One can calculate that the absolute value of b_{p^k} grows less fast than $|A_k|$. This implies that the power series representing h is not convergent on $D(0,1)$. This contradiction ends the proof. □

Corollary 7.4.2 *The map*

$$H^1(\mathbb{A}^1_{et}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(\mathbb{A}^{1\ an}_{etale}, \mathbb{Z}/p\mathbb{Z})$$

is injective but not surjective. In particular Theorem 7.3.2 is not valid for sheaves consisting of p-torsion.

Corollary 7.4.3 *Let $n > 1$ with $p|\phi(n)$ and with n not divisible by p. There is a locally constant sheaf S on $\mathbb{A}^{1\ an}_{étale}$ with stalk $\mathbb{Z}/n\mathbb{Z}$, which is not of the form $\epsilon_* T$.*

Proof. We consider the p-cyclic analytic covering $\psi : Y \to \mathbb{A}^{1\ an}$ of Lemma 7.4.1. Let σ denote the generator of the Galois group G of this extension. Let M denote the constant étale sheaf on Y with stalk $\mathbb{Z}/n\mathbb{Z}$. Let $a \in \mathbb{Z}/n\mathbb{Z}^*$ be an element of order p. We define an action G on $\mathbb{Z}/n\mathbb{Z}$ by $\sigma(i) = ai$. This induces an action of G on $\mathbb{Z}/n\mathbb{Z} \times Y$ by $\sigma((i,y)) = (ai, \sigma(y))$. The quotient by this group action is a sheaf S on $\mathbb{A}^{1\ an}_{étale}$ which is locally the constant sheaf with stalk $\mathbb{Z}/n\mathbb{Z}$. (And of course $\psi^* S$ is the constant sheaf on Y). However, there is no étale covering $\{Y_i\}$ of $\mathbb{A}^1$ which trivializes S. Indeed, such an étale covering would give a trivialization of $Y \to \mathbb{A}^{1\ an}$. □

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